Minimal lamellar structures in a periodic FitzHugh-Nagumo system (Nonlinear Analysis **194** (2020), 111436, 1–13)

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Abstract

A singular limit of a FitzHugh-Nagumo system leads to a nonlocal geometric variational problem with periodic boundary conditions. We study the stationary lamellar set and give a criterion to select out the one with the lowest energy. Such an optimal structure is called a minimal lamella. While the empty set or the full torus is a global minimizer for appropriate parameter regimes, the minimal lamellae beat both in other circumstances. The concept of minimal lamella points out that a preferred 1D mesh size is universal.
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1. Introduction

Pattern formation is of great interest in many fields of science. In physical and biological systems, these patterns are robust in the sense that they are stable and exist for a wide range of parameters. Periodic structures are observed in experiments in certain di-block copolymer melts [15, 18]. Such stationary patterns, resulting from orderly outcomes of self-organization principles, also appeared in various models [2, 13, 17] arisen in material sciences.

Recent advance in mathematical studies leads to a deeper understanding of the self-organized mechanism in the generation of localized structures, typically from a delicate balance between gain and loss in free energy. A simple example is

$$\mathcal{I}(u) = \int_D \Big(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{u^2(u-1)^2}{4} \Big)\, dx, \tag{1.1}$$

which is associated with the Allen-Cahn equation [3]. Replacing $\frac{1}{4}u^2(u-1)^2$ by a unbalanced double-well potential leads to the Nagumo equation. In both cases, lamellar solutions are not stable patterns.

To establish stable lamellar patterns, we need an inhibition mechanism in the free energy that prevents unlimited growth or spreading. A geometric variational problem, derived in [8, 9] from a singular limit of the FitzHugh-Nagumo system, takes the form

$$J(E) = \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2} \int_E \mathcal{N}_E \, dx \tag{1.2}$$

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as an action functional. Here σ and α are two positive parameters, $\mathbb{T}$ is a N-dimensional square torus, which can be thought of as the box $[-T/2, T/2]^N \subset \mathbb{R}^N$ with T-periodic boundary conditions. In (1.2), E is a measurable subset of $\mathbb{T}$ with $|E|$ as its Lebesgue measure. The admissible set of J is

$$\mathcal{A} = \big\{E \subset \mathbb{T} : \ E \text{ is Lebesgue measurable}\big\}. \tag{1.3}$$

The (possibly infinite) perimeter of E in $\mathbb{T}$ is denoted by $\mathcal{P}_{\mathbb{T}}(E)$. If E is of class C^1, $\mathcal{P}_{\mathbb{T}}(E)$ is the length of the boundary of $\partial E \cap \mathbb{T}$. For a general subset E of $\mathbb{T}$, $\mathcal{P}_{\mathbb{T}}(E)$ is finite when the characteristic function χ_E, which takes values 0 or 1, is of bounded variation.

The nonlocal interaction term containing σ represents an inhibition effect that favors an identically zero solution as a minimizer, while α measures the driving force towards a non-zero state. Many interesting patterns emerge from homogeneous media through destabilization by a spatial modulation, and we are interested in the structure of global or local minimizers of (1.2).

For the integral term in (1.2), $\mathcal{N}$ is an operator that assigns each E the solution of the following modified Helmholtz equation:

$$-\Delta \mathcal{N}_E + \mathcal{N}_E = \chi_E \text{ in } \mathbb{T}; \qquad \mathcal{N}_E \text{ is periodic in } \mathbb{T}\ , \tag{1.4}$$

which is also the unique $\mathbb{T}$-periodic minimizer of

$$v \mapsto \int_{\mathbb{T}} \Big(\frac{|Dv|^2}{2} + \frac{v^2}{2} - v\chi_E\Big)\, dx\ . \tag{1.5}$$

Remark 1.1. *We list some easy properties of $\mathcal{N}_E$. By minimality one sees that necessarily $\mathcal{N}_E \geq 0$, and since $\mathcal{N}_{\mathbb{T}\setminus E} = 1 - \mathcal{N}_E$ also that $\mathcal{N}_E \leq 1$. From (1.4), by the divergence theorem one gets*

$$\int_{\mathbb{T}} \mathcal{N}_E\, dx = |E|\ .$$

We thus have, writing E' for the complement $\mathbb{T} \setminus E$ of E,

$$\int_E \mathcal{N}_E\, dx = \int_{\mathbb{T}} \mathcal{N}_E\, dx - \int_{E'} \mathcal{N}_E\, dx = |E| - \int_{E'} (1 - \mathcal{N}_{E'})\, dx = |E| - |E'| + \int_{E'} \mathcal{N}_{E'}\, dx\ . \tag{1.6}$$

This implies

$$J(E) = J(E') + \Big(\frac{\sigma}{2} - \alpha\Big)(|E| - |E'|)\ . \tag{1.7}$$

A classical stationary set of J has a C^2 interface that satisfies the Euler-Lagrange equation

$$\mathcal{K}(\partial E \cap \mathbb{T}) - \alpha + \sigma \mathcal{N}_E = 0 \text{ on } \partial E \cap \mathbb{T} \tag{1.8}$$

where $\mathcal{K}$ denotes the Gaussian curvature, see for example [8, 9].

Remark 1.2. *The trivial states are given by $E = \emptyset$ and $E = \mathbb{T}$, and since the corresponding functions $\mathcal{N}_E$ are the constants* 0 *and* 1, *the trivial state energies are simply*

$$J(\emptyset) = 0\ , \qquad J(\mathbb{T}) = \Big(\frac{\sigma}{2} - \alpha\Big)T^N = \frac{\sigma}{2}\Big(1 - \frac{2\alpha}{\sigma}\Big)T^N\ .$$

It is convenient to name the constant between braces: we set

$$c = c(\alpha, \sigma) := 1 - \frac{2\alpha}{\sigma} \tag{1.9}$$

so the line above reads

$$J(\emptyset) = 0 \ , \qquad J(\mathbb{T}) = \frac{\sigma}{2} c T^N \ ;$$

the sign of the "fullness parameter" c determines whether the empty torus is more (when $c > 0$) or less ($c < 0$) energetically favorable than the full torus, and not only that, we may now rewrite (1.7) as

$$J(E) = J(E') + \frac{\sigma}{2} c(|E| - |E'|) \ . \tag{1.10}$$

This implies the following:

Remark 1.3. *If* $|E| < |\mathbb{T}|/2 < |\mathbb{T} \setminus E|$ *then*

$$\begin{array}{l} \text{if } c > 0, \ \text{then } J(E) < J(\mathbb{T} \setminus E), \\ \text{if } c < 0, \ \text{then } J(\mathbb{T} \setminus E) < J(E). \end{array}$$

Consequently when $c > 0$ *global minimizers of* J *all have measure less than* $|\mathbb{T}|/2$*, and the reverse is true if* $c < 0$*.*

For the moment it is convenient to rewrite J in order to manifest its dependence on c and σ: as $\alpha = \sigma/2 - c\sigma/2$, we have

$$J(E) = \mathcal{P}_{\mathbb{T}}(E) + (c-1)\frac{\sigma}{2}|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E \, dx =: J_{\sigma,c}(E)$$

so that by (1.6)

$$\frac{2}{\sigma}[J_{\sigma,c}(E) - J_{\sigma,-c}(E')] = (c-1)|E| - (-c-1)|E'| + \int_E \mathcal{N}_E \, dx - \int_{E'} \mathcal{N}_{E'} \, dx = c|\mathbb{T}| \ .$$

Set

$$G_{\sigma,c}(E) = J_{\sigma,c}(E) - c\frac{\sigma}{4}|\mathbb{T}|,$$

a functional which shares the same minimizers as J since the difference is only an additive constant, we have the following:

Remark 1.4. *For the functional* G *defined above,*

$$G_{\sigma,c}(E) = G_{\sigma,-c}(\mathbb{T} \setminus E)$$

so that the (global or local) minimizers of J *with* $c > 0$ *are the complements of those with* $c < 0$*.*

The search for global minimizers of J is not easy, but we can draw some conclusions for certain ranges of the parameters. Recall the isoperimetric inequality in a torus in $\mathbb{R}^N$

$$\mathcal{P}_{\mathbb{T}}(E) \geq c_N \min\{|E|, |\mathbb{T} \setminus E|\}^{1-1/N} \tag{1.11}$$

for some positive constant c_N.

Proposition 1.5. *Let* $\sigma > 0$ *and* c_N *be the isoperimetric constant in* (1.11)*. If* $\alpha < \sigma/2$ *satisfies* $\alpha \leq c_N \sqrt[N]{2}/T$ *the unique global minimizer of* J *is the empty set, while if* $\alpha > \sigma/2$ *satisfies* $\alpha \geq \sigma - c_N \sqrt[N]{2}/T$ *the unique global minimizer is the full torus.*

Proof: We first investigate a sufficient condition when the full torus is the absolute minimizer of J; by Remark 1.2 we may confine ourselves to the case $c < 0$, that is $\alpha > \sigma/2$, and consider only sets whose measure is greater than $|\mathbb{T}|/2$.

Writing F for the complement of E we have, by repeatedly using the properties of $\mathcal{N}_E$,

$$\begin{aligned} J(E) - J(\mathbb{T}) &= \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E\, dx + \alpha|\mathbb{T}| - \frac{\sigma}{2}|\mathbb{T}| \\ &= \mathcal{P}_{\mathbb{T}}(F) + \alpha|F| + \frac{\sigma}{2}\left(|E| - |F| + \int_F \mathcal{N}_F\, dx\right) - \frac{\sigma}{2}|\mathbb{T}| \\ &= \mathcal{P}_{\mathbb{T}}(F) + (\alpha - \sigma)|F| + \frac{\sigma}{2}\int_F \mathcal{N}_F\, dx\ . \end{aligned}$$

If $\alpha \geq \sigma$ the sum is strictly positive unless $F = \emptyset$, that is $E = \mathbb{T}$. For $\sigma/2 < \alpha < \sigma$ since $|E| \geq |\mathbb{T}|/2 \geq |F|$, using the isoperimetric inequality one gets

$$\begin{aligned} J(E) - J(\mathbb{T}) &\geq \left[c_N + (\alpha - \sigma)|F|^{1/N}\right]|F|^{1-1/N} + \frac{\sigma}{2}\int_F \mathcal{N}_F\, dx \\ &\geq \left[c_N - (\sigma - \alpha)T/\sqrt[N]{2}\right]|F|^{1-1/N} + \frac{\sigma}{2}\int_F \mathcal{N}_F\, dx \end{aligned}$$

which is positive for $E \neq \mathbb{T}$ provided

$$\alpha \geq \sigma - c_N \sqrt[N]{2}/T\ .$$

The above inequality can be cast as

$$c \leq -1 + 2c_N \sqrt[N]{2}/\sigma T\ . \tag{1.12}$$

Now suppose $c > 0$. In view of Remark 1.4 simply replace c with $-c$ in (1.12) so that

$$c \geq 1 - 2c_N \sqrt[N]{2}/\sigma T\ , \tag{1.13}$$

which yields the sufficient condition $\alpha \leq c_N \sqrt[N]{2}/T$ for the minimality of the empty set. □

Corollary 1.6. *Suppose $c_N \sqrt[N]{2}/T < \sigma/2$, then there exists $\sigma/2 \geq \alpha_\emptyset \geq c_N \sqrt[N]{2}/T$ such that whenever $\alpha \leq \alpha_\emptyset$, a global minimizer of J is the empty set, and if $\alpha > \alpha_\emptyset$, all global minimizers are not the empty set. The empty set is also the unique global minimizer when $\alpha < \alpha_\emptyset$.*

Proof: We show that whenever the empty set is a global minimizer for some $\alpha = \bar{\alpha}$, then the empty set is also the unique global minimizer for all $\alpha < \bar{\alpha}$. Indeed to highlight dependence on α write

$$H_\alpha(E) = J(E) = \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E\, dx\ ;$$

if $H_{\bar{\alpha}}(E) \geq 0$ for every non-empty E, then for all $\alpha < \bar{\alpha}$ we immediately have $H_\alpha(E) > 0$. □

Since $\sigma - c_N \sqrt[N]{2}/T > \sigma/2$ is the same as $c_N \sqrt[N]{2}/T < \sigma/2$, the same reasoning gives:

Corollary 1.7. *Suppose $c_N \sqrt[N]{2}/T < \sigma/2$, then there exists $\sigma/2 \leq \alpha_{\mathbb{T}} \leq \sigma - c_N \sqrt[N]{2}/T$ such that whenever $\alpha \geq \alpha_{\mathbb{T}}$, a global minimizer of J is the full torus, and if $\alpha < \alpha_{\mathbb{T}}$, all global minimizers are not the full torus. The full torus is also the unique global minimizer when $\alpha > \alpha_{\mathbb{T}}$.*

Both the empty set and the full torus are the homogeneous states of (1.2). With suitable parameters α, σ in a large torus, it will be shown that a lamella has a lower energy than both the empty set and the full torus. In Section 2 we employ (1.8) to study with multiple lamellar solutions with the same physical parameters in a fixed torus. Of particular interest is to count the number of lamellae and the distance between them. In fact, among all such solutions there is one with the lowest energy; that is, we seek a k-lamellae in a torus which are optimal among all multi-lamellar configurations. Such a ground state is referred to as a minimal lamellae and is expected to be the most stable pattern among all lamellae. By measuring the thickness of lamellae, one can therefore determine the physical parameters. To the best of our knowledge, the concept of minimal lamellae seems to be new. Detailed analysis will be carried out in Section 3.

The FitzHugh-Nagumo system is a well-known activator-inhibitor type reaction-diffusion model which exhibits rich phenomena [5, 6, 7, 12, 21] giving rise to a great diversity in structures. When the physical parameters are suitably ordered, its stationary solutions are the critical points of the functional

$$\Phi_\epsilon(u) = \int_{\mathbb{T}} \Big(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{u^2(u-1)^2}{4} - \epsilon\alpha u + \frac{\epsilon\sigma u}{2} \mathcal{N}_D u \Big)\, dx, \tag{1.14}$$

here ϵ^2 represents the ratio of diffusivity of the activator u to that of the inhibitor v. Unless they are local minimizers, the patterns associated with these solutions are always unstable [10, 11]. Restricting our focus to minimizers of Φ_ϵ, it is natural to employ the tool of Γ-convergence to aid our analysis. It is known that $\epsilon^{-1}\Phi_\epsilon$ Γ-converges in $L^1(\mathbb{T})$ norm to J in (1.2) as $\epsilon \to 0$; the minimizer u_ϵ of Φ_ϵ then converges to the minimizer u_0 of J in $L^1(\mathbb{T})$ and sharp gradient in u_ϵ becomes a discontinuous jump in u_0 [8, 9, 14]. In a forthcoming paper the minimality of critical points with positive second variation will be studied. The lamellae are one dimensional structures. It is also interesting to demonstrate multi-dimensional stationary configurations in future work.

2. Lamellae

We devote this section to the study of lamellar solutions to (1.8), that is, sets E which beside being $\mathbb{T}$-periodic are also invariant by translations orthogonal to a certain direction $\mathbf{v}$.

Remark 2.1. *For any lamellar set in a 2-dimensional torus,* $\mathbf{v}$ *must have a rational slope* $0 \le m/n \le 1$ *with respect to one of the axes (m and n coprime), and the set is then periodic in the* $\mathbf{v}$ *direction with period* $T/\sqrt{m^2+n^2}$*; the situation is a bit more intricate if the original torus was not a square but a rectangle (unless the slope is zero).*

Temporarily in this section we use $(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$ to denote a point in the torus. To simplify analysis we orient the coordinate axes so that $\mathbf{v}$ points in the x-direction. A lamellar solution is therefore independent of y. Let $0 < x_0 < T$ and let $E = L_{x_0} = [0, x_0] \times [0, T]^{N-1}$ be a single lamella with a thickness x_0 in the torus $\mathbb{T}$. With $\mathcal{P}_{\mathbb{T}}(L_{x_0}) = 2T^{N-1}$, it is immediate that

$$\lim_{x_0 \to 0^+} J(L_{x_0}) = 2T^{N-1} \quad \text{and} \quad \lim_{x_0 \to T^-} J(L_{x_0}) = 2T^{N-1} + J(\mathbb{T}) \, . \tag{2.1}$$

We study multi-lamellar configurations which are stationary points for the energy; later on we need to introduce an energy on a rectangular torus (a slice of the original one), but the changes are easy, so we stick to the square case up to the end, see Remark 2.8. Consider a k-lamellar configuration $\mathbb{L}$, which is composed of k "vertical" lamellae (where $\chi_{\mathbb{L}} = 1$) separated by wedges (where $\chi_{\mathbb{L}} = 0$) and with the first lamella beginning at the left boundary of $\mathbb{T}$. Let the i-th lamella be $[L_i, W_i] \times [0, T]^{N-1}$ so that the i-th wedge is $[W_i, L_{i+1}] \times [0, T]^{N-1}$ with $L_1 = 0$ and $L_{k+1} = T$, and of course $L_i \le W_i \le L_{i+1}$. Call

$$L = \bigcup_{i=1}^{k} [L_i, W_i] \; ; \qquad \mathbb{L} = L \times [0, T]^{N-1} \, .$$

The function $\mathcal{N}_{\mathbb{L}}$ appearing in the nonlocal term of (1.2) is the unique $\mathbb{T}$-periodic minimizer of the strictly convex energy (1.5). But replacing $\mathcal{N}_{\mathbb{L}}$ with its average in the y direction, by strict convexity we deduce that $\mathcal{N}_{\mathbb{L}}$ depends only on x.

Remark 2.2. *If the set* $\mathbb{L}$ *is periodic in the* x *direction not only with period* T*, but also with a smaller period* T/k *for some* $k \in \mathbb{N}$*, then replacing* $\mathcal{N}_{\mathbb{L}}(x, y)$ *with* $k^{-1} \sum_{i=1}^{k} \mathcal{N}_{\mathbb{L}}(x + iT/k, y)$ *we deduce as before that also* $\mathcal{N}_{\mathbb{L}}(x, y)$ *is* T/k*-periodic with respect to* x*.*

Since not only $\mathbb{L}$, but also $\mathcal{N}_{\mathbb{L}}$ has a one-dimensional structure, we are going to drop all but the first variable and work in one dimension; we also use the simpler notation $u(x)$ in place of $\mathcal{N}_{\mathbb{L}}(x,y)$. It is then useful to introduce the one-dimensional analogues of (1.4) and(1.5), that is, equation

$$-u'' + u = \chi_L \tag{2.2}$$

and energy

$$\frac{1}{2}\int_0^T \big(|u'(x)|^2 + |u(x)|^2\big)\,dx - \int_L u(x)\,dx\ , \qquad u \text{ is } T\text{-periodic}\ . \tag{2.3}$$

Remark 2.3. *If $u \in W^{1,2}_{\mathrm{per}}[0,\tilde{T}]$ is the minimizer to* (2.3) *then u is continuous, hence $u'' = u - \chi_L$ is bounded so that $u \in \mathcal{C}^1$.*

Lemma 2.4. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.2) *and let u be the 1-dimensional function introduced above. Set $u_0 = u(L_1)$ and $d_0 = u'(L_1)$. Then for all i*

$$u(L_i) = u(W_i) \equiv u_0\ ,$$

the function u is symmetric inside each lamella and inside each wedge, and we have for all i

$$u'(L_i) = -u'(W_i) \equiv d_0\ .$$

Proof: Indeed, the first assertion follows since (1.8) holds and curvature of a lamella is zero; to prove symmetry, dropping indices for simplicity of notation, take a wedge $[\bar{W},\bar{L}]$ and replace u in $[\bar{W},\bar{L}]$ by its symmetrized version

$$\bar{u}(x) = \frac{1}{2}\big(u(x) + u(\bar{W} + \bar{L} - x)\big)$$

(which is admissible since $u(\bar{W}) = u(\bar{L}) = u_0$). As $\bar{u}$ is a periodic solution to (2.2), we contradict uniqueness unless u is symmetric inside this lamella. In particular the derivatives of u at W and L are opposite. Since the same applies inside each lamella, the result is proved. $\square$

Lemma 2.5. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.2) *and let u be the 1-dimensional function introduced above. Then $\min u > 0$ and $\max u < 1$. A minimum point is located in a wedge while a maximum point in a lamella.*

Proof: That $0 \le u \le 1$ follows from Remark 1.1 and we only have to prove the strict inequalities. Recall u is $\mathcal{C}^1$ and piecewisely $\mathcal{C}^2$. By integrating (2.2) on the interval $[0,T]$ we observe $\frac{1}{T}\int_0^T u\,dx = \frac{|L|}{T} < 1$; this implies $\min u < 1$. Since u is continuous and periodic it has an absolute minimum at a point x_m, and we assume $u(x_m) = 0$: if x_m belongs to a lamella $[\bar{L},\bar{W}]$ then $u(x) < 1$ near x_m, and at least on one side of x_m we get $u'' = u - 1 < 0$ which gives concavity; together with $u'(x_m) = 0$ this contradicts minimality at x_m. If x_m is inside a wedge, then $u'' = u$ in the whole wedge, which together with $u(x_m) = u'(x_m) = 0$ gives $u(x) = 0$ to the boundary of the wedge. This reduces to the previous case and we know it leads to a contradiction.

Next let $w = 1 - u$ so that $-w'' + w = \chi_{L'}$, where $L' = [0,T] \setminus L$. Therefore $\min w > 0$, which is equivalent $\max u < 1$. The last assertions in the Lemma comes from the governing equation (2.2). $\square$

Proposition 2.6. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.2)*. Then all lamellae have the same size and are equally spaced.*

Proof: Take a wedge $[\bar{W}, \bar{L}]$: by the lemmas above, in this wedge the function u solves

$$u(\bar{W}) = u_0 \ , \quad u'(\bar{W}) = -d_0 \ , \quad u'' = u \ , \quad u > 0$$

and we know that $u(\bar{L}) = u_0$. As u is strictly convex it is necessary that $d_0 > 0$ in order for a minimum to be attained in a wedge. This Cauchy problem has a unique solution defined in $[\bar{W}, +\infty)$ and this solution takes the value u_0 only once after $\bar{W}$, in the point $\bar{W} + \delta(u_0, d_0)$. As this holds in every wedge, we have that all wedges have the same width $\delta(u_0, d_0)$. The same reasoning on lamellae, where $u'' = u - 1 < 0$, gives that all lamellae have the same size too. □

Now that the structure of lamellar stationary points is clear, restricting ourselves to admissible (i.e., equally sized and spaced) configurations with vertical lamellae, we compute the energy of these one dimensional structure. First recall some easy formulas:

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b \ , \qquad \cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b \ ,$$

$$2 \sinh a \cosh b = \sinh(a+b) + \sinh(a-b) \ , \qquad 2 \sinh a \sinh b = \cosh(a+b) - \cosh(a-b) \ .$$

The one-dimensional set L associated to $\mathbb{L}$ is made of k equal lamella-intervals with total length x_0, divided by k equal spaces; the function u solves $u'' = u$ in the wedges, $u'' = u - 1$ in the lamellae, takes value u_0 at all endpoints and its derivative at endpoints is $\pm d_0$. We remark that on each interval $[\bar{L}, \bar{W}]$ we have

$$\int_{\bar{L}}^{\bar{W}} u(x)\, dx = \int_{\bar{L}}^{\bar{W}} (u'' + 1)\, dx = u'(\bar{W}) - u'(\bar{L}) + (\bar{W} - \bar{L}) = (\bar{W} - \bar{L}) - 2d_0 \ ,$$

so that by (1.2)

$$J(\mathbb{L}) = 2kT^{N-1} - \alpha T^{N-1} x_0 + \frac{\sigma}{2}(T^{N-1} x_0 - 2kT^{N-1} d_0) \ . \tag{2.4}$$

To compute (u_0 and) d_0, it is convenient to fix coordinates so that each lamella-interval has length 2ξ, each wedge space has length 2τ and $[\bar{L}, \bar{W}] = [-2\xi, 0]$. Then by the symmetry properties we proved, we have for some $\lambda, \mu > 0$

$$u(x) = 1 - \lambda \cosh(x + \xi) \quad \text{in } [-2\xi, 0] \ , \qquad u(x) = \mu \cosh(x - \tau) \quad \text{in } [0, 2\tau] \ ,$$

and we must match conditions at the origin only. Thus

$$(u_0 =) 1 - \lambda \cosh \xi = \mu \cosh \tau \ , \qquad (-d_0 =) - \lambda \sinh \xi = -\mu \sinh \tau$$

from which by canceling the right hand sides

$$\sinh \tau - \lambda(\sinh \tau \cosh \xi + \cosh \tau \sinh \xi) = 0 \ \Rightarrow \ \lambda = \frac{\sinh \tau}{\sinh(\tau + \xi)}$$

and finally

$$d_0 = \lambda \sinh \xi = \frac{\sinh \tau \sinh \xi}{\sinh(\tau + \xi)} = \frac{\cosh(\tau + \xi) - \cosh(\tau - \xi)}{2 \sinh(\tau + \xi)} \ .$$

Taking $2\xi = x_0/k$ and $2\tau = (T - x_0)/k$, by (2.4) we deduce :

Proposition 2.7. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.2)*, with lamellae of total width x_0. Then*

$$\begin{aligned} J(\mathbb{L}) &= kT^{N-1}\Big[2 + \Big(\frac{\sigma}{2} - \alpha\Big)\frac{x_0}{k} - \frac{\sigma}{\sinh \frac{T}{2k}} \sinh \frac{T - x_0}{2k} \sinh \frac{x_0}{2k}\Big] \\ &= kT^{N-1}\Big[2 + \Big(\frac{\sigma}{2} - \alpha\Big)\frac{x_0}{k} - \frac{\sigma}{2 \sinh \frac{T}{2k}}\Big(\cosh \frac{T}{2k} - \cosh \frac{T - 2x_0}{2k}\Big)\Big] \ . \end{aligned} \tag{2.5}$$

We will be interested in u_0, for which we have

$$u_0 = 1 - \frac{\sinh\tau\cosh\xi}{\sinh(\tau+\xi)} = \frac{\cosh\tau\sinh\xi}{\sinh(\tau+\xi)}$$

or, substituting the values of ξ and τ,

$$u_0 = \frac{1}{\sinh\frac{T}{2k}}\cosh\frac{T-x_0}{2k}\sinh\frac{x_0}{2k} = \frac{1}{2\sinh\frac{T}{2k}}\left(\sinh\frac{T}{2k} - \sinh\frac{T-2x_0}{2k}\right), \tag{2.6}$$

whereas

$$d_0 = \frac{1}{\sinh\frac{T}{2k}}\sinh\frac{T-x_0}{2k}\sinh\frac{x_0}{2k}. \tag{2.7}$$

Remark 2.8. *If the torus $\mathbb{T}$ was not a square, but an n-dimensional rectangle, with side of length T in the direction orthogonal to lamellae and of measure T' in the other, the only change needed in formula* (2.5) *is replacing the initial T^{N-1} with T'.*

Not all k-lamellae with equal size and spacing are stationary points of energy (1.2), since (1.8) must be satisfied. But, with our notations, (1.8) reads $u_0 = \alpha/\sigma$, which by (2.6) gives

$$\frac{1}{2\sinh\frac{T}{2k}}\left(\sinh\frac{T}{2k} - \sinh\frac{T-2x_0}{2k}\right) = \frac{\alpha}{\sigma},$$

that is

$$\sinh\frac{T-2x_0}{2k} = \left(1 - \frac{2\alpha}{\sigma}\right)\sinh\frac{T}{2k}. \tag{2.8}$$

Recalling the definition (1.9) of c, the above says

$$\sinh\frac{T-2x_0}{2k} = c\sinh\frac{T}{2k} \qquad \Rightarrow \qquad \cosh\frac{T-2x_0}{2k} = \sqrt{1+c^2\sinh^2\frac{T}{2k}}$$

(we included last equality for use in the sequel). As $0 \le x_0 \le T$, we have $-T \le T - 2x_0 \le T$, so (2.8) may hold only if $-1 \le c \le 1$, i.e., only if $\alpha \le \sigma$. Solving (2.8) for x_0 we immediately deduce the following (where we summarize the results of this section).

Theorem 2.9. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.2)*, with lamellae of total width x_0. Then it is necessary and sufficient that $\alpha \le \sigma$, all lamellae have the same width*

$$\frac{x_0}{k} = \frac{T}{2k} - \operatorname{arc\,sinh}\left(c\sinh\frac{T}{2k}\right) \tag{2.9}$$

and are equally spaced, and the corresponding energy is

$$\begin{aligned} J(\mathbb{L}) \;&=\; kT^{N-1}\left\{2 + c\frac{\sigma}{2}\left[\frac{T}{2k} - \operatorname{arc\,sinh}\left(c\sinh\frac{T}{2k}\right)\right]\right. \\ &\qquad \left. - \frac{\sigma}{2\sinh\frac{T}{2k}}\left(\cosh\frac{T}{2k} - \sqrt{1+c^2\sinh^2\frac{T}{2k}}\right)\right\}. \end{aligned} \tag{2.10}$$

Remark 2.10. *One may wonder if the problem is trivial, in the sense that minimal energy is attained at a trivial state, so we may now show that at least for certain values of the parameters α and σ this is not the case: take $\alpha = \sigma/2$, so that trivial states have zero energy by Remark 1.2. Take any $x_0 > 0$ and $T = 2x_0$, one immediately sees that for $k = 1$ and $\sigma \to +\infty$ the content of the square bracket in* (2.5) *becomes negative. Thus there will be a non trivial global minimizer (which is not necessarily a lamella).*

Remark 2.11. *As we see, for stationary k-lamellae the proportion of lamella thickness to intermediate space depends on the fullness parameter c: when positive, lamellae are thinner than spaces, and the opposite is true when $c < 0$, a generalization of Remark 1.2.*

3. Minimal lamellae

Let $\alpha \le \sigma$ so that $-1 \le c \le 1$ and stationary lamellar solution exists. We now seek minimal lamellae, i.e., k-lamellae in a torus which are optimal among all multi-lamellar configurations. We still do not know if they exist, so we investigate the properties of the energy, beginning with some notation and an easy but interesting remark. We set

$$\mathcal{A}(c,t) = \operatorname{arc\,sinh}\bigl(c\sinh(t)\bigr)\ , \qquad \mathcal{B}(c,t) = \frac{\cosh t - \sqrt{1 + c^2 \sinh^2 t}}{\sinh t}\ ,$$

$$\mathcal{L}(c,t) = c\bigl(t - \mathcal{A}(c,t)\bigr) - \mathcal{B}(c,t)$$

and

$$\mathcal{E}(\sigma,c,t) = \frac{1}{t}\Bigl(2 + \frac{\sigma}{2}\mathcal{L}(c,t)\Bigr)\ ,$$

so that by (2.10)

$$J_{\mathbb{T}}(\mathbb{L}) = \frac{T^N}{2}\mathcal{E}\Bigl(\sigma, c, \frac{T}{2k}\Bigr)\ .$$

We list some properties of the functions we introduced, sketching the proof where needed; to avoid cluttering with symbols, primes will always denote derivatives with respect to t (we do not need derivatives with respect to c). Remember that $-1 \le c \le 1$, and that everything is quite trivial when $c = 0$ or $c = \pm 1$, so we will not mention these cases.

Proposition 3.1. *The function $\mathcal{B}(c,t)$ satisfies:*

i) $\mathcal{B}(c,t) = \dfrac{(1-c^2)\sinh t}{\cosh t + \sqrt{1 + c^2\sinh^2 t}}$;

ii) $\mathcal{B}(c,0) = 0\ ,\ \mathcal{B}(c,+\infty) = 1 - |c|$;

iii) *$\mathcal{B}$ is even with respect to c (and odd with respect to t);*

iv) $\mathcal{B}'(c,t) = \dfrac{1-c^2}{\Bigl(\cosh t + \sqrt{1+c^2\sinh^2 t}\Bigr)\sqrt{1+c^2\sinh^2 t}}$;

v) *$\mathcal{B}$ is increasing and $\mathcal{B}'$ is decreasing with respect to t* ;

vi) $\mathcal{B}'(c,0) = \dfrac{1-c^2}{2}\ ,\ \mathcal{B}'(c,+\infty) = 0^+$.

Proof: The first point is an easy algebraic manipulation, from which points 2, 3 follow. Point 4 is just a computation plus algebraic manipulation, from which one deduces that $\mathcal{B}' > 0$ and points 5, 6 follow (and in particular $\mathcal{B}$ is a concave function of t). □

We employ the notation ω_t to denote any function (which may change from line to line) vanishing as $t \to +\infty$.

Proposition 3.2. *The function $\mathcal{A}(c,t)$ satisfies*

i) $\mathcal{A}(c,0) = 0$, *$\mathcal{A}$ is odd respect to c (and respect to t);*

ii) $\mathcal{A}'(c,t) = \dfrac{c\cosh t}{\sqrt{1+c^2\sinh^2 t}} = c\sqrt{1 + \dfrac{1-c^2}{c^2 + (1/\sinh^2 t)}}$;

iii) $\mathcal{A}'(c,0) = c\ ,\ \mathcal{A}'(c,+\infty) = \operatorname{sign}(c)$;

iv) *$\mathcal{A}'$ always has the same sign as c, and is increasing with respect to t if $c > 0$, decreasing with respect to t if $c < 0$ (thus $-c\mathcal{A}$ is always decreasing);*

v) $\mathcal{A}''(c,t) = \dfrac{c(1-c^2)\sinh t}{(1+c^2\sinh^2 t)^{3/2}}$;

vi) *as* $t \to +\infty$

$$t - \mathcal{A}(c,t) = \begin{cases} -\log c + \omega_t & \text{if } c > 0 \\ 2t + \log|c| + \omega_t & \text{if } c < 0\ . \end{cases}$$

PROOF: Recalling that $\operatorname{arc\,sinh} x = \log\bigl(x + \sqrt{1+x^2}\,\bigr)$ and with some manipulations point 2 follows, so also point 4 becomes clear, using the second version of $\mathcal{A}'$; we will not use point 5, but we included it since the formula is nice (as opposite to the second derivative of $\mathcal{B}$). As for point 6 one has for large x (positive or negative)

$$\phi(x) := x + \sqrt{1+x^2} = \begin{cases} 2x\bigl(1 + o(1/x)\bigr) & \text{if } x \to +\infty \\ \dfrac{1}{2|x|}\bigl(1 + o(1/x)\bigr) & \text{if } x \to -\infty\ , \end{cases}$$

so that writing $2\sinh t = e^t\bigl(1 + o(e^{-t})\bigr)$

$$\phi(c\sinh t) = \begin{cases} ce^t(1+\omega_t) & \text{if } c > 0 \\ \dfrac{1}{|c|e^t}(1+\omega_t) & \text{if } c < 0\ , \end{cases}$$

and the assertion follows. □

Collecting these results we obtain the properties of $\mathcal{L}$.

Proposition 3.3. *The function* $\mathcal{L}(c,t)$ *satisfies*

i) $\mathcal{L}'(c,t) = -\dfrac{(1-c)(1-c^2)}{\Bigl(\cosh t + \sqrt{1+c^2\sinh^2 t}\Bigr)\Bigl(c\cosh t + \sqrt{1+c^2\sinh^2 t}\Bigr)}$;

ii) $\mathcal{L}'(c,t)$ *is negative and it is increasing with respect to* t *(and with respect to* c*);*

iii) $\mathcal{L}$ *is decreasing and convex as a function of* t*, for any* c*;*

iv) $\mathcal{L}'(c,0) = -(1-c)^2/2$, $\mathcal{L}'(c,+\infty) = 0^-$ *if* $c > 0$*, whereas* $\mathcal{L}'(c,+\infty) = 2c$ *if* $c < 0$;

v) $\mathcal{L}(c,0) = 0$, *and* $\mathcal{L}(c,+\infty) = c - 1 - c\log c$ *if* $c > 0$ *whereas if* $c < 0$

$$\mathcal{L}(c,t) = 2ct + c\log|c| + |c| - 1\ .$$

PROOF: Once one checks point 1 (again, just algebraic manipulation) everything follows. One has to take care, in the case $c < 0$, of the factor

$$c\cosh t + \sqrt{1+c^2\sinh^2 t} = \sqrt{1+c^2\sinh^2 t} - |c|\cosh t = \frac{1-c^2}{\sqrt{1+c^2\sinh^2 t} + |c|\cosh t}$$

so that

$$\begin{aligned} \mathcal{L}'(c,t) &= -\frac{(1-c)\bigl(\sqrt{1+c^2\sinh^2 t} + |c|\cosh t\bigr)}{\cosh t + \sqrt{1+c^2\sinh^2 t}} \\ &= -(1-c)\Bigl(1 - (1-|c|)\frac{\cosh t}{\cosh t + \sqrt{1+c^2\sinh^2 t}}\Bigr) \\ &= -(1-c) + (1-c^2)\frac{1}{1 + \sqrt{1-(1-c^2)\tanh^2 t}}\ . \end{aligned}$$

□

Before concluding with the properties of $\mathcal{E}$ we summarize the relevant properties of the function $t\mathcal{E}(\sigma,c,t) = 2 + (\sigma/2)\mathcal{L}$, where we re-include the easy cases.

Proposition 3.4. *For $-1 \le c \le 1$ the function $2 + (\sigma/2)\mathcal{L}(c,t)$ satisfies:*

i) *it is strictly convex and decreasing (except it is 2 for $c = 1$ and $2 - \sigma t$ for $c = -1$);*

ii) *for $t = 0$ it takes the value 2;*

iii) *if $c > 0$ its limit as $t \to +\infty$ is*

$$2 + \frac{\sigma}{2}(c - 1 - c\log c) \;;$$

iv) *if $c < 0$ it has as an asymptote as $t \to +\infty$ the function*

$$\sigma ct + \left[2 + \frac{\sigma}{2}(|c| - 1 - |c|\log|c|)\right] \;;$$

v) *if $c = 0$ its limit as $t \to +\infty$ is $2 - \sigma/2$.*

A crucial role is played by the threshold function

$$\Gamma(c) = |c| - 1 - |c|\log|c| \;, \qquad |c| \le 1 \tag{3.1}$$

extended by continuity at $c = 0$ by $\Gamma(0) = -1$, since points 3 and 4 may be rewritten as

$$\text{as } t \to +\infty \;, \qquad 2 + (\sigma/2)\mathcal{L} \sim \begin{cases} 2 + \sigma\Gamma(c)/2 & \text{if } c \ge 0, \\ \sigma ct + 2 + \sigma\Gamma(c)/2 & \text{if } c \le 0. \end{cases}$$

We remark that $\Gamma(0) = -1$, $\Gamma(\pm 1) = 0$ and Γ (which is an even function) decreases if $c < 0$ and increases if $c > 0$. Note that when $c \le 0$, the asymptote has an intercept of $2 + \sigma\Gamma(c)/2$ at the vertical axis.

Since $f(t)/t$ is the slope of the line connecting $(0,0)$ with $\big(t, f(t)\big)$, we obtain the main result of this section.

Theorem 3.5. *(a) If $2 + \sigma\Gamma(c)/2 \ge 0$ then $\mathcal{E}(\sigma,c,t)$ is a decreasing function of t;*
(b) if $2 + \sigma\Gamma(c)/2 < 0$ then there is a unique point $t_0 = t_0(c,\sigma) > 0$ such that $\mathcal{E}(\sigma,c,t)$ is strictly decreasing for $0 < t \le t_0$ and strictly increasing afterwards.

Proof: It all depends on point 1 in the crucial Proposition 3.4 and the fact that the slope of the line from the origin to a point on the graph of $t\mathcal{E}(\sigma,c,t)$ will tend to zero if $c \ge 0$ and to σc if $c < 0$. We examine 4 different cases.
(i) If $c \ge 0$ and $2 + \sigma\Gamma(c)/2 \ge 0$ then the graph of $t\mathcal{E}(\sigma,c,t)$ always stays above 0. Thus $\mathcal{E}$ is a strictly decreasing function of t with $\mathcal{E} \to 0$ as $t \to \infty$.
(ii) If $c < 0$ and $2 + \sigma\Gamma(c)/2 \ge 0$, then $\mathcal{E}(\sigma,c,t)$ decreases monotonically from a positive value to a negative value σc as $t \to \infty$.
(iii) If $c \ge 0$ and $2 + \sigma\Gamma(c)/2 < 0$, the graph of $t\mathcal{E}(\sigma,c,t)$ starts above but ends below 0 with a horizontal asymptote. Then there is a unique point $t_0 = t_0(c,\sigma) > 0$ satisfying (3.2) such that $\mathcal{E}(\sigma,c,t)$ is strictly decreasing for $0 < t \le t_0$ and strictly increasing afterwards.
(iv) If $c < 0$ and $2 + \sigma\Gamma(c)/2 < 0$, then there is a point $t_0 = t_0(c,\sigma) > 0$ such that $\mathcal{E}(\sigma,c,t)$ is strictly decreasing for $0 < t \le t_0$ and strictly increasing afterwards. Thus $\mathcal{E}(\sigma,c,t_0) < \sigma c < 0$. □

We now determine the minimum location t_0 in Cases (iii) and (iv). Observe that the tangent line at $t = t_0$ to the graph $y = 2 + \frac{\sigma}{2}\mathcal{L}(c,t)$ passes through the origin in both cases, it follows that

$$\frac{\sigma}{2}\mathcal{L}'(c,t_0)t_0 = 2 + \frac{\sigma}{2}\mathcal{L}(c,t_0) \;. \tag{3.2}$$

Define $g(t) = \mathcal{L}'(c,t)t - \mathcal{L}(c,t)$. It is readily checked that $g'(t) = t\mathcal{L}''(c,t) > 0$, $g(c,0) = 0$ and $\lim_{t\to\infty} g(t) = -\Gamma(c)$ whenever $c \ne 0$. With g being a strictly increasing function and $4/\sigma < -\Gamma(c)$, this $t_0 = t_0(c,\sigma)$ is the unique root of $g(t_0) = 4/\sigma$. It is clear that t_0 is a strictly decreasing function of σ (for fixed c), moreover

$$t_0 \to 0 \text{ as } \sigma \to \infty, \quad \text{and } t_0 \to \infty \text{ as } \sigma \to -4/\Gamma(c) \;. \tag{3.3}$$

Recall $J(\emptyset) = 0$, $2J(\mathbb{T})/T^N = \sigma c$ and $2J(\mathbb{L})/T^N = \mathcal{E}$. The points where the lines $y = 0$, $y = \sigma ct$ and $y = \mathcal{E}t$ cut the graph $y = 2+\sigma\mathcal{L}(c,t)/2$ determine the pecking order of the energy associated with the empty set, the full torus and the minimal lamella, respectively. Let t_f be the unique intersection point, if exist, of the straight line $y = \sigma ct$ and the curve $y = 2 + \sigma\mathcal{L}(c,t)/2$, and t_e be the unique point, if exist, at which $0 = 2 + \sigma\mathcal{L}(c,t_e)/2$. As can be seen in the 4 cases in the proof of Theorem 3.5:
Case (i): t_f exists ($t_e = \infty$); lowest energy is empty torus if $T > 0$.
Case (ii): t_e exists. ($t_f = \infty$); lowest energy is the full torus if $T > 0$.
Case (iii): $t_f \le t_e < t_0$; lowest energy can be the minimal lamella if $T > 2t_e$.
Case (iv): $t_e < t_f < t_0$; lowest energy can be the minimal lamella if $T > 2t_f$.

We now show that for sufficiently large T, the lowest energy configuration is the one associated with $\max\{t_0, t_e, t_f\}$. Thus in Cases (iii) and (iv) the lowest energy configuration is the minimal torus.

Corollary 3.6. *Given a torus with side T, if $2 + \sigma\Gamma(c)/2 \ge 0$ the minimal energy among stationary k-lamellae is attained for $k = 1$; moreover if $c > 0$ (Case (i)) the trivial empty state has lower energy than all lamellae and the full torus, whereas if $c < 0$ (Case (ii)) the full torus has lower energy than all lamellae and the empty state. The case $2 + \sigma\Gamma(c)/2 \ge 0$ will thus never see a lamellar configuration having less energy than both trivial states.*

If instead $2+\sigma\Gamma(c)/2 < 0$ (Cases (iii) and (iv)) then the minimal energy configuration among k-lamellae will divide the torus in k bands with mesh size close to $T_0 = 2t_0(c,\sigma)$, and precisely for $k = T/T_0$ if this is an integer, or (if T/T_0 is not an integer) for k equal to either the integer part of T/T_0, or the integer part plus one. We remark that $2 + \sigma\Gamma(c)/2 < 0$ is possible only when $\sigma > 4$; then the inequality will be satisfied for c in a neighbourhood of 0.

Comparison with the trivial "full torus" state and with the empty state, in the case $2 + \sigma\Gamma(c)/2 < 0$, is less neat. Indeed, in the latter case even when $c > 0$ there are values of t (all values beyond the point t_e where $\mathcal{E}(\sigma,c,t) = 0$) for which the slope of $\mathcal{E}(\sigma,c,t)/t$ is negative, but if the torus is small ($T < t_e$) one can never have $T/k > t_e$ as $k \ge 1$; together with $\mathcal{E}(\sigma,c,\cdot)$ being decreasing on the interval $[0,t_e]$, the empty torus wins. Instead, if the torus is large ($T > t_e$) some stationary lamella can beat the empty torus. Similar considerations hold for the comparison of lamellae and full torus if $c < 0$ (otherwise the full torus has higher energy than the already considered empty one).

In fact when T is very large, we can pick k so that $T/k := t_k \approx t_0$, where t_0 is defined just above Corollary 3.6; Cases (iii) and (iv) then dictate that $t_k > \max\{t_e, t_f\}$. Thus there exists a stationary lamellar solution which has a lower energy than both trivial states.

Remark 3.7. *For Cases (iii) and (iv) the function $t \mapsto \mathcal{E}(\sigma,c,t)$ has a single global minimum point at $t = t_0$. Then for the torus $\mathbb{T}_0$ with side $T_0 = 2t_0$ the minimal configuration among all (multi-)lamellae exists and is a single lamella, with thickness $x_0 = t_0 - \mathcal{A}(c,t_0)$. Moreover for the torus with side kT_0 the minimal configuration will be a k-lamella, with each strip of thickness x_0 (thus $k\mathbb{T}_0$ will be divided in k bands, all equal to the band appearing in $\mathbb{T}_0$); and if a torus has side $kT_0 < T < (k+1)T_0$ the minimal configuration will be either a k- or a $(k+1)$-lamella.*

Thus the Fitzhugh-Nagumo energy (1.2) *induces a preferred width (and spacing) of lamellae, to which minimal configurations try to conform when the torus is large. The preferred mesh size T_0 is thus "universal".*

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