RIGIDITY OF EINSTEIN MANIFOLDS WITH POSITIVE YAMABE INVARIANT

L. Branca[1], G. Catino[2], D. Dameno [3], P. Mastrolia[4].

Abstract. We provide optimal pinching results on closed Einstein manifolds with positive Yamabe invariant in any dimension, extending the optimal bound for the scalar curvature due to Gursky and LeBrun in dimension four. We also improve the known bounds of the Yamabe invariant *via* the $L^{\frac{n}{2}}$-norm of the Weyl tensor for low-dimensional Einstein manifolds. Finally, we discuss some advances on an algebraic inequality involving the Weyl tensor for dimensions 5 and 6.

1. Introduction and main results

The study of Riemannian functionals has proven to be widely important in the context of Riemannian Geometry and Geometric Analysis: indeed, many of the so-called *special* (or *canonical*) Riemannian metrics arise as critical points of certain functionals, i.e. metrics which are solutions of the associated Euler-Lagrange equations. Given a closed smooth manifold M of dimension n, a classical example of such special cases is provided by Einstein metrics, which can be characterized as critical points of the celebrated *Einstein-Hilbert functional*

$$\mathfrak{S}(g) = \mathrm{Vol}_g(M)^{-\frac{n-2}{n}} \int_M S_g \, d\mu_g, \tag{1.1}$$

where S_g and $\mathrm{Vol}_g(M)$ are, respectively, the scalar curvature and the volume with respect to the metric g. Other famous examples can be found if we consider the case $n = 4$: for instance, the critical points of the *Weyl functional*

$$\mathfrak{W}(g) = \int_M |\mathrm{W}_g|_g^2 d\mu_g$$

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[1]Università degli Studi di Milano, Italy. Email: letizia.branca@unimi.it

[2]Politecnico di Milano, Italy. Email: giovanni.catino@polimi.it.

[3]Università degli Studi di Milano, Italy. Email: davide.dameno@unimi.it.

[4]Università degli Studi di Milano, Italy. Email: paolo.mastrolia@unimi.it.

in dimension four are exactly the so-called *Bach-flat metrics*, which have been intensively studied for many decades, due to their connection with General Relativity ([3]). The definition of the Weyl functional can be extended to higher dimensional cases, defining

$$\mathfrak{W}(g) = \int_M |\mathrm{W}_g|_g^{\frac{n}{2}}\, d\mu_g, \tag{1.2}$$

although Bach-flat metrics are no longer critical points if $n \neq 4$. It is worth to note that, for every n, (1.2) is conformally invariant, i.e. it does not change under conformal changes of metric (see Section 2 below): therefore, since, for any $n \geq 4$, Einstein metrics are also Bach-flat, this implies that a *conformally Einstein metric*, i.e. a Riemannian metric whose conformal class contains an Einstein metric, is a critical point of (1.2) as well, if $n = 4$.

Even though the existence of Einstein metrics requires, in general, strict topological conditions on M, it is always possible to find "non-obstructed" metrics by using (1.1): indeed, given a Riemannian metric g on M and its conformal class $[g]$, one can consider the so-called *Yamabe invariant* $Y(M, [g])$, which is defined as the infimum of (1.1) over the metrics $\widetilde{g} \in [g]$. It is well-known that this infimum is always attained for every conformal class $[g]$ on M and the metrics which actually achieve the minimum are *constant scalar curvature metrics* (this is closely related to the so-called *Yamabe problem*, see Section 2).

Hypotheses on the sign of the Yamabe invariant may lead to surprising conclusions, especially in the four-dimensional case: for instance, a massive contribution was given by Gursky, who proved a sharp topological lower bound for the self-dual part of the Weyl functional, assuming the non-negativity of $Y(M, [g])$ and the existence of a positive eigenvalue for the intersection form of M ([24]). Later, this result was extended by the same author to half harmonic Weyl manifolds with positive Yamabe invariant ([25]); moreover, strong rigidity results for four-manifolds with positive Yamabe invariant were proven in [16], assuming additional curvature bounds. The same inequality obtained by Gursky was proven by LeBrun for conformal classes of symplectic type on a Del Pezzo surface, removing the hypothesis on the sign of $Y(M, [g])$ ([34]). We also mention the result obtained by Chang, Gursky and Yang, who managed to prove that, given a closed Riemannian four-manifold (M, g) with positive Yamabe invariant, there always exists a metric $\widetilde{g} \in [g]$ such that the Ricci tensor is strictly positive, provided that the integral of $\sigma_2(A)$, the second elementary symmetric function of the Schouten tensor, is positive ([15]). While, on one hand, all these results hold on closed four-manifolds, on the other hand some rigidity theorems can also be proven for compact manifolds of dimension four with boundary (see, for instance, [14]).

In this paper, we are interested in sharp pinching results for Einstein closed manifolds of dimension $n \geq 4$ with positive Yamabe invariant: in particular, we are interested in conformally invariant curvature inequalities of the form

$$Y(M, [g]) \leq A(n) \left(\int_M |\mathrm{W}|^{\frac{n}{2}}\, d\mu_g \right)^{\frac{2}{n}}.$$

In dimension four, the optimal result was proven by Gursky and LeBrun for the self-dual part of the Weyl tensor, with constant $A(4) = \sqrt{6}$ ([25], [26], see Remark 1.3). As far as higher dimensional cases are concerned, Hebey and Vaugon proved that, for a Riemannian metric g on a closed manifold M such that $[g]$ contains an Einstein metric or a locally conformally flat metric, either the Yamabe invariant, which is assumed to be positive, is bounded above by the $L^{\frac{n}{2}}$-norm of the tensor $Z = \mathrm{W} + \mathring{\mathrm{Ric}}$ or (M, g) is isometric to a quotient of the standard sphere $\mathbb{S}^n$ ([27]); their method relies on the classical Bochner-Weitzenböck formula and on the Yamabe-Sobolev inequality. A similar approach was used before by Singer to prove that, if (M, g) is a n-dimensional Einstein manifold with positive scalar curvature, then (M, g) is isometric to a quotient of the standard sphere, assuming that the $L^{\frac{n}{2}}$-norm of the Weyl curvature satisfies a pinching condition ([40]). The result due to Hebey and Vaugon was improved by the second and the fourth author, exploiting a method based on the Weitzenböck formula for the Weyl tensor ([12]). Moreover, we recall that Tran generalized and improved the previous bounds on closed manifolds with harmonic Weyl curvature ([42]).

We point out that the aforementioned result are not sharp if $n > 4$, meaning that the constants $A(n)$ are not the optimal ones. In this direction, a remarkable work due to Bour and Carron provides many answers about sharp pinching results for n-dimensional closed Riemannian manifolds with positive Yamabe invariant, under topological assumptions; the proofs are obtained *via* an integral version of the Bochner-Weitzenböck formula on differential forms and a clever modification of the Yamabe invariant, which we will exploit as well throughout this paper ([6]). In order to obtain better inequalities of the desired form, we rely on the classical Weitzenböck formula for the Weyl tensor, holding on every harmonic Weyl manifold, that is

$$\frac{1}{2}\Delta|\mathrm{W}|^2 = |\nabla \mathrm{W}|^2 + \frac{2}{n}S|\mathrm{W}|^2 - 2Q, \tag{1.3}$$

where

$$Q := 2W_{pqrs}W_{ptru}W_{qtsu} + \frac{1}{2}W_{pqrs}W_{pqtu}W_{rstu} \tag{1.4}$$

(here, W_{pqrs} are the components of the Weyl tensor with respect to a local orthonormal coframe). For some useful applications, see e.g. [11, 13, 16, 19, 25, 27, 42, 44]. The first step, which also is the main result of the paper, is to obtain a sharp upper bound of the Yamabe invariant with respect to a conformally invariant functional, involving W and Q. Namely, we are able to prove the following

Theorem 1.1. *Let (M, g) be a closed (conformally) Einstein manifold of dimension $n \geq 4$ with positive Yamabe invariant. Then, either (M, g) is locally conformally flat (hence, a quotient of the round sphere) or, if $n \neq 5$ and $\mathrm{W} \not\equiv 0$,*

$$Y(M, [g]) \leq n\left(\int_M |Q|^{\frac{n}{2}}|\mathrm{W}|^{-n}d\mu_g\right)^{\frac{2}{n}}. \tag{1.5}$$

Moreover, equality holds in (1.5) *if and only if* (M, g) *is locally symmetric. If* $n = 5$ *and* $\mathrm{W} \not\equiv 0$*, then*

$$Y(M,[g])\left(1+\mathrm{Vol}_g^{-\frac{2}{5}}(M)\frac{\frac{1}{15}\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\right)\leq \frac{16}{3}\left(\int_M |Q|^{\frac{5}{2}}|\mathrm{W}|^{-5}d\mu_g\right)^{\frac{2}{5}} \tag{1.6}$$

and equality holds if and only if (M, g) *is locally symmetric. In particular, the following strict inequality holds:*

$$Y(M,[g]) < \frac{16}{3}\left(\int_M |Q|^{\frac{5}{2}}|\mathrm{W}|^{-5}d\mu_g\right)^{\frac{2}{5}}. \tag{1.7}$$

As a consequence (see Section 2), we have the following lower bound for the $L^{\frac{n}{2}}$-norm of the Weyl curvature, improving the previous results in [27] for $5 \leq n \leq 9$ and in [42] for $n = 5, 6$:

Corollary 1.2. *Let* (M, g) *be a closed (conformally) Einstein manifold of dimension* $n \geq 4$ *with positive Yamabe invariant. Then, either* (M, g) *is locally conformally flat or*

$$Y(M,[g]) \leq A(n)\left(\int_M |\mathrm{W}|^{\frac{n}{2}}d\mu_g\right)^{\frac{2}{n}}, \tag{1.8}$$

where $A(4) = \sqrt{6}$*,* $A(5) = \frac{64}{3\sqrt{10}}$*,* $A(6) = \sqrt{210}$ *and* $A(n) = \frac{5}{2}n$ *for* $n \geq 7$*; if* $n = 5$*,* (1.8) *is a strict inequality.*

Remark 1.3. If $n = 4$, the result is sharp: in fact, $\mathbb{CP}^2$ endowed with the Fubini-Study metric realizes the equality in (1.8). We point out that Gursky and LeBrun provided the optimal pinching result, exploiting the peculiarities of 4-dimensional manifolds ([25], [26]): indeed, in this case the Weyl tensor can be regarded as a self-adjoint operator

$$\mathcal{W} : \Lambda^2 \longrightarrow \Lambda^2,$$

where Λ^2 is the bundle of 2-forms on M; moreover, in dimension four, the Hodge operator $\star$ induces the well-known decomposition

$$\Lambda^2 = \Lambda_+ \oplus \Lambda_-,$$

where Λ_+ (resp. Λ_-) is the subbundle of self-dual (resp. anti-self-dual) 2-forms. This splitting leads to the decomposition of the Weyl operator into a self-dual and an anti-self-dual part, namely

$$\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-,$$

which, in turn, provides the well-known decomposition of the Weyl tensor

$$\mathrm{W} = \mathrm{W}^+ + \mathrm{W}^-$$

(for a detailed description, see for instance [5], [38] and [39]). Then, if (M, g) is a closed Einstein manifold of dimension 4 with positive Yamabe invariant and such that $\mathrm{W}^+ \not\equiv 0$, we have

$$Y(M, [g]) \leq \sqrt{6}\left(\int_M \left|\mathrm{W}^+\right|^2 d\mu_g\right)^{\frac{1}{2}}, \tag{1.9}$$

with equality if and only if $\nabla \mathrm{W}^+ \equiv 0$; the same results holds if we replace W^+ with W^-. Note that both $\mathbb{CP}^2$ with the standard orientation and the Fubini-Study metric (which is a *self-dual* manifold, i.e. $\mathrm{W}^- \equiv 0$) and $\mathbb{S}^2 \times \mathbb{S}^2$ with the standard product metric realize the equality in (1.9). By the classification of irreducible symmetric spaces ([8], [9]), we get that equality in (1.9) is only realized by these manifolds, up to quotients.

Remark 1.4. If $n = 5$, the constant $A(5)$ in (1.8) was $\frac{80}{3}$ in [27] and it is the same we obtained in [42]; in our case, the estimate is strict. If $n = 6$, the constant $A(6)$ in (1.8) was 25 in [27] and 15 in [42]; note that $\sqrt{210} < 15$. If $n \geq 7$, we recover the result in [42], therefore improving the pinching in [27] for $7 \leq n \leq 9$.

The paper is organized as follows: in Section 2, we review some well-known facts about Riemannian manifolds and the Yamabe problem, recalling the classical definitions and some modifications of the Yamabe functional; after having fixed the notation, we proceed with the proof of the main results in Section 3. Finally, in Section 4, we provide some remarks about inequality (2.7), exhibiting a lower bound for the optimal constant in dimension 6 using a twistorial example; then, we describe a numerical approach simulating the Lagrange multiplier argument used to find the sharp constant in dimension 4.

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2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. It is well-known that the Riemann curvature tensor Riem admits the decomposition

$$\mathrm{Riem} = \mathrm{W} + \frac{1}{n-2}\,\mathrm{Ric} \owedge g - \frac{S}{2(n-1)(n-2)}\, g \owedge g, \tag{2.1}$$

where W, Ric and S denote the Weyl curvature tensor, the Ricci tensor and the scalar curvature, respectively, and $\owedge$ is the Kulkarni-Nomizu product (see, for instance, [5]).

With respect to a local orthonormal frame, (2.1) reads as

(2.2)
$$R_{ijkt} = W_{ijkt} + \frac{1}{n-2}(R_{ik}\delta_{jt} - R_{it}\delta_{jk} + R_{jt}\delta_{ik} - R_{jk}\delta_{it}) - \frac{S}{(n-1)(n-2)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}),$$

where $R_{ij} = R_{ikjk}$ and $S = R_{ii}$; throughout the paper, when dealing with local tensorial computations, we adopt Einstein's summation convention over repeated indices.

When (M, g) is an *Einstein manifold*, i.e. when there exists $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$, (2.1) becomes

$$\mathrm{Riem} = \mathrm{W} + \frac{S}{2n(n-1)} g \owedge g;$$

therefore, the curvature of any Einstein manifold of dimension $n \geq 4$ is encoded in the Weyl tensor and in the value of the scalar curvature S (which is necessarily constant). Taking the squared norms of the tensors, we immediately obtain that, on any Einstein manifold,

$$|\mathrm{Riem}|^2 = |\mathrm{W}|^2 + \frac{2S^2}{n(n-1)}. \tag{2.3}$$

This equation will be constantly used in the first part of Section 4; we point out that our convention for the squared norm of a (r, s)-tensor field T is

$$|T|^2 = T^{j_1\dots j_r}_{i_1\dots i_s} T^{j_1\dots j_r}_{i_1\dots i_s}.$$

For a Riemannian manifold (M, g) of dimension $n \geq 4$, the Weyl tensor is the totally trace-free part of Riem, while, on any 3-dimensional Riemannian manifold, the Weyl tensor identically vanishes. One of the main properties of W resides in its behaviour under conformal deformations of the metric g: we recall that a *conformal deformation* of g is a new metric $\widetilde{g}$ obtained by rescaling g *via* a smooth positive function f, i.e.

$$\widetilde{g} = f^2 g, \tag{2.4}$$

and that the *conformal class* of g is defined as

$$[g] = \{\widetilde{g} \in \mathcal{M} : \exists f \in C^\infty(M), f > 0, \text{ s.t } \widetilde{g} = f^2 g\},$$

where $\mathcal{M}$ denotes the space of smooth Riemannian metrics on M. For our purposes, it will be useful to choose $f = u^{2/(n-2)}$ throughout the paper, where $u \in C^\infty(M)$, $u > 0$.

Under the transformation (2.4), the $(1, 3)$-version of W does not change, i.e. the Weyl tensor is *conformally invariant*. This important feature leads to a well-known characterization of a class of special Riemannian metrics: indeed, a Riemannian metric g is *locally conformally flat* if, for every $p \in M$, there exist an open neighborhood U_p of p and a smooth positive function f such that $(U, f^2 g)$ is a flat submanifold of M. If $n \geq 4$, the celebrated Weyl-Schouten Theorem (see e.g. [28] or [35]) states that this condition is equivalent to the vanishing of W on M. In the next section we will need the transformed components of the $(0, 4)$-version of W, $\widetilde{W}_{ijkt}$, which satisfy

$$u^{\frac{4}{n-2}} \widetilde{W}_{ijkt} = W_{ijkt}, \tag{2.5}$$

and the expression of the transformed scalar curvature, i.e.

$$u^{\frac{4}{n-2}}S_{\widetilde{g}} = S_g - \frac{4(n-1)}{n-2}\frac{\Delta_g u}{u}, \tag{2.6}$$

where Δ_g is the Laplace-Beltrami operator of the metric g (for a full list of transformed curvature quantities, see e.g. [12]).

An important class of metrics, generalizing the locally conformally flat ones, is represented by *harmonic Weyl metrics*, i.e. Riemannian metrics whose Weyl tensor satisfies

$$\operatorname{div} \mathrm{W} \equiv 0, \text{ on } M,$$

where div is the divergence operator. It is well-known that this condition, for $n \geq 4$, is equivalent to the vanishing of the so-called *Cotton tensor* C and, of course, it is satisfied by every locally conformally flat metric; moreover, a straightforward computation shows that all Einstein metrics are harmonic Weyl.

As we mentioned in the Introduction, every harmonic Weyl manifold satisfies the Weitzenböck formula (1.3): we highlight the fact that, in dimension four, the same formula holds for W^+ and W^-, and, in this case, one can prove that

$$Q^{\pm} := 2W^{\pm}_{pqrs}W^{\pm}_{ptru}W^{\pm}_{qtsu} + \frac{1}{2}W^{\pm}_{pqrs}W^{\pm}_{pqtu}W^{\pm}_{rstu} = 36\det\nolimits_{\Lambda_{\pm}}\mathcal{W}_{\pm},$$

where $\det_{\Lambda_{\pm}}\mathcal{W}_{\pm}$ is the determinant of the linear operator $\mathcal{W}_{\pm}$ from $\Lambda_{\pm}$ to itself (see Remark 1.3). In general, we can find estimates for Q in terms of $|\mathrm{W}|^3$: namely,

$$|Q| \leq C(n)|\mathrm{W}|^3, \tag{2.7}$$

with $C(4) = \frac{\sqrt{6}}{4}$, $C(5) = \frac{4}{\sqrt{10}}$, $C(6) = \frac{\sqrt{70}}{2\sqrt{3}}$ and $C(n) = \frac{5}{2}$ for $n \geq 7$. We recall that (2.7) is an algebraic inequality, therefore an analogous estimate holds for every *algebraic* Weyl curvature tensor W', i.e. a $(0,4)$-tensor which is totally trace free and satisfies the same symmetries as Riem.

The constants in dimension 4 and 6 were obtained by Huisken ([29]), exploiting a Lagrange multiplier argument and an idea due to Tachibana ([41]). We point out that $C(4)$ is the optimal constant, since equality in (2.7) for $n = 4$ is achieved by quotients of $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{RP}^4$. On the other hand, the constant $C(5)$ was obtained by Tran ([42]). Now, we recall again the definition of the classical *Yamabe invariant*:

$$Y(M,[g]) = \inf_{\widetilde{g}\in[g]}\mathfrak{S}(g) = \inf_{\widetilde{g}\in[g]}\mathrm{Vol}_{\widetilde{g}}(M)^{-\frac{n-2}{n}}\int_M S_{\widetilde{g}}\,d\mu_{\widetilde{g}}; \tag{2.8}$$

as we briefly mentioned in the Introduction, the question of finding a metric $\widehat{g} \in [g]$ that attains the minimum of (2.8) is closely related to the so-called *Yamabe problem*, i.e., the problem of finding a constant scalar curvature metric in any conformal class $[g]$ on any closed smooth manifold, whose final resolution was given by the joint efforts of Yamabe, Trudinger, Aubin and Schoen (for a detailed survey concerning the Yamabe problem, see for instance [36]). It is well-known that a *Yamabe minimizer*, i.e. a metric that attains the minimum of (2.8), is a metric with constant scalar curvature, whose sign is the same of $Y(M,[g])$. A fundamental tool for the understanding and the resolution of the Yamabe problem is given by the *conformal Laplacian operator*, that is

$$\mathcal{L}_g = -\frac{4(n-1)}{n-2}\Delta_g + S_g;$$

we observe that $\mathcal{L}_g u$ represents the conformal change in (2.6), with $\widetilde{g} = u^{\frac{4}{n-2}}g$. In [25], Gursky introduced a modified version of the conformal Laplacian, involving the Weyl tensor:

$$\mathcal{L}_g^t = -\frac{4(n-1)}{n-2}\Delta_g + S_g - t|\mathrm{W}_g|_g, \tag{2.9}$$

where $t \in \mathbb{R}$. Starting from (2.9), he also defined a modified version of the Yamabe invariant, that is

$$\widehat{Y}(M,[g]) = \inf_{\widetilde{g}\in[g]} \mathrm{Vol}_{\widetilde{g}}(M)^{-\frac{n-2}{n}} \int_M \left(S_{\widetilde{g}} - t|\mathrm{W}_{\widetilde{g}}|_{\widetilde{g}}\right) d\mu_{\widetilde{g}}; \tag{2.10}$$

note that $\widehat{Y}(M,[g])$ is indeed a conformal invariant, since (see [25])

$$S_{\widetilde{g}} - t|\mathrm{W}_{\widetilde{g}}|_{\widetilde{g}} = u^{-\frac{n+2}{n-2}}\mathcal{L}_g^t u. \tag{2.11}$$

Moreover, when $\widehat{Y}(M,[g]) \leq 0$, the modified Yamabe problem always admits a solution in every conformal class, as shown in [25], which means that, for every $[g]$, there always exists a metric $\widehat{g} \in [g]$ which attains the minimum of (2.10). We just mention that the remaining case $\widehat{Y}(M,[g]) > 0$ was studied by Itoh ([30]).
For the proof of Theorem (1.1), we introduce a slightly different version of (2.10), depending on S, Q and W: namely, we define the following *modified Yamabe invariant*:

$$\overline{Y}^t(M,[g]) = \inf_{\widetilde{g}\in[g]} \mathrm{Vol}_{\widetilde{g}}(M)^{-\frac{3}{5}} \int_M \left(S_{\widetilde{g}} - tQ_{\widetilde{g}}|\mathrm{W}_{\widetilde{g}}|_{\widetilde{g}}^{-2}\right) d\mu_{\widetilde{g}}, \tag{2.12}$$

where $t \in \mathbb{R}$. It can be shown that

$$\overline{Y}^t(M,[g]) = \inf_{\substack{u\in C^\infty(M)\\ u\neq 0}} \frac{\int_M u\mathfrak{L}_g^t u\, d\mu_g}{\left(\int_M u^{\frac{10}{3}} d\mu_g\right)^{\frac{3}{5}}}, \tag{2.13}$$

where $\mathfrak{L}^t$ is defined as

$$\mathfrak{L}_g^t = -\frac{4(n-1)}{n-2}\Delta_g + S_g - tQ_g|\mathrm{W}_g|_g^{-2}. \tag{2.14}$$

To show (2.13), we perform the aforementioned conformal change of the metric: then, by (2.5) and (2.6), the quantity $S - tQ|\mathrm{W}|$ transforms as

$$S_{\widetilde{g}} - tQ_{\widetilde{g}}|\mathrm{W}_{\widetilde{g}}|^{-2} = u^{-\frac{4}{n-2}}\left[S - \frac{4(n-1)}{n-2}\frac{\Delta_g u}{u} - tQ_g|\mathrm{W}_g|^{-2}\right] = u^{-\frac{n+2}{n-2}}\mathfrak{L}^t u,$$

while the conformal change for the volume form is given by

$$\mu_{\widetilde{g}} = u^{\frac{2n}{n-2}}\mu_g.$$

It follows that

$$\mathrm{Vol}_{\widetilde{g}}(M)^{-\frac{n-2}{n}}\int_M S_{\widetilde{g}}-t|\mathrm{W}_{\widetilde{g}}|_{\widetilde{g}}d\mu_{\widetilde{g}}=\frac{\int_M u^{-\frac{n+2}{n-2}}\mathfrak{L}_g^t u\cdot u^{\frac{2n}{n-2}}\,d\mu_g}{\left(\int_M u^{\frac{2n}{n-2}}d\,\mu_g\right)^{\frac{n-2}{n}}}=\frac{\int_M u\mathfrak{L}_g^t u\,d\mu_g}{\left(\int_M u^{\frac{2n}{n-2}}\,d\mu_g\right)^{\frac{n-2}{n}}},$$

which implies (2.13).

3. Proof of Theorem 1.1

For the sake of simplicity, we will omit to write the dependance from g, when it is not necessary. Assume that $|\mathrm{W}|^2\not\equiv 0$ on M and let

$$\alpha=\frac{n-3}{2(n-1)}.$$

We want to find integral estimates for a suitable second order operator applied to $|\mathrm{W}|^{2\alpha}$: in order to do so, we exploit a strategy similar to the ones in [6] and [25] to deal with the points at which $|\mathrm{W}|$ vanishes, where smoothness of $|\mathrm{W}|^{2\alpha}$ fails. Let $\varepsilon>0$ and

$$f_\varepsilon:=\left(|\mathrm{W}|^2+\varepsilon^2\right)^\alpha:$$

by a straightforward computation and (1.3), we have that

$$\begin{aligned}\Delta f_\varepsilon&=\alpha(\alpha-1)f_\varepsilon^{1-\frac{2}{\alpha}}\left|\nabla|\mathrm{W}|^2\right|^2+\alpha f_\varepsilon^{1-\frac{1}{\alpha}}\Delta|\mathrm{W}|^2=\\&=4\alpha(\alpha-1)f_\varepsilon^{1-\frac{2}{\alpha}}|\mathrm{W}|^2|\nabla|\mathrm{W}||^2+\alpha f_\varepsilon^{1-\frac{1}{\alpha}}\left(2|\nabla\mathrm{W}|^2+\frac{4}{n}S|\mathrm{W}|^2-4Q\right).\end{aligned}\tag{3.1}$$

Now we exploit a refined Kato inequality for Einstein metrics (see [4] for a proof), which reads as

$$|\nabla|\mathrm{W}||\leq\sqrt{\frac{n-1}{n+1}}|\nabla\mathrm{W}|;\tag{3.2}$$

then, from (3.1) we deduce

$$\Delta f_\varepsilon\geq\alpha\left[4(\alpha-1)|\mathrm{W}|^2f_\varepsilon^{1-\frac{2}{\alpha}}+\frac{2(n+1)}{n-1}f_\varepsilon^{1-\frac{1}{\alpha}}\right]|\nabla|\mathrm{W}||^2+4\alpha f_\varepsilon^{1-\frac{1}{\alpha}}\left(\frac{S}{n}|\mathrm{W}|^2-Q\right).$$

Moreover, since, by definition, $f_\varepsilon\geq|\mathrm{W}|^{2\alpha}$, we get

$$\Delta f_\varepsilon\geq 2\alpha\left(2(\alpha-1)+\frac{n+1}{n-1}\right)|\mathrm{W}|^2|\nabla|\mathrm{W}||^2f_\varepsilon^{1-\frac{2}{\alpha}}+4\alpha f_\varepsilon^{1-\frac{1}{\alpha}}\left(\frac{S}{n}|\mathrm{W}|^2-Q\right),$$

and, by our choice of α, we conclude that

$$\Delta f_\varepsilon\geq 4\alpha\left(\frac{S}{n}|\mathrm{W}|^2-Q\right)f_\varepsilon^{1-\frac{1}{\alpha}}.\tag{3.3}$$

Now we adapt an idea due to Bour and Carron, defining the following operator:

$$\mathscr{L}^\beta=-\frac{4(n-1)}{n-2}\Delta+\beta S-\beta nQ|\mathrm{W}|^{-2}.\tag{3.4}$$

By (3.3), (3.4) and the definition of f_ε, we obtain the estimate

$$
\begin{aligned}
f_\varepsilon \mathscr{L}^\beta f_\varepsilon &= -\frac{4(n-1)}{n-2} f_\varepsilon \Delta f_\varepsilon + \beta S f_\varepsilon^2 - \beta n Q |\mathrm{W}|^{-2} f_\varepsilon^2 \leq \\
&\leq -\frac{8(n-3)}{n-2}\left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right)|\mathrm{W}|^2 f_\varepsilon^{2-\frac{1}{\alpha}} + \beta S f_\varepsilon^2 - \beta n Q|\mathrm{W}|^{-2} f_\varepsilon^2 = \\
&= -\frac{8(n-3)}{n-2}\left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right)\left(1 - \varepsilon^2 f_\varepsilon^{-\frac{1}{\alpha}}\right) f_\varepsilon^2 + \beta S f_\varepsilon^2 - \beta n Q |\mathrm{W}|^{-2} f_\varepsilon^2 = \\
&= \left(\beta - \frac{8(n-3)}{n(n-2)}\right) S f_\varepsilon^2 + \left(\frac{8(n-3)}{n-2} - \beta n\right) Q|\mathrm{W}|^{-2} f_\varepsilon^2 + \frac{8(n-3)}{n-2}\varepsilon^2\left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right) f_\varepsilon^{2-\frac{1}{\alpha}}.
\end{aligned}
$$

Since, for $n = 4$ and $n \geq 6$,

$$
\frac{8(n-3)}{n(n-2)} \leq 1,
$$

we can choose $\overline{\beta} = \frac{8(n-3)}{n(n-2)}$ in order to have

$$
f_\varepsilon \mathscr{L}^{\overline{\beta}} f_\varepsilon \leq \frac{8(n-3)}{n-2}\varepsilon^2 \left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right) f_\varepsilon^{2-\frac{1}{\alpha}}.
$$

Finally, since $2 - \frac{1}{\alpha} < 0$ for $n \geq 4$ and $f_\varepsilon \geq \varepsilon^{2\alpha}$, we get

$$
\varepsilon^2 f_\varepsilon^{2-\frac{1}{\alpha}} \leq \varepsilon^2 \cdot \varepsilon^{2\alpha\left(2-\frac{1}{\alpha}\right)} = \varepsilon^{4\alpha} \to 0, \text{ as } \varepsilon \to 0. \tag{3.5}
$$

Now, we adapt a modification of the Yamabe invariant (again due to Bour and Carron, see [6]), in the following way:

$$
\overline{Y}_g(\beta) := \inf_{\substack{\phi \in C^\infty(M) \\ \phi \neq 0}} \frac{\int_M \phi \mathscr{L}^\beta \phi \, d\mu_g}{\left(\int_M \phi^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}} = \inf_{\substack{\phi \in C^\infty(M) \\ \phi \neq 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla \phi|^2 + \beta S \phi^2 - \beta n Q |\mathrm{W}|^{-2}\phi^2\right) d\mu_g}{\left(\int_M \phi^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}}, \tag{3.6}
$$

for all $\beta \geq 0$. Observe that $\overline{Y}_g(1)$ is equal to (2.12) with $t = n$ and, since M is closed, $\overline{Y}_g(0) = 0$: moreover, (3.6) is the infimum of affine functions of β, therefore it is concave and, for $\beta \in [0,1]$,

$$
(1-\beta)\overline{Y}_g(0) + \beta \overline{Y}_g(1) \leq \overline{Y}_g(\beta),
$$

which implies that

$$
\beta \overline{Y}^n(M, [g]) \leq \overline{Y}_g(\beta). \tag{3.7}
$$

Note that, by (3.5) and the definition of f_ε, we get

$$\overline{Y}_g(\overline{\beta}) \leq \frac{\int_M f_\varepsilon \mathscr{L}^{\overline{\beta}} f_\varepsilon \, d\mu_g}{\left(\int_M f_\varepsilon^{\frac{2n}{n-2}} \, d\mu_g\right)^{\frac{n-2}{n}}} \leq \frac{8(n-3)}{n-2} \frac{\int_M \left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right) f_\varepsilon^{2-\frac{1}{\alpha}} \, d\mu_g}{\left(\int_M f_\varepsilon^{\frac{2n}{n-2}} \, d\mu_g\right)^{\frac{n-2}{n}}} \varepsilon^2 \leq$$

$$\leq \frac{8(n-3)}{n-2} \frac{\int_M \left(\frac{S}{n} - Q|\mathrm{W}|^{-2}\right) \varepsilon^2 f_\varepsilon^{2-\frac{1}{\alpha}} \, d\mu_g}{\left(\int_M u^{\frac{2n}{n-2}} \, d\mu_g\right)^{\frac{n-2}{n}}} \xrightarrow[\varepsilon \to 0]{} 0,$$

which implies that

$$\overline{Y}_g(\overline{\beta}) \leq 0.$$

Since $n \neq 5$, $\overline{\beta} \in (0,1)$, therefore, by (3.7), we obtain

$$\overline{\beta}\,\overline{Y}^n(M,[g]) \leq \overline{Y}_g(\overline{\beta}),$$

which implies that

$$\overline{Y}^n(M,[g]) \leq 0.$$

Now, following the same line of reasoning in [25] (i.e., adapting the argument of [36, Proposition 4.4]), we know that there exists a unique metric $\widehat{g} \in [g]$ which attains the minimum of (2.12), that is

$$\mathrm{Vol}_{\widehat{g}}(M)^{-\frac{2}{n}} \overline{Y}^n(M,[g]) = \widehat{S} - n\widehat{Q}\left|\widehat{\mathrm{W}}\right|^{-2},$$

where $\widehat{S}$, $\widehat{Q}$ and $\widehat{\mathrm{W}}$ are relative to the metric $\widehat{g}$. Then, since $\overline{Y}(M,[g]) \leq 0$ and g is an Einstein metric (therefore, it attains the minimum of $Y(M,[g])$), since $\widehat{g} \in [g]$, we have that

$$Y(M,[g]) = \mathrm{Vol}_g(M)^{-\frac{n-2}{n}} \int_M S \, d\mu_g \leq \mathrm{Vol}_{\widehat{g}}(M)^{-\frac{n-2}{n}} \int_M \widehat{S} \, d\mu_{\widehat{g}} \leq \mathrm{Vol}_{\widehat{g}}(M)^{-\frac{n-2}{n}} \int_M n\widehat{Q}\left|\widehat{\mathrm{W}}\right|^{-2} d\mu_{\widehat{g}}.$$

By Hölder inequality, we obtain

$$\int_M \widehat{Q}\left|\widehat{\mathrm{W}}\right|^{-2} d\mu_{\widehat{g}} \leq \left(\int_M \left|\widehat{Q}\right|^{\frac{n}{2}} \left|\widehat{\mathrm{W}}\right|^{-n} d\mu_{\widehat{g}}\right)^{\frac{2}{n}} \mathrm{Vol}_{\widehat{g}}(M)^{\frac{n-2}{n}},$$

which implies that

$$Y(M,[g]) \leq n\left(\int_M \left|\widehat{Q}\right|^{\frac{n}{2}} \left|\widehat{\mathrm{W}}\right|^{-n} d\mu_{\widehat{g}}\right)^{\frac{2}{n}}.$$

Now, by formula (2.5), we know that the right-hand side of the previous inequality is conformally invariant; therefore,

$$Y(M,[g]) \leq n\left(\int_M |Q|^{\frac{n}{2}} |\mathrm{W}|^{-n} d\mu_g\right)^{\frac{2}{n}}$$

and inequality (1.5) is proven.

Now, suppose that the equality in (1.5) holds: then,

$$\int_M \widehat{S} \, d\mu_{\widehat{g}} = \int_M n\left|\widehat{Q}\right|\left|\widehat{\mathrm{W}}\right|^{-2} d\mu_{\widehat{g}},$$

which immediately implies that, by definition, $\overline{Y}^n(M,[g])=0$. Moreover, since

$$Y(M,[g])=\mathrm{Vol}_{\widehat{g}}(M)^{-\frac{n-2}{n}}\int_M \widehat{S}\,d\mu_{\widehat{g}},$$

we observe that $\widehat{g}$ attains the minimum $Y(M,[g])$ and, therefore, $\widehat{g}$ is a solution of the Yamabe problem in $[g]$, which implies that $\widehat{S}$ is constant: hence, since g is an Einstein metric in $[g]$, we can exploit a well-known result due to Obata ([37]) in order to conclude that $\widehat{g}=g$ and, as a consequence, $\widehat{S}=S$, $\widehat{Q}=Q$ and $\widehat{\mathrm{W}}=\mathrm{W}$. This implies that

$$S-nQ|\mathrm{W}|^{-2}=0\Longrightarrow Q=\frac{S}{n}|\mathrm{W}|^2; \tag{3.8}$$

therefore, integrating (1.3), we get

$$0=\int_M\left(|\nabla\mathrm{W}|^2+\frac{2}{n}S|\mathrm{W}|^2-2Q\right)d\mu_g=\int_M|\nabla\mathrm{W}|^2d\mu_g,$$

which implies that $|\nabla\mathrm{W}|^2\equiv 0$ on M, i.e. (M,g) is locally symmetric. The converse is trivial, since, by (1.3), we immediately obtain (3.8).

Now, for $n=5$, we consider again (2.12), choosing $t=\frac{16}{3}$; also, in order to simplify the notation, we write $\overline{Y}(M,[g])=\overline{Y}^{\frac{16}{3}}(M,[g])$ and $\mathfrak{L}=\mathfrak{L}^{\frac{16}{3}}$. Exploiting the same technique used for $n\neq 5$ and noting that $\alpha=\frac{1}{4}$ in this case, we obtain the estimate

$$f_\varepsilon\mathfrak{L}f_\varepsilon\leq-\frac{1}{15}Sf_\varepsilon^2+\frac{16}{3}\varepsilon^2\left(\frac{S}{5}-Q|\mathrm{W}|^{-2}\right)f_\varepsilon^{-2}. \tag{3.9}$$

Note that, since $f_\varepsilon>\varepsilon^{\frac{1}{2}}$ for $n=5$, we can deduce

$$\varepsilon^2f_\varepsilon^{-2}\leq\varepsilon\to 0,\ \text{ as }\varepsilon\to 0. \tag{3.10}$$

Hence, by (3.9) and the definition of f_ε, we deduce

$$\overline{Y}(M,[g])\leq\frac{\int_M f_\varepsilon\mathfrak{L}f_\varepsilon\,d\mu_g}{\left(\int_M f_\varepsilon^{\frac{10}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\leq-\frac{\frac{1}{15}S\int_M f_\varepsilon^2\,d\mu_g}{\left(\int_M f_\varepsilon^{\frac{10}{3}}\,d\mu_g\right)^{\frac{3}{5}}}+\frac{16}{3}\frac{\int_M\left(\frac{S}{5}-Q|\mathrm{W}|^{-2}\right)f_\varepsilon^{-2}\,d\mu_g}{\left(\int_M f_\varepsilon^{\frac{10}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\varepsilon^2. \tag{3.11}$$

Note that, since $(|\mathrm{W}|^2+1)^{\frac{1}{4}}$ is integrable on M, by the dominated convergence theorem and (3.10) we obtain

$$-\frac{\frac{1}{15}S\int_M f_\varepsilon^2\,d\mu_g}{\left(\int_M f_\varepsilon^{\frac{10}{3}}\,d\mu_g\right)^{\frac{3}{5}}}+\frac{16}{3}\frac{\int_M\left(\frac{S}{5}-Q|\mathrm{W}|^{-2}\right)f_\varepsilon^{-2}\,d\mu_g}{\left(\int_M f_\varepsilon^{\frac{10}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\varepsilon^2\xrightarrow[\varepsilon\to 0]{}-\frac{\frac{1}{15}S\int_M|\mathrm{W}|\,d\mu_g}{\left(\int_M|\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}},$$

implying

$$\overline{Y}(M,[g])\leq-\frac{\frac{1}{15}S\int_M|\mathrm{W}|\,d\mu_g}{\left(\int_M|\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\leq 0.$$

Arguing as in the case $n \neq 5$, we have that there exists a unique metric $\widehat{g} \in [g]$ attaining the minimum of the modified Yamabe invariant $\overline{Y}(M,[g])$, i.e.

$$\mathrm{Vol}_{\widehat{g}}(M)^{-\frac{2}{5}}\overline{Y}(M,[g]) = \widehat{S} - \frac{16}{3}\widehat{Q}\left|\widehat{\mathrm{W}}\right|^{-2},$$

where $\widehat{S}$, $\widehat{Q}$ and $\widehat{\mathrm{W}}$ are relative to the metric $\widehat{g}$. Moreover, since g attains the minimum of $Y(M,[g])$ and $\widehat{g} \in [g]$, we have the inequalities

$$\begin{aligned} Y(M,[g]) = \mathrm{Vol}_g(M)^{-\frac{3}{5}}\int_M S\,d\mu_g &\leq \mathrm{Vol}_{\widehat{g}}(M)^{-\frac{3}{5}}\int_M \widehat{S}\,d\mu_{\widehat{g}} \leq \\ &\leq \mathrm{Vol}_{\widehat{g}}(M)^{-\frac{3}{5}}\int_M \frac{16}{3}\widehat{Q}\left|\widehat{\mathrm{W}}\right|^{-2}d\mu_{\widehat{g}} - \frac{\frac{1}{15}S\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}}. \end{aligned}$$

By Hölder inequality, we deduce

$$Y(M,[g]) \leq \frac{16}{3}\left(\int_M \left|\widehat{Q}\right|^{\frac{5}{2}}\left|\widehat{\mathrm{W}}\right|^{-5}d\mu_{\widehat{g}}\right)^{\frac{2}{5}} - \frac{\frac{1}{15}S\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}},$$

that is, since $S = \mathrm{Vol}_g(M)^{\frac{2}{5}}Y(M,[g])$,

(3.12)

$$Y(M,[g])\left(1 + \mathrm{Vol}_g(M)^{-\frac{2}{5}}\frac{\frac{1}{15}\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}}\right) \leq \frac{16}{3}\left(\int_M \left|\widehat{Q}\right|^{\frac{5}{2}}\left|\widehat{\mathrm{W}}\right|^{-5}d\mu_{\widehat{g}}\right)^{\frac{2}{5}} = \frac{16}{3}\left(\int_M |Q|^{\frac{5}{2}}|\mathrm{W}|^{-5}d\mu_g\right)^{\frac{2}{5}}.$$

It follows that inequality (1.6) holds. If equality in (1.6) is attained, then

$$\mathrm{Vol}_{\widehat{g}}(M)^{-\frac{3}{5}}\int_M \widehat{S}\,d\mu_{\widehat{g}} = \mathrm{Vol}_{\widehat{g}}(M)^{-\frac{3}{5}}\int_M \frac{16}{3}\left|\widehat{Q}\right|\left|\widehat{\mathrm{W}}\right|^{-2}d\mu_{\widehat{g}} - \frac{\frac{1}{15}S\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}},$$

which implies

$$\overline{Y}(M,[g]) = -\frac{\frac{1}{15}S\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}}. \tag{3.13}$$

Moreover, as for the case $n \neq 5$, $\widehat{g}$ is a solution of the Yamabe problem, implying again that $\widehat{g} = g$. Note that equality holds in Hölder's estimate: therefore, $Q|\mathrm{W}|^{-2}$ is constant on M. Moreover, $Q|\mathrm{W}|^{-2} \neq 0$ on M, otherwise, integrating (1.3), we would conclude that (M,g) is locally conformally flat, which contradicts our hypothesis. Now, since $|\mathrm{W}| \neq 0$ on M, we can repeat the initial argument of the proof replacing f_ε with $|\mathrm{W}|^{\frac{1}{2}}$ in order to get

$$\mathfrak{L}|\mathrm{W}|^{\frac{1}{2}} \leq -\frac{S}{15}|\mathrm{W}|^{\frac{1}{2}}; \tag{3.14}$$

therefore, by (3.14), we deduce

$$\overline{Y}(M,[g]) \leq \frac{\int_M |\mathrm{W}|^{\frac{1}{2}}\mathfrak{L}|\mathrm{W}|^{\frac{1}{2}}d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\right)^{\frac{3}{5}}} \leq -\frac{\frac{1}{15}S\int_M |\mathrm{W}|\,d\mu_g}{\left(\int_M |\mathrm{W}|^{\frac{5}{3}}\,d\mu_g\right)^{\frac{3}{5}}},$$

and, since (3.13) holds, we obtain

$$\int_M |\mathrm{W}|^{\frac{1}{2}}\mathfrak{L}|\mathrm{W}|^{\frac{1}{2}}d\mu_g = -\frac{S}{15}\int_M |\mathrm{W}|\,d\mu_g.$$

By the previous equality and (3.14), it is easy to observe that we must have a pointwise equality, namely

$$\mathfrak{L}|\mathrm{W}|^{\frac{1}{2}} = -\frac{S}{15}|\mathrm{W}|^{\frac{1}{2}}. \tag{3.15}$$

Using (2.14) and integrating (3.15), we conclude

$$\left(S - 5Q|\mathrm{W}|^{-2}\right)\int_M |\mathrm{W}|^{\frac{1}{2}}d\mu_g = 0;$$

since $|\mathrm{W}| \neq 0$ on M, the claim follows by (1.3).

Conversely, if (M,g) is locally symmetric, equality in (1.6) follows by observing that $|\mathrm{W}|$ is a constant function and by using (1.3) and (2.8). ☐

Proof of Corollary 1.2. Recall that, for $n \geq 4$, inequality (2.7) holds with the following constants

$$C(4) = \frac{\sqrt{6}}{4},\ C(5) = \frac{4}{\sqrt{10}},\ C(6) = \frac{\sqrt{70}}{2\sqrt{3}},\ C(n) = \frac{5}{2} \text{ for } n \geq 7.$$

Therefore, using (2.7) in (1.5) with these constants, the claim is proven. ☐

4. Remarks on the sharp estimate for Q

As we mentioned in Section 2, Huisken ([29]) exploited a standard Lagrange multiplier in order to obtain a sharp constant in the estimate (2.7): indeed, using the decomposition described in Remark 1.3 and pointwise diagonalizing both $\mathcal{W}_+$ and $\mathcal{W}_-$, the Lagrange multiplier problem reduces to solving a system of six polynomial equations in the eigenvalues of $\mathcal{W}_\pm$. As we observed before, Theorem 1.1 and (2.7) in dimension four partially recover the well-known pinching result due to Gursky and LeBrun ([25], [26]). As far as higher dimensional cases are concerned, the problem of finding the optimal constant $A(n)$ in

$$Y(M,[g]) \leq A(n)\left(\int_M |\mathrm{W}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{4.1}$$

is still open, also due to the fact that Einstein manifolds are far less understood in these cases than in the four-dimensional one. In view of Theorem 1.1, it seems apparent that this problem is closely related to the existence of a sharp constant in the estimate (2.7), since, in this case, the optimal pinching (4.1) would be a straightforward consequence of (1.5).

In order to improve the investigation on the best constants in (2.7) and (4.1), it is natural to check the most classical examples, such as locally symmetric, irreducible manifolds $M = G/K$, which also happen to be Einstein. Recall that, by (1.3), if (M, g) is locally symmetric Einstein, then

$$Q = \frac{S}{n}|\mathrm{W}|^2; \tag{4.2}$$

therefore, if $C_M \in \mathbb{R}$ is such that $Q = C_M|\mathrm{W}|^3$ on M, then

$$C_M = \frac{S}{n|\mathrm{W}|}.$$

Also, let us denote $A_M \in \mathbb{R}$ the constant such that $S = A_M|\mathrm{W}|$ on M. By Cartan's classification of classical Riemannian symmetric spaces and some curvature results contained in [23, Table III], using (2.3) and (4.2) we are able to describe these cases as follows:

Table 1

Classical 5-dimensional symmetric spaces						
G	K	S	$\|\mathrm{W}\|^2$	Q	C_M	A_M
$SU(3)$	$SO(3)$	30	210	1260	$\frac{\sqrt{210}}{35}$	$\frac{\sqrt{210}}{7}$

Table 2

Classical 6-dimensional symmetric spaces						
G	K	S	$\|\mathrm{W}\|^2$	Q	C_M	A_M
$SO(4)$	$\{I_4\}$	24	$\frac{288}{5}$	$\frac{1152}{5}$	$\frac{\sqrt{10}}{6}$	$\sqrt{10}$
$SO(5)$	$SO(2) \times SO(3)$	18	$\frac{312}{5}$	$\frac{936}{5}$	$\frac{\sqrt{390}}{52}$	$\frac{3\sqrt{390}}{26}$
$U(4)$	$U(1) \times U(3)$	24	$\frac{288}{5}$	$\frac{1152}{5}$	$\frac{\sqrt{10}}{6}$	$\sqrt{10}$
$SO(6)$	$U(3)$	24	$\frac{288}{5}$	$\frac{1152}{5}$	$\frac{\sqrt{10}}{6}$	$\sqrt{10}$
$Sp(2)$	$U(2)$	36	$\frac{1248}{5}$	$\frac{7488}{5}$	$\frac{\sqrt{390}}{52}$	$\frac{3\sqrt{390}}{26}$

Table 3

Classical 8-dimensional symmetric spaces						
G	K	S	$\|W\|^2$	Q	C_M	A_M
$SU(3)$	$\{I_3\}$	96	$\frac{6096}{7}$	$\frac{73152}{7}$	$\frac{\sqrt{2667}}{127}$	$8\frac{\sqrt{2667}}{127}$
$SO(6)$	$SO(2) \times SO(4)$	32	$\frac{864}{7}$	$\frac{3456}{7}$	$\frac{\sqrt{42}}{18}$	$\frac{4\sqrt{42}}{9}$
$U(4)$	$U(2) \times U(2)$	32	$\frac{864}{7}$	$\frac{3456}{7}$	$\frac{\sqrt{42}}{18}$	$\frac{4\sqrt{42}}{9}$
$U(5)$	$U(1) \times U(4)$	40	$\frac{720}{7}$	$\frac{3600}{7}$	$\frac{\sqrt{35}}{12}$	$\frac{2\sqrt{35}}{3}$
$Sp(3)$	$Sp(1) \times Sp(2)$	64	$\frac{1888}{7}$	$\frac{15104}{7}$	$\frac{\sqrt{826}}{59}$	$\frac{8\sqrt{826}}{59}$

Table 4

Classical 9-dimensional symmetric spaces						
G	K	S	$\|W\|^2$	Q	C_M	A_M
$SO(6)$	$SO(3) \times SO(3)$	36	180	720	$\frac{2\sqrt{5}}{15}$	$\frac{6\sqrt{5}}{5}$
$SU(4)$	$SO(4)$	72	720	5760	$\frac{2\sqrt{5}}{15}$	$\frac{6\sqrt{5}}{5}$

Note that, in every table, we excluded all the space forms appearing in the classification of classical symmetric spaces (for instance, $SO(n+1)/SO(n) \cong \mathbb{S}^n$). Moreover,

- $U(n)/(U(1)\times U(n))$, where $n \in \mathbb{N}$, can be regarded as $\mathbb{CP}^n$; also, $SO(6)/U(3)$ is $\mathbb{CP}^3$, while $SO(4) \cong \mathbb{S}^3 \times \mathbb{RP}^3$ has $SU(2) \times SU(2) \cong \mathbb{S}^3 \times \mathbb{S}^3$ as its universal cover;
- there are no 7-dimensional classical irreducible symmetric spaces which are not space forms.

We can also obtain locally symmetric spaces by taking into account Cartesian products $M \times N$ of irreducible symmetric Einstein manifolds, with the product metric $g = g_M + \beta g_N$, where β is chosen in such a way that g is also an Einstein metric, which is unique, up to rescaling. Exploiting the computations in [23] again, we derive the following tables, where manifolds are listed up to quotients:

Table 5

5-dimensional symmetric Einstein product spaces					
Type	S	$\|W\|^2$	Q	C_M	A_M
$\mathbb{S}^2 \times \mathbb{S}^3$	5	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{\sqrt{2}}{3}$	$\frac{5\sqrt{2}}{3}$

Table 6

6-dimensional symmetric Einstein product spaces					
Type	S	$\lvert W\rvert^2$	Q	C_M	A_M
$\mathbb{S}^2 \times \mathbb{S}^4$	24	$\frac{1024}{15}$	$\frac{4096}{15}$	$\frac{\sqrt{15}}{8}$	$\frac{3\sqrt{15}}{4}$
$\mathbb{S}^2 \times \mathbb{CP}^2$	6	$\frac{104}{15}$	$\frac{104}{15}$	$\frac{\sqrt{390}}{52}$	$\frac{3\sqrt{390}}{26}$
$\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$	12	$\frac{192}{5}$	$\frac{384}{5}$	$\frac{\sqrt{15}}{12}$	$\frac{\sqrt{15}}{2}$
$\mathbb{S}^3 \times \mathbb{S}^3$	48	$\frac{1152}{5}$	$\frac{9216}{5}$	$\frac{\sqrt{10}}{6}$	$\sqrt{10}$

Table 7

7-dimensional symmetric Einstein product spaces					
Type	S	$\lvert W\rvert^2$	Q	C_M	A_M
$\mathbb{S}^3 \times \mathbb{S}^4$	56	$\frac{640}{3}$	$\frac{5120}{3}$	$\sqrt{\frac{3}{10}}$	$\frac{7\sqrt{30}}{10}$
$\mathbb{S}^3 \times \mathbb{CP}^2$	14	24	48	$\frac{\sqrt{6}}{6}$	$\frac{7\sqrt{6}}{6}$
$\mathbb{S}^3 \times \mathbb{S}^2 \times \mathbb{S}^2$	14	$\frac{104}{3}$	$\frac{208}{3}$	$\sqrt{\frac{3}{26}}$	$7\sqrt{\frac{3}{26}}$
$\mathbb{S}^2 \times \mathbb{S}^5$	7	$\frac{25}{6}$	$\frac{25}{6}$	$\frac{\sqrt{6}}{5}$	$\frac{7\sqrt{6}}{5}$
$\mathbb{S}^2 \times (SU(3)/SO(3))$	14	40	80	$\frac{\sqrt{10}}{10}$	$\frac{7\sqrt{10}}{10}$

Table 8

8-dimensional symmetric Einstein product spaces					
Type	S	$\lvert W\rvert^2$	Q	C_M	A_M
$\mathbb{S}^4 \times \mathbb{S}^4$	48	$\frac{192}{7}$	$\frac{576}{7}$	$\frac{\sqrt{21}}{8}$	$\sqrt{21}$
$\mathbb{S}^4 \times \mathbb{CP}^2$	72	$\frac{360}{7}$	$\frac{1080}{7}$	$\frac{\sqrt{70}}{20}$	$\frac{2\sqrt{70}}{5}$
$\mathbb{S}^4 \times \mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \times \mathbb{CP}^2$	24	$\frac{528}{7}$	$\frac{1584}{7}$	$\frac{\sqrt{231}}{44}$	$\frac{2\sqrt{231}}{11}$
$\mathbb{CP}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$	24	$\frac{696}{7}$	$\frac{2088}{7}$	$\frac{\sqrt{1218}}{116}$	$\frac{2\sqrt{1218}}{29}$
$\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$	16	$\frac{384}{7}$	$\frac{768}{7}$	$\frac{\sqrt{42}}{24}$	$\frac{\sqrt{42}}{3}$
$\mathbb{S}^3 \times \mathbb{S}^5$	16	$\frac{90}{7}$	$\frac{180}{7}$	$\frac{\sqrt{70}}{15}$	$\frac{8\sqrt{70}}{15}$
$\mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{CP}^3$	8	$\frac{54}{7}$	$\frac{54}{7}$	$\frac{\sqrt{42}}{18}$	$\frac{4\sqrt{42}}{9}$
$\mathbb{S}^3 \times (SU(3)/SO(3))$	16	$\frac{760}{21}$	$\frac{1520}{21}$	$\sqrt{\frac{21}{190}}$	$4\sqrt{\frac{42}{95}}$

Table 9

9-dimensional symmetric Einstein product spaces					
Type	S	$\lvert W\rvert^2$	Q	C_M	A_M
$\mathbb{S}^5\times\mathbb{S}^4$	36	$\frac{140}{3}$	$\frac{560}{3}$	$2\sqrt{\frac{3}{35}}$	$18\sqrt{\frac{3}{35}}$
$\mathbb{S}^5\times\mathbb{CP}^2$	36	$\frac{268}{3}$	$\frac{1072}{3}$	$2\sqrt{\frac{3}{67}}$	$18\sqrt{\frac{3}{67}}$
$\mathbb{S}^5\times\mathbb{S}^2\times\mathbb{S}^2$	36	132	628	$\frac{2\sqrt{33}}{33}$	$\frac{6\sqrt{33}}{11}$
$(\mathbb{S}^2\times\mathbb{S}^3)\times(\mathbb{S}^2\times\mathbb{S}^2)$	9	$\frac{51}{4}$	$\frac{51}{4}$	$\frac{2\sqrt{51}}{51}$	$\frac{6\sqrt{51}}{17}$
$\mathbb{S}^2\times\mathbb{S}^3\times\mathbb{S}^4$	9	$\frac{89}{12}$	$\frac{89}{12}$	$2\sqrt{\frac{3}{89}}$	$18\sqrt{\frac{3}{89}}$
$\mathbb{S}^2\times\mathbb{S}^3\times\mathbb{CP}^2$	9	$\frac{121}{12}$	$\frac{121}{12}$	$\frac{2\sqrt{3}}{11}$	$\frac{18\sqrt{3}}{11}$
$(SU(3)\,/SO(3))\times\mathbb{S}^2\times\mathbb{S}^2$	54	507	3042	$\frac{2\sqrt{3}}{13}$	$\frac{18\sqrt{3}}{13}$
$(SU(3)\,/SO(3))\times\mathbb{S}^4$	54	315	1890	$\frac{2\sqrt{35}}{35}$	$\frac{18\sqrt{35}}{35}$
$(SU(3)\,/SO(3))\times\mathbb{CP}^2$	54	411	2466	$\frac{2\sqrt{411}}{137}$	$\frac{18\sqrt{411}}{137}$
$\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$, $\mathbb{S}^3\times\mathbb{CP}^3$	72	432	3456	$\frac{2\sqrt{3}}{9}$	$2\sqrt{3}$

Note that, comparing the Tables of irreducible spaces and product manifolds, we get that the maximal constants C_M and A_M for $5\leq n\leq 9$ are given by

- $\mathbb{S}^{\frac{n-1}{2}}\times\mathbb{S}^{\frac{n-1}{2}+1}$ with the standard product metric for $n=5,7,9$;
- $\mathbb{CP}^3$ with the Fubini-Study metric and $\mathbb{S}^3\times\mathbb{S}^3$ with the standard product metric for $n=6$;
- $\mathbb{S}^4\times\mathbb{S}^4$ with the standard product metric for $n=8$.

4.1. **A lower bound in the six-dimensional case.** In this section, we provide a lower bound for the optimal constant $C(6)$ which realizes (2.7). Recall that, in dimension 4, the equality in (2.7) is achieved by $\mathbb{CP}^2$, endowed with the Fubini-Study metric g_{FS}. However, this does not hold in higher dimensional cases in general: in fact, we prove that, if $n=6$, the equality in (2.7) cannot realized by a symmetric space.

We recall that, if (M,g) is an oriented four-dimensional Riemannian manifold, the *twistor space* Z associated to (M,g) ([2]) can be defined as the set of all pairs (p,J), where $p\in M$ and J is an orthogonal complex structure on the tangent space T_pM: alternatively, one can consider the representation of the group $U(2)$ in $SO(4)$ and define Z as the $(SO(4)/U(2))$-bundle

$$Z=O(M)_-\times_{SO(4)}SO(4)\,/U(2)\,,$$

where $O(M)_-$ is the negatively oriented orthonormal frame bundle of M. More clearly, since $O(M)_-\longrightarrow M$ is a principal $SO(4)$-bundle, Z can be regarded as the associated fiber bundle: therefore, standard theory of principal bundles ([32]) allows us to define the twistor space as

$$Z = O(M)_- / U(2)$$

and, therefore, as the sphere bundle of 2-forms in Λ_- of norm $\sqrt{2}$ (for a more complete dissertation about this construction of twistor spaces, see, for instance, [18], [31] and [38]. We also refer the reader to the useful surveys [17] and [33] and the references therein). There exists a 1-parameter family of Riemannian metrics g_t on Z, where $t > 0$, defined as the pullback of Riemannian metrics h_t on $O(M)_-$ *via* the $U(2)$-bundle $\sigma : O(M)_- \longrightarrow Z$ so that σ is a Riemannian submersion with totally geodesic fibers. Moreover, (Z, g_t) becomes an almost Hermitian manifold, since it can be endowed with two almost complex structures J_+ ([2]) and J_- ([20]).

It can be shown that, when $M = \mathbb{S}^4$, its twistor space Z is in fact $\mathbb{CP}^3$ ([2]). It is well-known that, if we fix $t > 0$ and we consider the rescaling of the round metric $\frac{1}{t^2} g_{\mathbb{S}^4}$ on $\mathbb{S}^4$, (Z, g_t, J_+) is a Kähler-Einstein manifold ([22]), where g_t happens to be the Fubini-Study metric, up to rescaling: this case has already been covered in Table 2, since (Z, g_t, J_+) is a compact, irreducible symmetric space. However, by choosing $\frac{1}{2t^2} g_{\mathbb{S}^4}$ on $\mathbb{S}^4$, we obtain another Einstein metric g_t on $Z = \mathbb{CP}^3$ ([21]), which is the so-called *squashed metric* g_{sq} (see [5], [43] and [45]). In this setting, (Z, g_t, J_+) is a strictly nearly Kähler manifold with positive holomorphic bisectional curvature ([1]) and it also is a 3-symmetric space (for the classification of homogeneous nearly Kähler manifold, see [7]); however, it can be shown that Z is not symmetric, since the only locally symmetric twistor spaces associated to half conformally flat manifolds are $\mathbb{CP}^3$ with the standard Kähler-Einstein structure and $M \times \mathbb{S}^2$, if M is flat ([10]).

An explicit expression for the non-zero components of the Weyl tensor of $(Z, g_t) = (\mathbb{CP}^3, g_{sq})$, viewed as the twistor space of $(\mathbb{S}^4, \frac{1}{2t^2} g_{\mathbb{S}^4})$, can be obtained by the formulas listed in the appendix B of [10], with respect to a local orthonormal frame. Choosing $t = 1$, we get:

$$\begin{aligned}
&W_{1212} = W_{3434} = \frac{1}{4}, \quad W_{1234} = \frac{1}{8}, \\
&W_{1313} = W_{4242} = W_{1414} = W_{2323} = \frac{1}{16}, \quad W_{1342} = W_{1423} = -\frac{1}{16}, \\
&W_{a5b5} = W_{a6b6} = -\frac{3}{16}\delta_{ab}, \text{ for } a, b = 1, ..., 4, \\
&W_{1526} = W_{3546} = -W_{1625} = -W_{3645} = \frac{3}{16}, \\
&W_{1256} = W_{3456} = \frac{3}{8}, \quad W_{5656} = \frac{3}{4}.
\end{aligned} \tag{4.3}$$

First, it is easy to observe that, although Z is not symmetric, the squared norm of the Weyl tensor $|\mathrm{W}|^2$ is a constant function equal to $\frac{15}{2}$: since the scalar curvature of Z is constant and equal to $\frac{15}{2}$, by (1.4) we have

$$C_M = \frac{Q_{sq}}{|\mathrm{W}|_{sq}^3} = \sqrt{\frac{3}{10}} > \frac{\sqrt{10}}{6},$$

where the right-hand side is the value of C_M for $(\mathbb{CP}^3, g_{FS})$ and $(\mathbb{S}^3 \times \mathbb{S}^3, \alpha g_{\mathbb{S}^3} + \beta g_{\mathbb{S}^3})$, where $\alpha, \beta > 0$, as it is shown in Table 2. Hence, if we define

$$C_{min} = \inf\{C \in \mathbb{R} : Q' \leq C|\mathrm{W}'|^3 \text{ for every algebraic Weyl curvature tensor } \mathrm{W}'\},$$

we can conclude that

$$C_{min} \geq \sqrt{\frac{3}{10}}.$$

Although this counterexample shows that irreducible symmetric 6-spaces do not achieve the equality in (2.7), the same may be not true for the constant A_M: indeed, we have

$$A_{sq} = \frac{S_{sq}}{|\mathrm{W}|_{sq}} = \sqrt{\frac{15}{2}} < \sqrt{10},$$

where the right-hand side is the value of A_M for $(\mathbb{CP}^3, g_{FS})$ and $(\mathbb{S}^3 \times \mathbb{S}^3, \alpha g_{\mathbb{S}^3} + \beta g_{\mathbb{S}^3})$. Coupling this observation with the optimal result obtained by Bour and Carron [6, Theorem C], we may guess that, given a closed (conformally) Einstein 6-manifold, the following integral pinching holds:

$$Y(M,[g]) \leq \sqrt{10}\left(\int_M |\mathrm{W}|^3 d\mu_g\right)^{\frac{1}{3}},$$

where equality holds if and only if (M, g) is $\mathbb{CP}^3$ with the Fubini-Study metric or $\mathbb{S}^3 \times \mathbb{S}^3$ endowed with the product metric $\alpha g_{\mathbb{S}^3} + \beta g_{\mathbb{S}^3}$, up to quotients. However, we know that, in general, the estimate

$$\int_M S^3 \, d\mu_g \leq 10\sqrt{10} \int_M |\mathrm{W}|^3 d\mu_g$$

is not true: indeed, if, for instance, $(M, g) = (\mathbb{S}^2 \times \mathbb{S}^4, g_{\mathbb{S}^2} + \beta g_{\mathbb{S}^4})$, where

$$\beta > \frac{\sqrt{15} - 3\sqrt{10}}{3\sqrt{10} - 6\sqrt{15}},$$

one can immediately observe that the opposite estimate holds (we highlight the fact that such a manifold is locally symmetric, but not Einstein).

4.2. **A numerical approach for the sharp constant in (2.7).** The classical Lagrange multiplier exploited to find the optimal constant in (2.7) when $n = 4$ is rather hard to extend to higher-dimensional cases, due to the rapidly increasing number of independent variables in the linear system. Therefore, we decided to reproduce these computations *via* a numerical method [5], in order to obtain a reasonable guess for the value of the constants $C(n)$.

[5] The algorithm is available under request, by sending an e-mail to any of the authors.

Our approach is the following: first, we define the Weyl tensor as a vector $\mathrm{W} \in \mathbb{R}^{n^4}$. In order to do so, we construct the vector by labeling the components W_{ijkl} as $\mathrm{W}[x]$, where

$$x = (i-1) \cdot n^3 + (j-1) \cdot n^2 + (k-1) \cdot n + (l-1); \tag{4.4}$$

at this point, $i, j, k, l = 1, ..., n$, without any symmetry condition on W. Then we construct the linear constraints given by the well-known symmetries of the Weyl curvature tensors, i.e., skew-symmetry with respect to the first two and the last two indices, the first Bianchi identity and the totally trace-free condition. Hence, the constraints are encoded in the rows of a matrix A with n^4 columns; we recall that, given any algebraic Weyl curvature tensor W', the number of independent components of W' is $m = \frac{n(n+1)(n+2)(n-3)}{12}$, therefore $A \in \ M_{n-m,n}(\mathbb{R})$. After that, we define the function

$$\begin{aligned} f : \mathbb{R}^{n^4} &\longrightarrow \mathbb{R} \\ \mathrm{W} &\longmapsto 2W_{pqrs}W_{ptru}W_{qtsu} + \frac{1}{2}W_{pqrs}W_{pqtu}W_{rstu}, \end{aligned}$$

writing every component of W as in (4.4). After defining $|\mathrm{W}|$ as usual and setting an upper and a lower bound for the entries of $\mathrm{W}[x]$ (i.e., for instance, $\mathrm{W}[x] \in [-1, 1]$ for every x defined as in (4.4)), we minimize the function $-f(\mathrm{W})/|\mathrm{W}|^3$, using the Sequential Least Squared Programming (SLSQP), an iterative method which, starting at a random vector $\mathrm{W}_0 \in \mathbb{R}^{n^4}$, after some iterations gives a numerical estimate of the maximum point of the function, also providing an approximation of the maximum. Namely, we are able to obtain a numerical estimate of the following quantity:

$$\min_{\mathrm{W} \in \mathbb{R}^{n^4}} -\frac{f(\mathrm{W})}{|\mathrm{W}|^3},$$

under the constraints given by $A \cdot \mathrm{W} = 0$ and $\mathrm{W}[x] \in [-1, 1]$. However, due to the heavy computational cost, we could not manage to perform this Lagrange multiplier argument if $n > 6$ for now; on the other hand, we verified the correctness of the algorithm, by recovering the sharp constant $\frac{\sqrt{6}}{4}$ in dimension four.

After many attempts, also starting from many different initial data, the algorithm hints that the sharp constant in dimension 5 and 6 might be the same as in dimension 4: namely,

$$C(4) = \frac{\sqrt{6}}{4} \text{ and } C(5), C(6) \approx \frac{\sqrt{6}}{4}.$$

We also checked the convergence of the algorithm, using standard numerical analysis arguments, in order to verify the effectiveness of our procedure: an example is given in Figure 1 below, where it is apparent that, starting from different initial random vectors, the error converges to zero.

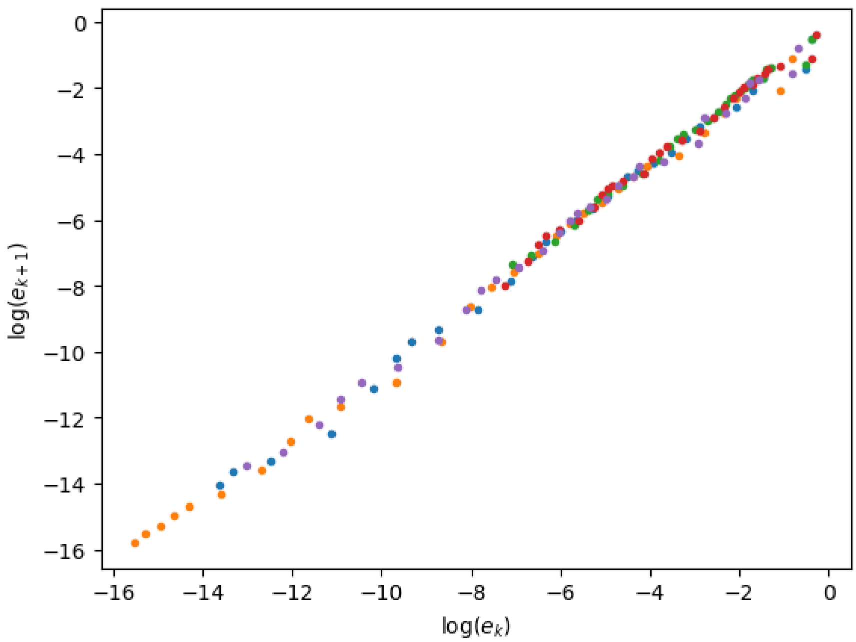

Figure 1. Estimates for the order of convergence of $\left|\dfrac{f(\mathrm{W})}{|\mathrm{W}|^3} - \dfrac{\sqrt{6}}{4}\right|$ for $n = 5$. Here, $\log(e_k) = \log\left|\dfrac{f(\mathrm{W}_k)}{|\mathrm{W}_k|^3} - \dfrac{\sqrt{6}}{4}\right|$, where W_k is the iteration at the k-th step. The scale of both axes are logarithmic.

This numerical result leads us to conjecture that the estimate

$$Q \leq C(n)|\mathrm{W}|^3$$

holds with $C(n) = \frac{\sqrt{6}}{4}$ for every n: however, if, on one hand, the equality is achieved when $n = 4$ by an algebraic Weyl tensor which is, in fact, the actual Weyl tensor of a metric g on a smooth manifold M, on the other hand, in higher-dimensional cases equality in (2.7) might be realized by some algebraic Weyl curvature tensor which does not derive from a Riemannian metric.

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