MINIMIZING MOVEMENTS FOR THE GENERALIZED POWER MEAN CURVATURE FLOW

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Abstract. Motivated by a conjecture of De Giorgi, we consider the Almgren-Taylor-Wang scheme for mean curvature flow, where the volume penalization is replaced by a term of the form

$$\int_{E\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx$$

for f ranging in a large class of strictly increasing continuous functions. In particular, our analysis covers the case

$$f(r) = r^\alpha, \qquad r \ge 0, \quad \alpha > 0,$$

considered by De Giorgi. We show that the generalized minimizing movement scheme converges to the geometric evolution equation

$$f(v) = -\kappa \quad \text{on } \partial E(t),$$

where $\{E(t)\}$ are evolving subsets of $\mathbb{R}^n$, v is the normal velocity of $\partial E(t)$, and κ is the mean curvature of $\partial E(t)$. We extend our analysis to the anisotropic setting, and in the presence of a driving force. We also show that minimizing movements coincide with the smooth classical solution as long as the latter exists. Finally, we prove that in the absence of forcing, mean convexity and convexity are preserved by the weak flow.

1. Introduction

Denote by $|\cdot|$ and by $\mathcal{H}^{n-1}$ the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively, and let $B_\rho(x)$ be the open ball centered at $x \in \mathbb{R}^n$ of radius $\rho > 0$. In [17] De Giorgi poses the following two conjectures[1], related to some results in [2, 20, 23], and more generally, to mean curvature flow.

Conjecture 1.1. *Let* $\mathrm{Conv_b}(\mathbb{R}^n)$ *be the class of all open convex bounded subsets of* $\mathbb{R}^n$*, endowed with the metric* $d(\mathcal{K}_1, \mathcal{K}_2) = |\mathcal{K}_1 \Delta \mathcal{K}_2|$*. Given* $\alpha \in (0, +\infty)$ *and* $\mathcal{K}_0 \in \mathrm{Conv_b}(\mathbb{R}^n)$*, set*

$$\mathcal{G}(\mathcal{K}_2; \mathcal{K}_1, \tau, k) = \begin{cases} |\mathcal{K}_1 \Delta \mathcal{K}_0| & \text{if } k \le 0, \\ \mathcal{H}^{n-1}(\partial\mathcal{K}_2) + \dfrac{1}{\tau^\alpha}\displaystyle\int_{\mathcal{K}_2\Delta\mathcal{K}_1} \mathrm{d}_{\mathcal{K}_1}(x)^\alpha\,dx & \text{if } k > 0, \end{cases}$$

where $\mathrm{d}_F(\cdot) := \mathrm{dist}(\cdot, \partial F)$*. Then there exists a unique minimizing movement* $E(\cdot)$ *in* $\mathrm{Conv_b}(\mathbb{R}^n)$*, associated to* $\mathcal{G}$ *and starting from* $\mathcal{K}_0$ *and, in the case* $\alpha = 1$*,* $\partial E(t)$ *moves along its mean curvature.*

Date: April 3, 2024.
2020 *Mathematics Subject Classification.* 35K93, 53E10, 35D30, 49Q20.
Key words and phrases. Affine curvature flow, power mean curvature flow, Almgren-Taylor-Wang functional, minimizing movements.

[1]We have slightly modified the notation with respect to the original conjectures.

Conjecture 1.2. *Let*

$$\mathcal{S} := \Big\{E \subset \mathbb{R}^n :\ |E| < +\infty,\ x \in E \ \Leftrightarrow\ \lim_{\rho\to 0^+} \rho^{-n}|B_\rho(x) \setminus E| = 0\Big\}, \tag{1.1}$$

endowed with the metric $d(E_1, E_2) = |E_1 \Delta E_2|$. *Given* $\alpha \in (0, +\infty)$, $E_0 \in \mathcal{S}$ *such that* $\mathcal{H}^{n-1}(\partial E_0) < +\infty$ *and* $g \in L^1(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$, *set*

$$\widetilde{\mathcal{G}}(E_2; E_1, \tau, k) = \begin{cases} |E_1 \Delta E_0| & \text{if } k \le 0,\\ \mathcal{H}^{n-1}(\partial E_2) + \displaystyle\int_{E_2} g(x)\ dx + \frac{1}{\tau^\alpha}\int_{E_2\Delta E_1} \mathrm{d}_{E_1}(x)^\alpha\ dx & \text{if } k > 0. \end{cases}$$

Then there exists a generalized minimizing movement in $\mathcal{S}$, *associated to* $\widetilde{\mathcal{G}}$ *and starting from* E_0.

Clearly, if $g \equiv 0$, then $\mathcal{G}(\mathcal{K}_2; \mathcal{K}_1, \tau, k) = \widetilde{\mathcal{G}}(\mathcal{K}_2; \mathcal{K}_1, \tau, k)$ for $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Conv_b}(\mathbb{R}^n)$.

In this paper we prove Conjecture 1.2 in a more general form (see Theorem 1.4). We also establish weaker versions of Conjecture 1.1: (a) minimizing $\widetilde{\mathcal{G}}$ in $\mathcal{S}$ (rather than in $\mathrm{Conv_b}(\mathbb{R}^n)$) we obtain a unique minimizing movement (Theorem 1.6); (b) minimizing $\widetilde{\mathcal{G}}$ in $\mathrm{Conv_b}(\mathbb{R}^n)$ we obtain the existence of generalized minimizing movements (Theorem 1.7). Of course, these two results are related; in fact, for $\alpha \in (0, 1]$, using the convexity of the map $x \mapsto \mathrm{dist}(x, \partial\mathcal{K})$ for convex sets $\mathcal{K}$ and the methods of [11], we can show that there exists a unique minimizing movement in $\mathrm{Conv_b}(\mathbb{R}^n)$ which is also the unique minimizing movement in $\mathcal{S}$ (Corollary 8.2). At the moment we miss the proof of the uniqueness of generalized minimizing movements for $\alpha > 1$.

The gradient flow of $\widetilde{\mathcal{G}}$ in case $g \equiv 0$, leads to the equation

$$v = -\kappa^{1/\alpha}, \tag{1.2}$$

where v and κ are the normal velocity and the mean curvature of an evolving family $t \to \partial E(t)$ of smooth closed hypersurfaces in $\mathbb{R}^n$; (1.2) is called $\kappa^{1/\alpha}$ (or power of mean curvature) flow [30, 31], and is meaningful also in the case of nonconvex sets provided $\alpha \in \mathbb{N}$ is odd [6]. When $\alpha = 1$, the evolution equation (1.2) is the classical mean curvature flow. When $\alpha = 3$ and $\partial E(t)$ are evolving curves in $\mathbb{R}^2$, (1.2) is called the *affine curvature flow* (see e.g. [1, 3, 5, 6, 14, 15, 21, 29] and references therein) because of the invariance of the flow with respect to affine transformations of coordinates; this equation has applications in image processing [3, 10]. Depending on α, various phenomena may occur. If $\alpha < 8$, the only embedded homothetically shrinking solutions are circles, except when $\alpha = 3$, where some self-shrinking ellipses occur, while if $\alpha \ge 8$, a new family of symmetric curves resembling either circles or polygons arise (see e.g. [1, 5, 15]). The flow (1.2) exists, for instance, in the case of bounded convex initial subsets of $\mathbb{R}^n$ [30, 31], with finite-time extinction towards a point. See also [19], for an asymptotic study of a time-fractional Allen-Cahn equation and its convergence to the power mean curvature flow.

A by-product of our analysis is the study of a generalization of (1.2) of the form

$$f(v) = -\kappa_\varphi - g, \tag{1.3}$$

where φ is an (even) anisotropy in $\mathbb{R}^n$, κ_φ is the anisotropic mean curvature of $\partial E(t)$, and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous odd surjective function. Clearly, in general we cannot expect invariance of solutions to (1.3) with respect to affine transformations; we refer to (1.3) as the *generalized power mean curvature flow*.

In this paper we study minimizing movement solutions corresponding to (1.3) under quite general assumptions on f and g. Following [2, 12, 17] we introduce an Almgren-Taylor-Wang type functional – generalizing the functional[2] in Conjecture 1.2:

$$\mathcal{F}_{\varphi,f,g}(E;F,\tau) := P_\varphi(E) + \int_{E\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx + \int_E g(x)\,dx$$

with domain

$$\mathcal{S}^* := BV(\mathbb{R}^n;\{0,1\}) \cap \mathcal{S},$$

where $\tau > 0$ and

$$P_\varphi(E) := \int_{\partial^* E} \varphi^o(\nu_E)\,d\mathcal{H}^{n-1} \tag{1.4}$$

is the φ-perimeter of E. Here $\partial^* E$ and ν_E are the reduced boundary and the generalized outer unit normal of E. When $\varphi(\cdot) = |\cdot|$ is the Euclidean norm, we write P in place of P_φ; localization of the perimeter in a Borel set A is denoted by $P(\cdot, A)$. When φ is Euclidean and $f(r) = r$ we recover the standard Almgren-Taylor-Wang functional with a driving force [12]. In what follows, if no confusion is possible,

$$\text{we shorthand } \mathcal{F}_{\varphi,f,g} \text{ with the symbol } \mathcal{F}. \tag{1.5}$$

Note that when φ is Euclidean and $g \equiv 0$, we have, for $B_\varrho = \{\xi \in \mathbb{R}^n : |\xi| < \varrho\}$,

$$\mathcal{F}(B_r; B_{r_0}, \tau) = n\omega_n r^{n-1} + n\omega_n \int_0^r f\left(\tfrac{s-r_0}{\tau}\right) s^{n-1} ds =: n\omega_n \ell(r)$$

for any $r, r_0 > 0$. Since ℓ is differentiable,

$$\ell'(r) = (n-1)r^{n-2} + r^{n-1} f\left(\tfrac{r-r_0}{\tau}\right), \quad r > 0.$$

Thus, $r > 0$ is a critical point if and only if

$$\frac{n-1}{r} + f\left(\tfrac{r-r_0}{\tau}\right) = 0 \quad \Longleftrightarrow \quad \frac{r-r_0}{\tau} = -f^{-1}\left(\tfrac{n-1}{r}\right). \tag{1.6}$$

Using this in Theorem 4.4 we show that balls shrink self-similarly and their radii satisfy

$$R'(t) = -f^{-1}\left(\tfrac{n-1}{R(t)}\right) \quad \text{if } R(t) > 0,$$

consistently with (1.3). See Section 4 for more.

The next definition is a particular case of a definition given in [17].

Definition 1.3 (**Flat flows,** GMM **and** MM)**.** A family $\{E(t)\}_{t\in[0,+\infty)} \subset \mathcal{S}^*$ is called a *generalized minimizing movement* (shortly, GMM) in $\mathcal{S}^*$, associated to $\mathcal{F}$ and starting from $E_0 \in S^*$, if there exist a sequence $\tau_j \to 0^+$ and a family $\{E(\tau_j, k)\}_{j,k\ge 0}$ of sets, so-called flat flows, defined as $E(\tau_j, 0) = E_0$ and

$$E(\tau_j, k) \in \operatorname{argmin} \mathcal{F}(\cdot; E(\tau_j, k-1), \tau_j), \quad k, j \ge 1,$$

such that for any $t \ge 0$

$$E(\tau_j, \lfloor t/\tau_j \rfloor) \to E(t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } j \to +\infty, \tag{1.7}$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. When $E(\cdot)$ in (1.7) is independent of the sequence (τ_j), i.e., the limit holds as $\tau \to 0^+$, then it is called a minimizing movement (shortly MM) in $\mathcal{S}^*$, associated to $\mathcal{F}$ and starting from E_0. The set of GMM and MM in $\mathcal{S}^*$ will be denoted, respectively, as $\mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$ and $\mathrm{MM}(\mathcal{F}, \mathcal{S}^*, E_0)$.

[2]We slightly abuse the notation: in contrast to the functionals $\mathcal{G}$, $\widetilde{\mathcal{G}}$ in Conjectures 1.1 and 1.2, we do not highlight the dependence on k.

Now, we list a number of assumptions on f and g needed in the sequel.

Hypothesis (H).

(Ha) *$f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, continuous, surjective odd function;*

(Hb) *for any $\varsigma_0, \varsigma_1 > 0$ and any $\tau > 0$, the unique solution (ρ_τ, r_τ) of the system*

$$\begin{cases} \rho f\Big(\frac{\rho}{\tau}\Big) = \varsigma_0, \\ r f\Big(\frac{r+2\rho}{\tau}\Big) = \varsigma_1, \\ \rho, r > 0, \end{cases} \tag{1.8}$$

satisfies[3]

$$\liminf_{\tau \to 0^+} \frac{r_\tau}{\tau} \in (0, +\infty]; \tag{1.9}$$

(Hc) *$g \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \in [n, +\infty]$.*

When $\alpha = 1$, we have $\rho_\tau = \sqrt{\varsigma_0}\,\tau^{1/2}$ and $r_\tau = (\sqrt{\varsigma_0 + 4\varsigma_1} - \sqrt{\varsigma_0})\tau^{1/2}$. More generally, when $r > 0$ and $f(r) = r^\alpha$ for some $\alpha > 0$, we can explicitly find ρ_τ in (1.8) : $\rho_\tau = \varsigma_0^{\frac{1}{1+\alpha}} \tau^{\frac{\alpha}{1+\alpha}}$. Moreover, $r_\tau > \varsigma_1^{1/\alpha} \tau^{\frac{\alpha}{1+\alpha}}$ and by the Hölder inequality and the explicit expression of ρ_τ

$$\varsigma_1^{1/\alpha} \tau = r_\tau^{\frac{1+\alpha}{\alpha}} + 2\rho r_\tau^{1/\alpha} \le r_\tau^{\frac{1+\alpha}{\alpha}} + \tfrac{2\alpha\epsilon}{1+\alpha}\tau + \tfrac{1}{(1+\alpha)\epsilon} r_\tau^{\frac{1+\alpha}{\alpha}}$$

and thus, choosing $\epsilon > 0$ small enough we find $r_\tau \ge \varsigma_2 \tau^{\frac{\alpha}{1+\alpha}}$ for some $\varsigma_2 > 0$. In particular r_τ satisfies (1.9). See also Example 2.1.

Our first result is the following

Theorem 1.4 (**Existence, time-continuity and bounds of** GMM)**.** *Assume Hypothesis* (H) *and use the notation* (1.5)*. Then:*

(i) *For any $E_0 \in \mathcal{S}^*$, $\mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$ is nonempty. Moreover, there exists a continuous strictly increasing function $\omega : [0, +\infty) \to [0, +\infty)$ (depending only on f, n, the constants c_φ and C_φ in* (2.1)*, $P_\varphi(E_0)$ and $\|g^-\|_{L^1(\mathbb{R}^n)}$) with $\omega(0) = 0$ such that for any $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$*

$$|E(t)\Delta E(s)| \le \omega(|t - s|), \quad s, t > 0. \tag{1.10}$$

If, additionally, $|\partial E_0| = 0$, then (1.10) *holds for all $s, t \ge 0$. Furthermore, if*

$$\exists \lim_{r \to +\infty} \frac{f(r)}{r^\alpha} \in (0, +\infty)$$

for some $\alpha > 0$, then ω can be chosen locally $\frac{\alpha}{1+\alpha}$-Hölder continuous.

(ii) *If E_0 is bounded and*

$$g^-(x) \le f(c_{g^-}(1 + |x|)) \quad \text{for all } |x| > c_{g^-} \tag{1.11}$$

for some sufficiently large constant $c_{g^-} > 0$, then each GMM *is locally bounded, i.e., for any $T > 0$ there exists $R_T > 0$ such that for any $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$*

$$E(t) \subset B_{R_T}(0) \quad \text{for all } t \in [0, T].$$

Some comments are in order.

[3] Note that r_τ depends on ς_0, ς_1 and ρ_τ.

- The oddness of f allows to reduce $\mathcal{F}$ to a prescribed mean curvature functional: when f is odd, we can write
$$\int_{E\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx = \int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx - \int_F f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx \tag{1.12}$$
whenever $E\cap F$ has finite measure, and sd_F is the signed distance from ∂F *positive outside* F. Indeed, one can readily check that $\chi_G \mathrm{sd}_G \in L^1(\mathbb{R}^n)$ for any G with $|G| < +\infty$. Thus,
$$\begin{aligned}\int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx - \int_F f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx &= \int_{E\setminus F} f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx - \int_{F\setminus E} f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx\\ &= \int_{E\setminus F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx + \int_{F\setminus E} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx.\end{aligned}$$
Therefore, similarly to the classical Almgren-Taylor-Wang functional,
$$\mathcal{F}(E;F,\tau) = P_\varphi(E) + \int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx + \int_E g(x)\,dx + c_F, \tag{1.13}$$
where c_F is a constant independent of E. In view of (1.13), the minimization problem for $\mathcal{F}(\cdot;F,\tau)$ is equivalent to the minimization of the prescribed mean curvature functional $E\in\mathcal{S}^*\mapsto P_\varphi(E) + \int_E h_{f,F,g}\,dx$, with
$$h_{f,F,g} := f\left(\tfrac{\mathrm{sd}_F}{\tau}\right) + g. \tag{1.14}$$
- (Ha) and (Hc) suffice for the well-definiteness and the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuity of $\mathcal{F}$, as well as for the existence of minimizers, and in particular, of flat flows (Lemma 3.1).
- Assumption $g\in L^p(\mathbb{R}^n)$ in (Hc) is used to establish the uniform density estimate
$$\frac{P(E,B_r(x))}{r^{n-1}} \ge \theta_1, \quad x\in\partial E, \quad r\in(0,r_\tau], \tag{1.15}$$
for minimizers E of $\mathcal{F}(\cdot;F,\tau)$, where $\theta_1 > 0$ is independent of E, F and τ (see Sections 3.1, 3.2 and the inequality (3.15)).
- Assumption (1.9) together with (1.15) and Lemma A.1 applied with $\ell=\tau$ and $p>0$, imply that any flat flow $\{E(\tau,k)\}$ satisfies
$$|E(\tau,k)\Delta E(\tau,k-1)| \le Cp^\sigma \tau P_\varphi(E(\tau,k-1)) + \frac{1}{f(p)}\int_{E(\tau,k)\Delta E(\tau,k-1)} f\left(\tfrac{\mathrm{d}_{E(\tau,k-1)}}{\tau}\right)\,dx$$
for all $k\ge 2$ and for some $\sigma\in\{1,n\}$, where C is a coefficient depending only on n, φ, θ and the liminf of r_τ/τ in (1.9). This estimate, with a suitable choice of p (see (3.23)) yields the almost uniform time-continuity of the flat flow $t\mapsto E(\tau,\lfloor t/\tau\rfloor)$ (see (3.24)), which in turn implies the existence of GMM and the validity of (1.10).
- If $f(r)=r^\alpha$ for $r>0$ and for some $\alpha>0$, then ω in (1.10) can be chosen as $\omega(t)=t^{\frac{\alpha}{1+\alpha}}$, see also (3.25).
- The strict monotonicity and surjectivity of f are needed for the unique solvability of system (1.8). At the moment, we do not know what happens if these assumptions are dropped.
- Without assumption (1.9) our method fails to apply for the existence of GMM (see Remark 3.2). However, we do not know whether GMM exists or not without this assumption.
- If $f(r)\sim r^\alpha$ as $r\to+\infty$ for some $\alpha>0$, then f satisfies (Hb), whereas if f has exponential growth, then (1.9) may fail (see Examples 2.1 and 2.2).

To prove the existence and uniform time continuity of GMM in Theorem 1.4 we follow the standard Almgren-Taylor-Wang method in [2, 12, 25]. However, the presence of f requires some extra care in techniques. Furthermore, unlike these papers, for the existence of GMM we do not assume a priori the boundedness of the initial sets and also of g. Finally, the uniform boundedness of GMM will be done employing the isoperimetric properties of (anisotropic) balls. This can be also done following [12, Lemma 3.9], where a time-dependent bounded g is considered.

It is well-known [8, Theorem 12] that the classical mean curvature flow preserves mean convexity; our next main qualitative result is a similar preservation (Section 5), which to some extent generalizes [13].

Theorem 1.5 (**Mean convex evolution**). *Suppose that f satisfies (Ha), (Hb) and $g \equiv 0$. Then any $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$ starting from a bounded δ-mean convex set $E_0 \Subset \Omega$ (in some open set Ω) is itself a flow of δ-mean convex sets in Ω. Moreover, the maps $t \mapsto E(t)$ and $t \mapsto P_\varphi(E(t))$ are nonincreasing.*

To prove this theorem we partially follow the ideas of [13]; in order to show the δ-mean convexity of minimizers of $\mathcal{F}$, the authors of [13] used the so-called Chambolle scheme for mean curvature flow and the Anzelotti-pairings, while here we employ comparison properties for the prescribed mean curvature functional. Classical mean curvature flow even preserves convexity [23]. Also, the GMM starting from a convex set is unique (which positively answers to the last assertion of Conjecture 1.1 when $\alpha = 1$), see [8]. The following result shows the validity of this property also in the generalized power mean curvature flow setting (see Section 7).

Theorem 1.6 (**Evolution of convex sets and stability**). *Let φ be the Euclidean norm, $g \equiv 0$ and $f(r) = r^\alpha$ for $r > 0$, with $\alpha > 0$. Let $\mathcal{K}_0 \in \mathrm{Conv_b}(\mathbb{R}^n)$. Then:*

(i) $\mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, \mathcal{K}_0) = \mathrm{MM}(\mathcal{F}, \mathcal{S}^*, \mathcal{K}_0) = \{\mathcal{K}(\cdot)\}$ *and $\mathcal{K}(t)$ is convex for any $t \geq 0$. Moreover, if $\mathcal{K}_0$ is smooth, then $\mathcal{K}(\cdot)$ is the smooth convex power mean curvature evolution starting from $\mathcal{K}_0$ (see Theorem 7.1);*

(ii) *Let $\mathcal{K}_0 \in \mathrm{Conv_b}(\mathbb{R}^n)$ and let $(\mathcal{K}_{0h})$ be any sequence of sets such that $\partial\mathcal{K}_{0h} \to \partial\mathcal{K}_0$ in the Kuratowski sense and $\{P(\mathcal{K}_{0h})\}$ is uniformly bounded. Let $\mathcal{K}_h(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, \mathcal{K}_{0h})$, and let $\{\mathcal{K}(\cdot)\}$ be the unique minimizing movement starting from $\mathcal{K}_0$. Then*
$$\lim_{h\to+\infty} |\mathcal{K}_h(t)\Delta\mathcal{K}(t)| = 0 \quad \textit{for any } t \geq 0.$$

Since any convex set is also mean convex, from Theorem 1.5 it follows that each $\mathcal{K}(t)$ is mean convex and $t \mapsto \mathcal{K}(t)$ is nonincreasing.

We expect similar uniqueness and stability properties for generalized curvature flow of convex sets also in the anisotropic case with more general f. However, we leave this problem for future investigations.

Theorem 1.6 is a weak formulation of Conjecture 1.1, where the minimization problem for $\mathcal{F}$ is conducted in the larger class $\mathcal{S}^*$ rather than in the class $\mathrm{Conv_b}(\mathbb{R}^n)$. Indeed, in $\mathrm{Conv_b}(\mathbb{R}^n)$ we cannot apply the cutting-filling with balls argument used in the proof of Theorem 1.4; rather using minimal cutting properties of convex sets we can show that the flat flows in $\mathrm{Conv_b}(\mathbb{R}^n)$ and their perimeter have the following monotonicity: $E(\tau, 0) \supset E(\tau, 1) \supset \ldots$ and $P(E(\tau, 0)) \geq P(E(\tau, 1)) \geq \ldots$. Thus, the map $t \mapsto P(E(\tau, \lfloor t/\tau \rfloor))$ is bounded and nonincreasing, and therefore, sequentially compact (w.r.t. τ) in $L^1_{\mathrm{loc}}(\mathbb{R}^+_0)$. Now using convexity we conclude that every limit point is indeed a GMM (Section 8):

Theorem 1.7 (GMM **in the class** $\mathrm{Conv_b}(\mathbb{R}^n)$)**.** *Assume φ is an anisotropy in $\mathbb{R}^n$, f satisfies Hypothesis (Ha) and (Hb) and $g \equiv 0$. For any $\mathcal{K}_0 \in \mathrm{Conv_b}(\mathbb{R}^n)$, $\mathrm{GMM}(\mathcal{F}, \mathrm{Conv_b}(\mathbb{R}^n), \mathcal{K}_0)$ is nonempty. Moreover, if φ is Euclidean and $f(r) = r^\alpha$ for $r > 0$ and some $\alpha \in (0,1]$, then $\mathrm{GMM}(\mathcal{F}, \mathrm{Conv_b}(\mathbb{R}^n), \mathcal{K}_0)$ is a singleton and concides with the unique minimizing movement in $\mathrm{MM}(\mathcal{F}, \mathcal{S}^*, \mathcal{K}_0)$.*

We observe that we miss the proof of uniqueness of the minimizing movement in $\mathrm{Conv_b}(\mathbb{R}^n)$ for $\alpha > 1$.

Our last result is consistency of GMM with the smooth classical solution of (1.3)[4]. Assuming the latter exists and is stable (in the sense of Definition 6.1), following some of the ideas of [2] and [24] we show:

Theorem 1.8 (**Consistency**)**.** *Suppose that $f \in C^\beta(\mathbb{R})$ and $g \in C^\beta(\mathbb{R}^n)$ for some $\beta \in (0,1]$. Assume that φ is an elliptic C^3-anisotropy and (1.3) admits a unique smooth stable solution $\{S(t)\}_{t\in[0,T)}$. Then for every $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, S(0))$ we have*

$$E(t) = S(t) \quad \textit{for all } t \in [0,T).$$

As it happens for the classical consistency proof for the Almgren-Taylor-Wang functional, the proof of this nontrivial theorem heavily relies on the stability of the flow, strong comparison principles and discrete comparison principles.

The paper is organized as follows. In Section 2, after setting the notation, we quickly shows some examples of interesting functions f. In Section 3 we prove Theorem 1.4. Various comparison results are studied in Section 4. The evolution of mean convex sets and convex sets are considered in Sections 5, 7, and 8. The consistency of GMM with smooth solutions is proven in Section 6. Finally, we conclude the paper with two appendices, where we establish some technical results, needed in various proofs.

Shortly after the conclusion of this paper, we became aware of the paper [16], where the author addresses a similar problem, showing existence of level set solutions to the power mean curvature flow, via the minimizing movements. That paper appears to be completely independent of the present paper.

Acknowledgements. G. Bellettini acknowledges support from PRIN 2022PJ9EFL "Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations", and from GNAMPA (INdAM). Sh. Kholmatov acknowledges support from the Austrian Science Fund (FWF) Stand-Alone project P 33716.

2. Notation and examples of functions f

We write $\mathbb{R}^+ := (0,+\infty)$, $\mathbb{R}^+_0 := [0,+\infty)$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. The inclusion $E \Subset F$ means that E is bounded, $E \subset F$ and $\mathrm{dist}(\partial E, \partial F) > 0$; we also set $E^c := \mathbb{R}^n \setminus E$. Unless otherwise stated, $B_r := B_r(0)$. In view of the definition of $\mathcal{S}$ in (1.1), every set E we consider coincides with the set $E^{(1)}$ of its points of density 1. Therefore $\partial E = \overline{\partial^* E}$ and $\mathrm{d}_E(\cdot) = \mathrm{dist}(\cdot, \partial E) = \mathrm{dist}(\cdot, \partial^* E)$. We denote by $\overline{co}\,(E)$ the closed convex hull of E. An anisotropy $\varphi : \mathbb{R}^n \to [0,+\infty)$ is a positively one-homogeneous even convex function equivalent to the Euclidean norm; its dual is defined as

$$\varphi^o(\eta) = \sup_{\varphi(\xi)=1} \xi\cdot\eta = 1,$$

so that φ^o is an anisotropy and

$$c_\varphi|\xi| \le \varphi^o(\xi) \le C_\varphi|\xi|, \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

[4]Thus, in this case, GMM is unique for short times.

for some $C_\varphi \ge c_\varphi > 0$.

Recall the following anisotropic isoperimetric inequality [26]:

$$P_\varphi(E) \ge c_{n,\varphi}|E|^{\frac{n-1}{n}}, \quad c_{n,\varphi} := \frac{P_\varphi(B_\varphi)}{|B_\varphi|^{\frac{n-1}{n}}}, \tag{2.2}$$

where P_φ is defined in (1.4) and[5] $B_\varphi := \{\varphi \le 1\}$. In the Euclidean case $\varphi(\cdot) = |\cdot|$, $c_{n,|\cdot|} = n\omega_n^{1/n}$, $B = B_{|\cdot|}$, and $\kappa_\varphi = \kappa$. Our conventions are: the mean curvature is positive for (uniformly) convex sets, and the normal velocity is increasing for expanding sets.

Let us give some examples of function f.

Example 2.1 (*f* **a power**)**.** Assume that f is strictly increasing, continuous and

$$c_1 r^{\alpha_1} \le f(r) \le c_2 r^{\alpha_2} \quad \text{for all large } r > 1 \tag{2.3}$$

for some $0 < c_1 \le c_2$ and $0 < \alpha_1 \le \alpha_2 \le \alpha_1 + 1$. Then for any $\tau > 0$ the system (1.8) is solvable and r_τ satisfies (1.9). Indeed, solvability of (1.8) follows from the continuity and the monotonicity of f. In particular, by the inverse monotone function theorem both maps $\tau \mapsto \rho_\tau$ and $\tau \mapsto r_\tau$ are continuous and increasing. Moreover, clearly, $\rho_\tau, r_\tau \to 0$ as $\tau \to 0^+$. Now by (2.3) and the equality $\rho_\tau f(\rho_\tau/\tau) = C_0$

$$c_1'\tau^{\frac{\alpha_2}{1+\alpha_2}} \le \rho_\tau < c_2'\tau^{\frac{\alpha_1}{1+\alpha_1}} \tag{2.4}$$

for all small $\tau > 0$ and for some $c_2' \ge c_1' > 0$ independent of τ. Similarly, from the equality $r_\tau f(\frac{r_\tau + 2\rho_\tau}{\tau}) = C_1$ and estimates (2.3) and (2.4) we get

$$c_1''\tau \le r_\tau^{1/\alpha_2}(r_\tau + 2c_2'\tau^{\frac{\alpha_1}{1+\alpha_1}}) = r_\tau^{1+1/\alpha_2} + c_2' r_\tau^{1/\alpha_2}\tau^{\frac{\alpha_1}{1+\alpha_1}}$$

for all small $\tau > 0$ and for some $c_1'' > 0$. This inequality implies either $\frac{\tau}{r_\tau^{1+1/\alpha_2}}$ or $\frac{\tau}{r_\tau^{1/\alpha_2}\tau^{\frac{\alpha_1}{1+\alpha_1}}}$ is uniformly bounded. Since $\alpha_1 \le \alpha_2 \le 1 + \alpha_1$, either condition implies $r_\tau \ge c\tau$ for some $c > 0$ and all small $\tau > 0$. Therefore, (1.9) holds.

Example 2.2 (*f* **an exponential**)**.** Let $f(r) = e^r$ for large $r > 0$. Then ρ_τ in (1.8) satisfies

$$\rho_\tau = -\tau \ln \rho_\tau + \tau \ln \varsigma_0. \tag{2.5}$$

For sufficiently small $\tau > 0$ setting $\rho_\tau = \tau \ln(\tau^{-1})(1+u)$ in (2.5) we get

$$F(u,x,y) := u + x + [\ln(1+u) - \ln \varsigma_0]y = 0, \quad x := \tfrac{\ln\ln(\tau^{-1})}{\ln(\tau^{-1})}, \quad y := \tfrac{1}{\ln(\tau^{-1})}.$$

This analytic implicit function admits a unique solution $u = w(x,y)$, where $w(\cdot,\cdot)$ is some real analytic function in a neighborhood of $(0,0)$, and thus, for sufficiently small $\tau > 0$,

$$\rho_\tau = \tau \ln(\tau^{-1})\Big[1 + w\Big(\tfrac{\ln\ln(\tau^{-1})}{\ln(\tau^{-1})}, \tfrac{1}{\ln(\tau^{-1})}\Big)\Big]. \tag{2.6}$$

Moreover, since r_τ satisfies the equation

$$\frac{r_\tau + 2\rho_\tau}{\tau} = \ln \varsigma_1 + \ln(r_\tau^{-1}),$$

we have

$$\frac{r_\tau}{\tau \ln(r_\tau^{-1})} + \frac{\rho_\tau}{\tau\ln(\rho_\tau^{-1})}\frac{2\ln(\rho_\tau^{-1})}{\ln(r_\tau^{-1})} = 1 + \frac{\ln \varsigma_1}{\ln(r_\tau^{-1})}. \tag{2.7}$$

In view of (2.5)

$$\lim_{\tau\to 0^+} \frac{\rho_\tau}{\tau\ln(\rho_\tau^{-1})} = 1.$$

[5]Sometimes B_φ is called Wulff shape.

Therefore, using $\frac{r_\tau}{\tau \ln(r_\tau^{-1})} \ge 0$ and recalling that $r_\tau \to 0^+$, from (2.7) we get

$$\limsup_{\tau\to 0} \frac{2\ln(\rho_\tau^{-1})}{\ln(r_\tau^{-1})} \le 1.$$

In particular, for for sufficiently small $\tau > 0$,

$$\frac{2\ln(\rho_\tau^{-1})}{\ln(r_\tau^{-1})} \le \frac{3}{2}.$$

This and (2.6) imply

$$\frac{r_\tau}{\tau} \le \frac{\rho_\tau^{4/3}}{\tau} = \tau^{1/3}[\ln(\tau^{-1})(1+w)]^{4/3} \to 0$$

as $\tau \to 0^+$. Hence, f does not satisfy (1.9).

Throughout the paper we always assume $n \ge 2$. We refer to [28] for the case $n = 1$.

3. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Assume as usual the convention (1.5).

Lemma 3.1. *Suppose that f satisfies (Ha) and that $g \in L^1(\mathbb{R}^n)$. Then for any $F \in \mathcal{S}^*$ and $\tau > 0$ there exists a minimizer of $\mathcal{F}(\cdot; F, \tau)$ in $\mathcal{S}^*$.*

As in the case of prescribed mean curvature functional, the assumption $g \in L^1(\mathbb{R}^n)$ can be relaxed to $g^- \in L^1(\mathbb{R}^n)$.

Proof. Let us study the minimum problem

$$\inf_{E\in\mathcal{S}^*} \ \mathcal{F}(E; F, \tau).$$

Let $(E_i) \subset \mathcal{S}^*$ be a minimizing sequence. We may assume

$$\mathcal{F}(E_i; F, \tau) \le \mathcal{F}(F; F, \tau) \quad \text{for all } i \ge 1.$$

Then

$$P_\varphi(E_i) + \int_{E_i \Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\, dx \le \mathcal{F}(F; F, \tau) + \int_{\mathbb{R}^n} |g|\, dx := C.$$

In particular, by the nonnegativity of f in $\mathbb{R}_0^+$, the sequence $(P_\varphi(E_i))$ is bounded, and hence, by compactness, there exists $E \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\})$ such that, up to a not relabelled subsequence, $E_i \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $i \to +\infty$. By the L^1_{loc}-lower semicontinuity of the anisotropic perimeter,

$$P_\varphi(E) \le \liminf_{i\to+\infty} P_\varphi(E_i) \le C < +\infty.$$

Moreover, by the L^1_{loc}-convergence and the isoperimetric inequality (2.2), for any bounded open set $U \subset \mathbb{R}^n$

$$|U \cap E| = \lim_{i\to+\infty} |U\cap E_i| \le \liminf_{i\to+\infty} |E_i| \le (c_{n,\varphi}^{\frac{n}{n-1}})^{-1} \liminf_{i\to+\infty} P_\varphi(E_i)^{\frac{n}{n-1}} \le \frac{C^{\frac{n}{n-1}}}{c_{n,\varphi}^{\frac{n}{n-1}}}.$$

Thus, letting $U \nearrow \mathbb{R}^n$ we get $|E| < +\infty$ and hence, $E \in \mathcal{S}^*$. By the L^1_{loc}-lower semicontinuity of $\mathcal{F}(\cdot; F, \tau, k)$, E is a minimizer. □

3.1. **L^∞-bound for d_F.** Assume Hypothesis (H). For $F \in \mathcal{S}^*$ and $\tau > 0$, let F_τ be a minimizer of $\mathcal{F}(\cdot; F, \tau)$. In (3.10), on the basis of (3.6), we establish an upper bound for

$$\sup_{F_\tau \Delta F} \mathrm{d}_F,$$

which is necessary to get the density estimates in Section 3.2 centered at points of ∂F_τ.

Take any $x \in \overline{F_\tau} \setminus F$ and set

$$r_x := \mathrm{d}_F(x) > 0, \qquad B_r := B_r(x).$$

We observe that $B_r \cap F = \emptyset$ for all $r \in (0, r_x)$, so that $F \setminus F_\tau = F \setminus (F_\tau \setminus B_r)$. Then by the minimality of F_τ,

$$\begin{aligned} 0 \le\ & \mathcal{F}(F_\tau \setminus B_r; F, \tau) - \mathcal{F}(F_\tau; F, \tau) \\ & = P_\varphi(F_\tau \setminus B_r) - P_\varphi(F_\tau) - \int_{F_\tau \cap B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy - \int_{F_\tau \cap B_r} g dy. \end{aligned} \tag{3.1}$$

Using the properties of the reduced boundary [26, Theorem 16.3], (2.1) and the Euclidean isoperimetric inequality, for a.e. $r \in (0, r_x)$ we get

$$\begin{aligned} P_\varphi(F_\tau \setminus B_r) - P_\varphi(F_\tau) = & \int_{F_\tau \cap \partial B_r} \varphi^o(-\nu_{B_r}) d\mathcal{H}^{n-1} - P_\varphi(F_\tau, B_r) \\ = & \int_{F_\tau \cap \partial B_r} \Big(\varphi^o(-\nu_{B_r}) + \varphi^o(\nu_{B_r})\Big) d\mathcal{H}^{n-1} - P_\varphi(F_\tau \cap B_r) \\ \le & 2C_\varphi \mathcal{H}^{n-1}(F_\tau \cap \partial B_r) - c_\varphi n \omega_n^{1/n} |F_\tau \cap B_r|^{\frac{n-1}{n}}. \end{aligned} \tag{3.2}$$

Since $\mathrm{d}_F \ge r_x - r$ in B_r, by the increasing monotonicity of f we obtain

$$-\int_{F_\tau \cap B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le -f\Big(\frac{r_x - r}{\tau}\Big) |F_\tau \cap B_r|.$$

Finally, if we also assume $r < \gamma_g$, with $\gamma_g > 0$ as in (A.3), then

$$\int_{F_\tau \cap B_r} g dy \le \frac{c_\varphi n \omega_n^{1/n}}{4} |F_\tau \cap B_r|^{\frac{n-1}{n}}. \tag{3.3}$$

Inserting (3.2)-(3.3) in (3.1) we get

$$2C_\varphi \mathcal{H}^{n-1}(F_\tau \cap \partial B_r) \ge f\Big(\frac{r_x - r}{\tau}\Big) |F_\tau \cap B_r| + \frac{3c_\varphi n \omega_n^{1/n}}{4} |F_\tau \cap B_r|^{\frac{n-1}{n}} \tag{3.4}$$

for a.e. $r \in (0, r_x \wedge \gamma_g)$. Since $r_x > r$ and $f > 0$ in $\mathbb{R}^+$, this inequality implies

$$\mathcal{H}^{n-1}(F_\tau \cap \partial B_r) \ge \frac{3c_\varphi n \omega_n^{1/n}}{8C_\varphi} |F_\tau \cap B_r|^{\frac{n-1}{n}},$$

or, after integration,

$$|F_\tau \cap B_r| \ge \left(\tfrac{3c_\varphi}{8C_\varphi}\right)^n \omega_n r^n, \quad r \in (0, r_x \wedge \gamma_g]. \tag{3.5}$$

Plugging this volume density estimate in (3.4) and using the positivity of the last term, we get

$$f\Big(\frac{r_x - r}{\tau}\Big) \left(\tfrac{3c_\varphi}{8C_\varphi}\right)^n \omega_n r^n \le 2C_\varphi \mathcal{H}^{n-1}(F_\tau \cap \partial B_r) \le 2C_\varphi n \omega_n r^{n-1},$$

and hence, by the continuity of f,

$$r f\Big(\frac{r_x - r}{\tau}\Big) \le C_1 := 2C_\varphi n \Big(\tfrac{8C_\varphi}{3c_\varphi}\Big)^n, \quad r \in [0, r_x \wedge \gamma_g]. \tag{3.6}$$

Now assume $x \in \overline{F}\setminus F_\tau$ and set again $r_x := \mathrm{d}_F(x) > 0$. In this case for a.e. $r \in (0, r_x \wedge \gamma_g)$, we use the properties of the reduced boundary, (2.1) and the Euclidean isoperimetric inequality to get

$$\begin{aligned} P_\varphi(F_\tau \cup B_r) - P_\varphi(F_\tau) =& \, 2\int_{F_\tau^c \cap \partial^* B_r} \Big(\varphi^o(\nu_{B_r}) + \varphi^o(-\nu_{B_r})\Big) d\mathcal{H}^{n-1} - P_{\varphi^\#}(F_\tau \cap B_r) \\ \le & \, 2C_\varphi \mathcal{H}^{n-1}(F_\tau^c \cap \partial^* B_r) - c_\varphi n\omega_n^{1/n} |F_\tau^c \cap B_r|^{\frac{n-1}{n}}. \end{aligned}$$

Moreover, since $\mathrm{d}_F \ge r_x - r$ in B_r (recall that $B_r \subset F$ by the choice of r_x), by the monotonicity of f

$$-\int_{F_\tau^c \cap B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le -f\Big(\frac{r_x - r}{\tau}\Big)|F_\tau^c \cap B_r|.$$

Finally, using $r < \gamma_g$ and the Morrey-type extimate (A.3) we find

$$\int_{F_\tau^c \cap B_r} g dy \le \frac{c_\varphi n \omega_n^{1/n}}{4} |F_\tau^c \cap B_r|^{\frac{n-1}{n}}.$$

Inserting these estimates in

$$\begin{aligned} 0 \le & \, \mathcal{F}(F_\tau \cup B_r; F, \tau) - \mathcal{F}(F_\tau; f, \tau) \\ & = P_\varphi(F_\tau \cup B_r) - P_\varphi(F) - \int_{F_\tau^c \cap B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\, dy + \int_{F_\tau^c \cap B_r} g\, dy, \end{aligned}$$

we get

$$2C_\varphi \mathcal{H}^{n-1}(F_\tau^c \cap \partial B_r) \ge f\Big(\frac{r_x - r}{\tau}\Big)|F_\tau^c \cap B_r| + \frac{3c_\varphi n\omega_n^{1/n}}{4}|F_\tau^c \cap B_r|^{\frac{n-1}{n}},$$

and hence, repeating the same aguments above, we find the same inequality as (3.6). Thus, for any $x \in \overline{F_\tau \Delta F}$ we have, inverting (3.6),

$$r_x \le r + \tau f^{-1}\Big(\tfrac{C_1}{r}\Big), \qquad r_x := \mathrm{d}_F(x) > 0, \quad r \in (0, r_x \wedge \gamma_g). \tag{3.7}$$

For any $\tau > 0$ let $\rho_\tau := \rho_\tau(C_1) > 0$ be the unique number (compare (Hb)) such that

$$\rho_\tau f\Big(\frac{\rho_\tau}{\tau}\Big) = C_1. \tag{3.8}$$

Clearly, $\tau \mapsto \rho_\tau$ is continuous in $\mathbb{R}^+$ and $\rho_\tau \to 0^+$ as $\tau \to 0^+$. In particular

$$\exists \tau_0 = \tau_0(f, g) > 0 : \rho_\tau \le \gamma_g \quad \text{for all } \tau \in (0, \tau_0). \tag{3.9}$$

Let us estimate r_x with a multiple of ρ_τ. For $\tau \in (0, \tau_0)$, if $r_x \le \rho_\tau$, we are done. If $\rho_\tau < r_x$, by the choice of τ_0, we can apply (3.7) with $r = \rho_\tau$ and the equality (3.8) to get

$$r_x \le \rho_\tau + \tau f^{-1}\Big(\tfrac{C_1}{\rho_\tau}\Big) = 2\rho_\tau.$$

Thus we finally get the desired L^∞-bound

$$\sup_{x \in \overline{F_\tau \Delta F}} \mathrm{d}_F(x) \le 2\rho_\tau, \quad \tau \in (0, \tau_0). \tag{3.10}$$

This estimate will be used also in Theorem 4.3.

3.2. **Density estimates.** Assume Hypothesis (H) and for $F \in \mathcal{S}^*$, $\tau_0 > 0$ (defined in (3.9)) and $\tau \in (0, \tau_0)$ let F_τ be a minimizer of $\mathcal{F}(\cdot; F, \tau)$. Let $\gamma_g > 0$ be as in (A.3). We establish uniform density estimates for minimizers F_τ in balls $B_r(x)$, centered at

$$x \in \partial F_\tau$$

(see (3.13), (3.14) and (3.15)). Basically, we use the arguments of Section 3.1 and the function $\tau \to \rho_\tau$ in (3.8), but some extra care is needed because now

$$B_r = B_r(x)$$

may intersect the boundary of F and we need to estimate the differences

$$\begin{aligned}\int_{(F_\tau\setminus B_r)\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy - \int_{F_\tau\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \\ = - \int_{B_r\cap(F_\tau\setminus F)} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy + \int_{B_r\cap F_\tau\cap F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le \int_{B_r\cap F_\tau\cap F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy\end{aligned}$$

and

$$\begin{aligned}\int_{(F_\tau\cup B_r)\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy - \int_{F_\tau\Delta F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \\ = - \int_{B_r\cap(F_\tau^c\setminus F^c)} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy + \int_{B_r\cap F_\tau^c\cap F^c} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le \int_{B_r\cap F_\tau^c\cap F^c} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy.\end{aligned}$$

Note that if $x \in \partial F_\tau$ and $y \in B_r \cap F_\tau \cap F$ (resp. $y \in B_r \cap F_\tau^c \cap F^c$), by the 1-lipschitzianity of d_F and (3.10)

$$\mathrm{d}_F(y) \le \mathrm{d}_F(x) + |y - x| \le 2\rho_\tau + r,$$

and hence, by the monotonicity of f

$$\int_{B_r\cap F_\tau\cap F} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau|$$

and similarly

$$\int_{B_r\cap F_\tau^c\cap F^c} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dy \le f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau^c|.$$

For any $\tau \in (0, \tau_0)$ let $r_\tau > 0$ be the unique number (compare (Hb)) satisfying

$$r_\tau f\left(\tfrac{r_\tau+2\rho_\tau}{\tau}\right) = \frac{c_\varphi n}{2}. \tag{3.11}$$

By the continuity of f, $\tau \mapsto r_\tau$ is continuous and $r_\tau \to 0$ as $\tau \to 0^+$.

Possibly decreasing τ_0 in (3.9), we assume

$$r_\tau < \gamma_g \qquad \text{for all } \tau \in (0, \tau_0). \tag{3.12}$$

Now, by the increasing monotonicity of $r \mapsto rf((r + 2\rho_\tau)/\tau)$ and the definition of r_τ, for any $r \in (0, r_\tau]$ we have

$$f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau| = \omega_n^{1/n} r f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau|^{\frac{n-1}{n}} \le \frac{c_\varphi n\omega_n^{1/n}}{2}|B_r \cap F_\tau|^{\frac{n-1}{n}}$$

and

$$f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau^c| = \omega_n^{1/n} r f\left(\tfrac{r+2\rho_\tau}{\tau}\right)|B_r \cap F_\tau^c|^{\frac{n-1}{n}} \le \frac{c_\varphi n\omega_n^{1/n}}{2}|B_r \cap F_\tau^c|^{\frac{n-1}{n}}.$$

Recalling the inequality $r_\tau < \gamma_g$ and the estimates (3.2) and (3.3), for a.e. $r \in (0, r_\tau]$ we have

$$0 \le \mathcal{F}(F_\tau \setminus B_r; F, \tau) - \mathcal{F}(F_\tau; F, \tau) \le 2C_\varphi \mathcal{H}^{n-1}(F_\tau \cap B_r) - c_\varphi n \omega_n^{1/n} |F_\tau \cap B_r|^{\frac{n-1}{n}} + \frac{c_\varphi n \omega_n^{1/n}}{2} |F_\tau \cap B_r|^{\frac{n-1}{n}} + \frac{c_\varphi n \omega_n^{1/n}}{2} |F_\tau \cap B_r|^{\frac{n-1}{n}}.$$

This implies, similarly to (3.5),

$$|F_\tau \cap B_r| \ge \left(\tfrac{3c_\varphi}{8C_\varphi}\right)^n \omega_n r^n, \quad r \in (0, r_\tau]. \tag{3.13}$$

Analogously,

$$|F_\tau^c \cap B_r| \ge \left(\tfrac{3c_\varphi}{8C_\varphi}\right)^n \omega_n r^n, \quad r \in (0, r_\tau]. \tag{3.14}$$

From (3.13), (3.14), and the relative isoperimetric inequality in balls, we obtain

$$P(F_\tau, B_r) \ge \theta_1 r^{n-1}, \quad \tau \in (0, \tau_0), \quad r \in (0, r_\tau), \tag{3.15}$$

with $\theta_1 := 2^{\frac{n-1}{n}} \omega_{n-1} \left(\frac{c_\varphi}{8C_\varphi}\right)^{n-1}$.

3.3. **Existence of GMM.** As already mentioned in the introduction, the proof runs along a well-established path due to Almgren-Taylor-Wang [2] and Luckhaus-Sturzenhecker [25]; however, here various nontrivial technical modifications are required mainly due to the presence of f (and of g).

Let us assume Hypothesis (H). Fix $E_0 \in \mathcal{S}^*$, and $\tau \in (0, \tau_0)$ (recall that τ_0, given in (3.9) and (3.12)), and using Lemma 3.1 define the flat flows $\{E(\tau, k)\}_{k \ge 0}$ inductively as follows: $E(\tau, 0) = E_0$ and

$$E(\tau, k) \in \operatorname{argmin} \mathcal{F}(\cdot; E(\tau, k-1), \tau), \quad k \ge 1.$$

From the inequality

$$\mathcal{F}(E(\tau, k); E(\tau, k-1), \tau) \le \mathcal{F}(E(\tau, k-1); E(\tau, k-1), \tau), \quad k \ge 1,$$

we get the standard estimate

$$\int_{E(\tau,k)\Delta E(\tau,k-1)} f\left(\tfrac{\mathrm{d}_{E(\tau,k-1)}}{\tau}\right) dx \le \mathfrak{p}_{k-1} - \mathfrak{p}_k, \quad k \ge 1, \tag{3.16}$$

where

$$\mathfrak{p}_k := P_\varphi(E(\tau, k)) + \int_{E(\tau,k)} g\, dx, \quad k \ge 0,$$

and hence the sequence $(\mathfrak{p}_k)$ is nonincreasing. In particular,

$$P_\varphi(E(\tau, k)) = \mathfrak{p}_k - \int_{E(\tau,k)} g\, dx \le \mathfrak{p}_0 + \int_{E(\tau,k)} |g|\, dx$$

and therefore,

$$P_\varphi(E(\tau, k)) \le \mathfrak{p}_0 + \int_{\mathbb{R}^n} |g|\, dx := C_2 \quad \text{for all } k \ge 0. \tag{3.17}$$

Consider the family $\{E(\tau, \lfloor t/\tau \rfloor)\}_{t \ge 0}$. In view of (3.17), compactness in $BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\})$ and a diagonal argument, we can find a sequence $\tau_j \to 0^+$ such that for every rational $t \ge 0$ there exists a set $E(t) \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\})$ such that

$$E(\tau_j, \lfloor t/\tau_j \rfloor) \to E(t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } j \to +\infty. \tag{3.18}$$

In view of (3.17) and the isoperimetric inequality (2.2), the measure of $E(\tau_j, \lfloor t/\tau_j \rfloor)$ is bounded uniformly in j and t, and therefore $|E(t)|$ is bounded for each rational $t \ge 0$, in particular $E(t) \in \mathcal{S}^*$.

Now we will prove that the convergence (3.18) holds for any $t \ge 0$ (without passing to a further subsequence). To this aim, let us establish a sort of uniform continuity of flat flows $\{E(\tau,k)\}$.

In view of (3.15) each $E(\tau,k)$ with $k \ge 1$ satisfies a uniform lower perimeter density estimate for radii $r \in (0, r_\tau]$. Thus, we can apply Lemma A.1 with

$$r_0 = r_\tau \tag{3.19}$$

and $\vartheta = \theta_1$, to estimate the measure $|E(\tau,k-1)\Delta E(\tau,k)|$. In view of the expression of the "distance"-term in $\mathcal{F}$, we apply that lemma with

$$\ell = \tau \tag{3.20}$$

and some $p > 0$ to be chosen later (see (3.23)).

Remark 3.2 (**Necessity of assumption** (1.9))**.** Assumption (1.9) implies

$$\limsup_{\tau\to 0^+} \frac{\tau}{r_\tau} \in [0,+\infty).$$

If this limsup is infinite (for instance, in the case $f(r) = e^r$ for large r, see Example 2.2), an application of Lemma A.1 would give a large coefficient $\frac{\tau^{n-1}}{r_\tau^{n-1}}$ in the first inequality in estimate (A.2), which seems hard to handle. To avoid such an issue, we assume the validity of (1.9), which is used only here.

For any $p > 0$ and small $\tau > 0$ we have by (3.19) (3.20), and (A.2),

$$|E(\tau,k)\Delta E(\tau,k-1)| \le C_3 p^\sigma \tau P_\varphi(E(\tau,k-1)) + \frac{1}{f(p)} \int_{E(\tau,k)\Delta E(\tau,k-1)} f\left(\tfrac{\mathrm{d}_{E(\tau,k-1)}}{\tau}\right)\, dx$$

for all $k \ge 2$, where $C_3 > 0$ and $\sigma \in \{1, n\}$. By the uniform perimeter bound (3.17) and inequality (3.16) we can estimate further

$$|E(\tau,k)\Delta E(\tau,k-1)| \le C_4 p^\sigma \tau + \frac{\mathfrak{p}_{k-1} - \mathfrak{p}_k}{f(p)},$$

where $C_4 := C_2 C_3$. Summing these inequalities in $k = m_1 + 1, \dots, m_2$ for $1 \le m_1 < m_2$ we obtain

$$|E(\tau,m_2)\Delta E(\tau,m_1)| \le C_4 p^\sigma \tau (m_2 - m_1) + \frac{\mathfrak{p}_{m_1} - \mathfrak{p}_{m_2}}{f(p)}. \tag{3.21}$$

Since

$$\mathfrak{p}_{m_1} - \mathfrak{p}_{m_2} \le \mathfrak{p}_0 - \mathfrak{p}_{m_2} \le \mathfrak{p}_0 + \int_{E(\tau,m_2)} g^-(x)\, dx \le C_2,$$

(3.21) estimates further as

$$|E(\tau,m_1)\Delta E(\tau,m_2)| \le C_4 p^\sigma (m_2 - m_1)\tau + \frac{C_2}{f(p)}. \tag{3.22}$$

Now we come back to our chosen flat flows $\{E(\tau_j, \lfloor t/\tau_j \rfloor)\}_{t\ge0}$. Fix any $t_2 > t_1 > 0$ and, for τ_0 as in (3.9), (3.12), let $\tau_j \in (0,\tau_0)$ be so small that $\min\{t_1, t_2 - t_1\} > 10\tau_j$. Now we choose p. Let $u : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be any continuous increasing function[6] with $u(0) = 0$ and

$$\limsup_{s\to 0^+} \frac{s}{u(s)} = 0.$$

[6]In the case of the standard Almgren-Taylor-Wang functional one can choose $u(s) = s^{1/2}$ for any $s \ge 0$ and get the 1/2-Hölder continuity of the flow (best exponent).

We set

$$p := \begin{cases} \frac{1}{u(t_2-t_1)^{1/n}} & \text{if } \liminf\limits_{j\to+\infty} \frac{r_{\tau_j}}{\tau_j} \in (0,+\infty), \\ \frac{1}{u(t_2-t_1)} & \text{if } \liminf\limits_{j\to+\infty} \frac{r_{\tau_j}}{\tau_j} = +\infty. \end{cases} \tag{3.23}$$

Applying (3.22) with such a p, $m_1 = \lfloor t_1/\tau_j \rfloor$, $m_2 = \lfloor t_2/\tau_j \rfloor$ using $\tau_j(m_2 - m_1) \le t_2 - t_1 + \tau_j$ and recalling $p^\sigma = 1/u(t_2 - t_1)$, we get

$$|E(\tau_j, \lfloor t_2/\tau_j \rfloor)\Delta E(\tau_j, \lfloor t_1/\tau_j \rfloor)| \le \omega(t_2 - t_1) + \frac{C_4\tau_j}{u(t_2 - t_1)}, \tag{3.24}$$

where, for $s > 0$,

$$\omega(s) = \begin{cases} \frac{C_4 s}{u(s)} + \frac{C_2}{f(u(s)^{-1/n})} & \text{if } \liminf\limits_{j\to+\infty} \frac{r_{\tau_j}}{\tau_j} \in (0,+\infty), \\ \frac{C_4 s}{u(s)} + \frac{C_2}{f(u(s)^{-1})} & \text{if } \liminf\limits_{j\to+\infty} \frac{r_{\tau_j}}{\tau_j} = +\infty. \end{cases} \tag{3.25}$$

Now if both t_1 and t_2 are rational, then letting $j \to +\infty$ in (3.24) and recalling (3.18) we deduce

$$|E(t_2)\Delta E(t_1)| \le \omega(t_2 - t_1). \tag{3.26}$$

By assumptions on w and f, $\omega(s) \to 0$ as $s \to 0^+$. Thus, ω provides a modulus of continuity, which is for the moment defined only for rational times. Since

$$\sup_{t\ge 0,\,\text{rational}} P_\varphi(E(t)) \le C_2,$$

by standard arguments, we can uniquely extend $E(\cdot)$ for all $t \ge 0$ still satisfying the uniform continuity (3.26) for all $0 < t_1 < t_2$.

We claim that $E(\cdot)$ is a GMM, i.e., the convergence (3.18) holds for all $t \ge 0$. Indeed, consider an irrational t, and take any rational $\bar t > t$. Then for any open set $U \subset \mathbb{R}^n$,

$$\begin{aligned} |U \cap (E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta E(t))| &\le |U \cap (E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta U \cap (E(\tau_j, \lfloor \bar t/\tau_j \rfloor))| \\ &\quad + |U \cap (E(\tau_j, \lfloor \bar t/\tau_j \rfloor)\Delta E(\bar t))| + |E(\bar t)\Delta E(t)| =: a_j + b_j + c. \end{aligned}$$

Since $\bar t$ is rational, by (3.18)

$$\limsup_{j\to+\infty} b_j = 0.$$

Moreover, by (3.24) and (3.26)

$$\limsup_{j\to+\infty} a_j \le \omega(\bar t - t) \quad \text{and} \quad c \le \omega(\bar t - t).$$

Therefore,

$$\limsup_{j\to+\infty} |U \cap (E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta E(t))| \le 2\omega(\bar t - t).$$

Now letting $\bar t \searrow t$ and recalling that $\omega(\bar t - t) \to 0$ we deduce the validity of (3.18) for all times $t \ge 0$. Whence, by definition, $E(\cdot)$ is a GMM.

Remark 3.3. In the proof of the existence we started with an arbitrary sequence and constructed a subsequence for which the L^1_{loc}-convergence (3.18) holds.

Finally, let us prove that every $\{E(t)\}_{t\ge 0} \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$ starting from E_0 is uniformly continuous. Let $\{E(t)\}$ be defined as an L^1_{loc}-limit of flat flows $\{E(\tau_j, \lfloor t/\tau_j \rfloor)\}$ for some $\tau_j \to 0$. Then these flat flows necessarily satisfy (3.24), and hence, after passing to the limit we see that $E(\cdot)$ also satisfies (3.26).

3.4. **Uniform continuity of GMM up to $t=0$.** Let us assume that $|\partial F| = 0$ and let F_τ be a minimizer of $\mathcal{F}(\cdot; F, \tau)$. By minimality

$$P_\varphi(F_\tau) + \int_{F_\tau \Delta F_0} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)\,dx + \int_{F_\tau} g\,dx \le P_\varphi(F) + \int_F g\,dx, \tag{3.27}$$

and hence, recalling $g \in L^1(\mathbb{R}^n)$, the family $\{P_\varphi(F_\tau)\}$ is uniformly bounded with respect to τ. By compactness, for any sequence $\tau_k \searrow 0$, $F_{\tau_k} \overset{L^1_{\rm loc}}{\to} F_0$ (up to a not relabelled subsequence) where $F_0 \in \mathcal{S}^*$. By (3.27), the assumption $|\partial F| = 0$, and Fatou's lemma, $|F_0 \Delta F| = 0$, and hence, $F_0 = F$. Thus,

$$F_\tau \to F \tag{3.28}$$

in $L^1_{\rm loc}(\mathbb{R}^n)$ as $\tau \to 0$. Of course, using (3.27) one can also show

$$\lim_{\tau\to 0} P_\varphi(F_\tau) = P_\varphi(F) \quad \text{and} \quad \lim_{\tau\to0}\int_{F_\tau\Delta F_0} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)dx = 0.$$

Now to prove the uniform continuity of GMM for $s, t \ge 0$ we proceed as in the standard Almgren-Taylor-Wang case: applying (3.24) with $t_2 = t > 0$ and $t_1 = \tau_j$, as well as (3.28) for any open set $U \subset \mathbb{R}^n$ we find

$$|U \cap (E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta E_0)| \le |E(\tau_j, \lfloor t/\tau_j\rfloor)\Delta E(\tau_j, 1)| + |U \cap (E(\tau_j,1)\Delta E_0)| \le \omega(t-\tau_j) + o(1)$$

as $j \to +\infty$. This implies $|U \cap (E(t)\Delta E_0)| \le \omega(t)$, and hence, letting $U \to \mathbb{R}^n$ we get $|E(t)\Delta E_0| \le \omega(t)$.

3.5. **Hölder continuity of GMM.** Suppose

$$\lim_{r\to+\infty} \frac{f(r)}{r^\alpha} = C_5 \in (0, +\infty) \tag{3.29}$$

for some $\alpha \in (0, +\infty)$. Then using (3.8) and recalling $\rho_\tau/\tau \to +\infty$ as $\tau \to 0^+$ (see (2.4)) we get

$$\limsup_{\tau\to0^+} \frac{C_1\tau^\alpha}{\rho_\tau^{1+\alpha}} = \limsup_{\tau\to0^+}\frac{f(\rho_\tau/\tau)}{(\rho_\tau/\tau)^\alpha} = C_5,$$

and hence,

$$\liminf_{\tau\to0^+}\frac{\rho_\tau}{\tau^{\frac{\alpha}{1+\alpha}}} = \frac{C_1^{\frac{1}{1+\alpha}}}{C_5}.$$

Similarly, as $\frac{r_\tau+2\rho_\tau}{\tau} \to +\infty$ (see Example 2.1), by (3.11)

$$C_5 = \limsup_{\tau\to0^+}\frac{f((r_\tau+2\rho_\tau)/\tau)}{[(r_\tau+2\rho_\tau)/\tau]^\alpha} = \limsup_{\tau\to0^+}\frac{c_\varphi n}{2r_\tau[(r_\tau+2\rho_\tau)/\tau]^\alpha}.$$

Thus,

$$\left(\tfrac{2C_5}{c_\varphi n}\right)^{1/\alpha} = \liminf_{\tau\to0}\left(\frac{r_\tau^{\frac{\alpha+1}{\alpha}}}{\tau} + \frac{2r_\tau^{\frac{1}{\alpha}}\rho_\tau}{\tau}\right),$$

and therefore,

$$C_6 := \liminf_{\tau\to0^+}\frac{r_\tau}{\tau^{\frac{\alpha}{1+\alpha}}} < +\infty.$$

This implies

$$\liminf_{\tau\to0^+}\frac{r_\tau}{\tau} = +\infty.$$

Now choosing $w(s) = s^{\frac{1}{1+\alpha}}$, in (3.22) we can represent the function ω in (3.26) as

$$\omega(s) = C_4 s^{\frac{\alpha}{1+\alpha}} + \frac{C_2}{f(s^{-\frac{1}{1+\alpha}})}. \tag{3.30}$$

In view of (3.29) and the strict monotonicity of f, there exists $s_0 > 0$ such that

$$f(s^{-\frac{1}{1+\alpha}}) \geq \frac{C_5}{2}\, s^{-\frac{\alpha}{1+\alpha}} \quad \text{for all } s \in (0, s_0).$$

Thus, for such s, from (3.30) we conclude

$$\omega(s) \leq \Big(C_4 + \frac{2C_2}{C_5}\Big) s^{\frac{\alpha}{1+\alpha}}.$$

In view of (3.26) this implies the local $\frac{\alpha}{1+\alpha}$-continuity of GMM.

3.6. **Boundedness of** GMM**.** The next lemma shows that boundedness of an initial set implies boundedness of minimizers of $\mathcal{F}$.

Lemma 3.4. *Suppose Hypothesis* (H)*. Then, for any bounded $F \in \mathcal{S}^*$ and any $\tau > 0$, every minimizer of $\mathcal{F}(\cdot; F, \tau)$ is bounded.*

Proof. Let E be a minimizer of $\mathcal{F}(\cdot; F, \tau)$ and let $\gamma_g > 0$ be given by (A.3). Writing $B_r := B_r(0)$, let $r_0 > 0$ be such that $F \subset B_{r_0}$ and $|E \setminus B_r| < \omega_n \gamma_g^n$ for all $r > r_0$ (the last inequality is possible since $|E| < +\infty$). For any $r > r_0$, by the minimality of E and the inequality $F \setminus E = F \setminus [E \cap B_r]$

$$\begin{aligned} 0 &\leq \mathcal{F}(E \cap B_r; F, \tau) - \mathcal{F}(E; F, \tau) \\ &= P_\varphi(E \cap B_r) - P_\varphi(E) - \int_{E\setminus B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dx - \int_{E\setminus B_r} g dx. \end{aligned} \tag{3.31}$$

By the standard properties of the reduced boundary, (2.1) and the Euclidean isoperimetric inequality

$$\begin{aligned} &P_\varphi(E \cap B_r) - P_\varphi(E) = \int_{E\cap\partial B_r} \varphi^o(\nu_{B_r}) d\mathcal{H}^{n-1} - P_\varphi(E, B_r^c) \\ &= \int_{E\cap\partial B_r} \Big[\varphi^o(\nu_{B_r}) + \varphi^o(-\nu_{B_r})\Big] d\mathcal{H}^{n-1} - P_\varphi(E\setminus B_r) \leq 2C_\varphi \mathcal{H}^{n-1}(E\cap\partial B_r) - c_\varphi n\omega_n^{1/n}|E\setminus B_r| \end{aligned}$$

for a.e. $r > r_0$. Moreover, since $\mathrm{d}_F \geq r - r_0$ in $E \setminus B_r$ and f is increasing,

$$f\left(\tfrac{\mathrm{d}_F}{\tau}\right) \geq f\Big(\tfrac{r-r_0}{\tau}\Big) \quad \text{a.e. in } E \setminus B_r$$

and therefore,

$$\int_{E\setminus B_r} f\left(\tfrac{\mathrm{d}_F}{\tau}\right) dx \geq f\Big(\tfrac{r-r_0}{\tau}\Big)|E \setminus B_r|.$$

By the choice of r_0 and (A.3)

$$-\int_{E\setminus B_r} g dx \leq \tfrac{c_\varphi n\omega_n^{1/n}}{4}|E \setminus B_r|^{\frac{n-1}{n}}.$$

Inserting the above estimates in (3.31) we find

$$2C_\varphi \mathcal{H}^{n-1}(E \cap B_r) + \tfrac{c_\varphi n\omega_n^{1/n}}{4}|E \setminus B_r|^{\frac{n-1}{n}} \geq c_\varphi n\omega_n^{1/n}|E \setminus B_r|^{\frac{n-1}{n}} + f\Big(\tfrac{r-r_0}{\tau}\Big)|E \setminus B_r|.$$

In particular,

$$\mathcal{H}^{n-1}(E\cap B_r^c) \ge \tfrac{3c_\varphi n\omega_n^{1/n}}{8C_\varphi}|E\cap B_r^c|^{\frac{n-1}{n}} \quad \text{for a.e. } r>r_0.$$

If $|E\cap B_r^c|>0$ for all $r>0$, then integrating this differential inequality in (r_0,r) we get

$$\tfrac{3c_\varphi}{8C_\varphi}(r-r_0) \le |E\cap B_{r_0}^c|^{1/n} - |E\cap B_r^c|^{1/n}.$$

However, letting $r\to+\infty$ and using $|E|<+\infty$, we obtain $|E\cap B_{r_0}^c|=+\infty$, a contradiction. Thus, $|E\cap B_r^c|=0$ for some $r>r_0$, i.e., $E\subset B_r$. □

From this lemma, we deduce that each flat flow starting from a bounded set is bounded. However, when passing to the limit as $\tau_j\to0$, we may loose this boundedness, and therefore, we need this lemma in some stronger form and for this we need stronger assumptions on g. In what follows we write, for $\varrho>0$ and $x\in\mathbb{R}^n$,

$$W_\varrho(x)=\{\xi\in\mathbb{R}^n:\varphi(\xi-x)<\varrho\}, \qquad W_\varrho=W_\varrho(0).$$

Lemma 3.5 (**Growth of g^- controlled by f**). *Assume Hypothesis (H) and that g satisfies (1.11). Then there exist constants $C_6,C_7>0$ depending only on $c_{g^-},c_\varphi,C_\varphi$ and γ_g (see (A.3)) such that the following holds. For any $F\in\mathcal{S}^*$ and $\tau>0$ let F_τ be a minimizer of $\mathcal{F}(\cdot;F,\tau)$. Suppose that $F\subset W_{r_0}$ for some $r_0>C_6$. Then $F_\tau\subset W_{r_\tau}$ for any $\tau<\frac{1}{2C_7}$, where $r_\tau=(1+C_7\tau)r_0+C_7\tau$.*

Proof. By (2.1), (1.11) and the strict monotonicity of f

$$g^-(x)\le f\Big(c_{g^-}+\tfrac{c_{g^-}}{c_\varphi}\varphi(x)\Big),\quad \varphi(x)>\tfrac{c_{g^-}}{c_\varphi}. \tag{3.32}$$

Let $F\subset W_{r_0}$ for some $r_0>C_6:=c_{g^-}/c_\varphi$; by Lemma 3.4 a minimizer F_τ is bounded, and let W_{r_τ} be the smallest Wulff shape containing F_τ. Thus, $\partial F_\tau\cap\partial W_{r_\tau}\ne\emptyset$. We may assume $r_\tau>r_0$.

Fix small $\epsilon\in(0,r_\tau-r_0)$ and consider the difference

$$\begin{aligned}0\le \mathcal{F}(F_\tau\cap W_{r_\tau-\epsilon};F,\tau)-\mathcal{F}(F_\tau;F,\tau) = P_\varphi(F_\tau\cap W_{r_\tau-\epsilon})-P_\varphi(F_\tau)\\ -\int_{F_\tau\setminus W_{r_\tau}} f\left(\tfrac{\mathrm{d}_F}{,\tau}\right)dx-\int_{F_\tau\setminus W_{r_\tau-\epsilon}} g\,dx.\end{aligned} \tag{3.33}$$

By (2.1) $\mathrm{d}_F(x)\ge\frac{1}{C_\varphi}\mathrm{d}_F^\varphi(x)\ge\frac{r_\tau-\epsilon-r_0}{C_\varphi}$ in $F_\tau\setminus W_{r_\tau-\epsilon}$, where

$$\mathrm{d}_F^\varphi(x):=\inf\{\varphi(x-y):\ y\in\partial^*F\}$$

is the anisotropic distance. Thus, by the monotonicity of f,

$$-\int_{F_\tau\setminus W_{r_\tau}} f\left(\tfrac{\mathrm{d}_F}{\tau}\right)dx\le -f\Big(\frac{r_\tau-\epsilon-r_0}{C_\varphi\tau}\Big)|F_\tau\setminus W_{r_\tau-\epsilon}|.$$

In view of (3.32), using $r_0<r_\tau-\epsilon<\varphi(\cdot)\le r_\tau$ in $E\setminus W_{r_\tau-\epsilon}$ and assumption (3.32) we have

$$\begin{aligned}-\int_{F_\tau\setminus W_{r_\tau-\epsilon}} g dx\le \int_{F_\tau\setminus W_{r_\tau-\epsilon}} g^- dx\\ \le \int_{F_\tau\setminus W_{r_\tau-\epsilon}} f\Big(c_{g^-}+\tfrac{c_{g^-}}{c_\varphi}\varphi(x)\Big)dx\le f\Big(c_{g^-}+\tfrac{c_{g^-}}{c_\varphi}r_\tau\Big)|E\setminus W_{r_\tau-\epsilon}|.\end{aligned}$$

Finally, by the convexity of $W_{r_\tau-\epsilon}$,

$$P_\varphi(F_\tau\cap W_{r_\tau-\epsilon})-P_\varphi(F_\tau)\le 0.$$

Inserting these estimates in (3.33) we get

$$0 \le -f\Big(\frac{r_\tau - \epsilon - r_0}{C_\varphi \tau}\Big) + f\Big(c_{g^-} + \tfrac{c_{g^-}}{c_\varphi} r_\tau\Big),$$

thus, by the strict monotonicity of f and the arbitrariness of ϵ we get

$$\frac{r_\tau - r_0}{C_\varphi \tau} \le c_{g^-} + \frac{c_{g^-}}{c_\varphi} r_\tau.$$

Now assuming $\frac{c_{g^-} C_\varphi}{c_\varphi}\tau < \frac{1}{2}$ we can write

$$r_\tau \le \frac{r_0 + c_{g^-} C_\varphi \tau}{1 - \frac{c_{g^-} C_\varphi}{c_\varphi}\tau} \le \Big(1 + \frac{2c_{g^-} C_\varphi}{c_\varphi}\tau\Big) r_0 + c_{g^-} C_\varphi \tau \Big(1 + \frac{2c_{g^-} C_\varphi}{c_\varphi}\tau\Big) \le (1 + C_7\tau) r_0 + C_7 \tau,$$

where $C_7 := \max\{\frac{2c_{g^-} C_\varphi}{c_\varphi}, 2c_{g^-} C_\varphi\}$. □

Now we prove the local uniform boundedness of GMM, as stated in (ii). Fix $\tau \in (0, \frac{1}{2C_7})$ and a bounded $E_0 \in \mathcal{S}^*$, and let $\{E(\tau,k)\}$ be a flat flow starting from E_0. We may assume $E_0 \subset W_{r_0}$ for some $r_0 > C_6$. Let $\{r(\tau,k)\}_k$ be a nondecreasing sequence such that $r(\tau,0) = r_0$, $E(\tau,k) \subset W_{r(\tau,k)}$ and for any $k \ge 1$ either $r(\tau,k) = r(\tau,k-1)$, or by Lemma 3.5 $r(\tau,k) \le (1 + C_7\tau) r(\tau,k-1) + C_7\tau$. Thus, there is no loss of generality in assuming

$$r(\tau,k) \le (1 + C_7\tau) r(\tau,k-1) + C_7\tau, \quad k \ge 1.$$

Then[7]

$$r(\tau,k) \le (1 + C_7\tau)^k r_0 + C_7\tau\,\frac{(1 + C_7\tau)^k - 1}{C_7\tau} < (1 + C_7\tau)^k (r_0 + 1).$$

Thus, for any $t \ge 0$

$$r(\tau, \lfloor t/\tau \rfloor) \le (1 + C_7\tau)^{\lfloor t/\tau \rfloor}(r_0 + 1) \le (1 + C_7\tau)^{t/\tau}(r_0 + 1) \le e^{C_7 t}(r_0 + 1) =: R(t).$$

This implies $E(\tau, \lfloor t/\tau \rfloor) \subset W_{R(t)}$ for all $t \ge 0$ and therefore, for any $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, E_0)$ we have $E(t) \subset W_{R(t)}$.

4. Rescalings of GMM and comparison with balls

Let us study how minimizers are related in rescaling.

Lemma 4.1 (Rescaling). *Assume that f satisfies (Ha) and (Hb), and $g \equiv 0$. Suppose that $E_0 \in \mathcal{S}^*$ contains the origin in its interior and $\lambda > 0$. Then $\lambda E(\cdot) \in \mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, \lambda E_0)$ if and only if $E(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\varphi,\lambda f,0}, \mathcal{S}^*, E_0)$.*

Proof. As usual, $\mathcal{F} = \mathcal{F}_{\varphi,f,0}$. Since

$$\begin{aligned}\mathcal{F}(\lambda G; \lambda E_0, \tau) &= P_\varphi(\lambda G) + \int_{\lambda G} f\left(\tfrac{\mathrm{sd}_{\lambda E_0}}{\tau}\right)\,dx \\ &= \lambda^{n-1} P_\varphi(G) + \lambda^n \int_G f\left(\tfrac{\mathrm{sd}_{E_0}}{\tau}\right)\,dx = \lambda^{n-1}\mathcal{F}_{\varphi,\lambda f,0}(G; E_0, \tau), \quad (4.1)\end{aligned}$$

[7] Let $A \ge 1$, $B \ge 0$ and $\{\alpha_k\}_{k\ge 0} \subset \mathbb{R}_0^+$ be such that $\alpha_{k+1} \le A\alpha_k + B$ for any $k \ge 0$. Then $\alpha_m \le A^m\alpha_0 + B\,\frac{A^m - 1}{A - 1}$ for any $m \ge 1$. Indeed

$$\begin{aligned}\alpha_m &\le A\alpha_{m-1} + B \le A(A\alpha_{m-2} + B) + B = A^2\alpha_{m-2} + B(1 + A) \\ &\le \ldots \le A^m\alpha_0 + B(1 + A + \ldots + A^{m-1}) = A^m\alpha_0 + B\,\frac{A^m - 1}{A - 1}.\end{aligned}$$

λG is a minimizer of $\mathcal{F}(\cdot;\lambda E_0,\tau)$ if and only if G is a minimizer of $\mathcal{F}_{\varphi,\lambda f,0}(\cdot;E_0,\tau)$.

Let $\{\lambda E(\tau,k)\}$ be flat flows starting from λE_0, associated to $\mathcal{F}$. Then by (4.1), the family $\{E(\tau,k)\}$ is a flat flow starting from E_0, associated to $\mathcal{F}_{\varphi,\lambda f,0}$. This implies the thesis. □

Corollary 4.2 (**Power case**). *Assume $g\equiv 0$, that*

$$f(r)=r^\alpha,\quad r\ge 0,$$

for some $\alpha>0$, and $E_0\in\mathcal{S}^*$ contains the origin in its interior. Then $E(\cdot)\in \mathrm{GMM}(\mathcal{F},\mathcal{S}^*,E_0)$ if and only if $\lambda E(t\lambda^{-1/\alpha})\in \mathrm{GMM}(\mathcal{F},\mathcal{S}^*,\lambda E_0)$.

Proof. Since $\lambda f(r)=f(\lambda^{1/\alpha}r)$, the equality (4.1) becomes

$$\mathcal{F}(\lambda G;\lambda E_0,\tau)=\lambda^{n-1}P_\varphi(G)+\lambda^{n-1}\int_G f\left(\tfrac{\mathrm{sd}_{E_0}}{\tau\lambda^{-1/\alpha}}\right)\,dx=\mathcal{F}(G;E_0,\tau\lambda^{-1/\alpha}).$$

Thus λG is a minimizer of $\mathcal{F}(\cdot;\lambda E_0,\tau)$ if and only if G is a minimizer of $\mathcal{F}(\cdot;E_0,\tau\lambda^{-1/\alpha})$.

Now if $\tau_j\to 0$ and flat flows $\lambda E(\tau_j,k)$, starting from λE_0, associated to $\mathcal{F}_{\varphi,f,0}$ and with the time step τ, are such that

$$\lim_{j\to+\infty}\lambda E(\tau_j,\lfloor t/\tau_j\rfloor)=\lambda E(t)\quad\text{for all } t\ge 0 \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{4.2}$$

Then $E(\tau_j,k)$ are flat flows starting from E_0, associated to $\mathcal{F}_{\varphi,f,0}$, but with the time step equal to $\tau\lambda^{-1/\alpha}$. Thus, by (4.2)

$$\lim_{j\to+\infty}E(\tau_j,\left\lfloor t/(\lambda^{-1/\alpha}\tau_j)\right\rfloor)=\lim_{j\to+\infty}E(\tau_j,\left\lfloor t\lambda^{1/\alpha}/\tau_j\right\rfloor)=E(t\lambda^{1/\alpha})\quad\text{for all } t\ge 0\text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

The converse assertion is done in a similar manner. See also [8]. □

Theorem 4.3 (**Comparison with balls**). *Suppose Hypothesis (H) and $g\in L^\infty(\mathbb{R}^n)$. Given $E_0\in\mathcal{S}^*$ and $\tau>0$, let $\{E(\tau,k)\}$ be flat flows starting from E_0. Let $r_0>0$. Then there exist $\widehat{\tau}_1>0$ and $C_8>0$ depending only on n, φ, f, r_0 and $\|g\|_\infty$, such that*

$$W_{r_0}(x_0)\subset E_0\quad\Longrightarrow\quad W_{r_0-C_8k\tau}(x_0)\subset E(\tau,k) \tag{4.3}$$

and

$$W_{r_0}(x_0)\cap E_0=\emptyset\quad\Longrightarrow\quad W_{r_0-C_8k\tau}(x_0)\cap E(\tau,k)=\emptyset \tag{4.4}$$

for all $\tau\in(0,\widehat{\tau}_1)$ and $0\le k\tau\le \frac{r_0}{2C_8}$.

Proof. Let τ_0 be given by (3.9), where we recall that γ_g is given in (A.3). We prove only (4.3), the relation (4.4) being similar. For shortness, we assume $x_0=0$ and let $W_{r_0}\subset E_0$.

Step 1. Since $\rho_\tau\to 0$ as $\tau\to 0^+$, there exists $\tau_1\in(0,\tau_0)$ (depending only on r_0) such that such that $\rho_{\tau_1}<r_0/10$. For $\tau\in(0,\tau_1)$ let E_τ be a minimizer of $\mathcal{F}(\cdot;E_0,\tau)$. By the choice of τ_1 and the L^∞-bound (3.10), we have $W_{4r_0/5}\subset E_\tau$. Let

$$\bar r:=\sup\{r>0:\ W_r\subset E_\tau\}\ge 4r_0/5.$$

We want to estimate $\bar r$ from below (see (4.9)), thus, there is no loss of generality in assuming $\bar r<r_0$. Fix $\epsilon\in(0,r_0-\bar r)$ and consider the difference

$$\begin{aligned}0&\le \mathcal{F}(E_\tau\cup W_{\bar r+\epsilon};E_0,\tau)-\mathcal{F}(E_\tau;E_0,\tau)\\&=P_\varphi(E_\tau\cup W_{\bar r+\epsilon})-P_\varphi(E_\tau)+\int_{W_{\bar r+\epsilon}\setminus E_\tau}f\left(\tfrac{\mathrm{sd}_{E_0}}{\tau}\right)\,dx+\int_{W_{\bar r+\epsilon}\setminus E_\tau}g\,dx.\end{aligned} \tag{4.5}$$

Using $W_{\bar r+\epsilon}\subset W_{r_0}\subset E_0$ we find $-\mathrm{sd}_{E_0}=\mathrm{d}_{E_0}\ge \frac{1}{C_\varphi}\mathrm{d}^\varphi_{E_0}\ge \frac{r_0-\bar r-\epsilon}{C_\varphi}$ in $W_{\bar r+\epsilon}\setminus E_\tau$, where d^φ_F stands for the φ-distance from the set F. Therefore, by the strict monotonicity and oddness of f,

$$-\int_{W_{\bar r+\epsilon}\setminus E_\tau} f\left(\tfrac{\mathrm{sd}_{E_0}}{\tau}\right)\,dx\ge f\Big(\tfrac{r_0-\bar r-\epsilon}{C_\varphi\tau}\Big)|W_{\bar r+\epsilon}\setminus E_\tau|. \tag{4.6}$$

Moreover, by the boundedness of g,

$$\int_{W_{\bar r+\epsilon}\setminus E_\tau} g\,dx\le \|g\|_\infty |W_{\bar r+\epsilon}\setminus E_\tau|. \tag{4.7}$$

Finally, using the anisotropic isoperimetric inequality (2.2) for a.e. $\epsilon>0$ we have

$$\begin{aligned} P_\varphi(E_\tau\cup W_{\bar r+\epsilon})-P_\varphi(E_\tau)=&P_\varphi(W_{\bar r+\epsilon})-P_\varphi(E_\tau\cap W_{\bar r+\epsilon})\\ \le& c_{n,\varphi}\Big(|W_{\bar r+\epsilon}|^{\frac{n-1}{n}}-|W_{\bar r+\epsilon}\cap E_\tau|^{\frac{n-1}{n}}\Big)\\ =&c_{n,\varphi}|W_{\bar r+\epsilon}|^{\frac{n-1}{n}}\Big(1-\Big|1-\frac{|W_{\bar r+\epsilon}\setminus E_\tau|}{|W_{\bar r+\epsilon}|}\Big|^{\frac{n-1}{n}}\Big)\\ \le&\frac{c_{n,\varphi}}{|B_\varphi|^{1/n}(\bar r+\epsilon)}|W_{\bar r+\epsilon}\setminus E_\tau|, \end{aligned} \tag{4.8}$$

where in the last inequality we used $(1-x)^\alpha\ge 1-x$ for any $x,\alpha\in(0,1)$. Inserting (4.6), (4.7), (4.8) in (4.5) and using the arbitrariness of ϵ we get

$$f\Big(\frac{r_0-\bar r}{C_\varphi\tau}\Big)\le \frac{c_{n,\varphi}}{|B_\varphi|^{1/n}\,\bar r}+\|g\|_\infty.$$

Thus, recalling $\bar r\ge 4r_0/5$ we get $\frac{r_0-\bar r}{C_\varphi\tau}\le f^{-1}\Big(\frac{5c_{n,\varphi}}{4|B_\varphi|^{1/n}\,r_0}+\|g\|_\infty\Big)$, or equivalently

$$\bar r\ge r_0-C_\varphi\tau f^{-1}\Big(\frac{5c_{n,\varphi}}{4|B_\varphi|^{1/n}\,r_0}+\|g\|_\infty\Big). \tag{4.9}$$

Notice that this inequality holds also in case $\bar r\ge r_0$.

Step 2. Let $\{E(\tau,k)\}$ be a flat flow starting from E_0 and let $\{F(\tau,k)_*\}$ be a flat flow starting from $F_0:=W_{r_0}$ and consisting of the minimal minimizers. By comparison (Theorem B.2 (c)), $F(\tau,k)_*\subset E(\tau,k)$ for all $k\ge 0$.

Consider the sequence $r_0=r(\tau,0)\ge r(\tau,1)\ge\ldots$ of radii defined as follows: we assume $W_{r(\tau,k)}\subset F(\tau,k)_*$ and if $r(\tau,k-1)>r(\tau,k)$, then $W_{r(\tau,k)}$ is the largest Wulff shape contained in $F(\tau,k)_*$. Let $k_0\ge 1$ be such that $r(\tau,k_0)\ge r_0/2$ and let $\tau_1>\tau_2>\ldots>\tau_{k_0}>0$ be given by step 1 applied with $r_0:=r(\tau,k)$ for $k=1,\ldots,k_0$. Thus, for any $\tau\in(0,\tau_k)$ one has $r(\tau,k)>4r(\tau,k-1)/5$. From step 1

$$r(\tau,k)\ge r(\tau,k-1)-C_\varphi\tau f^{-1}\Big(\frac{5c_{n,\varphi}}{4|B_\varphi|^{1/n}r(\tau,k-1)}+\|g\|_\infty\Big),\quad 1\le k\le k_0.$$

Now by the choice of k_0,

$$r(\tau,k)\ge r(\tau,k-1)-C_\varphi\tau f^{-1}\Big(\frac{5c_{n,\varphi}}{2|B_\varphi|^{1/n}r_0}+\|g\|_\infty\Big),\quad 1\le k\le k_0,$$

and hence

$$r(\tau,k)\ge r_0-C_\varphi f^{-1}\Big(\frac{5c_{n,\varphi}}{2|B_\varphi|^{1/n}r_0}+\|g\|_\infty\Big)k\tau,\quad 0\le k\le k_0.$$

Now if we assume

$$C_8 := C_8(n,\varphi,f,r_0,\|g\|_\infty) := C_\varphi f^{-1}\Big(\frac{5c_{n,\varphi}}{2|B_\varphi|^{1/n}r_0} + \|g\|_\infty\Big)$$

and $k\tau \le \frac{r_0}{2C_8}$, we get $r(\tau,k)\ge r_0/2$ and $r(\tau,k)\ge r_0 - C_8 k\tau$. □

It turns out that GMMs for generalized power mean curvature flow share several properties with GMMs in the anisotropic mean curvature flow, as we now see. Let us study GMM starting from a Euclidean ball centered at origin.

Theorem 4.4 (**Evolution of balls**). *Assume that φ is Euclidean and $g\equiv 0$. Suppose also that $r\mapsto f^{-1}((n-1)/r)$ is such that the ordinary differential equation*

$$\begin{cases} r'(t) = -f^{-1}\Big(\frac{n-1}{r(t)}\Big) & \text{if } r(t)>0,\\ r(0)=r_0 \end{cases} \tag{4.10}$$

admits a unique C^1 solution (for instance, $r\mapsto f^{-1}((n-1)/r)$ is convex in $(0,+\infty)$). Then $\mathrm{GMM}(\mathcal F,\mathcal S^*,B_{r_0})$ *is a singleton $\{B_{r(t)}\}_{t\ge0}$, where $r(\cdot)$ is a nonincreasing function satisfying* (4.10).

Proof. We can write

$$\mathcal F(E;E_0,\tau) := P(E) + \int_E f\left(\tfrac{\mathrm{sd}_{E_0}}{\tau}\right)\,dx + c_{E_0}.$$

Step 1: properties of minimizers. Fix $r_0>0$. Let τ_0 and ρ_τ be as in (3.9), (3.10) so that

$$\sup_{E_\tau\Delta B_{r_0}} \mathrm{d}_{B_{r_0}} \le 2\rho_\tau,\quad \tau\in(0,\tau_0),$$

for any minimizer E_τ of $\mathcal F(\cdot;B_{r_0},\tau)$. Since $\rho_\tau\to0^+$ as $\tau\to0^+$, there exists $\tau_{r_0}\in(0,\tau_0)$ such that $B_{r_0/2}\subset E_\tau\subset B_{3r_0/2}$ for all $\tau\in(0,\tau_{r_0})$.

Let us study the minimal and maximal minimizers. Owing to

$$f\left(\tfrac{\mathrm{sd}_{B_{r_0}}}{\tau}\right) = f\Big(\tfrac{|x|-r_0}{\tau}\Big),\quad x\in\mathbb R^n,$$

the volume term of $\mathcal F$ is radially symmetric. Therefore, the minimal and maximal minimizers of $\mathcal F(\cdot;B_{r_0},\tau)$ are radially symmetri c, i.e., both of them are balls. By the choice of τ_{r_0}, their radii are in the interval $(r_0/2,3r_0/2)$. In particular, the radii of minimal and maximal minimizers satisfy (1.6). By the assumptions on f, $r\mapsto f^{-1}\Big(\frac{n-1}{r}\Big)$ strictly decreases, convex and positive in $(0,+\infty)$. Therefore, using the linearity of $r\mapsto\frac{r-r_0}{\tau}$, we find that for any $\tau\in(0,\tau_{r_0})$ there exists a unique minimum point r_τ of f in the interval $(r_0/2,3r_0/2)$. Clearly, $r_\tau\in(r_0/2,r_0)$. In particular, the minimal and maximal minimizers coincide, i.e., $\mathcal F(\cdot;B_{r_0},\tau)$ has a unique minimizer B_{r_τ}.

Without loss of generality, we may assume $r_0\mapsto\tau_{r_0}$ is increasing.

Step 2: some properties of flat flows. Given $r_0>0$ and $\epsilon\in(0,1/4)$, let $\bar\tau_\epsilon := \tau_{\epsilon r_0/2}>0$ be given by step 1. For any $\tau\in(0,\bar\tau_\epsilon)$ let $r(\tau,k)$, $k\ge0$, be defined inductively as follows: $r(\tau,0)=r_0$, and for $k\ge1$, if $r(\tau,k)>\epsilon r_0$, the ball $B_{r(\tau,k)}$ is the unique minimizer of $\mathcal F(\cdot;B_{r(\tau,k-1)},\tau)$. In view of (1.6) these radii satisfy the equation

$$\frac{r(\tau,k)-r(\tau,k-1)}{\tau} = -f^{-1}\Big(\frac{n-1}{r(\tau,k)}\Big). \tag{4.11}$$

Thus, the sequence $k \mapsto r(\tau,k)$ strictly decreases. By (4.11) there exists a unique $\bar k_{\tau,\epsilon} > 0$ such that $r(\tau,\bar k_{\tau,\epsilon}+1) \le \epsilon r_0 < r(\tau,\bar k_{\tau,\epsilon})$. One can readily check that both maps $\tau \mapsto k_{\tau,\epsilon}$ and $\epsilon \mapsto k_{\tau,\epsilon}$ are decreasing. Let

$$T_\epsilon := \liminf_{\tau\to 0^+} \tau k_{\tau,\epsilon}.$$

By the monotonicity of $k_{\tau,\epsilon}$, $\epsilon \mapsto T_\epsilon$ is nondecreasing. Let us show that it is uniformly away from 0. Indeed, since $B_{r(\tau,k)}$, $0 \le k \le k_{\tau,\epsilon}$ are flat flows, applying (3.22) with $m_1 = 1$ and $m_2 = k_{\tau,\epsilon}$ we get

$$|B_{r(\tau,1)} \setminus B_{r(\tau,k_{\epsilon,\tau}+1)}| \le C_4 p^\sigma (\tau k_{\tau,\epsilon} + \tau - \tau) + \frac{1}{f(p)}$$

for some $p>0$ and suitable $\sigma \in \{1,n\}$. Now letting $\tau \to 0^+$ and recalling $B_{r(\tau,1)} \to B_{r_0}$, we get

$$|B_{r_0} \setminus B_{\epsilon r_0}| \le C_4 p^\sigma T_\epsilon \frac{1}{f(p)}.$$

Now taking p large enough (depending only on r_0) we deduce $T_\epsilon > C_{11}$ for some C_{11} depending only on C_4 and r_0. Let us denote

$$T_0 := \sup_{\epsilon\in(0,1/4)} T_\epsilon.$$

By the definition, $T_\epsilon \le T_0$.

Step 3: properties of GMM. Let $\tau_j \to 0^+$ be any sequence such that

$$B_{r(\tau_j,\lfloor t/\tau_j\rfloor)} \to B_{r(t)} \quad \text{in } L^1(\mathbb{R}^n) \text{ as } j\to+\infty \quad \forall t\ge 0,$$

where we assume $r(\tau,k) := 0$ if it does not satisfy (1.6). Equivalently,

$$\lim_{j\to+\infty} r(\tau_j,\lfloor t/\tau_j\rfloor) = r(t), \quad t \ge 0. \tag{4.12}$$

Fix any $\bar T < T_0$ and let $\bar\epsilon > 0$ be such that $T_{\bar\epsilon} \in (\bar T, T_0)$. Then for any $\tau_j < t < \bar T$ so that $1 \le k := \lfloor t/\tau_j \rfloor < \lfloor T_{\bar\epsilon}/\tau_j\rfloor$ and $r(\tau_j,\lfloor t/\tau_j\rfloor) \ge \bar\epsilon r_0 > 0$, we can apply (4.11):

$$\frac{r(\tau_j,\lfloor t/\tau_j\rfloor) - r(\tau_j,\lfloor (t-\tau_j)/\tau_j\rfloor)}{\tau_j} = -f^{-1}\Big(\frac{n-1}{r(\tau_j,\lfloor t/\tau_j\rfloor)}\Big).$$

Passing to the limit in this difference equation and using (4.12) we obtain

$$r'(t) = -f^{-1}\Big(\frac{n-1}{r(t)}\Big), \quad t\in(0,\bar T), \tag{4.13}$$

which admits a unique solution. By the definition of T_ϵ we can show that $\lim_{\bar T\nearrow T_0} r(\bar T) = 0$. Since $\bar T < T_0$ is arbitrary, the radii of balls in each GMM necessarily satisfy (4.13). Now by uniqueness, $\mathrm{GMM}(\mathcal F,\mathcal S^*,B_{r_0})$ is a singleton $\{B_{r(t)}\}_{t\ge0}$, where we extend $r(t) := 0$ for $t \ge T_0$. □

Remark 4.5. Let $f(r) = r^\alpha$ in $(0,+\infty)$ for $\alpha > 0$. Then the ODE in (4.10) reads as

$$r' = -\frac{(n-1)^{1/\alpha}}{r^{1/\alpha}}.$$

Thus, $\mathrm{GMM}(\mathcal F,\mathcal S^*,B_{r_0}) = \{B_{r(t)}\}$, where

$$r(t) = \begin{cases} \Big(r_0^{1+\frac1\alpha} - \big(1+\frac1\alpha\big)(n-1)^{\frac1\alpha} t\Big)^{\frac{\alpha}{1+\alpha}} & \text{if } t\in\Big[0, \frac{r_0^{1+1/\alpha}}{\big(1+1/\alpha\big)(n-1)^{1/\alpha}}\Big),\\ 0 & \text{otherwise.}\end{cases}$$

5. Evolution of mean convex sets: proof of Theorem 1.5

In this section we mostly follow the ideas of [13]. Apart from some technical points due to the presence of the function f, one difference here is in the proof of the δ-convexity preservation of minimizers (Proposition 5.3): we apply directly the prescribed mean curvature functional in place of 1-harmonic functions and Anzellotti-type arguments as done in [13]. We assume here that $g \equiv 0$, $\tau > 0$ and $k \ge 0$.

Given an anisotropy φ in $\mathbb{R}^n$, $\delta \ge 0$ and a (nonempty) open set $\Omega \subset \mathbb{R}^n$, a set $E \Subset \Omega$ is called *δ-mean convex* (in Ω) if

$$P_\varphi(E) - \delta|E| \le P_\varphi(F) - \delta|F| \quad \text{for any } F \Subset \Omega \text{ with } E \subset F.$$

When $\delta = 0$, we simply say E is mean convex. Repeating the density estimate arguments (Section 3.2) we can readily show the existence of $r_0, \vartheta > 0$ depending only on n, c_φ, C_φ and f such that every δ-mean convex set E satisfies

$$\frac{|B_r(x) \setminus E|}{|B_r(x)|} \ge \vartheta > 0$$

for all $r \in (0, r_0)$ and for all $x \in \partial E$. In particular, $E = E^{(1)}$ is open. Let us recall some more properties of δ-mean convex sets.

Proposition 5.1 ([13, 18])**.** *Let $\delta \ge 0$.*

(a) *Suppose there exists a δ-mean convex set $E \Subset \Omega$ in Ω. Then $\emptyset$ is δ-mean convex in Ω.*
(b) *$E \Subset \Omega$ is δ-mean convex in Ω if and only if*

$$P_\varphi(E) \le P_\varphi(E \cup F) - \delta|F \setminus E|, \quad F \Subset \Omega.$$

(c) *$E \Subset \Omega$ is δ-mean convex in Ω if and only if*

$$P_\varphi(E \cap F) \le P_\varphi(F) - \delta|F \setminus E|, \quad F \Subset \Omega.$$

(d) *Let $E_h \Subset \Omega$, $h = 1, 2, \ldots$, be δ-mean convex in Ω with $E_h \to E$ in $L^1(\Omega)$ for some $E \Subset \Omega$. Then E is δ-mean convex.*

Since we are mostly interested in bounded sets, in view of (1.12) we can write, for a constant c_F independent of E,

$$\mathcal{F}(E; F, \tau) := P_\varphi(E) + \int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\, dx + c_F$$

if F is bounded, see (1.13).

Lemma 5.2 (**Inclusion of minimizers**)**.** *Let $F \Subset \Omega$ be an open δ-mean convex set in Ω for some $\delta \ge 0$. Suppose that f satisfies (Ha), (Hb), and that $g \equiv 0$. Let E be a minimizer of $\mathcal{F}(\cdot; F, \tau)$. Then:*

(i) *$\overline{E} \subset \overline{F}$; if $\delta > 0$, then $E \subset F$,*
(ii) *if $\delta > 0$,*

$$\bigcup_{|\eta| < f^{-1}(\delta)\tau} (E + \eta) \subset \overline{F} \quad \text{and} \quad \mathrm{sd}_E \ge \mathrm{sd}_F + f^{-1}(\delta)\tau \ \text{ in } \mathbb{R}^n \tag{5.1}$$

provided $\mathrm{dist}(F + \delta\tau, \partial\Omega) > 0$,
(iii) *E is mean convex.*

Proof. As we have seen in the proof of Theorem 1.4, under the assumptions of the lemma every minimizer E of $\mathcal{F}(\cdot; F, \tau)$ satisfies $E \subset \Omega$.

(i) By the δ-mean convexity of F in Ω

$$P_\varphi(F) \le P_\varphi(F \cup E) - \delta|E \setminus F|, \tag{5.2}$$

and by the minimality of E

$$P_\varphi(E) + \int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\, dx \le P_\varphi(E \cap F) + \int_{E\cap F} f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\, dx.$$

Summing these inequalities we get

$$\int_{E\setminus F} f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\, dx + \delta|E \setminus F| \le P_\varphi(E \cup F) + P_\varphi(E \cap F) - P_\varphi(E) - P_\varphi(F) \le 0, \tag{5.3}$$

where in the last inequality we used the submodularity of P_φ. Since $\mathrm{sd}_F > 0$ in $\mathbb{R}^n \setminus \overline{F}$ and $f > 0$ in $\mathbb{R}^+$, from (5.3) we deduce $|E \setminus \overline{F}| = 0$, i.e., $E \subset \overline{F}$, hence, $\overline{E} \subset \overline{F}$. Note that if $\delta > 0$, then $E \subset F$.

(ii) Assume $\delta > 0$ and $\eta \in B_{f^{-1}(\delta)\tau}$. Clearly, $F + \eta$ is δ-mean convex in $\Omega + \eta$ and $E + \eta$ is a minimizer of $\mathcal{F}(\cdot; F + \eta, \tau)$. By the choice of τ, $E + \eta \Subset \Omega$ and hence we can apply (5.2) with $E := E + \eta$. Moreover,

$$P_\varphi(E + \eta) + \int_{E+\eta} f\left(\tfrac{\mathrm{sd}_{F+\eta}}{\tau}\right)\, dx \le P_\varphi([E + \eta] \cap F) + \int_{[E+\eta]\cap F} f\left(\tfrac{\mathrm{sd}_{F+\eta}}{\tau}\right)\, dx.$$

Summing this and (5.2) (applied with $E = E + \eta$) we deduce

$$\int_{[E+\eta]\setminus F} \left(f\left(\tfrac{\mathrm{sd}_{F+\eta}}{\tau}\right) + \delta\right) dx \le P_\varphi([E + \eta] \cup F) + P_\varphi([E + \eta] \cap F) - P_\varphi(F) - P_\varphi(E + \eta) \le 0.$$

Thus,

$$\int_{[E+\eta]\setminus F} \left(f\left(\tfrac{\mathrm{sd}_{F+\eta}}{\tau}\right) + \delta\right) dx = \int_{[E+\eta]\setminus F} \left(f\left(\tfrac{\mathrm{sd}_{F}}{\tau}\right) + \delta\right) dx \le 0. \tag{5.4}$$

Note that if $x \in [E + \tau] \setminus F$, then by (a) $x - \eta \in E \subset F$, and hence

$$\mathrm{sd}_F(x - \eta) = -\mathrm{d}_F(x - \eta) \ge -|x - (x - \eta)| = -|\eta| > -f^{-1}(\delta)\tau.$$

Then by the strict monotonicity of f,

$$f\left(\tfrac{\mathrm{sd}_F}{\tau}\right) + \delta > 0 \quad \text{on } [E + \tau] \setminus F$$

and therefore, by (5.4) $E + \eta \subset \overline{F}$.

Let us prove $\mathrm{sd}_E \ge \mathrm{sd}_F + f^{-1}(\delta)\tau$. We know that $E \subset F$ and

$$\inf_{x\in\partial F,\, y\in\partial E} |x - y| \ge f^{-1}(\delta)\tau. \tag{5.5}$$

Take $x \in F^c$ and let $y \in \partial E$ be such that $\mathrm{d}_E(x) = |x - y|$. Then there exists $z \in [x, y] \cap \partial F$, and by the definition of d_F and (5.5)

$$\mathrm{sd}_E(x) = \mathrm{d}_E(x) = |x - y| = |x - z| + |z - y| \ge \mathrm{d}_F(x) + f^{-1}(\delta)\tau = \mathrm{sd}_F(x) + f^{-1}(\delta)\tau.$$

Now assume that $x \in F \setminus E$ and let $y \in \partial F$ and $z \in \partial E$ such that $\mathrm{d}_F(x) = |x - y|$ and $\mathrm{d}_E(x) = |x - z|$. Then by (5.5)

$$\mathrm{d}_E(x) + \mathrm{d}_F(x) = |x - y| + |x - z| \ge |y - z| \ge f^{-1}(\delta)\tau,$$

and therefore,

$$\mathrm{sd}_E(x) = \mathrm{d}_E(x) \ge -\mathrm{d}_F(x) + f^{-1}(\delta)\tau = \mathrm{sd}_F(x) + f^{-1}(\delta)\tau.$$

Finally, assume that $x \in E$ and let $y \in \partial F$ be such that $\mathrm{d}_F(x) = |y-x|$. Then there exists $z \in [x,y] \cap \partial E$, and hence, as above

$$-\mathrm{sd}_F(x) = \mathrm{d}_F(x) = |x-z| + |z-y| \ge \mathrm{d}_E(x) + f^{-1}(\delta)\tau = -\mathrm{sd}_E(x) + f^{-1}(\delta)\tau.$$

(iii) We claim that $P_\varphi(E) \le P_\varphi(G)$ for any $G \Subset \Omega$ with $E \subset G$. Indeed, by the δ-mean convexity of F and Proposition 5.1 (c)

$$P_\varphi(F \cap G) \le P_\varphi(G) - \delta|G \setminus F|. \tag{5.6}$$

Moreover, by the minimality of E,

$$P_\varphi(E) + \int_E f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx \le P_\varphi(F\cap G) + \int_{F\cap G} f\left(\tfrac{\mathrm{sd}_F}{\tau}\right)\,dx.$$

Since $E \subset F \cap G$ and f is odd, this inequality becomes

$$P_\varphi(E) + \int_{[F\cap G]\setminus E} f\Big(-\tfrac{\mathrm{sd}_F}{\tau}\Big)\,dx \le P_\varphi(F \cap G).$$

The integral in this inequality is nonnegative. Therefore, by (5.6)

$$P_\varphi(E) \le P_\varphi(F\cap G) \le P_\varphi(G) - \delta|G\setminus F| \le P_\varphi(G).$$

□

The following proposition improves the last assertion of Lemma 5.2.

Proposition 5.3 (**Mean convexity of minimal and maximal minimizers**). *Suppose that f satisfies (Ha), (Hb), and that $g \equiv 0$. Let $E_0 \Subset \Omega$ be an open δ-mean convex set in Ω for some $\delta > 0$. Then the minimal and maximal minimizers of $\mathcal{F}(\cdot; E_0, \tau)$ (in the sense of Corollary B.3) are δ-mean convex in Ω.*

Proof. For any $s > 0$ let E_s be a minimizer of $\mathcal{F}(\cdot; E_0, s)$.

Step 1: For any $0 < s' < s''$

$$E_{s''} \subset E_{s'} \subset E_0.$$

The inclusion $E_{s''}, E_{s'} \subset \overline{E_0}$ follows from Lemma 5.2 (i). To prove the first inclusion it is enough to observe that $s \mapsto \frac{\mathrm{sd}_{E_0}}{s}$ is strictly decreasing in E_0. Now the inclusion follows from the comparison principle in Corollary B.4 for the prescribed mean curvature functional.

Step 2:

$$\lim_{s\searrow 0} |E_0 \setminus E_s| = 0.$$

Note that this assertion was already shown in (3.28) under the extra assumption $|\partial E_0| = 0$. Here we do not have such a regularity. By minimality

$$P_\varphi(E_s) + \int_{E_s} f\left(\tfrac{\mathrm{sd}_{E_0}}{s}\right)\,dx \le P_\varphi(E_0) + \int_{E_0} f\left(\tfrac{\mathrm{sd}_{E_0}}{s}\right)\,dx,$$

and using $E_s \subset E_0$,

$$P_\varphi(E_s) + \int_{E_0\setminus E_s} f\left(\tfrac{\mathrm{d}_{E_0}}{s}\right)\,dx \le P_\varphi(E_0).$$

This, the monotonicity of $s \mapsto E_s$ and the openness of E_s and E_0 imply $E_s \overset{L^1}{\to} E_0$ and $P_\varphi(E_s) \to P_\varphi(E_0)$ as $s \searrow 0$.

Step 3.

$$\lim_{s\nearrow\tau} |E_s \setminus E_\tau^*| = 0 \quad \text{and} \quad \lim_{s\searrow\tau} |E_{\tau *} \setminus E_s| = 0.$$

We start by proving the first equality. Consider any sequence $s_i \nearrow \tau$. Since $P_\phi(E_{s_i}) \le P_\phi(E_0)$ and $E_{s_i} \subset E_0$ for any $i \ge 1$, there exists $Q \subset E_0$ such that, up to a further not relabelled subsequence, $E_{s_i} \searrow Q$ in $L^1(\mathbb{R}^n)$. By the L^1-lower semicontinuity of $\mathcal{F}(\cdot; E_0, \tau)$, Q is its minimizer. Moreover, since $E_{s_i} \supset E_{s_{i+1}}$, we have also $E_{s_i} \supset Q$ for any $i \ge 1$. By step 1, $E_{s_i} \supset E_\tau^*$, and hence, we cannot have $|E_\tau^* \setminus Q| > 0$. Now arbitrariness of s_i implies $E_s \searrow E_\tau^*$ as $s \nearrow \tau$.

The proof of the second equality is similar.

Step 4. Now we prove the δ-mean convexity of the minimal and maximal minimizers.

Fix any $G \in \mathcal{S}^*$ with $G \Subset \Omega$, $\epsilon \in (0, \delta)$ and $0 < \underline{s} < \overline{s}$. For any $N > 1$ let

$$s_i := \underline{s} + \frac{(\overline{s} - \underline{s})i}{N}, \quad i = 0, \dots, N.$$

Possibly slightly perturbing G we may assume that

$$\sum_{i=0}^{N} \mathcal{H}^{n-1}(\partial^* G \cap \partial^* E_{s_i}) = 0, \quad G_i := G \cap [E_{s_{i-1}} \setminus E_{s_i}], \quad i = 1, \dots, N.$$

Since E_0 is bounded and f is continuous, there exists $N > 1$ such that

$$f\Big(\tfrac{\mathrm{d}_{E_0}(x)}{s_i}\Big) \ge f\Big(\tfrac{\mathrm{d}_{E_0}(x)}{s_{i-1}}\Big) - \epsilon, \quad x \in E_0, \quad i = 1, \dots, N. \tag{5.7}$$

Thus, by the minimality of E_{s_i} and (5.7)

$$\begin{aligned} P_\varphi(E_{s_{i-1}} \cap G) - P_\varphi(E_{s_i} \cap G) &= P_\varphi(E_{s_i} \cup G_i) - P_\varphi(E_{s_i}) \\ &\ge \int_{G_i} f\left(\tfrac{\mathrm{d}_{E_0}}{s_i}\right)\, dx \ge \int_{G_i} f\left(\tfrac{\mathrm{d}_{E_0}}{s_{i-1}}\right)\, dx - \epsilon|G_i|. \end{aligned}$$

By (5.1)

$$f\Big(\tfrac{\mathrm{d}_{E_0}(x)}{s_{i-1}}\Big) = f\Big(\tfrac{-\mathrm{sd}_{E_0}(x)}{s_{i-1}}\Big) \ge f\Big(\tfrac{-\mathrm{sd}_{E_{s_{i-1}}}(x) + f^{-1}(\delta) s_{i-1}}{s_{i-1}}\Big) \ge \delta \quad \text{for } x \in E_{s_{i-1}},$$

and therefore,

$$P_\varphi(E_{s_{i-1}} \cap G) - P_\varphi(E_{s_i} \cap G) \ge (\delta - \epsilon)|G_i|, \quad i = 1, \dots, N.$$

Summing these inequalities we get

$$P_\varphi(E_{\underline{s}} \cap G) - P_\varphi(E_{\overline{s}} \cap G) \ge (\delta - \epsilon)|G \cap [E_{\underline{s}} \setminus E_{\overline{s}}]|. \tag{5.8}$$

Moreover, since $E_{\underline{s}} \subset E_0$ is mean convex and E_0 is δ-mean convex, applying Proposition 5.1 (c) twice (first with $E_{\underline{s}}$ and $\delta = 0$ and then with E_0 and δ) we obtain

$$P_\varphi(E_{\underline{s}} \cap G) = P_\varphi(E_{\underline{s}} \cap [E_0 \cap G]) \le P_\varphi(E_0 \cap G) \le P_\varphi(G) - \delta|G \setminus E_0|.$$

Inserting this in (5.8)

$$P_\varphi(G) - \delta|G \setminus E_0| \ge P_\varphi(E_{\overline{s}} \cap G) + (\delta - \epsilon)|G \cap [E_{\underline{s}} \setminus E_{\overline{s}}]|,$$

and hence, letting $\epsilon, \underline{s} \to 0^+$ and using step 2 we get

$$P_\varphi(G) \ge P_\varphi(E_{\overline{s}} \cap G) + \delta|G \setminus E_{\overline{s}}|.$$

Finally, letting $\overline{s} \searrow \tau$ and $\overline{s} \nearrow \tau$, using step 3 and the L^1-lower semicontinuity of P_φ, we get

$$P_\varphi(G) \ge P_\varphi(E_{\tau*} \cap G) + \delta|G \setminus E_{\tau*}| \quad \text{and} \quad P_\varphi(G) \ge P_\varphi(E_\tau^* \cap G) + \delta|G \setminus E_\tau^*|.$$

Thus, both E_τ^* and $E_{\tau*}$ are δ-mean convex by Proposition 5.1. □

Corollary 5.4 (**Mean convexity of minimizers**). *Let $E_0 \Subset \Omega$ be δ-mean convex and $\tau > 0$. Then every minimizer E_τ of $\mathcal{F}(\cdot; E_0, \tau)$ is δ-mean convex.*

Proof. If $\delta = 0$, the assertion follows from Lemma 5.3 (c), so we assume $\delta > 0$ and consider the minimal and maximal minimizers $E_{\tau*} \subset E_\tau \subset E_\tau^*$. Then, for any $G \Subset \Omega$, by the δ-mean convexity of E_τ^* (Proposition 5.3),

$$P_\varphi(G) \ge P_\varphi(E_\tau^* \cap G) + \delta|G \setminus E_\tau^*|. \tag{5.9}$$

Moreover, possibly slightly perturbing G we assume that $\mathcal{H}^{n-1}(\partial^* G \cap \partial^* E_\tau) = 0$ so that by the minimality of $E_\tau \subset E_\tau^*$

$$P_\varphi(E_\tau^* \cap G) - P_\varphi(E_\tau \cap G) = P_\varphi(E_\tau \cup [G \cap (E_\tau^* \setminus E_\tau)]) - P_\varphi(E_\tau) \ge \int_{G\cap(E_\tau^*\setminus E_\tau)} f\left(\tfrac{\mathrm{d}_{E_0}}{\tau}\right)\, dx.$$

Moreover, by (5.1)

$$f\left(\tfrac{\mathrm{d}_{E_0}}{\tau}\right) = f\left(\tfrac{-\mathrm{sd}_{E_0}(x)}{\tau}\right) \ge f\left(\tfrac{-\mathrm{sd}_{E_\tau^*}(x) + f^{-1}(\delta)\tau}{\tau}\right) \ge \delta \quad \text{for } x \in E_\tau^*,$$

and hence,

$$P_\varphi(E_\tau^* \cap G) \ge P_\varphi(E_\tau \cap G) + \delta|G \cap (E_\tau^* \setminus E_\tau)|.$$

Adding this to (5.9) we get

$$P_\varphi(G) \ge P_\varphi(E_\tau \cap G) + \delta|G \setminus E_\tau|,$$

i.e., E_τ is δ-mean convex. □

Proof of Theorem 1.5. Let $\{E(\tau_j, k)\}$ be a family of flat flows starting from E_0 and satisfying

$$\lim_{j\to+\infty} |E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta E(t)| = 0 \quad \text{for any } t \ge 0 \tag{5.10}$$

for some $E(\cdot) \in \mathrm{GMM}(\mathcal{F}, E_0)$. By Lemma 5.2 (a)

$$\overline{E_0} \supset \overline{E(\tau_j, 1)} \supset \overline{E(\tau_j, 2)} \supset \ldots$$

and by Corollary 5.4 each $E(\tau_j, k)$ is δ-mean convex. Hence, $t \mapsto E(\tau_j, \lfloor t/\tau_j \rfloor)$ is a nonincreasing map of δ-mean convex sets. Then by (5.10) and Proposition 5.1 (d) each $E(t)$ is δ-mean convex and the map $t \mapsto E(t)$ is nonincreasing. Therefore, by the definition of mean convexity, so is $t \mapsto P_\varphi(E(t))$. □

6. Consistency with smooth flows: proof of Theorem 1.8

If, for an anisotropy φ, the map $\xi \mapsto \varphi(\xi) - \lambda|\xi|$ is also an anisotropy in $\mathbb{R}^n$ for some $\lambda > 0$, we say φ is elliptic.

Suppose φ is a $C^{3+\beta}$-elliptic anisotropy, and functions f and g satisfy Hypothesis (H), $f \in C^\beta(\mathbb{R})$ and $g \in C^\beta(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, for some $\beta \in (0, 1]$.

Definition 6.1 (**Stable smooth flow**).

(a) A C^1-in time family $\{S(t)\}_{t\in[0,T^\dagger)}$ of C^2-subsets of $\mathbb{R}^n$ is called a generalized power smooth mean curvature flow with driving force g starting from S_0, if

$$\begin{cases} f(v_{S(t)}(x)) = -\kappa_{S(t)}^\varphi(x) - g(x) & \text{for } t \in (0, T^\dagger) \text{ and } x \in \partial S(t), \\ S(0) = S_0, \end{cases}$$

where as usual $v_{S(t)}$ and $\kappa_{S(t)}^\varphi$ are the normal velocity and the anisotropic mean curvature of $\partial S(t)$, respectively.

(b) The family $\{S(t)\}_{t\in[0,T^\dagger)}$ is called *stable* if for any $T \in (0, T^\dagger)$ there are $\rho = \rho(T) > 0$, $\sigma = \sigma(T) > 0$ such that for any $a \in [0, T)$ there exist families $L^\pm[r, s, a, t]$ for $r \in [0, \rho]$, $s \in [0, \sigma]$ and $t \in [a, T]$ of $C^{2+\beta}$-subsets of $\mathbb{R}^n$ smoothly depending[8] on r, s, a, t, such that
 – $L^\pm[0, 0, a, t] = S(t)$ for all $t \in [a, T]$,
 – $L^\pm[r, s, a, a] = \{x \in \mathbb{R}^n : \ \mathrm{sd}_{S(a)}(x) < \pm(r + s)\}$ for all $r \in [0, \rho]$ and $s \in [0, \sigma]$,
 – for any $r \in [0, \rho]$ and $s \in [0, \sigma]$,

$$f(v_{L^\pm[r,s,a,t]}(x)) = -\kappa^\varphi_{L^\pm[r,s,a,t]}(x) - g(x) \pm s \quad \text{for } t \in [a, T] \text{ and } x \in \partial L^\pm[r, s, a, t]. \tag{6.1}$$

Using the signed distance functions we can rewrite (6.1) as

$$f\Big(\frac{\partial}{\partial t}\mathrm{sd}_{L^\pm[r,s,a,t]}(x)\Big) = -\kappa^\varphi_{L^\pm[r,s,a,t]}(x) - g(x) \pm s \quad \text{for } t \in [a, T] \text{ and } x \in \partial L^\pm[r, s, a, t],$$

where $\kappa^\varphi_{L^\pm[r,s,a,t]}$ stands for the φ-mean curvature of $L^\pm[r, s, a, t]$.

Proposition 6.2 (**Properties of stable flows**). *Let $\{S(t)\}_{t\in[0,T^\dagger)}$ be a stable flow as above starting from a bounded set S_0 and for $T \in (0, T^\dagger)$ let ρ, σ, $L^\pm[r, s, a, t]$ be as in Definition 6.1 (b).*

(a) *Assume that $r', r'' \in [0, \rho]$, $r' \le r''$ and $s', s'' \in [0, \sigma]$, $s' \le s''$ with $r' + s' < r'' + s''$. Then*

$$L^-[r'', s'', a, t] \Subset L^-[r', s', a, t] \quad \text{and} \quad L^+[r', s', a, t] \Subset L^+[r'', s'', a, t]$$

for any $t \in [a, T]$.

(b) *For any $s \in (0, \sigma)$ there exists $\tau_2 \in (0, T/2)$ such that for any $\tau \in (0, \tau_2)$, $r \in [0, \rho]$ and*[9] *$t \in [a + \tau, T]$*

$$f\Big(\frac{\mathrm{sd}_{L^+[r,s,a,t-\tau]}}{\tau}\Big) > -\kappa^\varphi_{L^+[r,s,a,t]} - g + \frac{s}{2} \quad \text{on } \partial L^+[r, s, a, t]$$

and

$$f\Big(\frac{\mathrm{sd}_{L^-[r,s,a,t-\tau]}}{\tau}\Big) < -\kappa^\varphi_{L^-[r,s,a,t]} - g - \frac{s}{2} \quad \text{on } \partial L^-[r, s, a, t].$$

(c) *There exists $t^* = t^*(T, \rho, \sigma, S(\cdot)) \in (0, \rho/64)$ such that*

$$L^-[\rho, s, a, a + t'] \subset L^-[\rho/2 + t', s, a, a] \quad \text{and} \quad L^+[\rho/2 - t', s, a, a] \subset L^+[\rho, s, a, a + t']$$

for all $s \in [0, \sigma]$, $a \in [0, T)$ and $t' \in [0, t^*]$ with $a + t' \le T$.

(d) *There exists a continuous increasing function $h : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $h(0) = 0$ such that for all $s \in [0, \sigma]$, $a \in [0, T)$, $t \in [a, T]$*

$$\sup_{x\in\partial L^\pm[0,s,a,t]} \mathrm{dist}(x, \partial L^\pm[0, 0, a, t]) \le h(s).$$

Proof. By smoothness, the family $\{S(t)\}$ is uniformly bounded. Therefore, assertion (a) follows from the strong comparison principle (see e.g. [27, Chapter 2]). The remaning assertions follow from the smooth dependence of L on its variables, the Hölder regularity of f and the continuity and boundedness of g. We refer to [2, Corollary 7.2] for more details in the mean curvature setting. □

Remark 6.3. As in the standard mean curvature case, using the Hamilton-type arguments [27, Chapter 2], one can show the following comparison principle: if $A_0 \Subset B_0$, and $\{A(t)\}_{t\in[0,T)}$ and $\{B(t)\}_{t\in[0,T)}$ are generalized power smooth mean curvature flows starting from A_0 and B_0, respectively, then $A(t) \Subset B(t)$ for any $t \in [0, 1)$.

[8]For instance, $(r, s, a, t) \mapsto \mathrm{sd}_{L^\pm[r,s,a,t]}$ smoothly varies, see also [2, Corollary 7.2].

[9]If $a + \tau > T$ the statement becomes trivial.

Let $E(\cdot)$ be any GMM starting from the smooth bounded set $E_0 = S_0$ and let a sequence $\tau_j \to 0^+$ and flat flows $E(\tau_j, k)$ be such that

$$\lim_{j\to+\infty} |E(\tau_j, \lfloor t/\tau_j \rfloor)\Delta E(t)| = 0 \quad \text{for all } t \geq 0. \tag{6.2}$$

In order to show Theorem 1.8, it suffices to prove that for any $T \in (0, T^\dagger)$

$$E(t) = S(t), \quad t \in [0, T). \tag{6.3}$$

Given $T \in (0, T^\dagger)$ and $a \in [0, T)$, let $\rho, \sigma > 0$ and $L^\pm[r, s, a, t]$ be as in Definition 6.1 (b), and given $s \in [0, \sigma]$, let $\tau_2 := \tau_2(s)$ be given by Proposition 6.2 (b). Let also t^* be as in Proposition 6.2 (c).

The proof of (6.3) basically follows applying inductively the following auxiliary lemma.

Lemma 6.4. *Assume that $a \in [0, T)$ and $s \in (0, \sigma)$ are such that*

$$L^-[0, s, a, a] \subset E(\tau_j, k_0) \subset L^+[0, s, a, a], \tag{6.4}$$

where $k_0 := \lfloor a/\tau_j \rfloor$. Then there exists $\bar t \in (0, t^*]$ depending only on t^* and ρ, such that

$$L^-[0, s, a, a + k\tau_j] \subset E(\tau_j, k_0 + k) \subset L^+[0, s, a, a + k\tau_j] \tag{6.5}$$

for all $j \geq 1$ with $\tau_j \in (0, \tau_2(s))$ and $k = 0, 1, \ldots, \lfloor \bar t/\tau_j \rfloor$ provided that $a + k\tau_j < T$. Moreover, if $a + \bar t < T$, for any $s \in (0, \sigma]$ with $h(2s) < \sigma/4$ there exists $j(s) > 1$ such that

$$L^-[0, 4h(2s), a + \bar t, a + \bar t] \subset E(\tau_j, k_0 + \bar k_j) \subset L^+[0, 4h(2s), a + \bar t, a + \bar t], \tag{6.6}$$

whenever $j > j(s)$ and $\bar k_j := \lfloor \bar t/\tau_j \rfloor$.

Proof. We closely follow the arguments of the proof of the consistency in [2]. In view of (6.4) and Proposition 6.2 (a)

$$L^-[\rho/64, s, a, a] \subset L^-[0, s, a, a] \subset E(\tau_j, k_0) \subset L^+[0, s, a, a] \subset L^+[\rho/64, s, a, a]. \tag{6.7}$$

Thus, by the definition of $L^\pm[\rho, s, a, a]$

$$\begin{cases} B_{\rho/64}(x) \subset E(\tau_j, k_0) & \text{if } x \in L^-[\rho/64, s, a, a], \\ B_{\rho/64}(x) \cap E(\tau_j, k_0) = \emptyset & \text{if } x \notin L^+[\rho/64, s, a, a]. \end{cases}$$

Thus, applying Theorem 4.3 we find a constant $C_8 > 1$ depending only on n, φ, f, ρ and $\|g\|_\infty$ such that

$$\begin{cases} B_{\rho/64 - C_8 i\tau_j}(x) \subset E(\tau_j, k_0 + i) & \text{if } x \in L^-[\rho/64, s, a, a], \\ B_{\rho/64 - C_8 i\tau_j}(x) \cap E(\tau_j, k_0 + i) = \emptyset & \text{if } x \notin L^+[\rho/64, s, a, a] \end{cases} \tag{6.8}$$

for all $0 \leq i\tau_j \leq \frac{\rho}{128 C_8}$. By (6.7) and (6.8) for such i

$$L^-[\rho/32 - C_8 i\tau_j, s, a, a] \subset E(\tau_j, k_0 + i) \subset L^+[\rho/32 - C_8 i\tau_j, s, a, a]. \tag{6.9}$$

Now using the inequality

$$\frac{\rho}{32} - C_8 i\tau_j \leq \frac{\rho}{2} - i\tau_j$$

(recall that $C_8 > 1$) and the definition of $L^\pm[r, s, a, a]$ in (6.9), we find

$$L^-[\rho/2 - i\tau_j, s, a, a] \subset E(\tau_j, k_0 + i) \subset L^+[\rho/2 - i\tau_j, s, a, a]. \tag{6.10}$$

Now if $i\tau_j \leq t^*$, by Proposition 6.2 (c) and (6.10)

$$L^-[\rho, s, a, a + i\tau_j] \subset E(\tau_j, k_0 + i) \subset L^+[\rho, s, a, a + i\tau_j]. \tag{6.11}$$

Let us define
$$\bar t := \min\Big\{t^*, \tfrac{\rho}{128C_8}\Big\}.$$
By (6.11)
$$L^-[\rho,s,a,a+i\tau_j]\subset E(\tau_j,k_0+i)\subset L^+[\rho,s,a,a+i\tau_j],\quad i=0,1,\dots,\lfloor \bar t/\tau_j\rfloor$$
provided $a+i\tau_j<T$. We claim that for any $j>1$ with $\tau_j\in(0,\tau_2(s))$
$$L^-[0,s,a,a+i\tau_j]\subset E(\tau_j,k_0+i)\subset L^+[0,s,a,a+i\tau_j],\quad i=0,1,\dots,\lfloor \bar t/\tau_j\rfloor\,, \tag{6.12}$$
with $a+i\tau_j<T$. Indeed, let
$$\bar r := \inf\Big\{r\in[0,\rho]:\ E(\tau_j,k_0+i)\subset L^+[r,s,a,a+i\tau_j],\quad i=0,\dots,\lfloor \bar t/\tau_j\rfloor\,,\ a+i\tau_j<T\Big\}.$$
To prove the claim we need to show that
$$\bar r=0. \tag{6.13}$$

In view of (6.11) the infimum is taken over a nonempty set. By contradiction, assume that $\bar r>0$. By the continuity of $L^+[\cdot,s,a,a+i\tau_j]$ at $r=\bar r$, there exists the smallest integer $k\le\lfloor \bar t/\tau_j\rfloor$ (clearly, $k>0$ by (6.7) and the assumption $\bar r>0$) for which
$$\partial E(\tau_j,k_0+k)\cap\partial L^+[\bar r,s,a,a+k\tau_j]\neq\emptyset. \tag{6.14}$$
Moreover, by the minimality of k and the definition of $\bar r$
$$E(\tau_j,k_0+k-1)\subset L^+[\bar r,s,a,a+(k-1)\tau_j],\quad E(\tau_j,k_0+k)\subset L^+[\bar r,s,a,a+k\tau_j].$$
By Proposition 6.2 (b) (recall that $k_0=\lfloor a/\tau_j\rfloor$)
$$f\Big(\frac{\mathrm{sd}_{L^+[\bar r,s,a,a+k\tau_j-\tau_j]}}{\tau_j}\Big) > -\kappa^{\varphi}_{L[\bar r,s,a,a+k\tau_j]}-g+\frac{s}{2}\quad\text{on }\partial L[\bar r,s,a,a+k\tau_j].$$
Thus, applying Lemma B.1 (a) with $E:=F(\tau_j,k_0+k-1)$, $E_\tau:=F(\tau_j,k_0+k)$, $F:=L^+[\bar r,s,a,a+(k-1)\tau_j]$ and $F_\tau:=L^+[\bar r,s,a,a+k\tau_j]$ we obtain
$$\partial E(\tau_j,k_0+k)\cap\partial L^+[\bar r,s,a,a+k\tau_j]=\emptyset,$$
which contradicts (6.14). Thus (6.13) is proven. Analogous contradiction argument based on Lemma B.1 (b) and Proposition 6.2 (b) for $s<0$ shows the validity of (6.12). This concludes the proof of (6.5).

Now, let us prove (6.6). By construction $L^-[0,2s,a,a]\Subset L^-[0,s,a,a]$ and $L^+[0,s,a,a]\Subset L^+[0,2s,a,a]$. Thus, by the comparison principle in Remark 6.3, $L^-[0,2s,a,t]\Subset L^-[0,s,a,t]$ and $L^+[0,s,a,t]\Subset L^+[0,2s,a,t]$ for all $t\in[a,T]$. Since $L^\pm[0,\cdot,\cdot,\cdot]$ continuously varies, there is $j(s)>1$ such that for any $j>j(s)$
$$\begin{aligned}L^-[0,2s,a,a+\bar t]&\subset L^-[0,s,a,a+\bar k_j\tau_j]\\&\subset E(\tau_j,\bar k_j)\subset L^+[0,s,a,a+\bar k_j\tau_j]\subset L^+[0,2s,a,a+\bar t],\end{aligned} \tag{6.15}$$
where we recall that $\bar k_j:=\lfloor \bar t/\tau_j\rfloor$, and we used $\bar k_j\tau_j=\lfloor \bar t/\tau_j\rfloor\tau_j\nearrow\bar t$. Let the function h be given by Proposition 6.2 (d) so that
$$\max_{x\in\partial L^\pm[0,2s,a,a+\bar t]}\mathrm{dist}(x,\partial L^\pm[0,0,a,a+\bar t])\le h(2s).$$
Since, by the definition of $L^\pm$, we have $L^\pm[0,0,a,a+\bar t]=E(a+\bar t)=L^\pm[0,0,a+\bar t,a+\bar t]$ and, by the definition of h,
$$\mathrm{dist}(\partial L^\pm[0,h(2s),a+\bar t,a+\bar t],\partial L^\pm[0,0,a+\bar t,a+\bar t])=h(2s),$$

it follows that

$$L^-[0,4h(2s),a+\bar t,a+\bar t]\subset L^-[0,2s,a,a+\bar t] \quad \text{and} \quad L^+[0,2s,a,a+\bar t]\subset L^+[0,4h(2s),a+\bar t,a+\bar t].$$

Using this and (6.15) we get

$$L^-[0,4h(2s),a+\bar t,a+\bar t]\subset E(\tau_j,k_0+\bar k_j)\subset L^+[0,4h(2s),a+\bar t,a+\bar t].$$

□

Proof of Theorem 1.8. Let $\bar t$ be given by Lemma 6.4,

$$N:=\lfloor T/\bar t\rfloor+1$$

and let $\sigma_0\in(0,\sigma/16)$ be such that the numbers

$$\sigma_l=4h(2\sigma_{l-1}),\quad l=1,\dots,N,$$

satisfy $\sigma_l\in(0,\sigma/16)$. By the monotonicity and continuity of h together with $h(0)=0$, and the finiteness of N, such a choice of σ_0 is possible (indeed, it is enough to observe that if $\sigma_0\to0$, then all $\sigma_l\to0$).

Fix any $s\in(0,\sigma_0)$ and let

$$a_0(s):=s,\quad a_l(s):=4h(2a_{l-1}(s)),\quad l=1,\dots,N.$$

Note that $a_l(s)\in(0,\sigma_l)$. In particular, the numbers $j(a_l(s))$, given by the last assertion of Lemma 6.4, are well-defined. Let also

$$j_l^s:=\max\{j\ge1:\ \tau_j\notin(0,\tau_2(a_l(s)))\}$$

and

$$j_s:=1+\max_{l=0,\dots,N}\max\{j(a_l(s)),j_l^s\}.$$

By the definition of $L^\pm$,

$$L^-[0,s,0,0]\subset S(0)=E_0=E(\tau_j,0)\subset L^+[0,s,0,0]$$

for all $j>j_s$ (basically, this is true for all j). Therefore, by Lemma 6.4 applied with $a=0$ (so that $k_0:=\lfloor a/\tau_j\rfloor=0$) we find

$$L^-[0,s,0,k\tau_j]\subset E(\tau_j,k)\subset L^+[0,s,0,k\tau_j],\quad k=0,1,\dots,\bar k_j,$$

where $\bar k_j:=\lfloor \bar t/\tau_j\rfloor$. Moreover, since $s\in(0,\sigma_0,)$ by the last assertion of Lemma 6.4 and the definition of $a_l(s)$

$$L^-[0,a_1(s),\bar t,\bar t]\subset E(\tau_j,\bar k_j)\subset L^+[0,a_1(s),\bar t,\bar t]$$

for all $j\ge j_s$. Hence, we can reapply Lemma 6.4 with $s:=a_1(s)$, $a=\bar t$ and $k_0=\bar k_j$, to find

$$L^-[0,a_1(s),\bar t,\bar t+k\tau_j]\subset E(\tau_j,\bar k_j+k)\subset L^+[0,a_1(s),\bar t,\bar t+k\tau_j],\quad k=0,1,\dots,\bar k_j.$$

In particular, since $j>j_s>j(a_1(s))$, again by the last assertion of Lemma 6.4 we deduce

$$L^-[0,a_2(s),2\bar t,2\bar t]\subset E(\tau_j,2\bar k_j)\subset L^+[0,a_2(s),2\bar t,2\bar t].$$

Repeating this argument at most N times, for all $j>j_s$ we find

$$L^-[0,a_l(s),l\bar t,l\bar t+k\tau_j]\subset E(\tau_j,l\bar k_j+k)\subset L^+[0,a_l(s),l\bar t,l\bar t+k\tau_j],\quad k=0,1,\dots,\bar k_j \tag{6.16}$$

whenever $l=0,\dots,N$ with $l\bar t+k\tau_j<T$.

Now take any $t\in(0,T)$, and let $l:=\lfloor t/\bar t\rfloor$ and $k=\lfloor t/\tau_j\rfloor-l\bar k_j$ so that $l\bar k_j+k=\lfloor t/\tau_j\rfloor$. By means of l and k, as well as the definition of $\bar k_j$ we represent (6.16) as

$$L^-\Big[0,a_l(s),l\bar t,l\bar t+\tau_j\Big\lfloor\tfrac{t}{\tau_j}\Big\rfloor-l\tau_j\Big\lfloor\tfrac{\bar t}{\tau_j}\Big\rfloor\Big]$$

$$\subset E\Big(\tau_j, \Big\lfloor \tfrac{t}{\tau_j} \Big\rfloor \Big) \subset L^+\Big[0, a_l(s), l\bar t, l\bar t + \tau_j \Big\lfloor \tfrac{t}{\tau_j} \Big\rfloor - l\tau_j \Big\lfloor \tfrac{\bar t}{\tau_j} \Big\rfloor \Big] \quad (6.17)$$

for all $j > j_s$. Since

$$\lim_{j\to+\infty} \Big(l\bar t + \tau_j \Big\lfloor \tfrac{t}{\tau_j} \Big\rfloor - l\tau_j \Big\lfloor \tfrac{\bar t}{\tau_j} \Big\rfloor \Big) = t,$$

by the continuous dependence of $L^\pm$ on its parameters, as well as the convergence (6.2) of the flat flows, letting $j \to +\infty$ in (6.17) we obtain

$$L^-[0, a_l(s), l\bar t, t] \subset E(t) \subset L^+[0, a_l(s), l\bar t, t], \quad (6.18)$$

where, due to the L^1-convergence, the inclusions in (6.18) hold possibly up to some negligible set. Now we let $s \to 0^+$ and recalling that $a_l(s) \to 0$ (by the continuity of h and assumption $h(0) = 0$), from (6.18) we deduce

$$L^-[0, 0, l\bar t, t] \subset E(t) \subset L^+[0, 0, l\bar t, t].$$

Now recalling $L^\pm[0, 0, a, t] = S(t)$ for $t \in [a, T]$ we get

$$E(t) = L^\pm[0, 0, l\bar t, t] = S(t).$$

□

7. Evolution of convex sets: proof of Theorem 1.6

In this section we prove Theorem 1.6; thus we assume φ is Euclidean, $g \equiv 0$ and $f(r) = \mathrm{sign}(r)|r|^\alpha$ for some $\alpha > 0$. We need the following result, proven in [30].

Theorem 7.1. *Let $E_0 \subset \mathbb{R}^n$ be a bounded $C^{2+\beta}$-convex set for some $\beta \in (0, 1]$. Then there exist $T^* > 0$ and a unique C^1-in time flow $\{E(t)\}_{t\in[0,T^*)}$ starting from E_0 such that each $E(t)$ is C^2, convex and*

$$v = -\kappa^{1/\alpha} \quad \text{on } \partial E(t)$$

for all $t \in [0, T^*)$. Moreover, $\{E(t)\}$ is stable in the sense of Definition 6.1 and $|E(t)| \to 0$ as $t \nearrow T^*$.

Proof of Theorem 1.6(i). If $\mathrm{Int}(\mathcal{K}_0) = \emptyset$, then by convexity, $|\mathcal{K}_0| = 0$, and we are done. Otherwise, since $g \equiv 0$, translating if necessary, we assume that $\mathrm{Int}(\mathcal{K}_0)$ contains the origin. Suppose that $\mathrm{GMM}(\mathcal{F}, \mathcal{S}^*, \mathcal{K}_0)$ contains at least two different GMM's, say, $\mathcal{K}'(\cdot)$ and $\mathcal{K}''(\cdot)$ so that there exist $T > 0$ and $\epsilon_0 > 0$ such that

$$|\mathcal{K}'(T) \Delta \mathcal{K}''(T)| \ge 2\epsilon_0. \quad (7.1)$$

Also, by the uniform time-continuity (see Theorem 1.4(i)) of $\mathcal{K}'(\cdot), \mathcal{K}''(\cdot)$, possibly decreasing T if necessary, we may further assume that $|\mathcal{K}'(T)|, |\mathcal{K}''(T)| \ge 2\epsilon_0$. Fix any $\lambda \in (0, 1)$ sufficiently close to 1 so that

$$|\lambda^{-1}\mathcal{K}'(\lambda^{1/\alpha}T) \setminus \lambda\mathcal{K}'(\lambda^{-1/\alpha}T)| < \epsilon_0, \quad |\lambda^{-1}\mathcal{K}''(\lambda^{1/\alpha}T) \setminus \lambda\mathcal{K}''(\lambda^{-1/\alpha}T)| < \epsilon_0. \quad (7.2)$$

Let us define $E_0^\lambda := \lambda\mathcal{K}_0$ and $F_0^\lambda := \lambda^{-1}\mathcal{K}_0$, so that $E_0^\lambda \Subset \mathcal{K}_0 \Subset F_0^\lambda$. Now we choose smooth convex sets P_0 and Q_0 such that

$$E_0^\lambda \Subset P_0 \Subset \mathcal{K}_0 \Subset Q_0 \Subset F_0^\lambda.$$

If we define the corresponding flat flows, then by Corollary B.4

$$E^\lambda(\tau, k) \Subset P(\tau, k) \Subset C(\tau, k) \Subset Q(\tau, k) \Subset F^\lambda(\tau, k) \quad (7.3)$$

for any $\tau > 0$ and $k \ge 0$ with $|E^\lambda(\tau, k)| > 0$.

Since P_0 and Q_0 are smooth and convex, by Theorem 7.1 the corresponding smooth flows exist and disappear at a maximal time (however, GMM starting from them exists for all times). In particular, by consistency (Theorem 1.8) both $\mathrm{GMM}(\mathcal{F},\mathcal{S}^*,P_0)$ and $\mathrm{GMM}(\mathcal{F},\mathcal{S}^*,Q_0)$ are singletons, say, $\{P(\cdot)\}$ and $\{Q(\cdot)\}$.

Let $\tau_j'\searrow 0$ and $\tau_j''\searrow 0$ be sequences for which

$$|\mathcal{K}(\tau_j',\lfloor t/\tau_j'\rfloor)\Delta\mathcal{K}'(t)|\to 0 \quad \text{and} \quad |\mathcal{K}(\tau_j'',\lfloor t/\tau_j''\rfloor)\Delta\mathcal{K}''(t)|\to 0$$

as $j\to+\infty$ for all $t\ge 0$. In view of Corollary 4.2, as $j\to+\infty$ we have

$$|E^\lambda(\tau_j',\lfloor t/\tau_j'\rfloor)\Delta[\lambda\mathcal{K}'(\lambda^{-1/\alpha}t)]|\to 0 \quad \text{and} \quad |F^\lambda(\tau_j',\lfloor t/\tau_j'\rfloor)\Delta[\lambda^{-1}\mathcal{K}''(\lambda^{-1/\alpha}t)]|\to 0$$

and

$$|E^\lambda(\tau_j'',\lfloor t/\tau_j''\rfloor)\Delta(\lambda\mathcal{K}'(\lambda^{-1/\alpha}t))|\to 0 \quad \text{and} \quad |F^\lambda(\tau_j'',\lfloor t/\tau_j''\rfloor)\Delta[\lambda^{-1}\mathcal{K}''(\lambda^{1/\alpha}t)]|\to 0.$$

Now applying (7.3) with τ_j' we deduce

$$\lambda\mathcal{K}'(\lambda^{-1/\alpha}t)\subset P(t)\subset\mathcal{K}'(t)\subset Q(t)\subset\lambda^{-1}\mathcal{K}'(\lambda^{1/\alpha}t) \quad \text{for all } t\in[0,T], \tag{7.4}$$

and appyling it with τ_j'' we deduce

$$\lambda\mathcal{K}''(\lambda^{-1/\alpha}t)\subset P(t)\subset\mathcal{K}''(t)\subset Q(t)\subset\lambda^{-1}\mathcal{K}''(\lambda^{1/\alpha}t) \quad \text{for all } t\in[0,T]. \tag{7.5}$$

By (7.2)

$$|Q(T)\setminus P(T)|\le|[\lambda^{-1}C'(\lambda^{1/\alpha}T)]\setminus[\lambda C'(\lambda^{-1/\alpha}T)]|\le\epsilon_0.$$

However, in view of (7.4) and (7.5) as well as of (7.1),

$$2\epsilon_0\le|\mathcal{K}''(T)\Delta\mathcal{K}'(T)|\le|Q(T)\setminus P(T)|\le\epsilon_0,$$

a contradiction. □

Proof of Theorem 1.6(ii). By the Kuratowski convergence of $(\partial\mathcal{K}_{0h})$ and comparison principles, given $\lambda\in(0,1)$, as in the proof of (i),

$$\lambda\mathcal{K}(t\lambda^{-1/\alpha})\subset\mathcal{K}_h(t)\subset\lambda^{-1}\mathcal{K}(t\lambda^{1/\alpha}) \quad \text{for all } t\ge 0,$$

provided $h\in\mathbb{N}$ is large enough depending only on λ. Since the sequence $(P(\mathcal{K}_h(t)))$ is bounded (by the supremum of $P(\mathcal{K}_{0h})$), up to a subsequence, $\mathcal{K}_h(t)\to\mathcal{K}'(t)$ in $L^1(\mathbb{R}^n)$ as $i\to+\infty$. Then

$$\lambda\mathcal{K}(t\lambda^{-1/\alpha})\subset\mathcal{K}'(t)\subset\lambda^{-1}\mathcal{K}(t\lambda^{1/\alpha}) \quad \text{for all } t\ge 0.$$

Now letting $\lambda\to 1$ we get $\mathcal{K}'(t)=\mathcal{K}(t)$, i.e., the limit of $(\mathcal{K}_h)$ is independent of the subsequence. Thus, the thesis follows. □

8. Minimizing movements in the class $\mathrm{Conv_b}(\mathbb{R}^n)$: proof of Theorem 1.7

In this section we suppose $g\equiv 0$. Motivated by Conjecture 1.2, we study GMM in the class $\mathrm{Conv_b}(\mathbb{R}^n)$. Notice that, due to the convexity constraint, the first variation of $\mathcal{F}$ is nonlocal, and thus, in general we cannot write a pointwise Euler-Lagrange equation; therefore, the nature of $\mathrm{GMM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)$ seems not clear. Moreover, to prove Theorem 1.7 we cannot apply the techniques used in the proof of Theorem 1.4, based on cutting and filling with balls (because they lead to the lost of convexity).

Proof of Theorem 1.7. Let $\mathcal{K}_0\in\mathrm{Conv_b}(\mathbb{R}^n)$. Without loss of generality we assume that the interior of $\mathcal{K}_0$ is nonempty. By the L^1_{loc}-closedness of $\mathrm{Conv_b}(\mathbb{R}^n)$ and the L^1_{loc}-lower semicontinuity of $\mathcal{F}(\cdot;\mathcal{K},\tau)$ for any $\mathcal{K}\in\mathrm{Conv_b}(\mathbb{R}^n)$, there exists a minimizer of $\mathcal{F}(\cdot;\mathcal{K},\tau)$ in

$\mathrm{Conv_b}(\mathbb{R}^n)$. By truncation, we can readily show that every minimizer $\mathcal{K}_\tau$ satisfies $\mathcal{K}_\tau \subset \mathcal{K}$. Now we define flat flows $\{\mathcal{K}(\tau,k)\}$; clearly,

$$\mathcal{K}_0 = \mathcal{K}(\tau,0) \supset \mathcal{K}(\tau,1) \supset \dots \quad \text{and} \quad P_\varphi(\mathcal{K}_0) = P_\varphi(\mathcal{K}(\tau,0)) \geq P_\varphi(\mathcal{K}(\tau,1)) \geq \dots. \tag{8.1}$$

For any $\tau > 0$, define

$$h_\tau(t) := P_\varphi(\mathcal{K}(\tau, \lfloor t/\tau \rfloor)), \quad t \geq 0.$$

By (8.1) $\{h_\tau\}$ are nonincreasing nonnegative functions satisfying $h_\tau(0) = P(\mathcal{K}_0)$. Thus, Helly's selection theorem implies the existence of $\tau_j \to 0^+$ and a nonincreasing function $h_0 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $h_{\tau_j} \to h_0$ in $\mathbb{R}_0^+$. Since h_0 is monotone, its discontinuity set $J \subset \mathbb{R}_0^+$ is at most countable. Let $Q \subset \mathbb{R}_0^+ \setminus J$ be any countable dense set in $\mathbb{R}_0^+$. By compactness in BV, passing to a further not relabelled sequence τ_j and using a diagonal argument, for any $s \in Q \cup J$, we can define a $\mathcal{K}(s) \subset \mathbb{R}^n$ such that

$$\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor) \to \mathcal{K}(s) \quad \text{in } L^1(\mathbb{R}^n) \text{ as } j \to +\infty.$$

Since each $\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)$ is convex, so is $\mathcal{K}(s)$ and hence,

$$\lim_{j\to+\infty} h_{\tau_j}(s) = \lim_{j\to+\infty} P_\varphi(\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)) = P_\varphi(\mathcal{K}(s)) = h_0(s).$$

Now take any $0 < s \in \mathbb{R}_0^+ \setminus [Q \cup J]$ so that h_0 is continuous at s. By density of Q in $\mathbb{R}_0^+$, we can choose sequences $Q \ni a_k \nearrow s$ and $Q \ni b_k \searrow s$. By (8.1) the maps $k \mapsto \mathcal{K}(a_k)$ and $k \mapsto P_\varphi(\mathcal{K}(a_k))$ are nonincreasing and the maps $k \mapsto \mathcal{K}(b_k)$ and $k \mapsto P_\varphi(\mathcal{K}(b_k))$ are nondecreasing. Let

$$\bigcap_{k\geq1} \mathcal{K}(a_k) =: \mathcal{K}(s)^* \supset \mathcal{K}(s)_* = \bigcup_{k\geq1} \mathcal{K}(b_k).$$

By the continuity of g at s, both $P_\varphi(\mathcal{K}(a_k))$ and $P_\varphi(\mathcal{K}(b_k))$ converges to $g(s)$, and therefore, $P_\varphi(\mathcal{K}(s)^*) = P_\varphi(\mathcal{K}(s)_*)$. Thus, by the convexity of both $\mathcal{K}(s)_*$ and $\mathcal{K}(s)^*$, it follows

$$\mathcal{K}(s)_* = \mathcal{K}(s)^* =: \mathcal{K}(s)$$

and $|\mathcal{K}(a_k)\Delta\mathcal{K}(s)| \to 0$. Let us show that

$$\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor) \to \mathcal{K}(s) \quad \text{in } L^1(\mathbb{R}^n) \text{ as } j \to +\infty. \tag{8.2}$$

Notice that

$$\limsup_{j\to+\infty} \limsup_{k\to+\infty} (h_{\tau_j}(a_k) - h_{\tau_j}(b_k)) = 0, \tag{8.3}$$

otherwise, h_0 cannot be continuous at s. Therefore, in view of the inclusion

$$\mathcal{K}(\tau_j, \lfloor a_k/\tau_j \rfloor) \supset \mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor) \supset \mathcal{K}(\tau_j, \lfloor b_k/\tau_j \rfloor)$$

we have

$$\bigcap_{k\geq1} \mathcal{K}(\tau_j, \lfloor a_k/\tau_j \rfloor) \supset \mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor) \supset \bigcup_{k\geq1} \mathcal{K}(\tau_j, \lfloor b_k/\tau_j \rfloor)$$

and by (8.3) and convexity of these sets, for any ϵ,

$$|\mathcal{K}(\tau_j, \lfloor a_k/\tau_j \rfloor) \setminus \mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)| < \epsilon$$

provided that $k > k_\epsilon > 0$. Hence, letting $j \to +\infty$ in the estimate

$$\begin{aligned}&|\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)\Delta\mathcal{K}(s)| \\ &\qquad \leq |\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)\Delta\mathcal{K}(\tau_j, \lfloor a_k/\tau_j \rfloor)| + |\mathcal{K}(\tau_j, \lfloor a_k/\tau_j \rfloor)\Delta\mathcal{K}(a_k)| + |\mathcal{K}(a_k)\Delta\mathcal{K}(s)|,\end{aligned}$$

and then $k \to +\infty$ and $\epsilon \to 0^+$ in

$$\limsup_{j\to+\infty} |\mathcal{K}(\tau_j, \lfloor s/\tau_j \rfloor)\Delta\mathcal{K}(s)| \leq \epsilon + |\mathcal{K}(a_k)\Delta\mathcal{K}(s)|,$$

we get (8.2). Thus, by definition the family $\{\mathcal{K}(s)\}_{s\geq 0}$ is a GMM. □

Suppose (compare with the next proposition) that for any bounded convex set $\mathcal{K}_0$, $\mathcal{F}(\cdot;\mathcal{K}_0,\tau)$ admits a convex minimizer in $\mathcal{S}^*$. In this case, the flat flows $\mathcal{K}(\tau,k)$ consisting of those convex sets are also flat flows in $\mathrm{Conv_b}(\mathbb{R}^n)$; by Theorem 1.6 $\mathcal{K}(\tau,\lfloor t/\tau\rfloor)\to\mathcal{K}(t)$ as $\tau\to 0^+$ (because $\mathrm{GMM}(\mathcal{F},\mathcal{S}^*,\mathcal{K}_0)$ is a singleton). Then clearly, $\mathcal{K}(\cdot)\in\mathrm{GMM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)$ (in fact is a minimizing movement). However, it is not clear whether $\mathrm{GMM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)$ contains other generalized minimizing movements.

Proposition 8.1. *Suppose that φ is an elliptic C^3-anisotropy and that $g\equiv 0$. Assume that f is concave in $(0,+\infty)$. Then for any bounded convex open set $\mathcal{K}_0$ and $\tau>0$ the minimal and maximal minimizers of $\mathcal{F}(\cdot;\mathcal{K}_0,\tau)$ in $\mathcal{S}^*$ are convex.*

Proof. We follow the notation of [11]. Since $\mathcal{K}_0$ is convex, $\mathrm{d}_{\mathcal{K}_0}$ is concave in $\mathcal{K}_0$, and hence, so is $u_0:=f(\mathrm{d}_{\mathcal{K}_0}/\tau)$. Consider

$$\mathcal{E}(v):=\begin{cases}\displaystyle\int_{\mathcal{K}_0}\varphi^o(Dv)+\int_{\mathcal{K}_0}\frac{(v+u_0)^2}{2}\,dx & \text{if } v\in L^2(\mathcal{K}_0)\cap BV(\mathcal{K}_0),\\ +\infty & \text{if } L^2(\mathcal{K}_0)\setminus BV(\mathcal{K}_0),\end{cases}$$

where $\phi^o(Dv)$ is the ϕ^o-total variation of v. One checks that $\mathcal{E}$ admits a unique minimizer $v_0\in L^2(\mathcal{K}_0)\cap BV(\mathcal{K}_0)$. Since $\mathcal{E}$ is convex, by [7] there exists $z\in L^\infty(\mathcal{K}_0,\mathbb{R}^n)$ with $\mathrm{div} z\in L^2(\mathcal{K}_0)$ such that [10]

$$-\mathrm{div} z+v_0+u_0=0,\quad \varphi(z)\leq 1,\quad \int_{\mathcal{K}_0}z\cdot Dv_0=\int_{\mathcal{K}_0}\varphi^o(Dv_0).$$

Repeating the same arguments of [11, Lemma 5.1] we can show that for a.e. $s<0$ the set $\{v_0<s\}$ is a minimizer of

$$\mathcal{E}_s(F):=P_\varphi(F)-\int_F(v_0+s)\,dx,\qquad F\subseteq\mathcal{K}_0,\quad s\in\mathbb{R}.$$

If v_0 is twice continuously differentiable in $\mathcal{K}_0$ with $|\nabla v_0|>0$. Then v_0 is a viscosity solution of the equation

$$-\mathrm{div}\,\nabla\varphi^o(Dv_0)+v_0+u_0=0.$$

Since u_0 is concave, using [4, Theorem 1] we find that v_0 is convex. In case v_0 is not sufficiently smooth, we can approximate the equation as in [11] and again get that v_0 is convex. In particular, $\mathcal{E}_s$ admits a convex minimizer $\{v_0<s\}$. One can readily check that the sets

$$\mathcal{K}_*:=\bigcup_{s<0}\{v_0<s\}\quad\text{and}\quad\mathcal{K}^*:=\bigcap_{s>0}\{v_0\leq s\}$$

are the minimal and maximal (convex) minimizers of the functional $\mathcal{E}_0(\cdot)=\mathcal{F}(\cdot;\mathcal{K}_0,\tau)$. □

As a corollary, we obtain the validity of Conjecture 1.1 under a restriction on $\alpha>0$.

Corollary 8.2 (Conjecture 1.1 for $\alpha\in(0,1]$). *Let φ be Euclidean, $f(r)=r^\alpha$ for $r>0$,*

$$\alpha\in(0,1],$$

and $g\equiv 0$. Then for any bounded convex $\mathcal{K}_0\subset\mathbb{R}^n$, $\mathrm{GMM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)$ is a singleton and coincides with the unique minimizing movement in $\mathrm{GMM}(\mathcal{F},\mathcal{S}^*,\mathcal{K}_0)$.

[10] Concerning the notation $z\cdot Dv_0$ and $\varphi^o(Dv_0)$, see [11].

Proof. Let $\mathcal{K}(\tau,k)_*$ and $\mathcal{K}(\tau,k)^*$ be flat flows in $\mathcal{S}^*$ consisting of the minimal and maximal minimizers of $\mathcal{F}$ (in $\mathcal{S}^*$), starting from $\mathcal{K}_0$. By Theorem 1.6 we have

$$\lim_{\tau\to0^+}\mathcal{K}(\tau,\lfloor t/\tau\rfloor)_*=\mathcal{K}(t)\quad\text{and}\quad\lim_{\tau\to0^+}\mathcal{K}(\tau,\lfloor t/\tau\rfloor)^*=\mathcal{K}(t)\quad\text{in } L^1(\mathbb{R}^n)\text{ for all } t\ge0,$$

where $\{\mathcal{K}(t)\}=\mathrm{MM}(\mathcal{F},\mathcal{S}^*,\mathcal{K}_0)$. By Proposition 8.1 both $\mathcal{K}(\tau,k)_*$ and $\mathcal{K}(\tau,k)^*$ are convex. Note that they are the minimal and maximal minimizers of $\mathcal{F}$ also in $\mathrm{Conv_b}(\mathbb{R}^n)$. Therefore, $\mathrm{GMM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)=\mathrm{MM}(\mathcal{F},\mathrm{Conv_b}(\mathbb{R}^n),\mathcal{K}_0)=\{\mathcal{K}(\cdot)\}$. □

Appendix A. Volume-distance inequality

In this appendix we establish a couple of technical results needed in various proofs. In particular, the next result is crucial, and is an easy modification of the volume-distance inequality of Almgren-Taylor-Wang [2] (see also [25]).

Lemma A.1 (Volume-distance inequality). *Let $r_0>0$, and $F\in\mathcal{S}^*$ satisfy*

$$P(F,B_r(x))\ge\vartheta r^{n-1},\quad x\in\partial F,\ \ r\in(0,r_0],\tag{A.1}$$

for some $\vartheta>0$. Then for any measurable $E\subset\mathbb{R}^n$ and any $p,\ell>0$ one has

$$|E\Delta F|\le\begin{cases}\dfrac{cp^n\ell^n}{r_0^{n-1}}\,P_\varphi(F)+\dfrac{1}{f(p)}\displaystyle\int_{E\Delta F}f\left(\tfrac{\mathrm{d}_F}{\ell}\right)\,dx & \text{if } \ell>r_0,p\ell>r_0,\\ cp^n\ell\,P_\varphi(F)+\dfrac{1}{f(p)}\displaystyle\int_{E\Delta F}f\left(\tfrac{\mathrm{d}_F}{\ell}\right)\,dx & \text{if } \ell\in(0,r_0] \text{ and } p\ell>r_0,\\ cp\ell\,P_\varphi(F)+\dfrac{1}{f(p)}\displaystyle\int_{E\Delta F}f\left(\tfrac{\mathrm{d}_F}{\ell}\right)\,dx & \text{if } p\ell\in(0,r_0],\end{cases}\tag{A.2}$$

where $c:=\frac{10^n\omega_n}{c_\varphi\vartheta}$.

Proof. Define

$$\mathcal{X}:=\{x\in E\Delta F:\ \mathrm{d}_F(x)\ge p\ell\},\quad\mathcal{Y}:=\{x\in E\Delta F:\ \mathrm{d}_F(x)<p\ell\}.$$

Since f is strictly increasing, we have

$$\mathcal{X}=\{x\in E\Delta F:\ f(\mathrm{d}_F(x)/\ell)\ge f(p)\},$$

and hence, by the Chebyshev inequality

$$|\mathcal{X}|\le\frac{1}{f(p)}\int_P f\left(\tfrac{\mathrm{d}_F}{\ell}\right)\,dx\le\frac{1}{f(p)}\int_{E\Delta F}f\left(\tfrac{\mathrm{d}_F}{\ell}\right)\,dx.$$

On the other hand, we cover $\mathcal{Y}$ with balls $B_{2p\ell}$ of radius $2p\ell$ centered at points of ∂F. By the Vitali covering lemma we can take a countable subfamily $\{B'_{10p\ell}\}$ still covering $\mathcal{Y}$ with a pairwise disjoint family $\{B'_{p\ell}\}$.

If $\ell>r_0$ and $p\ell>r_0$, then by the disjointness of $\{B'_{r_0}\}$ and the estimate (A.1) (we cannot apply it with ℓ)

$$|\mathcal{Y}|\le\sum_{B'_{10p\ell}}\omega_n(10p\ell)^n=\tfrac{10^n\omega_np^n\ell^n}{\vartheta r_0^{n-1}}\sum_{B'_{r_0}}\vartheta r_0^{n-1}\le\tfrac{10^n\omega_np^n\ell^n}{\vartheta r_0^{n-1}}\sum_{B'_{r_0}}P(F,B'_{r_0})\le\tfrac{10^n\omega_np^n\ell^n}{\vartheta r_0^{n-1}}\,P(F)$$

On the other hand, if $\ell\le r_0<p\ell$, by disjointness of $\{B'_\ell\}$

$$|\mathcal{Y}|\le\sum_{B'_{10p\ell}}\omega_n(10p\ell)^n=\tfrac{10^n\omega_np^n\ell}{\vartheta}\sum_{B'_{p\ell}}\vartheta\ell^{n-1}\le\tfrac{10^n\omega_np\ell}{\vartheta}\sum_{B'_{p\ell}}P(F,B'_\ell)\le\tfrac{10^n\omega_np\ell}{\vartheta}\,P(F).$$

Finally, if $p\ell \le r_0$ using the disjointness of $\{B'_{p\ell}\}$,

$$|\mathcal{Y}| \le \sum_{B'_{10p\ell}} \omega_n (10p\ell)^n = \tfrac{10^n\omega_n p\ell}{\vartheta} \sum_{B'_{p\ell}} \vartheta p^{n-1}\ell^{n-1} \le \tfrac{10^n\omega_n p\ell}{\vartheta} \sum_{B'_{p\ell}} P(F, B'_{p\ell}) \le \tfrac{10^n\omega_n p\ell}{\vartheta} P(F).$$

Now use (2.1) and $|E\Delta F| = |\mathcal{X}| + |\mathcal{Y}|$ to conclude the proof of estimate (A.2). □

Lemma A.2 (**Morrey-type estimate**). *Suppose that g satisfies (Hc). Then there exists $\gamma_g > 0$ such that*

$$\sup_{0<|A|<\omega_n\gamma_g^n} \tfrac{1}{|A|^{\frac{n-1}{n}}} \int_A |g|\,dx \le \tfrac{c_\varphi n \omega_n^{1/n}}{4}, \tag{A.3}$$

where we write $\gamma_g^n = (\gamma_g)^n$.

Proof. If $p = +\infty$, then

$$\int_A |g|\,dx \le \|g\|_\infty |A|^{\frac1n} |A|^{\frac{n-1}{n}} \le \tfrac{c_\varphi n\omega_n^{1/n}}{4} |A|^{\frac{n-1}{n}}$$

provided $|A| \le \omega_n\gamma_g^n$ with $\gamma_g := \frac{c_\varphi n}{4(1+\|g\|_\infty)}$. If $p \in (n, +\infty)$, then by the Hölder inequality

$$\int_A |g|\,dx \le \|g\|_{L^p(\mathbb{R}^n)} |A|^{\frac{p-1}{p}} \le \tfrac{c_\varphi n\omega_n^{1/n}}{4} |A|^{\frac{n-1}{n}}$$

provided $|A| \le \omega_n\gamma_g^n$ with $\gamma_g := \omega_n^{-1/n}\big(\frac{c_\varphi n}{4(1+\|g\|_{L^p})}\big)^{\frac{p}{p-n}}$. Finally, if $p = n$, then by the Hölder inequality for any $A \subset \mathbb{R}^n$

$$\int_A |g|\,dx \le \Big(\int_A |g|^n\,dx\Big)^{\frac1n} |A|^{\frac{n-1}{n}}.$$

By the absolute continuity of the Lebesgue integral, there exists $\gamma_g > 0$ such that if $|A| < \omega_n\gamma_g^n$, then

$$\Big(\int_A |g|^n\,dx\Big)^{\frac1n} \le \frac{c_\varphi n\omega_n^{1/n}}{4}.$$

□

Appendix B. Comparison principles

The next lemma is a generalization of [2, Lemma 7.3].

Lemma B.1. *Let φ be a C^3-elliptic anisotropy, f, g satisfy Hypothesis (H) and let g be continuous. Let $E \in \mathcal{S}^*$, $\tau > 0$, $k \ge 0$ and E_τ be a minimizer of $\mathcal{F}(\cdot; E, \tau)$. Let F and F_τ be C^2-sets.*

(a) *Let $E \subset F$, $E_\tau \subset F_\tau$ and*

$$f\Big(\tfrac{\mathrm{sd}_F}{\tau}\Big) > -\kappa^\varphi_{F_\tau} - g \quad \text{on } \partial F_\tau. \tag{B.1}$$

Then $\partial E_\tau \cap \partial F_\tau = \emptyset$.

(b) *Let $F \subset E$, $F_\tau \subset E_\tau$ and*

$$f\Big(\tfrac{\mathrm{sd}_F}{\tau}\Big) < -\kappa^\varphi_{F_\tau} - g \quad \text{on } \partial F_\tau.$$

Then $\partial E_\tau \cap \partial F_\tau = \emptyset$.

Proof. We prove only (a), assertion (b) being similar. By contradiction, assume that $x_0 \in \partial E_\tau \cap \partial F_\tau$. Since φ is smooth and elliptic, repeating the arguments of [22, Chapter 17] we can show that singular minimal cones locally minimizing φ-perimeter cannot contain a halfspace. By the smoothness of F_τ, $x_0 \in \partial^* E_\tau$. Moreover, by elliptic regularity, $\partial^* E_\tau$ is C^2, and hence, computing the first variation of $\mathcal{F}(\cdot; E, \tau)$ at E_τ we get

$$f\Big(\tfrac{\mathrm{sd}_E(x_0)}{\tau}\Big) = -\kappa^{\varphi}_{F_\tau}(x_0) - g(x_0). \tag{B.2}$$

By the choice of x_0, $\kappa^{\varphi}_{E_\tau}(x_0) \ge \kappa^{\varphi}_{F_\tau}(x_0)$ and by assumption $E \subset F$, $\mathrm{sd}_E(x_0) \ge \mathrm{sd}_F(x_0)$. Therefore, combining (B.2) and (B.1) we get

$$f\Big(\tfrac{\mathrm{sd}_F(x_0)}{\tau}\Big) > -\kappa^{\varphi}_{F_\tau}(x_0) - g(x_0) \ge -\kappa^{\varphi}_{E_\tau}(x_0) - g(x_0) = f\Big(\tfrac{\mathrm{sd}_E(x_0)}{\tau}\Big),$$

a contradiction. □

Comparison principles for (1.13) are well-established provided that the prescribed mean curvatures are comparable.

Theorem B.2 (**Comparison principles**). *Let f_1, f_2, g_1, g_2 satisfy (Ha) and (Hc). Let $F_1, F_2 \in \mathcal{S}^*$ and $\tau > 0$. The following properties hold:*

(i) *If $F_1 \subset F_2$, $f_1 \ge f_2$ and $g_1 > g_2$ a.e. in $\mathbb{R}^n$, then minimizers F^i_τ of $\mathcal{F}_{\varphi,f_i,g_i}(\cdot; F_i, \tau)$ satisfy $F^1_\tau \subset F^2_\tau$.*
(ii) *If $F_1 \Subset F_2$, $f_1 \ge f_2$ and $g_1 \ge g_2$ a.e. in $\mathbb{R}^n$, then minimizers F^i_τ of $\mathcal{F}_{\varphi,f_i,g_i}(\cdot; F_i, \tau)$ satisfy $F^1_\tau \subset F^2_\tau$.*
(iii) *If $F_1 \subset F_2$, $f_1 \ge f_2$ and $g_1 \ge g_2$ a.e. in $\mathbb{R}^n$, then there exist minimizers $F^1_{\tau*}$ of $\mathcal{F}_{\varphi,f_1,g_1}(\cdot; F_1, \tau)$ and F^{2*}_τ of $\mathcal{F}_{\varphi,f_2,g_2}(\cdot; F_2, \tau)$ such that $F^1_{\tau*} \subset F^2_\tau$ and $F^1_\tau \subset F^{2*}_\tau$ for all minimizers F^i_τ of $\mathcal{F}_{\varphi,f_i,g_i}(\cdot; F_i, \tau)$.*

Indeed, assumptions of (i) and (ii) imply that $h_{\varphi,f_1,g_1} > h_{\varphi,f_2,g_2}$ a.e. in $\mathbb{R}^n$, while (iii) implies $h_{\varphi,f_1,g_1} \ge h_{\varphi,f_2,g_2}$, see (1.14). Thus the proof follows from standard arguments (see e.g. [9] and references therein).

Corollary B.3 (**Minimal and maximal minimizers**). *Let f satisfy (Ha) and g satisfy (Hc), and let $\tau > 0$. Then for any $F \in \mathcal{S}^*$ there exist minimizers $F_{\tau*}, F^*_\tau$ (called the minimal and maximal minimizer) of $\mathcal{F}(\cdot; F, \tau)$ such that for every minimizer F_τ,*

$$F_{\tau*} \subseteq F_\tau \subseteq F^*_\tau.$$

Corollary B.4. *Assume that $g \equiv 0$, $F_1 \Subset F_2$ with $\mathrm{dist}(\partial F_1, \partial F_2) = \epsilon > 0$. Then the minimizers F^i_τ of $\mathcal{F}(\cdot; F_i, \tau)$ satisfy $F^1_\tau \Subset F^2_\tau$ and $\mathrm{dist}(\partial F^1_\tau, \partial F^2_\tau) \ge \epsilon$.*

Indeed, since $g \equiv 0$, $\mathcal{F}(E; F, \tau) = \mathcal{F}(E + \xi; F + \xi, \tau)$ for any $\xi \in \mathbb{R}^n$. By assumptions on F_1 and F_2, for any $\xi \in B_\epsilon(0)$ we have $F_1 + \xi \Subset F_2$, and hence $F^1_\tau + \xi \Subset F^2_\tau$ by Theorem B.2 (b). Thus, $\mathrm{dist}(\partial F^1_\tau, \partial F^2_\tau) \ge \epsilon$.

As in [9] we can introduce a comparison principle for two GMMs.

Theorem B.5. *Let f_1, f_2, g_1, g_2 satisfy Hypothesis* (H) *with $f_1 \ge f_2$ and $g_1 \ge g_2$. Let $F^0_1, F^0_2 \in \mathcal{S}^*$ be such that $F^0_1 \subset F^0_2$. Then:*

(a) *for any $F_1(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\varphi,f_1,g_1}, \mathcal{S}^*, F^0_1)$ there exists $F^*_2(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\varphi,f_2,g_2}, \mathcal{S}^*, F^0_2)$ such that*
$$F_1(t) \subset F^*_2(t), \quad t \ge 0;$$
(b) *for any $F_2(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\varphi,f_2,g_2}, \mathcal{S}^*, F^0_2)$ there exists $F_{1*}(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\varphi,f_1,g_1}, \mathcal{S}^*, F^0_1)$ such that*
$$F_{1*}(t) \subset F_2(t), \quad t \ge 0.$$

We sketch the proof of only (a). Let $\{F_1(\tau_j,k)\}$ be flat flows starting from F_1^0 such that

$$\lim_{j\to+\infty} |F_1(\tau_j,\lfloor t/\tau_j\rfloor)\Delta F_1(t)| = 0.$$

Let also $F_2^*(\tau_j,k)$ be the flat flows starting from F_2^0 consisting of the maximal minimizers of $\mathcal{F}_{\varphi,f_2,g_2}$. By Theorem B.2 (d)

$$F_1(\tau_j,k) \subset F_2^*(\tau_j,k), \quad k\ge 0,\ j\ge 1. \tag{B.3}$$

Now consider the sequence $(F_2^*(\tau_j,\lfloor t/\tau_j\rfloor))$. In the proof of Theorem 1.4 we have constructed a not relabelled subsequence and a family $F_2^*(\cdot)\in \mathrm{GMM}(\mathcal{F}_{\varphi,f_2,g_2},\mathcal{S}^*,F_2^0)$ such that

$$\lim_{j\to+\infty} |F_2^*(\tau_j,\lfloor t/\tau_j\rfloor)\Delta F_2^*(t)| = 0 \qquad \forall t\ge 0$$

(see also Remark 3.3). Now the inclusion $F_1(\cdot)\subset F_2^*(\cdot)$ follows from (B.3).

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