AN OVERDETERMINED PROBLEM IN 2D LINEARISED HYDROSTATICS

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ABSTRACT. In two spatial dimensions, we discuss the relation between the solvability of Schiffer's overdetermined problem and the optimisation, among sets of prescribed area, of the first eigenvalue in the buckling problem for a clamped plate and that of the first eigenvalue of the Stokes operator. For the latter, we deduce that the minimisers under area constraint that are smooth and simply connected must be discs from the fact that a pressureless velocity is a necessary condition of optimality.

1. INTRODUCTION

Let Ω denote a planar open set of finite area with Lipschitz boundary. We compare the eigenvalues of the buckling problem for a clamped plate

$$\begin{cases} \Delta^2\psi + \lambda\Delta\psi = 0\,, & \text{in } \Omega, \\ \psi = 0\,, & \text{on } \partial\Omega, \\ \partial_N\psi = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

the eigenvalues of interest when considering the steady Stokes equations, i.e.,

$$\begin{cases} \Delta u + \lambda u = \nabla p\,, & \text{in } \Omega, \\ \nabla \cdot u = 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and those relative to a third spectral variational problem, namely

$$\begin{cases} \Delta w + \lambda w = g\,, & \text{in } \Omega, \\ \Delta g = 0\,, & \text{in } \Omega, \\ \displaystyle\int_\Omega wf\,dx = 0\,, & \text{for all } f \in C(\overline{\Omega}) \text{ with } \Delta f = 0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

The sets of all $\lambda > 0$ for which the problems (1.1), (1.2), and (1.3) admit a non-trivial solution are denoted by $\mathfrak{S}^B(\Omega), \mathfrak{S}^S_p(\Omega), \mathfrak{S}^H_g(\Omega)$, respectively.

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Proposition 1. *We have*

$$\mathfrak{S}^B(\Omega) = \bigcup_{\substack{g\in C(\overline{\Omega})\,,\\ \Delta g=0\ \text{in}\ \Omega}} \mathfrak{S}^H_g(\Omega) \subseteq \bigcup_{p\in H^1(\Omega)} \mathfrak{S}^S_p(\Omega)\,.$$

If Ω is simply connected, then the second inclusion holds as an equality.

After proving Proposition 1, we discuss the distinguished role of the special value

$$\lambda_1^B(\Omega) = \min_{\psi\in H^2_0(\Omega)} \frac{\displaystyle\int_\Omega (\Delta\psi)^2\,dx}{\displaystyle\int_\Omega |\nabla\psi|^2\,dx}\,, \tag{1.4}$$

i.e., the first eigenvalue in (1.1), in the overdetermined problem

$$\begin{cases} -\Delta w = \lambda w\,, & \text{in } \Omega,\\ w \equiv \text{const}\,, & \text{on } \partial\Omega,\\ \partial_N w = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

By $\mathfrak{S}^D(\Omega) = \{\lambda_n^D(\Omega)\}_{n\geq 1}$ we denote the set of all Dirichlet eigenvalues of the Laplacian.

Theorem 1. *Let $\Omega\subset\mathbb{R}^2$ be a smooth and bounded open set. Then, (1.5) admits non-trivial solutions if and only if $\lambda\in\mathfrak{S}^B(\Omega)$ and (1.1) has a non-zero solution ψ for which $\Delta\psi$ is constant along $\partial\Omega$. In that case, either $\lambda > \lambda_1^B(\Omega)$ and $\lambda\in\mathfrak{S}^D(\Omega)$, with $\lambda > \lambda_2^D(\Omega)$, or Ω must be a disc.*

It is well known [17] that the boundary of the bounded Lipschitz open sets that support non-zero solutions of the overdetermined problem (1.5) must be analytic. Hence, the smoothness assumption on Ω is redundant.

As for the last statement of Theorem 1, it is obtained in [2] as a consequence of Weinstein's inequality, an isoperimetric property conjectured in [16] and proved in [12, 6] (see Eq. (1.11) below). Also, it rephrases another known fact: if the minimiser of λ_1^B under area constraint, which exists [1] in the class of simply connected open sets, is smooth then it must be a disc; we refer the reader to the related discussion in [8, Chap. 11], where a detailed description of this idea, ascribed to N.B. Willms and H.F. Weinberger, is provided; see also [10]. We just recall an expedient fact in both applications, that if Ω is a smooth open set which is not a disc, then by acting with infinitesimal rotations on solutions of the overdetermined problem (1.5) one may produce Dirichlet eigenfunctions relative to λ with three nodal domains at least; in view of Faber-Krahn inequality, of the equality case in the universal Ashbaugh-Benguria inequality, and of that in Weinstein's inequality, that would yield a contradiction unless $\lambda > \lambda_1^B(\Omega)$.

We will prove the rest of Theorem 1 in Section 2.

Remark 1. Incidentally, we recall that $\Delta\psi\big|_{\partial\Omega}\equiv$ const is a necessary condition of optimality for the minimisation of $\lambda_1^B(\Omega)$ among sets of given area, assuming simplicity. This implies a restriction [5]. This condition is relevant also to merely critical shapes, see [3, 4].

Remark 2. Any solution w of the overdetermined problem (1.5) must be orthogonal in $L^2(\Omega)$ to harmonic functions in Ω. Indeed, if $\Delta h = 0$ then

$$\lambda \int_\Omega w\, h\, dx = -\int_\Omega h \Delta w\, dx = \int_\Omega \nabla h \cdot \nabla w\, dx = \int_{\partial\Omega} w \partial_N h\, d\mathcal{H}^1 = \text{const} \cdot \int_\Omega \Delta h\, dx = 0\,,$$

where we also used the three equations in (1.5) for the first equality, for the second one, and for the fourth one, respectively. Thus, (1.5) can be seen as the pairing of problem (1.3), with $g = 0$, and of an additional Dirichlet boundary condition.

Given $\nu \in (0, +\infty)$, the conditions in Theorem 1 relate to the geometric rigidity of some special cellular flows confined within contractible rigid walls solving the 2D Navier-Stokes equations

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = \nu \Delta v - \nabla p\,, & \text{in } \Omega, \\ \nabla \cdot v = 0\,, & \text{in } \Omega, \\ v = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The pressure gradient in (1.6) is often thought of as a Lagrange multiplier arising with the incompressibility constraint $\nabla \cdot v = 0$, like in the case of steady Stokes equations. When solving for the pressure function, it is immediate to recognise that it is a harmonic function. Since it is defined up to additive constant, one might expect the equations for p to be automatically supplemented with boundary conditions that only involve ∇p, such as Neumann conditions, which may look natural at a first glance. Notwithstanding, such requirements for the pressure are extremely rigid, as one can imply from the following statement.

Proposition 2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and simply connected open set and let $\lambda > 0$. A necessary and sufficient condition that either condition (and hence both) in Theorem 1 be valid is that there exist a vector field u for which (1.2) holds with pressure p satisfying*

$$\partial_N p = 0\,, \qquad \text{on } \partial\Omega. \tag{1.7}$$

In fact, if (1.2) holds, then the Neumann boundary condition (1.7) is equivalent to condition

$$p \equiv \text{const}\ , \qquad \text{in } \Omega. \tag{1.8}$$

Eventually, this happens if and only if for any $\nu \in (0, +\infty)$ the function

$$v(x,t) = e^{-\nu\lambda t} u(x)\,, \tag{1.9}$$

is a solution of (1.6) with $\nabla p = 0$.

Note that in Proposition 2 the primitive variable at hand is a velocity field u solving (1.2). The assumption that Ω is smooth assures that u is smooth all the way up to the boundary, by Solonnikov's estimates [14, 15]. The assumption that Ω is simply connected, in turn, assures that we can write $u = \nabla^\perp \psi$ for a scalar function and then the boundary equations $\psi = \partial_N \psi = 0$ are inherited from the no-slip condition $u = 0$ on $\partial\Omega$.

Then, we consider the least eigenvalue for the problem (1.2), viz.

$$\lambda_1^S(\Omega) = \min_{u \in H_0^1(\Omega;\mathbb{R}^2)} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx} \,:\, \nabla\cdot u = 0 \text{ in } \Omega \right\}, \tag{1.10}$$

which was considered in the recent paper [9] from the perspective of spectral optimisation. As an immediate consequence of Lemma 3 and Lemma 4, proved in Section 3, we can prove the following fact. The equality case is already pointed out in [9, Sect. 1.3].

Proposition 3. *$\lambda_1^B(\Omega) \geq \lambda_1^S(\Omega)$, with equality if $\Omega \subset \mathbb{R}^2$ is simply connected.*

We recall *Weinstein's inequality* which gives a lower bound for the first eigenvalue λ_1^B in (1.1) in terms of the second Dirichlet eigenvalue λ_2^D of the Laplacian: for all planar open sets Ω one has

$$\lambda_1^B(\Omega) \geq \lambda_2^D(\Omega)\,, \tag{1.11}$$

with equality if and only if $\Omega = \Omega_\star$, where $\Omega_\star$ is the disc contouring as much area as Ω. As a consequence of the inequality in Proposition 3 and of the equality case in Weinstein's inequality, the isoperimetric property of the disc for λ_1^S, were it valid, would confirm the Pólya-Szegö conjecture for λ_1^B, i.e., that *the disc minimises the least buckling eigenvalue among open sets of given area.* The local optimality of the disc proved in [9, Theorem 2] is consistent with this long standing conjecture.

Remark 3 (Optimality conditions for $\lambda_1^S(\Omega)$)**.** If Ω is a bounded and smooth simply connected open set for which $\lambda_1^S(\Omega)$ simple and minimal among open sets of given area then the corresponding eigenfunctions u solve (1.2) with a pressure for which (1.8) (or, equivalently, with (1.7)) holds.

The necessary condition of optimality indicated in Remark 3 can be deduced from Theorem 1 and from the equality case of Proposition 3 on simply connected domains (see also Remark 1). Since the least Stokes eigenvalue is simple on smooth open sets [9, Theorem 4], we have the following.

Corollary 1. *Let Ω be a minimiser for $\lambda_1^S(\Omega)$ under area constraint. If Ω is smooth, bounded, and simply connected, then it must be a disc.*

A quasi-open set that minimises λ_1^S under area constraint is known to exist [9, Theorem 1]. Yet, the difficult regularity issue is still open, nonetheless. Note the difference with the three-dimensional case, in which case the ball is not a minimiser [9, Theorem 3].

Here is a final comment on the rigidity of requirement (1.7) for the pressure in Stokes equations (1.2).

Corollary 2. *If $\Omega \subset \mathbb{R}^2$ is a smooth, bounded, and simply connected open set and (1.2) with $\lambda = \lambda_1^S(\Omega)$ admits solutions for which the pressure satisfies the Neumann boundary conditions (1.7), then Ω must be a disc.*

It is not difficult to see that (1.7) may hold if Ω describes an infinite straight channel. Hence, the assumption that Ω be bounded cannot be removed.

2. Overdetermined problems

We will make repeatedly use of the following Lemma. We refer to [13] for a more general statement in the case of smooth open sets.

Lemma 1. *Let Ω be Lipschitz and $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$. Then $\psi \in H_0^2(\Omega)$ if and only if*

$$\int_\Omega f \Delta\psi \, dx = 0 \,, \qquad \text{for all } f \in C(\overline{\Omega}) \text{ with } \Delta f = 0 \text{ in } \Omega. \tag{2.1}$$

Proof. Integrating by parts twice yields (2.1) for $\psi \in C_0^\infty(\Omega)$, and hence for all $\psi \in H_0^2(\Omega)$ by a density argument. Conversely, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be orthogonal to all harmonic functions. Then, the vector field $\nabla\psi$ has divergence in $L^2(\Omega)$ and its normal trace at the boundary is a well defined element $\partial_N\psi$ of $L^2(\partial\Omega)$. If $f \in C(\overline{\Omega})$ is harmonic,

$$\int_{\partial\Omega} f \partial_N \psi \, d\mathcal{H}^1 = \int_\Omega \nabla f \cdot \nabla\psi \, dx = 0 \,,$$

where we also used, in the first integration by parts, that $\Delta\psi$ is orthogonal to f and, in the second one, that the boundary trace of ψ is zero. By the solvability of the Dirichlet problem for the Laplace equation for all boundary values in the trace space $H^{1/2}(\partial\Omega)$, we deduce that the boundary trace $\partial_N\psi$ is orthogonal to all elements of a dense subset of $L^2(\partial\Omega)$ and hence must be zero. □

2.1. **Proof of Proposition 1.** We split the proof into steps.

2.1.1. $\mathfrak{S}^B(\Omega) \subseteq \mathfrak{S}_0^H(\Omega)$. Let $\lambda \in \mathfrak{S}^B(\Omega)$ and let ψ be a solution of (1.1). By Lemma 1, $w = \Delta\psi$ defines an element of $H^1(\Omega)$ orthogonal to all harmonic functions and clearly we have $\Delta w + \lambda w = 0$. Thus, (1.3) holds with $g = 0$.

2.1.2. $\mathfrak{S}_g^H(\Omega) \subseteq \mathfrak{S}^B(\Omega)$ *for all harmonic g.* Let g be harmonic, let $\lambda \in \mathfrak{S}_g^H(\Omega)$, and let w solve (1.3). We let $\psi \in H_0^1(\Omega)$ be the solution of Poisson's equation

$$\begin{cases} \Delta\psi = \lambda^{-1}(g - \Delta w) \,, & \text{in } \Omega, \\ \psi = 0 \,, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then, by [7, Theorem 8.12], one has in fact $\psi \in H^2(\Omega)$. By inserting the equation for w in (2.2) we also see that $w = \Delta\psi$. Hence, by Lemma 1, the orthogonality of w against harmonic functions implies that $\psi \in H_0^2(\Omega)$, too. Therefore, ψ solves (1.1).

2.1.3. $\mathfrak{S}^B(\Omega) \subseteq \mathfrak{S}_p^S(\Omega)$ *for an appropriate* p. Let ψ be a solution of (1.1). By Lemma 1, $w = \Delta\psi$ must be orthogonal in $L^2(\Omega)$ to all harmonic functions in Ω. Also, the equation $\Delta w + \lambda w = 0$ holds. Then, we notice that

(2.3) $\quad h := \Delta\psi + \lambda\psi$ is the harmonic extension of the boundary trace of $\Delta\psi$.

Hence, $u = \nabla^\perp \psi$ is a solution of (1.2) with $\nabla p = \nabla^\perp h$. Equivalently, h is the harmonic conjugate of p. This means that p is defined up to an additive constant as the harmonic conjugate of h, i.e., the complex function whose real and imaginary parts are p and h, respectively, is holomorphic.

2.1.4. *The last statement.* We assume that Ω is simply connected. Then, we can find ψ with $u = \nabla^\perp\psi$ where u is a given solution of (1.2), with $p \in H^1(\Omega)$ and $\lambda \in \mathfrak{S}_p^S(\Omega)$. We prove that $\lambda \in \mathfrak{S}_0^H(\Omega)$. Indeed, $\psi \in H_0^2(\Omega)$ by (1.2) and so $w = \partial_{x_1} u^2 - \partial_{x_2} u^1$ is orthogonal to harmonic functions by Lemma 1. As $w = \Delta\psi$, after taking the curl of the first equation in (1.2), we arrive at (1.3) with $g = 0$. □

Henceforth in this section, we will sometimes understand the solutions in the classical sense. In the case of a smooth domain Ω, the global estimates due to Solonnikov [14, 15, 11] for the Dirichlet problem in linearised hydrostatics (1.2) imply the regularity up to the boundary of $u = \nabla^\perp\psi$ whenever $\psi \in H_0^2(\Omega)$ is a weak solution of (1.1), and hence $w = \partial_{x_1} u^2 - \partial_{x_1} u^1$ is also smooth up to the boundary; then, so is ψ because of the regularity of the forcing term in Poisson's equation $\Delta\psi = w$, with $\psi = 0$ on $\partial\Omega$.

2.2. **Proof of Theorem 1.** For the last statement, we refer to the proof of [8, Theorem 11.3.7], where the idea is credited to Weinberger and Wills.

If $\lambda > 0$ is such that (1.5) has a non-trivial solution w which takes the constant value c on the boundary, then $\psi = w - c$ satisfies the boundary conditions in (1.1) and $\Delta\psi + \lambda\psi$ is constant, hence harmonic. Also, in view of Remark 2, we have $\lambda \in \mathfrak{S}_0^H(\Omega)$ and that is contained in $\mathfrak{S}^B(\Omega)$ by Proposition 1.

Conversely, if $\lambda \in \mathfrak{S}^B(\Omega)$ and ψ solves (1.1), then $w := \Delta\psi$ satisfies the first equation in (1.5). If w is constant on $\partial\Omega$, then the harmonic exstension $h = w + \lambda\psi$ of the boundary values of w must be constant as well by the maximum principle, and so will be its harmonic conjugate p. This proves $u := \nabla^\perp\psi$ to solve (1.2) with constant pressure. Therefore, both the tangential and the normal component of Δu vanish at the boundary. As $\nabla^\perp w = -\Delta u$, we have $\nabla w = 0$ along $\partial\Omega$, and that gives the last two equations in (1.5). □

2.3. **Proof of Proposition 2.** Let u solve (1.2) for an appropriate pressure function p. Recall that both u and p are smooth functions up to the boundary. Since we are assuming Ω to be simply connected, there exists ψ with $u = \nabla^\perp\psi$.

The harmonic function $h = \Delta\psi + \lambda\psi$ is constant if, and only if, $\nabla p = \nabla^\perp h$ is identically zero, because Ω is connected. Also, h is the harmonic function with the same boundary values as $\Delta\psi$, hence h is constant in Ω if and only if $\Delta\psi$ is constant on $\partial\Omega$.

Conversely, the homogeneous Neumann boundary conditions (1.7) for the pressure imply that $\nabla^\perp(\Delta\psi)\cdot N = \Delta u \cdot N = 0$ along $\partial\Omega$, whence it follows that $\Delta\psi$ is constant on $\partial\Omega$, because the boundary of a simply connected *planar* set is connected.

We set $w = \Delta\psi$. Recalling that $\nabla^\perp w = -\Delta u$, in view of (1.2) we have $(u\cdot\nabla)w = (\nabla p - \lambda u)\cdot\nabla\psi$. Hence, recalling that $u = \nabla^\perp\psi$, we see that $\nabla p = 0$ implies that the convective term disappears from (1.6) if v is defined by (1.9), and the latter obviously defines a solution of the heat equation. Thus, we have seen that solutions of (1.2) with constant pressure make (1.9) solve (1.6).

Conversely, let v be defined by (1.9) and let (1.6) be valid. As $\partial_t v = \nu\Delta v$, the convective term $(v\cdot\nabla)v$ must clear off, which happens only if ∇p and $\nabla\psi$ are orthogonal, where $u = \nabla^\perp\psi$. In view of the condition $u = 0$ on $\partial\Omega$, that implies $\nabla p = 0$ on $\partial\Omega$. □

3. Relations between the least eigenvalues

Lemma 2. *Let Ω be a planar open set of finite area. Let $\psi \in H_0^2(\Omega)$ and $u = \nabla^\perp\psi$. Then $u \in H_0^1(\Omega;\mathbb{R}^2)$, with $\nabla\cdot u = 0$, and*

$$\frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx} = \frac{\displaystyle\int_\Omega (\Delta\psi)^2\,dx}{\displaystyle\int_\Omega |\nabla\psi|^2\,dx}$$

Proof. One uses that $\partial^2_{xy}\psi - \partial^2_{xx}\psi\partial^2_{yy}\psi$ integrates to zero in Ω. □

Lemma 3. *Let Ω be a planar open set. Then*

$$\lambda_1^B(\Omega) \geq \lambda_1^S(\Omega)\,.$$

Proof. Given $\varepsilon > 0$, we can find $\psi \in C_0^\infty(\Omega)$ with

$$\lambda_1^B(\Omega) + \varepsilon \geq \frac{\displaystyle\int_\Omega (\Delta\psi)^2\,dx}{\displaystyle\int_\Omega |\nabla\psi|^2\,dx}\,.$$

Note that $u = \nabla^\perp\psi$ is admissible for the definition of $\lambda_1^S(\Omega)$. Then, the conclusion follows by Lemma 2. □

Lemma 4. *Let Ω be simply connected. Then*

$$\lambda_1^S(\Omega) \geq \lambda_1^B(\Omega)\,.$$

Proof. Let $\varepsilon > 0$. Recall that $C_0^\infty(\Omega;\mathbb{R}^2)$ is dense in $H_0^1(\Omega;\mathbb{R}^3)$. Then, by arguing as done in Lemma 3, we find $u \in C_0^\infty(\Omega;\mathbb{R}^2)$, with $\nabla\cdot u = 0$,

$$\lambda_1^S(\Omega) + \varepsilon \geq \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx}\,.$$

By assumption, there exists $\psi \in H_0^2(\Omega)$ with $\nabla^\perp\psi = u$ and we conclude by Lemma 2. □

We end this section by observing some direct consequences of Theorem 1 concerning a variational description of the buckling eigenvalues λ, with eigenfunction ψ, in terms of the laplacian $w = \Delta\psi$.

Remark 4. If ψ solves (1.1), then the function $w = \Delta\psi$ and the harmonic extension

$$h = w + \lambda\psi$$

of the boundary values of w are smooth up to the boundary and satisfy the identity

$$\int_\Omega |\nabla h|^2\,dx = \int_{\partial\Omega} w\partial_N w\,d\mathcal{H}^1 = \int_\Omega |\nabla w|^2\,dx - \lambda\int_\Omega w^2\,dx\,. \tag{3.1}$$

Hence, whenever we start from a buckling eigenfunction ψ with eigenvalue λ we arrive at an eigenpair (w,λ) for the eigenvalue problem (1.3) with

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx - \int_{\partial\Omega} w\partial_N w\,d\mathcal{H}^1}{\displaystyle\int_\Omega w^2\,dx} \le \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_\Omega w^2\,dx}\,, \tag{3.2}$$

with equality if, and only if, $\partial_N w = 0$ identically on $\partial\Omega$, i.e., $\Delta\psi \equiv c$ on $\partial\Omega$. In the equality case of (3.2), we have $\lambda \ge \lambda_2^D(\Omega)$. In fact, whether or not the equality holds, it is known that either $\lambda = \lambda_1^B(\Omega)$ and Ω is a disc, so that $\lambda_1^B(\Omega) = \lambda_2^D(\Omega)$, or else λ is also a Dirichlet eigenvalue of the Laplacian, with $\lambda > \lambda_2^D(\Omega)$.

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