POSITIVE MASS AND ISOPERIMETRY FOR CONTINUOUS METRICS WITH NONNEGATIVE SCALAR CURVATURE

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Abstract. This paper deals with the positive mass theorem and the existence of isoperimetric sets on 3-manifolds endowed with continuous complete metrics having nonnegative scalar curvature in a suitable weak sense.

We prove that if the manifold has an end that is C^0-locally asymptotically flat, and the metric is the local uniform limit of smooth metrics with vanishing lower bounds on the scalar curvature outside a compact set, then Huisken's isoperimetric mass is nonnegative. This addresses a version of a recent conjecture of Huisken about positive isoperimetric mass theorems for continuous metrics satisfying $R_g \geq 0$ in a weak sense. As a consequence, any fill-in of a truncation of a Schwarzschild space with negative ADM mass has nonnegative isoperimetric mass.

Moreover, in case the whole manifold is C^0-locally asymptotically flat and the metric is the local uniform limit of smooth metrics with vanishing lower bounds on the scalar curvature outside a compact set, we prove that, as a large scale effect, isoperimetric sets with arbitrarily large volume exist.

Contents

1. Introduction

1.1. **Statement of the result.** The classical positive mass theorem (PMT) in 3 dimensions, originally due to Schoen-Yau [77], asserts that the ADM mass of a 3-dimensional smooth complete asymptotically flat manifold with nonnegative scalar curvature is nonnegative. This seminal result has been later rediscovered, generalized, improved and exploited in a number of ways. Without

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any attempt to be complete, we refer to [83, 78, 1, 16] for different proofs and extensions to higher dimensions, and to [45, 15] for its main refinement, i.e., the Riemannian Penrose inequality.

In this paper we prove an extension of the PMT for continuous metrics with weakly nonnegative scalar curvature on 3-manifolds. First, we introduce the notions of *mass* and *nonnegative scalar curvature* we are going to adopt. We say that (M, g) is a C^0*-Riemannian manifold* if M is a smooth differentiable manifold, and g is a C^0-metric on M. For a Borel set $E \subset M$, let $|E|, P(E)$ denote the volume, and perimeter of E, see Section 2.2 for precise definitions. The following notion has been introduced by Huisken [43].

Definition 1.1 (Isoperimetric mass, Huisken [43])**.** Let (M, g) be a C^0-Riemannian 3-manifold, possibly with boundary. Then, its *isoperimetric mass* is

$$\begin{aligned} \mathfrak{m}_{\mathrm{iso}} := \sup \Bigg\{ \limsup_{j\to+\infty} \frac{2}{P(\Omega_j)} \left(|\Omega_j| - \frac{P(\Omega_j)^{3/2}}{6\sqrt{\pi}} \right) \ &: \ \Omega_j \subset M, P(\Omega_j) < +\infty \ \forall j, \\ &P(\Omega_j) \to +\infty \Bigg\}. \end{aligned} \tag{1.1}$$

For asymptotically flat 3-manifolds with nonnegative scalar curvature, the isoperimetric mass coincides with the ADM mass whenever the boundary is minimal: see, e.g., [47, Theorem 3], [26, Theorem C.2]. In [12, Theorem 4.13] such an equality is proved in the $C^{1,1/2+\varepsilon}$-asymptotically flat regime, which is the sharp one to show that the ADM mass is independent of the chart at infinity [9, 27]. The fact that the ADM mass can be detected through the isoperimetric deficit of large coordinate spheres was first observed in the Schwarzschild space in [14], and in asymptotically flat manifolds in [33, Corollary 2.3].

Huisken has conjectured [44] that a weak (isoperimetric) PMT should hold true for continuous Riemannian metrics if one interprets the notion of nonnegative scalar curvature (shortly, $R_g \geq 0$) in an appropriate weak sense, see Section 1.3.1 for more details. In this paper we prove a version of this conjecture under the following weak notion of nonnegative scalar curvature.

Definition 1.2 ($R_g \geq 0$ in the approximate sense)**.** Let (M, g) be a complete C^0-Riemannian manifold without boundary, and let $\Omega \subset M$ be an open set. We say that $R_g \geq 0$ *in the approximate sense on* Ω if there exist smooth complete Riemannian metrics g_j on M, such that:

(1) g_j converges to g locally uniformly on M;
(2) There exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ of positive numbers such that $\varepsilon_j \to 0$, and $R_{g_j} \geq -\varepsilon_j$ on Ω.

By a result of Gromov, recalled below in Theorem 1.7, a smooth Riemannian manifold has nonnegative scalar curvature if and only if $R_g \geq 0$ holds in the approximate sense on M. Before stating the main theorems of this paper, we make clear what it means that an end of a C^0-manifold is C^0_{loc}-asymptotic to $\mathbb{R}^n$.

Definition 1.3. Let $K \subset M$ be a compact set (possibly empty) of a C^0-Riemannian manifold (M, g). We say that an unbounded connected component Ω of $M \setminus K$ is C^0_{loc}*-asymptotic to a* C^0*-Riemannian manifold* (N, h) if the following holds. For every diverging sequence $\Omega \ni p_i \to +\infty$ there exists a point $o \in N$ such that $(M, g_i, p_i) \to (N, h, o)$ in the C^0-sense, see Definition 2.3.

Theorem 1.4. *Let (M, g) be a complete 3-dimensional C^0-Riemannian manifold without boundary. Let $K \subset M$ be a compact set, and let Ω be an unbounded connected component $M \setminus K$. Assume that Ω is C^0_{loc}-asymptotic to $\mathbb{R}^3$, see Definition 1.3, and that $R_g \geq 0$ in the approximate sense on $\Omega \setminus K'$, see Definition 1.2, where $K' \subset M$ is a compact set.*

Then $\mathfrak{m}_{\mathrm{iso}} \geq 0$.

The PMT for the isoperimetric mass in the C^0_{loc}-asymptotically flat case with nonnegative scalar curvature only outside a compact set in a given end, is, to our knowledge, a new result even in the smooth setting. In fact, one can immediately draw out of Theorem 1.4 the following surprising consequence.

Remark 1.5. Any fill-in of the truncated spacelike slice of a Schwarzschild spacetime of *negative* ADM mass has *nonnegative* isoperimetric mass. Namely, any complete fill-in of the noncompact, scalar-flat, asymptotically flat manifold with boundary $[s_0, +\infty) \times \mathbb{S}^2$, for $s_0 > 0$, endowed with the metric

$$g = \frac{1}{1 - \mathfrak{m}s^{-1}} \,\mathrm{d}s \otimes \,\mathrm{d}s + s^2 g_{\mathbb{S}^2},$$

with $\mathfrak{m} < 0$, enjoys $\mathfrak{m}_{\mathrm{iso}} \geq 0$. For such manifolds, we presently do not know neither the shape of isoperimetric regions nor the exact value of $\mathfrak{m}_{\mathrm{iso}}$.

The observation above shows that without some additional assumption on the boundary, like minimality (see [12, Theorem 4.13]), the identification between the two notions of mass in general fails. Moreover, this is coherent with the cross-sections being unstable constant mean curvature surfaces, a property showed in [29, Section 5], where the question about the shape of isoperimetric regions is also raised.

Theorem 1.4 applies also to manifolds possibly having more than one end, for which there is a distinguished end that has $R_g \geq 0$ (outside a compact set), and that is C^0_{loc}-asymptotic to $\mathbb{R}^3$. Positive mass theorems on manifolds with arbitrary ends, except for a *globally* asymptotically flat one, have been recently considered e.g. in [57, 55, 25, 28], see also references therein. Our distinguished end might not be globally asymptotically flat: for example, our result applies to asymptotically conical ends, and in particular to ALE ends.

On smooth 3-dimensional asymptotically flat manifolds the condition $R_g \geq 0$, which governs the *local* isoperimetric structure around each point, has also a *global effect* in the existence of isoperimetric sets for arbitrarily large volumes (see, e.g., [32, Theorem 1.2], [23, Proposition K.1], [26, Theorem 1.1], and references therein). This can be understood as an interpolation effect between $R_g \geq 0$ and the positivity of the mass at infinity. The next result confirms this heuristic idea also in the continuous setting.

Theorem 1.6. *Let (M, g) be a complete 3-dimensional C^0-Riemannian manifold without boundary, and assume that $R_g \geq 0$ in the approximate sense on $M \setminus \mathcal{C}$, where $\mathcal{C}$ is a compact set, see Definition 1.2. Assume in addition that M is C^0_{loc}-asymptotic to $\mathbb{R}^3$, see Definition 1.3.*

Then there exists a sequence of isoperimetric sets $(E_j)_{j\in\mathbb{N}}$ on M such that $|E_j| \to +\infty$.

Very much like Theorem 1.4, even in the smooth setting Theorem 1.6 strongly weakens the asymptotic assumptions for the existence of isoperimetric sets. The literature on the subject is very vast, in particular in relation to the study of canonical foliations of stable CMC: we refer to the seminal [46] and to [31, 32, 70, 26, 85], as well as to the references therein, for a fairly complete picture.

In the setting of Theorem 1.6 it is an interesting open problem to analyze existence of isoperimetric sets for *any* (large) volume, uniqueness or foliation properties of such isoperimetric sets, as in [70, 26, 85] (see also the survey [10]). Under the additional assumption that the isoperimetric profile I is strictly increasing, we can actually prove existence of isoperimetric sets for every volume, see Proposition 4.3.

1.2. **Strategy.** We briefly discuss the strategy of the proof of Theorem 1.4, and Theorem 1.6, referring the reader to Section 3, and Section 4 for more details.

The starting point in the proof of Theorem 1.4 is the following consequence of a result due to Shi [79], after important insights by Brendle–Chodosh [18]: in a smooth 3-dimensional asymptotically flat manifold with nonnegative scalar curvature the level sets of the weak inverse mean curvature flow (shortly, IMCF, see Definition 3.1) issuing from a point satisfy a reverse Euclidean isoperimetric inequality with the sharp constant, as long as their boundaries are connected. Hence the isoperimetric deficit appearing on the right hand side of (1.1) is nonnegative when computed on such sets, and this directly implies the nonnegativity of the isoperimetric mass by letting the level sets of the IMCF exhaust the manifold.

In fact, we will push this heuristic to give an isoperimetric PMT for metrics that are only continuous. The crux of the argument is to show that on the approximating manifolds (M, g_j) one can define a well-behaved weak IMCF w_j on arbitrarily large punctured balls. This is done by taking scaled limits of the logarithms of p-Green functions on punctured balls, for $p \to 1^+$, as pioneered by R. Moser [66]. In fact, we build on the sharp gradient estimate obtained in [51], and on the Harnack inequality with explicit constants in [74], to get a weak IMCF w_j on punctured balls B_j in (M, g_j) that is bounded below explicitly with constants that stay bounded when B_j is C^0-close enough to a Euclidean ball. This is the content of the new quantitative existence result for local IMCFs in Theorem 3.6. The latter allows to pass to the limit the level sets of these IMCFs, obtaining sets satisfying the (sharp) reverse Euclidean isoperimetric inequality with arbitrarily large perimeters, see Proposition 4.2.

The proof of the existence of isoperimetric sets in Theorem 1.6 is not constructive, and it is based on a novel contradiction argument. One notices that if after a certain volume threshold isoperimetric sets do not exist, then the isoperimetric profile is strictly increasing for large volumes: this is a consequence of a generalized existence theorem for the isoperimetric problem, see Theorem 2.16. In addition, arguing as in the previous paragraph, we construct sets that satisfy the reverse sharp Euclidean isoperimetric inequality with arbitrarily large volumes and perimeters, and that avoid any compact set. This is enough to show, again using Theorem 2.16 and that the isoperimetric profile is increasing, that isoperimetric sets with arbitrarily large volumes must exist, thus resulting in a contradiction.

1.3. **Comments and comparison with related literature.**

We collect here some comments and perspectives on the main results of this paper.

1.3.1. *Other notions of weak scalar curvature bounds, and relations with the works [19, 20].* Several notions of scalar curvature lower bounds for smooth manifolds endowed with C^0-Riemannian metrics have been proposed in the recent years. In [37] Gromov proposes a definition based on nonexistence of suitable small polyhedra on the manifold, see the recent works [58, 17] motivated by this study; a definition by using a regularization through Ricci flow has been suggested by Burkhardt-Guim in [19]; and a definition using volumes of small balls has been hinted by Huisken [44]. Let us compare our result in Theorem 1.4 with the framework in [19, 20].

Assume that a complete C^0-Riemannian manifold (M, g) is *globally* C^0-asymptotic to $\mathbb{R}^3$ outside a compact set $K \subset M$, see Definition 1.8. Hence, for $\beta \in (0, 1/2)$, we claim that if $R_g \geq 0$ in the β-weak sense [20, Definition 2.3] on $M \setminus K$, then $R_g \geq 0$ in the approximate sense on $M \setminus K$, up to possibly enlarging K. Indeed, this is due to the fact that under the C^0-asymptotic hypothesis at infinity one can first define a Ricci-deTurck flow (M, g_t) starting from g which C^0-converges to (M, g) locally uniformly by using [81, Theorem 1.1]. Then, since $R_g \geq 0$ in the β-weak sense on $M \setminus K$, [20, Lemma 5.1] implies that, up to possibly enlarging K, $R_{g_t} \geq -o(1)$ on $M \setminus K$ as $t \to 0$, which in turn implies that $R_g \geq 0$ in the approximate sense on $M \setminus K$ by definition. In the case M is a compact manifold, the previous reasoning has been explicitly recorded in [19, Corollary 1.6]. Related to this, it would be interesting to understand whether a non-compact C^0-manifold (M, g) that has $R_g \geq 0$ in the approximate sense on M admits a smooth metric $\tilde{g}$ with $R_{\tilde{g}} \geq 0$. The compact case has been settled in [19, Corollary 1.8]. Taking into account the discussion above, this seems likely to hold when (M, g) is, in addition, C^0-asymptotic to $\mathbb{R}^3$ (Definition 1.8).

As a consequence of the above discussion, under the stronger assumption that (M, g) is C^0-asymptotic to $\mathbb{R}^3$ (Definition 1.8), one gets that $\mathfrak{m}_{\rm iso} \geq 0$ provided that $R_g \geq 0$ in the β-weak sense outside a compact set, for some $\beta \in (0, 1/2)$ ([20, Definition 2.3]). It is yet to be understood whether, in such setting, the isoperimetric mass coincides with the (weak notion of) ADM mass defined in the work [20], see [20, Question 1]. If the latter holds, then Theorem 1.4 would answer the question about the nonnegativity of such mass [20, Question 2] in the positive.

As a side note, we point out that as of today it is not known whether the definitions in [37] and [19] are equivalent, compare with the discussion [19, Page 1707-1708], nor any relation with Huisken's notion in [44] has been investigated yet.

We conclude this section by mentioning a result according to which the notion of $R_g \geq 0$ in the approximate sense is consistent with $R_g \geq 0$ for smooth metrics. In fact, as a consequence of a result of Gromov [37, page 1118] (see [8] for a proof using the Ricci flow) the following holds.

Theorem 1.7 ([37, 8])**.** *Let M be a smooth manifold and $\kappa : M \to \mathbb{R}$ be a continuous function on M. Suppose g_i is a sequence of C^2-metrics on M that converges locally uniformly to a C^2-metric g on M. If $R_{g_i} \geq \kappa$ on M, then $R_g \geq \kappa$ everywhere on M as well.*

1.3.2. *Comparison with the literature related to weak PMT.* Weak positive mass theorems (PMT) have been thoroughly studied in the last two decades. We briefly discuss some contributions without the aim of being exhaustive, and refer the interested reader to the references therein.

In [64, Theorem 1] the author proves a PMT for asymptotically flat Lipschitz metrics that are smooth away from a compact hypersurface, and such that $R_g \geq 0$ in an appropriate weak sense (i.e., a condition on the exterior/interior mean curvatures of the hypersurface is required). See also [63, Theorem 1] for a proof of the result in [64, Theorem 1] using Ricci flow. Then, in [80, Theorem 1.3], the authors prove a PMT for n-dimensional manifolds that carry an asymptotically flat $C^0 \cap W^{1,n}$ metric that is smooth away from a compact set Σ of codimension at least 2, and for which $R_g \geq 0$ outside Σ. See also [48, Theorem 1.1] for an improvement of [80, Theorem 1.3], where a weaker condition on the codimension of Σ (depending on the integrability of the derivatives of the metric) is asked. Similar results, but with global L^∞ metrics which satisfy appropriate conditions close to Σ, are in [59, Theorem 1.8, Theorem 1.9].

In all of the papers discussed above, the manifolds considered are smooth and with nonnegative scalar curvature outside a compact set, and thus, in dimension 3, Theorem 1.4 implies they enjoy nonnegative isoperimetric mass. Finally, in [54, Theorem 1.1] the authors prove a PMT for a carefully generalized ADM mass in the setting of $C^0 \cap W^{1,n}$ metrics with weak decays at infinity, and for which a notion of distributional scalar curvature is nonnegative. By means of a mollification procedure, see, e.g., [48, Lemma 2.6], these metrics have nonnegative scalar curvature in the approximate sense. Moreover, their asymptotic constraints imply C^0-asymptotic flatness. In particular, Theorem 1.4 holds in this case as well. To our knowledge, it is not known whether the isoperimetric mass and the ADM mass agree in the setting of the results described above.

1.3.3. *Another notion of isoperimetric mass.* On a C^0-Riemannian 3-manifold (M, g), it is also common to consider another notion of isoperimetric mass given in terms of exhaustions [47, 26, 12], namely

$$\tilde{\mathfrak{m}}_{\text{iso}} := \sup \left\{ \limsup_{j \to +\infty} \frac{2}{P(\Omega_j)} \left(|\Omega_j| - \frac{P(\Omega_j)^{3/2}}{6\sqrt{\pi}} \right) \ : \ (\Omega_j)_{j \in \mathbb{N}} \text{ is an exhaustion of } M, \right.$$
$$\left. P(\Omega_j) < +\infty \ \forall\, j \right\}.$$

As a by-product of the proof in [47, Proposition 37] one gets that if (M, g) satisfies a global Euclidean-like isoperimetric inequality, i.e., $P(E) \geq C|E|^{2/3}$ for some $C > 0$ for any $|E| < +\infty$, then $\tilde{\mathfrak{m}}_{\text{iso}} = \mathfrak{m}_{\text{iso}}$. This holds, for instance, when M is globally C^0-asymptotic to $\mathbb{R}^3$, see Definition 1.8 for the precise definition. In particular, Theorem 1.4 implies that on 3-dimensional C^0-Riemannian manifolds that are C^0-asymptotic to $\mathbb{R}^3$ (Definition 1.8) and that satisfy $R_g \geq 0$ in the approximate sense out of a compact set, there holds $\tilde{\mathfrak{m}}_{\text{iso}} \geq 0$.

1.4. **The question of rigidity.** In the smooth PMT [77], one gets that the mass is zero if and only the metric is flat. In this paper, we do not address the issue of rigidity in Theorem 1.4. We pose it as a problem, which is likely to require the use of new techniques tailored for the C^0-setting, or some refined approximation procedure. We state it in the most basic asymptotically flat case.

Definition 1.8 (Manifold C^0-asymptotic to $\mathbb{R}^n$)**.** We say that an n-dimensional C^0-Riemannian manifold (M, g) is C^0*-asymptotic to* $\mathbb{R}^n$ if there exists a compact set $K \subset M$ such that the following holds. There exists $R > 0$, and a diffeomorphism $\Phi : \mathbb{R}^n \setminus \overline{B}^{\mathbb{R}^n}_R(0) \to M \setminus K$, such that

$$|(\Phi^* g)_{ij} - \delta_{ij}|_x = o(1) \quad \text{as } |x| \to +\infty.$$

Question 1.9. *Let (M, g) be a complete C^0-manifold of dimension 3 without boundary. Assume that M has $R_g \geq 0$ in the approximate sense everywhere, and that M is C^0-asymptotic to $\mathbb{R}^3$. If $\mathfrak{m}_{\mathrm{iso}} = 0$, is it true that (M, g) is isometric to $(\mathbb{R}^3, g_{\mathrm{eu}})$?*

Notice one cannot expect rigidity in the general formulation of Theorem 1.4, not even in the smooth case: indeed, it can be showed by a direct computation that any metric that is flat outside a compact set in $\mathbb{R}^3$ has $\mathfrak{m}_{\mathrm{iso}} = 0$. We finally point out that Question 1.9 may be related to the study of the stability of the PMT with respect to nonsmooth notions of convergence. Stability results in this direction have been proved, for instance, in [30] with respect to pointed measure Gromov–Hausdorff convergence, in [56] for rotationally symmetric metrics, and in [41, 42] for graphs with respect to (intrinsic) convergence in the sense of currents (see [82] and Conjecture 10.1 therein for more details on the problem).

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2. Preliminaries

We gather in this section a number of fundamental properties of C^0-Riemannian manifolds, possibly endowed with C^0-asymptotics.

2.1. Basic definitions and properties of C^0-metrics.

Definition 2.1 (C^0-Riemannian manifold)**.** A C^0*-Riemannian manifold* is a couple (M, g), where M is a smooth n-dimensional differentiable manifold (possibly with boundary) and g is a continuous metric tensor. More precisely, in any local chart (U, φ) on M, the metric tensor in local coordinates is represented by a symmetric positive definite matrix with components $g_{ij} \in C^0(\varphi(U))$.

We briefly recall some metric properties of C^0-Riemannian manifolds (M, g), and we refer the reader to [21, 72] for a more detailed discussion. As in the smooth case, the continuous Riemannian metric g defines a length structure on absolutely continuous curves on M, which in turn gives raise to a distance d that induces the manifold topology. We always assume that (M, d) is a complete metric space, so that by Hopf–Rinow Theorem the distance d is geodesic, and every closed ball is compact.

The volume form induced by g in a local chart induces integration with respect to $\mathrm{vol} := \sqrt{\det g}\,\mathcal{L}^n$, where $\mathcal{L}^n$ denotes Lebesgue measure. It can be proved that $\mathrm{vol} = \mathcal{H}^n$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure relative to d, and that $\mathcal{H}^n$ is a Radon measure. Hence $(M, \mathsf{d}, \mathcal{H}^n)$ is a complete and separable metric measure space. We will often denote $|E| := \mathcal{H}^n(E)$. In the following, if E is a measurable set, integrals over E are tacitly understood to be taken with respect to $\mathcal{H}^n$.

Given $f \in C^\infty(M)$, one can define ∇f as in the smooth case by setting $g(\nabla f, X) = \mathrm{d}f(X)$ for every vector field X on M. Hence, in a local chart $\{\partial_i\}_{i=1,\dots,n}$, we have $|\nabla f|^2 := g(\nabla f, \nabla f) = g^{ij}\partial_i f \partial_j f$. Notice that by the classical Rademacher Theorem applied in chart, if $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$, then ∇f exists $\mathcal{H}^n$-almost everywhere.

From now on we will always assume that C^0-Riemannian manifolds are complete. We recall that a map between metric spaces $F : (X, \mathsf{d}_X) \to (Y, \mathsf{d}_Y)$ is said to be *L-biLipschitz*, with $L \geq 1$, when $L^{-1}\mathsf{d}_X(a, b) \leq \mathsf{d}_Y(F(a), F(b)) \leq L\mathsf{d}_X(a, b)$ for every $a, b \in X$.

Lemma 2.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ such that the following holds. Let (M, g) be a C^0-Riemannian manifold and let $o \in M$ and $R > 0$. Denote by d the Riemannian distance on M. Let N be a differentiable manifold and let $\Omega \subset N$ be an open set. Suppose that there exists a diffeomorphism $F : B_{10R}(o) \to \Omega$ and a C^0-Riemannian metric h on Ω such that $|(F^*h - g)(v, v)| \leq \varepsilon g(v, v)$ for any $v \in T_xM$ and any $x \in B_{10R}(o)$.*

Then, letting

$$\tilde{\mathsf{d}}(x, y) := \inf\left\{\int_0^1 |\gamma'|_h \ : \ \gamma : [0, 1] \to \Omega, \gamma(0) = x, \gamma(1) = y\right\},$$

the map $F|_{B_R(o)} : (B_R(o), \mathsf{d}) \to (F(B_R(o)), \tilde{\mathsf{d}})$ is $(1 + \delta)$-biLipschitz with its image.

Proof. Denote $o' := F(o)$. Let $x, y \in B_R(o)$. A constant speed geodesic $\gamma : [0, 1] \to M$ from x to y has image contained in $B_{5R}(o)$. Hence we can estimate

$$\tilde{\mathsf{d}}(F(x), F(y)) \leq \int_0^1 h\left((F\circ\gamma)', (F\circ\gamma)'\right)^{\frac{1}{2}}\,\mathrm{d}t \leq \sqrt{1+\varepsilon}\int_0^1 g(\gamma', \gamma')^{\frac{1}{2}}\,\mathrm{d}t \leq \sqrt{1+\varepsilon}\,\mathsf{d}(x, y).$$

Denoting $G = F^{-1}$, by assumptions we have that

$$|G^*g(w, w) - h(w, w)| \leq \varepsilon G^*g(w, w),$$

for any $w \in T_zN$ and $z \in \Omega$. Taking now $p, q \in F(B_R(o))$ and a constant speed curve $\sigma : [0, 1] \to \Omega$ from p to q such that $\int_0^1 |\sigma'|_h \leq (1 + \eta)\tilde{\mathsf{d}}(p, q)$ for $\eta \in (0, 1)$, we can similarly estimate

$$\mathsf{d}(G(p), G(q)) \leq \int_0^1 g((G\circ\sigma)', (G\circ\sigma)')^{\frac{1}{2}}\,\mathrm{d}t \leq \frac{1}{\sqrt{1-\varepsilon}}\int_0^1 h(\sigma', \sigma')^{\frac{1}{2}}\,\mathrm{d}t \leq \frac{1+\eta}{\sqrt{1-\varepsilon}}\tilde{\mathsf{d}}(p, q).$$

Sending $\eta \to 0$, for ε small enough the claim follows. □

Definition 2.3 (C^0-convergence)**.** We say that pointed C^0-Riemannian manifolds (M_i, g_i, p_i) *C^0-converge to* (M, g, p) if the following holds. For every $R, \varepsilon > 0$ there exist $i_0 := i_0(R, \varepsilon)$ and an open set $\Omega \subset M$ such that we have:

(1) $\overline{B}_R(p) \subset \Omega$;
(2) for every $i \geq i_0$ there exists an embedding $F_i : \Omega \to M_i$ such that
   - $F_i(p) = p_i$;
   - $\overline{B}_R(p_i) \subset F_i(\Omega)$;
   - $|(F_i^*g_i - g)_x(v, v)| \leq \varepsilon g_x(v, v)$ for every $x \in \overline{B}_R(p)$ and $v \in T_xM$.

Remark 2.4. In the notation of Definition 2.3, arguing as in the proof of Lemma 2.5, it follows that for any $R, \varepsilon > 0$ the map $F_i : (B_R(p), \mathsf{d}) \to (F_i(B_R(p)), \mathsf{d}_i)$ is $(1+\varepsilon)$-biLipschitz with its image for any i large enough.

Lemma 2.5. *Let (M, g) be a C^0-Riemannian manifold, let $x_0 \in M \setminus \partial M$, and denote by d the Riemannian distance on M. Then the following holds.*

- *The metric tangent space of M at x_0 is isometric to the Euclidean space $\mathbb{R}^n$, i.e., the rescalings $(M, \delta^{-2}g, x_0)$ converges to $(\mathbb{R}^n, g_{\mathrm{eu}}, 0)$ in C^0-sense as $\delta \searrow 0$.*
- *For any $\delta > 0$ there is $r = r(x_0, \delta) > 0$ and a local chart $\varphi : (B_r(x_0), \mathsf{d}) \to (\varphi(B_r(x_0)), \mathsf{d}_{\mathrm{eu}}) \subset \mathbb{R}^n$ such that φ is $(1+\delta)$-biLipschitz with its image, where d_{eu} denotes Euclidean distance.*

Proof. Passing in local coordinates we can identify a neighborhood of x_0 in M with (Ω, g), where $\Omega \subset \mathbb{R}^n$ is open and $g = (g_{ij})$ is the metric in local coordinates. Also we can identify x_0 with the origin $0 \in \Omega$. Let $r_k \searrow 0$. For the first part of the statement, it is sufficient to prove that $(\Omega, r_k^{-2}g, 0)$ C^0_{loc}-converges to $(\mathbb{R}^n, g_0, 0)$, where g_0 is the constant metric given by g evaluated at the origin. The diffeomorphism $\Phi_k : (r_k^{-1}\Omega, g_0) \to (\Omega, r_k^{-2}g)$ given by $\Phi_k(x) = r_k x$ satisfies $(\Phi_k^*(r_k^{-2}g))_x(v,v) = g_{r_k x}(v,v)$ for any tangent vector v. Hence, given any $R > 0$ and $\varepsilon > 0$, for k large enough we have that

$$1 - \varepsilon \le \frac{(\Phi_k^*(r_k^{-2}g))_x(v,v)}{g_0(v,v)} \le 1 + \varepsilon,$$

for any $x \in B_R^{g_0}(0) \subset (r_k^{-1}\Omega, g_0)$ and any tangent vector $v \neq 0$. This establishes the convergence in C^0 to $\mathbb{R}^n$ as $k \to \infty$.

Concerning the second part of the statement, let $r > 0$ to be chosen small and fix a local chart $\varphi : B_{10r}(x_0) \to \varphi(B_{10r}(x_0)) =: \Omega \subset \mathbb{R}^n$ such that $\varphi(x_0) = 0$, and denote $g_0 := g_{\mathrm{eu}}$. We identify $B_{10r}(x_0)$ with (Ω, g) in the local chart. Let $\tilde{\mathsf{d}}$ be as in Lemma 2.2 with $N = \mathbb{R}^n$. By continuity of the metric tensor, for any $\varepsilon > 0$ we can take r so small that $|g_x(v,v) - g_0(v,v)| \le \varepsilon g_0(v,v)$ for any $x \in \Omega$ and any $v \in \mathbb{R}^n$. For ε small enough, it follows from Lemma 2.2 that the identity $\mathrm{id} : (B_r^g(0), \mathsf{d}) \to (B_r^g(0), \tilde{\mathsf{d}})$ is $(1+\delta)$-biLipschitz. It remains to observe that $\tilde{\mathsf{d}} = \mathsf{d}_{\mathrm{eu}}$ on $B_r^g(0)$. Indeed, fix $p, q \in B_r^g(0)$ and let $\gamma_i : [0,1] \to \Omega$ be a sequence of constant speed curves such that $\int_0^1 |\gamma_i'|_{\mathrm{eu}} \to \tilde{\mathsf{d}}(p,q)$. It follows that $\gamma_i([0,1]) \subset B_{9r}^g(0)$ for large i. Indeed, taking $\sigma : [0,1] \to B_{5r}^g(0)$ a constant speed geodesic for d from p to q, we know that

$$\tilde{\mathsf{d}}(p,q) \le \int_0^1 |\sigma'|_{\mathrm{eu}} \le \frac{1}{\sqrt{1-\varepsilon}} \int_0^1 |\sigma'|_g = \frac{1}{\sqrt{1-\varepsilon}} \mathsf{d}(p,q) \le 2r \frac{1}{\sqrt{1-\varepsilon}}.$$

Therefore, if $\gamma_i([0,1]) \not\subset B_{9r}^g(0)$ for some i, we would get

$$\int_0^1 |\gamma_i'|_{\mathrm{eu}} \ge \frac{1}{\sqrt{1+\varepsilon}} \int_0^1 |\gamma_i'|_g \ge \frac{1}{\sqrt{1+\varepsilon}} 16r,$$

leading to a contradiction for large i, provided ε is small enough. Hence we can pass to the limit γ_i to a curve $\gamma : [0,1] \to \overline{B}_{9r}^g(0)$ such that $\tilde{\mathsf{d}}(p,q) = \int_0^1 |\gamma'|_{\mathrm{eu}}$. Hence γ is a critical point for the length functional on $(B_{10r}^g(0), g_{\mathrm{eu}})$. Then γ is a straight segment from p to q and in particular $\tilde{\mathsf{d}}(p,q) = \mathsf{d}_{\mathrm{eu}}(p,q)$. □

2.2. **BV functions and sets of finite perimeter.** Let (M, g) be a complete C^0-Riemannian manifold of dimension n. Then we can consider the complete and separable metric measure space $(M, \mathsf{d}, \mathcal{H}^n)$. Following [65, 3], we define the *total variation* $|Df|(B) \in [0, +\infty]$ of a function $f \in L^1_{\mathrm{loc}}(M)$ in a Borel set $B \subseteq M$ as

$$|Df|(B) \coloneqq \inf_{B \subseteq \Omega \text{ open}} \inf \left\{ \liminf_{n \to \infty} \int_\Omega |\nabla f_n| \;\middle|\; (f_n)_{n \in \mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ f_n \to f \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\}. \tag{2.1}$$

Remark 2.6. On a C^0-Riemannian manifold, the previous definition of total variation is equivalent to the usual one adopted on metric measure spaces. More precisely, defining the slope of a locally Lipschitz function f by

$$\operatorname{lip} f(x) := \limsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(y,x)}, \tag{2.2}$$

then plugging $\operatorname{lip} f_n$ in place of $|\nabla f_n|$ in (2.1) yields the same quantity. Indeed, if $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$, then $\operatorname{lip} f = |\nabla f|$ almost everywhere, see Lemma A.1.

If for some open cover $(\Omega_n)_{n\in\mathbb{N}}$ of M we have that $|Df|(\Omega_n) < +\infty$ holds for every $n \in \mathbb{N}$, then $|Df|$ is a *locally finite Borel measure on M*. We say that a Borel set $E \subseteq M$ is *of locally finite perimeter* if $P(E,\cdot) := |D\chi_E|$ is a locally finite measure, called the *perimeter measure* of E. When $P(E) := P(E,M) < +\infty$, we say that E is *of finite perimeter*. If $f \in L^1(M)$ satisfies $|Df|(M) < +\infty$, then we say that $f \in BV(M)$.

Given any $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$, it holds that $|Df|$ is a locally finite measure and $|Df| = \operatorname{lip} f\mathcal{H}^n = |\nabla f|\mathcal{H}^n$, see Lemma A.1. Moreover, we recall the coarea formula in our setting,

Theorem 2.7 (Coarea formula [65, Proposition 4.2])**.** *Let (M,g) be a complete C^0-Riemannian manifold. Let $f \in L^1_{\mathrm{loc}}(M)$ be such that $|Df|$ is a locally finite measure. Fix a Borel set $E \subseteq M$. Then $\mathbb{R} \ni t \mapsto P(\{f<t\},E) \in [0,+\infty]$ is a Borel measurable function and it holds that*

$$|Df|(E) = \int_{\mathbb{R}} P(\{f<t\},E)\,\mathrm{d}t.$$

Let us finally introduce the notion of isoperimetric profile.

Definition 2.8. Let (M,g) be a complete C^0-Riemannian manifold with $|M| = +\infty$. The *isoperimetric profile* function is the function $I : (0,+\infty) \to [0,+\infty]$ defined as follows

$$I(V) := \inf\{P(E) : E \subset M, \mathcal{H}^n(E) = V\}.$$

2.3. **Technical Lemmas on C^0-Riemannian manifolds.** In this section we prove several technical lemmas about C^0-Riemannian manifolds we are going to use throughout the paper. Notably, we prove: a precompactness theorem for BV functions on converging sequences of continuous Riemannian manifolds (Lemma 2.11); the fact that every continuous Riemannian manifold is PI on every ball (Lemma 2.12); continuity of the isoperimetric profile of continuous Riemannian manifolds with C^0-controlled geometry at infinity (Corollary 2.15); and a generalized existence theorem for the isoperimetric problem on continuous Riemannian manifolds with C^0-controlled geometry at infinity (Theorem 2.16).

Lemma 2.9. *Let (M,g) be a C^0-Riemannian manifold. For any $p \in M$, $r > 0$ and $\delta > 0$ there exists a Riemannian manifold (N,g^δ) with smooth metric g^δ such that the inclusion $\iota : (B_r(p),\mathsf{d}) \to (N,\mathsf{d}^\delta)$ is well defined and it is $(1+\delta)$-biLipschitz with its image, where $\mathsf{d},\mathsf{d}^\delta$ denote Riemannian distance on (M,g), (N,g^δ) respectively.*

Proof. For any $\varepsilon \in (0,1)$ let g^ε be a smooth Riemannian metric on $\overline{B}^g_{20r}(p)$ such that $|g(v,v) - g^\varepsilon(v,v)| \le \varepsilon g(v,v)$ for any $v \in T_xM$ and $x \in B_{20r}(p)$. Gluing the boundary of a smooth connected open domain D such that $\overline{B}^g_{10r}(p) \subset D \subset\subset B^g_{20r}(p)$ with a half-cylinder $[0,+\infty)\times\partial D$ and suitably extending the metric g^ε, we obtain a smooth complete Riemannian manifold (N,g^ε). Denote by d^ε the Riemannian distance on (N,g^ε) and by $\tilde{\mathsf{d}}$ the distance defined in Lemma 2.2 with $F = \iota : B^g_{10r}(p) \to B^g_{10r}(p) \subset (N,g^\varepsilon)$. Arguing as in the second part of the proof of Lemma 2.5, it follows that $\tilde{\mathsf{d}} = \mathsf{d}^\varepsilon$ on $B^g_r(p)$. Indeed, as in the proof of Lemma 2.5 we find that for any $x,y \in B^g_r(p)$ there exists a constant speed curve $\gamma : [0,1] \to B^g_{10r}(p)$ such that $\tilde{\mathsf{d}}(x,y) = \int_0^1 |\gamma'|_{g^\varepsilon}$, hence γ is a geodesic for the metric g^ε. For every $x \in B^g_r(p)$, we have that for almost every $y \in B^g_r(p)$ there exists a unique, hence minimizing, geodesic in (N,g^ε) from x to y. Then, for every $x \in B^g_r(p)$, we have that for almost every $y \in B^g_r(p)$ the curve γ just obtained must be the minimizing geodesic from x to y in (N,g^ε). Hence, given $x \in B^g_r(p)$, there holds $\tilde{\mathsf{d}}(x,y) = \mathsf{d}^\varepsilon(x,y)$ for almost every

$y \in B^g_r(p)$. Thus by continuity $\tilde{\mathsf{d}} = \mathsf{d}^\varepsilon$ on $B^g_r(p)$. Therefore choosing ε small enough and eventually renaming g^ε into g^δ, the conclusion follows from Lemma 2.2. □

Remark 2.10 (Representation of the perimeter)*.* Let (M, g) be an n-dimensional C^0-Riemannian manifold, and let E be a set of finite perimeter. Then $P(E, \cdot) = \mathcal{H}^{n-1} \llcorner \partial^e E$, being $\partial^e E := M \setminus (E^{(0)} \cup E^{(1)})$ the so-called *essential boundary*, where

$$E^{(1)} := \left\{ x \in M \;\middle|\; \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 1 \right\}, \qquad E^{(0)} := \left\{ x \in M \;\middle|\; \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 0 \right\}. \tag{2.3}$$

Indeed, fix on a compact ball $B \subset M$ a sequence g_i of smooth metrics such that $g_i \to g$ uniformly on B (see, e.g., Lemma 2.9). Then, up to subsequences, $\partial^e E \cap B$ does not depend on the metrics g_i, g chosen in the definition (2.3) (compare also with item (1) in Lemma 2.12). Moreover, by the very definition of Hausdorff measure, $(1 - o_i(1))\mathcal{H}^{n-1}_{g_i} \llcorner B \le \mathcal{H}^{n-1}_g \llcorner B \le (1 + o_i(1))\mathcal{H}^{n-1}_{g_i} \llcorner B$. From the classical De Giorgi-Federer's theorem, $|D\chi_E|_i(\Omega) = \mathcal{H}^{n-1}_{g_i}(\partial^e E \cap \Omega)$ for every $\Omega \subset\subset B$. Thus, taking into account Lemma A.1 one has that $|D\chi_E|_i(\Omega) \to |D\chi_E|(\Omega)$ for every Borel $\Omega \subset\subset B$, and thus we conclude $|D\chi_E| \llcorner B = \mathcal{H}^{n-1} \llcorner (\partial^e E \cap B)$. Since B was arbitrary, we get the sought claim. In the following, if E is a set of locally finite perimeter, integrals over its essential boundary are tacitly understood to be taken with respect to the perimeter measure.

Lemma 2.11. *Let (M_i, g_i, p_i) be a sequence of pointed C^0-Riemannian manifolds of dimension n converging in C^0-sense to a pointed C^0-manifold (M, g, p). Denote by d, d_i the Riemannian distances on $(M, g), (M_i, g_i)$ respectively. Let $f_i \in BV(M_i)$ be such that $\sup_i \|f_i\|_{L^1(M_i)} + |Df_i|(M_i) < +\infty$. Then, up to subsequence, there exist $f \in BV(M)$, $R_i \nearrow +\infty$ and $(1 + 1/i)$-biLipschitz embeddings $F_i : (B_{R_i}(p), \mathsf{d}) \to (M_i, \mathsf{d}_i)$ with $F_i(p) = p_i$ such that the functions $f_i \circ F_i$ converge to f in $L^1_{\mathrm{loc}}(M)$. Moreover*

$$|Df|(M) \le \liminf_i |Df_i|(M_i).$$

If also $\mathrm{spt} f_i \subset B_R(p_i)$ for some $R > 0$ and for every i, then the convergence occurs in $L^1(M)$.

Proof. By a diagonal argument, up to passing to a subsequence, by Remark 2.4 there exist $R_i \nearrow +\infty$ and $(1 + 1/i)$-biLipschitz embeddings $F_i : B_{R_i}(p) \to M_i$ with $F_i(p) = p_i$. Denote $h_i := f_i \circ F_i$. For any $r > 0$, for i large we have that $h_i \in BV(B_r(p))$ with $\sup_i \|h_i\|_{L^1(B_r(p))} + |Dh_i|(B_r(p)) < +\infty$. If we show that h_i admits a subsequence converging in $L^1(B_{r/2}(p))$, the first part of the statement follows. Indeed, by Lemma 2.9 we can find a smooth Riemannian manifold (N, g^δ) such that it is well defined the inclusion $\iota : (B_r(p), \mathsf{d}) \to (N, \mathsf{d}^\delta)$ and ι is 2-biLipschitz with its image. Hence h_i can be seen as an equibounded sequence in $BV(B^g_r(p), g^\delta)$. By classical precompactness we can extract a subsequence converging in $L^1(B^g_{r/2}(p), \mathcal{H}^n_{g^\delta})$. Since $\mathcal{H}^n_{g^\delta}$ and $\mathcal{H}^n_g$ are equivalent, the subsequence converges in $L^1(B_{r/2}(p))$ as well.

The lower semicontinuity inequality readily follows since, for any $r > 0$ for i large enough we have

$$|Dh_i|(B^g_r(p)) \le (1 + o(1))|Df_i|(B^{g_i}_{2(1+1/i)r}(p_i)) \le (1 + o(1))|Df_i|(M_i),$$

where $o(1) \to 0$ as $i \to \infty$. □

Lemma 2.12. *Let (M, g) be a C^0-Riemannian manifold of dimension n. Fix $R > 0$, $p \in M$. Then there exists $C := C(p, R) > 1$ such that the following hold.*

(1) For any $x \in B_R(p)$ and $0 < r \le R$ there holds

$$\frac{|B_{2r}(x)|}{|B_r(x)|} \le C, \qquad C^{-1} r^n \le |B_r(x)| \le C r^n.$$

(2) For any $x \in B_R(p)$, any $r \le R$, and any $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$ there holds

$$\fint_{B_r(x)} \left| f - \fint_{B_r(x)} f \right| \le C r \fint_{B_{2r}(x)} |\nabla f|. \tag{2.4}$$

(3) For any $E \subset\subset B_R(p)$ there holds

$$|E|^{\frac{n-1}{n}} \leq CP(E). \tag{2.5}$$

Proof. The claims are well-known to hold true on manifolds with smooth Riemannian metrics as a consequence of the fact that one can always find a lower bound for the Ricci curvature on compact sets. In fact, on a smooth manifold, (1) is a consequence of the Bishop–Gromov comparison theorem [71, Lemma 7.1.4], (2) follows from [22, 73], and (3) is a consequence of (1) and (2) by [38, Theorem 9.7] and [2, Theorem 4.3, Remark 4.4]. Therefore claims (1) and (3) in the setting of the statement readily follow from Lemma 2.9 applied with r sufficiently large.

It remains to prove (2). Let (N, g^δ) be given by Lemma 2.9 with $\delta = 1/4$ and radius equal to $3R$. Letting now $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$, for $x \in B_R(p)$ and $r \leq R$, we have $h := f \circ \iota^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\iota(B_{2r}(p)), \mathsf{d}^\delta)$, for ι as in Lemma 2.9 where d^δ is the distance on (N, g^δ). Since $|\nabla h(y)|_{g^\delta} \leq 2|\nabla f(y)|_g$ for any $y \in B_{2r}(p)$, we find

$$\begin{aligned}
\int_{B^g_r(x)} \left| f - \fint_{B^g_r(x)} f \, \mathrm{d}\mathcal{H}^n_g \right| \mathrm{d}\mathcal{H}^n_g &\leq 2 \int_{B^g_r(x)} \left| f - \fint_{B^{g^\delta}_{\frac{5}{4}r}(x)} h \, \mathrm{d}\mathcal{H}^n_{g^\delta} \right| \mathrm{d}\mathcal{H}^n_g \\
&\leq C \int_{B^{g^\delta}_{\frac{5}{4}r}(x)} \left| h - \fint_{B^{g^\delta}_{\frac{5}{4}r}(x)} h \, \mathrm{d}\mathcal{H}^n_{g^\delta} \right| \mathrm{d}\mathcal{H}^n_{g^\delta} \\
&\leq Cr \int_{B^{g^\delta}_{\frac{5}{4}r}(x)} |\nabla h|_{g^\delta} \, \mathrm{d}\mathcal{H}^n_{g^\delta} \\
&\leq Cr \int_{B^g_{2r}(x)} |\nabla f(y)|_g \, \mathrm{d}\mathcal{H}^n_g,
\end{aligned}$$

where in the third inequality we applied a Poincaré inequality as in (2.4), recalling that on smooth manifolds it is possible to take the integral on the right hand side on the ball of the same radius that appears on the left hand side [76, Corollary 5.3.5]. □

What observed so far implies that C^0-Riemannian manifolds locally asymptotic at infinity to space forms are PI spaces, see the forthcoming Corollary 2.14. We recall their definition, leaving the interested reader to the seminal [38, 24] and to the survey [50].

Definition 2.13 (PI space)**.** Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric measure space, where $\mathfrak{m}$ is a Radon measure. We say that $\mathfrak{m}$ is *uniformly locally doubling* if for every $R > 0$ there exists $C_D(R) > 0$ such that the following holds

$$\mathfrak{m}(B_{2r}(x)) \leq C_D(R)\mathfrak{m}(B_r(x)), \qquad \forall x \in \mathrm{X} \ \forall r \leq R.$$

We say that a *weak local* $(1,1)$*-Poincaré inequality* holds on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ if there exists λ such that for every $R > 0$ there exists $C_P(R)$ such that for every $f \in \mathrm{Lip}(X)$, the following inequality holds:

$$\fint_{B_r(x)} |f - \overline{f}(x)| \, \mathrm{d}\mathfrak{m} \leq C_P(R) r \fint_{B_{\lambda r}(x)} \mathrm{lip} f \, \mathrm{d}\mathfrak{m},$$

for every $x \in \mathrm{X}$ and $r \leq R$, where $\overline{f}(x) := \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m}$, and

$$\mathrm{lip} f(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(y, x)}, \tag{2.6}$$

if x is an accumulation point, or $\mathrm{lip} f(x) = 0$ if x is not an accumulation point.

We say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a *PI space* when $\mathfrak{m}$ is uniformly locally doubling and a weak local $(1,1)$-Poincaré inequality holds on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$.

Corollary 2.14. *Let (M,g) be a C^0-Riemannian manifold of dimension n that is $C^0_{\rm loc}$-asymptotic to the n-dimensional simply connected complete model $\mathbb{H}^n_K$ of constant sectional curvature $K\le 0$*[1]*. Then $(M,\mathsf{d},\mathcal{H}^n)$ is a PI space.*

Proof. Let $R>0$ and fix $o\in M$. By assumptions and recalling Remark 2.4, we can fix $\rho>R$ such that for any $x\in M\setminus B_\rho(o)$ there exists a diffeomorphism $F_x:(B_{3R}(x),g)\to(\mathbb{H}^n_K,g_K)$ that is 2-biLipschitz with its image, where g_K is the metric on $\mathbb{H}^n_K$, and whose image contains $B_R(F_x(x))$. Since $\mathbb{H}^n_K$ is PI, arguing as in the proof of Lemma 2.12 it follows that there exist $\overline{C}>0$ such that

$$\frac{|B_{2r}(x)|}{|B_r(x)|}\le\overline{C},\qquad \fint_{B_r(x)}\left|f-\fint_{B_r(x)}f\right|\le\overline{C}r\fint_{B_{2r}(x)}|\nabla f|,$$

for any $x\in M\setminus B_\rho(o)$, any $r\le R$, and any $f\in{\rm Lip}_{\rm loc}(M)$.

Next apply Lemma 2.12 with $p=o$ and $R=2\rho$. Hence for any $x\in M$ we have that either $x\in B_{2\rho}(o)$, or $x\in M\setminus B_\rho(o)$. Hence the fact that $(M,\mathsf{d},\mathcal{H}^n)$ is PI follows putting together the previous inequalities with those given by Lemma 2.12. □

The next corollary states the local Hölder continuity of the isoperimetric profile of C^0-Riemannian manifolds that are C^0-locally asymptotic to a model of constant curvature. The proof essentially follows a classical path, see e.g. [5, Lemma 2.23]. However in this context we do not have an explicit asymptotic rate for the perimeter of balls of infinitesimal radii, which are often used to perturb competitors. In place of ball, we shall employ images of Euclidean balls through biLipschitz maps into the manifold, so to get a one-parameter increasing family of sets whose perimeter has an explicit rate as the parameter goes to zero.

Corollary 2.15. *Let (M,g) be a C^0-Riemannian manifold of dimension n that is $C^0_{\rm loc}$-asymptotic to the n-dimensional simply connected complete model $\mathbb{H}^n_K$ of constant sectional curvature $K\le 0$. Denote by I (resp., I_K) the isoperimetric profile of M (resp., $\mathbb{H}^n_K$).*

Then $I\le I_K$, and I is locally $\frac{n-1}{n}$-Hölder continuous on $(0,+\infty)$.

Proof. We start by proving that $I\le I_K$. Let $B^K_t(0)\subset\mathbb{H}^n_K$ be a ball with volume $V\in(0,+\infty)$ in $\mathbb{H}^n_K$. If $q_i\in M$ is a diverging sequence of points, for large i, up to subsequence, there exist diffeomorphisms $F:(B_{2t}(0),g_K)\to F(B_{2t}(0))\subset(M,g)$ such that $F(0)=q_i$ and F is $(1+1/i)$-biLipschitz. Hence, there exists t_i such that $t_i\to t$, $E_i:=F(B_{t_i}(0))$ has volume V in M, and $P(E_i)\to P(B^K_t(0))=I_K(V)$. Therefore $I(V)\le\lim_i P(E_i)=I_K(V)$.

Fix $o\in M$. Since M is $C^0_{\rm loc}$-asymptotic to $\mathbb{H}^n_K$, there exists $r>0$ such that for any $p\in M\setminus B_r(o)$ there exists a 2-biLipschitz map F_p from a Euclidean ball of sufficiently small radius to M with $F_p(0)=p$. Combining with Lemma 2.9, we conclude that there exists $\overline{r}\in(0,1)$ such that for any $p\in M$ there exists a 2-biLipschitz map $F_p:(B^{\rm eu}_{\overline{r}}(0),\mathsf{d}_{\rm eu})\to F_p(B^{\rm eu}_{\overline{r}}(0))\subset M$ such that $F_p(0)=p$. Hence we define the one-parameter family of sets $E_t(p):=F_p(B^{\rm eu}_t(0))$, for any $p\in M$ and $t\in(0,\overline{r}]$. In particular, recalling also the representation of the perimeter Remark 2.10 and the fact that the essential boundary is biLipschitz invariant,

$$P(E_t(p))\le\overline{C}t^{n-1},\qquad \frac{1}{\overline{C}}t^n\le|E_t(p)|\le\overline{C}t^n, \tag{2.7}$$

for any $p\in M$ and $t\in(0,\overline{r}]$, for some $\overline{C}$ independent of p,t.

Combining again the fact that M is $C^0_{\rm loc}$-asymptotic to $\mathbb{H}^n_K$ with item (1) in Lemma 2.12 we get that for any $R>0$ there exists a constant $C_R>0$ such that $C_R^{-1}r^n\le|B_r(x)|\le C_Rr^n$ for any $x\in M$ and $r\in(0,R)$. Hence recalling that $(M,\mathsf{d},\mathcal{H}^n)$ is also PI by Corollary 2.14, then it is well-known that there holds a relative isoperimetric inequality in balls of M, see [38, Theorem 5.1] and [2, Remark 4.4]. Since also $\inf_{x\in M}|B_1(x)|>0$ thanks to the asymptotic assumption, it is readily checked that the proof of [5, Lemma 2.10] can be repeated in our setting, yielding that: for

[1] $\mathbb{H}^n_K=\mathbb{R}^n$ if $K=0$, while $\mathbb{H}^n_K$ is the n-dimensional hyperbolic space of constant sectional curvature K if $K<0$.

any $R>0$ there exists $\widetilde{C}=\widetilde{C}(R)>0$ such that for any $E\subset M$ with $|E|\in(0,+\infty)$ there exists x_E such that

$$|E\cap B_r(x_E)|\geq \widetilde{C}\min\left\{\frac{|E|^n}{P(E)^n},r^n\right\}, \tag{2.8}$$

for any $r\in(0,R]$.

Fix now $\overline{V}\in(0,+\infty)$ and $\eta\in(0,\overline{V})$. Let also $V_0\in(\max\{\overline{V}-1,\eta\},\overline{V}+1)$ and for any $\varepsilon>0$ let $E\subset M$ be a bounded set such that $|E|=V_0$ and $P(E)\leq I(V_0)+\varepsilon$ (it is readily checked that, arguing as in [4, Lemma 2.17], the isoperimetric profile is achieved by bounded sets).

Let

$$v:=\min\left\{\min_{x\in M}|E_{\overline{r}}(x)|,1\right\}\geq\min\{\overline{C}^{-1}\overline{r}^n,1\}>0.$$

Since E is bounded, for any $V\in[V_0,V_0+v)$ there exist $x_V\in M$ and $r_V\leq\overline{r}$ such that $|E\cup E_{r_V}(x_V)|=V$ and $|E_{r_V}(x_V)|=V-V_0$. Therefore

$$I(V)\leq P(E)+P(E_{r_V}(x_V))\leq I(V_0)+\varepsilon+\overline{C}r_V^{n-1}.$$

Since $r_V^n\leq\overline{C}(V-V_0)$, letting $\varepsilon\to0$ we conclude that

$$I(V)\leq I(V_0)+\overline{C}^{2-\frac{1}{n}}(V-V_0)^{\frac{n-1}{n}}. \tag{2.9}$$

We next consider volumes smaller than V_0. By (2.8) we know that there exists a constant $\widetilde{C}=\widetilde{C}(\overline{r})$ independent of E and a point x_E such that

$$|E\cap E_t(x_E)|\geq|E\cap B_{\frac{t}{2}}(x_E)|\geq\frac{\widetilde{C}}{2^n}\min\left\{\frac{|E|^n}{P(E)^n},t^n\right\}\geq\frac{\widetilde{C}}{2^n}\min\left\{\frac{V_0^n}{(I_K(V_0)+\varepsilon)^n},t^n\right\}=\frac{\widetilde{C}}{2^n}t^n,$$

for any $t\leq t_0$ for some $t_0=t_0(\overline{V},\eta,n,K,\overline{r})\in(0,\overline{r})$. Let $\widetilde{C}_2:=\widetilde{C}2^{-n}t_0^n>0$. Up to decrease t_0, we can assume that $\widetilde{C}_2<\overline{V}$.

If $V_0>\overline{V}-\widetilde{C}_2$, let $V\in(\max\{\overline{V}-1,\eta,\overline{V}-\widetilde{C}_2\},\overline{V}+1)$ such that $V<V_0$. Hence there exists $\overline{t}\in(0,t_0)$ such that $|E\setminus E_{\overline{t}}(x_E)|=V$. Similarly as before, we have

$$I(V)\leq P(E)+P(E_{\overline{t}}(x_E))\leq I(V_0)+\varepsilon+\overline{C}\overline{t}^{n-1}\leq I(V_0)+\varepsilon+\frac{2^{n-1}\overline{C}}{\widetilde{C}^{\frac{n-1}{n}}}(V_0-V)^{\frac{n-1}{n}},$$

which letting $\varepsilon\to0$ yields

$$I(V)\leq I(V_0)+\frac{2^{n-1}\overline{C}}{\widetilde{C}^{\frac{n-1}{n}}}(V_0-V)^{\frac{n-1}{n}}. \tag{2.10}$$

Putting together (2.9) and (2.10), we have proved that there exist $v,C_H(M)>0$ and $\widetilde{C}_2=\widetilde{C}_2(M,\overline{V},\eta)$ such that for any $V_0\in(\max\{\overline{V}-1,\eta,\overline{V}-\widetilde{C}_2\},\overline{V}+1)$, for any $V\in(\max\{\overline{V}-1,\eta,\overline{V}-\widetilde{C}_2\},\min\{\overline{V}+1,V_0+v\})$ there holds

$$I(V)\leq I(V_0)+C_H|V-V_0|^{\frac{n-1}{n}}. \tag{2.11}$$

The dependence of the previous constants imply that there exists a neighborhood U of $\overline{V}$ such that (2.11) holds for any choices of $V,V_0\in U$. This implies the desired local Hölder continuity. □

The next theorem is based on a concentration-compactness argument that has been used several times in the literature, applied to the study of the isoperimetric problem in noncompact manifolds. In the smooth setting it has been first obtained in [69, Theorem 2]. See also [75, Theorem 4.48] and references therein for a complete account. Results analogous to Theorem 2.16 have been worked out also in the setting of nonsmooth spaces with bounds below on the curvature, see [4, Theorem 4.6] and [5, Theorem 3.3 & Theorem 1.1].

Theorem 2.16 (Asymptotic mass decomposition under C^0_{loc}-asymptotic assumptions)**.** *Let (M,g) be an n-dimensional complete C^0-Riemannian manifold and assume that M is C^0_{loc}-asymptotic to the n-dimensional simply connected complete model $\mathbb{H}^n_K$ of constant sectional curvature $K\leq0$.*

Fix $o \in M$. Let $V > 0$ and let $E_i \subset M$ be a sequence of bounded sets such that $|E_i| = V$ for any i and $\lim_i P(E_i) = I(V)$.

Then, up to subsequence, one of the following alternatives holds true.

- *The sequence E_i converges in $L^1(M)$ to an isoperimetric set E of volume V.*
- *There exist two sequences of radii $R_i, r_i \nearrow +\infty$ with $R_i < r_i$ and a diverging sequence of points $p_i \in M \setminus B_{R_i(o)}$ such that $E_i^c := E_i \cap B_{R_i}(o)$ converges in $L^1(M)$ to a (possibly empty) isoperimetric set E, and $E_i^d := E_i \cap B_{r_i}(p_i) \setminus B_{R_i}(o)$ converges to a ball $B \subset \mathbb{H}^n_K$ in the sense that*
$$\lim_i P(E_i^d) = P(B), \qquad \lim_i |E_i^d| = |B|.$$
Moreover
$$V = |E| + |B|, \qquad I(V) = P(E) + P(B).$$

Proof. Since the proof of Theorem 2.16 is standard and follows closely the strategy of [4, 5] we just sketch it.

At first, one gets the analogue of Ritoré–Rosales' result [5, Theorem 3.3] in the setting of Theorem 2.16. Indeed, the proof of [5, Theorem 3.3] uses the coarea formula, the precompactness of BV in L^1_{loc}, and the existence, around every point $p \in M$, of a one-parameter family $\{\mathcal{B}_{p,r}\}_{r\in(0,\varepsilon)}$ of sets such that $r \mapsto |\mathcal{B}_{p,r}|$ is continuous and vanishing as $r \to 0$, and $P(\mathcal{B}_{p,r}) \to 0$ as $r \to 0$. The first two come from Lemma 2.11, and Theorem 2.7, while for the last one it suffices to take pre-images of small Euclidean balls under the map in the second item of Lemma 2.5, as it has been done in the proof of Corollary 2.15. Once this is done, one follows verbatim the proof of [4, Theorem 4.6], which additionally needs that $(M, \mathsf{d}, \mathcal{H}^n)$ is PI, and $|B_r(p)| \geq \zeta r^3$ for every $r \in (0, R]$, and every $p \in M$, where $\zeta, R > 0$ are constants depending on M: these two properties come from Corollary 2.14 and from its proof (compare also with item (1) in Lemma 2.12).

Following [4, Theorem 4.6] until (4.20), and then jumping to Step 5 in there, one finally gets the following. There is $\overline{N} \in \mathbb{N} \cup \{+\infty\}$ and radii $R_i \to +\infty$, $T_{i,j} \to_i +\infty$ for $1 \leq j < \overline{N} + 1$, and there are mutually (with respect to j) diverging $p_{i,j} \in M \setminus B_{R_i}(o)$ such that
$$E_i^c \xrightarrow[i]{} E \text{ in } L^1(M), \qquad (E_i \setminus B_{R_i}(o)) \cap B_{T_{i,j}}(p_{i,j}) \xrightarrow[i]{} B_j \text{ in } L^1(M),$$
where B_j is a ball in $\mathbb{H}^n_K$ for any $j < \overline{N} + 1$, and E is an isoperimetric set in M. The convergence in L^1 to the B_j's has to be intended as in the statement of Lemma 2.11, after the composition with biLipschitz embeddings. Moreover one has
$$|E| + \sum_{j=1}^{\overline{N}} |B_j| = V, \qquad P(E) + \sum_{j=1}^{\overline{N}} I_K(|B_j|) = P(E) + \sum_{j=1}^{\overline{N}} P(B_j) = I(V),$$
where I_K is the isoperimetric profile of $\mathbb{H}^n_K$. Hence either $\overline{N} = 0$ and the first item holds, or $\overline{N} \geq 1$. In the latter case, we want to show that $\overline{N} = 1$, completing the proof of the second item. By coarea formula we can fix a sequence $\rho_i \nearrow +\infty$ such that $P(E \cap B_{\rho_i}(o)) \leq P(E) + 1/i$. For any i we find balls $B_{s_i}(q_i) \subset M \setminus B_{\rho_i+1}(o)$ such that $|B_{s_i}(q_i)| = V - |E \cap B_{\rho_i}(o)|$ and such that $B_{s_i}(q_i)$ converges to a ball $B \subset \mathbb{H}^n_K$ with $\lim_i P(B_{s_i}(q_i)) = P(B)$ and $|B| = V - |E|$. If by contradiction $\overline{N} > 1$, since the isoperimetric profile I_K is a strictly subadditive function, we get
$$\begin{aligned} P(E) + \sum_{j=1}^{\overline{N}} I_K(|B_j|) = P(E) + \sum_{j=1}^{\overline{N}} P(B_j) = I(V) &\leq \liminf_i P(E \cap B_{\rho_i}(o)) + P(B_{s_i}(q_i)) \\ &= P(E) + I_K(|B|) \leq P(E) + I_K\left(\sum_{j=1}^{\overline{N}} |B_j|\right) < P(E) + \sum_{j=1}^{\overline{N}} I_K(|B_j|), \end{aligned}$$
which is a contradiction. $\square$

3. Local inverse mean curvature flow

We start by recalling the definition of weak inverse mean curvature flow (IMCF) as introduced in [45].

Definition 3.1 (Weak IMCF - Level set formulation)**.** Let (M, g) be a smooth Riemannian manifold. Given a precompact $K \subset M$, a locally Lipschitz function $u : M \to \mathbb{R}$, and a set of locally finite perimeter E, define

$$J_u^K(E) := P(E, K) - \int_{E \cap K} |\nabla u|.$$

Let $\Omega \subset M$ be an open set. A function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is called a *weak solution* (resp., subsolution, supersolution) to the inverse mean curvature flow (IMCF) in Ω if

$$J_u^K(\{u < t\}) \leq J_u^K(E),$$

for all $t \in \mathbb{R}$, all $K \subset\subset \Omega$, and all sets E (resp., $E \supset \{u < t\}$, $E \subset \{u < t\}$) such that $E \Delta \{u < t\} \subset K$.

Remark 3.2. By virtue of [45, Lemma 1.1], a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a weak solution the IMCF in Ω if and only if

$$\int_K |\nabla u| + u|\nabla u| \leq \int_K |\nabla v| + v|\nabla u|, \tag{3.1}$$

for all $K \subset\subset \Omega$, and all $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\{u \neq v\} \subset K$.

The aim of this section is to show Theorem 3.6, stating that on every punctured ball B centered at o on a smooth complete Riemannian manifold one can define a weak IMCF that is bounded from below explicitly in terms of constants that will nicely behave on metrics C^0-close to the flat one. We also gather useful properties of this flow in Section 3.1, and Section 3.2.

Definition 3.3. Let (M, g) be an n-dimensional complete C^0-Riemannian manifold, and let $1 \leq p < n$. Let $\Omega \subset M$ be an open set. We say that Ω *supports a* (p, p^*)*-Sobolev inequality* if there exists a constant $C > 0$ such that

$$\left(\int_M |\psi|^{\frac{np}{n-p}} \right)^{\frac{n-p}{n}} \leq C \int_M |\nabla \psi|^p \quad \text{for all } \psi \in \mathrm{Lip}_c(\Omega). \tag{3.2}$$

We denote $C_{p,\mathrm{Sob}}(\Omega)$ the smallest constant C for which the latter inequality holds.

Remark 3.4. It is known that (3.2) with $p = 1$ is equivalent to

$$|E|^{\frac{n-1}{n}} \leq C P(E), \quad \text{for all bounded measurable } E \subset\subset \Omega.$$

Indeed, one implication readily comes from the very definition in (2.1), while the other comes from an application of the coarea formula in Theorem 2.7. The latter implication is classical and dates back at least to works of Federer–Fleming and Maz'ya in the 60s, see, e.g., [34, page 488].

We now provide solutions to the weak IMCF in punctured balls $B_R(o) \setminus \{o\}$. The weak IMCF is obtained in the limit, as $p \to 1^+$, of functions $w_p^R := -(p-1) \log G_p^R$, where G_p^R, for $p \in (1, n)$ denotes the p-harmonic Green function on $B_R(o)$ with Dirichlet boundary conditions. Namely, G_p^R is the solution to

$$\begin{cases} -\Delta_p G_p^R = |\mathbb{S}^{n-1}| \left(\frac{n-p}{p-1} \right)^{p-1} \delta_o & \text{on } B_R(o), \\ \quad G_p^R = 0 & \text{on } \partial B_R(o), \end{cases} \tag{3.3}$$

where δ_o is the Dirac delta supported at o, and $|\mathbb{S}^{n-1}|$ is the measure of the $(n-1)$-dimensional unit sphere. With the above choice of normalization, it follows from the blow-up procedure leading to [49, Theorem 1.1] that

$$\left| \frac{G_p^R(x)}{r(x)^{-(n-p)/(p-1)}} - 1 \right| \to 0, \qquad \left| \frac{\left|\nabla G_p^R(x)\right|}{r(x)^{-(n-1)/(p-1)}} - \frac{n-p}{p-1} \right| \to 0 \tag{3.4}$$

as $r(x) \to 0$, where we let $r(x) := \mathsf{d}(o,x)$. We recall that the relative p-capacity of a compact $K \Subset B_R(o)$ is defined as

$$\mathrm{Cap}_p(K, B_R(o)) := \inf\left\{ \int_{B_R(o)\setminus K} |\nabla v|^p \ : \ v \in \mathrm{Lip_c}(B_R(o)), v \geq \chi_K \right\}. \tag{3.5}$$

The following lemma is well known, and consists essentially in [40, Lemma 3.8].

Lemma 3.5. *Let (M,g) be a smooth complete Riemannian manifold, and let $o \in M$, $R > 0$. Let $E_t^R := \{w_p^R \leq t\}$ for $p \in (1,n)$, where $w_p^R := -(p-1)\log G_p^R$ and G_p^R solves (3.3). Then*

$$\mathrm{Cap}_p(E_t^R, B_R(o)) = \left(\frac{n-p}{p-1}\right)^{p-1} \left|\mathbb{S}^{n-1}\right| e^t, \tag{3.6}$$

for every $t \in \mathbb{R}$.

Proof. In this proof, let $G_p := G_p^R$, and $w_p := w_p^R$ for the ease of notation. It is well-known that the p-capacity defined in (3.5) is attained computing the L^p-norm of the gradient of the p-harmonic function with Dirichlet data equal to 1 on ∂K, and equal to 0 on $\partial B_R(o)$ (see [39] for a thorough account on nonlinear potential theory). In particular, one gets, for every $t \in (0, +\infty)$,

$$\mathrm{Cap}_p(\{G_p \geq t\}, B_R(o)) = \frac{1}{t^p}\int_{\{G_p<t\}} |\nabla G_p|^p = \frac{1}{t^p}\int_0^t \int_{\{G_p=s\}} |\nabla G_p|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s. \tag{3.7}$$

On the other hand, a straightforward application of the divergence theorem combined with the p-harmonicity of G_p (see e.g. the computations in [11, Proposition 2.8]) yields that $\int_{\{G_p=s\}} |\nabla G_p|^{p-1}$ attains the same value for almost every $s \in (0, +\infty)$. Such constant is computed using (3.4) as

$$\int_{\{G_p=s\}} |\nabla G_p|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1} = \left(\frac{n-p}{p-1}\right)^{p-1} \left|\mathbb{S}^{n-1}\right|,$$

for almost every $s \in (0, +\infty)$. Plugging it into (3.7) leaves us with

$$\mathrm{Cap}_p(\{G_p \geq t\}, B_R(o)) = \frac{1}{t^{p-1}}\left(\frac{n-p}{p-1}\right)^{p-1} \left|\mathbb{S}^{n-1}\right|, \tag{3.8}$$

for any $t \in (0, +\infty)$. Rewriting it in terms of w_p as stated in (3.6) completes the proof. □

We denote with $C_P(B_R(o))$ the *Poincaré constant of* $B_R(o)$, defined as the smallest constant C such that

$$\fint_{B_r(x)} \left| f - \fint_{B_r(x)} f \right| \leq Cr \fint_{B_{2r}(x)} |\nabla f| \tag{3.9}$$

holds for every $x \in B_R(o)$, $r \leq R$, and $f \in \mathrm{Lip_{loc}}(M)$. We denote with $C_A(B_R(o))$ the *Ahlfors constant of* $B_R(o)$, defined as the smallest number $C \geq 1$ such that

$$C^{-1} r^n \leq |B_r(x)| \leq Cr^n, \tag{3.10}$$

for every $x \in B_R(o)$, and every $0 < r \leq R$.

Finally, denoting $A_{\rho_1,\rho_2}(o) := B_{\rho_2}(o) \setminus \overline{B}_{\rho_1}(o)$ for $\rho_2 > \rho_1$, we denote

$$\begin{aligned} C_{\mathrm{cov}}(B_\rho(o)) := \min\Big\{ N \in \mathbb{N} \ : \ & A_{3r/4,5r/4}(o) \text{ is covered by } N \text{ open balls of radius } r/2 \\ & \text{with centers in } A_{3r/4,5r/4}(o) \text{ for any } 0 < r \leq \rho \Big\}. \end{aligned} \tag{3.11}$$

Observe that on any complete smooth Riemannian manifold (M,g) of dimension $n \geq 2$, for any $o \in M$ and $R > 0$ there exists $\rho \in (0, R/2]$ such that

$$\forall 0 < r \leq \rho, \ \forall p, q \in \partial B_r(o) \ \exists \text{ continuous curve } \gamma \subset A_{3r/4,5r/4}(o) \text{ connecting } p \text{ and } q. \tag{3.12}$$

Theorem 3.6. *Let (M,g) be a complete smooth Riemannian manifold of dimension $n \geq 2$. Fix $o \in M$, and $R > 0$, and let $p \in (1,n)$. Let $w_p^{2R} = -(p-1)\log G_p^{2R}$, with G_p^{2R} as in (3.3). Let $\rho \in (0, R/2]$ be such that (3.12) is satisfied. Then, the following hold.*

(1) The sequence of functions w_p^{2R} converges, up to subsequence, locally uniformly in $B_{2R}(o) \setminus \{o\}$ as $p \to 1^+$ to a weak solution w of the IMCF on $B_{2R}(o) \setminus \{o\}$.

(2) The function w satisfies

$$w(x) \geq (n-1)\log r(x) - C, \qquad \text{for all } x \in \overline{B}_R(o) \setminus \{o\}, \tag{3.13}$$

where $C = C\Big(n, C_{1,\mathrm{Sob}}(B_{2R}(o)), C_P(B_{2R}(o)), C_A(B_{2R}(o)), R/\rho, C_{\mathrm{cov}}(B_\rho(o))\Big)$, and $r(x) := \mathsf{d}(o,x)$.

(3) It holds $w(x) \to -\infty$ as $x \to o$.

(4) For every $r \leq R$, letting $T_r := (n-1)\log r - C - 1$, there holds $\{w \leq T_r\} \subset B_r(o)$.

Remark 3.7. If (M,g) in Theorem 3.6 supports a $(1,1^*)$-Sobolev inequality, then the existence of a *global* proper weak IMCF issuing from a point follows from a careful modification of the proof of [84]. Yet, we do not know how to obtain the quantitative estimate (3.13) through the procedure introduced there.

The proof of Theorem 3.6 requires some preliminary steps. The following result is a direct consequence of [51, Theorem 1.1].

Theorem 3.8. *Let (M,g) be a complete smooth Riemannian manifold, and let u_p be a positive p-harmonic function defined in an open set $U \subset M$, for $p \in (1,2)$. Let $w_p = -(p-1)\log u_p$. Then, for any compact subset $K \subset U$, we have*

$$|\nabla w_p| \leq C(K), \tag{3.14}$$

where $C(K)$ does not depend on $p \in (1,2)$.

The constant $C(K)$ in Theorem 3.8 depends on sectional curvature bounds for g on K. The following observation will be useful also for estimating the asymptotic behavior of the Hawking mass at o along the IMCF, see Proposition 3.13 below.

Remark 3.9. For $p \in (1,2)$, if u_p is a positive p-harmonic function on $B_{2R}(o) \setminus \{o\}$, and $\mathrm{Sec} \geq -k^2$ on $B_{2R}(o)$, then there exist two constants $\eta := \eta(k,n)$ and $\zeta := \zeta(k,n)$ such that for every $x \in B_{2R}(o) \setminus \{o\}$ with $\mathsf{d}(x,o) < \min\{\eta, R\}$ there holds

$$|\nabla w_p|(x) \leq \frac{\zeta}{\mathsf{d}(o,x)}. \tag{3.15}$$

Indeed, it suffices to apply [51, Equation (1.5)] on balls $B(x, \mathsf{d}(o,x)/2)$. We remark that both η, ζ can be chosen to be uniform with respect to $p \to 1^+$.

The following result yields the uniform lower bound on w_p^{2R} that will result in (3.13). Its core is in the Harnack inequality for p-harmonic functions that comes with the sharp dependence with respect to p. The estimate (3.17) below was suggested to the authors by Luca Benatti.

Theorem 3.10. *Let (M,g) be a complete smooth Riemannian manifold of dimension $n \geq 2$. Fix $o \in M$, and $R > 0$, and let $p \in (1,n)$. Let $w_p^{2R} = -(p-1)\log G_p^{2R}$, with G_p^{2R} as in (3.3). Let $\rho \in (0, R/2]$ be such that (3.12) holds.*

Then

$$w_p^{2R}(x) \geq (n-p)\log r(x) - C, \qquad \text{for all } x \in B_{3R/2}(o) \setminus \{o\}, \tag{3.16}$$

for $C = C\Big(n, C_{1,\mathrm{Sob}}(B_{2R}(o)), C_P(B_{2R}(o)), C_A(B_{2R}(o)), R/\rho, C_{\mathrm{cov}}(B_\rho(o))\Big)$, where $r(x) := \mathsf{d}(o,x)$.

Proof. Within this proof let $G_p^{2R} =: G_p$, and $w_p^{2R} =: w_p$ for the ease of notation. Let $m(r) = \max_{\partial B_r(o)} w_p$ for $r \in (0, 3R/2)$. Notice that $w_p(x) \to -\infty$ as $x \to o$. Then, by the maximum principle, $B_r(o) \subset \{w_p \leq m(r)\}$, the monotonicity of the p-capacity with respect to inclusion and (3.6) give

$$\mathrm{Cap}_p(\overline{B}_r(o), B_{2R}(o)) \leq \mathrm{Cap}_p(\{w_p \leq m(r)\}, B_{2R}(o)) = \left(\frac{n-p}{p-1}\right)^{p-1} \left|\mathbb{S}^{n-1}\right| e^{m(r)},$$

that results in

$$m(r) \geq \log(\mathrm{Cap}_p(\overline{B}_r(o), B_{2R}(o))) - (p-1)\log\left(\frac{n-p}{p-1}\right) - \log\left|\mathbb{S}^{n-1}\right|. \tag{3.17}$$

It is now well-known that capacities can be estimated exploiting isoperimetric inequalities. More precisely, setting $C = 1/C_{1,\mathrm{Sob}}(B_{2R}(o))$, we can apply [36, Eq. (7)] to get

$$\begin{aligned}
\mathrm{Cap}_p(\overline{B}_r(o), B_{2R}(o)) &\geq \left(\int_{|B_r(o)|}^{|B_{2R}(o)|} \frac{1}{Cv^{\frac{p(n-1)}{n(p-1)}}}\right)^{1-p} \\
&= \left(C\frac{(n-p)}{n(p-1)}\right)^{p-1} |B_r(o)|^{\frac{n-p}{n}} \left[1 - \left(\frac{|B_r(o)|}{|B_{2R}(o)|}\right)^{\frac{n-p}{n(p-1)}}\right]^{1-p} \\
&\geq \left(C\frac{(n-p)}{n(p-1)}\right)^{p-1} |B_r(o)|^{\frac{n-p}{n}}.
\end{aligned}$$

We can now estimate $|B_r(o)|^{\frac{n-p}{n}} \geq C_A(B_{2R}(o))^{\frac{n-p}{n}} r^{n-p}$, and so we get from (3.17) that

$$m(r) \geq (n-p)\log r - C \tag{3.18}$$

where now $C = C\Big(n, C_{1,\mathrm{Sob}}(B_{2R}(o)), C_A(B_{2R}(o))\Big)$, and it is independent of p as $p \to 1^+$.

Let us consider now $y \in \partial B_r(o)$ for $r \in (0, \frac{3}{2}R)$. We aim at estimating $w_p(y)$ combining (3.18) with a Harnack inequality. The Harnack inequality for positive p-harmonic functions [74, Theorem 1.28] applied to G_p reads

$$G_p(y) \leq C_H^{\frac{1}{p-1}}(B_s(z)) G_p(x), \tag{3.19}$$

for any $x, y \in B_s(z) \subset\subset B_{2R}(o) \setminus \{o\}$, and the Harnack constant $C_H(B_s(z))$ can be estimated from above in terms of $n, C_P(B_{2R}(o)), C_A(B_{2R}(o))$ and $C_{p,\mathrm{Sob}}(B_{2R}(o))$. The interested reader might consult [62, Theorem 3.4 and Remark 3.5] for the full computations leading to the explicit constant in the Harnack inequality. Since $C_{p,\mathrm{Sob}}(B_{2R}(o)) \to C_{1,\mathrm{Sob}}(B_{2R}(o))$ as $p \to 1^+$, then

$$C_H(B_s(z)) \leq C = C\Big(n, C_P(B_{2R}(o)), C_A(B_{2R}(o)), C_{1,\mathrm{Sob}}(B_{2R}(o))\Big)$$

for any $B_s(z) \subset\subset B_{2R}(o) \setminus \{o\}$.

Fix $x, y \in \partial B_r(o)$ such that

$$G_p(x) = \min_{\partial B_r(o)} G_p, \qquad G_p(y) = \max_{\partial B_r(o)} G_p.$$

We first consider the case $r \leq \rho$, where ρ is as in the assumptions. Hence by (3.12) we can connect x and y with a curve $\gamma \subset A_{3r/4, 5r/4}(o)$. Letting $N := C_{\mathrm{cov}}(B_\rho(o))$, we can fix a family of at most N balls $\{B_{r/2}(z_j)\}$ such that $z_j \in A_{3r/4,5r/4}(o)$ and $A_{3r/4,5r/4}(o) \subset \cup_j B_{r/2}(z_j)$. The existence of γ implies that we can find a sequence of points $y := x_1, \ldots, x_{N'} =: x$ such that $N' \leq N+1$, $x_i, x_{i+1} \in \overline{B}_{r/2}(z_{j_i})$ for every $i = 1, \ldots, N'-1$ for some z_{j_i}. Thus applying iteratively (3.19) we deduce (the value of the constant C might change from line to line)

$$\max_{\partial B_r(o)} G_p \leq C^{\frac{N'}{p-1}} \min_{\partial B_r(o)} G_p \leq C^{\frac{1}{p-1}} \min_{\partial B_r(o)} G_p, \tag{3.20}$$

for $C = C\Big(n, C_P(B_{2R}(o)), C_A(B_{2R}(o)), C_{1,\mathrm{Sob}}(B_{2R}(o)), C_{\mathrm{cov}}(B_\rho(o))\Big)$.

If instead $r \in (\rho, \frac{3}{2}R)$, then we consider $p_1, p_2 \in \partial B_\rho(o)$ such that p_1 (resp. p_2) belongs to the intersection of $\partial B_\rho(o)$ with a minimizing geodesic from y to o (resp. from o to x). By (3.20) we already know that

$$G_p(p_1) \leq C^{\frac{1}{p-1}} G_p(p_2).$$

Applying (3.19) with $s = \rho/4$ iteratively along the geodesic from y to o we find

$$G_p(y) \leq C^{\frac{N''}{p-1}} G_p(p_1),$$

with $N'' \in \mathbb{N}$ such that $N'' \leq \frac{3}{2}R/(\frac{\rho}{8}) + 1$. Arguing analogously along the geodesic from o to x, we finally get that

$$\max_{\partial B_r(o)} G_p \leq C^{\frac{1}{p-1}} \min_{\partial B_r(o)} G_p, \tag{3.21}$$

for $C = C\Big(n, C_P(B_{2R}(o)), C_A(B_{2R}(o)), C_{1,\mathrm{Sob}}(B_{2R}(o)), C_{\mathrm{cov}}(B_\rho(o)), R/\rho\Big)$. Rewriting (3.21) in terms of w_p and combining with (3.18) yields

$$\min_{\partial B_r(o)} w_p = w_p(y) \geq w_p(x) - C = \max_{\partial B_r(o)} w_p - C \geq (n-p)\log r - C,$$

for $C = C\Big(n, C_{1,\mathrm{Sob}}(B_{2R}(o)), C_P(B_{2R}(o)), C_A(B_{2R}(o)), R/\rho, C_{\mathrm{cov}}(B_\rho(o))\Big)$. □

In order to pass to the limit $w_p = -(p-1)\log G_p$ as $p \to 1^+$ on compact sets $K \subset B_{2R}(o) \setminus \{o\}$, we need also some uniform upper bound on w_p. This is a well-known consequence of the Laplacian comparison theorem and of the asymptotics of the Green function at the pole in (3.4), see [52, Theorem 1.2].

Proposition 3.11. *Let (M, g) be an n-dimensional smooth complete Riemannian manifold, and let $p \in (1, n)$. Let G_p be the solution of (3.3) with Ω in place of $B_R(o)$. Let $r(x) := \mathsf{d}(o, x)$. Fix $R > 0$, and assume $B_R(o) \subset \Omega$. Suppose that $\mathrm{Ric} \geq -(n-1)a$ holds on $B_R(o)$, for some $a > 0$.*
Then

$$G_p(x) \geq \int_{r(x)}^R (v_a(t))^{-\frac{1}{p-1}}\, \mathrm{d}t, \tag{3.22}$$

for any $x \in B_R(o)\setminus\{o\}$, where $v_a(t) := c_{n,p}\left((\sqrt{a})^{-1}\sinh(\sqrt{a}t)\right)^{n-1}$, for suitable $c_{n,p} > 0$. Moreover $c_{n,p} \to c_n > 0$ as $p \to 1^+$.

Remark 3.12. In Proposition 3.11, the function

$$\int_{r(x)}^R (v_a(t))^{-\frac{1}{p-1}}\, \mathrm{d}t$$

is the solution of (3.3) in the space form of constant curvature a.

Proof of Theorem 3.6. Within this proof let $G_p^{2R} =: G_p$, and $w_p^{2R} =: w_p$ for the ease of notation. Let $K \subset B_{2R}(o) \setminus \{o\}$ be compact. We first show that w_p is equibounded on K as $p \to 1^+$. Indeed by (3.16) we know that $w_p \geq (n-p)\log(r(x)) - C$ on $B_{3R/2}(o) \setminus \{o\}$ for C as in Theorem 3.10. On the other hand, taking $\overline{r} \in (0, 2R)$ such that $K \subset B_{\overline{r}}(o) \setminus \{o\} \subset\subset B_{2R}(o)$, if $\mathrm{Ric} \geq -(n-1)a$ on $B_{2R}(o)$ for some $a > 0$, by (3.22) we have that for all $x \in B_{\overline{r}}(o) \setminus \{o\}$ there holds

$$w_p(x) \leq -\log\left(\int_{r(x)}^{\overline{r}} (v_a(t))^{-\frac{1}{p-1}}\, \mathrm{d}t\right)^{(p-1)} \leq -\log\left(|r(x) - \overline{r}|^{p-2}\int_{r(x)}^{\overline{r}} (v_a(t))^{-1}\, \mathrm{d}t\right). \tag{3.23}$$

Hence the above upper bound is uniform with respect to $p \to 1^+$ for any $x \in K$. Thus w_p is bounded on $K \cap B_{3R/2}(o)$ uniformly with respect to $p \to 1^+$. Therefore, applying the gradient bound (3.14) on K, we deduce that w_p is bounded on K uniformly with respect to $p \to 1$. Exhausting $B_{2R}(o) \setminus \{o\}$ with a sequence of increasing compact sets, by Ascoli–Arzelà and by a diagonal argument, we get that w_p converges to some function w locally uniformly on $B_{2R}(o) \setminus \{o\}$, up to passing to a subsequence with respect to p.

We claim that w satisfies the weak formulation of the IMCF on $B_{2R}(o) \setminus \{o\}$. Indeed, arguing as in [66, Equation (9)], one gets that $|\nabla w_p|^p \mathcal{H}^n$ converges to $|\nabla w| \mathcal{H}^n$ in duality with bounded continuous functions on compact subsets contained in $B_{2R}(o)\setminus\{o\}$, along the suitable sequence $p_i \to 1^+$ for which w_{p_i} converges. Using again [66, Equation (9)] this is enough to conclude that w is a weak solution of the IMCF on $B_{2R}(o)\setminus\{o\}$.

The lower bound (3.16) passes to the limit and gives (3.13). The upper bound (3.23) is preserved on compact subsets of $B_{3R/2}(o)\setminus\{o\}$ for the limit w as well, hence it shows that $w\to-\infty$ as $x\to o$.

Therefore we proved items (1), (2) and (3) of the statement. It remains to show that $\{w\leq (n-1)\log r - C - 1\}\subset B_r(o)$ for every $r\leq R$. Fix $r\in(0,R]$. By Theorem 3.10 there exists

$$C = C\Big(n, C_{1,\mathrm{Sob}}(B_{2R}(o)), C_P(B_{2R}(o)), C_A(B_{2R}(o)), R/\rho, C_{\mathrm{cov}}(B_\rho(o))\Big),$$

such that $w_p\geq (n-p)\log r - C$ on $\partial B_r(o)$. Denoting $T_r := (n-1)\log r - C - 1$, it follows that $w_p \geq T_r + \frac{1}{2}$ on $\partial B_r(o)$ for any p sufficiently close to 1. By (3.13), it follows that $\{w\leq T_r\}\cap\partial B_r(o)=\emptyset$. Suppose by contradiction that there exists $z\in B_{2R}(o)\setminus B_r(o)$ such that $w(z)\leq T_r$. Then $w_{p_i}(z)\leq T_r+\frac{1}{4}$ for a sequence $p_i\to 1^+$ such that w_{p_i} converges to w, and for i large enough. Since $w_{p_i}\geq T_r+\frac{1}{2}$ on $\partial B_r(o)$ and $w_{p_i}(x)\to+\infty$ as $r(x)\to 2R$, then w_{p_i} would have an interior minimum on $B_{2R}(o)\setminus\overline{B}_r(o)$, which is a contradiction to the maximum principle for p-harmonic functions. □

3.1. **Properties of the weak IMCF.** Let us collect in this section few known properties on the weak IMCF constructed in Theorem 3.6.

Let (M,g) be a smooth complete Riemaniann manifold. Let $o\in M, R>0$ and let w be given by Theorem 3.6. Let $T\in\mathbb{R}$ be such that $\{w\leq T\}\subset B_R(o)$ (such a T exists by Theorem 3.6). For every $t\in(-\infty,T]$, set $\Omega_t:=\{w<t\}$. Then the following hold.

(1) If $n\leq 7$, then for every $t\in(-\infty,T]$ both the set Ω_t and the set $\{w\leq t\}$ have $C^{1,\alpha}$ boundary, for some $\alpha>0$. Indeed, as a direct consequence of Definition 3.1, the previous sets are local (Λ,r_0)-minimizers in $B_R(o)\setminus\{o\}$, see, e.g., [61, Example 21.2]; thus by (the Riemannian analogue of) [61, Theorem 21.8] one gets the sought claim.
It follows that $\partial\Omega_t=\partial\{w\leq t\}=\{w=t\}$ for almost every $t\leq T$, and that $\partial\Omega_s\to\partial\Omega_t$ in C^1 as $s\nearrow t$, for every $t\leq T$, cf. [45, Eq. (1.10)].
Moreover, it is meaningful to speak about the weak mean curvature H on $\partial\Omega_t$, see, e.g., [45, page 16]. Moreover, $H=|\nabla w|>0$ $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega_t$ for almost every $t\in(-\infty,T]$, see [45, Equation (1.12)], and [45, Lemma 5.1].
(2) For almost every $t\in(-\infty,T]$, the boundary $\partial\Omega_t$ has weak second fundamental form A (see [67, Definition 1.3], or [45, pages 401–405]), and

$$\int_{\partial\Omega_t}|A|^2<+\infty.$$

The last inequality is a consequence of [67, Theorem 1.1(vi)].

The following Proposition gathers some key properties of the weak IMCF in dimension 3 constructed through the procedure of the previous section; most notably the Geroch monotonicity formula of Huisken–Ilmanen through such flow.

Proposition 3.13. *Let (M,g) be a smooth complete Riemaniann manifold of dimension $n=3$. Let $o\in M, R>0$ and let w be given by Theorem 3.6. Denote $\Omega_t:=\{w<t\}$. Let T be such that $\{w\leq T\}\subset B_R(o)$. Let H be the weak mean curvature of the boundary $\partial\Omega_t$. Then the following hold.*

- *There exist $\bar t\in\mathbb{R}$, $C_1>0$ such that*

$$P(\Omega_t)=4\pi e^t \quad \text{for all } t\in(-\infty,T], \quad \text{and} \quad \int_{\partial\Omega_t}H^2\leq C_1, \quad \text{for almost every } t\in(-\infty,\bar t). \tag{3.24}$$

- *For $t \in (-\infty, T]$, define the* Hawking mass

$$\mathfrak{m}_H(\partial\Omega_t) := \frac{P(\Omega_t)^{1/2}}{(16\pi)^{3/2}} \left(4\pi - \int_{\partial\Omega_t} \frac{H^2}{4}\right). \tag{3.25}$$

Then $\mathfrak{m}_H(\partial\Omega_t) \to 0$ for almost every $t \to -\infty$. Moreover, for every $-\infty < r < s \leq T$, if $\partial\Omega_t$ is connected for every $t \in [r, s]$, then

$$\mathfrak{m}_H(\partial\Omega_s) \geq \mathfrak{m}_H(\partial\Omega_r) + \frac{1}{(16\pi)^{3/2}} \int_r^s P(\Omega_t)^{1/2} \int_{\partial\Omega_t} R_g \,\mathrm{d}t. \tag{3.26}$$

Proof. Denote $E_t := \{w \leq t\}$ and let $E_t^p := \{w_p^{2R} \leq t\}$ for w_p^{2R} as in Theorem 3.6. We shall omit the superscript $2R$ in the sequel, as R is fixed. We know from Theorem 3.6 that w_p converges to w locally uniformly on $B_{2R}(o) \setminus \{o\}$ along a sequence $p_i \to 1$.

We claim that for any $\sigma \in (-\infty, T), \varepsilon > 0$ there exists $\tilde{p} > 1$ such that $E_{t-\varepsilon}^{p_i} \subset E_t \subset E_{t+\varepsilon}^{p_i}$ for any $p_i \in (1, \tilde{p})$ and $t \in [\sigma, T]$.
We prove the containment $E_{t-\varepsilon}^{p_i} \subset E_t$ first. Assume by contradiction that there exist $\sigma \in (-\infty, T)$, $\varepsilon > 0$ such that, up to subsequence, there exist $t_i \in [\sigma, T]$ such that $E_{t_i-\varepsilon}^{p_i} \not\subset E_{t_i}$ for any i. Then there exist points $x_i \in B_{2R}(o) \setminus \{o\}$ such that $w_{p_i}(x_i) \leq t_i - \varepsilon$ but $w(x_i) > t_i$ for any i. Hence $\liminf_i \mathsf{d}(x_i, o) > 0$, for otherwise $-\infty = \liminf_i w(x_i) \geq \sigma$ by Theorem 3.6. Moreover, there exists $\eta > 0$ such that $w_{p_i} > T + \eta$ on $\partial B_R(o)$ for large i, for otherwise $\{w \leq T\} \cap \partial B_R(o)$ would be nonempty. Hence $x_i \in B_R(o)$ for large i, for if $x_i \in B_{2R}(o) \setminus \overline{B}_R(o)$, since $w_{p_i}(x_i) \leq T - \varepsilon$ and $w_{p_i}(x) \to +\infty$ as $r(x) \to 2R$, then w_{p_i} would have an interior minimum on $B_{2R}(o) \setminus \overline{B}_R(o)$, contradicting the maximum principle for p-harmonic functions. Hence, up to subsequence, $t_i \to \tau \in [\sigma, T]$ and $x_i \to x \in \overline{B}_R(o) \setminus \{o\}$. But this contradicts the local uniform convergence, as this implies $\tau - \varepsilon \geq \lim_i w_{p_i}(x_i) = w(x) = \lim_i w(x_i) \geq \tau$.
The containment $E_t \subset E_{t+\varepsilon}^{p_i}$ follows by an analogous contradiction argument. This time the contradicting sequence of points $x_i \in B_{2R}(o) \setminus \{o\}$ satisfies $w_{p_i}(x_i) > t_i + \varepsilon$ and $w(x_i) \leq t_i$, for $t_i \in [\sigma, T]$. Hence $x_i \in \{w \leq T\} \subset B_R(o)$. Also $x_i \not\to o$ because of the uniform upper bound (3.23). Hence we have again the convergence $x_i \to x \in \overline{B}_R(o) \setminus \{o\}$, up to subsequence, and one derives a contradiction as in the previous case.

Recalling Lemma 3.5, for any $\sigma \in (-\infty, T), \varepsilon > 0$ there exists $\tilde{p} > 1$ such that

$$\begin{aligned} 4\pi \left(\frac{3-p_i}{p_i - 1}\right)^{p_i - 1} e^{t-\varepsilon} &= \mathrm{Cap}_{p_i}(E_{t-\varepsilon}^{p_i}, B_{2R}(o)) \leq \mathrm{Cap}_{p_i}(E_t, B_{2R}(o)) \leq \mathrm{Cap}_{p_i}(E_{t+\varepsilon}^{p_i}, B_{2R}(o)) \\ &= 4\pi \left(\frac{3-p_i}{p_i - 1}\right)^{p_i - 1} e^{t+\varepsilon}, \end{aligned}$$

for any $p_i \in (1, \tilde{p})$ and $t \in [\sigma, T]$. Letting first $p_i \to 1$ and then $\varepsilon \to 0$ and $\sigma \to -\infty$, we find that

$$\lim_i \mathrm{Cap}_{p_i}(E_t, B_{2R}(o)) = 4\pi e^t,$$

for any $t \in (-\infty, T]$. Then the first equality in (3.24) will follow if we prove that

$$\lim_{p \to 1} \mathrm{Cap}_p(E_t, B_{2R}(o)) = P(\Omega_t), \tag{3.27}$$

for any $t \in (-\infty, T]$.
Fix $t \in (-\infty, T]$. It follows from the very definition of weak IMCF as in [45, Minimizing Hull Property 1.4] that E_t is strictly outward minimizing relatively to $B_{2R}(o)$; meaning that whenever $E_t \subsetneq F \subset\subset B_{2R}(o)$, then $P(E_t) < P(F)$. It is readily checked that the proof of [35, Theorem 1.2] can be localized in the ball $B_{2R}(o)$, thus yielding that

$$\lim_{p \to 1} \mathrm{Cap}_p(E_t, B_{2R}(o)) = P(E_t).$$

Finally, as in [45, Minimizing Hull Property 1.4(iv)], there holds $P(E_t) = P(\Omega_t)$, so that (3.27) follows.

We show the second property in (3.24). By applying (3.23) with $\overline{r} = 2r(x)$ and sending $p = p_i \to 1^+$, we get that there exists $\tilde{\eta}, \vartheta > 0$ such that if $r(x) < \tilde{\eta}$, then $w(x) \leq 2\log(r(x)) + \vartheta$. On the other hand, for $\bar{t}$ small enough we have $\Omega_t \subset B_{e^{(t+C)/2}}(o)$ for every $t < \bar{t}$, see item (4) in Theorem 3.6. Up to possibly taking a smaller $\bar{t}$, for every $t < \bar{t}$ we have $\partial\Omega_t \subset \{w = t\} \subset M \setminus B_{e^{(t-\vartheta)/2}}$. Thus

$$\operatorname{esssup}_{\partial\Omega_t} |\nabla w| \leq \operatorname{esssup}_{\partial\Omega_t} \frac{\zeta}{\mathsf{d}(o,x)} \leq \zeta e^{\vartheta/2} e^{-t/2},$$

for almost every $t < \bar{t}$, where the first inequality comes from the fact that (3.15) passes to the limit as $p_i \to 1^+$. Thus, for almost every $t \in (-\infty, \bar{t})$, there holds

$$\int_{\partial\Omega_t} |\nabla w|^2 \leq |\partial\Omega_t| \zeta^2 e^{\vartheta} e^{-t} = 4\pi\zeta^2 e^{\vartheta} =: C_1 < +\infty,$$

and then the first item is proved recalling that $H = |\nabla w|$ $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega_t$ for almost every $t \in (-\infty, \bar{t})$.

The fact that $\mathfrak{m}_H(\partial\Omega_t) \to 0$ for almost every $t \to -\infty$ is a direct consequence of the first item and the definition of Hawking mass. The last part of the second item is a consequence of the analogue of [45, Geroch Monotonicity Formula 5.8] in our setting. A detailed proof the Geroch Monotonicity Formula for weak IMCFs defined through limits of p-harmonic functions as in our setting will be provided in the forthcoming [13]. □

3.2. **Connectedness of level sets of the weak IMCF.** In the following lemma we collect some facts about connectedness of level set of the weak IMCF. This is analogous to [45, Connectedness Lemma 4.2], we provide a proof for the convenience of the reader.

Lemma 3.14. *Let Ω be an open set with Lipschitz boundary in a smooth complete n-dimensional Riemannian manifold (M, g) with $n \leq 7$, and let $o \in \Omega$. Let $\Omega \subset\subset U$, where U is an open set, and let $w \in \mathrm{Lip}_{\mathrm{loc}}(U \setminus \{o\})$ be a weak solution of the IMCF on $U \setminus \{o\}$, see Definition 3.1. Then the following hold.*

(1) *Let $t \in \mathbb{R}$. Then every connected component of $\{w < t\} \cap \Omega$ (resp., $\{w > t\} \cap \Omega$) is not relatively compact in $\Omega \setminus \{o\}$.*
(2) *Assume further that $\partial\Omega$ is connected, and there exist $t_0 < t_1$ such that:*
    - *$\{w < t_1\} \subset\subset \Omega$;*
    - *there exists an open set $\Omega'' \ni o$, such that $\Omega''\setminus\{o\}$ is connected, and $\Omega''\setminus\{o\} \subset \{w < t_0\}$.*

    Then for every $t \in (t_0, t_1)$ we have that both $\{w < t\}$ and $\{w > t\} \cap \Omega$ are connected.
(3) *Let the hypotheses of (2) above be satisfied. Assume further that Ω is connected, and $H_1(\Omega; \mathbb{Z}) = \{0\}$. Then $\partial\{w < t\}$ is connected for every $t \in (t_0, t_1)$.*

Proof of Lemma 3.14. The proof of item (1) follows verbatim as in [45, Connectedness Lemma 4.2(i)]. Let us repeat it here for the ease of the reader. Assume by contradiction a connected component C of $\{w > t\} \cap \Omega$ is relatively compact in $\Omega \setminus \{o\}$. Then consider the function $v := w$ on $U \setminus (C \cup \{o\})$ and $v := t$ on C. By (3.1) we get

$$\int_C |\nabla w| + w|\nabla w| \leq \int_C t|\nabla w|.$$

Since $C \subset \{w > t\}$, the latter implies that $|\nabla w| = 0$ on C, and thus $w = t$ on C, which is a contradiction.

Now assume by contradiction a connected component C' of $\{w < t\} \cap \Omega$ is relatively compact in $\Omega \setminus \{o\}$. Take $\bar{t} := \min_{\overline{C'}} w$. Then for $0 < \eta < 1$ small enough there is a connected component C'' of $\{w < \bar{t} + \eta\} \cap \Omega$ inside C'. Notice that $w \geq \bar{t} > \bar{t} + \eta - 1$ on C''. Repeating the previous argument with $\bar{t} + \eta$ in place of t and C'' in place of C gives again a contradiction.

The item (2) follows from item (1). First, by the assumption in the first bullet, for $t \in (t_0, t_1)$ every connected component of $\{w < t\}$ stays away from $\partial\Omega$. Then, from item (1), for every $t \in (t_0, t_1)$, o is in the closure of any connected component of $\{w < t\}$. Then every connected component of

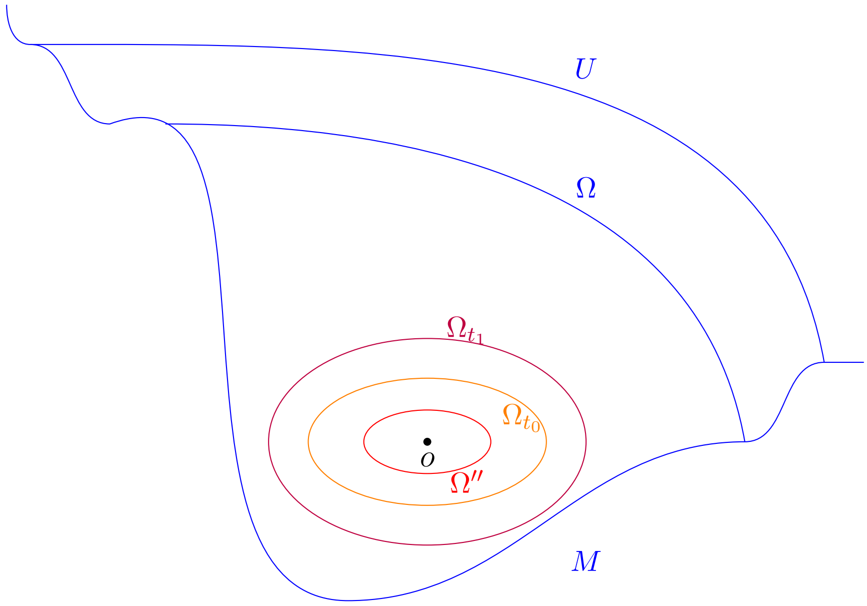

FIGURE 1. The picture sketches the assumptions of Lemma 3.14.

$\{w < t\}$ intersects $\Omega'' \setminus \{o\}$. Since $\Omega'' \setminus \{o\}$ is connected, and $\Omega'' \setminus \{o\} \subset \{w < t_0\} \subset \{w < t\}$, thus every connected component of $\{w < t\}$ contains $\Omega'' \setminus \{o\}$. Thus, there exists at most one connected component of $\{w < t\}$, because every connected component contains $\Omega'' \setminus \{o\}$, which is itself connected. Similarly, from the hypotheses of the second bullet, for every $t \in (t_0, t_1)$, we have $\partial\Omega \subset \{w > t\}$, and $\{w > t\} \cap \Omega \subset \Omega \setminus \overline{\Omega''}$. Thus, every connected component of $\{w > t\} \cap \Omega$ avoids o, and then, from item (1), its closure must then intersect $\partial\Omega$. Since $\partial\Omega$ is connected, there exists at most one connected component of $\{w > t\} \cap \Omega$, arguing as before.

The item (3) is inspired by [45, Connectedness Lemma 4.2(ii)]. Analogous arguments have appeared in [68, Lemma 2.3], [60, Lemma 6.1], and [53, Lemma 4.46]. We give here a self-contained proof using item (2) and the Mayer–Vietoris sequence.

Recall from item (1) of Section 3.1 that $\partial\{w < t\}$ is $C^{1,\alpha}$ for every $t \in (t_0, t_1)$, and that $\partial\{w < t\} = \{w = t\}$ for almost every $t \in (t_0, t_1)$. It is sufficient to prove that $\partial\{w < t\}$ is connected for t such that $\partial\{w < t\} = \{w = t\}$. Indeed, for $\tau \in (t_0, t_1)$ such that $\partial\{w < \tau\} \neq \{w = \tau\}$, there exists a sequence $t_i \nearrow \tau$ such that $\partial\{w < t_i\} = \{w = t_i\}$. From item (1) of Section 3.1 we know that $\partial\{w < t_i\} \to \partial\{w < \tau\}$ in C^1, hence connectedness will be preserved in the limit.

Hence we can assume by contradiction that there exists $t \in (t_0, t_1)$ such that $\partial\{w < t\} = \{w = t\}$ is not connected. Since $\partial\{w < t\}$ is $C^{1,\alpha}$, it has a finite number $m \geq 2$ of connected components. Since $\bigcap_{\eta>0}\{t - \eta < w < t + \eta\} = \{w = t\} = \partial\{w < t\}$, there exists η small enough such that $[t - \eta, t + \eta] \subset (t_0, t_1)$ and $\{t - \eta < w < t + \eta\}$ has $m' \geq 2$ connected components. Call $A := \{w < t + \eta\} \cup \{o\}$ and $B := \{w > t - \eta\} \cap \Omega$. Notice that A and B are open, and by item (2) they are both connected.

Finally notice that $A \cap B = \{t - \eta < w < t + \eta\}$ is not connected, and $A \cup B = \Omega$. The Mayer–Vietoris exact sequence (where homology is understood with integer coefficients) ends with

$$\ldots \to H_1(\Omega) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(\Omega) \to 0.$$

Recall that for a topological space X there holds $H_0(X; \mathbb{Z}) \cong \mathbb{Z}^\ell$, where ℓ is the number of connected components of X. Thus by using the assumptions of item (3) the previous exact sequence becomes

$$\ldots \to 0 \to \mathbb{Z}^{m'} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0,$$

from which $\mathbb{Z} \cong \mathbb{Z}^2/\mathbb{Z}^{m'}$, and thus $m' = 1$, which results in a contradiction. $\square$

4. Proof of the main results

In this section we prove the main theorems Theorem 1.6, Theorem 1.4.

4.1. **Producing a set satisfying the reverse Euclidean isoperimetric inequality.**

In this section we show how, in the hypotheses of Theorem 1.4, we can produce sets with arbitrarily large perimeter and volume that satisfy the Euclidean reserve isoperimetric inequality with sharp constant, see Proposition 4.2.

Lemma 4.1. *Let (M, g) be a smooth complete Riemannian manifold of dimension 3. Let $o \in M$ and $R > 0$. Let w be given by Theorem 3.6. Let $T \in \mathbb{R}$ be such that $\{w < T\} \subset\subset B_\rho(o)$ for some $\rho \leq R$. Suppose that $\partial\{w < t\}$ is connected for any $t < T$. If $R_g \geq -\delta$ on $B_\rho(o)$, for some $\delta > 0$, then*

$$|\Omega_t| \geq \frac{1}{\sqrt{1+\frac{2}{3}\delta e^T}} \frac{P(\Omega_t)^{3/2}}{6\sqrt{\pi}} \qquad \forall t < T,$$

where $\Omega_t := \{w < t\}$.

Proof. We recall that Ω_t has $C^{1,\alpha}$ boundary, $\partial\Omega_t$ admits weak mean curvature H for all $t < T$, and $H = |\nabla w| > 0$ $\mathcal{H}^2$-a.e. for a.e. $t \in (-\infty, T)$, see item (1) of Section 3.1. The Hawking mass of a set Ω with C^1 boundary $\partial\Omega$ possessing weak mean curvature H is given by

$$\mathfrak{m}_H(\partial\Omega) = \frac{P(\Omega)^{1/2}}{(16\pi)^{3/2}} \left(4\pi - \int_{\partial\Omega} \frac{H^2}{4}\right).$$

As $\{w \leq T'\} \subset\subset B_\rho(o)$ for any $T' < T$, we can apply Proposition 3.13, which yields that $P(\Omega_t) = 4\pi e^t$ for any $t < T$, and that $\mathfrak{m}_H(\partial\Omega_t) \to 0$ for almost every $t \to -\infty$. Hence the Geroch Monotonicity formula (3.26) with $r \to -\infty$ and $s = t$, together with the fact that $R_g \geq -\delta$ on $B_\rho(o)$, gives that for all $t \in (-\infty, T)$ there holds

$$\mathfrak{m}_H(\partial\Omega_t) \geq -\frac{\delta}{(16\pi)^{3/2}} \int_{-\infty}^t P(\Omega_t)^{3/2} = -\frac{\delta}{(16\pi)^{3/2}} \int_{-\infty}^t (4\pi)^{3/2} e^{3t/2} = -\frac{\delta}{12} e^{3t/2}.$$

Since $\mathfrak{m}_H(\partial\Omega_t) = \frac{2\pi^{1/2} e^{t/2}}{64\pi^{3/2}} \left(4\pi - \int_{\partial\Omega_t} \frac{H^2}{4}\right)$, for every $t \in (-\infty, T)$, we find

$$\int_{\partial\Omega_t} H^2 \leq 16\pi + \frac{32}{3}\delta\pi e^t. \tag{4.1}$$

By Hölder inequality we get that for almost every $t \in (-\infty, T)$ there holds

$$P(\Omega_t) \leq \left(\int_{\partial\Omega_t} |\nabla w|^2\right)^{1/3} \left(\int_{\partial\Omega_t} \frac{1}{|\nabla w|}\right)^{2/3}. \tag{4.2}$$

Hence, by recalling that $\int_{\partial\Omega_t} |\nabla w|^2 = \int_{\partial\Omega_t} H^2$ is finite by (4.1) for almost all $t \in (-\infty, T)$, recalling that $|\nabla w| > 0$ $\mathcal{H}^2$-a.e. on $\partial\Omega_t$ for a.e. $t \in (-\infty, T)$, and by using the coarea formula together with (4.2) and (4.1), for any $t < T$ we obtain

$$\begin{aligned}
|\Omega_t| &\geq \int_{\Omega_t \cap \{|\nabla w| > 0\}} \frac{1}{|\nabla w|} |\nabla w| = \int_{-\infty}^t \left(\int_{\partial\Omega_\tau \cap \{|\nabla w| > 0\}} \frac{1}{|\nabla w|}\right) \mathrm{d}\tau \\
&= \int_{-\infty}^t \left(\int_{\partial\Omega_\tau} \frac{1}{|\nabla w|}\right) \mathrm{d}\tau \geq \int_{-\infty}^t \left(P(\Omega_\tau)^{3/2} \left(\int_{\partial\Omega_\tau} |\nabla w|^2\right)^{-1/2}\right) \\
&\geq \int_{-\infty}^t \frac{2\pi e^{3t/2}}{\sqrt{1+\frac{2}{3}\delta e^t}} \geq \int_{-\infty}^t \frac{2\pi e^{3t/2}}{\sqrt{1+\frac{2}{3}\delta e^T}} = \frac{1}{\sqrt{1+\frac{2}{3}\delta e^T}} \frac{4}{3}\pi e^{3t/2} = \frac{1}{\sqrt{1+\frac{2}{3}\delta e^T}} \frac{P(\Omega_t)^{3/2}}{6\sqrt{\pi}}.
\end{aligned}$$

□

In the following proposition we construct sets with arbitrarily large perimeter and volume that satisfy the reverse Euclidean isoperimetric inequality. This represents the crucial step for the proof of the main results.

Proposition 4.2. *Let (M, g) be a 3-dimensional complete C^0-Riemannian manifold without boundary, let $K \subset M$ be a compact set, and let Ω be an unbounded connected component of $M \setminus K$. Assume that Ω is C^0_{loc}-asymptotic to $\mathbb{R}^3$ (Definition 1.3), and that $R_g \geq 0$ in the approximate sense on $\Omega \setminus K'$ (Definition 1.2), where $K' \subset M$ is a compact set. Then, there exists a universal constant $\vartheta \in (0, 1)$ such that the following holds.*

For every $\mathscr{P} > 0$, there exists a set of finite perimeter $E \subset\subset \Omega \setminus K'$ such that
$$\vartheta \mathscr{P} \leq P(E) \leq \mathscr{P},$$
and
$$|E| \geq \frac{1}{6\sqrt{\pi}} \mathscr{P}^{3/2} \geq \frac{1}{6\sqrt{\pi}} P(E)^{3/2}. \tag{4.3}$$

Proof. Let $\rho > 1$, $\eta < 1/2$ to be chosen. The choice of ρ, only depending on $\mathscr{P}$ and on geometric constants on $\mathbb{R}^3$, will be made clear during the proof in (4.8). We do not insist on the precise choice of η for the sake of readability; however, it will be clear from the proof that choosing $\eta < 10^{-3}$ is sufficient. In this proof we will repeatedly use the elementary metric results recorded in Lemma 2.2, and in the last part of the proof of Lemma 2.12.

By applying a contradiction argument and Remark 2.4 (see also the beginning of the proof of Corollary 2.14) we can find a compact set $\mathcal{C} \supset (K \cup K')$ such that for every $x \in \Omega \setminus \mathcal{C}$ there exists $F_x : (B_{64\rho+64}(x), g) \to (\mathbb{R}^3, g_{\mathrm{eu}})$ which is a $(1 + \eta)$-biLipschitz diffeomorphism with its image, with $F_x(x) = 0$, and whose image contains $B^{\mathbb{R}^3}_{32\rho+32}(0)$. Let us now fix o such that
$$B_{64\rho+64}(o) \subset\subset \Omega \setminus \mathcal{C}.$$
Let g_i be a sequence of smooth Riemannian metrics on M converging to g locally uniformly with $R_{g_i} \geq -\varepsilon_i$ on $\Omega \setminus \mathcal{C}$. We will frequently pass to subsequences with respect to i in the course of the proof, without relabeling. We will denote $B_s(p), B^i_s(p)$ the open balls of radius s and center p in the metrics g, g_i, respectively. Up to passing to a subsequence with respect to i, we can assume that
$$B^i_{\rho-1}(o) \subset\subset B_\rho(o) \subset F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0)) \subset\subset B^i_{4\rho+4}(o) \subset\subset \Omega \setminus \mathcal{C} \text{ for all } i \in \mathbb{N}. \tag{4.4}$$

Notice that (see Definition 2.3) the compact $\mathcal{C}$ can be chosen such that, additionally, we have
$$|(g - F_o^* g_{\mathrm{eu}})_x(v, v)| \leq \eta (F_o^* g_{\mathrm{eu}})_x(v, v), \tag{4.5}$$
for every $x \in B_{64\rho+64}(o)$, and every $v \in T_x M$. Then, for i large enough, we have
$$|(g^i - F_o^* g_{\mathrm{eu}})_x(v, v)| \leq 2\eta (F_o^* g_{\mathrm{eu}})_x(v, v), \tag{4.6}$$
for every $x \in B^i_{32\rho+32}(o)$, and every $v \in T_x M$. As a consequence, the map $F_x : (B^i_{16\rho+16}(x), g_i) \to (\mathbb{R}^3, g_{\mathrm{eu}})$ is $(1 + 3\eta)$-biLipschitz with its image. Notice also that the image of this map contains the ball $B^{\mathbb{R}^3}_{8\rho+8}(0)$.

As a consequence of (4.6), if η is small enough, the constants
$$C_{1,\mathrm{Sob}}(B^i_{4\rho+4}(o)), C_P(B^i_{4\rho+4}(o)), C_A(B^i_{4\rho+4}(o))$$
appearing in Theorem 3.6 (and defined in (3.2), (3.9), (3.10)), are uniformly bounded from above by a universal constant multiplied by the value of the corresponding constants on $\mathbb{R}^3$, which are independent on ρ. Moreover, since $F_x : (B^i_{16\rho+16}(x), g_i) \to (\mathbb{R}^3, g_{\mathrm{eu}})$ is $(1 + 3\eta)$-biLipschitz with its image, and contains the ball $B^{\mathbb{R}^3}_{8\rho+8}(0)$, the following holds: for every $r \leq \rho + 1$ we can connect any two points $p, q \in \partial B^i_r(o)$ with a continuous curve γ in the annulus $A^i_{3r/4,5r/4}(o)$. Indeed, it suffices to connect $F_x(p)$ with $F_x(q)$ with a curve $\tilde{\gamma} \subset \mathbb{R}^3$ in the annulus $A^{\mathbb{R}^3}_{7r/8,9r/8}(0)$, and take $\gamma := F_x^{-1}(\tilde{\gamma})$. Moreover, with the same reasoning, the constant $C_{\mathrm{cov}}(B_{\rho+1}(o))$ defined in (3.11) is

bounded above by a universal constant which only depends on the following universal constant independent on ρ:

$$\tilde{C} := \min\Big\{ N \in \mathbb{N} \ : A^{\mathbb{R}^3}_{r/2,3r/2}(0) \text{ is covered by } N \text{ open balls of radius } r/4 \\ \text{with centers in } A^{\mathbb{R}^3}_{r/2,3r/2}(o) \text{ for any } 0 < r \leq \rho + 1 \Big\}. \tag{4.7}$$

Hence, we can apply Theorem 3.6 on the ball $B^i_{2R}(o) := B^i_{4\rho+4}(o)$, where the radius $\rho+1$ in this proof corresponds to the number ρ in the statement of Theorem 3.6. As a result of the discussion above, the constant C appearing in Theorem 3.6 when applied to the ball $B^i_{4\rho+4}(o)$ is bounded above by a universal constant $\xi > 1$ independent of i, ρ, for i large enough.

Now let w_i be the weak IMCF issuing from o in $B^i_{4\rho+4}(o)$, given by Theorem 3.6. Denote $\Omega^i_t := \{w_i < t\}$. Let $T_{\rho,\xi} := 2\log(\rho-1) - \xi - 1$, and take $\rho > 1$ such that

$$4\pi e^{T_{\rho,\xi}} = 4\pi e^{2\log(\rho-1)-\xi-1} = \mathscr{P}. \tag{4.8}$$

Notice that the choice of ρ only depends on $\mathscr{P}$ and on the universal constant ξ. By the last assertion of Theorem 3.6, applied with $r := \rho - 1$, we have that

$$\{w_i < T_{\rho,\xi}\} = \Omega^i_{T_{\rho,\xi}} \subset B^i_{\rho-1}(o) \subset\subset B_\rho(o).$$

Now we aim at applying item (3) of Lemma 3.14 with the choices $U := B^i_{4\rho+4}(o)$, $\Omega := F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0))$, and $t_1 := T_{\rho,\xi}$. We stress that $\{w_i < T_{\rho,\xi}\} \subset\subset F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0))$, and

$$F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0)), \text{ and } \partial F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0)) = F_o^{-1}(\partial B^{\mathbb{R}^3}_{2\rho}(0)) \text{ are connected.}$$

Moreover, for any arbitrary $t_0 < T_{\rho,\xi}$, one can define Ω'' to be a sufficiently small ball so that all the hypotheses of item (2) in Lemma 3.14 are met, since we have $w_i(x) \to -\infty$ as $x \to o$. Finally noticing that $H_1(F_o^{-1}(B^{\mathbb{R}^3}_{2\rho}(0)); \mathbb{Z}) = \{0\}$, one gets that all the hypotheses of item (2) and (3) in Lemma 3.14 are met, and thus $\partial\{w_i < t\}$ is connected for every $t < T_{\rho,\xi}$.

Therefore, recalling that $R_{g_i} \geq -\varepsilon_i$ on $B^i_{4\rho+4}(o)$, by Lemma 4.1 we get

$$|\Omega^i_{T_{\rho,\xi}}|_i \geq \frac{1}{\sqrt{1+\frac{2}{3}\varepsilon_i e^{T_{\rho,\xi}}}} \frac{P_i(\Omega^i_{T_{\rho,\xi}})^{3/2}}{6\sqrt{\pi}} = \frac{1}{\sqrt{1+\frac{2}{3}\varepsilon_i e^{T_{\rho,\xi}}}} \frac{(4\pi e^{T_{\rho,\xi}})^{3/2}}{6\sqrt{\pi}} = \frac{1}{\sqrt{1+\frac{2}{3}\varepsilon_i \frac{\mathscr{P}}{4\pi}}} \frac{\mathscr{P}^{3/2}}{6\sqrt{\pi}}, \tag{4.9}$$

where we used that $P_i(\Omega^i_t) = 4\pi e^t$, see Proposition 3.13, where $P_i(\cdot)$ denotes perimeter computed with respect to the metric g_i.

Now, since $\Omega^i_{T_{\rho,\xi}} \subset\subset B_\rho(o)$, and the perimeters $P_i(\Omega^i_{T_{\rho,\xi}})$ are equibounded, we can use the precompactness and lower semicontinuity result in Lemma 2.11 to get a set $E_\rho \subset B_\rho(o)$ such that $\Omega^i_{T_{\rho,\xi}} \to E_\rho \subset\subset \Omega \setminus \mathcal{C}$ in L^1, $P(E_\rho) \leq \mathscr{P}$, and, by passing (4.9) to the limit, such that

$$|E_\rho| \geq \frac{\mathscr{P}^{3/2}}{6\sqrt{\pi}}, \tag{4.10}$$

which completes the proof of (4.3).

Finally, notice that $E_\rho \subset\subset B_{\rho+1}(o)$. Moreover (4.5) holds. This implies that, if $\eta < 1$ is small enough, on $B_{\rho+1}(o)$ there holds a $(1,1^*)$-Sobolev inequality (3.2) with $C_{1,\mathrm{Sob}}(B_{\rho+1}(o))$ bounded from above by a universal constant only depending on the constant in the Euclidean $(1,1^*)$-Sobolev inequality. By Remark 3.4, this implies that $P(E) \geq \tilde{\vartheta}|E|^{2/3}$ for a universal $\tilde{\vartheta}$, for every $E \subset\subset B_{\rho+1}(o)$. Applying the latter inequality on E_ρ, and using again (4.10), we finally get $P(E_\rho) \geq \vartheta \mathscr{P}^{2/3}$ for a universal ϑ, concluding the proof. □

4.2. **Proof of the main results and consequences.** We are now ready to prove the main results of the paper.

Proof of Theorem 1.4. It is a direct consequence of Proposition 4.2 and the definition of isoperimetric mass in Definition 1.1. □

Proof of Theorem 1.6. Suppose by contradiction that there exists $v_0 > 0$ such that for every $v \in (v_0, +\infty)$ there are no isoperimetric sets of volume v in M. We claim that then I is strictly increasing on $(2v_0, +\infty)$.

Let $v \in (2v_0, +\infty)$. By Theorem 2.16 there is an isoperimetric set $E \subset M$ with $|E| \leq v_0$ (possibly empty) and a ball $B \subset \mathbb{R}^3$ with $|B| \geq v - v_0 \geq v/2$ such that

$$v = |E| + |B|_{\mathrm{eu}}, \qquad I(v) = P(E) + P_{\mathrm{eu}}(B).$$

Let now $\varepsilon > 0$ be such B_ε is a ball in $\mathbb{R}^3$ concentric to B and with volume $|B_\varepsilon|_{\mathrm{eu}} = |B|_{\mathrm{eu}} - \varepsilon$. Notice that, by approximating $B_\varepsilon \subset \mathbb{R}^3$ with sets diverging along the manifold, arguing as in the proof of the upper bound for the isoperimetric profile in Corollary 2.15, we get $I(v-\varepsilon) \leq P(E) + P_{\mathrm{eu}}(B_\varepsilon)$. Thus, taking $\varepsilon \to 0$, we find

$$\frac{I(v) - I(v-\varepsilon)}{\varepsilon} \geq \frac{P_{\mathrm{eu}}(B) - P_{\mathrm{eu}}(B_\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 2\left(\frac{4\pi}{3}\right)^{1/3} |B|_{\mathrm{eu}}^{-1/3} \geq 2\left(\frac{4\pi}{3}\right)^{1/3} v^{-1/3}.$$

Thus we get that the lower left Dini derivative satisfies

$$\overline{D^-} I(v) := \liminf_{\varepsilon \to 0^+} \frac{I(v) - I(v-\varepsilon)}{\varepsilon} \geq 2\left(\frac{4\pi}{3}\right)^{1/3} v^{-1/3} > 0, \qquad \forall v \in (2v_0, +\infty).$$

Since I is continuous by Corollary 2.15, the latter implies that I is strictly increasing on $(2v_0, +\infty)$. Thus the sought claim is proved.

We aim now at showing that there exists an isoperimetric set with volume strictly greater than $2v_0$, thus reaching a contradiction. Fix $v > 2v_0$. By Theorem 2.16 there is an isoperimetric set $E \subset M$ with $|E| \leq v_0$ and a ball $B \subset \mathbb{R}^3$ with $|B| \geq v - v_0 > 0$ such that

$$v = |E| + |B|_{\mathrm{eu}}, \qquad I(v) = P(E) + P_{\mathrm{eu}}(B).$$

By Theorem A.2 we know that E is bounded.

We apply Proposition 4.2 with $K = \emptyset$ and $K' = \overline{B}_r(o)$ for some ball $B_r(o)$ with $r > 1$ such that $E \cup \mathcal{C} \subset B_{r-1}(o)$. Then there is a set $F \subset M$ such that $F \subset\subset M \setminus K'$, $P(F) \leq P_{\mathrm{eu}}(B)$, and

$$|F| \geq \frac{1}{6\sqrt{\pi}} P_{\mathrm{eu}}(B)^{3/2} = |B|_{\mathrm{eu}}.$$

Thus the set $E \cup F$ is such that

$$|E \cup F| = |E| + |F| \geq |E| + |B|_{\mathrm{eu}} = v.$$

Hence, since I is strictly increasing on $(2v_0, +\infty)$, one gets

$$I(|E \cup F|) \geq I(v) = P(E) + P_{\mathrm{eu}}(B),$$

but at the same time

$$I(|E \cup F|) \leq P(E \cup F) = P(E) + P(F) \leq P(E) + P_{\mathrm{eu}}(B).$$

Hence all the inequalities in the previous formula are equalities, and then $E \cup F$ is an isoperimetric set with volume $> 2v_0$, which is a contradiction. $\square$

In the proof of Theorem 1.6 we argued that, if for some $v_0 > 0$ no isoperimetric sets exist for volumes $v \geq v_0$, then the isoperimetric profile is strictly increasing for large volumes. This implied that some isoperimetric set of large volume must exist, resulting in a contradiction. In fact, if the isoperimetric profile is strictly increasing, then one can deduce that isoperimetric sets exist for any volume. This is what happens in a smooth asymptotically flat 3-manifold of nonnegative scalar curvature that is complete with no closed minimal surfaces or endowed with a horizon boundary. For this reason, the following result can be seen as a generalization of the existence result for any volume obtained in [23, Proposition K.1] (see also [10, Theorem 3.6]). Moreover, when in addition the isoperimetric profile diverges at infinity, such isoperimetric sets realize the isoperimetric mass in the sense of (4.11). The isoperimetric profile diverges at infinity for instance when a global

Euclidean-like isoperimetric inequality is in force. The latter happens, for example, when the manifold is C^0-asymptotic to $\mathbb{R}^3$ according to Definition 1.8.

Proposition 4.3. *Let (M, g) be a C^0-Riemannian manifold that is C^0_{loc}-asymptotic to $\mathbb{R}^3$, and such that $R_g \geq 0$ in the approximate sense on $M \setminus K$, where $K \subset M$ is a compact set. Assume that its isoperimetric profile $I(v)$ is strictly increasing. Then, for every volume v, there exists an isoperimetric set E_v of volume v on M. If in addition $\lim_{v\to+\infty} I(v) = +\infty$, then*

$$\mathfrak{m}_{\mathrm{iso}} = \limsup_{v\to+\infty} \frac{2}{P(E_v)} \left(|E_v| - \frac{P(E_v)^{3/2}}{6\sqrt{\pi}} \right). \tag{4.11}$$

Proof. We assume by contradiction that, for some $v > 0$, there exists no isoperimetric sets of volume v. By Theorem 2.16, we have $I(v) = P(E) + P_{\mathrm{eu}}(B)$ for a possibly empty $E \subset M$ realizing $P(E) = I(|E|)$ and $B \subset \mathbb{R}^3$ a nonempty ball, such that $|E| + |B|_{\mathrm{eu}} = v$. The proof of Theorem 1.6 shows that we can find a set $F \subset\subset M \setminus \overline{B}_{r+1}(o)$, for some ball such that $E \subset B_r(o)$, with $P(F) \leq P_{\mathrm{eu}}(B)$ and with volume $|F| \geq |B|_{\mathrm{eu}}$. If $|F| = |B|_{\mathrm{eu}}$, then $E \cup F$ is an isoperimetric set of volume v, giving a contradiction. If $|F| > |B|_{\mathrm{eu}}$, we derive a contradiction with strict monotonicity of I. Indeed, on the one hand we would have

$$I(|E| + |F|) > I(|E| + |B|_{\mathrm{eu}}) = I(v),$$

and on the other hand there holds

$$I(|E| + |F|) \leq P(E) + P(F) \leq P(E) + P_{\mathrm{eu}}(B) = I(v).$$

This proves the existence of isoperimetric sets of any volume.

We are left to prove (4.11). Observe that, since the isoperimetric profile diverges at infinity, the isoperimetric sets E_v have perimeter diverging to infinity as $v \to +\infty$, and thus they are valid competitors in the definition (1.1) of $\mathfrak{m}_{\mathrm{iso}}$. Hence

$$\mathfrak{m}_{\mathrm{iso}} \geq \limsup_{v\to+\infty} \frac{2}{P(E_v)} \left(|E_v| - \frac{P(E_v)^{3/2}}{6\sqrt{\pi}} \right).$$

On the other hand, let $(\Omega_j)_{j\in\mathbb{N}}$ be any other sequence of finite perimeter sets such that $P(\Omega_j) \to +\infty$. Let E_j be an isoperimetric set of volume $V_j = |\Omega_j|$. Then, the sequence $(E_j)_{j\in\mathbb{N}}$ satisfies

$$\frac{2}{P(\Omega_j)} \left(|\Omega_j| - \frac{P(\Omega_j)^{3/2}}{6\sqrt{\pi}} \right) \leq \frac{2}{P(E_j)} \left(|E_j| - \frac{P(E_j)^{3/2}}{6\sqrt{\pi}} \right),$$

implying (4.11). $\square$

Appendix A. Auxiliary results

In this appendix we collect two technical results we used in the paper.

Lemma A.1. *Let (M, g) be an n-dimensional complete C^0-Riemannian manifold. Then the following hold.*

- *Let $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$. Then $\mathrm{lip}\, f = |\nabla f|$ almost everywhere.*
- *Let $\Omega \subset M$ be an open set and suppose that g_i is a sequence of smooth Riemannian metrics converging to g uniformly on Ω. Then for any $f \in L^1_{\mathrm{loc}}(\Omega, g)$ there holds*

  $$\lim_i |Df|_i(\Omega) = |Df|(\Omega),$$

  where $|Df|_i$ denotes the total variation of f as a function in $L^1_{\mathrm{loc}}(\Omega, g_i)$.
- *For any $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$, there holds $|Df| = \mathrm{lip}\, f\, \mathcal{H}^n = |\nabla f|\, \mathcal{H}^n$. In particular*

  $$|D\mathsf{d}_{x_0}| = \mathcal{H}^n,$$

  for any $x_0 \in M$, where d_{x_0} denotes distance from x_0.

Proof. Let $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$. Recall that the identity $\mathrm{lip}\, f = |\nabla f|$ almost everywhere readily follows on smooth Riemannian manifolds exploiting the exponential map. Fix $x \in M$ and let g_i be a sequence of smooth Riemannian metrics converging to g in C^0-sense on a neighborhood A of x. We can write $(1-\varepsilon_i)^2 g(v,v) \le g_i(v,v) \le (1+\varepsilon_i)^2 g(v,v)$ for any tangent vector v on A, for some $\varepsilon_i \to 0$. Denote by d_i the distance function on (A, g_i) defined by taking infimum of lengths of curves contained in A. For any i, let $y_j \in A$ such that $\lim_j \mathsf{d}_i(x, y_j) = 0$ and

$$\mathrm{lip}_i f(x) = \lim_j \frac{|f(x) - f(y_j)|}{\mathsf{d}_i(x, y_j)},$$

where $\mathrm{lip}_i f$ denotes the slope of f as a function in (A, g_i). For j large, $\mathsf{d}_i(x, y_j)$ is realized by a curve contained in Ω. Hence $\mathsf{d}_i(x, y_j) \ge (1-\varepsilon_i)\mathsf{d}(x, y_j)$ for any j large. Thus

$$\mathrm{lip} f(x) \ge \limsup_j \frac{|f(x) - f(y_j)|}{\mathsf{d}(x, y_j)} \ge (1-\varepsilon_i) \lim_j \frac{|f(x) - f(y_j)|}{\mathsf{d}_i(x, y_j)} = (1-\varepsilon_i)\mathrm{lip}_i f(x).$$

A symmetric argument implies that $\lim_i \mathrm{lip}_i f(x) = \mathrm{lip}\, f(x)$. Since $\mathrm{lip}_i f = |\nabla^{g_i} f|_{g_i}$ almost everywhere, and norms of gradients clearly pass to the limit, then $\mathrm{lip}\, f(x) = \lim_i |\nabla^{g_i} f|_{g_i} = |\nabla f|$ almost everywhere.

Let now f, g_i be as in the second item. We can write again that $(1-\varepsilon_i)^2 g(v,v) \le g_i(v,v) \le (1+\varepsilon_i)^2 g(v,v)$ for any tangent vector v on Ω, for some $\varepsilon_i \to 0$. Denote by d_i the distance function on (Ω, g_i) defined by taking infimum of lengths of curves contained in Ω and by $|D(\cdot)|_i$ the corresponding total variation. Let $f_k \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a sequence converging to f in L^1_{loc} on (Ω, g) such that $|Df|(\Omega) = \lim_k \int_\Omega \mathrm{lip} f_k$. As before, one estimates

$$\mathrm{lip} f_k(x) \ge (1-\varepsilon_i)\mathrm{lip}_i f_k(x).$$

Therefore

$$|Df|(\Omega) = \lim_k \int_\Omega \mathrm{lip} f_k \ge (1-\tilde{\varepsilon}_i) \liminf_k \int_\Omega \mathrm{lip}_i f_k \,\mathrm{d}\mathcal{H}^n_{g_i} \ge (1-\tilde{\varepsilon}_i)|Df|_i(\Omega),$$

for suitable $\tilde{\varepsilon}_i \to 0$. Hence $|Df|(\Omega) \ge \limsup_i |Df|_i(\Omega)$. An analogous argument shows that $|Df|(\Omega) \le \liminf_i |Df|_i(\Omega)$.

Now let $f \in \mathrm{Lip}_{\mathrm{loc}}(M)$. For any $x_0 \in M$ there exist $r_0 > 0$ and a sequence of smooth metrics g_i on $B_{r_0}(x_0)$ uniformly converging to g on $B_{r_0}(x_0)$. The above argument also shows that $\mathrm{lip}_i f \to \mathrm{lip} f$ pointwise on $B_{r_0}(x_0)$, hence in L^1_{loc} on $(B_{r_0}(x_0), g)$, being f locally Lipschitz. Using the second item, and since $|Df|_i = \mathrm{lip}_i f\, \mathcal{H}^n_{g_i}$ we have

$$|Df|(B_{r_0}(x_0)) = \lim_i \int_{B_{r_0}(x_0)} \mathrm{lip}_i f \,\mathrm{d}\mathcal{H}^n_{g_i} = \int_{B_{r_0}(x_0)} \mathrm{lip} f \,\mathrm{d}\mathcal{H}^n.$$

Taking into account the first item, the third one follows as well. □

Boundedness of isoperimetric sets usually follows from a suitable deformation lemma, see e.g. [4, Theorem B.1]. It is not completely clear how to work out such an argument in the C^0 setting, yet one can slightly modify the argument, taking advantage of the asymptotic behavior imposed, to obtain the following weaker version, that still suffices for our aims.

Theorem A.2. *Let (M, g) be an n-dimensional complete C^0-Riemannian manifold that is C^0_{loc}-asymptotic to $\mathbb{R}^n$. Assume that for some $V > 0$ there exist $E \subset M$ and a nonempty Euclidean ball $B \subset \mathbb{R}^n$ such that $|E| + |B|_{\mathrm{eu}} = V$ and $I(V) = P(E) + P_{\mathrm{eu}}(B)$. Then E has a bounded representative.*

Proof. Without loss of generality, we can assume that E has positive volume. Take $p \in M$. Let $V(r) = |E \setminus B_r(p)|$. Call $A(r) = P(E, M \setminus B_r(p))$. By coarea (see Theorem 2.7), and since by Lemma A.1 we have $|D\mathsf{d}_p| = \mathcal{H}^n$, notice that $V'(r) = -P(B_r(p), E)$. Moreover, a Euclidean-like isoperimetric inequality holds for small volumes. Indeed, the proof of Corollary 2.14 (compare also with item (1) in Lemma 2.12) shows that there is a constant ζ such that $|B_r(p)| \ge \zeta r^3$ for every

$r \in (0, 1]$, and every $p \in M$. Thus [6, Proposition 3.20] gives that $I(v) \geq \xi v^{2/3}$ for every $v \in (0, \eta)$ for some $\eta, \xi > 0$. Now arguing verbatim as after [4, Equation (B.1)], we can reduce ourselves to show that, for some constant C, there holds

$$A(r) \leq -V'(r) + CV(r), \tag{A.1}$$

for almost all large enough r. Assuming by contradiction that $V(r) > 0$ for any $r > 0$, the latter would result in a contradiction by ODE comparison on $V(r)$, see [4, Theorem B.1]. To this aim, let B^r be a smooth deformation of $B \subset \mathbb{R}^n$ of volume $|B| + V(r)$ such that

$$P_{\mathrm{eu}}(B^r) \leq P_{\mathrm{eu}}(B) + CV(r), \tag{A.2}$$

where C only depends on B. Notice it is enough to take B^r to be a ball containing B. Then, $|E \cap B_r(p)| + |B^r|_{\mathrm{eu}} = V$ and thus, by using the analogue of [4, Proposition 3.2], which can be proved analogously as in [4] (again exploiting that, arguing as in [4, Lemma 2.17], the isoperimetric profile is achieved by bounded sets), we get

$$P(E) + P_{\mathrm{eu}}(B) = I(V) \leq I(|E| - V(r)) + I_{\mathbb{R}^n}(|B| + V(r)) \leq P(E \cap B_r(p)) + P_{\mathrm{eu}}(B^r). \tag{A.3}$$

On the other hand, for almost every radius, we have (see [7, Proposition 2.6], which can be applied thanks to Corollary 2.14) that

$$P(E \cap B_r(p)) \leq P(E) - P(E, M \setminus B_r(p)) + P(B_r(p), E). \tag{A.4}$$

Plugging (A.2) and (A.4) into (A.3), we are left with (A.1). So the proof is concluded. □

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