Convergence analysis of controlled particle systems arising in deep learning: from finite to infinite sample size

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Abstract

This paper deals with a class of neural SDEs and studies the limiting behavior of the associated sampled optimal control problems as the sample size grows to infinity. The neural SDEs with N samples can be linked to the N-particle systems with centralized control. We analyze the Hamilton–Jacobi–Bellman equation corresponding to the N-particle system and establish regularity results which are uniform in N. The uniform regularity estimates are obtained by the stochastic maximum principle and the analysis of a backward stochastic Riccati equation. Using these uniform regularity results, we show the convergence of the minima of objective functionals and optimal parameters of the neural SDEs as the sample size N tends to infinity. The limiting objects can be identified with suitable functions defined on the Wasserstein space of Borel probability measures. Furthermore, quantitative algebraic convergence rates are also obtained.

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1 Introduction

In recent years, neural networks have been shown to be very effective modelling complicated data sets. For the situations where large amounts of samples are observed, it is important to ensure the convergence of optimal parameters as the number of samples goes to infinity, i.e., the generality of the neural network. Such problems are studied in [17, 25], and [44]. Motivated by these studies, we investigate a mathematical model concerning so-called *neural SDEs* with N samples and establish quantitative results on the convergence of the minima of the corresponding objective functionals and the optimal parameters to suitable limit objects.

Our research concerns with the following neural SDEs

$$\begin{cases} dX_N^{\theta,i}(t) = f\left(t, \theta(t), X_N^{\theta,i}(t), \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \delta_{X_N^{\theta,j}(t)}\right)dt + \sigma dW^0(t), \\ X_N^{\theta,i}(0) = x_i, \ i = 1, \dots, N, \end{cases} \tag{1.1}$$

where $\theta : [0,T] \to \Theta$ is a stochastic process that represents the trainable parameters (valued in a given control set Θ). Here $f : [0,T] \times \Theta \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is a nonlinear function that governs the feed forward dynamics. $T > 0$ is a given time horizon, $(W^0(t))_{t\in[0,T]}$ is a given Brownian motion on $\mathbb{R}$ with intensity $\sigma \in \mathbb{R}$, and the neural SDEs are initiated with samples $x_i \in \mathbb{R}$, $i = 1, \dots, N$. The standing assumptions on the data and the set up for the specific sampled optimal control problems, including the description of the objective function is given in Section 2 (see (2.1) and (2.4)).

The neural SDE (1.1) describes the deep learning from a dynamical systems viewpoint and relying on this, our results make it possible to analyze the convergence of trainable parameters obtained from samples

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with size N. The dynamical system approach to deep learning was proposed in [22, 32], and studied later in [4, 19, 25, 37, 52], etc. See also, for instance, [2, 12, 16, 47], for the application of such approach. The intuition of the dynamical system approach to deep learning is as follows. For models such as residual networks, recurrent neural networks and normalizing flows, the typical feed-forward propagation with T layers can be presented as

$$x(t+1) = x(t) + f\big(x(t), \theta(t)\big), \quad t = 0, 1, \dots, T-1,$$

where $x(0)$, $x(T) \in \mathbb{R}^d$ are the input and output, respectively, and $\theta(t)$ is the weight matrix. Given multiple samples of the input $x(0)$, the goal of learning is to tune the trainable parameters $\theta(t)$, $t = 0, 1, \dots, T-1$, so that the outputs $x(T)$ minimize a given objective function. As the layer number T tends to infinity, after an appropriate rescaling, the above discrete system is then turned into an ODE

$$\dot{x}(t) = f\big(x(t), \theta(t)\big), \quad t \in [0, T].$$

In a more general setting, the feed-forward propagation might depend on the distribution of the input (see for instance [2, 12, 26]) and systemic noise, the aforementioned continuous idealization is then naturally generalized to the neural SDE below:

$$dx(t) = f\big(x(t), \mathcal{L}(x(t)), \theta(t)\big)dt + \sigma dW_t^0, \quad t \in [0, T],$$

where $\mathcal{L}(x(t))$ stands for the law of $x(t)$ and $(W_t^0)_{t\in[0,T]}$ is a given Brownian motion. So the goal is to tune the stochastic control $\theta(t)$ so that a given objective function is minimized. Another situation where the distribution of samples enters the feed-forward propagation is the so-called batch normalization (see [36]), for example

$$f(x, \mu, \theta) = \tilde{f}\left(\frac{x - \int y\mu(dy)}{\sqrt{\int y^2\mu(dy) + \epsilon}}, \theta\right),$$

for some function $\tilde{f}$, where the variable μ in the above corresponds to the distribution of samples, and $\epsilon > 0$ is a given parameter.

Since the distribution $\mathcal{L}(x(t))$ is practically hard to observe, this is usually replaced with the empirical measure of the samples. As a result, we obtain (1.1) as well as the empirical risk minimization Problem 2.2 (see the details in Section 2).

Inspired by [22, 25, 32], in this manuscript we treat (1.1) from an optimal control point of view. The dynamics in (1.1) can be viewed as the N-particle systems with centralized control. Indeed, we may view each $X_N^{\theta,i}(t)$ as the process driving particle i and $\theta(t)$ the control process. However, we note here that the controlled particle system (1.1) is different from the usual mean field type, typically studied in the literature, in the sense that every particle $X_N^{\theta,i}(t)$ in the system shares the same control $\theta(t)$ rather than having their own $\theta^i(t)$. In summary, our problem relates to the convergence of the value functions and optimal controls of the controlled particle systems, i.e., the propagation of chaos or the law of large numbers. As the number of particles grows to infinity, we explore sufficient conditions that ensure the aforementioned convergence. Such convergences are possible thanks to the presence of an L^2-regularizer in the objective functional. Furthermore, quantitative results on the convergence rate are also obtained.

Similar convergence problems for mean field control have been extensively studied recently. To name a few, we refer to [6, 7, 11, 20, 21, 28, 29, 40, 41, 43, 50, 51], see also [13, 14, 15, 18, 31], as well as the references therein for the ones with uncontrolled particle systems. The limit of the value functions in the aforementioned convergence is a function whose state variable is a probability measure. For literature on such limit, see for instance [28, 29, 43, 46, 53].

As mentioned before, our model (1.1) is significantly different from the ones above in terms of the form of control, which thus results in a very different Hamilton–Jacobi–Bellman (HJB) system. Although similar models are studied in [23, 24, 33, 34, 35, 38], their emphasis is on the analysis of the corresponding algorithm. The models and results in [4, 5, 25] are the closest ones to the present paper, where the convergence of both value functions and optimal controls are investigated. In [4, 5] the law of large numbers is obtained where there is no quantitative results. In [25] on the other hand, the authors focus on the deterministic control and obtain quantitative results on large deviations, but the state dynamics f therein is required to be independent of the distribution of particles. Here, we study models with more general state dynamics f that could depend on the distribution of particles and obtain quantitative results. More specifically, besides the law of large numbers, we further show the corresponding convergence rate: as the sample size N grows to infinity, the minima of the objective functional, i.e. V_N and the optimal feedback function θ_N^* converge, at certain algebraic rates, to a value function and a feedback function whose state variable is the empirical measure of the samples. As a result, we show that the optimal parameters also converge at certain algebraic rate. We obtain two kinds of convergence results: the short time convergence and the global convergence, both accompanied with a precise convergence rates.

The HJB equation written for the value function V_N associated to our main control problem, i.e. Problem 2.2, formally reads as

$$\begin{cases} \partial_t V_N + \dfrac{\sigma^2}{2} \displaystyle\sum_{i,j=1}^N \partial^2_{x_i x_j} V_N + \inf_{\theta\in\Theta} \left\{ \dfrac{\lambda}{2} |\theta|^2 + \displaystyle\sum_{i=1}^N f\left(t,\theta,x_i,\dfrac{1}{N}\sum_{j=1}^N \delta_{x_j}\right) \partial_{x_i} V_N + \dfrac{1}{N}\sum_{i=1}^N L(x_i) \right\} = 0, \\ \quad (t,x)\in[0,T)\times\mathbb{R}^N, \\ V_N(T,x_1,\ldots,x_N) = \dfrac{1}{N}\displaystyle\sum_{i=1}^N U(x_i), \quad x\in\mathbb{R}^N, \end{cases} \tag{1.2}$$

where L and U are the suitably chosen loss function and final cost function, respectively, and the control functions are valued in some control set Θ and $\sigma\in\mathbb{R}$ and $\lambda>0$ are given further parameters.

To study the convergence of the value functions and optimal feedback functions of Problem 2.2, the main ingredients are the regularity results on the HJB equations which are uniform and decay suitably with respect to N – the dimension of the input variables (Theorem 3.17 and Theorem 3.18). This idea is in the spirit of [29] where the control is of the mean field type. However, in contrast to this, in our problem we are faced with the same common control for each particle and the dynamics of each particle could be nonlinear with respect to the control variable. Hence the resulting Hamiltonian is significantly different in structure. Moreover, in our problem there is a common Brownian motion in the dynamics of each particle. Therefore the method in [29] is no longer applicable directly in our situation. Instead, here we rely more on a probabilistic approach to analyze the HJB equation and to obtain the desired regularity results.

The *first main contribution* of this paper is the uniform (in N) estimates on the degenerate PDE systems describing V_N, as well as $\nabla_x V_N$, $\nabla^2_{xx} V_N$. Such uniform estimates will imply the convergence rate of $V_N(t,x)$ and the corresponding feedback functions $\theta_N^*(t,x)$. In order to obtain the desired uniform estimates, we apply the nonlinear Feymann–Kac representation and focus on the stochastic processes corresponding to V_N, $\nabla_x V_N$ and $\nabla^2_{xx} V_N$, respectively. Because of the degenerate nature of the problem, we need to introduce regularizations at several levels: these will be via involving non-degenerate idiosyncratic noise as well as some suitable cut-off procedures to handle the growth properties of the data. Our estimates will turn out to be independent of these regularization parameters. It is well known that $\nabla_x V_N$ corresponds to the adjoint process in the stochastic maximum principle. As a result of this, we apply the stochastic maximum principle and obtain that each entry of $\nabla_x V_N$ decays at the rate of $O(N^{-1})$. However, the analysis of the systems involving $\nabla^2_{xx} V_N$ is more subtle. It turns out that the suitable approximations of $\nabla^2_{xx} V_N$ introduced above, are related to matrix-valued processes (Y_t in (3.60) and (3.59)) that satisfy backward stochastic Riccati equations. We first analyze the processes Y_t using the contraction mapping principle and obtain short time estimates for each entry of Y_t: the $(i,j)-$entry of Y_t has a decay rate of $O(\delta_{ij}N^{-1}+N^{-2})$. As for the global estimates, we make further suitable convexity assumptions on the data and analyze the eigenvalues of Y_t utilizing the Riccati (i.e. quadratic) feature of the corresponding BSDE. Under these extra assumptions, each eigenvalue of Y_t decays at the rate of $O(N^{-1})$ for arbitrary long time horizon T. These convexity assumptions are similar in spirit to displacement convexity (used in [3, 9, 29]), however, they are not covered by the existing literature (not even by the displacement monotonicity conditions introduced in [1, 30]), because the state dynamics given by f is allowed to have a measure dependence. We note here that such measure dependence of f has been investigated in [21, 40, 45] within the framework of standard mean field games and control.

Our *second main contribution* is the convergence analysis on $V_N(t,x)$ and $\theta_N(t,x)$ on a quantitative level. We use a variational approach to show that V_N and θ_N^* are both finite dimensional projection of certain functions $\mathcal{V}$ and θ^* whose state variables are probability measures. Furthermore, thanks to the previous uniform estimates, we show that, both $\mathcal{V}$ and θ^* are Lipschitz continuous with respect to their state variables. Under our two sets of different assumptions, the previous results hold for a short time horizon or global in time, respectively. Such convergence of $V_N(t,x)$ and $\theta_N(t,x)$ has two major implications on neural SDE. First, the convergence $V_N(t,x)$ translates to the convergence of minima of objective functionals. Second, the convergence of $\theta_N(t,x)$ would yield pathwise convergence results that translate to the convergence of optimal parameters obtained via neural SDEs (see Proposition 4.7 and Proposition 4.12).

Some concluding remarks. The limit function $\mathcal{V}$ is formally associated to a second order HJB equation set on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$. This formally read as

$$\begin{cases} \partial_t \mathcal{V}(t,\mu) + \dfrac{\sigma^2}{2}\left\{\displaystyle\int_{\mathbb{R}} \partial_{y\mu}\mathcal{V}(t,\mu)(y)\mu(dy) + \int_{\mathbb{R}^2} \partial_{\mu\mu}\mathcal{V}(t,\mu)(y,y')\mu(dy)\mu(dy')\right\} \\ \qquad + \inf_{\theta\in\Theta}\left\{\dfrac{\lambda}{2}|\theta|^2 + \displaystyle\int_{\mathbb{R}} f\Big(t,\theta,y,\mu\Big)\partial_\mu\mathcal{V}(t,\mu)(y)\mu(dy) + \int_{\mathbb{R}} L(y)\mu(dy)\right\} = 0, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}), \\ \mathcal{V}(T,\mu) = \displaystyle\int_{\mathbb{R}} U(y)\mu(dy), \quad \mu\in\mathcal{P}_2(\mathbb{R}). \end{cases} \tag{1.3}$$

We would like to underline at this stage that studying the quantitative decay estimates with respect to N of second order spacial derivatives of V_N (that we perform in this paper) results in the fact that $\partial_\mu\mathcal{V}$ exists and it is Lipschitz continuous in a suitable sense. The very same analysis that we perform on these objects could be pushed further, to study quantitative third order derivative estimates for V_N, which would result in twice differentiability of $\mathcal{V}$, and hence in the fact that $\mathcal{V}$ is a classical solution to the HJB equation (1.3). This would be very much in the flavor of the $C^{2,1,w}(\mathcal{P}_2(\mathbb{R}))$ type estimates from [29]. However, because of the technical burden behind such estimates, we do not pursue the question of classical solutions to (1.3) in this paper.

The specific choice for L, U and f in the above setting is motivated by the concrete applications in deep neural networks we have described above. In our analysis, in fact one would be able to allow more general measure dependent functions in (1.3).

We would like to emphasize once more that connections between equations of type (1.2) and (1.3), and the corresponding quantitative rates of convergence as $N \to +\infty$ have received a great attention in the past 2-3 years in the works [6, 8, 20]. However, these works were seeking relationship and convergence rates for viscosity solutions to the corresponding HJB equations. The results of these papers differ significantly from ours, as their motivation is quite different. In particular, in those works the authors have always considered non-degenerate idiosyncratic noise and no common noise. In our models, we consider purely common noise coming from centralized control problems. Also, our analysis is based on finite dimensional approximations and a careful combination of parabolic PDE techniques and stochastic analysis of FBSDE systems, while the mentioned papers relied on viscosity solutions techniques and regularization procedures for semi-concave and Lipschitz continuous functions defined on the Wasserstein space.

The remainder of the paper is organized as follows. In Section 2 we describe the the model and the main problem of interest. In Section 3 we first introduce the auxiliary problems and study the regularity of the corresponding value functions. Then we establish the estimate on the derivatives of the value function as well as the verification results associated to the original problem. In Section 4 we show that the value function V_N in Problem 2.2 is the finite dimensional projection of a function $\mathcal{V}$ whose state variable is in the space of probability measure, and establish the results on the convergence rate. In Section 5, as a concrete example, we consider a linear quadratic model which falls into our framework and for which closed form solutions are available.

2 The model problem and standing assumptions

Let $T > 0$ be a given time horizon. Let $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$ be an augmented filtered probability space satisfying the usual conditions, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by a sequence of independent Brownian motions $\{W^i\}_{i=0}^{\infty}$.

The $[0,T] \ni t \mapsto X_N^{\theta,i}(t),\ i = 1,\dots,N$, in (1.1) is a sequence of controlled diffusion processes coupled with the common noise $W^0(t)$ and the mean field term $\frac{1}{N}\sum_{j=1}^N \delta_{X_N^{\theta,j}(t)}$. The control $[0,T] \ni t \mapsto \theta(t)$ in (1.1), which is understood as the weight process in deep learning, is shared among the dynamics of all $X_N^{\theta,i}(t)$.

Given $x_1, x_2, \dots,$ we consider the optimization problem over the admissible set $\mathcal{U}^{ad}$, consisting of the tuple $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F}, \{W^i\}_{i=0}^{+\infty}, \theta)$ such that

- θ is predictable, $\theta(t) \in \Theta$, $t \in [0,T]$;
- for each $N \geq 1$, $\big(\{x_i\}_{i=1}^N, \{W^i\}_{i=0}^N, \theta\big)$ is a weak solution to (1.1).

When there is no ambiguity, we use θ to denote the admissible control.

Given a control $\theta \in \mathcal{U}^{ad}$ and N inputs $(x_1,\dots,x_N) \in \mathbb{R}^N$, we can further define the objective function $J_N : \mathcal{U}^{ad} \times [0,T] \times \mathbb{R}^N \to \mathbb{R}$ as follow

$$\begin{aligned} &J_N(\theta, t, x_1, \dots, x_N) \\ &\quad := \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N \int_0^T L\big(X_N^{\theta,i}(t)\big)dt + \frac{1}{N}\sum_{i=1}^N U\big(X_N^{\theta,i}(T)\big) + \frac{\lambda}{2}\int_0^T |\theta(t)|^2\, dt\Big], \end{aligned} \tag{2.1}$$

The third term on the right hand side of (2.1) is the regularizer. It is straightforward but notationally cumbersome to generalize our results to the case where $\Theta = \mathbb{R}^d$ and $x_i \in \mathbb{R}^m$. For the ease of notations and convenience in this paper we choose $d = m = 1$.

In our analysis we consider the space of Borel probability measures, supported in Euclidean spaces $\mathbb{R}^m$. We work on the specific subset of these measures, which have finite second moment, and denote this by $\mathcal{P}_2(\mathbb{R}^m)$. We equip this subset with the classical 2-Wasserstein distance $\mathcal{W}_2$.

Here we make the following technical assumptions on parameters.

Assumption 2.1. *Assume that*

1. *The function $[0,T] \times \Theta \ni (t,\theta) \mapsto f(t,\theta,0,\delta_{\{0\}})$ is continuous, where $\Theta = \mathbb{R}$;*
2. *the function $f : [0,T] \times \Theta \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is such that $\partial_t f$ is bounded and has bounded derivatives with respect to (θ, x, μ) up to the second order;*
3. *For $\varphi \in \{L, U\}$, $\varphi \geq 0$, and there exist constants C_{11}^{φ}, C_{10}^{φ}, C_{20}^{φ} such that*

$$|\varphi'(x)| \leq C_{11}^{\varphi}|x| + C_{10}^{\varphi},\ \ |\varphi''(x)| \leq C_{20}^{\varphi}. \tag{2.2}$$

In Assumption 2.1, by derivative with respect to μ we mean the intrinsic derivative, the so-called Wasserstein derivative (see for instance [7, Definition 2.2.2] or [10, Chapter 5] and the discussion therein). In particular, when we say differentiability with respect to the measure variable, we always mean the so-called fully C^1, C^2, etc. classes (see [10, Chapter 5]). In what follows we use the notation ∂_μ to denote this intrinsic Wasserstein derivative. We denote by $\tilde{x}$ the new variable arising after applying ∂_μ, and we display this after the measure variable as $\partial_\mu g(\mu, \tilde{x})$, for any $g \in C^1(\mathcal{P}_2(\mathbb{R}^M))$.

We note here that the optimization of J_N under the constraint (1.1) can be understood as a learning process with neural SDEs. Suppose we are to determine a system with dynamics

$$dX(t) = g\big(t, X(t), \mathcal{L}(X(t))\big)dt + \sigma d\tilde{W}^0(t), \tag{2.3}$$

where the diffusion coefficient σ has been observed but the drift $g(t,x,\mu)$ is unknown. The motivation of determining such $g(t,x,\mu)$ could be mimicking the genuine dynamics of certain processes (e.g. [16]) or finding an optimal feedback function (e.g. [2, 12, 47]). The logic behind the learning process is to approximate $g(t,x,\mu)$ with the candidate function chosen from the family $f(t,\theta(t),x,\mu)$ where $\theta(t)$ is the parameter to be determined. Given inputs $x_1,x_2,\dots,x_N$, we may use the dynamics in (1.1) to approximate (2.3) according to appropriate performance functionals. Abstractly speaking, the training process is equivalent to obtaining the optimal control θ^* of the following optimization problem:

Problem 2.2. *Minimizing* (2.1) *over* $\mathcal{U}^{ad}$.

Denote the value function to Problem 2.2 by

$$V_N(t,x_1,\dots,x_N) := \inf_{\theta\in\mathcal{U}^{ad}} J_N(\theta,t,x_1,\dots,x_N), \tag{2.4}$$

and $\theta_N^*(t,x_1,\dots,x_N)$ one of the optimal feedback functions (if exists). Suppose that

$$\mathcal{W}_2\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k},\mu\right) \longrightarrow 0 \quad \text{as} \quad N\to+\infty,$$

where $\mu\in\mathcal{P}_2(\mathbb{R})$ is a given probability measure. We are interested in the convergence as well as the convergence rate of both $V_N(t,x_1,\dots,x_N)$ and $\theta_N^*(t,x_1,\dots,x_N)$ to their corresponding limits. To the questions above, we give our positive answers in Section 4.

3 The auxiliary problems and corresponding uniform estimates

In order to study the aforementioned convergence as well as the convergence rate, we establish uniform derivative estimates on V_N as the number of variables increases, which is different from the usual PDE estimates. Our results include the uniform estimates on the first and the second order derivatives of V_N. These estimates are used in Section 4. It turns out (as we will see in the next section) that the estimates on the first order derivatives yield the convergence rate of $V_N(t,x_1,\dots,x_N)$, while the estimates on the second order derivatives yield the convergence rate of $\theta_N^*(t,x_1,\dots,x_N)$.

3.1 The auxiliary problems and the estimates on the first order derivatives

To solve (2.4), the dynamic programming principle yields the HJB equation

$$\begin{cases} \partial_t V_N + \dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N \partial^2_{x_ix_j}V_N + \inf_{\theta\in\Theta}\left\{\dfrac{\lambda}{2}|\theta|^2 + \displaystyle\sum_{i=1}^N f\left(t,\theta,x_i,\dfrac{1}{N}\displaystyle\sum_{j=1}^N\delta_{x_j}\right)\partial_{x_i}V_N + \dfrac{1}{N}\displaystyle\sum_{i=1}^N L(x_i)\right\} = 0, \\ \qquad (t,x)\in[0,T)\times\mathbb{R}^N, \\ V_N(T,x_1,\dots,x_N) = \dfrac{1}{N}\displaystyle\sum_{i=1}^N U(x_i), \quad x\in\mathbb{R}^N. \end{cases} \tag{3.1}$$

The equation (3.1) is degenerate parabolic, as the Fourier symbol of the second order differential operator is given by

$$\frac{\sigma^2}{2}\sum_{i,j=1}^N \xi_i\xi_j = \frac{\sigma^2}{2}\left(\sum_{i=1}^N \xi_i\right)^2.$$

Hence the classical solution to (3.1) is not guaranteed by standard results.

In order to study (3.1), we introduce the following auxiliary equation with parameters $R=(R_1,R_2)$ and ε:

$$\begin{cases} \partial_t V_N^{\varepsilon,R} + \dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N \partial^2_{x_i x_j} V_N^{\varepsilon,R} + \dfrac{\varepsilon^2}{2}\sum_{i=1}^N \partial^2_{x_i x_i} V_N^{\varepsilon,R} \\ \qquad + \displaystyle\inf_{\theta\in\Theta_{R_2}} \left\{ \dfrac{\lambda}{2}|\theta|^2 + \sum_{i=1}^N f\Big(t,\theta,x_i,\dfrac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\partial_{x_i} V_N^{\varepsilon,R} \dfrac{1}{N}\sum_{i=1}^N L_{R_1}(x_i) \right\} = 0, \\ V_N^{\varepsilon,R}(T,x_1,\ldots,x_N) = \dfrac{1}{N}\displaystyle\sum_{i=1}^N U_{R_1}(x_i), \end{cases} \tag{3.2}$$

where $\Theta_{R_2} := \Theta \cap B_{R_2}(0)$ and for $\varphi \in \{L,U\}$ we have defined the smooth truncated version φ_{R_1} satisfying

1. $\varphi_{R_1}(x) = \varphi(x)$ on $x \in B_{R_1}(x)$, $|\varphi_{R_1}(x)| \le |\varphi(x)|$;
2. $\varphi_{R_1},\ \nabla_x \varphi_{R_1},\ \nabla_x^2 \varphi_{R_1}$ are bounded;
3. The derivatives satisfy

$$|\varphi'_{R_1}(x)| \le C_{11}^{\varphi}|x| + C_{10}^{\varphi},\ \ |\varphi''_{R_1}(x)| \le C_{20}^{\varphi}. \tag{3.3}$$

These derivative bounds and growth rates on the truncated functions can be guaranteed because of the main assumptions on L,U, which we imposed in Assumption 2.1.

The equation above corresponds to the auxiliary optimization problem with he underlying training processes

$$dX_N^{\varepsilon,\theta,i}(t) = f\Big(t,\theta(t),X_N^{\varepsilon,\theta,i}(t),\frac{1}{N}\sum_{j=1}^N \delta_{X_N^{\varepsilon,\theta,j}(t)}\Big)dt + \varepsilon dW^i(t) + \sigma dW^0(t),\ \ i=1,\ldots,N, \tag{3.4}$$

and the admissible set $\mathcal{U}^{ad}_{R_2}$ consists of $\theta \in \mathcal{U}^{ad}$ with $|\theta(t)| \le R_2$, $t\in[0,T]$, as well as the objective function $J_N^{\varepsilon,R_1} : \mathcal{U}^{ad}_{R_2} \times [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is defined as

$$\begin{aligned} &J_N^{\varepsilon,R_1}(\theta,t,x_1,\ldots,x_N) \\ &\quad := \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N \int_0^T L_{R_1}\big(X_N^{\theta,i}(t)\big)dt + \frac{1}{N}\sum_{i=1}^N U_{R_1}\big(X_N^{\theta,i}(T)\big) + \frac{\lambda}{2}\int_0^T |\theta(t)|^2\,dt\Big]. \end{aligned}$$

Suppose that Assumption 2.1 takes place. Then we have

$$V_N^{\varepsilon,R}(t,x_1,\ldots,x_N) = \inf_{\theta\in\mathcal{U}^{ad}_{R_2}} J_N^{\varepsilon,R_1}(\theta,t,x_1,\ldots,x_N). \tag{3.5}$$

Using the corresponding variational representations, it is straightforward to show the following convergences

$$\lim_{R_2\to+\infty} V_N^{\varepsilon,R}(t,x_1,\ldots,x_N) = \inf_{\theta\in\mathcal{U}^{ad}} J_N^{\varepsilon,R_1}(\theta,t,x_1,\ldots,x_N) =: V_N^{\varepsilon,R_1}(t,x_1,\ldots,x_N), \tag{3.6}$$

$$\lim_{R_1\to+\infty} V_N^{\varepsilon,R_1}(t,x_1,\ldots,x_N) = \inf_{\theta\in\mathcal{U}^{ad}} J_N^{\varepsilon}(\theta,t,x_1,\ldots,x_N) =: V_N^{\varepsilon}(t,x_1,\ldots,x_N), \tag{3.7}$$

$$\lim_{\substack{R_1,R_2\to+\infty\\ \varepsilon\to 0}} V_N^{\varepsilon,R}(t,x_1,\ldots,x_N) = \lim_{\substack{R_1\to+\infty\\ \varepsilon\to 0}} V_N^{\varepsilon,R_1}(t,x_1,\ldots,x_N) = V_N(t,x_1,\ldots,x_N), \tag{3.8}$$

where for the training processes in (3.4) we have introduced yet another objective function $J_N^{\varepsilon} : \mathcal{U}^{ad}\times[0,T]\times\mathbb{R}^N \to \mathbb{R}$ is defined as

$$J_N^{\varepsilon}(\theta,t,x_1,\ldots,x_N)$$

$$:= \mathbb{E}\bigg[\frac{1}{N}\sum_{i=1}^{N}\int_0^T L\big(X_N^{\varepsilon,\theta,i}(t)\big)dt + \frac{1}{N}\sum_{i=1}^{N} U\big(X_N^{\varepsilon,\theta,i}(T)\big) + \frac{\lambda}{2}\int_0^T |\theta(t)|^2\, dt\bigg].$$

After some modification of standard results on parabolic PDEs (that we detail below), we can show that the HJB equation (3.2) admits a solution $V_N^{\varepsilon,R} \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)\cap C\left([0,T]\times\mathbb{R}^N\right)$. In this section, we establish uniform estimates on $V_N^{\varepsilon,R}$ and its first order derivatives, especially uniform in (ε, N). Different from the usual PDE estimates, the estimates here are focused more on the dimension of variables since the dimension, which corresponds to the number of samples, is now changing. We begin with the existence and uniqueness of classical solution to (3.2).

Lemma 3.1. *Suppose that Assumption 2.1 takes place. Then the HJB equation* (3.2) *admits a unique bounded solution* $V_N^{\varepsilon,R} \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)\cap C\left([0,T]\times\mathbb{R}^N\right)$ *where* $0<\gamma<1$ *and* $\partial_t V_N^{\varepsilon,R}$, $\partial_{x_i} V_N^{\varepsilon,R}$, $\partial^2_{x_ix_j} V_N^{\varepsilon,R}$, $1\le i,j\le N$ *are bounded.*

Proof. Notice that L_{R_1} and U_{R_1} as well as their derivatives are all bounded. According to Theorem 4.4.3, Theorem 4.7.2 and Theorem 4.7.4 in [39], the value function $V_N^{\varepsilon,R}$ defined in (3.5) is the weak solution (in the distributional sense) to (3.2), furthermore, $V_N^{\varepsilon,R}$ and its weak derivatives $\partial_t V_N^{\varepsilon,R}$, $\partial_{x_i} V_N^{\varepsilon,R}$, $\partial^2_{x_ix_j} V_N^{\varepsilon,R}$, $1\le i,j\le N$ are all bounded. Note also that for $(t,x)\in(0,T)\times\mathbb{R}^N$

$$\partial_t V_N^{\varepsilon,R}(t,x) + \frac{\sigma^2}{2}\sum_{i,j=1}^{N}\partial^2_{x_ix_j}V_N^{\varepsilon,R}(t,x) + \frac{\varepsilon^2}{2}\sum_{i=1}^{N}\partial^2_{x_ix_i}V_N^{\varepsilon,R}(t,x) = g(t,x), \tag{3.9}$$

where

$$g(t,x) := -\inf_{\theta\in\Theta_{R_2}}\bigg\{\frac{\lambda}{2}|\theta|^2 + \sum_{i=1}^{N} f\bigg(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\bigg)\partial_{x_i}V_N^{\varepsilon,R} + \frac{1}{N}\sum_{i=1}^{N}L_{R_1}(x_i)\bigg\}.$$

As is shown above, $\partial^2_{x_ix_j}V_N^{\varepsilon,R}$ is bounded. Moreover, we have from Corollary 4.7.8 of [39] that for $0<\gamma<1$, $\nabla_x V_N^{\varepsilon,R}(t,x)$ is $\frac{\gamma}{2}$-Hölder with respect to t (uniformly in x). Hence $g(t,x)$ is locally Lipschitz continuous with respect to x and Hölder continuous with respect to t. Let us view $V_N^{\varepsilon,R}$ as the solution to PDE (3.9) with constant coefficients, where the terminal conditions are the same as (3.2). Standard results then yield that $\partial_t V_N^{\varepsilon,R}$, $\partial_{x_i} V_N^{\varepsilon,R}$, $\partial^2_{x_ix_j} V_N^{\varepsilon,R} \in C_{loc}^{\frac{\gamma}{2},\gamma}\left([0,T)\times\mathbb{R}^N\right)$, $1\le i,j\le N$.

As for the uniqueness, we can use the stochastic control interpretation to (3.2) and show that any solution $V_N^{\varepsilon,R}$ equals the value function in (3.5) by the standard verification results. □

Notice that at the moment the bound on $V_N^{\varepsilon,R}$, $\partial_{x_i} V_N^{\varepsilon,R}$, $\partial^2_{x_ix_j} V_N^{\varepsilon,R}$, $1\le i,j\le N$ might depend on ε, R and N. Before establishing uniform estimates with respect to ε, R and N, we need a refined analysis on the sample path.

Lemma 3.2. *Let* $x_i\in\mathbb{R}$, $1\le i\le N$, $\theta(t)$ *be an admissible control, and* $X_N(t)$ *be the associated sample path in* (3.4). *Then there exists a constant* $\tilde{C}_1 = \tilde{C}_1(f,T)$ *(depending only on* f,T*, independent of* $N,\varepsilon,\sigma,R_1,R_2$*), increasing in* T*, such that*

$$\mathbb{E}|X_N^{\varepsilon,\theta,i}(t)|^2 \le \tilde{C}_1\bigg(1+|x_i|^2+\mathbb{E}\int_0^t \big|f\big(s,\theta(s),0,\delta_{\{0\}}\big)\big|^2 ds + \frac{1}{N}\sum_{j=1}^{N}|x_j|^2\bigg). \tag{3.10}$$

Proof. For $x_1,\ldots,x_N\in\mathbb{R}$, denote by

$$\tilde{f}_i(s,\theta,x_1,\ldots,x_N) := f\bigg(s,\theta,x_i,\frac{1}{N}\sum_{k=1}^{N}\delta_{x_k}\bigg),$$

then

$$\partial_{x_j}\tilde{f}_i\big(s,\theta,x_1,\dots,x_N\big)=\delta_{ij}f_x\bigg(s,\theta,x_i,\frac{1}{N}\sum_{k=1}^N\delta_{x_k}\bigg)+\frac{1}{N}\partial_\mu f\bigg(s,\theta,x_i,\frac{1}{N}\sum_{k=1}^N\delta_{x_k},x_j\bigg),$$

where δ_{ij} stands for the Kronecker symbol. We represent the dynamics of $X_N^{\varepsilon,\theta,i}(t)$ in (3.4) in such a way that

$$\begin{aligned}X_N^{\varepsilon,\theta,i}(t)&=x_i+\int_0^t\tilde{f}_i\big(s,\theta(s),0,\dots,0\big)ds+\sum_{j=1}^N\int_0^t\Delta_j\tilde{f}_i\big(s,\theta(s),,X_N^{\varepsilon,\theta,1}(s),\dots,X_N^{\varepsilon,\theta,N}(s)\big)ds\\&+\sigma W^0(t)+\varepsilon W^i(t),\end{aligned}$$

where for $j=1,\dots,N$,

$$\begin{aligned}&\Delta_j\tilde{f}_i\big(s,\theta(s),X_N^{\varepsilon,\theta,1}(s),\dots,X_N^{\varepsilon,\theta,N}(s)\big)\\:=\ &\tilde{f}_i\big(s,\theta(s),\underbrace{0,0,\dots,0,}_{(j-1)\text{ - times}}X_N^{\varepsilon,\theta,j}(s),\dots,X_N^{\varepsilon,\theta,N}(s)\big)-\tilde{f}_i\big(s,\theta(s),\underbrace{0,0,\dots,0,}_{j\text{ - times}}X_N^{\varepsilon,\theta,j+1}(s),\dots,X_N^N(s)\big).\end{aligned}$$

According to the Lipschitz continuity, we can deduce

$$\big|\Delta_j\tilde{f}_i\big(s,\theta(s),X_N^{\varepsilon,\theta,1}(s),\dots,X_N^{\varepsilon,\theta,N}(s)\big)\big|\le\big(\delta_{ij}\|\partial_x f\|_\infty+N^{-1}\|\partial_\mu f\|_\infty\big)\big|X_N^j(s)\big|.$$

Therefore there exist constant $C_1=C_1(f)$ and the corresponding matrix valued process $A_N(s)$ satisfying $A_N(s)\in M_N(C_1)$ such that

$$\begin{aligned}X_N^{\varepsilon,\theta}(t)&=x+\int_0^t f\big(s,\theta(s),0,\delta_{\{0\}}\big)\mathbf{1}ds+\int_0^t A_N(s)X_N^{\varepsilon,\theta}(s)ds\\&+\varepsilon\mathbf{W}^N(t)+\sigma\mathbf{1}W^0(t),\end{aligned}$$

where

$$\mathbf{W}^N(t):=(W^1(t),\dots,W^N(t))^\top,\mathbf{1}:=(1,\dots,1)^\top.$$

Here for the brevity of expression we have introduced the subset $M_N(C)\subset\mathbb{R}^{N\times N}$ such that

$$A\in M_N(C)\quad\text{if and only if}\quad|A_{ij}|\le C(\delta_{ij}+N^{-1}),\ 1\le i,j\le N.\tag{3.11}$$

Solving the linear SDE above, we have

$$\begin{aligned}X_N^i(t)&=\big(\Phi_N^+(t)x\big)_i+\int_0^t f\big(s,\theta(s),0,\delta_{\{0\}}\big)\big(\Phi_N^+(t)\Phi_N^-(s)\mathbf{1}\big)_i ds\\&+\varepsilon\bigg(\int_0^t\Phi_N^+(t)\Phi_N^-(s)d\mathbf{W}^N(s)\bigg)_i+\sigma\int_0^t\big(\Phi_N^+(t)\Phi_N^-(s)\mathbf{1}\big)_i dW^0(s),\end{aligned}\tag{3.12}$$

where the matrix valued processes $\Phi_N^\pm(s)$ solve

$$\Phi_N^+(t)=I_N+\int_0^t A_N(s)\Phi_N^+(s)ds,\quad\Phi_N^-(t)=I_N-\int_0^t\Phi_N^-(s)A_N(s)ds.$$

Note that

$$\frac{d}{dt}\big[\Phi_N^-(t)\Phi_N^+(t)\big]=0\quad\text{and}\quad\Phi_N^-(0)\Phi_N^+(0)=I_N,$$

thus $\Phi_N^-(t)\Phi_N^+(t)=\Phi_N^-(0)\Phi_N^+(0)=I_N$.

According to Lemma A.2, $\Phi_N^\pm(s)\in M_N(C_2)$ for some $C_2=C_2(f,T)$ because $A_N(s)\in M_N(C_1)$. Moreover, $\Phi_N^+(t)\Phi_N^-(s)\in M_N(C_2)$ due to Lemma A.1. An application of Burkholder–Davis–Gundy inequality (see e.g. [54]) to the i-th component in (3.12) gives the estimate (3.10). □

Remark 3.3. *Take $\theta(t) \equiv 0$ (which is admissible since $0 \in \Theta$), then (3.10) can be rephrased as*

$$\mathbb{E}|X_N^{\varepsilon,0,i}(t)|^2 \leq \tilde{C}_1\left(1 + |x_i|^2 + \frac{1}{N}\sum_{j=1}^N |x_j|^2\right). \tag{3.13}$$

Based on Lemma 3.2, we can go on with the estimates on the first derivatives. In the context below, the values of constants C_k, $\tilde{C}_k$, $k \geq 1$, might vary, but their dependence on the model parameters remains the same.

For $(t, p, q, z, \theta) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R}^N$, define the following Hamiltonian

$$H_N^{R_1}(t, x, p, \theta) := \lambda\big|\theta\big|^2 + \sum_{i=1}^N f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right)p_i + \frac{1}{N}\sum_{i=1}^N L_{R_1}(x_i), \tag{3.14}$$

as well as, for later use,

$$H_N(t, x, p, \theta) := \lambda\big|\theta\big|^2 + \sum_{i=1}^N f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right)p_i + \frac{1}{N}\sum_{i=1}^N L(x_i). \tag{3.15}$$

With the preparation above, we show the following estimates on $V_N^{\varepsilon,R}$ in (3.2).

Lemma 3.4. *Let $x_i \in \mathbb{R}$, $1 \leq i \leq N$ and $V_N^{\varepsilon,R}$ be the solution to (3.2). Then there exists a constant $\tilde{C}_2 = \tilde{C}_2(f, \lambda^{-\frac{1}{2}}, T)$, increasing in $T, \lambda^{-\frac{1}{2}}$, independent of N, σ, ε and R such that for $1 \leq i \leq N$,*

$$\big|\partial_{x_i} V_N^{\varepsilon,R}(t, x)\big| \leq \frac{\tilde{C}_2(C_{11}^L + C_{11}^U)}{N}\left(1 + |x_i|^2 + \frac{1}{N}\sum_{j=1}^N |x_j|^2\right)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L + C_{10}^U)}{N}. \tag{3.16}$$

Proof. We begin by showing the existence of a constant $\hat{C}_2 = \hat{C}_2(f, T)$ such that

$$\begin{aligned}\Big|\partial_{x_i} V_N^{\varepsilon,R}(t, x)\Big| \leq{}& \frac{\hat{C}_2(C_{11}^L + C_{11}^U)}{N}\left(1 + |x_i|^2 + \mathbb{E}\left[\int_0^T |f\big(s, \theta^*(s), 0, \delta_{\{0\}}\big)|ds\right] + \frac{1}{N}\sum_{j=1}^N |x_j|^2\right)^{\frac{1}{2}} \\ &+ \frac{\hat{C}_2(C_{10}^L + C_{10}^U)}{N},\end{aligned} \tag{3.17}$$

where θ^* is the optimal control process.

It suffices to show the existence of such $\hat{C}_2$ for $t = 0$. For other $t \in [0, T]$ the proof and $\hat{C}_2$ can be deduced in the same way.

In view of Lemma 3.1, the HJB equation (3.2) admits a classical solution. Following the standard verification procedure (see e.g. [27]), one can show the existence of an optimal control (in the weak sense) and the corresponding optimal path. Hence we may denote by $\theta^*(t)$ and $(X_N^*(t), Y_N^*(t))$ the optimal control, optimal path as well as the adjoint process. According to the stochastic maximum principle, we have the adjoint equation (in the weak sense) as follow

$$\begin{cases} dY_N^{*,i}(t) = -\partial_{x_i} H_N^{R_1}\left(t, X_N^*(t), Y_N^*(t), \theta^*(t)\right) dt + \sum_{j=0}^N Z_N^{ij}(t) dW^j(t), \\ dX_N^{*,i}(t) = \partial_{p_i} H_N^{R_1}\left(t, X_N^*(t), Y_N^*(t), \theta^*(t)\right) dt + \varepsilon dW^i(t) + \sigma dW^0(t), \\ X_N^*(0) = x, \quad Y_N^{*,i}(T) = \frac{1}{N} U'_{R_1}\left(X_N^{*,i}(T)\right). \end{cases} \tag{3.18}$$

Here $Y_N^*,\ X_N^* \in \mathbb{R}^N,\ Z_N \in \mathbb{R}^{N\times N}$, and recall that $H_N^{R_1}(t,x,p,\theta)$ is given in (3.14). Rewrite (3.18) in the following manner:

$$\begin{cases} dY_N^{*,i}(t) = -\Big[\sum_{j=1}^N A_{ij}^N(t)Y_N^{*,j}(t) + \frac{1}{N}L'_{R_1}(X_N^{*,i}(t))\Big]dt + \sum_{j=0}^N Z_N^{ij}(t)dW^j(t) \\ dX_N^{*,i}(t) = f\Big(t,\theta^*(t),X_N^{*,i}(t),\frac{1}{N}\sum_{j=1}^N \delta_{X_N^{*,j}(t)}\Big)dt + \varepsilon dW^i(t) + \sigma dW^0(t), \\ X_N^*(0) = x, \quad Y_N^{*,i}(T) = \frac{1}{N}U'_{R_1}\big(X_N^{*,i}(T)\big), \end{cases} \tag{3.19}$$

where for $1 \le i,j \le N$,

$$A_{ij}^N(t) := \delta_{ij}f_x\Big(t,\theta^*(t),X_N^{*,j}(t),\frac{1}{N}\sum_{k=1}^N \delta_{X_N^{*,k}(t)}\Big) + \frac{1}{N}\partial_\mu f\Big(t,\theta^*(t),X_N^{*,j}(t),\frac{1}{N}\sum_{k=1}^N \delta_{X_N^{*,k}(t)},X_N^{*,i}(t)\Big). \tag{3.20}$$

Consider the matrix valued processes $\Phi_N^\pm(t) \in \mathbb{R}^{N\times N}$ solving

$$\Phi_N^+(t) = I_N - \int_0^t A^N(s)\Phi_N^+(s)ds, \tag{3.21}$$

$$\Phi_N^-(t) = I_N + \int_0^t \Phi_N^-(s)A^N(s)ds. \tag{3.22}$$

Then

$$\Phi_N^-(t)Y_N^*(t) = \frac{1}{N}\mathbb{E}_t\big[\Phi_N^-(T)U'_{R_1}\big(X_N^*(T)\big)\big] + \frac{1}{N}\mathbb{E}_t\Big[\int_t^T \Phi_N^-(s)L'_{R_1}\big(X_N^*(s)\big)ds\Big], \tag{3.23}$$

where

$$U'_{R_1}(X_N^*(T)) := \Big(U'_{R_1}(X_N^{*,1}(T)),\ldots,U'_{R_1}(X_N^{*,N}(T))\Big)^\top,$$
$$L'_{R_1}(X_N^*(s)) := \Big(L'_{R_1}(X_N^{*,1}(s)),\ldots,L'_{R_1}(X_N^{*,N}(s))\Big)^\top.$$

In particular,

$$Y_N^*(0) = \frac{1}{N}\mathbb{E}\left[\Phi_N^-(T)U'\left(X_N^*(T)\right)\right] + \frac{1}{N}\mathbb{E}\left[\int_0^T \Phi_N^-(s)L'_{R_1}\left(X_N^*(s)\right)ds\right],$$

and thus

$$Y_N^{*,i}(0)^2 \le \frac{2}{N^2}\big|\mathbb{E}\big(\Phi_N^-(T)U'_{R_1}(X_N^*(T))\big)_i\big|^2 + \frac{2T}{N^2}\int_0^T \big|\mathbb{E}\big(\Phi_N^-(s)L'_{R_1}(X_N^*(s))\big)_i\big|^2 ds \tag{3.24}$$

According to (3.20), $A_N \in M_N(C_1)$ with $C_1 = C_1(f)$. In view of Lemma A.2 and (3.22), for $A_N(t) \in M_N(C_1)$, it follows that

$$\mathbb{E}\Phi_N^-(s) \in M_N(C_2),\ C_2 = C_2(f,T),\ s \in [0,T].$$

Note (2.2) and (3.3), for the i-th entry of $\Phi_N^-(T)U'_{R_1}\left(X_N^*(T)\right)$:

$$\begin{aligned}&\big|\mathbb{E}\big(\Phi_N^-(T)U'_{R_1}(X_N^*(T))\big)_i\big|^2 \\ &\le 2\mathbb{E}\Big[\big(\Phi_N^-(T)\big)_{ii}^2\Big]\mathbb{E}\Big[U'_{R_1}\big(X_N^{*,i}(T)\big)^2\Big] + 2\mathbb{E}\Big[\sum_{\substack{j=1\\ j\neq i}}^N \big(\Phi_N^-(T)\big)_{ij}^2\Big]\mathbb{E}\Big[\sum_{\substack{j=1\\ j\neq i}}^N U'_{R_1}\big(X_N^{*,j}(T)\big)^2\Big]\end{aligned}$$

$$\le C_2\mathbb{E}\Big[U'_{R_1}\big(X_N^{*,i}(T)\big)^2\Big]+\frac{C_2}{N}\mathbb{E}\Big[\sum_{\substack{j=1\\ j\neq i}}^N U'_{R_1}\big(X_N^{*,j}(T)\big)^2\Big]$$
$$\le C_2(C_{11}^U)^2\bigg(1+\mathbb{E}\big[|X_N^{*,i}(T)|^2\big]+\frac{1}{N}\sum_{j=1}^N\mathbb{E}\big[|X_N^{*,j}(T)|^2\big]\bigg)+C_2(C_{10}^U+1)^2. \tag{3.25}$$

Similarly,

$$\big|\mathbb{E}\big(\Phi_N^-(s)L'_{R_1}(X_N^*(s))\big)_i\big|^2$$
$$\le C_2(C_{11}^U)^2\bigg(1+\mathbb{E}\big[|X_N^{*,i}(s)|^2\big]+\frac{1}{N}\sum_{j=1}^N\mathbb{E}\big[|X_N^{*,j}(s)|^2\big]\bigg)+C_2(C_{10}^U+1)^2. \tag{3.26}$$

In view of Lemma 3.2,

$$\mathbb{E}\big[\big|X_N^{*,i}(t)\big|\big]\le\tilde{C}_1\bigg(|x_i|^2+\mathbb{E}\int_0^t\big|f\big(s,\theta^*(s),0,\delta_{\{0\}}\big)\big|ds+\frac{1}{N}\sum_{j=1}^N|x_j|^2\bigg). \tag{3.27}$$

Plugging (3.25), (3.26) and (3.27) into (3.24), we obtain (3.17).

To further prove (3.16), it suffices to prove that there exist constant $\check{C}=\check{C}(f,\lambda^{-\frac{1}{2}},T)$ (increasing in $\lambda^{-\frac{1}{2}}$) such that

$$\mathbb{E}\int_0^t\big|f\big(s,\theta^*(s),0,\delta_{\{0\}}\big)\big|ds\le\check{C}\bigg(1+\frac{1}{N}\sum_{i=1}^N|x_i|^2\bigg)^{\frac{1}{2}}.$$

In fact,

$$\mathbb{E}\int_0^t\big|f\big(s,\theta^*(s),0,\delta_{\{0\}}\big)\big|ds\le\int_0^t\big|f\big(s,0,0,\delta_{\{0\}}\big)\big|ds+\|f_\theta\|_\infty\mathbb{E}\int_0^t|\theta^*(s)|ds.$$

And we notice that

$$\mathbb{E}\int_0^t|\theta^*(s)|ds\le T^{\frac{1}{2}}\mathbb{E}\bigg(\int_0^T|\theta^*(s)|^2ds\bigg)^{\frac{1}{2}}\le(2T)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}J_N(\theta^*,0,x_1,\ldots,x_N)^{\frac{1}{2}}$$
$$\le(2T)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}J_N(0,0,x_1,\ldots,x_N)^{\frac{1}{2}}\le\check{C}\bigg(1+\frac{1}{N}\sum_{i=1}^N|x_i|^2\bigg)^{\frac{1}{2}}, \tag{3.28}$$

where the last inequality holds because of (3.13). Hence we may deduce (3.16) from the estimates above. ☐

The next lemma shows that, thanks to Lemma 3.4, we may drop the parameter R_2 in (3.2) and consider

$$\begin{cases}\partial_t V_N^{\varepsilon,R_1}+\dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}+\dfrac{\varepsilon^2}{2}\displaystyle\sum_{i=1}^N\partial^2_{x_ix_i}V_N^{\varepsilon,R_1}+\tilde{H}_N^{R_1}(t,x,\nabla_x V_N^{\varepsilon,R_1})=0,\\[2ex] V_N^{\varepsilon,R_1}(T,x_1,\ldots,x_N)=\dfrac{1}{N}\displaystyle\sum_{i=1}^N U_{R_1}(x_i),\end{cases} \tag{3.29}$$

where for $(t,x,p)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^N$,

$$\tilde{H}_N^{R_1}(t,x,p):=\inf_{\theta\in\mathbb{R}}\bigg\{\frac{\lambda}{2}\big|\theta\big|^2+\sum_{i=1}^N f\bigg(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^N\delta_{x_j}\bigg)p_i+\frac{1}{N}\sum_{i=1}^N L_{R_1}(x_i)\bigg\}. \tag{3.30}$$

Lemma 3.5. *The equation* (3.29) *admits a unique classical solution* $V_N^{\varepsilon,R_1} \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right) \cap C\left([0,T]\times\mathbb{R}^N\right)$ *where for* $0<\gamma<1$ *and* V_N^{ε,R_1}, $\partial_t V_N^{\varepsilon,R_1}$, $\partial_{x_i} V_N^{\varepsilon,R_1}$, $\partial^2_{x_ix_j} V_N^{\varepsilon,R_1}$, $1\le i,j\le N$ *are bounded. Moreover, the derivatives* $\partial_{x_i} V_N^{\varepsilon,R_1}$, $1\le i\le N$, *satisfy*

$$\left|\partial_{x_i} V_N^{\varepsilon,R_1}(t,x)\right| \le \frac{\tilde{C}_2(C_{11}^L+C_{11}^U)}{N}\left(1+|x_i|^2+\frac{1}{N}\sum_{j=1}^N |x_j|^2\right)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L+C_{10}^U)}{N}, \tag{3.31}$$

where the constant $\tilde{C}_2$ *is from Lemma 3.4.*

Proof. We recall Lemma 3.1 saying that (3.2) admits classical solutions $V_N^{\varepsilon,R}$. Moreover, since L_{R_1} and U_{R_1} both have bounded derivatives, in view of Lemma 3.4, $\left|\nabla_x V_N^{\varepsilon,R}\right|$ is bounded by a constant independent of R_2. Therefore, for sufficiently large R_2, we have

$$\begin{aligned}&\inf_{\theta\in\Theta_{R_2}}\left\{\frac{\lambda}{2}|\theta|^2+\sum_{i=1}^N f\left(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^N\delta_{x_j}\right)\partial_{x_i}V_N^{\varepsilon,R}(t,x)+\frac{1}{N}\sum_{i=1}^N L_{R_1}(x_i)\right\}\\ &=\inf_{\theta\in\mathbb{R}}\left\{\cdots\right\} = \tilde{H}_N^{R_1}\left(t,x,\nabla_x V^{\varepsilon,R}(t,x)\right).\end{aligned} \tag{3.32}$$

In other words, for those R_2 satisfying (3.32), $V_N^{\varepsilon,R}$ solves (3.29). Choose an arbitrary R_2 such that $V_N^{\varepsilon,R}$ satisfies (3.32) and denote it by V_N^{ε,R_1}. We thus have by Lemma 3.1 that $V_N^{\varepsilon,R} \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right) \cap C\left([0,T]\times\mathbb{R}^N\right)$ and $V_N^{\varepsilon,R}, \partial_t V_N^{\varepsilon,R}$, $\partial_{x_i}V_N^{\varepsilon,R}$, $\partial^2_{x_ix_j}V_N^{\varepsilon,R}$ are bounded. We can show the uniqueness via the variational arguments described in Lemma 3.1. We can also obtain (3.31) from Lemma 3.4 since it is satisfied by any $V_N^{\varepsilon,R}$. □

3.2 The estimates on the second order derivatives

In this section we establish uniform estimates on the second order derivatives $\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}$, $1\le i,j\le N$ for the solution to (3.29) where the parameter R_2 has been dropped. To do so, our idea is to formally take the derivatives with respect to x_i and x_j in (3.29) and obtain the PDE system on $\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}$, $1\le i,j\le N$. The above differentiation requires further analysis on the differentiability of Hamiltonian in (3.29). It then turns out that the aforementioned analysis involves the uniform estimates on the first order derivatives in (3.31). We can see from (3.31) that the first order derivatives therein are only locally bounded in general. In our forthcoming analysis, we propose some technical assumptions so as to deal with this non-global boundedness.

Denote by $\mathcal{A}_N$ the set consisting of real numbers $p_1, \ldots, p_N, x_1, \ldots, x_N$ satisfying

$$|p_i| < \frac{\tilde{C}_2(C_{11}^L+C_{11}^U)}{N}\left(1+|x_i|^2+\frac{1}{N}\sum_{j=1}^N |x_j|^2\right)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L+C_{10}^U)}{N}. \tag{3.33}$$

In other words,

$$\mathcal{A}_N := \left\{(x,p)\in\mathbb{R}^N\times\mathbb{R}^N : (x,p) \text{ satisfies } (3.33)\right\}. \tag{3.34}$$

We assume that the following hold in the remaining of the paper.

Hypothesis (R) Suppose the following

1. There exists $\lambda_0>0$, such that for any $(\theta,x,p)\in\Theta\times\mathcal{A}_N$, $N\ge 1$

$$\frac{\lambda_0}{N} \ge \partial^2_{\theta\theta} f\left(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^N\delta_{x_j}\right)p_i. \tag{3.35}$$

2. There exists $C^Q > 0$ such that for any $(\theta, x, p) \in \Theta \times \mathcal{A}_N$, $N \geq 1$ and for $\varphi \in \{|\partial^2_{x\theta} f|, |\partial^2_{xx} f|\}$ and $\phi \in \{|\partial^2_{x\mu} f|,\ |\partial^2_{\theta\mu} f|\}$

$$C^Q > \varphi\Big(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big) \cdot N p_i + \phi\Big(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}, x_i\Big) \cdot N p_i. \tag{3.36}$$

3. The coefficient λ is taken such that $\lambda > \lambda_0$.

Remark 3.6. *In terms of* **Hypothesis (R)***, we have the following examples.*

1. *For an LQ model with uncontrolled diffusion, $\partial^2_{x\theta} f, \partial^2_{xx} f, \partial^2_{x\mu} f, \partial^2_{\theta\mu} f = 0$ and (3.35), (3.36) holds trivially.*

2. *For f, L, U with bounded derivatives, $C^L_{11} + C^U_{11} = 0$ and (3.35), (3.36) holds trivially.*

Given (3.35), for $(x,p) \in \mathcal{A}_N$, the corresponding $H^{R_1}_N(t,x,p,\theta)$ in (3.14) is strictly convex in θ. Hence the unique minimizer $\theta^{R_1}_N \in \Theta$ can be defined as a function of (t,x,p) in such a way that

$$\theta^{R_1}_N(t,x,p) := \arg\min_{\theta\in\Theta} H^{R_1}_N(t,x,p,\theta). \tag{3.37}$$

In light of the definition above, an optimal control $\theta^*(t)$ in (3.18) can be represented as

$$\theta^*(t) = \theta^{R_1}_N(t, X^*_N(t), Y^*_N(t)).$$

Thanks to **Hypothesis (R)**, we can now show the Lipschitz continuity of the feedback function $\theta^{R_1}_N(t,x,p)$.

Lemma 3.7. *Suppose* **Hypothesis (R)**. *Then $\theta^{R_1}_N(t,x,p)$ is smooth with respect to $(x,p) \in \mathcal{A}_N$ with derivatives*

$$\left|\partial_{x_k}\theta^{R_1}_N(t,x,p)\right| \leq \frac{(\lambda-\lambda_0)^{-1}C^Q}{N},\ \left|\partial_{p_k}\theta^{R_1}_N(t,x,p)\right| \leq (\lambda-\lambda_0)^{-1}\|f_\theta\|_\infty,\ k = 1,\dots,N. \tag{3.38}$$

Proof. We postpone the proof of this result to Appendix A. ☐

We have the following estimates on the coefficients based on Lemma 3.7.

Lemma 3.8. *Suppose* **Hypothesis (R)***, then there exists a constant $\tilde{C}_3 = \tilde{C}_3(f, \lambda^{\frac{1}{2}}, T, L, (\lambda-\lambda_0)^{-1})$, increasing in T, $\lambda^{-\frac{1}{2}}$, $(\lambda-\lambda_0)^{-1}$, such that for $(x,p) \in \mathcal{A}_N$,*

$$\begin{aligned}
&\left|\partial_{x_i}\tilde{H}^{R_1}_N(t,x,p)\right|, \left|\partial^2_{x_ip_j}\tilde{H}^{R_1}_N(t,x,p)\right| \leq \tilde{C}_3 N^{-1}, \quad \left|\partial^2_{x_ix_j}\tilde{H}^{R_1}_N(t,x,p)\right| \leq \tilde{C}_3 N^{-1}(\delta_{ij} + N^{-1}),\\
&\left|\partial_{p_i}\tilde{H}^{R_1}_N(t,x,p)\right|, \left|\partial^2_{p_ip_j}\tilde{H}^{R_1}_N(t,x,p)\right| \leq \tilde{C}_3, \quad 1 \leq i,j \leq N.
\end{aligned} \tag{3.39}$$

Proof. Recall (3.30) and (3.37),

$$\tilde{H}^{R_1}_N(t,x,p) = H^{R_1}_N\big(t,x,p,\theta^{R_1}_N(t,x,p)\big), \quad (x,p) \in \mathcal{A}_N.$$

Hence we can obtain the above estimates via (3.38). ☐

3.2.1 Short time estimates

As is mentioned before, with the preparation above, we may take partial derivatives in (3.29) and derive the equation satisfied by $V^{\varepsilon,kl}_N := \partial^2_{x_kx_l} V^{\varepsilon,R_1}_N$. We begin with a regularity results which validates the differentiation.

Lemma 3.9. *Suppose* **Hypothesis (R)**. *The equation* (3.29) *admits a unique classical solution* $V_N^{\varepsilon,R_1} \in C\left([0,T]\times\mathbb{R}^N\right)$ *where* $V_N^{\varepsilon,R_1}, \partial_{x_i}V_N^{\varepsilon,R_1}, \partial^2_{x_ix_j}V_N^{\varepsilon,R_1}, 1\le i,j\le N$ *are bounded. Moreover for* $0<\gamma<1$, $V_N^{\varepsilon,R_1}, \partial_{x_i}V_N^{\varepsilon,R_1}, \partial^2_{x_ix_j}V_N^{\varepsilon,R_1} \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right), 1\le i,j\le N$.

Proof. In Lemma 3.5 we have shown that the solution to (3.29) $V_N^{\varepsilon,R_1}\in C([0,T]\times\mathbb{R}^N)$ has bounded derivatives $V_N^{\varepsilon,R_1}, \partial_{x_i}V_N^{\varepsilon,R_1}, \partial^2_{x_ix_j}V_N^{\varepsilon,R_1}, 1\le i,j\le N$. In order to show the higher regularity of V_N^{ε,R_1}, let R_2 be sufficiently large and take ∂_{x_i} $(1\le i\le N)$ in (3.9) to obtain the linear PDE satisfied by $\partial_{x_i}V_N^{\varepsilon,R_1}$. Notice that when R_2 is sufficiently large,

$$\partial_{x_i}g(t,x)=\partial_{x_i}\left(\tilde{H}_N(x,\nabla_x V_N^{\varepsilon,R_1})\right)\in C_{loc}^{\frac{\gamma}{2},\gamma}\left([0,T)\times\mathbb{R}^N\right).$$

So we have by the standard results on linear PDE that $\partial_{x_i}V_N^{\varepsilon,R_1}\in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right), 1\le i\le N$. In view of Lemma 3.8, we may repeat the previous procedure once more, i.e., take $\partial^2_{x_ix_j}$ $(1\le i,j\le N)$ in (3.9) and show that $\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}\in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right), 1\le i,j\le N$. □

Denote by $V_N^{\varepsilon,R_1,kl}=\partial^2_{x_kx_l}V_N^{\varepsilon,R_1}$, $1\le k,l\le N$. By direct calculation, applying $\partial^2_{x_kx_l}$ to the equation (3.29), one obtains

$$\begin{cases} \partial_t V_N^{\varepsilon,R_1,kl}+\dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N \partial^2_{x_ix_j}V_N^{\varepsilon,kl}+\dfrac{\varepsilon^2}{2}\sum_{i=1}^N\partial^2_{x_ix_i}V_N^{\varepsilon,kl}+\partial^2_{x_kx_l}\tilde{H}_N^{R_1}(t,x,\nabla_x V_N^{\varepsilon,R_1}) \\ \qquad +\displaystyle\sum_{i=1}^N\partial_{p_i}\tilde{H}_N^{R_1}(t,x,\nabla_xV_N^{\varepsilon,R_1})\partial_{x_i}V_N^{\varepsilon,R_1,kl}+\sum_{i,j=1}^N\partial^2_{p_ip_j}\tilde{H}_N^{R_1}(t,x,\nabla_xV_N^{\varepsilon,R_1})V_N^{\varepsilon,R_1,ki}V_N^{\varepsilon,R_1,jl} \\ \qquad +\displaystyle\sum_{i=1}^N\partial^2_{x_lp_i}\tilde{H}_N^{R_1}(t,x,\nabla_xV_N^{\varepsilon,R_1})V_N^{\varepsilon,R_1,ki}+\sum_{i=1}^N\partial^2_{x_kp_i}\tilde{H}_N^{R_1}(t,x,\nabla_xV_N^{\varepsilon,R_1})V_N^{\varepsilon,R_1,li} \\ \qquad =0, \\ \qquad V_N^{\varepsilon,R_1,kl}(T,x_1,\dots,x_N)=\dfrac{\delta_{kl}}{N}U''_{R_1}(x_k), \quad 1\le k,l\le N. \end{cases} \tag{3.40}$$

The equation above enables us to arrive to the results on the second order derivatives via nonlinear Feynman–Kac representation. In the current subsection we present the estimates on the second order derivatives for short time.

Proposition 3.10. *Suppose* **Hypothesis (R)**. *There exists a constant* $\tilde{c}=\tilde{c}(f,\lambda^{-\frac{1}{2}},L,U,(\lambda-\lambda_0)^{-1})$ *and* $\tilde{C}_4=\tilde{C}_4(f,L,U,(\lambda-\lambda_0)^{-1})$, $\tilde{C}_4$ *increasing in* $(\lambda-\lambda_0)^{-1}$, *such that for* $T<\tilde{c}$, *PDE* (3.40) *admits a unique bounded solution satisfying for* $1\le i,j\le N$,

$$\left|V_N^{\varepsilon,R_1,ij}(t,x)\right|\le \tilde{C}_4N^{-1}(\delta_{ij}+N^{-1}),\quad (t,x)\in[0,T]\times\mathbb{R}^N. \tag{3.41}$$

Proof. For $x=(x_1,\dots,x_N)\in\mathbb{R}^N$, consider

$$dX_t^i=\partial_{p_i}\tilde{H}_N^{R_1}\left(t,X_t,\nabla_xV_N^{\varepsilon,R_1}(t,X_t)\right)dt+\sigma dW_t^i+\varepsilon dW_t^0, X_0^i=x_i, \tag{3.42}$$

as well as

$$Y_t^{kl}=V_N^{\varepsilon,R_1,kl}(t,X_t),\quad 1\le k,l\le N. \tag{3.43}$$

According to Lemma 3.5, we can deduce the existence of a constant C depending on R_1 such that

$$\left|\nabla_xV_N^{\varepsilon,R_1}(t,X_t)\right|\le C.$$

The estimate above and the first order condition associated to (3.30) yields

$$\left|\partial_{p_i}\tilde{H}_N^{R_1}(t,X_t,\nabla_xV_N^{\varepsilon,R_1}(t,X_t))\right|\le C(1+|X_t|). \tag{3.44}$$

Hence SDE (3.42) admits a weak solution satisying

$$\mathbb{E}\bigg[\max_{0\le t\le T}|X_t|^m\bigg]\le C(1+|x|^m),\quad m\ge 1.$$

In view of Lemma 3.5,

$$|Y_t^{kl}|\le C,\quad 1\le k,l\le N,$$

where the constant C might depend on ε and R_1.

In view of (3.40) and the estimates above, we can infer from the nonlinear Feynman–Kac representation that the matrix process $Y(t)$ satisfies the backward stochastic Riccati equation

$$\begin{aligned}Y_t=\mathbb{E}_t\bigg\{\frac{1}{N}\tilde U(T)+\int_t^T\bigg[&\nabla_{xx}^2\tilde H_N^{R_1}\big(s,X_s,\nabla_x V_N^{\varepsilon,R_1}(s,X_s)\big)+Y_s\nabla_{xp}^2\tilde H_N^{R_1}\big(s,X_s,\nabla_x V_N^{\varepsilon,R_1}(s,X_s)\big)\\&+\nabla_{xp}^2\tilde H_N^{R_1}\big(s,X_s,\nabla_x V_N^{\varepsilon,R_1}(s,X_s)\big)Y_s+Y_s\nabla_{pp}^2\tilde H_N^{R_1}\big(s,X_s,\nabla_x V_N^{\varepsilon,R_1}(s,X_s)\big)Y_s\bigg]ds\bigg\},\end{aligned}\tag{3.45}$$

where the matrix $\tilde U(T)$ is given by

$$\tilde U^{ij}(T)=\delta_{ij}U''(X_T^i),\quad 1\le i,j\le N.\tag{3.46}$$

Next, define the mapping from the set of adapted matrix processes to itself

$$\Phi:\quad L^\infty\big(\Omega;C([0,T];\mathbb{R}^{N\times N})\big)\longrightarrow L^\infty\big(\Omega;C([0,T];\mathbb{R}^{N\times N})\big),\quad \Phi(Y)=\tilde Y,$$

such that for $t\in[0,T]$,

$$\begin{aligned}\tilde Y_t=\mathbb{E}_t\bigg[\frac{1}{N}\tilde U(T)+\int_t^T&\big[\nabla_{xx}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^{\varepsilon,R_1}(s,X_s))+Y_s\nabla_{xp}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^{\varepsilon,R_1}(s,X_s))\\&+\nabla_{xp}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^{\varepsilon,R_1}(s,X_s))Y_s+Y_s\nabla_{pp}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^{\varepsilon,R_1}(s,X_s))Y_s\big]ds\bigg].\end{aligned}$$

We can see that Y_t in (3.43) is a fixed point of Φ. Next we show that such fixed point is unique. In fact, let Y_t^* and Y_t^{**} be two bounded fixed points. And consider their norm of the following form

$$\max_{0\le t\le T}\big\|Y_t^*\big\|_\infty:=\max_{0\le t\le T}\max_{1\le i\le N}\sum_{j=1}^N\big|Y_t^{*,ij}\big|\le C,\quad \max_{0\le t\le T}\big\|Y_t^{**}\big\|_\infty\le C.$$

Then for $t\in[T-\delta,T]$ and $\tilde C=\tilde C_3$ depending only on $\tilde C_3$ from (3.39),

$$\begin{aligned}&\big\|Y_t^*-Y_t^{**}\big\|_\infty\\&\le\mathbb{E}_t\bigg[\int_t^T\big(\big\|Y_s^*\big\|_\infty+\big\|Y_s^{**}\big\|_\infty\big)\big\|\nabla_{pp}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^\varepsilon(s,X_s))\big\|_\infty\big\|Y_s^*-Y_s^{**}\big\|_\infty ds\bigg]\\&\quad+2\mathbb{E}_t\bigg[\int_t^T\big\|\nabla_{xp}^2\tilde H_N^{R_1}(s,X_s,\nabla_s V_N^\varepsilon(s,X_s))\big\|_\infty\big\|Y_s^*-Y_s^{**}\big\|_\infty ds\bigg]\\&\le 2(C+1)\tilde C\mathbb{E}_t\bigg[\int_t^T\big\|Y_s^*-Y_s^{**}\big\|_\infty ds\bigg]\le 2(C+1)\tilde C\delta\max_{T-\delta\le s\le T}\big\|Y_s^*-Y_s^{**}\big\|_\infty\quad a.s..\end{aligned}$$

Choose $2(C+1)\tilde C\delta<1$, then the inequality above implies that $Y_t^*=Y_t^{**}$ on $t\in[T-\delta,T]$. Repeat the above procedure, we can show that $Y_t^*=Y_t^{**}$ on $t\in[T-\delta,T]$, $[T-2\delta,T-\delta]$ and after finite times repetitions we obtain $Y_t^*=Y_t^{**}$ on $t\in[0,T]$. The uniqueness above thus tells that Y_t in (3.43) is the only bounded fixed point of Φ.

To continue, define the closed subset $\mathcal{B}(N,K)$ of adapted matrix processes in such a way that $Y \in \mathcal{B}(N,K)$ if and only if

$$\max_{t\in[0,T]} |Y_t^{ij}| \le KN^{-1}(\delta_{ij} + N^{-1}) \quad \text{a.s.}, \tag{3.47}$$

where the constant $K > 0$ is to be determined.

We claim that for appropriate K and $\tilde{c}$ (independent of N), Φ is invariant on $\mathcal{B}(N,K)$, and Φ is a contraction mapping on $\mathcal{B}(N,K)$ with $T < \tilde{c}$.

Let $Y_t^{(1)}$ and $Y_t^{(2)}$ be two inputs from $\mathcal{B}(N,K)$ and $\tilde{Y}_t^{(1)}$ and $\tilde{Y}_t^{(2)}$ be the associated outputs.

$$\begin{aligned}
&\big\|\tilde{Y}_t^{(1)} - \tilde{Y}_t^{(2)}\big\|_\infty = \max_{1\le i\le N} \sum_{j=1}^N \big|\tilde{Y}_t^{(1),ij} - \tilde{Y}_t^{(2),ij}\big| \\
&\le \mathbb{E}_t\bigg[\int_t^T \big(\|Y_s^{(1)}\|_\infty + \|Y_s^{(2)}\|_\infty\big)\big\|\nabla_{pp}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^\varepsilon(s,X_s))\big\|_\infty \big\|Y_s^{(1)} - Y_s^{(2)}\big\|_\infty ds\bigg] \\
&\quad + 2\mathbb{E}_t\bigg[\int_t^T \big\|\nabla_{xp}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^\varepsilon(s,X_s))\big\|_\infty \big\|Y_s^{(1)} - Y_s^{(2)}\big\|_\infty ds\bigg] \\
&\le (2K+1)\tilde{C}\mathbb{E}_t\bigg[\int_t^T \big\|Y_s^{(1)} - Y_s^{(2)}\big\|_\infty ds\bigg] \le (2K+1)\tilde{C}T \max_{0\le s\le T} \big\|Y_s^{(1)} - Y_s^{(2)}\big\|_\infty \quad \text{a.s.},
\end{aligned}$$

where $\tilde{C}$ is increasing in T because by Lemma 3.8 the constant $\tilde{C}_3$ is increasing in T. Let's further fix the parameter T in $\tilde{C}$ to be $T = 1$ and obtain $\tilde{C} = \tilde{C}(f, \lambda^{-\frac{1}{2}}, L, (\lambda-\lambda_0)^{-1})$. Hence for $T < 1$,

$$\max_{0\le t\le T} \big\|\tilde{Y}_t^{(1)} - \tilde{Y}_t^{(2)}\big\|_\infty \le (2K+1)\tilde{C}T \max_{0\le s\le T} \big\|Y_s^{(1)} - Y_s^{(2)}\big\|_\infty \quad \text{a.s..}$$

We thus have that if we choose K, $\tilde{c}$ satisfying

$$(2K+1)\tilde{C}\tilde{c} < 1, \quad \tilde{c} < 1,$$

then Φ is a contraction mapping on $\mathcal{B}(N,K)$ with $T < \tilde{c}$. Next we show that $\mathcal{B}(N,K)$ is invariant for appropriate K and $\tilde{c}$. Denote by $Y_t \in \mathcal{B}(N,K)$ the input and $\tilde{Y}_t$ the output, then Lemma 3.8 and direct calculation yield

$$\begin{aligned}
|\tilde{Y}_t^{ij}| &\le \mathbb{E}_t\bigg[\frac{1}{N}|\tilde{U}^{ij}(T)| + \int_t^T \bigg|\partial_{x_ix_j}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^{\varepsilon,R_1}(s,X_s)) + \sum_{k=1}^N Y_s^{ik}\partial_{p_kx_j}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^\varepsilon(s,X_s)) \\
&\quad \sum_{k=1}^N \partial_{x_ip_k}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^\varepsilon(s,X_s))Y_s^{kj} + \sum_{k,l=1}^N Y_s^{ik}\partial_{p_kp_l}^2 \tilde{H}_N^{R_1}(s, X_s, \nabla_s V_N^\varepsilon(s,X_s))Y_s^{lj}\bigg| ds\bigg] \\
&\le \delta_{ij}\tilde{C}N^{-1} + \tilde{C}\tilde{c}N^{-1}(\delta_{ij} + N^{-1}) + \tilde{C}\tilde{c}KN^{-2}\sum_{k=1}^N(\delta_{ik} + N^{-1}) + \tilde{C}\tilde{c}KN^{-2}\sum_{k=1}^N(\delta_{kj} + N^{-1}) \\
&\quad + \tilde{C}\tilde{c}K^2N^{-2}\sum_{k,l=1}^N (\delta_{ik} + N^{-1})(\delta_{lj} + N^{-1}) \\
&= \delta_{ij}\tilde{C}N^{-1} + \tilde{C}\tilde{c}N^{-1}(\delta_{ij} + N^{-1}) + 4\tilde{C}\tilde{c}KN^{-2} + 4\tilde{C}\tilde{c}K^2N^{-2}.
\end{aligned}$$

It is easy to see that we can choose K, $\tilde{c}$ such that

$$K = K\big(f, \lambda^{-\frac{1}{2}}, L, (\lambda-\lambda_0)^{-1}\big), \quad \tilde{c} = \tilde{c}\big(f, \lambda^{-\frac{1}{2}}, L, (\lambda-\lambda_0)^{-1}\big),$$

and

$$KN^{-1}(\delta_{ij} + N^{-1}) > \delta_{ij}\tilde{C}N^{-1} + \tilde{C}\tilde{c}N^{-1}(\delta_{ij} + N^{-1}) + 4\tilde{C}\tilde{c}KN^{-2} + 4\tilde{C}\tilde{c}K^2N^{-2}.$$

Then we have for such K, $\tilde{c}$ that $\mathcal{B}(N,K)$ is invariant.

Since Φ is contractive and invariant on $(\mathcal{B}(N,K), \|\cdot\|_\infty)$, which is a Banach space, it follows that Φ admits a fixed point in $\mathcal{B}(N,K)$ when $T < \tilde{c}$. Note that processes in $\mathcal{B}(N,K)$ are all bounded. Therefore the aforementioned fixed point in $\mathcal{B}(N,K)$ is nothing but the matrix process in (3.43) and we may take $\tilde{C}_4 = K$. Consider $t = 0$ in (3.43), then we have (3.41) from (3.47). □

We can see from the proof above that $\tilde{C}_4$ actually depends on U', U'' rather than U.

3.2.2 Global in time estimates

In this subsection we focus on the global estimates for any given $T > 0$ with sufficiently smooth data. As will be seen, the global estimates rely heavily on the convexity assumption (with respect to x) on the Hamiltonian $\tilde{H}_N(t,x,p)$ in (3.49). However, the truncation of L, U might break the convexity of $\tilde{H}_N(t,x,p)$. Therefore, we need to pass R_1 to infinity in (3.29) and consider

$$\begin{cases} \partial_t V_N^\varepsilon + \dfrac{\sigma^2}{2} \displaystyle\sum_{i,j=1}^N \partial^2_{x_i x_j} V_N^\varepsilon + \dfrac{\varepsilon^2}{2} \sum_{i=1}^N \partial^2_{x_i x_i} V_N^\varepsilon + \tilde{H}_N(t,x,\nabla_x V_N^\varepsilon) = 0, \\ V_N^\varepsilon(T, x_1, \dots, x_N) = \dfrac{1}{N} \displaystyle\sum_{i=1}^N U(x_i), \end{cases} \tag{3.48}$$

Here

$$\tilde{H}_N(t,x,p) := \inf_{\theta\in\mathbb{R}} \left\{ \frac{\lambda}{2}|\theta|^2 + \sum_{i=1}^N f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right) p_i + \frac{1}{N}\sum_{i=1}^N L(x_i) \right\}. \tag{3.49}$$

Similarly to (3.40), we would like to take $\partial^2_{x_k x_l}$ in (3.48) and analysis the resulting system. To do so, we show the validity of taking derivatives in the next proposition.

Proposition 3.11. *Suppose* **Hypothesis (R)**. *The PDE (3.48) admits a unique classical solution $V_N^{\varepsilon,R_1} \in C\left([0,T]\times\mathbb{R}^N\right)$ where $V_N^{\varepsilon,R_1}, \partial_t V_N^{\varepsilon,R_1}, \partial_{x_i} V_N^{\varepsilon,R_1}, \partial^2_{x_i x_j} V_N^{\varepsilon,R_1}, 1 \le i,j \le N$ are bounded. For $0 < \gamma < 1$ and $1 \le i,j \le N$, $V_N^\varepsilon, \partial_{x_i} V_N^\varepsilon, \partial^2_{x_i x_j} V_N^\varepsilon \in C_{loc}^{1+\frac{\gamma}{2}, 2+\gamma}\left([0,T)\times\mathbb{R}^N\right)$. And for $\varphi \in \{V_N^\varepsilon, \partial_t V_N^\varepsilon, \partial_{x_i} V_N^\varepsilon, \partial^2_{x_i x_j} V_N^\varepsilon\}$, $1 \le i,j \le N$, φ has polynomial growth in x:*

$$|\varphi(t,x)| \le \check{C}(1+|x|)^7, \quad (t,x) \in [0,T)\times\mathbb{R}^N. \tag{3.50}$$

Here the constant $\check{C}$ depends only on f, L, U, σ, ε. Moreover, the solution V_N^{ε,R_1} to (3.29) satisfies

$$\lim_{R_1\to+\infty} (V_N^{\varepsilon,R_1}, \partial_t V_N^{\varepsilon,R_1}, \partial_{x_i} V_N^{\varepsilon,R_1}, \partial^2_{x_i x_j} V_N^{\varepsilon,R_1})(t,x) = (V_N^\varepsilon, \partial_t V_N^\varepsilon, \partial_{x_i} V_N^\varepsilon, \partial^2_{x_i x_j} V_N^\varepsilon)(t,x),$$

where the convergence is locally uniform on $[0,T]\times\mathbb{R}^N$.

As a result, for the first order derivatives of V_N^{ε,R_1}, we also have

$$\left|\partial_{x_i} V_N^\varepsilon(t,x)\right| \le \frac{\tilde{C}_2(C_{11}^L + C_{11}^U)}{N}\left(1 + |x_i|^2 + \frac{1}{N}\sum_{j=1}^N |x_j|^2\right)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L + C_{10}^U)}{N}, \tag{3.51}$$

where the constant $\tilde{C}_2$ is from Lemma 3.4.

Proof. Let V_N^{ε,R_1} be the solution to (3.29) in Lemma 3.5. According to Theorem 4.7.2 and Theorem 4.7.4 in [39] as well as the growth condition (3.3), we have that for $\varphi \in \{V_N^{\varepsilon,R_1}, \partial_t V_N^{\varepsilon,R_1}, \partial_{x_i} V_N^{\varepsilon,R_1}, \partial^2_{x_i x_j} V_N^{\varepsilon,R_1}\}$, $1 \le i,j \le N$, φ has polynomial growth in x:

$$|\varphi(t,x)| \le \check{C}(1+|x|)^7, \quad (t,x) \in [0,T)\times\mathbb{R}^N,$$

where the constant $\breve{C}$ depends only on $f,\ L,\ U,\ \sigma,\ \varepsilon$ and is independent of R_1[1]

Similar to (3.9), we may view the solution of (3.29) as the solution of the constant coefficients PDE

$$\partial_t V_N^{\varepsilon,R_1}(t,x) + \frac{\sigma^2}{2}\sum_{i,j=1}^N \partial^2_{x_ix_j} V_N^{\varepsilon,R_1}(t,x) + \frac{\varepsilon^2}{2}\sum_{i=1}^N \partial^2_{x_ix_i} V_N^{\varepsilon,R_1}(t,x) = g^{R_1}(t,x), \tag{3.52}$$

where

$$g^{R_1}(t,x) := -\tilde{H}_N^{R_1}(t,x,\nabla_x V_N^{\varepsilon,R_1}).$$

In view of Corollary 4.7.8 in [39] as well as Lemma 3.5 and Lemma 3.8, $g^{R_1}(t,x)$ is locally Lipschitz continuous with respect to x with Lipschitz constant independent of R_1 while $g^{R_1}(t,x)$ is locally $\frac{\gamma}{2}-$Hölder continuous $(0<\gamma<1)$ with respect to t with Hölder constant independent of R_1. It then follows that $\partial_t V_N^{\varepsilon,R_1}(t,x)$ and $\partial^2_{x_ix_j} V_N^{\varepsilon,R_1}(t,x)$, $1\le i,j\le N$ are locally Hölder continuous in (t,x) with Hölder constant independent of R_1. According to Arzelà–Ascoli Theorem, we may pass R_1 to infinity in (3.29) and obtain the limit of V_N^{ε,R_1} as the solution $V_N^\varepsilon \in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)\cap C\left([0,T]\times\mathbb{R}^N\right)$ of (3.48). We remark that because of the uniqueness of solutions to this last problem, there is no need to consider sub-sequential limits in the Arzelà–Ascoli theorem. Moreover, we have (3.51) and (3.50) for $\varphi\in\{V_N^\varepsilon, \partial_t V_N^\varepsilon, \partial_{x_i}V_N^\varepsilon, \partial^2_{x_ix_j}V_N^\varepsilon\}$, $1\le i,j\le N$.

In order to show higher regularity of V_N^ε, we may take ∂_{x_i}, $(1\le i\le N)$ in (3.48) and obtain the PDE satisfies by $\partial_{x_i}V_N^\varepsilon$. Notice that $\partial_{x_i}\left(\tilde{H}_N(x,\nabla_x V_N^\varepsilon)\right)\in C_{loc}^{\frac{\gamma}{2},\gamma}\left([0,T)\times\mathbb{R}^N\right)$, then it follows that $\partial_{x_i}V_N^\varepsilon\in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)\cap C\left([0,T]\times\mathbb{R}^N\right)$. Thanks to Lemma 3.8 we may let R_1 go to infinity in (3.39) to obtain that $\partial^2_{x_ix_j}\left(\tilde{H}_N(x,\nabla_x V_N^\varepsilon)\right)$ is bounded and $\partial^2_{x_ix_j}\left(\tilde{H}_N(x,\nabla_x V_N^\varepsilon)\right)\in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)$. Thus we can further take $\partial^2_{x_ix_j}$, $(1\le i,j\le N)$ in (3.48) and repeat the previous procedure once more to show that $\partial^2_{x_ix_j}V_N^\varepsilon\in C_{loc}^{1+\frac{\gamma}{2},2+\gamma}\left([0,T)\times\mathbb{R}^N\right)\cap C\left([0,T]\times\mathbb{R}^N\right)$. $\square$

Now we make the following assumption on convexity of the data and set out for the global in time estimates.

Hypothesis (R1) Suppose **Hypothesis (R)** and the following:

1. U is convex;
2. $\tilde{H}_N$ in (3.49) is convex in $x\in\mathbb{R}^N$, for all (t,p).

Remark 3.12. *The second assumption in* **Hypothesis (R1)** *on the convexity of the Hamiltonian is quite common in mean field control problems. In our model, since the control is centralized and the dynamics of the particles is more complicated, this assumption could no longer be guaranteed in a simple way. According to direct calculation, with the notation* $\mu=\frac{1}{N}\sum_{j=1}^N\delta_{x_j}$, *we find*

$$\begin{aligned}\partial^2_{x_kx_l}\tilde{H}_N(t,x,p) = {}& \delta_{kl}\frac{1}{N}L''(x_k) + \partial^2_{\theta x}f(t,\theta^*,x_k,\mu)p_k\partial_{x_l}\theta^* + \delta_{kl}\partial^2_{xx}f(t,\theta^*,x_k,\mu)p_k + \frac{1}{N}\partial^2_{x\mu}f(t,\theta^*,x_k,\mu,x_l)p_k\\ &+\frac{1}{N}\partial^2_{x\mu}f(t,\theta^*,x_l,\mu,x_k)p_l + \frac{1}{N}\sum_{i=1}^N\partial^2_{\mu\theta}f(t,\theta^*,x_i,\mu,x_k)p_i\partial_{x_l}\theta^*\\ &+\delta_{kl}\frac{1}{N}\sum_{i=1}^N\partial^2_{\mu\tilde{x}}f(t,\theta^*,x_i,\mu,x_k)p_i + \frac{1}{N^2}\sum_{i=1}^N\partial^2_{\mu\mu}f(t,\theta^*,x_i,\mu,x_k,x_l)p_i.\end{aligned}$$

We can see from the above that one possible way to ensure the convexity of $\tilde{H}_N(t,x,p)$ *in* x *is to assume an affine structure on* f.

[1]According to Theorem 4.7.2 and Theorem 4.7.4 in [39] for L, U with growth rate $(1+|x|^m)$, the estimates on the second order derivatives are of growth rate $(1+|x|^{3m+1})$. Here in our case, since L and U grow at most quadratically, we have $m=2$.

Indeed, set the parameters in (1.1) *and* (2.1) *as follows*

$$f(t,\theta,x,\mu) = \theta + x + \int_{\mathbb{R}} y\mu(dy),\ L(x) = U(x) = x^2,\ \sigma = \lambda = 1.$$

Then **Hypothesis (R1)** *can be easily verified.*

More generally, if we suppose that $f(t,\theta,x,\mu)$ *is jointly convex in* (x,μ) *in the sense that*

$$f\big(t,\theta,sx_1+(1-s)x_2, Law(s\xi_1+(1-s)\xi_2)\big) \le sf\big(t,\theta,x_1,Law(\xi_1)\big) + (1-s)f\big(t,\theta,x_2,Law(\xi_2)\big),$$

for all $s\in[0,1]$ *and* ξ_1,ξ_2*, random variables, we can argue as follows. In this case,*

$$f^i(t,\theta,x) := f\bigg(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\bigg),$$

is convex in x *and for* $y\in\mathbb{R}^N$ *there exists* $\hat{f}^i(t,\theta,y)$ *convex in* y *such that*

$$\hat{f}^i(t,\theta,y) = \sup_{x\in\mathbb{R}^N}\Big\{-y\cdot x - f^i(t,\theta,x)\Big\},$$

as well as

$$f^i(t,\theta,x) = \sup_{y\in\mathbb{R}^N}\Big\{-y\cdot x - \hat{f}^i(t,\theta,y)\Big\}.$$

Hence (3.30) *yields*

$$\tilde{H}_N(t,x,p) = \inf_{\theta\in\mathbb{R}}\sup_{y^1,\dots,y^N\in\mathbb{R}^N}\bigg\{\lambda|\theta|^2 - \sum_{i=1}^N (x\cdot y)p_i - \sum_{i=1}^N \hat{f}^i(t,\theta,y)p_i + \frac{1}{N}\sum_{i=1}^N L(x_i)\bigg\}.$$

Suppose that $p_i\ge 0$ *and for all* (t,p,x) *the optimal* θ^* *is attained in a compact set, then according to the minimax theorem (see e.g. [48]),*

$$\tilde{H}_N(t,x,p) = \sup_{y^1,\dots,y^N\in\mathbb{R}^N}\inf_{\theta\in\mathbb{R}}\bigg\{\lambda|\theta|^2 - \sum_{i=1}^N (x\cdot y)p_i - \sum_{i=1}^N \hat{f}^i(t,\theta,y)p_i + \frac{1}{N}\sum_{i=1}^N L(x_i)\bigg\}, \tag{3.53}$$

and thus $H_N(t,x,p)$ *is convex. However, the constraint* $p_i\ge 0$ *requires that the value function* V_N^ε *is increasing in every component. Roughly speaking, one way to achieve this is to assume* $\partial_\mu f, L', U' \ge 0$ *then apply the theory on monotone dynamical systems (see e.g. [49]).*

Similar to the last subsection, the key estimate in this subsection is from the BSDE of Riccati type (3.59) below. The following lemma is devoted to estimating the terms appearing in (3.59).

Lemma 3.13. *Suppose* **Hypothesis (R1)***, then for* $\tilde{H}_N$ *in* (3.49) *and* $(x,p)\in\mathcal{A}_N$*,* $t\in[0,T]$*,*

$$\begin{aligned}&\big|\partial_{x_i}\tilde{H}_N(t,x,p)\big|, \big|\partial^2_{x_ip_j}\tilde{H}_N(t,x,p)\big| \le \tilde{C}_3N^{-1}, \quad \big|\partial^2_{x_ix_j}\tilde{H}_N(t,x,p)\big| \le \tilde{C}_3N^{-1}(\delta_{ij}+N^{-1}),\\ &\big|\partial_{p_i}\tilde{H}_N(t,x,p)\big|, \big|\partial^2_{p_ip_j}\tilde{H}_N(t,x,p)\big| \le \tilde{C}_3, \quad 1\le i,j\le N.\end{aligned} \tag{3.54}$$

As a result, there exists a constant $\tilde{C}_5 = \tilde{C}_5(f,\lambda^{-\frac{1}{2}},T,L,U,(\lambda-\lambda_0)^{-1})$ *such that*

$$0 \le \nabla_x^2\tilde{H}_N(t,x,p) \le \frac{\tilde{C}_5}{N}I_N, \quad (x,p)\in\mathcal{A}_N,\ t\in[0,T].$$

Proof. In view of Lemma 3.8, the constant $\tilde{C}_3$ in (3.39) is independent of R_1. Therefore we can let R_1 go to infinity and obtain (3.54) according to definitions in (3.30) and (3.49). Furthermore, in view of the second inequality in (3.54), we can deduce the existence of $\tilde{C}_5$ such that for any $\xi \in \mathbb{R}^N$, $(x,p) \in \mathcal{A}_N$, $t \in [0,T]$,

$$\sum_{i,j=1}^{N} \partial^2_{x_i x_j} \tilde{H}_N(t,x,p)\xi_i\xi_j \leq \frac{1}{2}\sum_{i,j=1}^{N} \partial^2_{x_i x_j} \tilde{H}_N(t,x,p)(\xi_i^2+\xi_j^2) \leq \frac{\tilde{C}_5}{N}|\xi|^2.$$

Combining the above with **Hypothesis (R1)** we have the last inequality. ☐

With the preparation above, if we further assume that $\partial^2_{x_i x_j} V_N^\varepsilon(t,x)$ $(1 \leq i,j \leq N)$ are bounded, we would then obtain a refined estimate on the bound of $\partial^2_{x_i x_j} V_N^\varepsilon(t,x)$ $(1 \leq i,j \leq N)$ in (3.55). This is obtained via the BSDE of Riccati type in (3.59) below where the convexity assumption plays a key role.

Lemma 3.14. *Suppose that there exist positive constants δ and $\breve{C}$ (which could depend on N and ε) such that for $(t,x) \in [T-\delta,T]\times\mathbb{R}^N$, it holds that $\partial^2_{x_i x_j} V_N^\varepsilon(t,x)$ $(1 \leq i,j \leq N)$ are bounded by constant $\breve{C}$. Then there exists a constant $\tilde{C}_6 = \tilde{C}_6(f, \lambda^{-\frac{1}{2}}, T, L, U, (\lambda-\lambda_0)^{-1})$ (independent of $\breve{C}$ and δ) such that for $\xi \in \mathbb{R}^N$ and $(t,x) \in [T-\delta,T]\times\mathbb{R}^N$,*

$$0 \leq \sum_{i,j=1}^{N} V_N^{\varepsilon,ij}(t,x)\xi_i\xi_j \leq \frac{\tilde{C}_6}{N}|\xi|^2, \tag{3.55}$$

In particular, $\partial^2_{x_i x_j} V_N^\varepsilon(t,x)$ $(1 \leq i,j \leq N)$ are bounded by $\frac{\tilde{C}_6}{N}$ for $(t,x) \in [T-\delta,T]\times\mathbb{R}^N$.

Proof. Without the loss of generality, we show (3.55) when $t = T-\delta$. For $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, consider

$$dX_t^i = \partial_{p_i} H(t, X_t, \nabla_x V_N^\varepsilon(t,X_t))dt + \sigma dW_t^i + \varepsilon dW_t^0, X_0^i = x_i, \tag{3.56}$$

as well as

$$Y_t^{kl} = V_N^{\varepsilon,kl}(t,X_t), \quad (t,x) \in [T-\delta,T]\times\mathbb{R}^N. \tag{3.57}$$

According to (3.51) and **Hypothesis (R1)**, it is easy to show that

$$\left|\partial_{p_i} H(t, X_t, \nabla_x V_N^\varepsilon(t,X_t))\right| \leq C(1+|X_t|). \tag{3.58}$$

for some constant C. Hence (3.56) admits a weak solution satisfying

$$\mathbb{E}\left[\max_{0\leq t\leq T}|X_t|^\kappa\right] \leq C(1+|x|^\kappa), \quad \forall \kappa \geq 1.$$

Moreover, since $t \in [T-\delta,T]$, we have by assumption that

$$|Y_t^{kl}| \leq \breve{C}, \quad 1 \leq k,l \leq N.$$

Given the estimates above and Proposition 3.11, we may differentiate (3.48) with respect to x_i, x_j $(1 \leq i,j \leq N)$ and obtain an analog of (3.40), then we can deduce from the assumption on the boundedness of the matrix process $Y(t)$ that it satisfies the Riccati type equation

$$\begin{aligned} Y_t = \mathbb{E}_t\bigg[\frac{1}{N}\tilde{U}(T) + \int_t^T &\left[\nabla^2_{xx}\tilde{H}_N\big(X_s, \nabla_x V_N^\varepsilon(s,X_s)\big) + Y_s\nabla^2_{xp}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)\right.\\ &\left.+ \nabla^2_{px}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)Y_s + Y_s\nabla^2_{pp}\tilde{H}_N\big(X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)Y_s\right]ds\bigg]. \end{aligned} \tag{3.59}$$

Here we recall that the first term $\frac{1}{N}\tilde{U}(T)$ on the right hand side is defined similarly to that in (3.46). Define Φ_s satisfying

$$\Phi_t = I_N - \int_{T-\delta}^{t} \Phi_s \Big[\frac{1}{2} Y_s \nabla^2_{pp}\tilde{H}_N(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)) + \nabla^2_{px}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big)\Big] ds, \quad t \in [T-\delta, T]. \tag{3.60}$$

Note here that Φ_t, $t \in [T-\delta, T]$ is bounded because Y_t, $\nabla^2_{pp}\tilde{H}_N$ and $\nabla^2_{px}\tilde{H}_N$ in the right hand side above are bounded. According to the above and (3.59), we may write the dynamics of Y_t, Φ_t as

$$\begin{aligned} dY_t = &-\Big[\nabla^2_{xx}\tilde{H}_N\big(X_t, \nabla_x V^\varepsilon_N(t, X_t)\big) + Y_t \nabla^2_{xp}\tilde{H}_N\big(t, X_t, \nabla_x V^\varepsilon_N(t, X_t)\big) \\ &+ \nabla^2_{px}\tilde{H}_N\big(t, X_t, \nabla_x V^\varepsilon_N(t, X_t)\big) Y_t + Y_t \nabla^2_{pp}\tilde{H}_N\big(X_t, \nabla_x V^\varepsilon_N(t, X_t)\big) Y_t\Big] dt + \sum_{i=0}^{N} Z^i_t dW^i(t), \\ d\Phi_t = &-\Phi_t \Big[\frac{1}{2} Y_t \nabla^2_{pp}\tilde{H}_N(t, X_t, \nabla_x V^\varepsilon_N(t, X_t)) + \nabla^2_{px}\tilde{H}_N\big(t, X_t, \nabla_x V^\varepsilon_N(t, X_t)\big)\Big] dt, \end{aligned}$$

where $\int_{T-\delta}^{t} Z^i_s dW^i(s)$ $(0 \le i \le N)$ are BMO martingales. Then Itô's formula gives

$$\begin{aligned} d(\Phi_t Y_t \Phi_t^\top) &= (d\Phi_t) Y_t \Phi_t^\top + \Phi_t (dY_t) \Phi_t^\top + \Phi_t Y_t (d\Phi_t^\top) \\ &= -\Phi_t \nabla^2_{xx}\tilde{H}_N\big(t, X_t, \nabla_x V^\varepsilon_N(t, X_t)\big) \Phi_t^\top dt + \sum_{i=0}^{N} \Phi_t Z^i_t \Phi_t^\top dW^i(t). \end{aligned}$$

Since Φ_t is bounded, we may present the above as

$$\Phi_t Y_t \Phi_t^\top = \mathbb{E}_t\Big[\frac{1}{N}\Phi_T \tilde{U}(T) \Phi_T^\top + \int_t^T \Phi_s \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big) \Phi_s^\top ds\Big]. \tag{3.61}$$

According to **Hypothesis (R1)**, we have

$$\tilde{U}(T) \ge 0, \quad \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big) \ge 0.$$

Here the ordering relation $\ge$ is used in the sense of positive semi-definite matrices. Hence

$$\frac{1}{N}\Phi_T \tilde{U}(T) \Phi_T^\top \ge 0, \quad \Phi_s \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big) \Phi_s^\top \ge 0, \quad s \in [t, T].$$

Moreover,

$$\mathbb{E}_t\Big[\frac{1}{N}\Phi_T \tilde{U}(T) \Phi_T^\top + \int_t^T \Phi_s \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big) \Phi_s^\top ds\Big] \ge 0. \tag{3.62}$$

In other words, the right hand side of (3.61) is a (random) positive semi-definite matrix. Now we may take $t = T-\delta$ in (3.61) and combine (3.60), (3.62) to obtain $Y_{T-\delta} \ge 0$. In view of (3.57), we have $\nabla^2_{xx} V^\varepsilon_N \ge 0$ and hence

$$\sum_{i,j=1}^{N} V^{\varepsilon, ij}_N(T-\delta, x)\xi_i \xi_j \ge 0.$$

One the other hand, according to **Hypothesis (R1)** and (3.49), we have

$$\nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big), \ -\nabla^2_{pp}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big) \ge 0.$$

Hence for any $\alpha \in \mathbb{R}^N$ satisfying $|\alpha| = 1$,

$$0 \le \alpha^\top Y_t \alpha \le \mathbb{E}_t\Big[\frac{1}{N}\alpha^\top \tilde{U}(T)\alpha + \int_t^T \Big[\alpha^\top Y_s \nabla^2_{xp}\tilde{H}_N\big(s, X_s, \nabla_x V^\varepsilon_N(s, X_s)\big)\alpha$$

$$+\alpha^\top \nabla^2_{px}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)Y_s\alpha + \alpha^\top \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)\alpha\Big]ds\Big].$$

Moreover, in view of Lemma 3.13 and $Y_s \geq 0$, we have

$$\begin{aligned}&\alpha^\top Y_s \nabla^2_{xp}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)\alpha \leq \big|Y_s\alpha\big| \cdot \big|\nabla^2_{xp}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)\alpha\big|\\ &\leq \tilde{C}_3\big|Y_s\alpha\big|\cdot\big|\alpha\big| \leq \tilde{C}_3 \sup_{|\beta|=1}\beta^\top Y_s\beta,\end{aligned}$$

as well as

$$\alpha^\top \nabla^2_{xx}\tilde{H}_N\big(s, X_s, \nabla_x V_N^\varepsilon(s,X_s)\big)\alpha \leq \frac{\tilde{C}_5}{N}|\alpha|^2 = \frac{\tilde{C}_5}{N}, \quad \frac{1}{N}\alpha^\top\tilde{U}(T)\alpha \leq \frac{C_{20}^U}{N},$$

where we recall that C_{20}^U is from (2.2). Hence

$$\sup_{|\beta|=1}\beta^\top Y_t\beta \leq \frac{C_{20}^U}{N} + \frac{\tilde{C}_5 T}{N} + 2\tilde{C}_3\mathbb{E}_t\Big[\int_t^T\Big(\sup_{|\beta|=1}\beta^\top Y_s\beta\Big)ds\Big].$$

Therefore we may deduce the existence of $\tilde{C}_6 = \tilde{C}_6(f, \lambda^{-\frac{1}{2}}, T, L, U, (\lambda-\lambda_0)^{-1})$ such that

$$\mathbb{E}\Big[\sup_{|\beta|=1}\beta^\top Y_t\beta\Big] \leq \frac{\tilde{C}_6}{N}, \quad t \in [T-\delta, T].$$

The inequality above implies that

$$\sum_{i,j=1}^N V_N^{\varepsilon,ij}(t,x)\xi_i\xi_j \leq \frac{\tilde{C}_6}{N}|\xi|^2, \quad \xi \in \mathbb{R}^N, \quad (t,x)\in[T-\delta,T]\times\mathbb{R}^N,$$

and we have completed the proof. □

Remark 3.15. 1. *To see why we confine ourselves to the case where $\partial^2_{x_ix_j}V_N^\varepsilon(t,x)$ $(1 \leq i,j \leq N)$ are bounded, one might turn to the definition of the matrix valued process Φ_t. If we do not assume that Y_t is bounded, then we cannot ensure the integrability of Φ_t. Without the integrability of Φ_t, we can not do the calculations in (3.59) and below, since they all involve taking conditional expectation.*

2. *For now, in this Lemma 3.14, the the existence of the constants δ and $\breve{C}$ is merely an assumption. But we know from Proposition 3.10 and Proposition 3.11 that δ indeed exists and is at least $\tilde{c}$, so does $\breve{C}$. In the next Proposition 3.16, we will use the refined estimate (3.55) to show that $\delta = T$. Moreover, showing that $\delta = T$ will then in turn gives us the refined estimate (3.55) on $[0,T]$.*

We finish this section with the next proposition where the extra assumption on boundedness in Lemma 3.14 is removed. The main idea is to take advantage of the refined estimate in (3.55) while utilizing a suitable 'continuity' method.

Proposition 3.16. *Suppose* **Hypothesis (R1)** *and λ is sufficiently large. There exists a constant $\tilde{C}_6 = \tilde{C}_6(f, \lambda^{-\frac{1}{2}}, T, L, U, (\lambda-\lambda_0)^{-1})$ such that for $1 \leq i,j \leq N$,*

$$0 \leq \sum_{i,j=1}^N V_N^{\varepsilon,ij}(t,x)\xi_i\xi_j \leq \frac{\tilde{C}_6}{N}|\xi|^2, \quad \xi\in\mathbb{R}^N,\ (t,x)\in[0,T]\times\mathbb{R}^N.$$

Proof. We have from Proposition 3.10 that, for $0 \leq T-t \leq \tilde{c}$ and $x \in \mathbb{R}^N$, $\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}(t,x)$, $1 \leq i,j \leq N$ are uniformly bounded by $\tilde{C}_4 + 1$ independent of R_1. In view of the convergence of $\partial^2_{x_ix_j}V_N^{\varepsilon,R_1}(t,x)$ to $\partial^2_{x_ix_j}V_N^\varepsilon(t,x)$, as $R_1 \to +\infty$ in Proposition 3.11, we obtain that $\partial^2_{x_ix_j}V_N^\varepsilon(t,x)$, $1 \leq i,j \leq N$ are bounded on $(t,x)\in[T-\tilde{c},T]\times\mathbb{R}^N$. Therefore we have both (3.51) and (3.55) on $(t,x)\in[T-\tilde{c},T]\times\mathbb{R}^N$ from Proposition 3.11 and Lemma 3.14.

Next we replace $\frac{1}{N}\sum_{i=1}^N U_{R_1}(x_i)$ with $\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x)$ in (3.29), (3.40) (we impose some specific properties on ρ below) and consider the following coupled PDE system on a time interval $(T-\tilde{c}-c, T-\tilde{c})$, where $c>0$ is a small number which will be specified later (written for $\hat{V}_N^{\varepsilon,R_1}$)

$$\begin{cases} \partial_t \hat{V}_N^{\varepsilon,R_1} + \dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N \partial^2_{x_ix_j}\hat{V}_N^{\varepsilon,R_1} + \dfrac{\varepsilon^2}{2}\sum_{i=1}^N \partial^2_{x_ix_i}\hat{V}_N^{\varepsilon,R_1} + \tilde{H}_N^{R_1}(t,x,\nabla_x \hat{V}_N^{\varepsilon,R_1}) = 0, \\ \hat{V}_N^{\varepsilon,R_1}(T-\tilde{c},x_1,\dots,x_N) = \rho\Big(\dfrac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x), \end{cases} \tag{3.63}$$

as well as

$$\begin{cases} \partial_t \hat{V}_N^{\varepsilon,R_1,kl} + \dfrac{\sigma^2}{2}\displaystyle\sum_{i,j=1}^N \partial^2_{x_ix_j}\hat{V}_N^{\varepsilon,kl} + \dfrac{\varepsilon^2}{2}\sum_{i=1}^N \partial^2_{x_ix_i}\hat{V}_N^{\varepsilon,kl} + \partial^2_{x_kx_l}\tilde{H}_N^{R_1}(x,\nabla_x \hat{V}_N^{\varepsilon,R_1}) \\ +\displaystyle\sum_{i=1}^N \partial_{p_i}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})\partial_{x_i}\hat{V}_N^{\varepsilon,R_1,kl} + \sum_{i,j=1}^N \partial^2_{p_ip_j}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})\hat{V}_N^{\varepsilon,R_1,ki}\hat{V}_N^{\varepsilon,R_1,jl} \\ +\displaystyle\sum_{i=1}^N \partial^2_{x_lp_i}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})\hat{V}_N^{\varepsilon,R_1,ki} + \sum_{i=1}^N \partial^2_{x_kp_i}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})\hat{V}_N^{\varepsilon,R_1,li} \\ = 0, \\ \hat{V}_N^{\varepsilon,R_1,kl}(T-\tilde{c},x) = \partial^2_{x_kx_l}\Big[\rho\Big(\dfrac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x)\Big], \quad 1\le k,l\le N. \end{cases} \tag{3.64}$$

Here ρ is any twice continuously differentiable function on $[0,+\infty)$ such that $\rho(x)=1$ if $x\in[0,1]$, $\rho(x)=0$ on $[\eta^{-1},+\infty)$, as well as $|\rho''(x)|+|\rho'(x)|\le e^{-\eta x^2}$ for some $0<\eta<1$. As a result,

$$\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x),\ \partial_{x_k}\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x),\ \partial^2_{x_kx_l}\Big[\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x)\Big],\ 1\le k,l\le N,$$

are all bounded. Moreover, the terminal condition $\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x)$ admits the uniform growth estimate (3.16) which is the counterpart to (2.2). We can also establish the first order estimate analogous to Lemma 3.4, which is uniform in (R_1,N) and possible with different coefficients. Note also that $\tilde{C}_2$ in Lemma 3.4 is decreasing in λ, then for sufficiently large λ (independent of (R_1,N)), we may show that $\partial_{x_i}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})$, $\partial^2_{x_ix_j}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})$, $\partial^2_{p_i}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})$, $\partial^2_{p_ip_j}\tilde{H}_N^{R_1}(x,\nabla_x\hat{V}_N^{\varepsilon,R_1})$ are well-defined and admit the same uniform estimates as Lemma 3.8 with possibly different coefficients.

Next, we may use a contraction method similar to the one in the proof of Proposition 3.10 to show the existence of $c>0$, depending on $\tilde{C}_2(C_{10}^L+C_{10}^U)$ in (3.51), $\tilde{C}_6$ in (3.55) as well as N, such that the solution $\hat{V}_N^{\varepsilon,R_1,kl}$ to (3.64) is unique and bounded on $(t,x)\in[T-\tilde{c}-c,T-\tilde{c}]\times\mathbb{R}^N$ uniformly in R_1. We may also argue similarly to Proposition 3.11 to obtain that

$$\lim_{R_1\to+\infty}\partial^2_{x_kx_l}V_N^{\varepsilon,R_1,kl}(t,x) = \partial^2_{x_kx_l}V_N^{\varepsilon,kl}(t,x),$$

where $(t,x)\in[T-\tilde{c}-c,T-\tilde{c}]\times\mathbb{R}^N$, $1\le k,l\le N$. In particular, we have shown that $\partial^2_{x_ix_j}V_N^\varepsilon(t,x)$, $1\le i,j\le N$ are bounded on $t\in[T-\tilde{c}-c,T]$. Then Proposition 3.11 and Lemma 3.14 again yield both (3.51) and (3.55) on $(t,x)\in[T-\tilde{c}-c,T-\tilde{c}]\times\mathbb{R}^N$.

It is important to notice that $\rho\Big(\frac{|x|}{2R_1}\Big)V_N^\varepsilon(T-\tilde{c},x)$, as the terminal condition of (3.64), is only used to show the boundedness of $\partial^2_{x_ix_j}V_N^\varepsilon(t,x)$ but not (3.55). We are relying the convexity of the final datum only after passing to the imit $R_1\to+\infty$.

Now we can replace $U_{R_1}(x)$ with $V_N^\varepsilon(T-\tilde{c}-c,x)$ in (3.29), (3.40) and repeat the procedure above to prove (3.51) and (3.55) on $(t,x)\in[T-\tilde{c}-2c,T-\tilde{c}-c]\times\mathbb{R}^N$. After finite such repetition we can show (3.55) on $(t,x)\in[0,T]\times\mathbb{R}^N$. □

3.3 Convergence of auxiliary problems

In this section, we study the original problem associated to the HJB equation (3.1). Thanks to the uniform estimates in the previous sections, we may obtain the desired solution by extracting convergence subsequence from the families $(V_N^{\varepsilon,R_1})_{\varepsilon,R_1}$ and $(V_N^{\varepsilon})_{\varepsilon}$ which solve (3.29) and (3.48), respectively. More importantly, the resulting limits inherit the estimates (uniform in N) satisfied by V_N^{ε,R_1} and V_N^{ε}.

For short time, we have the following result on the well-posedness of (3.1) as well as the corresponding estimates.

Theorem 3.17. *Suppose* **Hypothesis (R)**. *Let $\tilde{c} > 0$ given in Proposition 3.10. For $T < \tilde{c}$, the original HJB equation (3.1) admits a solution $V_N \in W_{loc}^{1,2,\infty}([0,T] \times \mathbb{R}^N)$, satisfying for $1 \le i,j \le N$, $(t,x) \in [0,T] \times \mathbb{R}^N$,*

$$|\partial_{x_i} V_N(t,x)| \le \frac{\tilde{C}_2(C_{11}^L + C_{11}^U)}{N}\left(1 + |x_i|^2 + \frac{1}{N}\sum_{k=1}^{N}|x_k|^2\right)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L + C_{10}^U)}{N}, \tag{3.65}$$

and

$$\left|\partial^2_{x_i x_j} V_N(t,x)\right| \le \tilde{C}_4 N^{-1}(\delta_{ij} + N^{-1}) \quad \text{a.e.} \tag{3.66}$$

Moreover, such solution V_N is characterized by the value function in (2.4) and thus it is unique. The unique optimal feedback function is

$$\begin{aligned}\theta_N^*(t,x) &:= \lim_{R_1 \to +\infty} \theta_N^{R_1}\left(t, x, \nabla_x V_N(t,x)\right) \\ &\in \arg\min_{\theta \in \Theta}\left\{\frac{\lambda}{2}\left|\theta\right|^2 + \sum_{i=1}^{N} f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j})\right)\partial_{x_i} V_N(t,x)\right\},\end{aligned} \tag{3.67}$$

where $\theta_N^{R_1}(t,p,q)$ is defined in (3.37).

Proof. Rewrite (3.29) as follow

$$\begin{aligned}-\frac{\varepsilon^2}{2}\sum_{i=1}^{N}\partial^2_{x_i x_i} V_N^{\varepsilon,R_1} &= \partial_t V_N^{\varepsilon,R_1} + \frac{\sigma^2}{2}\sum_{i,j=1}^{N}\partial^2_{x_i x_j} V_N^{\varepsilon,R} \\ &+ \inf_{\theta \in \Theta}\left\{\frac{\lambda}{2}\left|\theta\right|^2 + \sum_{i=1}^{N} f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\right)\partial_{x_i} V_N^{\varepsilon,R_1}\right\} + \frac{1}{N}\sum_{i=1}^{N} L_{R_1}(x_i).\end{aligned}$$

In view of the uniform estimates in Lemma 3.5 and Proposition 3.10 let

$$\varepsilon \to 0+,\ R_1 \to +\infty,$$

we immediately have the existence of $V_N \in W_{loc}^{1,2,\infty}([0,T) \times \mathbb{R}^N)$ such that on any compact subset of $[0,T) \times \mathbb{R}^N$, V_N^{ε,R_1} and $\nabla_x V_N^{\varepsilon,R_1}$ converge (up to a subsequence) uniformly to V_N and DV_N whereas $\partial_t V_N^{\varepsilon,R_1}$, $\nabla_x^2 V_N^{\varepsilon,R_1}$ converges weakly to $\partial_t V_N$, $\nabla_x^2 V_N$. Moreover, V_N also satisfies the corresponding local estimates (3.51), (3.41) of V_N^{ε,R_1}, hence (3.65) and (3.66) is valid.

According to (3.41),

$$\left|\frac{\varepsilon^2}{2}\sum_{i=1}^{N}\partial^2_{x_i x_i} V_N^{\varepsilon,R_1}\right| \le \frac{\varepsilon^2}{2}\tilde{C}_4.$$

Sending ε to $0+$, R_1 to $+\infty$ in (3.29), we get

$$\partial_t V_N + \frac{\sigma^2}{2}\sum_{i,j=1}^{N}\partial^2_{x_i x_j} V_N + \inf_{\theta \in \Theta}\left\{\frac{\lambda}{2}\left|\theta\right|^2 + \sum_{i=1}^{N} f\left(t, \theta, x_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\right)\partial_{x_i} V_N\right\} + \frac{1}{N}\sum_{i=1}^{N} L(x_i) = 0, \tag{3.68}$$

in the distributional sense.

To show the uniqueness, it suffices to establish the verification result that any solution $V_N \in W^{1,2,\infty}_{loc}([0,T]\times\mathbb{R}^N)$ satisfying (3.65) and (3.66) equals the value function in (2.4). Consider any $\theta \in \mathcal{U}^{ad}$, as well as the corresponding $X^{\theta,i}(t)$ in (1.1) and $X^{\varepsilon,\theta,i}(t)$ in (3.4). The generalized Itô's formula (see e.g. [39]) gives that, for any bounded domain $D \subset \mathbb{R}^N$, denoting by τ_D the corresponding exit time of $\mathbf{X}^{\varepsilon,\theta}_N(t)$,

$$
\begin{aligned}
&V_N(T\wedge\tau_D, \mathbf{X}^{\varepsilon,\theta}_N(T\wedge\tau_D))\\
&= V_N(0,\mathbf{X}^{\varepsilon,\theta}_N(0)) + \int_0^{T\wedge\tau_D} \mathcal{L}^\varepsilon_t V_N(t)dt + \sigma\sum_{i=1}^N \int_0^{T\wedge\tau_D} \partial_{x_i}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t))dW^0(t)\\
&\quad + \varepsilon\sum_{i=1}^N \int_0^{T\wedge\tau_D} \partial_{x_i}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t))dW^i(t)
\end{aligned}
$$

For the ease of notation, we have adopted the notation

$$
\begin{aligned}
\mathcal{L}^\varepsilon_t V_N(t) &= \partial_t V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t)) + \frac{\sigma^2}{2}\sum_{i,j=1}^N \partial^2_{x_ix_j}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t)) + \frac{\varepsilon^2}{2}\sum_{i=1}^N \partial^2_{x_ix_i}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t))\\
&+\sum_{i=1}^N f\Big(t,\theta(t),X^{\varepsilon,\theta,i}(t),\frac{1}{N}\sum_{j=1}^N \rho(X^{\varepsilon,\theta,j}(t))\Big)\partial_{x_i}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t)).
\end{aligned}
$$

According to Proposition 3.10 as well as the convergence of $\nabla_x V^{\varepsilon,R_1}_N(t,x)$ to $\nabla_x V_N(t,x)$, $\nabla_x V_N(t,x)$ is continuous and uniformly bounded on D. Hence

$$
\mathbb{E}\Big[V_N(T\wedge\tau_D,\mathbf{X}^{\varepsilon,\theta}_N(T\wedge\tau_D))\Big] = V_N(0,\mathbf{X}^{\varepsilon,\theta}_N(0)) + \mathbb{E}\Big[\int_0^{T\wedge\tau_D}\mathcal{L}^\varepsilon_t V_N(t)dt\Big]. \tag{3.69}
$$

According to Theorem 2.10.2 in [39] and (3.66), (3.68),

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^{T\wedge\tau_D}\mathcal{L}^\varepsilon_t V_N(t)dt\Big] &\le \frac{\varepsilon^2}{2}\mathbb{E}\Big[\int_0^{T\wedge\tau_D}\sum_{i=1}^N \partial^2_{x_ix_i}V_N(t,\mathbf{X}^{\varepsilon,\theta}_N(t))dt\Big]\\
&\qquad - \mathbb{E}\Big[\frac{\lambda}{2}\int_0^{T\wedge\tau_D}\big|\theta(t)\big|^2dt + \frac{1}{N}\sum_{i=1}^N\int_0^{T\wedge\tau_D}L(X^{\varepsilon,\theta,i}(t))dt\Big]\\
&\le \frac{\varepsilon^2}{2}\tilde{C}_4 - \mathbb{E}\Big[\frac{\lambda}{2}\int_0^{T\wedge\tau_D}\big|\theta(t)\big|^2dt + \frac{1}{N}\sum_{i=1}^N\int_0^{T\wedge\tau_D}L(X^{\varepsilon,\theta,i}(t))dt\Big].
\end{aligned} \tag{3.70}
$$

Plug (3.70) into (3.69), and let D extend to $\mathbb{R}^N$, the monotone convergence theorem yields that

$$
\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N U(X^{\varepsilon,\theta,i}(T)) + \frac{1}{N}\sum_{i=1}^N\int_0^T L(X^{\varepsilon,\theta,i}(t))dt + \frac{\lambda}{2}\int_0^T\big|\theta(t)\big|^2dt\right] \le V_N(0,\mathbf{X}^{\varepsilon,\theta}_N(0)) + \frac{\varepsilon^2}{2}\tilde{C}_4. \tag{3.71}
$$

Sending ε to $0+$ and noticing the convergence of $X^{\varepsilon,\theta,i}$ to $X^{\theta,i}$, we have

$$
\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N U(X^{\theta,i}(T)) + \frac{1}{N}\sum_{i=1}^N\int_0^T L(X^{\theta},i(t))dt + \frac{\lambda}{2}\int_0^T\big|\theta(t)\big|^2dt\right] \le V_N(0,\mathbf{X}^{\theta}_N(0)). \tag{3.72}
$$

On the other hand, consider the candidate optimal feedback control $\theta^*_N\big(t,x,\nabla_x V_N(t,x)\big)$. We first claim that the corresponding system

$$
dX^{*,i}_N(t) = f\Big(t,\theta^*_N(t,X^*_N),X^{*,i}_N(t),\frac{1}{N}\sum_{j=1}^N\delta_{X^{*,j}_N(t)}\Big)dt + \sigma dW^0(t),\ i=1,\dots,N. \tag{3.73}
$$

admits a unique solution for any initial data $x_1, \ldots, x_N$, $N \geq 1$. In fact, it is easy to see from (3.67) and Lemma 3.7 that $\theta_N^*(t,p,q)$ is locally Lipschitz continuous with respect to $(p,q) \in \mathcal{A}_N$. In the same time, $(x, \nabla_x V_N(t,x)) \in \mathcal{A}_N$ and $V_N \in W_{loc}^{1,2,\infty}([0,T]\times\mathbb{R}^N)$. Therefore, after composition,

$$x \mapsto f\Big(t, \theta_N^*(t,x,\nabla_x V_N(t,x)), x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big), \quad i = 1, \ldots, N,$$

is locally Lipschitz continuous. The local Lipschitz continuity then gives the strong uniqueness of the solution. Notice that we have got (3.65), the weak existence can be deduced from (3.67) and from the linear growth property that

$$\left| f\Big(t, \theta_N^*(t,x,\nabla_x V_N(t,x)), x_1, \frac{1}{N}\sum_{j=1}^N \rho(x_j)\Big)\right| \leq C_N(1+|x|),$$

for some constant C_N.

Having shown the well-posedness of (3.73), the first "$\leq$" in (3.70) becomes "$=$". The estimates in (3.66) then enable us to replace the "$\leq$" in (3.72) with "$\geq$", implying that θ_N^* is optimal and V_N is the value function. □

Using the same method as in Theorem 3.17 and combining with the uniform estimates in Proposition 3.11, Proposition 3.16, we can prove the following result for long time.

Theorem 3.18. *Suppose* **Hypothesis (R1)** *and λ is sufficiently large. The original HJB equation (3.1) admits a solution $V_N \in W_{loc}^{1,2,\infty}([0,T]\times\mathbb{R}^N)$, satisfying for $1 \leq i,j \leq N$, $(t,x) \in [0,T]\times\mathbb{R}^N$,*

$$|\partial_{x_i} V_N(t,x)| \leq \frac{\tilde{C}_2(C_{11}^L + C_{11}^U)}{N}\Big(1 + |x_i|^2 + \frac{1}{N}\sum_{k=1}^N |x_k|^2\Big)^{\frac{1}{2}} + \frac{\tilde{C}_2(C_{10}^L + C_{10}^U)}{N}, \tag{3.74}$$

and

$$0 \leq \sum_{i,j=1}^N \partial^2_{x_i x_j} V_N(t,x)\xi_i\xi_j \leq \frac{\tilde{C}_6}{N}|\xi|^2 \quad a.e.. \tag{3.75}$$

Moreover, such solution V_N is characterized by the value function in (2.4) and thus unique. An optimal feedback function is

$$\theta_N^*(t,x) \in \arg\min_{\theta\in\Theta}\Big\{\frac{\lambda}{2}|\theta|^2 + \sum_{i=1}^N f\Big(t,\theta,x_i,\frac{1}{N}\sum_{j=1}^N \delta_{x_j})\Big)\partial_{x_i} V_N(t,x)\Big\}.$$

4 Discussion on the convergence rate

In this section we discuss the convergence rate of convergence for the value functions V_N as well as the minimizer θ_N^* where the number of samples N goes to infinity. In terms of neural SDEs, the convergence of V_N above is instantly interpreted as the convergence of minima of objective functionals, while we may use the convergence of θ_N^* above to yield pathwise convergence results that imply the convergence of optimal parameters obtained via neural SDE with N samples (see Proposition 4.7 and Proposition 4.12 below). We recall that for sufficiently large N, the conclusion in Theorem 3.17 holds as long as $T < \tilde{c}$, while the conclusion in Theorem 3.18 holds for any $T > 0$.

We first show the interesting fact that the value function V_N of Problem 2.2 is actually the finite dimensional projection of a function $\mathcal{V}$ with probability measure as state variable.

Lemma 4.1. *Suppose* **Hypothesis (R)**. *Let V_N be the value function in (2.4). For samples $x_1, \dots, x_N \in \mathbb{R}$ and $y_1, \dots, y_M \in \mathbb{R}$, (for $M, N \in \mathbb{N}$) suppose that*

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} = \frac{1}{M}\sum_{i=1}^{M}\delta_{y_i},$$

then for $t \in [0,T]$, $T > 0$,

$$V_N(t, x_1, \dots, x_N) = V_M(t, y_1, \dots, y_M). \tag{4.1}$$

Proof. In view of (3.6) and (3.8), it suffices to show

$$V_N^{0,R}(t, x_1, \dots, x_N) = V_M^{0,R}(t, y_1, \dots, y_M),$$

for any $R_1, R_2 > 0$. Here we have defined the value function

$$V_N^{0,R}(t, x_1, \dots, x_N) := \inf_{\theta \in \mathcal{U}_{R_2}^{ad}} J_N^{0,R_1}(\theta, t, x_1, \dots, x_N). \tag{4.2}$$

Note that

$$\frac{1}{NM}\sum_{i=1}^{N} M\delta_{x_i} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} = \frac{1}{M}\sum_{i=1}^{M}\delta_{y_i} = \frac{1}{NM}\sum_{i=1}^{M} N\delta_{y_i}.$$

Since the left hand side and the right hand side have the same sample size, it holds that

$$\{x_1, \dots, x_N, x_1, \dots, x_N, \dots, x_1, \dots, x_N\} = \{y_1, \dots, y_M, y_1, \dots, y_M, \dots, y_1, \dots, y_M\}.$$

Here the left hand side above consists of M duplicates of $\{x_1, \dots, x_N\}$, while the right hand side above consists of N duplicates of $\{y_1, \dots, y_M\}$. According to the variational definition of $V_{MN}^{0,R}$, it is easy to check the symmetric feature that

$$V_{NM}^{0,R}(t, x_1\mathbf{1}_M^\top, \dots, x_N\mathbf{1}_M^\top) = V_{NM}^{0,R}(t, y_1, \dots, y_M, y_1, \dots, y_M, \dots, y_1, \dots, y_M),$$

where

$$\mathbf{1}_M := \underbrace{(1, \dots, 1)}_{M\ -\ \text{times}}{}^\top.$$

Therefore it suffices to show that

$$V_N^{0,R}(t, x_1, \dots, x_N) = V_{NM}^{0,R}(t, x_1\mathbf{1}_M^\top, \dots, x_N\mathbf{1}_M^\top). \tag{4.3}$$

For any continuous $\theta \in \mathcal{U}_{R_2}^{ad}$, define the following particle systems

$$\begin{cases} d\tilde{X}_{NM}^{(k-1)M+l}(t) = f\Big(t, \theta(t), \tilde{X}_{NM}^{(k-1)M+l}(t), \dfrac{1}{NM}\displaystyle\sum_{i=1}^{NM}\delta_{\tilde{X}_{NM}^i(t)}\Big)dt + \sigma dW^0(t), \\ \tilde{X}_{NM}^{(k-1)M+l}(s) = x_k, \quad 1 \le k \le N,\ 1 \le l \le M. \end{cases}$$

Now that θ is a bounded process, the solution admits strong uniqueness. Taking advantage of the symmetry and the strong uniqueness, it is easy to verify that for $t \in [s,T]$, the only solution to the above SDE satisfies

$$\tilde{X}_{NM}^{(k-1)M+l_1}(t) = \tilde{X}_{NM}^{(k-1)M+l_2}(t), \quad 1 \le k \le N,\ 1 \le l_1, l_2 \le M. \tag{4.4}$$

Denote by

$$X_N^k(t) := \tilde{X}_{NM}^{(k-1)M+1}(t), \quad 1 \le k \le N.$$

In view of (4.4), we have

$$\frac{1}{MN}\sum_{i=1}^{NM}\delta_{\tilde{X}^i_{NM}(t)}=\frac{1}{N}\sum_{i=1}^{N}\delta_{X^i_N(t)}.$$

Moreover, $(X^1_N(t),\ldots,X^N_N(t))$ uniquely solves

$$\begin{cases} dX^i_N(t)=f\Big(t,\theta(t),X^i_N(t),\frac{1}{N}\sum_{i=1}^{N}\delta_{X^i_N(t)}\Big)dt+\sigma dW^0(t),\\ X^i_N(s)=x_i,\quad 1\le i\le N.\end{cases}$$

Therefore

$$\begin{aligned} J^{0,R_1}_N(\theta,t,x_1,\ldots,x_N)&=\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T L_{R_1}\big(X^i_N(t)\big)dt+\frac{1}{N}\sum_{i=1}^{N}U_{R_1}\big(X^i_N(T)\big)+\frac{\lambda}{2}\int_0^T|\theta(t)|^2\,dt\Big]\\ &=\mathbb{E}\Big[\frac{1}{NM}\sum_{i=1}^{NM}\int_0^T L_{R_1}\big(\tilde{X}^i_{NM}(t)\big)dt+\frac{1}{NM}\sum_{i=1}^{NM}U_{R_1}\big(\tilde{X}^i_{NM}(T)\big)+\frac{\lambda}{2}\int_0^T|\theta(t)|^2\,dt\Big]\\ &=J^{0,R_1}_{NM}\big(\theta,t,x_1\mathbf{1}_M^\top,\ldots,x_N\mathbf{1}_M^\top\big).\end{aligned}$$

Since θ is taken arbitrarily from $\mathcal{U}^{ad}_{R_2}$, we have (4.3). □

In view of Lemma 4.1, it is easy to see the following definition is meaningful.

Definition 4.1. *For samples x_1, …, x_N, denote by μ^N the corresponding empirical measure $\mu^N=\frac{1}{N}\sum_{i=1}^N\delta_{x_i}$. Define*

$$\mathcal{V}(t,\mu^N):=V_N(t,x_1,\ldots,x_N). \tag{4.5}$$

In view of the estimates in (3.51), we may now show the Lipschitz continuity of $\mathcal{V}$ defined above.

Theorem 4.2. *Suppose* **Hypothesis (R)**. *Let μ_1 and μ_2 be two empirical measures, then*

$$|\mathcal{V}(t,\mu_1)-\mathcal{V}(t,\mu_2)|\le\tilde{C}_{71}\mathcal{W}_2(\mu_1,\mu_2)+\tilde{C}_{72}\Big[\mathcal{W}_2^2(\mu_1,\mu_2)+\Big(\int_{\mathbb{R}}y^2\mu_1(dy)+\int_{\mathbb{R}}y^2\mu_2(dy)\Big)\mathcal{W}_2(\mu_1,\mu_2)\Big],$$

where

$$\tilde{C}_{71}=\tilde{C}_2(C^L_{11}+C^U_{11}+C^L_{10}+C^U_{10}),\quad \tilde{C}_{72}=\frac{\tilde{C}_2(C^L_{11}+C^U_{11})}{2}. \tag{4.6}$$

As a result, $\mathcal{V}(t,\cdot)$ can be uniquely extended to a local Lipschitz function on $\mathcal{P}_2(\mathbb{R})$.

Proof. Up to a duplication, we may assume μ_1 and μ_2 admit the following representation

$$\mu_1=\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i},\ \ \mu_2=\frac{1}{N}\sum_{i=1}^{N}\delta_{y_i}.$$

Then

$$\mathcal{W}_2(\mu_1,\mu_2)=\min_\sigma\Big(\frac{1}{N}\sum_{i=1}^{N}|x_i-y_{\sigma(i)}|^2\Big)^{\frac{1}{2}},$$

where the minimum is taken over all permutation on $\{1,\dots,N\}$. Up to a permutation, we may further assume that

$$\mathcal{W}_2(\mu_1,\mu_2)=\left(\frac{1}{N}\sum_{i=1}^N|x_i-y_i|^2\right)^{\frac12}.$$

Denote by

$$\mathbf{x}_N:=(x_1,\dots,x_N),\ \ \mathbf{y}_N:=(y_1,\dots,y_N),$$

then

$$|\mathcal{V}(t,\mu_1)-\mathcal{V}(t,\mu_2)|=|V_N(\mathbf{x}_N)-V_N(\mathbf{y}_N)|.$$

Let $g:[0,1]\to\mathbb{R}$ be defined as

$$g(\gamma):=V_N(t,\gamma\mathbf{x}_N+(1-\gamma)\mathbf{y}_N).$$

Then according to (3.65),

$$|V_N(t,\mathbf{x}_N)-V_N(t,\mathbf{y}_N)|\le\int_0^1|g'(\gamma)|d\gamma\le\sum_{i=1}^N\int_0^1|\partial_{x_i}V_N(t,\gamma\mathbf{x}_N+(1-\gamma)\mathbf{y}_N)|\cdot|x_i-y_i|d\gamma, \tag{4.7}$$

where

$$\begin{aligned}&|\partial_{x_i}V_N(t,\gamma\mathbf{x}_N+(1-\gamma)\mathbf{y}_N)|\\ \le&\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{N}\left(1+|\gamma x_i+(1-\gamma)y_i|^2+\frac1N\sum_{j=1}^N|\gamma x_j+(1-\gamma)y_j|^2\right)^{\frac12}+\frac{\tilde C_2(C_{10}^L+C_{10}^U)}{N}\\ \le&\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{N}\left[1+|\gamma x_i+(1-\gamma)y_i|+\left(\frac1N\sum_{j=1}^N|\gamma x_j+(1-\gamma)y_j|^2\right)^{\frac12}\right]+\frac{\tilde C_2(C_{10}^L+C_{10}^U)}{N}.\end{aligned}$$

Direct calculation gives

$$\begin{aligned}&|x_i-y_i|\int_0^1|\gamma x_i+(1-\gamma)y_i|d\gamma=\frac{|x_i-y_i|}{2(x_i-y_i)}(|x_i|x_i-|y_i|y_i)\le\frac12|x_i-y_i|^2,\\ &|x_i-y_i|\int_0^1\left(\frac1N\sum_{j=1}^N|\gamma x_j+(1-\gamma)y_j|^2\right)^{\frac12}d\gamma\le|x_i-y_i|\left(\frac1N\sum_{j=1}^N\int_0^1|\gamma x_j+(1-\gamma)y_j|^2d\gamma\right)^{\frac12}\\ \le&|x_i-y_i|\left(\frac{1}{2N}\sum_{j=1}^N|x_j|^2+\frac{1}{2N}\sum_{j=1}^N|y_j|^2\right)^{\frac12}.\end{aligned}$$

We notice that in the previous computations we have assumed that $x_j\neq y_j$, otherwise the inequalities are trivially true. Plug the above inequalities into (4.7),

$$\begin{aligned}|\mathcal{V}(t,\mu_1)-\mathcal{V}(t,\mu_2)|\le&\frac{\tilde C_2(C_{11}^L+C_{11}^U+C_{10}^L+C_{10}^U)}{N}\sum_{i=1}^N|x_i-y_i|+\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{2N}\sum_{i=1}^N|x_i-y_i|^2\\ &+\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{N}\left(\sum_{i=1}^N|x_i-y_i|\right)\left(\frac{1}{2N}\sum_{j=1}^N|x_j|^2+\frac{1}{2N}\sum_{j=1}^N|y_j|^2\right)^{\frac12}\\ \le&\tilde C_2(C_{11}^L+C_{11}^U+C_{10}^L+C_{10}^U)\mathcal{W}_2(\mu_1,\mu_2)+\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{2}\mathcal{W}_2^2(\mu_1,\mu_2)\\ &+\frac{\tilde C_2(C_{11}^L+C_{11}^U)}{2}\mathcal{W}_2(\mu_1,\mu_2)\cdot\left(\int_{\mathbb{R}}y^2\mu_1(dy)+\int_{\mathbb{R}}y^2\mu_2(dy)\right).\end{aligned}$$

□

Corollary 4.3. *Suppose* **Hypothesis (R)** *or* **Hypothesis (R1)***. Let* $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \to \mu$ *in* $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)$*, as* $N \to +\infty$*. Then*

$$\lim_{N\to+\infty} V_N(t, x_1, \ldots, x_N) = \mathcal{V}(t, \mu),$$

at a rate

$$\begin{aligned}|V_N(t, x_1, \ldots, x_N) - \mathcal{V}(t, \mu)| &\le \tilde{C}_{71}\mathcal{W}_2(\mu^N, \mu)\\ &\quad + \tilde{C}_{72}\left[\mathcal{W}_2^2(\mu^N, \mu) + \left(\int_{\mathbb{R}} x^2\mu^N(dx) + \int_{\mathbb{R}} x^2\mu(dx)\right)\mathcal{W}_2(\mu_1, \mu_2)\right].\end{aligned}$$

In view of Theorem 4.2 and Corollary 4.3 above, the definition domain of $\mathcal{V}$ can be extended to $\mathcal{P}_2(\mathbb{R})$. Moreover, they reveal the convergence (at a specific rate) of $V_N(t, x_1, \ldots, x_N)$ to $\mathcal{V}(t, \mu)$ whenever $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ converges in $\mathcal{P}_2(\mathbb{R})$. It is thus natural to further consider the convergence of feedback control function $\theta^*(t, x_1, \ldots, x_N)$, as well as the corresponding convergence rate.

Consider empirical measure

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i},$$

and we introduce the notation

$$D_\mu^{(N)}\mathcal{V}(t, \mu^N, x_i) := N\partial_{x_i}V_N(t, x_1, \ldots, x_N),\ i = 1, \ldots, N. \tag{4.8}$$

In view of the symmetric property of V_N, $D_\mu^{(N)}\mathcal{V}(t, \mu^N, x_i)$ above is well-defined.

Next we show that $\mathcal{V}$ is differentiable in the measure variable at (t, μ^N) and

$$\partial_\mu\mathcal{V}(t, \mu^N, x_i) = D_\mu^{(N)}\mathcal{V}(t, \mu^N, x_i). \tag{4.9}$$

Lemma 4.4. *Suppose* **Hypothesis (R)***. Let* V_N *be the value function in* (2.4)*. For samples* $x_1, \ldots, x_N \in \mathbb{R}$ *and* $y_1, \ldots, y_M \in \mathbb{R}$*, suppose that*

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} = \frac{1}{M}\sum_{i=1}^M \delta_{y_i} =: \nu^M,$$

then for $t \in [0, T]$*,* $T > 0$ *and any bounded continuous function* $\varphi : \mathbb{R} \to \mathbb{R}$*, we have*

$$\int_{\mathbb{R}} \varphi(y)D_\mu^{(N)}\mathcal{V}(t, \mu^N, y)\mu^N(dy) = \int_{\mathbb{R}} \varphi(y)D_\mu^{(M)}\mathcal{V}(t, \nu^M, y)\nu^M(dy), \tag{4.10}$$

as well as

$$\lim_{\epsilon\to0+} \epsilon^{-1}\Big(\mathcal{V}\big(t, (I + \epsilon\varphi)\sharp\mu^N\big) - \mathcal{V}(t, \mu^N)\Big) = \int_{\mathbb{R}} \varphi(y)D_\mu^{(N)}\mathcal{V}(t, \mu^N, y)\mu^N(dy). \tag{4.11}$$

As a result, (4.9) *is valid.*

Proof. Similarly to the comments before (4.3), it suffices to show that

$$\sum_{i=1}^N \varphi(x_i)\partial_{x_i}V_N(t, x_1, \ldots, x_N) = \sum_{i=1}^{MN} \varphi(y_i)\partial_{y_i}V_{NM}(t, y_1, \ldots, y_{NM}), \tag{4.12}$$

where we have adopted the notation in (4.8) and

$$(y_1, \ldots, y_{NM}) = (x_1\mathbf{1}_M^\top, \ldots, x_N\mathbf{1}_M^\top),$$

in other words, $y_{(i-1)+k} = x_i$, $1 \le i \le N$. According to Lemma 4.1,

$$V_N(t, x_1, \dots, x_N) = V_{NM}(t, x_1 \mathbf{1}_M^\top, \dots, x_N \mathbf{1}_M^\top) = V_{NM}(t, y_1, \dots, y_{NM}).$$

Take the derivative with respect to x_i and obtain

$$\partial_{x_i} V_N(t, x_1, \dots, x_N) = \sum_{k=1}^{M} \partial_{y_{(i-1)+k}} V_{NM}(t, y_1, \dots, y_{NM}).$$

Using the equality above and noticing that $y_{(i-1)+k} = x_i$, $1 \le i \le N$, we can show that (4.12) is true.

To show (4.11), we may plug in (4.5) and (4.8). Then (4.11) and (4.9) follows. $\square$

According to Lemma 4.4, we may present the optimal feedback function θ_N^* in such a way that

$$\begin{aligned}\theta^*(t, \mu^N) &:= \theta_N^*\big(t, x, \nabla_x V_N(t, x)\big) \\ &= \arg\min_{\theta \in \Theta} \left\{ \frac{\lambda}{2} |\theta|^2 + \int_{\mathbb{R}} f(t, \theta, y, \mu^N) \partial_\mu \mathcal{V}(t, \mu^N, y) \mu^N(dy) \right\}.\end{aligned} \tag{4.13}$$

Similar to Theorem 4.2, we can show the Lipschitz continuity of $\theta^*(t, \mu^N)$ in (4.13), which implies the convergence rate of the optimal feedback function.

Theorem 4.5. *Suppose* **Hypothesis (R)**. *Let μ_1, μ_2 be two empirical measures and $\theta^*(t, \mu)$ be defined as in* (4.13). *Then for $T < \tilde{c}$, where $\tilde{c}$ is from Proposition 3.10,*

$$|\theta^*(t, \mu_1) - \theta^*(t, \mu_2)| \le \tilde{C}_8 \mathcal{W}_1(\mu_1, \mu_2). \tag{4.14}$$

Here

$$\tilde{C}_8 := (\lambda - \lambda_0)^{-1} C^Q + (\lambda - \lambda_0)^{-1} \|f_\theta\|_\infty \tilde{C}_6.$$

As a result, $\theta^*(t, \cdot)$ can be uniquely extended to a Lipschitz continuous mapping on $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_1)$.

Proof. Up to a duplication, we may assume that μ_1 and μ_2 have the same sample size. Denote by

$$\mu_1 = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \;\; \mu_1 = \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i}.$$

It is easy to see from (4.13) that $\theta^*(t, x_1, \dots, x_N)$ on the right hand side remains unchanged after a permutation of the input $\{x_1, \dots, x_N\}$. Hence up to a permutation, we may assume that

$$\mathcal{W}_1(\mu_1, \mu_2) = \frac{1}{N} \sum_{i=1}^{N} |x_i - y_i|.$$

Define

$$g(\gamma) := \theta^*(t, \gamma \mathbf{x}_N + (1 - \gamma) \mathbf{y}_N), \;\; \gamma \in [0, 1].$$

According to (3.38) and (3.66),

$$\begin{aligned}&|\theta^*(t, \mu_1) - \theta^*(t, \mu_2)| = |\theta^*(t, x_1, \dots, x_N) - \theta^*(t, y_1, \dots, y_N)| \\ &= |g(1) - g(0)| \le \int_0^1 |g'(\gamma)| d\gamma \\ &\le \sum_{l=1}^{N} |x_l - y_l| \int_0^1 \left| \frac{\partial}{\partial p_l} \theta^* + \sum_{k=1}^{N} \frac{\partial}{\partial q_k} \theta^* \cdot \partial_{kl}^2 V_N \right| (t, \gamma \mathbf{x}_N + (1 - \gamma) \mathbf{y}_N) d\gamma\end{aligned}$$

$$\leq \frac{(\lambda-\lambda_0)^{-1}C^Q + (\lambda-\lambda_0)^{-1}\|f_\theta\|_\infty \tilde{C}_6}{N} \sum_{l=1}^N |x_l - y_l|$$
$$= ((\lambda-\lambda_0)^{-1}C^Q + (\lambda-\lambda_0)^{-1}\|f_\theta\|_\infty \tilde{C}_6)\mathcal{W}_1(\mu_1,\mu_2).$$

□

Now we have the convergence rate of feedback function as the sample size grows to infinity.

Corollary 4.6. *Suppose that the assumptions of Theorem 4.5 take place and suppose that* $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \to \mu$ *in* $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)$*, as* $N \to +\infty$*. Then for* $T < \tilde{c}$ *where* $\tilde{c}$ *is from Proposition 3.10,*

$$\lim_{N\to+\infty} \theta_N^*\big(t, x, \nabla_x V_N(t,x)\big) = \theta^*(t,\mu),$$

at a rate

$$|\theta_N^*\big(t, x, \nabla_x V_N(t,x)\big) - \theta^*(t,\mu)| \leq \tilde{C}_8 \mathcal{W}_1(\mu^N, \mu).$$

Proof. This is directly from (4.13) and Theorem 4.5. □

Another consequence of Theorem 4.5 is the pathwise convergence with algebraic rate.

Proposition 4.7. *Let* $X_N^* = (X_N^{1,*}(t), \ldots, X_N^{N,*}(t))_{t\in[0,T]}$*,* $N \geq 1$ *be the optimal path of Problem 2.2, with initial values* $x_N^{(1)}, x_N^{(2)}, \ldots, x_N^{(N)}$*. Suppose that the assumptions of Theorem 4.5 take place and suppose that* $\frac{1}{N}\sum_{i=1}^N \delta_{x_N^{(i)}} \to \mu$ *in* $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)$*, as* $N \to +\infty$*. Then for* $T < \tilde{c}$ *where* $\tilde{c}$ *is from Proposition 3.10, there exists an adapted limit process* (θ^*, μ^*)*, where* $\theta^*(t) \in \Theta$ *and* $\mu^*(t) \in \mathcal{P}_1(\mathbb{R})$*,* $0 \leq t \leq T$*, such that* $\mu^*(0) = \mu$ *and*

$$\max_{s\in[0,T]} \mathcal{W}_1(\mu_N^*(s), \mu^*(s)) \leq \hat{C}_8 \mathcal{W}_1\big(\mu_N^*(0), \mu(0)\big),$$
$$\max_{s\in[0,T]} |\theta_N^*\big(s, X_N^*(s), \nabla_x V_N(t, X_N^*(s))\big) - \theta^*(s)| \leq \hat{C}_8 \mathcal{W}_1(\mu_N^*(0), \mu(0)). \tag{4.15}$$

Here $\mu_N^*(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_N^{i,*}(t)}$ *and* $\hat{C}_8 > 0$ *is a constant independent of* N*.*

Proof. In order to show the first inequality in (4.15), it suffices to first show that, for the sample paths X_N^* and X_M^*, which corresponds to sample number N and M respctively, it holds that

$$\max_{s\in[0,T]} \mathcal{W}_1(\mu_N^*(s), \mu_M^*(s)) \leq \hat{C}_8 \mathcal{W}_1\big(\mu_N^*(0), \mu_M^*(0)\big), \tag{4.16}$$

and then pass M to infinity. Here we have used the assumption that $\mu_N^*(0) = \frac{1}{N}\sum_{i=1}^N \delta_{x_N^{(i)}}$, $N \geq 1$, is a Cauchy sequence in $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)$.

According to Lemma 4.4 and (4.13), define

$$f^*(t,x,\mu) := f(t, \theta^*(t,\mu), x, \mu), \quad (t,x,\mu) \in [0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R}),$$

then the optimal path X_N^* satisfies

$$dX_N^{i,*}(t) = f^*(t, X_N^{i,*}(t), \mu_N^*(t))dt + \sigma dW(t), \quad 1 \leq i \leq N. \tag{4.17}$$

Moreover, according to Theorem 4.5, for $x_1, \ldots, x_N, \tilde{x}_1, \ldots, \tilde{x}_N \in \mathbb{R}$, it holds that

$$\left| f^*\left(t, x_i, \frac{1}{N}\sum_{ij=1}^N \delta_{x_j}\right) - f^*\left(t, \tilde{x}_i, \frac{1}{N}\sum_{j=1}^N \delta_{\tilde{x}_j}\right)\right| \leq \hat{C}_8|x_i - \tilde{x}_i| + \frac{\hat{C}_8}{N}\sum_{j=1}^N |x_j - \tilde{x}_j|, \quad 1 \leq i \leq N. \tag{4.18}$$

Given the dynamics in (4.17), we may proceed in a similar fashion to the proof in Lemma 4.1 and show that

$$\mu_N^*(t) = \tilde{\mu}_{NM}^*(t),$$

where

$$\tilde{\mu}_{NM}^*(t) := \frac{1}{NM}\sum_{i=1}^{NM}\delta_{\tilde{X}_{NM}^{i,*}(t)}, \quad \tilde{\mu}_{NM}^*(0) = \frac{1}{NM}\sum_{i=1}^{N} M\delta_{x_N^{(i)}},$$

with

$$d\tilde{X}_{NM}^{i,*}(t) = f^*(t, \tilde{X}_{NM}^{i,*}(t), \tilde{\mu}_{NM}^*(t))dt + \sigma dW(t), \quad 1 \le i \le NM.$$

Therefore, up to a duplication, showing (4.16) is equivalent to showing that

$$\max_{s\in[0,T]} \mathcal{W}_1(\mu_N^*(s), \tilde{\mu}_N^*(s)) \le \hat{C}_8 \mathcal{W}_1\big(\mu_N^*(0), \tilde{\mu}_N^*(0)\big), \quad N \ge 1, \tag{4.19}$$

where

$$\tilde{\mu}_N^*(t) := \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{X}_N^{i,*}(t)}, \quad \tilde{\mu}_N^*(0) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{x}_N^{(i)}},$$

with

$$d\tilde{X}_N^{i,*}(t) = f^*(t, \tilde{X}_N^{i,*}(t), \tilde{\mu}_N^*(t))dt + \sigma dW(t), \quad 1 \le i \le N.$$

Here $\tilde{x}_N^{(1)}, \dots, \tilde{x}_N^{(N)}$ are N arbitrary numbers from $\mathbb{R}$. But in view of (4.18), subtracting the above and (4.17) as well as the standard Grönwall's inequality then yields (4.19), which further implies (4.16).

Having obtained (4.16), we may use Theorem 4.5 to further show that for any $t \in [0,T]$,

$$\begin{aligned} &|\theta_N^*\big(t, X_N(t), \nabla_x V_N(t, X_N(t))\big) - \theta_M^*\big(t, X_M(t), \nabla_x V_M(t, X_M(t))\big)| \\ &= |\theta^*(t, \mu_N^*(t)) - \theta^*(t, \mu_M^*(t))| \le \hat{C}_8 \mathcal{W}_1(\mu_N^*(0), \mu_M(0)). \end{aligned}$$

Passing M to infinity in the above yields the second inequality in (4.15). ☐

The path $\theta_N^*(t, X_N(t), \nabla_x V_N(t, X_N(t)))$, $t \in [0,T]$ in Proposition 4.7 actually corresponds to the optimal parameters obtained via the neural SDE with N samples. Hence we may interpret Proposition 4.7 in such a way that the aforementioned parameters converge, at certain rate, as long as the empirical distributions of inputs converge as N tends to infinity.

In addition to the above convergence for short time, we can also obtain the global convergence under assumptions on convexity. We first do some preparation in Lemma 4.8 then present the main results in Theorem 4.10. Denote by

$$\mu^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \quad \nu^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{y_i}.$$

Lemma 4.8. *Suppose* **Hypothesis (R1)** *and* λ *is sufficiently large. Then*

$$\frac{1}{N}\sum_{i=1}^{N}\big|\partial_\mu \mathcal{V}(t, \mu^N, x_i) - \partial_\mu \mathcal{V}(t, \nu^N, y_i)\big|^2 \le \frac{\tilde{C}_6}{N}\sum_{i=1}^{N}|x_i - y_i|^2. \tag{4.20}$$

Proof. In view of Theorem 3.18, for $(t,x) \in [0,T] \times \mathbb{R}^N$ we have

$$\nabla_x^2 V_N(t,x) = \Big(\nabla_x^2 V_N(t,x)^{\frac{1}{2}}\Big)^\top \nabla_x^2 V_N(t,x)^{\frac{1}{2}},$$

for some matrix $\nabla_x^2 V_N(t,x)^{\frac{1}{2}} \in \mathbb{R}^{N\times N}$ such that for any $\alpha \in \mathbb{R}^N$ with $|\alpha| = 1$,

$$|\nabla_x^2 V_N(t,x)^{\frac{1}{2}}\alpha| \le \sqrt{\frac{\tilde{C}_6}{N}}|\alpha|.$$

Therefore for any $\alpha, x, y \in \mathbb{R}^N$,

$$\begin{aligned}&\langle \alpha, \nabla_x V_N(t,x) - \nabla_x V_N(t,y)\rangle \\ &= \int_0^1 \left\langle \nabla_x^2 V_N(t, y+s(x-y))^{\frac{1}{2}}\alpha, \nabla_x^2 V_N(t, y+s(x-y))^{\frac{1}{2}}(x-y)\right\rangle ds \\ &\le \frac{\tilde{C}_6}{N}|\alpha|\cdot|x-y|.\end{aligned}$$

The inequality above implies that

$$|\nabla_x V_N(t,x) - \nabla_x V_N(t,y)| \le \frac{\tilde{C}_6}{N}|x-y|,$$

which is (4.20) according to (4.8). ☐

Lemma 4.8 tells that we can extend the domain of $\partial_\mu \mathcal{V}(t,\nu,\cdot)$ from the set of all empirical measures to $\nu \in \mathcal{P}_2(\mathbb{R})$ in some weak sense, which is formalized as follows.

Corollary 4.9. *For each $t \in [0,T]$, there exists a Lipschitz continuous mapping Φ_t that maps empirical measures on $\mathbb{R}$ to $\mathcal{P}_2(\mathbb{R})$ in such a way that for any $\mu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$,*

$$\Phi_t(\mu) = \frac{1}{N}\sum_{i=1}^N \delta_{\partial_\mu \mathcal{V}(t,\mu,x_i)}.$$

Therefore, Φ_t can be uniquely extended to a Lipschitz continuous map $\Phi_t : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$.

Proof. Consider the following empirical measures

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \quad \nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}.$$

In view of the symmetric property, there is no loss of generality in assuming

$$\mathcal{W}_2(\mu^N,\nu^N) = \left(\frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2\right)^{\frac{1}{2}}.$$

Then we have from Lemma 4.8 that

$$\mathcal{W}_2\big(\Phi_t(\mu^N), \Phi_t(\nu^N)\big) \le \sqrt{\tilde{C}_6}\mathcal{W}_2(\mu^N,\nu^N).$$

Hence Φ_t is Lipschitz continuous. ☐

As a result of the distributional difference estimate in Lemma 4.8, we deduce the Lipschitz continuity of $\theta^*(t,\mu^N)$ for long time $T > 0$.

Theorem 4.10. *Suppose* **Hypothesis (R1)**. *Let $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, $\nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$. Then*

$$|\theta^*(t,\mu^N) - \theta^*(t,\nu^N)| \le \tilde{C}_9 \mathcal{W}_2(\mu^N,\nu^N), \tag{4.21}$$

for some $\tilde{C}_9 = \tilde{C}_9(f, \lambda^{\frac{1}{2}}, T, L, U, (\lambda-\lambda_0)^{-\frac{1}{2}})$.

Proof. Let θ^*, $\hat{\theta}^*$ denote the optimal feedback function corresponding to μ^N and ν^N. According to the first order condition,

$$\lambda\theta^* + \frac{1}{N}\sum_{i=1}^{N} f_\theta(t,\theta^*,x_i,\mu^N)\partial_\mu \mathcal{V}(t,\mu^N,x_i) = 0,$$
$$\lambda\hat{\theta}^* + \frac{1}{N}\sum_{i=1}^{N} f_\theta(t,\hat{\theta}^*,y_i,\nu^N)\partial_\mu \mathcal{V}(t,\nu^N,y_i) = 0.$$

Subtracting the above and utilizing (3.35), (3.36), (3.74) as well as (4.6), we have

$$\begin{aligned}(\lambda-\lambda_0)|\theta^*-\hat{\theta}^*| \leq{}& \frac{\|f_{\theta x}\|_\infty C^Q}{N}\sum_{i=1}^{N}|x_i-y_i| + \frac{\|f_{\mu\theta}\|_\infty \tilde{C}_{71}}{N}\sum_{i=1}^{N}|x_i-y_i| \\ &+ \frac{\|f_\theta\|_\infty}{N}\sum_{i=1}^{N}|\partial_\mu\mathcal{V}(t,\mu^N,x_i)-\partial_\mu\mathcal{V}(t,\hat{\mu}^N,y_i)|.\end{aligned}$$

In view of Lemma 4.8, by choosing appropriate (x_i,y_i), $i=1,\dots,N$, we can deduce (4.21) from the above. □

Parallel to Corollary 4.6 and Proposition 4.7, we estimate the convergence rate of feedback function and the optimal parameters for long time $T>0$ as follows.

Corollary 4.11. *Suppose that the assumptions of Theorem 4.10 take place and suppose that* $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \to \mu$ *in* $(\mathcal{P}_2(\mathbb{R}),\mathcal{W}_2)$, *as* $N\to+\infty$. *Then for* $T>0$,

$$\lim_{N\to+\infty}\theta_N^*\big(t,x,\nabla_x V_N(t,x)\big) = \theta^*(t,\mu),\ \ t\in[0,T],$$

at a rate

$$|\theta_N^*\big(t,x,\nabla_x V_N(t,x)\big)-\theta^*(t,\mu)| \leq \tilde{C}_9\mathcal{W}_2(\mu^N,\mu).$$

Proposition 4.12. *Let* $X_N^* = (X_N^{1,*}(t),\dots,X_N^{N,*}(t))_{t\in[0,T]}$, $N\geq 1$ *be the optimal path of Problem 2.2, with initial values* $x_N^{(1)},x_N^{(2)},\dots,x_N^{(N)}$. *Suppose that the assumptions of Theorem 4.10 take place and suppose that* $\frac{1}{N}\sum_{i=1}^N \delta_{x_N^{(i)}} \to \mu$ *in* $(\mathcal{P}_2(\mathbb{R}),\mathcal{W}_2)$, *as* $N\to+\infty$. *Then for* $T>0$, *there exists an adapted limit process* (θ^*,μ^*), *where* $\theta^*(t)\in\Theta$ *and* $\mu^*(t)\in\mathcal{P}_1(\mathbb{R})$, $0\leq t\leq T$, *such that* $\mu^*(0)=\mu$ *and*

$$\begin{aligned}&\max_{s\in[0,T]}\mathcal{W}_1(\mu_N^*(s),\mu^*(s)) \leq \hat{C}_9\mathcal{W}_1\big(\mu_N^*(0),\mu(0)\big),\\ &|\theta_N^*\big(t,X_N^*,\nabla_x V_N(t,X_N^*)\big)-\theta^*(t)| \leq \hat{C}_9\mathcal{W}_1(\mu_N^*(0),\mu(0)).\end{aligned} \tag{4.22}$$

Here $\mu_N^*(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_N^{i,*}(t)}$ *and* $\hat{C}_9>0$ *is a constant independent of* N.

5 A Linear-Quadratic example

As an example, covered by our main results, we study a linear-quadratic model in this section. See also [41] and [42] for other propagation of chaos results under linear-quadratic model.

Set the parameters in (1.1) and (2.1) as follows

$$f(t,\theta,x,\mu) = \theta + x + \int_{\mathbb{R}} y\mu(dy),\ \ L(x)=U(x)=x^2-x,\ \ \sigma=\lambda=1,\ \ T=2.$$

Then **Hypothesis (R1)** can be easily verified. Moreover (3.1) reduces to

$$\begin{cases} \partial_t V_N + \frac{\sigma^2}{2}\sum_{i,j=1}^N \partial_{ij}^2 V_N + \inf_{\theta\in\mathbb{R}}\Big\{ |\theta|^2 + \sum_{i=1}^N \Big(\theta + x_i + \frac{1}{N}\sum_{j=1}^N x_j\Big)\partial_i V_N + \frac{1}{N}\sum_{i=1}^N (x_i^2 - x_i)\Big\} = 0, \\ \hfill V_N(T, x_1, \dots, x_N) = \frac{1}{N}\sum_{i=1}^N (x_i^2 - x_i). \end{cases} \tag{5.1}$$

In view of the quadratic structure as well as the symmetry of the value function, we make the ansatz

$$V_N(t, x_1, \dots, x_N) = a_N(t)\Big(\int_{\mathbb{R}} y\mu^N(dy)\Big)^2 + b_N(t)\Big(\int_{\mathbb{R}} y\mu^N(dy)\Big) + c_N(t)\Big(\int_{\mathbb{R}} y^2\mu^N(dy)\Big) + d_N(t), \tag{5.2}$$

for some $a_N, b_N, c_N, d_N : [0,T] \to \mathbb{R}$, where

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}.$$

Suppose the previous ansatz, then according to Lemma 4.1 and (4.5),

$$\mathcal{V}(t, \mu^N) = V_N(t, x_1, \dots, x_N),$$

and we may further reduce the ansatz to

$$V_N(t, x_1, \dots, x_N) = a(t)\Big(\int_{\mathbb{R}} y\mu^N(dy)\Big)^2 + b(t)\Big(\int_{\mathbb{R}} y\mu^N(dy)\Big) + c(t)\Big(\int_{\mathbb{R}} y^2\mu^N(dy)\Big) + d(t), \tag{5.3}$$

for some $a, b, c, d : [0,T] \to \mathbb{R}$.

According to the first order condition, we can write the optimal control as

$$\theta_N^*(t, x_1, \dots, x_N) = -\big(a(t) + c(t)\big)\Big(\int_{\mathbb{R}} y\mu^N(dy)\Big) - \frac{1}{2}b(t). \tag{5.4}$$

Plugging (5.3), (5.4) into (5.1), we have that

$$\begin{cases} \dot{a}(t) + 4a(t) + 2c(t) - (a(t) + c(t))^2 = 0, \\ \dot{c}(t) + 2c(t) + 1 = 0, \\ \dot{b}(t) + \big(2b(t) - (a(t) + c(t))b(t)\big) - 1 = 0, \\ \dot{d}(t) - \frac{1}{4}b^2(t) + \big(a(t) + c(t)\big) = 0, \end{cases} \tag{5.5}$$

with the terminal condition $a(T) = d(T) = 0$, $-b(T) = c(T) = 1$. The ansatz (5.3) and (5.4) are verified once we show the well-posedness of (5.5).

In fact, we may first solve $c(t)$ according to the second equation in (5.5). Then we can add up the first and the second equation in (5.5) and obtain

$$\dot{g}(t) + 4g(t) - g^2(t) + 1 = 0, \quad g(T) = 1,$$

where $g(t) := a(t) + c(t)$. After a reverse of time, we find that the above is a Riccati equation whose global solution is guaranteed. Therefore we may solve $a(t)$ after obtaining $g(t)$. Given $a(t)$ and $c(t)$, we can solve $b(t)$ and $d(t)$ accordingly.

The following plots illustrate the convergence of the value functions and optimal parameters in (5.3) and (5.4) as $N \to +\infty$.

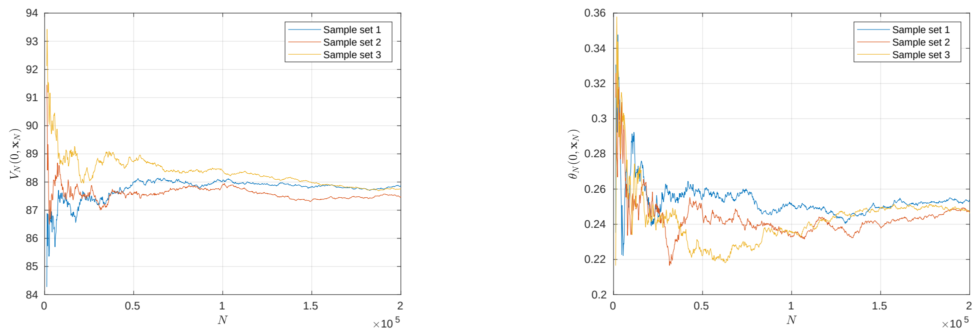

Figure 1: $V_N(0, \mathbf{x}_N)$ and $\theta_N(0, \mathbf{x}_N)$ versus N

Here we generate the first N samples $\mathbf{x}_N := (x_N, \dots, x_N) \in \mathbb{R}^N$ in such a way that $\frac{1}{N}\sum_{i=1}^N \delta_{x_N} \xrightarrow{d} N(0,1)$ (standard Gaussian distribution). The above generation of sample set is repeated 3 times and the results are reported in Figure 1. It is suggested in the plots that, as the number N tends to infinity, both the minima of objective functionals and the optimal parameters converge to certain value.

A Some technical results

First we recall the notation introduced in (3.11): $M_N(C) \subset \mathbb{R}^{N\times N}$ and

$$A \in M_N(C) \quad \text{if and only if} \quad |A_{ij}| \le C(\delta_{ij} + N^{-1}),\ 1 \le i, j \le N.$$

Next, for a matrix valued function $\tilde{A}$, where $\tilde{A}(t) \in \mathbb{R}^{N\times N}$, $t \in [0,T]$, we define the matrix $|\tilde{A}| \in \mathbb{R}^{N\times N}$ in such a way that

$$|\tilde{A}|_{ij} = \max_{t\in[0,T]} |\tilde{A}_{ij}(t)|, \quad 1 \le i, j \le N. \tag{A.1}$$

Further, define the norm $\|\tilde{A}\|_N$ by

$$\|\tilde{A}\|_N = \max_{1\le i,j\le N} N^{\delta_{ij}-1}|\tilde{A}|_{ij}. \tag{A.2}$$

Lemma A.1. *For $N \ge 1$, let $A_N \in M_N(C_1)$ and $B_N \in M_N(C_2)$, then $A_N B_N \in M_N(3C_1C_2)$.*

Proof. If $i \ne j$, then

$$\sum_{k=1}^N |a_{ik}^N b_{kj}^N| \le C_1 \cdot \frac{C_2}{N} + \frac{C_1}{N}\cdot C_2 + \frac{C_1}{N}\cdot\frac{C_2}{N}\cdot(N-2) < \frac{3C_1C_2}{N}.$$

If $i = j$, then

$$\sum_{k=1}^N |a_{ik}^N b_{kj}^N| \le C_1C_2 + \frac{C_1}{N}\cdot\frac{C_2}{N}\cdot(N-1) < 3C_1C_2.$$

Hence we have $A_N B_N \in M_N(3C_1C_2)$. □

Lemma A.2. *For each $N \geq 1$, let $(W(t))_{t\in(0,T)}$ be a real valued standard Brownian motion. Let the $\mathbb{R}^{N\times N}$-valued processes $(X(t))_{t\in(0,T)}$, $(A(t))_{t\in(0,T)}$, $(B(t))_{t\in(0,T)}$, satisfy*

$$X(t) = X(0) + \int_0^t X(s)A(s)ds + \int_0^t X(s)B(s)dW_s, \quad t \in [0,T]. \tag{A.3}$$

Suppose that $X(0)$ satisfies for $k = 1, 2, \ldots$

$$\mathbb{E}\left[|X_{ij}(0)|^{2k}\right] \leq \begin{cases} C_{0,k}, & i = j, \\ \dfrac{C_{0,k}}{N^{2k}}, & i \neq j, \end{cases}$$

and $|A|, |B| \in M_N(C)$ for some constant C. Then

$$\mathbb{E}\left[\max_{0\leq s\leq T} |X_{ij}(s)|^{2k}\right] \leq \begin{cases} \tilde{C}_k, & i = j, \\ \dfrac{\tilde{C}_k}{N^{2k}}, & i \neq j, \end{cases}$$

where $\tilde{C}_k = \tilde{C}_k(C_{0,k}, C, T)$ is increasing in T (but independent of N).

Proof. We show that (A.3) admits a unique solution X, with the required estimates, which is also the fixed point of the mapping $\Phi: \ X \mapsto \tilde{X}$ defined as follow:

$$\tilde{X}(t) = X_0 + \int_0^t X(s)A(s)ds + \int_0^t X(s)B(s)dW(s), \quad t \in [0,\delta], \tag{A.4}$$

where $\delta > 0$ is a small enough positive number. Consider to inputs $X^{(1)}$ and $X^{(2)}$, for $1 \leq p, q \leq N$,

$$\begin{aligned} &\Phi_{pq}\big(X^{(1)}\big)(t) - \Phi_{pq}\big(X^{(2)}\big)(t) \\ &= \int_0^t \sum_{k=1}^N \left[X_{pk}^{(1)}(s) - X_{pk}^{(2)}(s)\right] A_{kq}(s)ds + \int_0^t \sum_{k=1}^N \left[X_{pk}^{(1)}(s) - X_{pk}^{(2)}(s)\right] B_{kq}(s)dW(s), \end{aligned}$$

then according to Burkholder–Davis–Gundy inequality,

$$\begin{aligned} &\mathbb{E}\left[\max_{t\in[0,\delta]} |\Phi_{pq}\big(X^{(1)}\big)(t) - \Phi_{pq}\big(X^{(2)}\big)(t)|^{2k}\right] \\ &\leq C_k\mathbb{E}\left[\int_0^\delta \left|\sum_{\substack{i=1\\ i\neq q}}^N \left[X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\right] A_{iq}(s)\right|^{2k} ds\right] \\ &\quad + C_k\mathbb{E}\left[\int_0^\delta \left|\left[X_{pq}^{(1)}(s) - X_{pq}^{(2)}(s)\right] A_{qq}(s)\right|^{2k} ds\right] \\ &\quad + C_k\mathbb{E}\left[\int_0^\delta \left|\sum_{\substack{i=1\\ i\neq q}}^N \left[X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\right] B_{iq}(s)\right|^{2k} ds\right] \\ &\quad + C_k\mathbb{E}\left[\int_0^\delta \left|\left[X_{pq}^{(1)}(s) - X_{pq}^{(2)}(s)\right] B_{qq}(s)\right|^{2k} ds\right]. \end{aligned}$$

According to Jensen's inequality,

$$\left|\sum_{\substack{i=1\\ i\neq q}}^N \left[X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\right] A_{iq}(s)\right|^{2k} \leq \frac{C^{2k}}{N^{2k}} \cdot (N-1)^{2k} \cdot \left|\frac{1}{N-1} \sum_{\substack{i=1\\ i\neq q}}^N |X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)|\right|^{2k}$$

$$\leq C^{2k} \cdot \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq q}}^{N} \big|X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big|^{2k}. \tag{A.5}$$

Here we have used

$$\big|\big[X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big]A_{iq}(s)\big| \leq \begin{cases} C\big|X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big|, & i = q, \\ \frac{C}{N}\big|X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big|, & i \neq q. \end{cases}$$

Therefore

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1 \\ i \neq q}}^{N} \Big[X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\Big] A_{iq}(s)\Big|^{2k} ds\Big] \\
&\leq \frac{C^{2k}}{N-1} \sum_{\substack{i=1 \\ i \neq q}}^{N} \mathbb{E}\left[\int_0^\delta \big|X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big|^{2k} ds\right] \\
&\leq \frac{C^{2k}\delta}{N-1} \sum_{\substack{i=1 \\ i \neq q}}^{N} \mathbb{E}\left[\max_{0 \leq s \leq \delta} \big|X_{pi}^{(1)}(s) - X_{pi}^{(2)}(s)\big|^{2k}\right] \\
&\leq C^{2k}\delta \max_{1 \leq i,j \leq N} \mathbb{E}\left[\max_{0 \leq t \leq \delta} |X_{ij}^{(1)}(t) - X_{ij}^{(2)}(t)|^{2k}\right].
\end{aligned} \tag{A.6}$$

Similar estimates to (A.5) yields

$$\mathbb{E}\left[\max_{t \in [0,\delta]} |\Phi_{pq}\big(X^{(1)}\big)(t) - \Phi_{pq}\big(X^{(2)}\big)(t)|^{2k}\right] \leq 4\delta C_k C^{2k} \max_{1 \leq i,j \leq N} \mathbb{E}\left[\max_{0 \leq t \leq \delta} |X^{(1)}(t) - X^{(2)}(t)|^{2k}\right],$$

and thus

$$\begin{aligned}
&\max_{1 \leq i,j \leq N} \mathbb{E}\left[\max_{t \in [0,\delta]} |\Phi_{ij}\big(X^{(1)}\big)(t) - \Phi_{ij}\big(X^{(2)}\big)(t)|^{2k}\right] \\
&\leq 4\delta C_k C^{2k} \max_{1 \leq i,j \leq N} \mathbb{E}\left[\max_{0 \leq t \leq \delta} |X_{ij}^{(1)}(t) - X_{ij}^{(2)}(t)|^{2k}\right],
\end{aligned}$$

Consider

$$\delta < \frac{1}{8C_k C^{2k}}. \tag{A.7}$$

For the sake of later iterations, we note here that the choice of δ in (A.7) is independent of the bound C_0 of initial data.

In view of (A.7), Φ is a contraction mapping. Next, we claim that Φ maps the following set

$$\mathcal{X} := \left\{X : \ X \text{ is matrix valued process and } \mathbb{E}\left[\max_{0 \leq s \leq \delta} |X_{ij}(s)|^{2k}\right] \leq M_k(N^{-2k} + \delta_{ij})\right\}. \tag{A.8}$$

into itself for some M_k.

To see the claim, consider $1 \leq p, q \leq N$ and $p \neq q$,

$$\begin{aligned}
&\mathbb{E}\left[\max_{t \in [0,\delta]} |\Phi_{pq}\big(X\big)(t)|^{2k}\right] \\
&\leq C_k \mathbb{E}\big[\big|X_{pq}(0)\big|^{2k}\big] + C_k \mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1 \\ i \neq p,q}}^{N} X_{pi}(s) A_{iq}(s)\Big|^{2k} ds\Big] + C_k \mathbb{E}\Big[\int_0^\delta \big|X_{pp}(s) A_{pq}(s)\big|^{2k} ds\Big]
\end{aligned}$$

$$+ C_k\mathbb{E}\Big[\int_0^\delta \big|X_{pq}(s)A_{qq}(s)\big|^{2k}ds\Big] + C_k\mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1\\ i\neq p,q}}^N X_{pi}(s)B_{iq}(s)\Big|^{2k}ds\Big]$$

$$+ C_k\mathbb{E}\Big[\int_0^\delta \big|X_{pp}(s)B_{pq}(s)\big|^{2k}ds\Big] + C_k\mathbb{E}\Big[\int_0^\delta \big|X_{pq}(s)B_{qq}(s)\big|^{2k}ds\Big].$$

In view of (A.5) and (A.8),

$$\mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1\\ i\neq p,q}}^N X_{pi}(s)A_{iq}(s)\Big|^{2k}ds\Big] \le \frac{C^{2k}}{N-2}\mathbb{E}\Big[\sum_{\substack{i=1\\ i\neq p,q}}^N \int_0^\delta \big|X_{pi}(s)\big|^{2k}ds\Big]$$

$$\le \frac{\delta C^{2k}}{N-2}\sum_{\substack{i=1\\ i\neq p,q}}^N \mathbb{E}\Big[\max_{0\le s\le\delta}\big|X_{pi}(s)\big|^{2k}\Big] \le \frac{\delta C^{2k}M_k}{N^{2k}}.$$

Combining the two inequalities above together, we arrive at

$$\mathbb{E}\left[\max_{t\in[0,\delta]}|\Phi_{pq}\big(X\big)(t)|^{2k}\right] \le \frac{C_kC_0^{2k}}{N^{2k}} + \frac{6\delta C_kM_kC^{2k}}{N^{2k}}.$$

Similarly,

$$\mathbb{E}\left[\max_{t\in[0,\delta]}|\Phi_{pp}\big(X\big)(t)|^{2k}\right]$$

$$\le C_k\mathbb{E}\big[\big|X_{pp}(0)\big|^{2k}\big] + C_k\mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1\\ i\neq p}}^N X_{pi}(s)A_{ip}(s)\Big|^{2k}ds\Big] + C_k\mathbb{E}\Big[\int_0^\delta \big|X_{pp}(s)A_{pp}(s)\big|^{2k}ds\Big]$$

$$+ C_k\mathbb{E}\Big[\int_0^\delta \Big|\sum_{\substack{i=1\\ i\neq p}}^N X_{pi}(s)B_{ip}(s)\Big|^{2k}ds\Big] + C_k\mathbb{E}\Big[\int_0^\delta \big|X_{pp}(s)B_{pp}(s)\big|^{2k}ds\Big]$$

$$\le C_kC_0^{2k} + \frac{2\delta C_kM_kC^{2k}}{N^{2k}} + 2\delta C_kM_kC^{2k}.$$

Let δ and M_k satisfy

$$\delta < \frac{1}{12C_kC^{2k}}, \quad M_k > 3C_0^{2k}.$$

Note again that the choice of δ is still independent of C_0. Then estimate above implies that

$$\mathbb{E}\left[\max_{0\le s\le\delta}|\Phi(X)_{ij}(s)|^{2k}\right] \le M_k(N^{-2k}+\delta_{ij}).$$

In other words, contraction mapping Φ maps $\mathcal{X}$ into itself. Hence the only fixed point of Φ lies in $\mathcal{X}$.

To conclude the lemma, notice that the choice of δ is independent of C_0, therefore we can separate $[0,T]$ into $[0,\delta]$, $[\delta,2\delta]$, $[2\delta,3\delta]$, …, then go over the procedure above repeatedly and obtain the desired results. ☐

Proof of Lemma 3.7: Note that for each $(t,x,p)\in[0,T]\times\mathcal{A}_N$, $\theta^* := \theta_N^{R_1}(t,x,p)$ minimizes the strictly convex function $H_N^{R_1}(t,x,p,\theta)$ with respect to $\theta\in\Theta$, hence

$$\langle\partial_\theta H_N^{R_1}(t,x,p,\theta^*),\theta-\theta^*\rangle \ge 0,\ \theta\in\Theta.$$

Similarly, for another pair of $(\hat{x},\hat{p})\in\mathcal{A}_N$ and $\hat{\theta}^* := \theta_N^{R_1}(t,\hat{x},\hat{p})$,

$$\langle\partial_\theta H_N^{R_1}(t,\hat{x},\hat{p},\hat{\theta}^*),\theta-\hat{\theta}^*\rangle \ge 0,\ \theta\in\Theta.$$

Therefore we have by taking $\theta = \hat{\theta}^*,\ \theta^*$ that

$$0 \geq \left\langle \partial_\theta H_N^{R_1}(t,x,p,\theta^*) - \partial_\theta H_N^{R_1}(t,\hat{x},\hat{p},\hat{\theta}^*), \theta^* - \hat{\theta}^* \right\rangle. \tag{A.9}$$

on the other hand,

$$\begin{aligned} &\partial_\theta H_N(t,x,p,\theta^*) - \partial_\theta H_N(t,\hat{x},\hat{p},\hat{\theta}^*) \\ &= I + (\theta^* - \hat{\theta}^*) \cdot (\lambda + II), \end{aligned} \tag{A.10}$$

where

$$I := \sum_{i=1}^N f_\theta\Big(t,\theta^*,x_i,\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)p_i - \sum_{i=1}^N f_\theta\Big(t,\theta^*,\hat{x}_i,\frac{1}{N}\sum_{j=1}^N \delta_{\hat{x}_j}\Big)\hat{p}_i, \tag{A.11}$$

$$II := \sum_{i=1}^N \left[f_\theta\Big(t,\theta^*,\hat{x}_i,\frac{1}{N}\sum_{j=1}^N \delta_{\hat{x}_j}\Big) - f_\theta\Big(t,\hat{\theta}^*,\hat{x}_i,\frac{1}{N}\sum_{j=1}^N \delta_{\hat{x}_j}\Big)\right] \frac{\hat{p}_i}{\theta^* - \hat{\theta}^*}.$$

According to (3.35) in the assumption, it holds for some constant $\lambda_0 > 0$ that

$$\lambda_0 \geq |II|. \tag{A.12}$$

Plugging (A.10), (A.12) into (A.9), and using the Cauchy-Schwartz inequality, we have that

$$|\theta^* - \hat{\theta}^*| \leq (\lambda - \lambda_0)^{-1}|I|. \tag{A.13}$$

According to (A.11), I is the difference of the following function (w.r.t. $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$)

$$\sum_{i=1}^N f_\theta\Big(t,\hat{\theta}^*,x_i,\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)p_i,$$

which implies the local Lipschitz continuity of $\theta_N^{R_1}(t,x,p)$ with respect to $(x,p) \in \mathcal{A}_N$.

In view of the local Lipschitz continuity, $\theta_N^{R_1}(t,x,p)$ is differentiable almost everywhere. Furthermore, it follows from (A.13) that

$$\begin{aligned} &\left|\partial_{x_k}\theta_N^{R_1}(t,x,p)\right| \\ &\leq (\lambda-\lambda_0)^{-1}\left| f_{\theta x}\Big(t,\hat{\theta}^*,x_k,\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)p_k + \frac{1}{N}\sum_{i=1}^N \partial_\mu f_\theta\Big(t,\hat{\theta}^*,x_i,\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_k)p_i\right|. \end{aligned}$$

In view of (3.36),

$$\left|\partial_{x_k}\theta_N^{R_1}(t,x,p)\right| \leq \frac{2(\lambda-\lambda_0)^{-1}C^Q}{N}.$$

Similarly we also have

$$\left|\partial_{p_k}\theta_N^{R_1}(t,x,p)\right| \leq (\lambda-\lambda_0)^{-1}\|f_\theta\|_\infty.$$

□

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