universität wien

DISSERTATION / DOCTORAL THESIS

Titel der Disseratation / Title of the Doctoral Thesis

„Solutions for a two-phase free-boundary problem and for dynamic perfect elasto-plasticity“

verfasst von / submitted by

Randy Fernando Llerena Montenegro M.

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr.rer.nat.)

Wien, 2024 / Vienna, 2024

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Prof. Paolo Piovano, PhD Prof. Jean-François Babadjian, PhD

Yo soy yo, y mi circunstancia
y si no la salvo a ella no me salvo yo.
José Ortega y Gasset ("Meditaciones del Quijote")

Acknowledgments

I would like to express my deepest gratitude to my advisors, Prof. Paolo Piovano and Prof. Jean-François Babadjian, whose unwavering support, insightful guidance, and scholarly expertise have been instrumental in shaping this dissertation. Your mentorship has been a source of inspiration, and I am truly grateful for the opportunities you provided for intellectual growth and exploration.

I extend my sincere thanks to the reviewers of my doctoral manuscript, Professor Dorin Bucur and Professor Rodica Toader for kindly agreeing to examine and evaluate this thesis.

I am grateful to the Faculty of Mathematics of the University of Vienna for fostering a vibrant academic environment. Special thanks to Shokhrukh Kholmatov for engaging discussions, collaborative problem-solving, and the shared journey through the challenges of mathematical research. My sincere thanks to Fondation Mathématiques Jacques Hadamard for their financial support during my scientific visit to the Laboratoire de Mathématiques d'Orsay, where a significant portion of this thesis was written.

I extend my heartfelt gratitude to Amarun and all its members for their invaluable efforts in enhancing the field of mathematics in Ecuador. Their support has been instrumental in guiding and shaping my academic journey. Especially, I would like to thank Diego Chamorro and Oscar Jarrín. I am also deeply grateful to my dear friends residing in Paris, with whom I shared the journey of becoming a mathematician, Ana Julia, Alex David, Mónica, Sofía, María José and Quentin: your companionship added immense joy and support to this transformative experience. Furthermore, I would like to express my gratitude to all the friends that the mathematical community has brought into my life, with special acknowledgment to Lorenzo, Lorenza, and José. Their friendship has been particularly meaningful and supportive throughout this journey.

To my dear friends from Ecuador, I extend my heartfelt gratitude for always welcoming me and being there at any moment. Special thanks to Steven, André 1, Renato, Ghandi, André A, Grace, Pavlova, and Nora for their unwavering support and companionship.

For his time, his support, and his help, I would like to express my appreciation to my younger brother, David, mis palabras son breves, pero mi gratitud es infinita para lo que has representado en la escritura de esta tesis, ¡Gracias David y gracias Mica! I want to express my heartfelt gratitude to Daysi, Pablito, Sofía and Pablo for their unwavering support and love, even across thousands of kilometers. ¡Gracias totales!

Ich möchte mich herzlich bei Aida, Diana und Rudi bedanken, dass sie zu meiner zweiten Familie in Wien geworden sind. Ihre Freundschaft und Unterstützung haben meine Erfahrung hier zu etwas Unersetzlichem gemacht.

La redacción de esta tesis es un testimonio de la perseverancia de mis padres. Les agradezco profundamente, gracias a su amor infinito, soy la persona que soy. Su apoyo inquebrantable ha sido una fuente constante de fuerza. Lastly, I extend my heartfelt thanks to Melissa. You have been my steadfast support, and my rock throughout this journey. You are not only a witness to the most challenging moments on this path but also to the happiest ones. Your unwavering support has been a constant source of strength, and I deeply appreciate you always being there for me. ¡Gracias por tu amor y tu tiempo!

Abstract

The thesis relates to the research areas of the *calculus of variations*, *geometric measure theory*, and *PDEs* with a focus on models from *continuum mechanics* and aims at moving forward from the state of art with respect to the following two problems:

Part A: the existence of solutions for free-boundary problems in linear elasticity involving more phases [65, 66],

Part B: the existence and uniqueness of solutions for dynamic perfect elasto-plasticity [12],

which represent the two parts of the thesis.

In Part A both the static model introduced in [66] for the description of the morphology of two-phase continua with the feature of allowing for both coherent and incoherent interfaces, and the extension to the setting of film multilayers addressed in [65] are discussed. Such models are designed in the framework of the theory of Stress Driven Rearrangement Instabilities, which are characterized by the competition between elastic and surface effects. For both settings the existence of energy minimizers is established in the plane by means of the direct method of the calculus of variations under a constraint on the number of boundary connected components, and by prescribing a graph assumption for the underlying substrate phase in [66] and for the interfaces between film layers in [65]. Both the wetting and the dewetting regimes are included in the analysis.

In Part B the well-posedness of a dynamical model of perfect plasticity with mixed boundary conditions for general closed and convex elasticity sets is addressed in [12]. The proof is based on an asymptotic analysis for the perfect plasticity model with relaxed dissipative boundary conditions, on extending the measure theoretic duality pairing between stresses and plastic strains, as well as on a convexity inequality to a more general context where deviatoric stresses are not necessarily bounded. Complete answers are given in the pure Dirichlet and pure Neumann cases, while for general mixed boundary conditions in dimension 2 and 3 under additional geometric hypotheses on the elasticity sets and the reference configuration.

Kurzfassung

Die Thesis bezieht sich auf die Forschungsgebiete der *Variationsrechnung* und der *PDEs* mit einem Schwerpunkt auf Modellen aus der *Kontinuumsmechanik* und zielt darauf ab, den Stand der Technik in Bezug auf die folgenden zwei Probleme zu erweitern:

Teil A: die Existenz von Lösungen für freie Randbedingungen bei linearer Elastizität mit Beteiligung mehrerer Phasen [65, 66],

Teil B:. die Existenz und Einzigartigkeit der dynamischen perfekten Elasto-Plastizität [12],

die die beiden Teile der These darstellen.

In Teil A wird ein statisches Modell zur Beschreibung der Morphologie von Zweiphasenkontinua vorgestellt, das sowohl kohärente als auch inkohärente Grenzflächen berücksichtigt, wird in [66] vorgestellt. Das Modell wurde im Rahmen der Theorie der spannungsgesteuerten Umlagerungsinstabilitäten entwickelt, die durch die Konkurrenz zwischen elastischen und Oberflächeneffekten gekennzeichnet sind. Das Vorhandensein von Energieminimierern wird in der Ebene mit Hilfe der direkten Methode unter der Bedingung nachgewiesen, dass die Anzahl der zusammenhängenden Komponenten der zugrundeliegenden Phase, deren äußerer Rand vorgeschrieben ist, um eine Graphenannahme zu erfüllen, und der zweiphasigen zusammengesetzten Region begrenzt ist. Sowohl das Benetzungs- als auch das Entnetzungsregime werden in die Analyse einbezogen, und die Ausweitung auf die Situation von Folienmehrschichten wird in [65] untersucht.

In Teil B wird die Wohlgeformtheit eines dynamischen Modells der perfekten Plastizität mit gemischten Randbedingungen für allgemeine geschlossene und konvexe Elastizitätsmengen in [12] behandelt. Der Beweis basiert auf einer asymptotischen Analyse für das Modell der perfekten Plastizität mit entspannten dissipativen Randbedingungen, auf der Erweiterung der maßtheoretischen Dualitätspaarung zwischen Spannungen und plastischen Dehnungen sowie auf einer Konvexitätsungleichung für einen allgemeineren Kontext, in dem abweichende Spannungen nicht unbedingt begrenzt sind. Vollständige Antworten werden für den reinen Dirichlet- und den reinen Neumann-Fall gegeben, während für allgemeine gemischte Randbedingungen in den Dimensionen 2 und 3 unter zusätzlichen geometrischen Hypothesen über die Elastizitätsmengen und die Referenzkonfiguration.

Notations

In this manuscript, otherwise stated, we consider $n = 2, 3$. We denote by $\mathbb{S}^1$ the unit sphere, i.e., $\mathbb{S}^1 := \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Linear Algebra

Let $a, b \in \mathbb{R}^n$.

$a \cdot b$	Euclidean scalar product between a and b, i.e., $a \cdot b := \sum_{i=1}^n a_i b_i$
$\|a\|$	norm of a with respect to the Euclidean scalar product, i.e., $\|a\| := \sqrt{a \cdot a}$
$\mathbb{M}^n$	the set of $n \times n$ matrices
$\mathbb{M}^n_{sym}$	the space of symmetric $n \times n$ matrices.
$a \odot b$	the symmetric tensor product, i.e., $a \odot b := (ab^T + b^T a)/2 \in \mathbb{M}^n_{sym}$
π_i	The projections onto x_i-axis for $i = 1, \dots, n$, i.e., $\pi_i(x_1, \dots, x_i, \dots, x_n) = x_i$ for every $(x_1, \dots, x_i, \dots, x_n) \in \mathbb{R}^n$ and $i = 1, \dots, n$

Let $A, B \in \mathbb{M}^n$

Id	identity matrix of $\mathbb{M}^n$
$\mathrm{tr}\, A$	trace of A
$A : B$	Fröbenius scalar product, i.e., $A : B := \mathrm{tr}\Big(A^T B\Big)$
$\|A\|$	Fröbenius norm, i.e., $\|A\| := \sqrt{A : A}$.
A_D	deviatoric part of A, i.e., $A_D := A - \frac{1}{n}(\mathrm{tr}\, A)\mathrm{Id}$

If $n = 2$, we denote the orthonormal basis of $\mathbb{R}^2$ as follows:

$\mathbf{e_1}$	$(1, 0) \in \mathbb{R}^2$
$\mathbf{e_2}$	$(0, 1) \in \mathbb{R}^2$

Measure theory

Let μ, ν be two Radon measures.

$\nu \ll \mu$	ν is absolutely continuous with respect to μ
$\nu \perp \mu$	μ and ν are mutually singular
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\mathcal{H}^{n-1}$	$(n-1)$-dimensional Hausdorff measure
$\mu \llcorner E$	restriction of the measure μ to E
$\frac{d\mu}{d\nu}$	Radon-Nikodým derivative of μ respect to ν

Topology and metric space notation

Let $x, x_0 \in \mathbb{R}^n$, $\nu \in \mathbb{S}^1$ and $\rho > 0$.

$B_\rho(x)$	the open ball with center x and radius ρ
$\mathrm{Int}(E)$	topological interior of E
$\overline{E}$	topological closure of E
∂E	topological boundary of E
$Cl(F)$	the closure of a set F in E with respect to the relative topology in E
rE	$rE := \{rx : x \in E\}$
$Q_{\rho,\nu}(x)$	the open square of sidelength 2ρ whose sides are either perpendicular or parallel to ν
Q_ρ	$Q_\rho := Q_{\rho,\mathbf{e_2}}(0)$
I_ρ	$I_\rho := [-\rho, \rho]$
$\mathrm{dist}(\cdot, E)$	distance function from E
$\mathrm{sdist}(\cdot, \partial E)$	signed distance function from ∂E, i.e., $$\mathrm{sdist}(x, \partial E) := \begin{cases} \mathrm{dist}(x, E) & \text{if } x \in \mathbb{R}^n \setminus E, \\ -\mathrm{dist}(x, E) & \text{if } x \in E \end{cases}$$
σ_{ρ,x_0}	the *blow-up map* centered in x_0, i.e., $$\begin{array}{rccc} \sigma_{\rho,y_0}: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & x & \mapsto & \sigma_{\rho,x_0}(x) := \frac{x-x_0}{\rho} \end{array}$$
σ_ρ	$\sigma_\rho = \sigma_{\rho,0}$, where $0 = (0, \dots, 0) \in \mathbb{R}^n$

Notice that $\sigma_{\rho,x}\left(Q_{\rho,\nu}(x)\right) = Q_{1,\nu}(0)$ and $\sigma_{\rho,x}\left(\overline{Q_{\rho,\nu}(x)}\right) = \overline{Q_{1,\nu}(0)}$.

Convex Analysis

Let X be a topological vector space and X^* be the topological dual of X. Let $A \subset X$ be a convex set and $f, g : A \to [-\infty, \infty]$ two convex functions

I_A	indicator function of the convex set A, i.e., $I_A(x) = 0$ if $x \in A$ or $I_A(x) = \infty$ if $x \notin A$
f^*	convex conjugate of f, i.e., $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$
$f \Box g$	infimal convolution of f and g, i.e., $f \Box g(x) := \inf\{f(x-y) + g(y) : y \in X\}$

Functions and functional spaces

Let Ω be an open set of $\mathbb{R}^n$ and let Y be an Euclidean space.

Du	Distributional derivative of u
$E(u)$	Symmetric distributional derivative of u
$C^k(\Omega;\mathbb{R}^m)$	$\{u : \Omega \to \mathbb{R}^m : u$ is k-continuously differentiable in $U\}$
$C^\infty(\Omega;\mathbb{R}^m)$	$\bigcap_{k\in\mathbb{N}} C^k(\Omega;\mathbb{R}^m)$
$\mathcal{M}(\Omega;Y)$	space of Y-valued bounded Radon measures in X
$\mathcal{M}(X)$	space of real valued bounded Radon measures in X
$L^p(\Omega;\mathbb{R}^m)$	$\{u : \Omega \to \mathbb{R}^m :$ u is measurable and $\int_\Omega \lvert u\rvert^p \, dx < \infty\}$, for $p \in [1, \infty)$
$L^\infty(\Omega;\mathbb{R}^m)$	$\{u : \Omega \to \mathbb{R}^m :$ u is measurable and $\operatorname{ess\,sup} u < \infty\}$, for $p \in [1, \infty)$
$W^{1,p}(\Omega;\mathbb{R}^m)$	$\{u \in L^p(\Omega;\mathbb{R}^m) : Du \in L^p(\Omega;\mathbb{R}^{n\times m})\}$, for $p \in [1, \infty)$
$BV(\Omega;\mathbb{R}^m)$	$\{u \in L^1(\Omega;\mathbb{R}^m) : Du \in \mathcal{M}(\Omega;\mathbb{R}^m)\}$
$BD(\Omega)$	$\{u \in L^1(\Omega;\mathbb{R}^m) : Eu \in \mathcal{M}(\Omega;\mathbb{R}^m)\}$

Contents

1. Introduction

In this manuscript problems in *Materials Science* with a focus on models in the framework of *Continuum Mechanics* are addressed. Specifically, the thesis includes results both for the characterization of optimal crystalline morphologies as minimizers of new original variational models and for the evolution of elastic-plastic deformation of crystalline materials under critical stress. As such, this dissertation is divided in two parts containing the results introduced in the papers [12, 65, 66]:

-**Part A: Existence of solutions for free-boundary problems in linear elasticity involving more phases**;

-**Part B: Existence and uniqueness of solutions for dynamic perfect elasto-plasticity**.

Regarding the employed mathematical strategies, they are based on the fields of *partial differential equations, calculus of variations* and *geometric measure theory.* The methods include various variational techniques [4, 47, 68], such as the *direct method of the calculus of variations*, *blow-up techniques*, the fine properties of the space of *functions of bounded deformation*, and the variational treatment of *evolutionary dissipative processes.*

Regarding the mathematical modeling, the investigations find application in various settings of *Materials Science*, such as epitaxially-strained thin films, material fractures, and delamination phenomena. Moreover, models of elasto-plasticity have the capacity to predict the appearance of permanent deformations in materials as related critical stresses are reached. From a microscopic point of view, the so-called plastic deformations are the result of atomic defects due to inter-crystalline slips inside a lattice, called *dislocations.* It is experimentally observed that plastic flows occur on very thin zones called *slip bands*, on which there is a strain localization: these zones are macroscopically interpreted as discontinuity surfaces of the displacement. We refer to [9, 33, 55] for the theory of *stress driven rearrangement instabilities*, to [19] for the classical variational approach to fracture, and to the monograph [67] for an comprehensive presentation of elasto-plasticity models.

The state of the art, the results and the strategy used in this manuscript are described below distinguishing the two parts in two sections. In Part A, the results of [65, 66] are described, which relate to the static theory in dimension $n = 2$. In Part B the results of [12] are presented, which regard the evolutionary theory for both dimensions $n = 2, 3$. In both parts, we consider an open smooth set $\Omega \subset \mathbb{R}^n$ as the reference region where the elasto(-plastic) body is located. Furthermore, given a *set of finite perimeter* $U \subset \Omega$ we denote by $U^{(s)}$ the set of points in U with density $s \in \{0, 1\}$, and by $\nu_U(x)$ the outward unit normal of U at $x \in \partial^* U$, where $\partial^* U$ is the reduced boundary of U [4, 68].

1.1. Introduction to Part A

In this part, we tackle the challenge of establishing a variational framework to characterize the surface and elastic properties of multi-phase continua, by following Gibbs's notion of a well-defined interface [24, 51, 56] among phases. The interaction between two or more media can cause considerable tension due to the discrepancy in their crystalline order, which is responsible for various morphological destabilizations often referred to in the literature as *stress-driven rearrangement instabilities* (SDRI) [9, 33, 55, 58, 77]. These instabilities encompass features like rough crystalline boundaries, bulk material cracks, dislocations in crystalline lattices, and delamination at contact regions. Although the existing literature extensively studies these phenomena assuming that one phase acts as a rigid continuous medium, either underlying or constraining the other, there are scenarios where the hierarchy between phases is unclear. The complexity arises from integrating material accretion and deletion responsible for phase interface movement with the elasticity framework governing bulk deformation and fractures [26, 50], leading to conceptual difficulties, as highlighted by Gurtin [56]. A crucial modeling issue concerns the interplay between *coherency*, the microscopic arrangement of atoms in a homogeneous lattice with deformation as the sole stress relief mechanism [56], and *incoherency*, representing debonding between atoms and resulting in composite delamination at the two-phase interface [63].

First, in Chapter 3 the focus is on the two-phase scenario by presenting the results achieved in [66], and then Chapter 4 addresses the multi-phase results for film multilayers contained in [65]. In both these contexts the possibility of interfaces between phases presenting both coherent and incoherent portions is incorporated into the models. To do so, the approach introduced in [58, 59] for the setting of a fixed underlying substrate phase and based on the technique devised for addressing the Mumford-Shah problem in [30] is adopted. The strategy consists in initially imposing a fixed constraint on the number of connected components for the boundaries of the free phases in order to apply adaptations of Gołąb's Theorem [53] to obtain a compactness property for energy equibounded sequences with respect to a proper selected topology. However, extending the approach of [58] to a two-phase context requires significant adjustments to the model configuration. Concerning the characterization of the incoherent interface, we do not consider it as the discontinuous segment of the bulk displacement along the two-phase interface, as done in [58, 59, 60], since this option shows to be unfeasible in our case because of the irregularity of the two-phase interfaces exceeding that of Lipschitz manifolds, which was instead the assumption in [58, 59, 60]. To address this challenge, the energy is defined as dependent on set variables, which are not the regions occupied by the two phases (referred to as the *film* and the *substrate phase*), but rather the ones of the substrate phase and of the entire composite of both phases. In this framework, we characterize the incoherent interfaces as a segment of the boundary intersection of these variables. In Chapters 3 and 4 we obtain existence results comparable to the existence result established in [58], with the extension that also the possibility of a countable number of film islands on top of the substrate (or of islands of a film layer on top of another layer) is allowed, since the strategy eliminates the need to impose a constraint on the number of boundary components of the film phases.

In Chapter 3 the container $\Omega := (-l,l) \times_{\mathbb{R}^2} (-L,L) \subset \mathbb{R}^2$ represents the region where the composite material is located for two positive parameters l and L. In accordance with the SDRI theory [9, 33, 55], the total energy $\mathcal{F}$ is the sum of two contributions: the elastic energy $\mathcal{W}$ and the surface energy $\mathcal{S}$. More precisely, $\mathcal{F}$ is defined on triples $(A,S,u) \in \widetilde{\mathcal{C}}$, where u represents the *bulk displacement* of the composite material consisting of the two phases, and A and S are sets whose closures denote the *composite region* and the *substrate region*, respectively. Notice that the *film region* is instead represented by $\overline{A} \setminus S^{(1)}$. More specifically, the family of admissible configurations $\widetilde{\mathcal{C}}$ is defined as

$$\begin{aligned}\widetilde{\mathcal{C}} := \{(A,S,u) : \quad & A \text{ and } S \text{ are } \mathcal{L}^2\text{-measurable sets with } S \subset \overline{A} \subset \overline{\Omega} \text{ such that} \\ & \partial A \cap \operatorname{Int}(S) = \emptyset,\ \partial A \text{ and } \partial S \text{ are } \mathcal{H}^1\text{-rectifiable}, \\ & \mathcal{H}^1(\partial A) + \mathcal{H}^1(\partial S) < \infty, \text{ and } u \in H^1_{\mathrm{loc}}(\operatorname{Int}(A);\mathbb{R}^2)\},\end{aligned}$$

and $\mathcal{F} : \widetilde{\mathcal{C}} \to \mathbb{R}$ is given by

$$\mathcal{F}(A,S,u) := \mathcal{S}(A,S) + \mathcal{W}(A,u)$$

for every $(A, S, u) \in \widetilde{\mathcal{C}}$. The elastic energy is considered in the framework of the *theory of small deformations* of *linear elasticity* as

$$\mathcal{W}(A, u) := \int_A W(z, E(u) - E_0) dz,$$

where $Eu := (Du + Du^T)/2$ is the linearized *strain tensor* which takes its values in the set $\mathbb{M}^2_{sym}$ of symmetric 2×2 matrices, the elastic density $W(z, M) := \mathbb{C}(z)M : M$ is defined with respect to a positive-definite elasticity tensor $\mathbb{C}$, and $E_0 \in \mathbb{M}^2_{sym}$ is referred to as the *mismatch strain*. The inclusion of mismatch strain in the SDRI theory addresses the scenario where the two phases consist of potentially different crystalline materials, each with its own equilibrium lattice. The surface energy $\mathcal{S}(A, S)$ is given by

$$\mathcal{S}(A, S) := \int_{\Omega \cap (\partial A \cup \partial S)} \psi(x, \nu(x)) \, d\mathcal{H}^1(x),$$

where, by denoting with $\nu_U(z)$ the normal unit vector pointing outward to a set $U \subset \mathbb{R}^2$ with $\mathcal{H}^1$-rectifiable boundary at a point $x \in \partial U$,

$$\nu(x) := \begin{cases} \nu_A(x) & \text{if } z \in \partial A \setminus \partial S, \\ \nu_S(x) & \text{if } z \in \partial S, \end{cases}$$

and $\psi : \overline{\Omega} \times \mathbb{R}^2 \to [0, \infty]$ represents the *surface tension of the composite* of the two phases, which we allow to be anisotropic. In order to properly define ψ we consider the following three surface tensions φ_{F}, φ_{S}, $\varphi_{\mathrm{FS}} : \overline{\Omega} \times \mathbb{R}^2 \to [0, \infty]$, which are supposed to characterize the three interfaces in the two-phase setting: the interface between the film phase and the vapor, the interface between the substrate phase and the vapor, and the interface between the film and substrate. Moreover, to consider both the wetting and dewetting regimes, in [66] we introduce two auxiliary surface tensions, referred to as the *regime surface tensions*, and defined as follows:

$$\varphi := \min\{\varphi_{\mathrm{S}}, \varphi_{\mathrm{F}} + \varphi_{\mathrm{FS}}\} \qquad \text{and} \qquad \varphi' := \min\{\varphi_{\mathrm{S}}, \varphi_{\mathrm{F}}\}.$$

We define

$$\psi(x, \nu(x)) := \begin{cases} \varphi_{\mathrm{F}}(x, \nu(x)) & x \in \Omega \cap (\partial^* A \setminus \partial^* S), \\ \varphi(x, \nu(x)) & x \in \Omega \cap \partial^* S \cap \partial^* A, \\ \varphi_{\mathrm{FS}}(x, \nu(x)) & x \in \Omega \cap (\partial^* S \setminus \partial A), \\ (\varphi_{\mathrm{F}} + \varphi)(x, \nu(x)) & x \in \Omega \cap \partial^* S \cap \partial A \cap A^{(1)}, \\ 2\varphi_{\mathrm{F}}(x, \nu(x)) & x \in \Omega \cap \partial A \cap A^{(1)} \cap S^{(0)}, \\ 2\varphi'(x, \nu(x)) & x \in \Omega \cap \partial A \cap A^{(0)}, \\ 2\varphi_{\mathrm{FS}}(x, \nu(x)) & x \in \Omega \cap (\partial S \setminus \partial A) \cap \left(S^{(1)} \cup S^{(0)}\right) \cap A^{(1)}, \\ 2\varphi(x, \nu(x)) & x \in \Omega \cap \partial S \cap \partial A \cap S^{(1)}, \end{cases} \tag{1.1.1}$$

where the sub-regions of $\Omega \cap (\partial A \cup \partial S)$ appearing in (1.1.1) represents, by moving line by line, the *film free boundary*, the *substrate free boundary*, the *film-substrate coherent interface*, the *film-substrate incoherent interface*, the *film cracks*, the *exposed filaments*, the *substrate filaments and cracks* in the film-substrate coherent interface, and the *substrate cracks* in the film-substrate incoherent interface, respectively. For the physical justification of (1.1.1) we refer to Section 3.1 (see also [66]).

The existence of minimizers of $\mathcal{F}$ is established under a *two-phase volume constraint* and with a constraint on the number of boundary connected components by following the strategy of [58] for the setting of a fixed substrate region and of [30] for the Mumford-Shah problem. In particular, in [66] we restrict to the family $\mathcal{C}_{\mathbf{m}} \subset \widetilde{\mathcal{C}}$ for $\mathbf{m} := (m_0, m_1) \in \mathbb{N}^2$, where any configuration $(A, S, u) \in \mathcal{C}_{\mathbf{m}}$ has the following two properties:

- the number of boundary connected components of S and A are fixed to be at most m_0 and m_1, respectively,
- the substrate regions S satisfy an *exterior graph constraint* consisting in requiring that $\partial^* S \cup (\partial S \cap S^{(0)})$ is the graph of an upper semicontinuous function with pointwise bounded variation (while internal, also non-graph-like, substrate cracks are allowed).

More precisely, for any two volume parameters $\mathbb{v}_0, \mathbb{v}_1 \in [\mathcal{L}^2(\Omega)/2, \mathcal{L}^2(\Omega)]$ such that $\mathbb{v}_0 \leq \mathbb{v}_1$, we consider the problem:

$$\inf_{\substack{(A,S,u)\in\mathcal{C}_{\mathbf{m}} \\ \mathcal{L}^2(S)=\mathbb{v}_0,\, \mathcal{L}^2(A)=\mathbb{v}_1}} \mathcal{F}(A,S,u),$$

which we tackle by employing the *direct method* of the calculus of variations, for which we choose a proper topology in $\widetilde{\mathcal{C}}$ sufficiently weak to establish a compactness property for energy-equibounded sequences in $\mathcal{C}_{\mathbf{m}}$ and strong enough to prove the lower semicontinuity of $\mathcal{F}$ in $\mathcal{C}_{\mathbf{m}}$.

Finally, in Chapter 4 the extension of the results of [66] to the multi-phase setting of film multilayered composites, which is achieved in [65], is presented. In [65] analogous existence results to the one of [66] are achieved by means of the direct method of the calculus of variations for a model that represents the combination of the implementation to the multilayer setting of the model in [66] and of the variational models for singled-layer films previously addressed in [25, 34, 35, 46]. As such, the existence result of [65] extends literature results for single-layered films deposited on a fixed substrate in the following directions: by letting the substrate surface free, by addressing the presence of multiple layers of various materials, and by including into the analysis the possibility of a failure of the film coatings taking into account the delaminated portions both at the contact interface with the substrate and at the interface between film layers. The Reader is kindly referred to Section 4.2 for the detailed description of the model.

1.2. Introduction to Part B

By means of the variational principles of Hodge-Prager for the stress rate and of Greenberg for the velocity, a variational formulation of the mathematical models of plasticity that consists in highly-nonlinear hyperbolic systems, is possible, providing a more tractable setting to prove stress existence and uniqueness (see, e.g., [42, 68]). The corresponding evolution problem for the velocity (and the displacement) involves additional difficulties connected with the regularity of the strain tensor, which were addressed in [84, 85] for the *quasi-static* case and in [8] for the dynamical case by means of different types of visco-plastic regularizations. Furthermore, in [83] a proper functional space for the kinematically admissible displacements, which can exhibit discontinuity, was introduced by considering the space BD of *functions of bounded deformation* (see [89] for a comprehensive treatment). Moreover, the quasi-static case was revisited more recently in [32] within the general framework of variational evolutions of *rate independent processes*.

In the framework of small strain elasto-plasticity where the natural kinematic variable is the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^n$. In small strain elasto-plasticity, Eu decomposes additively in the following form:

$$Eu = e + p,$$

where $e : \Omega \times [0,T] \to \mathbb{M}^n_{\text{sym}}$ is the *elastic strain* and $p : \Omega \times [0,T] \to \mathbb{M}^n_{\text{sym}}$ the *plastic strain*. The elastic strain is related to the stress tensor $\sigma : \Omega \times [0,T] \to \mathbb{M}^n_{\text{sym}}$ by means of Hooke's law $\sigma := \mathbb{C}e$, where $\mathbb{C}$ is the symmetric fourth order elasticity tensor. In a dynamical framework and in the presence of an external body load $f : \Omega \times [0,T] \to \mathbb{R}^n$, the equation of motion is

$$\ddot{u} - \text{div}\sigma = f \quad \text{in } \Omega \times [0,T],$$

where $\dot{u}$ and $\ddot{u}$ denote the time derivative and the second time derivative of u, respectively. Plasticity is characterized by the existence of a *yield zone* beyond which permanent strains appear. The stress tensor is indeed constrained to belong to a fixed closed and convex subset $\mathbf{K}$ of $\mathbb{M}^n_{\text{sym}}$, i.e., $\sigma \in \mathbf{K}$. If σ lies inside the interior of $\mathbf{K}$, the material behaves elastically, so that unloading will bring back the body into its initial configuration ($\dot{p} = 0$). On the other hand, if σ reaches the boundary of $\mathbf{K}$, i.e., the *yield surface*, a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain p. Its evolution is described is expressed with the Prandtl-Reuss law, i.e., $\dot{p} \in N_{\mathbf{K}}(\sigma)$, where $N_{\mathbf{K}}(\sigma)$ denotes the normal cone to $\mathbf{K}$ at σ. From the theory of convex analysis, $N_{\mathbf{K}}(\sigma) = \partial I_{\mathbf{K}}(\sigma)$, *i.e.*, the sub-differential of the indicator function $I_{\mathbf{K}}$ of the set $\mathbf{K}$ ($I_{\mathbf{K}}(\sigma) = 0$ if $\sigma \in \mathbf{K}$, while $I_{\mathbf{K}}(\sigma) = +\infty$ otherwise). Hence, from convex duality, the flow rule can be equivalently written as

$$\sigma : \dot{p} = \max_{\tau \in \mathbf{K}} \tau : \dot{p} =: H(\dot{p}),$$

where $H : \mathbb{M}^n_{\text{sym}} \to \mathbb{R}$ is the support function of $\mathbf{K}$. This reformulation of the flow rule is nothing but Hill's principle of maximum plastic work, and $H(\dot{p})$ denotes the plastic dissipation.

Therefore the problem under investigation in this project's direction is to find a triplet $(u, e, p) : \Omega \times [0, T] \to \mathbb{R}^n \times \mathbb{M}^n_{\text{sym}} \times \mathbb{M}^n_{\text{sym}}$ satisfying

$$\begin{cases} Eu = e + p \text{ in } \Omega \times [0, T], \\ \sigma = \mathbb{C}e, \quad \sigma \in K \text{ in } \Omega \times [0, T], \\ \ddot{u} - \text{div}\sigma = f \text{ in } \Omega \times [0, T], \\ \sigma : \dot{p} = H(\dot{p}) \text{ in } \Omega \times [0, T], \\ (u(0), e(0), p(0)) = (u_0, e_0, p_0), \quad \dot{u}(0) = v_0 \text{ in } \Omega, \end{cases} \tag{1.2.1}$$

for a given *body force* $f : \Omega \times [0, T] \to \mathbb{R}^n$ and *initial data* (u_0, v_0, e_0, p_0), together with suitable *boundary conditions*. Well posedness of the problem under Dirichlet boundary conditions has been established in [11].

We notice that problem (1.2.1) can be interpreted as a constrained Friedrichs system using an entropic formulation (see [11] and [13] for the scalar and vectorial setting), which is inspired by Otto's formulation of initial-boundary-value scalar conservation laws [74] when the solution lies in a functional space where there is no notions of trace on the boundary. However, this generalized formulation, which did not explicitly appeal to the trace of the solution, requires to impose a class of admissible boundary conditions well adapted to the underlying hyperbolic structure of the problem. Following the seminal paper [44], one can see that all admissible *dissipative boundary conditions* can be written as

$$\sigma\nu + S\dot{u} = 0 \quad \text{ on } \partial\Omega \times [0, T], \tag{1.2.2}$$

for some suitable spatially dependent symmetric and positive definite matrix $S(x)$. The well posedness of the model of dynamical perfect plasticity (1.2.1) complemented by this so-called dissipative boundary condition (1.2.2) has been established in [11]. It turns out that the coupling between the stress constraint and the boundary condition leads the latter to accommodate the former through a relaxation phenomena which modifies the original boundary condition into

$$\sigma\nu + P_{-K\nu}(S\dot{u}) = 0 \quad \text{ on } \partial\Omega \times [0, T],$$

where $P_{-K\nu}$ denotes the projection in $\mathbb{R}^n$, with respect to a suitable scalar product, onto the closed and convex set $-K\nu(x)$.

Unfortunately, standard Dirichlet, Neumann and Mixed boundary conditions fall outside this general framework. In [12] we proved that it is possible to recover such boundary conditions through a limit procedure by letting the coefficients of the matrix S tend to 0 (for Neumann) and $+\infty$ (for Dirichlet). The argument is based in accurately analyze the energy estimates performed in [11] and check their dependence with respect to the matrix S. More precisely and in view of [32, 49, 61], we assume that $\partial\Omega$ can be decomposed into the disjoint union $\Gamma_D \cup \Gamma_N \cup \Sigma$, where Γ_D and Γ_N stand for the Dirichlet and Neumann boundary, respectively, and Σ is a $\mathcal{H}^{n-1}$-negligible set. In [12], when we consider the re-scaling $S = \lambda \mathbb{1}_{\Gamma_D} + \frac{1}{\lambda}\mathbb{1}_{\Gamma_N}$, we proved that it is possible to recover Dirichlet ($\Gamma_N = \emptyset$) and Neumann ($\Gamma_D = \emptyset$) boundary conditions through a limit procedure by letting λ tends to ∞.

2. Preliminaries

In this chapter, we provide an overview of the preliminaries required for comprehending the content presented in this manuscript. This collection of definitions and results are contained in [4, 29, 43, 47, 64, 68, 69, 75], for the proofs of the following results, we refer to the references enumerated before. To facilitate reader understanding, we have organized these results into distinct sections.

2.1. Functional spaces

2.1.1. Functions of bounded pointwise variation

Within this section, we present an introduction of pointwise bounded variation functions. The proof of the following results are stated in [4, 64].

Definition 2.1.1. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function. The *pointwise variation* of f on the interval I is

$$\operatorname{Var} u := \sup \left\{ \sum_{i=0}^{n} |u(x_i) - u(x_{i-1})| : \{x_i\}_{i=1}^{n} \text{ is a partition of } I \right\},$$

where the supremum is taken over all partitions of I. A function u has finite or bounded pointwise variation if $\operatorname{Var} u < \infty$.

A key property of the functions of pointwise bounded variation, stated in the following theorem, is that the set of discontinuities is countable.

Theorem 2.1.2. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be such that $\operatorname{Var} u < \infty$. Then for every $x \in I$ the limits*

$$\lim_{x \to x_0^+} u(x) =: u^+(x_0) \quad \textit{and} \quad \lim_{x \to x_0^-} u(x) =: u^-(x_0)$$

exist in $\mathbb{R}$, u has at most countably many discontinuity points and is differentiable $\mathcal{L}^1$-a.e. in I. Furthermore, u is bounded and the limits

$$\lim_{x \to (\inf I)^+} u(x) \quad \textit{and} \quad \lim_{x \to (\sup I)^-} u(x)$$

exist in $\mathbb{R}$.

2.1.2. Functions of bounded deformation

For more reference about the BD-space, we refer to [10, 89].

Definition 2.1.3. Let Ω be an open set. The space of bounded deformation, denoted as $BD(\Omega)$, is defined as

$$BD(\Omega) = \left\{ u \in L^1(\Omega; \mathbb{R}^n) : \, Eu := \frac{Du + Du^T}{2} \in \mathcal{M}(\Omega; \mathbb{M}^n_{sym}) \right\}.$$

As seen in [10], we introduce the notion of strict convergence in $BD(\Omega)$ which plays a key role in the continuity of the trace operator in $BD(\Omega)$.

Definition 2.1.4. We say that the sequence $\{u_k\} \subset BD(\Omega)$ converges *strictly* to $u \in BD(\Omega)$ if and only if

$$\begin{cases} u_k \to u \,\text{strongly in } L^1(\Omega;\mathbb{R}^n), \\ Eu_k \rightharpoonup Eu \,\text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym}), \\ |Eu_k|(\Omega) \to |Eu|(\Omega). \end{cases}$$

Theorem 2.1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. There exists a unique linear continuous mapping*

$$\gamma : BD(\Omega) \to L^1(\partial\Omega;\mathbb{R}^n)$$

such that the following integration by parts formula holds: for every $u \in BD(\Omega)$ and $\varphi \in C^1(\mathbb{R}^n)$,

$$\int_\Omega u \odot \nabla\varphi \, dx + \int_\Omega \varphi \, dEu = \int_{\partial\Omega} \gamma(u) \odot \nu \, \varphi \, d\mathcal{H}^{n-1},$$

where ν is the outer unit normal to $\partial\Omega$. In addition,

$$\gamma(u) \equiv u \llcorner \partial\Omega \quad \text{for all } u \in C^0(\overline{\Omega};\mathbb{R}^n) \cap BD(\Omega).$$

In the next result, we can notice that the trace operator is continuous with respect to the strict convergence of $BD(\Omega)$ but not with respect to the weak* convergence in $BD(\Omega)$. More precisely, by the construction presented in [10], we have that the trace mapping in $BD(\Omega)$ is continuous with respect to the strong convergence and we can not expect that the trace operator is weakly* continuous.

Proposition 2.1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in BD(\Omega)$ and $\{u_k\} \subset BD(\Omega)$ such that $u_k \to u$ strictly in $BD(\Omega)$. Then $\gamma(u_k) \to \gamma(u)$ strongly in $L^1(\partial\Omega;\mathbb{R}^n)$.*

2.1.3. H(div) space

For any $\Omega \subset \mathbb{R}^n$, We introduce the $H(\mathrm{div},\Omega)$ space, which was first introduced by Jacques Louis Lions in the context of studying the Laplacian equation with Neumann boundary conditions. For an extensive treatment of this space, we recommend [86, 88, 89] to the reader.

Definition 2.1.7. We define

$$H(\mathrm{div},\Omega) := \{u \in L^2(\Omega;\mathbb{R}^n) : \mathrm{div}\, u \in L^2(\Omega)\}.$$

We remark that in Part B, we make use of elastic stresses defined in the set of symmetric matrices, in that case, the space $H(\mathrm{div},\Omega)$ is defined as follows

$$H(\mathrm{div},\Omega) = \left\{\sigma \in L^2(\Omega;\mathbb{M}^n_{sym}) : \ \mathrm{div}\, \sigma \in L^2(\Omega;\mathbb{R}^n)\right\}.$$

The space $H(\mathrm{div},\Omega)$ admits the existence of a trace operator in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)$. The proof of the following theorem can be found in [88, Theorem 1.2] (or also in [89, Theorem 1.2, Chapter 1])

Theorem 2.1.8. *Let $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega$ is Lipschitz. Let $\sigma \in H(\mathrm{div},\Omega)$, then there exists $\sigma\nu \in H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)$ and*

$$\langle \sigma\nu, \psi \rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n), H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)} := \int_\Omega \psi \cdot \mathrm{div}\sigma \, dx + \int_\Omega \sigma : E\psi \, dx \,. \tag{2.1.1}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^n)$, where ν is the outer unit normal to Ω.

2.2. Geometric Measure theory

In this section, we introduce the tools used in Part A, where we present a free boundary problem and its variational approach. For an extensive reference of this topic, we refer [4, 44, 64, 68, 69].

2.2.1. Hausdorff and Net measures

Let $s \in [0, \infty]$, we recall that for any $E \subset \mathbb{R}^n$, the Hausdorff outer measure is defined as

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E),$$

where

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(E_i)^s : \, E \subset \bigcup_{i \in \mathbb{N}} E_i, \operatorname{diam}(E_i) \leq \delta \right\}.$$

Now, we introduce the definition of the net measure. To do this, we start by recalling the definition of dyadic cubes, more precisely, we denote by $U_{k,l}$ the (half-open) dyadic cube, i.e.,

$$U_{k,l} := [0, 2^{-k}) \times \ldots \times [0, 2^{-k}) + 2^{-k} l$$

for any $k \in \mathbb{N}$ and $l \in \mathbb{Z}^n$. We denote by $\mathcal{Q}$ the family of dyadic cubes and by $\mathcal{N}^s$ *the net s-dimensional outer measure*. More precisely, for any set E,

$$\mathcal{N}^s(E) := \liminf_{\delta \to 0} \mathcal{N}^s_\delta(E), \tag{2.2.1}$$

is the $\mathcal{N}^s$ outer measure of E, where

$$\mathcal{N}^s_\delta(E) := \inf\{\sum_{i \in I} \mathrm{d}iam(U_i)^s : \{U_i\}_{i \in I} \subset \mathcal{Q} \text{ is a countable disjoint covering of } E \text{ and } \mathrm{d}iam(U_i) \leq \delta\}. \tag{2.2.2}$$

Note that this measure does not coincide with the Hausdorff measure, however we have the following equivalence (see [69, Chapter 5])

$$\mathcal{H}^s(E) \leq \mathcal{N}^s(E) \leq 4^s n^{s/2} \mathcal{H}^s(E), \tag{2.2.3}$$

for any Borel set $E \subset \mathbb{R}^n$.

2.2.2. Rectifiable sets

This section summarizes the notions of rectifiability used in Part A.

Definition 2.2.1. Let $E \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set for $k = 1, \ldots, n-1$. We say that E is k-rectifiable, if there exists a family of Lipschitz maps $\{f_i\}$ such that $f_i : \mathbb{R}^k \to \mathbb{R}^n$ for every $i \in \mathbb{N}$ and

$$E \subset \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^k).$$

We say that E is countable $\mathcal{H}^k$-rectifiable if there exists a family of Lipschitz maps $\{f_i\}$, such that $f_i : \mathbb{R}^k \to \mathbb{R}^n$ for every $i \in \mathbb{N}$ and

$$\mathcal{H}^k \left(E \setminus \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^k) \right) = 0.$$

Finally, we say that E is $\mathcal{H}^k$-rectifiable if $\mathcal{H}^k(E) < \infty$ and E is countable $\mathcal{H}^k$-rectifiable.

There is another equivalent definition of rectifiability based on density properties of the k-dimensional Hausdorff measures ($\mathcal{H}^k$), thanks to the fact that for any Borel set $E \subset \mathbb{R}^n$ such that E is $\mathcal{H}^k$-rectifiable, we have that $\mathcal{H}^k \llcorner E$ is a Radon measure for any $k = 0, \ldots, n-1$. First, for any Borel set $E \subset \mathbb{R}^n$ we define the $\mathcal{H}^k$*-lower* and $\mathcal{H}^k$*-upper* densities respectively as follows

$$\theta_*(E,x) := \liminf_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B_r(x))}{\omega_k r^k} \quad \text{and} \quad \theta^*(E,x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B_r(x))}{\omega_k r^k},$$

for every $k = 0, \ldots, n-1$, where $\omega_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$ and $\Gamma : \mathbb{R} \to \mathbb{R}$ is the Γ function defined in any $t \in \mathbb{R}$ by

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds.$$

The following theorem shows the relation between rectifiable sets and the density of a set.

Theorem 2.2.2 (Besicovitch-Mastrand-Mattila)**.** *Let $E \subset \mathbb{R}^n$ be a Borel set such that $\mathcal{H}^k(E) < \infty$. Then, E is $\mathcal{H}^k$-rectifiable if and only if $\theta_*(E,x) = \theta^*(E,x) = 1$ for $\mathcal{H}^k$-a.e. $x \in E$.*

An important example of a rectifiable set used throughout Part A are the *parametrized curves*, more precisely, a *parametrized curve* in $\mathbb{R}^2$ is a continuous function $r : [a,b] \to \mathbb{R}^2$ injective in (a,b) for $a, b \in \mathbb{R}$ with $a < b$. The image of r, denoted by γ, is the *support of the curve.* If γ is $\mathcal{H}^1$-finite, by [44, Lemma 3.2 and Corollary 3.3] (or also [44, Lemma 3.5] in view of Theorem 2.2.2), we obtain that γ is $\mathcal{H}^1$ rectifiable.

2.2.3. Lebesgue–Besicovitch differentiation theorem

In this section, we introduce the main tool of the blow up technique (used in Part A) which is based on the Besicovith derivation theorem. For the proofs of the following results we make reference to [4, 47, 68].

Theorem 2.2.3 (Besicovith derivation theorem)**.** *Let μ and ν be two nonnegative Radon measures. Then there exists a Borel set $N \subset \mathbb{R}^n$, with $\mu(N) = 0$, such that for any $x \in \mathbb{R}^n \setminus N$,*

$$\frac{d\nu_{\mathrm{ac}}}{d\mu}(x) = \lim_{r \to 0^+} \frac{\nu(x + rC)}{\mu(x + rC)} \in \mathbb{R}$$

where

$$\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}}, \quad \nu_{\mathrm{ac}} << \mu, \quad \nu_{\mathrm{s}} \perp \mu,$$

and C is any bounded, convex closed set containing the origin in its interior.

One consequence of the Besicovith derivation theorem is that Lebesgue points exist for functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R})$ as stated by the following result.

Corollary 2.2.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^n; [0, \infty])$ and let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R})$. Then there exists a Borel set $N \subset \mathbb{R}^n$ with $\mu(N) = 0$ such that $R^n \setminus N \subset \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}$ and for any $x \in R^n \setminus N$,*

$$\lim_{r \to 0^+} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |u(y) - u(x)| \, d\mu(y) = 0. \tag{2.2.4}$$

Any point $\mathbb{R}^n$ satisfying (2.2.4) is called *Lebesgue point* of u. Thanks to Corollary 2.2.4, for any $E \subset \mathbb{R}^n$ Borel set, we define *the set of points of density α of E* by

$$E^{(\alpha)} := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha \right\}.$$

Remark 2.2.5. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. For a.e. $x \in E$ we have that $x \in E^{(1)}$, conversely, for a.e. $x \in \mathbb{R}^n \setminus E$ we have that $x \in E^{(0)}$. The set of points of density 1 are refereed as measure theoretical interior points, analogously, the set of points of density 0 are refereed as measure theoretical exterior points.

2.2.4. Sets of finite perimeter

Now, we introduce the notion of sets of finite perimeter (or *Cacciopoli sets*). In the following, we state important definitions and properties used throughout Part A about sets of finite perimeter. For an exhaustive treatment of this topic we refer to [4, 68].

Definition 2.2.6. Let E be a $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$. For any open set $A \subset \mathbb{R}^n$ the perimeter of E in Ω, denoted by $P(E, A)$, is the variation of $\mathbb{1}_E$ in A, i.e.,

$$P(E, A) := \sup \left\{ \int_A \mathrm{div}\varphi \, dx : \varphi \in [C_c^1(A)]^n, \ \|\varphi\|_\infty \leq 1 \right\}. \tag{2.2.5}$$

We say that E is a set of locally finite perimeter in A if $P(E, A) < \infty$.

It is very well known that if $|E \cap A| < \infty$, then E has finite perimeter in A if and only if $\mathbb{1}_E \in \mathrm{BV}(A)$. Now, we introduce a different notion of boundary.

Definition 2.2.7. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, the *reduced boundary of* E, denoted as $\partial^* E$, is defined by

$$\partial^* E := \left\{ x \in \mathbb{R}^2 : \exists \nu_E (x) := - \lim_{r \to 0} \frac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E|(B_r(x))}, |\nu_E(x)| = 1 \right\}, \tag{2.2.6}$$

furthermore, we call to $\nu_E(x)$ as the *measure-theoretical unit normal* at $x \in \partial E$.

Now, for a set $E \subset \mathbb{R}^n$ of finite perimeter, we state the relation that exists between the reduced boundary and $E^{(1/2)}$.

Theorem 2.2.8. *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter and let $x \in \partial^* E$, then*

$$\lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \frac{1}{2}$$

and

$$\lim_{r \to 0^+} \frac{P(E, B_r(x))}{\omega_{n-1} r^{n-1}} = 1.$$

In particular, $\partial^* E \subset E^{(1/2)}$.

The last notion of boundary of a Lebesgue measurable set $E \subset \mathbb{R}^n$ used in this manuscript is the *essential boundary* denoted as $\partial_* E$, more precisely,

$$\partial_* E := \mathbb{R}^n \setminus (E^{(1)} \cup E^{(0)}).$$

Theorem 2.2.9 (Federer)**.** *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter, then $\partial^* E \subset E^{(1/2)} \subset \partial_* E$ and*

$$\mathcal{H}^1(\partial_* E \setminus \partial^* E) = 0.$$

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, it follows the following decomposition of the topological boundary of E thanks to the definition of the essential boundary and by Federer's theorem,

$$\partial E = (E^{(1/2)} \cap \partial E) \cup ((E^{(1)} \cup E^{(0)}) \cap \partial E) \cup N_1 = (\partial^* E \cap \partial E) \cup ((E^{(1)} \cup E^{(0)}) \cap \partial E) \cup N_2, \quad (2.2.7)$$

where N_1 and N_2 are $\mathcal{H}^{n-1}$-negligibles sets. Let $x \in \partial E$, if $\nu_E(x)$ exists, we end this section by defining the *approximate tangent line* and the *half space* at x by $T_{x,\nu_E(x)} := \{y \in \mathbb{R}^2 : y \cdot \nu_E(x) = 0\}$ and $H_{x,\nu_E(x)} := \{y \in \mathbb{R}^2 : y \cdot \nu_E(x) \leq 0\}$, respectively.

2.3. Convex analysis

In this section, we introduce some tools needed in throughout this manuscript. For a proof of the following results, we refer to [4, 38, 39, 47, 54, 68, 75]. In the section we consider X and X^* as a vectorial space an its dual, respectively.

Definition 2.3.1. *Let $C \subset X$ be a set. We define the support function $H : X^* \to \mathbb{R}$ of C as*

$$H_C(y) := \sup_{x \in C} \langle x, y \rangle.$$

Notice that if $C \subset X$ is closed and convex, H_C is a convex, lower semicontinuous and positively 1-homogeneous function. Furthermore, for any $C \subset X$ convex and closed set in view of [75, Theorem 13.2], we have that

$$H_C^* = I_C,$$

where I_C is the indicator function of C.

We now state a result, which was proved by means of the Radon-Nikodym decomposition theorem in [38, 39, 54]

Theorem 2.3.2. *Let $\mu \in \mathcal{M}(X;Y)$ and let $f : Y \to [0,\infty]$ be a convex, positively one homogeneous function. Then, $f(\mu)$, defined by*

$$f(\mu) := f\left(\frac{d\mu}{d|\mu|}\right)|\mu|,$$

is a Borel measure.

Finally, we state the Reshentnyak lower semicontinuity theorem and an application to the case of anisotropies.

Theorem 2.3.3 (Reshetnyak lower semicontinuity)**.** *Let $A \subset \mathbb{R}^n$ be an open set. Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued finite Radon measures and let μ be a finite $\mathbb{R}^m$-valued Radon measure such that $\mu_k \to \mu$ weakly* in A. Then,*

$$\int_A f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|(x) \leq \liminf_{k\to\infty} \int_A f\left(x, \frac{\mu_k}{|\mu_k|}(x)\right) d|\mu_k|(x),$$

for every lower semicontinuous function $f : A \times \mathbb{R}^m \to [0,\infty]$, positively 1-homogeneous and convex in the second variable.

By applying the Reshetnyak lower semicontinuity theorem, we can prove that the for any set of finite perimeter, the anisotropic surface energy of the reduced boundary is lower semicontinuous, this is the purpose of the following theorem.

Theorem 2.3.4. *Let $A \subset \mathbb{R}^n$ be an open set. Let $\varphi : \mathbb{R}^n \to [0, \infty]$ be a one homogeneous, convex and lower semicontinuous anisotropy. Let $\{E_k\}_{k \in \mathbb{N}}$ and E be a sequence of locally finite perimeter sets and set of locally finite perimeter, respectively, such that $\mathbb{1}_{E_k} \to \mathbb{1}_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\nu_{E_k} \mathcal{H}^{n-1} \llcorner \partial^* E_k \overset{*}{\rightharpoonup} \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$. Then,*

$$\int_{A \cap \partial^* E} \varphi(\nu_E(x)) \, d\mathcal{H}^{n-1} \leq \liminf_{k \to \infty} \int_{A \cap \partial^* E_k} \varphi(\nu_{E_k}(x)) \, d\mathcal{H}^{n-1}.$$

Part A.

Multiphase Free Boundary Problems

3. Two-phase free boundary problem

In this chapter, the results contained in the following submitted paper are presented:

- R. Llerena, P. Piovano: *Existence of minimizers for a two-phase free boundary problem with coherent and incoherent interfaces*, submitted (2023).

3.1. Introduction

In this chapter the problem of providing a mathematical variational framework for the description of the morphology and the elastic properties of two-phase continua based on Gibbs's notion of a sharp phase-interface dividing them [24, 49, 56] is addressed. In the presence of two interacting media large stresses due to the different crystalline order of the two materials originate and, besides bulk deformation, various types of morphological destabilization may occur as a further strain relief mode. These are often referred to as the family of *stress driven rearrangement instabilities* (SDRI) [9, 33, 55, 58, 77], which include the roughness of the exposed crystalline boundaries, the formation of cracks in the bulk materials, the nucleation of dislocations in the crystalline lattices, and the delamination (as opposed to the adhesion) at the contact regions with the other material.

Literature provides extensive studies of these phenomena under the assumption that one phase is a rigid fixed continuous medium underlying, such as the substrates for epitaxially-strained thin films [25, 28, 46, 62], or constraining, such as crystal cavities [45] or the containers in capillarity problems [37], the other phase, which is instead let free, or by modeling the interactions with other media simply by means of fixed boundary conditions. There are though settings in which the hierarchy between the phases is not clear, or a rigidity ranking between them is not easily identifiable, since the interplay among the deformation and the interface instabilities affecting all phases is crucial, such as, in the shock-induced transformations and mechanical twinning [56] or in the deposition of film multilayers [65].

As described in [56] the extension of classical theories of continuum mechanics to two-phase deformable media is though “not as straightforward as it might appear”, since combining the accretion and deletion of material constituents responsible for the moving of the interface between the two phases and their boundaries, with the framework of elasticity related to bulk deformation and fractures [26, 50], by quoting [56], “leads to conceptual difficulties”. A critical modeling issue related to the interface between the two phases is the interplay between *coherency*, that is here intended as the microscopical arrangement of atoms of the two materials in a homogeneous lattice, with deformation being the solely stress relief mechanism [56], and *incoherency*, that instead refers to the debonding occurring between the atoms of the two materials [24], which results in the composite delamination at the two-phase interface [63]. This work seems to be, to the best of our knowledge, the first attempt to provide a mathematical framework able to simultaneously describe coherent and incoherent interfaces for a two-phase setting, which we carry out by also keeping in the picture the other features of SDRI, such as the dichotomy between the *wetting regime*, that is the setting in which it is more convenient for a phase to cover the surface of the other phase with an infinitesimal layer of atoms, and the *dewetting regime*, in which it is preferable to let such surface exposed to the vapor.

The studies for the setting of only coherent interfaces go back to Almgren [1], who was the first to formulate the problem in $\mathbb{R}^d$, $d > 1$, in the context without elasticity for surface tensions proportional among the various interfaces, by means of *integral currents* in geometric measure theory and by singling out a condition ensuring the lower semicontinuity of the overall surface energy with respect to the L^1-convergence of the sets in the partition. Then, Ambrosio and Braides in [2, 3] extended the setting to also non-proportional surface tensions by introducing a new integral condition referred to as *BV-ellipticity*, which they show to be both sufficient and necessary for the lower semicontinuity with respect to the L^1-convergence. As such condition is the analogous, for the setting of *Caccioppoli partitions*, of Morrey's *quasi-convexity*, it is difficult to check it in practice. In [2, 3] BV-ellipticity is proved to coincide with a simpler to check *triangle inequality condition* among surface tensions for the case of partitions in 3 sets (like the setting considered in this chapter, in which one element of the partition is always represented by the *vapor* outside the two phases), which was then confirmed to be the only case by [22]. Other conditions therefore have been introduced, such as *B-convexity* and *joint convexity*, with though BV-ellipticity remaining so far the only condition known to be both necessary and sufficient for lower semicontinuity apart from specific settings (see [23, 71] for more details). Recently, the BV-ellipticity has been extended in the context of *BD-spaces* in [52]. Finally, in [18] a variant of the *Ohta–Kawasaki model* is considered to model thin films of diblock copolymers in the unconfined case, which represents a recent example in the literature of a two-phase model in the absence of elasticity and of incoherent and crack interfaces, under a graph constraint for union of the two phases.

Regarding incoherent interfaces the problem is intrinsically related to the renowned *segmentation problem* in image reconstruction that was actually originally introduced by Mumford and Shah in [72] with a multi-phase formulation, as a partition problem of an original image, with the connected contours of the image areas characterized as discontinuity set of an auxiliary state function. Then, the approaches developed to tackle the problem led to the study of a single phase setting with the jump set of the state function representing internal interfaces, proven to satisfy *Ahlfors-type regularity* result [4, 30]. Such single-phase framework has been then extended to the context of linear elasticity in fracture mechanics with the state function being vectorial and representing the bulk displacement of a crystalline material and the energy replaced by the *Griffith energy* [26, 50]. The attempt to recover the original setting of [72] in a rigorous mathematical formulation (apart from some formulations with piecewise-constant state functions or numerical investigations) has been then addressed by Bucur, Fragalà, and Giacomini in [20] and [21] (see also [27] for a related multi-phase boundary problem in the context of reaction-diffusion systems). In [20] Ahlfors-type regularity is established for *ad hoc* notions of multi-phase local *almost-quasi minimizers* of an energy accounting for incoherent isotropic interfaces and disregarding the contribution of the coherent portions, while in [21] they introduce a multi-phase version of the Mumford-Shah problem by treating all the reduced phase boundaries as incoherent interfaces (like the internal jump sets) and by adding an extra (statistical) term, which induces multi-phase minimizers.

In order to finally include in the model both coherent and incoherent portions (possibly also on the same interface between the phases), in [66] we first restrict to the two-phase setting and follow a different direction than the one of [20, 21], which works for $d = 2$: We adopt the approach considered in [58, 59], that was relying on the strategy developed for the Mumford-Shah problem in [30]. Such approach consists in first imposing a fixed constraint on the number of connected components for the boundary of the free phases, in order then to employ adaptations of Golab's Theorem [53] for proving the compactness with respect to a proper selected topology, and then in studying the convergence of the solutions of the different minimum problems related to different constraints on the connected components, as such constraints tend to infinity. This second step has been performed for the one-phase setting in [59] (and for higher dimension in [60]) by means of density estimates.

In [66] we performed the first step in this program, reaching an existence result analogous to the one in [58]. However, the extension of [58] to the two-phase setting requires important modification in the model setting, since characterizing the incoherent interface as the jump portion of the bulk displacement on the two-phase interface as in [58, 59, 60] appears to be not feasible, as in our setting the two-phase interfaces need to be considered much less regular than Lipschitz manifolds like in [58, 59, 60]. To solve this issue the set variables of the energy are not considered to be the two phases, referred to as the *film* and the *substrate phase*, but the substrate phase and the whole region occupied by the composite of both the two phases, and the incoherent interfaces are characterized as the portion of the boundary intersection of such variables. As a byproduct of this strategy there is no need to impose a constraint on the number of boundary components of the film phases (but only with respect to the substrates and the composite regions), so that the physical relevant setting of countable separated isolated film islands forming on top of the substrate is included in our analysis, even though it was prevented by the formulation in [58]. Moreover, in [66] we can also extend [58] to the presence of adjacent materials for the Griffith-type model with mismatch strain and delamination [26, 50, 60].

In agreement with the SDRI theory [9, 33, 55] the total energy $\mathcal{F}$ is given by the sum of two contributions, namely the elastic energy $\mathcal{W}$ and the surface energy $\mathcal{S}$, and it is defined on triples $(A, S, u) \in \widetilde{\mathcal{C}}$, where u represent the *bulk displacement* of the composite material of the two phases, and A and S are sets whose closures represents the *composite region* and the *substrate region*, respectively, while the *film region* is given by $\overline{A} \setminus S^{(1)}$ (for $S^{(1)}$ denoting the points with density 1 in S). More precisely, given $\Omega := (-l, l) \times_{\mathbb{R}^2} (-L, L) \subset \mathbb{R}^2$ as the region where the composite material is located, which is defined for the two parameters $l, L > 0$ and that is referred to as the *container* in analogy to the notation of capillarity problems, in [66] we introduce

$$
\begin{aligned}
\widetilde{\mathcal{C}} := \{(A, S, u) : \quad & A \text{ and } S \text{ are } \mathcal{L}^2\text{-measurable sets with } S \subset \overline{A} \subset \overline{\Omega} \text{ such that} \\
& \partial A \cap \mathrm{Int}(S) = \emptyset,\ \partial A \text{ and } \partial S \text{ are } \mathcal{H}^1\text{-rectifiable}, \\
& \mathcal{H}^1(\partial A) + \mathcal{H}^1(\partial S) < \infty, \text{ and } u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A); \mathbb{R}^2)\}
\end{aligned}
$$

and we define $\mathcal{F} : \widetilde{\mathcal{C}} \to \mathbb{R}$ as

$$\mathcal{F}(A, S, u) := \mathcal{S}(A, S) + \mathcal{W}(A, u)$$

for every $(A, S, u) \in \widetilde{\mathcal{C}}$. The elastic energy $\mathcal{W}(A, u)$ is defined analogously to [34, 58, 59, 60] by

$$\mathcal{W}(A, u) := \int_A W\left(x, E\left(u\left(x\right) - E_0\left(x\right)\right)\right) dx,$$

where the elastic density W is determined by the quadratic form

$$W\left(x, M\right) := \mathbb{C}\left(x\right) M : M,$$

for a fourth-order tensor $\mathbb{C} : \Omega \to \mathbb{M}^2_{sym}$, E denotes the symmetric gradient, i.e., $E(v) := \frac{\nabla v + \nabla^T v}{2}$ for any $v \in H^1_{\mathrm{loc}}(\mathrm{Int}(A); \mathbb{R}^2)$, representing the *strain*, and E_0 is the *mismatch strain* $x \in \Omega \mapsto E_0\left(x\right) \in \mathbb{M}^2_{\mathrm{sym}}$ defined as

$$E_0 := \begin{cases} E(u_0) & \text{in } \Omega \setminus S, \\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\Omega; \mathbb{R}^2)$. The mismatch strain is included in the SDRI theory to represent the fact that the two phases are given by possibly different crystalline materials whose free-standing equilibrium lattice could present a lattice mismatch. In this context notice that $\mathbb{C}$ is allowed to present discontinuities at the interface between the two materials (see hypothesis (H3) in Section 3.2.2).

The surface energy $\mathcal{S}(A,S)$ is given by

$$\mathcal{S}(A,S) := \int_{\Omega\cap(\partial A\cup\partial S)} \psi(x,\nu(x))\,d\mathcal{H}^1(x),$$

where, by denoting with $\nu_U(z)$ the normal unit vector pointing outward to a set $U\subset\mathbb{R}^2$ with $\mathcal{H}^1$-rectifiable boundary at a point $x\in\partial U$,

$$\nu(x) := \begin{cases} \nu_A(x) & \text{if } z\in\partial A\setminus\partial S,\\ \nu_S(x) & \text{if } z\in\partial S,\end{cases}$$

and $\psi:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$ represents the *surface tension of the composite* of the two phases, which it is allowed to be anisotropic.

In order to properly define ψ the three surface tensions φ_{F}, φ_{S}, $\varphi_{\mathrm{FS}}:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$ characterizing the three possible interfaces for the two-phase setting, i.e., the interface between the film phase and the vapor, the interface between the substrate phase and the vapor, and the interface between the film and the substrate phases, are considered. Furthermore, to simultaneously treat both the wetting and the dewetting regime, in [66] we introduce two auxiliary surface tensions, to which we refer as the *regime surface tensions*, that we defined as:

$$\varphi := \min\{\varphi_{\mathrm{S}},\varphi_{\mathrm{F}}+\varphi_{\mathrm{FS}}\} \qquad \text{and} \qquad \varphi' := \min\{\varphi_{\mathrm{S}},\varphi_{\mathrm{F}}\},$$

since φ_{S} is the surface tension associated to the dewetting regime, as the substrate surface remains exposed to the vapor, while $\varphi_{\mathrm{F}}+\varphi_{\mathrm{FS}}$ and φ_{F} are both associated to the wetting regime, respectively, to the situation of an infinitesimal layer of film atoms covering the substrate surface (by being bonded to the substrate atoms), which is referred to as the *wetting layer*, or of simply a detached *film filament*. The surface tension ψ is then defined by

$$\psi(x,\nu(x)) := \begin{cases} \varphi_{\mathrm{F}}(x,\nu(x)) & x\in\Omega\cap(\partial^*A\setminus\partial^*S),\\ \varphi(x,\nu(x)) & x\in\Omega\cap\partial^*S\cap\partial^*A,\\ \varphi_{\mathrm{FS}}(x,\nu(x)) & x\in\Omega\cap(\partial^*S\setminus\partial A),\\ (\varphi_{\mathrm{F}}+\varphi)(x,\nu(x)) & x\in\Omega\cap\partial^*S\cap\partial A\cap A^{(1)},\\ 2\varphi_{\mathrm{F}}(x,\nu(x)) & x\in\Omega\cap\partial A\cap A^{(1)}\cap S^{(0)},\\ 2\varphi'(x,\nu(x)) & x\in\Omega\cap\partial A\cap A^{(0)},\\ 2\varphi_{\mathrm{FS}}(x,\nu(x)) & x\in\Omega\cap(\partial S\setminus\partial A)\cap\left(S^{(1)}\cup S^{(0)}\right)\cap A^{(1)},\\ 2\varphi(x,\nu(x)) & x\in\Omega\cap\partial S\cap\partial A\cap S^{(1)},\end{cases} \tag{3.1.1}$$

where ∂^*U and $U^{(\alpha)}$ denote, when well defined, the *reduced boundary* and the set of points of density $\alpha\in[0,1]$ for a set $U\subset\mathbb{R}^2$. We notice that the 8 subregions of the domain $\Omega\cap(\partial A\cup\partial S)$ in which the definition of ψ is distinguished are the counterpart of the 5 terms appearing in the surface energy of [58] for the two-phase setting (see Remark 3.2.13 for more details). Each subregion appearing in (3.1.1) represents, by moving line by line, the *film free boundary*, the *substrate free boundary*, the *film-substrate coherent interface*, the *film-substrate incoherent interface*, the *film cracks*, the *exposed filaments*, the *substrate filaments and cracks* in the film-substrate coherent interface, and the *substrate cracks* in the film-substrate incoherent interface, respectively.

Observe that the surface tensions associated to the film free boundary, the coherent substrate-film interface, and the substrate free boundary, are simply φ_{F}, φ_{FS}, and (to accommodate both wetting and dewetting regimes) φ respectively, while the surface tension associated to the incoherent film-substrate interface is chosen to be $\varphi_{\mathrm{F}}+\varphi$ in analogy with the film-substrate delamination or delaminated region in [58, 59, 60], since the incoherent interface coincides with the portion of the film-substrate interface in which there is no bonding between the film and the substrate surfaces. All remaining 4 terms are weighted double (in analogy to the lower-semicontinuity results previously obtained in [25, 34, 46, 58, 59, 60] for the one-phase setting) as they refer to either material filaments in the void or cracks in the composite bulk. In particular, notice that in the substrate bulk region represented by $S^{(1)}$ we distinguish between substrate cracks in the coherent and in the incoherent film-substrate interface, that are counted with weight $2\varphi_{\mathrm{FS}}$ and 2φ, respectively, while in the film bulk region $A^{(1)} \cap S^{(0)}$ we distinguish between substrate filaments that are not film cracks counted with $2\varphi_{\mathrm{FS}}$ and film cracks counted $2\varphi_{\mathrm{F}}$ (see Figure 3.1).

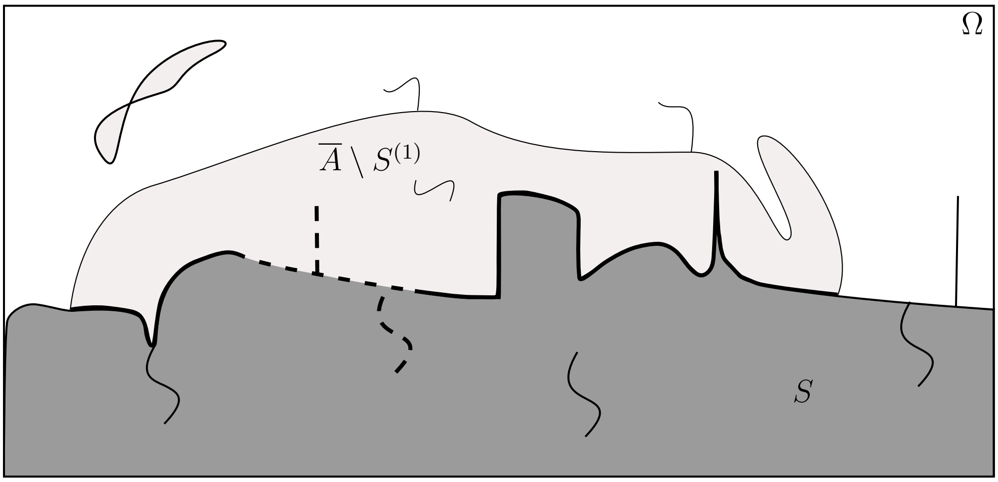

Figure (3.1): The admissible regions for an admissible configuration $(A, S, u) \in \mathcal{C}_{\mathbf{m}}$ (see Definition 3.2.2) are represented by indicating the substrate region $\overline{S}$ and the film region $\overline{A} \setminus S^{(1)}$ with a darker and a lighter gray, respectively. In particular, the film and the substrate free boundaries (with the film and substrate filaments) are indicated with a thinner line, while the film-substrate interface is depicted with a thicker line that is continuous or dashed to distinguish between its incoherent portions and its coherent portions (inclusive of substrate cracks and filaments that are not film cracks), respectively.

The main result of [66] consists in finding a physically relevant family of admissible configurations in $\widetilde{\mathcal{C}}$, which is denoted by $\mathcal{C}_{\mathbf{m}}$ for $\mathbf{m} := (m_0, m_1) \in \mathbb{N}^2$, in which we can prove that, under a *two-phase volume constraint*, $\mathcal{F}$ admits a minimizer. We find such a family $\mathcal{C}_{\mathbf{m}} \subset \widetilde{\mathcal{C}}$ by considering as admissible configurations $(A, S, u) \in \widetilde{\mathcal{C}}$ the ones for which (see Definition 3.2.2 for more details):

- the number of boundary connected components of S and A are fixed to be at most m_0 and m_1, respectively,
- the substrate regions S satisfy an *exterior graph constraint* consisting in requiring that $\partial^* S \cup (\partial S \cap S^{(0)})$ is the graph of an upper semicontinuous function with pointwise bounded variation (while internal, also non-graph-like, substrate cracks are allowed),

as shown in Figure 3.1. Notice that such an exterior graph constraint allows to have a more involved description of the substrate regions than the previously considered graph constraint in the literature for the one-phase setting [25, 28, 34, 46], which is indeed needed to achieve the compactness result contained in Theorem 3.2.11.

Therefore, for any two volume parameters $\mathbb{v}_0, \mathbb{v}_1 \in [\mathcal{L}^2(\Omega)/2, \mathcal{L}^2(\Omega)]$ such that $\mathbb{v}_0 \leq \mathbb{v}_1$, we consider the problem:

$$\inf_{\substack{(A,S,u)\in\mathcal{C}_{\mathbf{m}} \\ \mathcal{L}^2(S)=\mathbb{v}_0,\, \mathcal{L}^2(A)=\mathbb{v}_1}} \mathcal{F}(A, S, u), \tag{3.1.2}$$

which we tackle by employing the *direct method* of the calculus of variations, namely by equipping $\widetilde{\mathcal{C}}$ with a properly chosen topology $\tau_{\mathcal{C}}$ sufficiently weak to establish a compactness property for energy-equibounded sequences in $\mathcal{C}_{\mathbf{m}}$ and strong enough to prove the lower semicontinuity of $\mathcal{F}$ in $\mathcal{C}_{\mathbf{m}}$. The topology $\tau_{\mathcal{C}}$ is characterized by the convergence:

$$(A_k, S_k, u_k) \xrightarrow[k\to\infty]{\tau_{\mathcal{C}}} (A, S, u) \iff \begin{cases} \sup_{k\in\mathbb{N}} \mathcal{H}^1(\partial A_k) < \infty, \ \sup_{k\in\mathbb{N}} \mathcal{H}^1(\partial S_k) < \infty, \\ \operatorname{sdist}(\cdot, \partial A_k) \xrightarrow[k\to\infty]{} \operatorname{sdist}(\cdot, \partial A) \text{ locally uniformly in } \mathbb{R}^2, \\ \operatorname{sdist}(\cdot, \partial S_k) \xrightarrow[k\to\infty]{} \operatorname{sdist}(\cdot, \partial S) \text{ locally uniformly in } \mathbb{R}^2, \\ u_k \xrightarrow[k\to\infty]{} u \quad \text{a.e. in Int (A)}, \end{cases}$$

where the *signed distance function* is defined for any $E \subset \mathbb{R}^2$ as follows

$$\operatorname{sdist}(x, \partial E) := \begin{cases} \operatorname{dist}(x, E) & \text{if } x \in \mathbb{R}^2 \setminus E, \\ -\operatorname{dist}(x, E) & \text{if } x \in E. \end{cases}$$

The compactness property shared by energy-equibounded sequences $(A_k, S_k, u_k) \in \mathcal{C}_{\mathbf{m}}$ described in Theorem 3.2.11 consists in the existence, up to a subsequence, of a possibly different sequence $(\widetilde{A}_k, \widetilde{S}_k, \widetilde{u_k}) \in \mathcal{C}_{\mathbf{m}}$ compact in $\mathcal{C}_{\mathbf{m}}$ with respect to $\tau_{\mathcal{C}}$ such that

$$\liminf_{n\to\infty} \mathcal{F}(A_k, S_k, u_k) = \liminf_{n\to\infty} \mathcal{F}\left(\widetilde{A}_k, \widetilde{S}_k, \widetilde{u_k}\right).$$

This is achieved by both extending to the two-phase setting and to the situation with the exterior graph constraint the strategy used in [58, Theorem 2.7]. For the latter, we rely on the arguments already used in [25, 46], while for the former we used the Blaschke-type selection principle proved in [58, Proposition 3.1] together with the Golab's Theorem [53, Theorem 2.1] and we implement to the two-phase setting the construction of [58, Proposition 3.6]. Such construction is needed to take care of those connected components of A_k that separate in the limit in multiple connected components, e.g., in the case of neckpinches, in order to properly apply *Korn's inequality* just after having introduced extra boundary to create different components also at the level A_k (by passing to the sequence with composite regions $\widetilde{A}_k$). We notice though that the characterization of the delamination region introduced in this chapter allows for a simplification in the arguments used [58, Theorem 2.7], as the surface energy does not involve the bulk displacements, also yielding an extension of the result by including the situation with $S \neq \emptyset$ and $\mathbb{v}_1 := \mathcal{L}^2(\Omega)$.

The crucial point in proving the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{F}$ is the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{S}$, as the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{W}$ directly follows by convexity similarly to [46, 58]. In order to establish the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{S}$ in Proposition 3.4.13 we fix $(A_k, S_k, u_k) \in \mathcal{C}_{\mathbf{m}}$ and $(A, S, u) \in \mathcal{C}_{\mathbf{m}}$ such that $(A_k, S_k, u_k) \xrightarrow{\tau_{\mathcal{C}}} (A, S, u)$, and we associate the positive Radon measures μ_k and μ in $\mathbb{R}^2$ to the localized energy versions of $\mathcal{S}(A_k, S_k, u_k)$ and $\mathcal{S}(A, S, u)$, respectively. We have that

$$\liminf_{k\to+\infty} \mathcal{S}(A_k, S_k, u_k) \geq \mathcal{S}(A, S, u) \qquad \iff \qquad \liminf_{k\to+\infty} \mu_k(\mathbb{R}^2) \geq \mu(\mathbb{R}^2), \tag{3.1.3}$$

and since, up to a subsequence, μ_k weakly* converges to some positive Radon measure μ_0, and μ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner ((\partial A \cup \partial S) \cap \Omega)$, by proving the following estimate involving *Radon-Nikodym derivatives*:

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Omega \cap (\partial A \cup \partial S))} \geq \frac{d\mu}{d\mathcal{H}^1 \llcorner (\Omega \cap (\partial A \cup \partial S))} \quad \mathcal{H}^1\text{-a.e. on } \Omega \cap (\partial A \cup \partial S), \tag{3.1.4}$$

which implies that $\lim \mu_k(\mathbb{R}^2) = \mu_0(\mathbb{R}^2) \geq \mu(\mathbb{R}^2)$, in view of (3.1.3), the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{S}$ follows.

The proof of (3.1.4) is very involved and it is performed by separating $\Omega \cap (\partial A \cup \partial S)$ in 12 portions on which we apply a *blow-up technique* (see, e.g., [48]) together with *ad hoc* (apart from the 2 portions in which it turns out that we can use [68, Theorem 20.1]) results, i.e., Lemmas 3.4.7-3.4.12, which can be seen as the counterpart in the two-phase setting of [58, Lemmas 4.4 and 4.5] (see Table 3.1 for more details on the 12 blow-ups). In order to prove Lemmas 3.4.7-3.4.12, firstly we formalize the notions of *film islands*, *composite voids*, and *substrate grains* (see Definition 3.4.6), secondly we prove in Lemma 3.4.5 that the coherent interface associated to any configuration $(A, S, u) \in \mathcal{C}_{\mathbf{m}}$ can be regarded, up to an error and a modification of (A, S, u) by passing to the family $\widetilde{\mathcal{C}}$, as given by a finite number (depending on the initial configuration (A, S, u)) of connected components, and finally we design induction arguments (with respect to the number of such components) in which we are able to use the induction hypothesis by "shrinking" islands, "filling" voids, and "modifying grains in new voids" as depicted in Figures 3.3, 3.4, and 3.6, respectively, by means of employing the *anisotropic minimality of segments* [68, Remark 20.3].

The chapter is organized as follows: in Section 3.2 we introduce the model under consideration, present some preliminary results and state the main results of [66], i.e., the existence of a solution to (3.1.2) in Theorem 3.2.10 together with the compactness result of Theorem 3.2.11 and the lower semicontinuity result of Theorem 3.2.12, in Section 3.3, we prove Theorem 3.2.11, in Section 3.4 we prove Theorem 3.2.12, and finally in Section 3.5 we prove Theorem 3.2.10.

3.2. Mathematical setting and main results

In this section we present the model introduced in [66] with some preliminaries, and then we state the main results of the chapter outlining the consequences for the related one-phase setting of [58, 59, 60] and the multiple-phase setting of film multilayers considered in Chapter 4.

3.2.1. The two-phase model

Let $\Omega := (-l, l) \times_{\mathbb{R}^2} (-L, L) \subset \mathbb{R}^2$ for positive parameters $l, L \in \mathbb{R}$. We begin by introducing the family $\mathcal{C}_{\mathbf{m}}$ of admissible configurations and, in particular, the admissible substrate regions.

Roughly speaking, an admissible substrate region $S \subset \Omega$ is characterized as the subgraph of a upper semicontinuous height function h with finite pointwise variation to which we subtract a closed $\mathcal{H}^1$-rectifiable set K such that $\mathcal{H}^1(K) < \infty$, which represents the substrate internal cracks. More precisely, we consider the *family of admissible (substrate) heights* $\mathrm{AH}(\Omega)$ defined by

$$\mathrm{AH}(\Omega) := \{h : [-l, l] \to [0, L] : h \text{ is upper semicontinuous and } \operatorname{Var} h < \infty\} \tag{3.2.1}$$

and let S_h denote the closed subgraph with height $h \in \mathrm{AH}(\Omega)$, i.e.,

$$S_h := \{(x, y) : -l < x < l, y \leq h(x)\}. \tag{3.2.2}$$

We then define the *family of admissible (substrate) cracks* $\mathrm{AK}(\Omega)$ by

$$\mathrm{AK}(\Omega) := \{K \subset \Omega : \ K \text{ is a closed set in } \mathbb{R}^2, \mathcal{H}^1\text{-rectifiable and } \mathcal{H}^1(K) < \infty\} \tag{3.2.3}$$

and the *family of pairs of admissible heights and cracks* $\mathrm{AHK}(\Omega)$ by

$$\mathrm{AHK}(\Omega) := \{(h, K) \in \mathrm{AH}(\Omega) \times \mathrm{AK}(\Omega) : \ K \subset \overline{\mathrm{Int}(S_h)}\}. \tag{3.2.4}$$

Finally, given $(h, K) \in \mathrm{AHK}(\Omega)$ we refer to the set

$$S_{h,K} := (S_h \setminus K) \cap \Omega, \tag{3.2.5}$$

as the *substrate with height h and cracks K*, and we define the *family of admissible substrates* as

$$\mathrm{AS}(\Omega) := \{S \subset \Omega \ : \ S = S_{h,K} \text{ for a pair } (h, K) \in \mathrm{AHK}(\Omega)\}. \tag{3.2.6}$$

We observe that

$$\partial S_{h,K} = \partial S_h \cup K \tag{3.2.7}$$

for every $S_{h,K} \in \mathrm{AS}(\Omega)$, so that $\overline{S_{h,K}} = S_h$ and $\mathrm{Int}(S_{h,K}) = \mathrm{Int}(S_h) \setminus K$. We denote the *jumps points* and the *vertical filament points* of the graph of $h \in \mathrm{A}H$ by

$$J(h) := \{x \in (-l,l) : h^-(x) \neq h^+(x)\} \quad \text{and} \quad F(h) := \{x \in (-l,l) : h^+(x) < h(x)\}, \tag{3.2.8}$$

respectively. By [64, Corollary 2.23] and thanks to the fact that $h \in \mathrm{A}H$, $J(h)$ and $F(h)$ are countable. Moreover, it follows that ∂S_h is connected and, ∂S_h and $\partial S_{h,K}$ have finite $\mathcal{H}^1$-measure. By [44, Lemma 3.12 and Lemma 3.13], for any $h \in \mathrm{AH}$, ∂S_h is rectifiable and applying the Besicovitch-Marstrand-Mattila Theorem (see [4, Theorem 2.63]), ∂S_h is $\mathcal{H}^1$-rectifiable, and hence, $\partial S_{h,K}$ is $\mathcal{H}^1$-rectifiable. Furthermore, applying [59, Proposition A.1] S_h and $S_{h,K}$ are sets of finite perimeter.

The following result allows to interchangeably bound the pointwise variation of a function $h \in \mathrm{A}H$ from the $\mathcal{H}^1$-measure of ∂S_h, and vice versa.

Lemma 3.2.1. *Let $h \in AH$. Then*

$$\mathrm{Var}\, h \leq \mathcal{H}^1(\partial S_h) \leq 2l + 2\mathrm{Var}\, h, \tag{3.2.9}$$

where S_h is defined as in (3.2.2)*.*

Proof. The proof is divided in two steps.

Step 1. We prove the left inequality of (3.2.9). We proceed as in [25, Section A.2.3]. Let $m \in \mathbb{N}$ and let $\{l_i : i \in \{0, \dots, m\}, l_0 = -l, l_m = l \text{ and } l_i < l_{i+1}\}$ be a partition of $[-l, l]$. Take $i \in \{0, \dots, m-1\}$ and define by L_i the segment connecting $(l_i, h(l_i))$ with $(l_{i+1}, h(l_{i+1}))$. By [44, Lemma 3.12], there exists a parametrization $r_i : [0,1] \to \mathbb{R}^2$ of $\partial\mathrm{Int}(S_h) \cap ((l_i, l_{i+1}) \times_{\mathbb{R}^2} [0, L))$, whose support γ_i joins the points $(l_i, h(l_i))$ and $(l_{i+1}, h(l_{i+1}))$, furthermore, it follows that

$$|h(l_{i+1}) - h(l_i)| \leq \sqrt{|l_{i+1} - l_i|^2 + |h(l_{i+1}) - h(l_i)|^2} = \mathcal{H}^1(L_i) \leq \mathcal{H}^1(\gamma_i).$$

Moreover, repeating the same argument for any $i \in \{0, \dots, m-1\}$ we have that

$$\sum_{i=0}^{m-1} |h(l_{i+1}) - h(l_i)| \leq \sum_{i=0}^{m-1} \mathcal{H}^1(\gamma_i) \leq \mathcal{H}^1(\partial S_h),$$

where in the last inequality we have used that

$$\partial S_h = \partial\mathrm{Int}(S_h) \cup (\partial S_h \cap S_h^{(0)}) \cup N, \tag{3.2.10}$$

where N is a $\mathcal{H}^1$-negligible set. Taking the supremum aver all partitions of we obtain the left inequality of (3.2.9).

Step 2. In this step, we prove the right inequality of (3.2.9). We observe that

$$\mathrm{Int}(S_h) = \{(x,y) : -l < x < l, y < h^-(x)\}$$

and so,

$$\begin{array}{rccc} r: & [-l,l] & \to & \partial\mathrm{Int}(S_h) \\ & x & \mapsto & (x, h^-(x)) \end{array}$$

is a parametrization of $\partial\mathrm{Int}(S_h)$, whose support we denote by γ. Therefore, from [64, Definition 4.6 and Remark 4.20] it follows that

$$\mathcal{H}^1(\partial\mathrm{Int}(S_h)) = \sup\left\{\sum_{i=0}^{m-1} |r(x_{i+1}) - r(x_i)|\right\}, \tag{3.2.11}$$

where the supremum is taken over all partitions of $[-l, l]$. By definition of r and thanks to (3.2.11), we see that

$$\mathcal{H}^1(\partial \mathrm{Int}(S_h)) \leq \sup \left\{ \sum_{i=0}^{m-1} \left(|x_{i+1} - x_i| + |h^-(x_{i+1}) - h^-(x_i)| \right) \right\} \leq 2l + \mathrm{Var}\, h,$$

where we used the fact that $\mathrm{Var}\, h^- \leq \mathrm{Var}\, h$. Finally, by (3.2.10) we have that $\mathcal{H}^1(\partial S_h) = \mathcal{H}^1(\partial \mathrm{Int}(S_h)) + \mathcal{H}^1(\partial S_h \cap S_h^{(0)})$ and since $\mathcal{H}^1(\partial S_h \cap S_h^{(0)}) \leq \mathrm{Var}\, h$, by the fact that $\partial S_h \cap S_h^{(0)}$ is the union of vertical segments, we can deduce the right inequality of (3.2.9). □

We now introduce the family of admissible region pairs and configurations.

Definition 3.2.2 (Admissible regions and configurations)**.** We define the families of admissible pairs $\mathcal{B}(\Omega)$ and of admissible configurations $\mathcal{C}$ by

$$\begin{aligned}\mathcal{B}(\Omega) := \{(A, S) :\ & A \text{ is } \mathcal{L}^2\text{-measurable}, \partial A \text{ is } \mathcal{H}^1\text{-rectifiable}, \mathcal{H}^1(\partial A) < \infty, \\ & \text{there exists } (h, K) \in \mathrm{AHK}(\Omega),\ S = S_{h,K} \in \mathrm{AS}(\Omega), \\ & S_{h,K} \subset \overline{A} \subset \overline{\Omega} \text{ and } \partial A \cap \mathrm{Int}(S_{h,K}) = \emptyset\},\end{aligned}$$

and

$$\mathcal{C} := \left\{ (A, S, u) \in \widetilde{\mathcal{C}} : (A, S) \in \mathcal{B} \right\},$$

respectively.

In the following we also refer to the sets $\overline{A}$, $\overline{S}$, and $\overline{A} \setminus S^{(1)}$ with respect to an admissible pair $(A, S) \in \mathcal{B}$ as the *composite region*, the *substrate region*, and the *film region* of the admissible pair. Moreover, we refer to $S^{(1)}$ and $A^{(1)} \cap S^0$ as the *substrate* and the *film bulk regions*, respectively.

In Theorem 3.2.12 we will need to consider a natural extension of the families $\mathcal{B}$ and $\mathcal{C}$, which we denote as $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$, respectively.

Definition 3.2.3. We define the families of admissible pairs $\widetilde{\mathcal{B}}(\Omega)$ and of admissible configurations $\widetilde{\mathcal{C}}$ by

$$\begin{aligned}\widetilde{\mathcal{B}}(\Omega) := \{(A, S) :\ & A, S \text{ are } \mathcal{L}^2\text{-measurable}, \partial A, \partial S \text{ are } \mathcal{H}^1\text{-rectifiable}, \\ & \mathcal{H}^1(\partial A), \mathcal{H}^1(\partial S) < \infty,\ S \subset \overline{A} \subset \overline{\Omega} \text{ and } \partial A \cap \mathrm{Int}(S) = \emptyset\},\end{aligned}$$

and

$$\widetilde{\mathcal{C}} := \left\{ (A, S, u) : (A, S) \in \widetilde{\mathcal{B}}, u \in H^1_{\mathrm{loc}}\left(\mathrm{Int}(A); \mathbb{R}^2\right) \right\},$$

respectively.

We observe that $\mathcal{B} \subset \widetilde{\mathcal{B}}$, since for any $(A, S) \in \mathcal{B}$ there exists $(h, K) \in \mathrm{AHK}(\Omega)$ such that $S = S_{h,K} \in \mathrm{AS}(\Omega)$ and $\partial S_{h,K}$ is $\mathcal{H}^1$-rectifiable, and thus, $(A, S) \in \widetilde{\mathcal{B}}(\Omega)$.

Notice that, for simplicity, in the absence of ambiguity we omit the dependence on the set Ω in the notation $\widetilde{\mathcal{B}}(\Omega)$ and $\mathcal{B}(\Omega)$ by writing in the following only $\widetilde{\mathcal{B}}$ and $\mathcal{B}$, respectively.

Remark 3.2.4. We observe that any bounded $\mathcal{L}^2$-measurable set $A \subset \mathbb{R}^2$ such that $\mathcal{H}^1(\partial A) < \infty$ is a set of finite perimeter in $\mathbb{R}^2$ by [59, Proposition A.1].

We now equip the family $\widetilde{\mathcal{B}}$ with a topology.

Definition 3.2.5 ($\tau_{\mathcal{B}}$-convergence)**.** A sequence $\{(A_k, S_k)\} \subset \widetilde{\mathcal{B}}$ $\tau_{\mathcal{B}}$-converges to $(A, S) \in \widetilde{\mathcal{B}}$, if

- $\sup_{k\in\mathbb{N}} \mathcal{H}^1(\partial A_k) < \infty$,$\sup_{k\in\mathbb{N}} \mathcal{H}^1(\partial S_k) < \infty$,
- $\mathrm{sdist}(\cdot, \partial A_k) \to \mathrm{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ as $k \to \infty$,

- $\mathrm{sdist}\,(\cdot, \partial S_k) \to \mathrm{sdist}\,(\cdot, \partial S)$ locally uniformly in $\mathbb{R}^2$ as $k \to \infty$.

It will follow from Lemma 3.2.9 below that the τ_B-convergence is closed in the subfamily of admissible triples $\mathcal{B}_{\mathbf{m}} \subset \mathcal{B}$ whose definition depending on the vector $\mathbf{m} = (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ we now provide.

Definition 3.2.6. For any $\mathbf{m} := (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ the family $\mathcal{B}_{\mathbf{m}}$ is given by all pairs $(A, S) \in \mathcal{B}$ such that ∂A and ∂S have at most m_1 and m_0-connected components, respectively. Let us also define

$$\mathcal{C}_{\mathbf{m}} := \{(A, S, u) \in \mathcal{C} : (A, S) \in \mathcal{B}_{\mathbf{m}}\} \subset \mathcal{C}. \tag{3.2.12}$$

We denote the topology with which we equip the family $\widetilde{\mathcal{C}}$ by $\tau_{\mathcal{C}}$.

Definition 3.2.7 ($\tau_{\mathcal{C}}$-Convergence)**.** A sequence $\{(A_k, S_k, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}$ is said to $\tau_{\mathcal{C}}$-convergence to $(A, S, u) \in \mathcal{C}$, denoted as $(A_k, S_k, u_k) \xrightarrow{\tau_{\mathcal{C}}} (A, S, u)$, if

- $(A_k, S_k) \xrightarrow{\tau_{\mathcal{B}}} (A, S)$,
- $u_k \to u$ a.e. in $\mathrm{Int}\,(\mathrm{A})$.

We now state some properties of the topology $\tau_{\mathcal{C}}$.

Remark 3.2.8. We notice that:

(i) The following assertions are equivalent

 (i.1) $\mathrm{sdist}(\cdot, \partial E_k) \to \mathrm{sdist}(\cdot, \partial E)$ locally uniformly in $\mathbb{R}^2$.

 (i.2) $E_k \xrightarrow{\mathcal{K}} \overline{E}$ and $\mathbb{R}^2 \setminus E_k \xrightarrow{\mathcal{K}} \mathbb{R}^2 \setminus \mathrm{Int}(E)$.

 Moreover, these imply that $\partial E_k \xrightarrow{\mathcal{K}} \partial E$.

(ii) If there exist $(h_k, K_k) \in \mathrm{AHK}(\Omega)$ and $(h, K) \in \mathrm{AHK}(\Omega)$ such that $E_k = E_{h_k,K_k} \in \mathrm{A}S(\Omega)$ and $E = E_{h,K} \in AS$, for every $k \in \mathbb{N}$, we observe that Item (i.1) above is equivalent to

$$E_k = E_{h_k,K_k} \xrightarrow{\mathcal{K}} E_h \quad \text{and} \quad \mathbb{R}^2 \setminus E_k = \mathbb{R}^2 \setminus E_{h_k,K_k} \xrightarrow{\mathcal{K}} (\mathbb{R}^2 \setminus \mathrm{Int}(E_h)) \cup K,$$

 where E_{h_k,K_k}, $E_{h,K}$ are defined as in (3.2.5) and E_h is defined as in (3.2.2).

(iii) Let $\{(E_k, F_k)\} \subset \mathbb{R}^2 \times \mathbb{R}^2$ be a sequence of bounded sets and let $E, F \subset \mathbb{R}^2$ be two bounded sets such that $\partial E_k \xrightarrow{\mathcal{K}} \partial E$ and $\partial F_k \xrightarrow{\mathcal{K}} \partial F$. In view of the Kuratowski convergence (see [4, Section 6.1], [29, Chapter 4] or [58, Appendix A.1]), we observe that for every $x \in \partial E \setminus \partial F$ there exist $r := r(x) > 0$ and $k_{r,x} \in \mathbb{N}$ such that $B(x, r) \cap \partial F_k = \emptyset$ for any $k \geq k_{r,x}$. Similarly, for every $x \in \partial F \setminus \partial E$ there exists $r' := r'(x) > 0$ and $k_{r',x} \in \mathbb{N}$ such that $B(x, r) \cap \partial E_k = \emptyset$ for any $k \geq k_{r',x}$.

From the next result the closedness and the compactness (see Theorem 3.3.2) of the family $\mathcal{B}_{\mathbf{m}}$ with respect to the topology $\tau_{\mathcal{B}}$ follows for every $\mathbf{m} := (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$.

Lemma 3.2.9. *Let $\{E_k\}$ be a sequence of $\mathcal{L}^2$-measurable subsets of $\overline{\Omega}$ having $\mathcal{H}^1$-rectifiable boundaries ∂E_k with at most m-connected components such that*

- $\sup_k \mathcal{H}^1(\partial E_k) < \infty$,
- *$\mathit{sdist}(\cdot, \partial E_k) \to \mathit{sdist}(\cdot, \partial E)$ locally uniformly in $\mathbb{R}^2$ as $k \to \infty$ for a set $E \subset \mathbb{R}^2$.*

Then, ∂E is $\mathcal{H}^1$-finite, $\mathcal{H}^1$-rectifiable, and with at most m-connected components, and $E \subset \overline{\Omega}$ is $\mathcal{L}^2$-measurable. Furthermore, if $E_k = E_{h_k,K_k} \in \mathrm{AS}(\Omega)$ for every $k \in \mathbb{N}$ and for some $(h_k, K_k) \in$ $\mathrm{AHK}(\Omega)$*, then*

$$\sup_k (\mathcal{H}^1(K_k) + \mathrm{Var}\, h_k) < \infty \tag{3.2.13}$$

and there exists $(h,K)\in \mathrm{AHK}(\Omega)$ *such that* $E = E_{h,K}\in \mathrm{AS}(\Omega)$*.*

Proof. The fact that ∂E is $\mathcal{H}^1$-finite, $\mathcal{H}^1$-rectifiable, and with at most m-connected components, is a direct consequence of [58, Lemma 3.2]. Since $\mathcal{H}^1(\partial E) < \infty$, it follows that $\mathcal{L}^2(\overline{E}\setminus \mathrm{Int}(E)) = \mathcal{L}^2(\partial E) = 0$, by applying [17, Theorem 14.5] to $E\setminus\mathrm{Int}(E)\subset \overline{E}\setminus\mathrm{Int}(E)$ we infer that $E\setminus\mathrm{Int}(E)$ is $\mathcal{L}^2$-measurable and so, $\mathcal{L}^2(E\setminus\mathrm{Int}(E)) = 0$. Therefore, $E = E\setminus\mathrm{Int}(E)\cup\mathrm{Int}(E)$ is $\mathcal{L}^2$-measurable.

It remains to prove the last assertion of the statement. Let $(h_k, K_k)\in\mathrm{AHK}(\Omega)$ such that $E_k = E_{h_k,K_k}\in\mathrm{AS}(\Omega)$ for every k. We begin by observing that (3.2.13) is a direct consequence of (3.2.7), by applying (3.2.9) to h_k. To conclude the proof we proceed in 2 steps.

Step 1. We claim that $\overline{E} = E_h$, where h is the upper semicontinuous function defined by

$$h(x_1) := \sup\{\limsup_{k\to\infty} h_k(x_1^k) : x_1^k \to x_1\}.$$

Let $x=(x_1,x_2)\in\overline{E}$, by Remark 3.2.8-(i) we observe that there exists $x_k = (x_1^k, x_2^k)\in E_k$ such that $x_k\to x$. We deduce that

$$x_2 = \lim_{k\to\infty} x_2^k \le \limsup_{k\to\infty} h_k(x_1^k) \le h(x_1),$$

and by (3.2.2) we deduce that $\overline{E}\subset E_h$. Now let $x=(x_1,x_2)\in E_h$, by definition we observe that

$$x_2 \le h(x_1) := \sup\{\limsup_{k\to\infty} h_k(x_1^k) : x_1^k\to x_1\}.$$

Let $x_k = (x_1^k, x_2^k)\in\Omega$ such that $x_1^k\to x_1$, $h_k(x_1^k)\to h(x_1)$ and define $x_2^k := \min\{x_2, h_k(x_1^k)\}$. It follows that $x_k\in\overline{E_k}$ and $x_k\to x$, by Kuratowski convergence we have that $x\in\overline{E}$, therefore, $E_h\subset\overline{E}$.

Step 2. We claim that $\mathrm{Int}(E) = \mathrm{Int}(E_{h,K})$, where $K := \partial E\cap\overline{\mathrm{Int}(E_h)}$. Notice that K is a closed set in $\mathbb{R}^2$ and since ∂E is $\mathcal{H}^1$-rectifiable, we deduce that K is also $\mathcal{H}^1$-rectifiable. On one hand, we see that

$$\mathrm{Int}(E) = \mathrm{Int}(E)\setminus(\partial E\cap\overline{\mathrm{Int}(E_h)}) \subset \mathrm{Int}(E_h)\setminus(\partial E\cap\overline{\mathrm{Int}(E_h)}) =: \mathrm{Int}(E_h)\setminus K = \mathrm{Int}(E_{h,K}),$$

where in the first equality we used the fact that $\mathrm{Int}(E)\cap\partial E = \emptyset$ and in the inclusion we used Step 1 and the fact that $E\subset\overline{E} = E_h$. On the other hand, let $x\in\mathrm{Int}(E_{h,K}) = \mathrm{Int}(E_h)\setminus(\partial E\cap\overline{\mathrm{Int}(E_h)})$ and assume by contradiction that $x\notin\mathrm{Int}(E)$. This assumption implies that either $x\in\partial E$ or $x\in\Omega\setminus\overline{E}$, which is a contradiction by the facts that $\partial E\subset\overline{E} = E_h$ and

$$x\in\mathrm{Int}(E_h)\setminus(\partial E\cap\overline{\mathrm{Int}(E_h)}) = \mathrm{Int}(E_h)\setminus\partial E\subset E_h = \overline{E}.$$

Finally, observe that $(h,K)\in\mathrm{AHK}(\Omega)$. Thanks to the uniqueness of Kuratowski convergence, the facts that $\overline{E_{h,K}} = E_h$ and $\mathrm{Int}(E_{h,K}) = \mathrm{Int}(E_h)\setminus(\partial E\cap\overline{\mathrm{Int}(E_h)})$, and in view of Remark 3.2.8-(i) we conclude from the previous two steps that $E = E_{h,K}$. □

The total energy $\mathcal{F}:\widetilde{\mathcal{C}}\to[0,+\infty]$ of admissible configurations is given as the sum of two contributions, namely the surface energy $\mathcal{S}$ and the elastic energy $\mathcal{W}$, i.e.,

$$\mathcal{F}(A,S,u) := \mathcal{S}(A,S) + \mathcal{W}(A,u)$$

for any $(A,S,u)\in\widetilde{\mathcal{C}}$, where we observe that the surface energy does not depend on the displacements (as a difference from [58, 59]). The surface energy $\mathcal{S}$ is defined for any $(A,S)\in\widetilde{\mathcal{B}}$

by

$$\begin{aligned}\mathcal{S}(A,S) := &\int_{\Omega\cap(\partial^*A\setminus\partial^*S)} \varphi_{\mathrm{F}}(x,\nu_A(x))\,d\mathcal{H}^1 + \int_{\Omega\cap\partial^*S\cap\partial^*A} \varphi(x,\nu_A(x))\,d\mathcal{H}^1\\ &+\int_{\Omega\cap(\partial^*S\setminus\partial A)\cap A^{(1)}} \varphi_{\mathrm{FS}}(x,\nu_S(x))\,d\mathcal{H}^1 + \int_{\Omega\cap\partial^*S\cap\partial A\cap A^{(1)}} (\varphi_{\mathrm{F}}+\varphi)(x,\nu_A(x))\,d\mathcal{H}^1\\ &+\int_{\Omega\cap\partial A\cap A^{(1)}\cap S^{(0)}} 2\varphi_{\mathrm{F}}(x,\nu_A(x))\,d\mathcal{H}^1 + \int_{\Omega\cap\partial A\cap A^{(0)}} 2\varphi'(x,\nu_A(x))\,d\mathcal{H}^1\\ &+\int_{\Omega\cap(\partial S\setminus\partial A)\cap\left(S^{(1)}\cup S^{(0)}\right)\cap A^{(1)}} 2\varphi_{\mathrm{FS}}(x,\nu_S(x))\,d\mathcal{H}^1\\ &+\int_{\Omega\cap\partial S\cap\partial A\cap S^{(1)}} 2\varphi(x,\nu_A(x))\,d\mathcal{H}^1,\end{aligned} \tag{3.2.14}$$

where φ_{F}, $\varphi_{\mathrm{FS}}:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$ and, given also the function $\varphi_{\mathrm{S}}:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$, we define the functions φ and φ' in in $C(\overline{\Omega}\times\mathbb{R}^2;[0,\infty])$ by

$$\varphi := \min\{\varphi_{\mathrm{S}},\varphi_{\mathrm{F}}+\varphi_{\mathrm{FS}}\} \qquad \text{and} \qquad \varphi' := \min\{\varphi_{\mathrm{F}},\varphi_{\mathrm{S}}\}.$$

Notice that $\varphi_{\mathrm{F}},\varphi_{\mathrm{S}},\varphi_{\mathrm{FS}}$ represent the anisotropic surface tensions of the film/vapor, the substrate/vapor and the substrate/film interfaces, respectively, while φ and φ' are referred to as the anisotropic *regime surface tensions* and are introduced to include into the analysis the wetting and dewetting regimes. We refer the Reader to the Introduction for related explanation and for the motivation for the integral densities choice in (3.2.14).

Similarly to [34, 58, 59, 60], by also taking into account that in our setting the film and substrate regions are given as subsets of the composite regions, the elastic energy is defined for configurations $(A,S,u)\in\widetilde{\mathcal{C}}$ by

$$\mathcal{W}(A,u) := \int_A W\left(x, E\left(u\left(x\right) - E_0\left(x\right)\right)\right)dx,$$

where the elastic density W is determined by the quadratic form

$$W(x,M) := \mathbb{C}(x)\,M : M,$$

for a fourth-order tensor $\mathbb{C}:\Omega\to\mathbb{M}^2_{sym}$, E denotes the symmetric gradient, i.e., $E(v) := \frac{\nabla v+\nabla^T v}{2}$ for any $v\in H^1_{\mathrm{loc}}(\Omega)$ and E_0 is the mismatch strain $x\in\Omega\mapsto E_0(x)\in\mathbb{M}^2_{\mathrm{sym}}$ defined as

$$E_0 := \begin{cases} E(u_0) & \text{in } \Omega\setminus S,\\ 0 & \text{in } S,\end{cases}$$

for a fixed $u_0\in H^1(\Omega)$.

3.2.2. Main results

We state here the main results of the chapter and the connection to the one-phase and multiple-phase settings.

Fix $l,L>0$ and consider $\Omega := (-l,l)\times_{\mathbb{R}^2}(-L,L)$. Let $\varphi := \min\{\varphi_{\mathrm{S}},\varphi_{\mathrm{F}}+\varphi_{\mathrm{FS}}\}$, $\varphi' := \min\{\varphi_{\mathrm{F}},\varphi_{\mathrm{S}}\}$ for three functions $\varphi_{\mathrm{F}},\varphi_{\mathrm{S}},\varphi_{\mathrm{FS}}:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$. We assume throughout the chapter that:

(H1) $\varphi_{\mathrm{F}},\varphi_{\mathrm{FS}},\varphi,\varphi'\in C(\overline{\Omega}\times\mathbb{R}^2;[0,\infty])$ are Finsler norms such that

$$c_1|\xi| \le \varphi_{\mathrm{F}}(x,\xi),\varphi(x,\xi),\varphi_{\mathrm{FS}}(x,\xi) \le c_2|\xi|. \tag{3.2.15}$$

for every $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}^2$ and for two constants $0<c_1\le c_2$.

(H2) We have

$$\varphi(x,\xi) \geq |\varphi_{\mathrm{FS}}(x,\xi) - \varphi_{\mathrm{F}}(x,\xi)| \tag{3.2.16}$$

for every $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^2$.

(H3) $\mathbb{C} \in L^\infty(\Omega; \mathbb{M}^2_{sym})$ and there exists $c_3 > 0$ such that

$$\mathbb{C}\,(x)\, M : M \geq 2c_3 M : M \tag{3.2.17}$$

for every $M \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$.

We notice that under assumptions (H1)-(H3), the energy $\mathcal{F}(A,S,u) \in [0,\infty]$ for every $(A,S,u) \in \widetilde{\mathcal{C}}$.

The main result of the chapter is the following existence result.

Theorem 3.2.10 (Existence of minimizers)**.** *Assume (H1)-(H3) and let* $\mathbb{v}_0, \mathbb{v}_1 \in [\mathcal{L}^2(\Omega/2), \mathcal{L}^2(\Omega))$ *such that* $\mathbb{v}_0 \leq \mathbb{v}_1$. *Then for every* $\mathbf{m} = (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ *the volume constrained minimum problem*

$$\inf_{(A,S,u)\in\mathcal{C}_{\mathbf{m}},\, \mathcal{L}^2(A)=\mathbb{v}_1,\, \mathcal{L}^2(S_{h,K})=\mathbb{v}_0} \mathcal{F}(A,S,u) \tag{3.2.18}$$

and the unconstrained minimum problem

$$\inf_{(A,S,u)\in\mathcal{C}_{\mathbf{m}}} \mathcal{F}^\lambda(A,S,u) \tag{3.2.19}$$

have solution, where $\mathcal{F}^\lambda : \mathcal{C}_{\mathbf{m}} \to \mathbb{R}$ *is defined as*

$$\mathcal{F}^\lambda(A,S,u) := \mathcal{F}(A,S,u) + \lambda_1 \Big|\mathcal{L}^2(A) - \mathbb{v}_1\Big| + \lambda_0 \Big|\mathcal{L}^2(S) - \mathbb{v}_0\Big|,$$

for any $\lambda = (\lambda_0, \lambda_1)$, *with* λ_0, $\lambda_1 > 0$.

To prove Theorem 3.2.10 we apply the direct method of calculus of variations. On the one hand, in Section 3.3, we show that any energy equi-bounded sequence $\{(A_k, S_k, u_k)\} \subset \mathcal{C}_{\mathbf{m}}$ satisfy the following compactness property.

Theorem 3.2.11 (Compactness in $\mathcal{C}_{\mathbf{m}}$)**.** *Assume (H1) and (H3). Let* $\{(A_k, S_{h_k,K_k}, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}_{\mathbf{m}}$ *be such that*

$$\sup_{k\in\mathbb{N}} \mathcal{F}(A_k, S_{h_k,K_k}, u_k) < \infty. \tag{3.2.20}$$

Then, there exist an admissible configuration $(A,S,u) \in \mathcal{C}_{\mathbf{m}}$ *of finite energy, a subsequence* $\{\Big(A_{k_n}, S_{h_{k_n},K_{k_n}}, u_{k_n}\Big)\}_{n\in\mathbb{N}}$, *a sequence* $\Big\{\Big(\widetilde{A}_n, \widetilde{S}_n, u_{k_n}\Big)\Big\}_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathbf{m}}$ *and a sequence* $\{b_n\}_{n\in\mathbb{N}}$ *of piecewise rigid displacements associated to* $\widetilde{A}_n$ *such that* $\Big(\widetilde{A}_n, \widetilde{S}_n, u_{k_n} + b_n\Big) \xrightarrow{\tau_{\mathcal{C}}} (A,S,u)$, $\mathcal{L}^2(A_{k_n}) = \mathcal{L}^2(\widetilde{A}_n)$, $\mathcal{L}^2(S_{h_{k_n},K_{k_n}}) = \mathcal{L}^2(\widetilde{S}_n)$ *for all* $n \in \mathbb{N}$ *and*

$$\liminf_{n\to\infty} \mathcal{F}(A_{k_n}, S_{h_{k_n},K_{k_n}}, u_{k_n}) = \liminf_{n\to\infty} \mathcal{F}\left(\widetilde{A}_n, \widetilde{S}_n, u_{k_n} + b_n\right). \tag{3.2.21}$$

On the other hand, in Section 3.4 we show that $\mathcal{F}$ is lower semicontinuous in $\mathcal{C}_{\mathbf{m}}$ with respect to the topology $\tau_{\mathcal{C}}$.

Theorem 3.2.12 (Lower semicontinuity of $\mathcal{F}$)**.** *Assume (H1)-(H3). Let* $\{(A_k, S_{h_k,K_k}, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}_{\mathbf{m}}$ *and* $(A, S_{h,K}, u) \in \mathcal{C}_{\mathbf{m}}$ *be such that* $(A_k, S_{h_k,K_k}, u_k) \xrightarrow{\tau_{\mathcal{C}}} (A, S_{h,K}, u)$. *Then*

$$\mathcal{F}\,(A, S_{h,K}, u) \leq \liminf_{k\to\infty} \mathcal{F}\,(A_k, S_{h_k,K_k}, u_k). \tag{3.2.22}$$

We now describe the consequences for the one-phase setting of the results obtained in this chapter for the two-phase setting.

Remark 3.2.13 (Relation to literature models with fixed substrate)**.** The energy considered in this chapter can be seen as an extension of the energies previously considered in the literature, e.g., in [58, 46, 45], by "fixing the substrate regions". More precisely, if we consider the subfamily $\mathcal{B}' \subset \mathcal{B}$ where

$$\mathcal{B}' := \{(A, S) \in \mathcal{B} : \overline{\partial S \cap \Omega} \text{ is a Lipschitz 1-manifold}\}$$

and $\mathcal{C}' := \{(A, S, u) \in \mathcal{C} : (A, S) \in \mathcal{B}'\}$, then the energy $\mathcal{F}'$ defined for every $(A, S) \in \mathcal{B}'$ by

$$\begin{aligned}\mathcal{F}'(A, S, u) &:= \mathcal{F}(A, S, u) - \int_{\Omega\cap\partial^* S} \varphi_{\mathrm{FS}}(z, \nu_S(z))\, d\mathcal{H}^1 - \int_{\Omega\cap(\partial A\setminus\partial S)\cap A^{(0)}} 2\varphi'(z, \nu_A(z)) d\mathcal{H}^1 \\ &= \int_{\Omega\cap\partial^* A\setminus\partial S} \varphi_{\mathrm{F}}(z, \nu_A(z)) d\mathcal{H}^1 + \int_{\Omega\cap(\partial A\setminus\partial S)\cap\cup A^{(1)}} 2\varphi_{\mathrm{F}}(z, \nu_A(z)) d\mathcal{H}^1 \\ &\quad + \int_{\Omega\cap\partial^* S\cap\partial A\cap A^{(1)}} (\varphi_{\mathrm{F}} - \beta)(x, \nu_A(x))\, d\mathcal{H}^1 - \int_{\Omega\cap\partial^* S\cap\partial^* A} \beta(z, \nu_A(z)) d\mathcal{H}^1,\end{aligned}$$

where $\beta := \varphi_{\mathrm{FS}} - \varphi$, is analogous to the energy $\mathcal{F}'$ of [58, Theorem 2.9] (where the notation φ was referring to φ_{F}), which is an extension of the energies of [46, 45] as described in [58, Remark 2.10]. We notice though that, even in the situation of a fixed regular substrate, i.e., by considering the family $\mathcal{C}'' := \{(A, S, u) \in \mathcal{C} : S = S_0\} \subset \mathcal{C}'$ for a fixed admissible region S_0 such that $\overline{\partial S_0 \cap \Omega}$ is a Lipschitz 1-manifold, the setting considered in this chapter allows to include into the analysis the possibility of an uncountable number of film islands (or film voids) on top of the substrate (which was instead precluded in [58]), because of the crucial difference introduced in the setting of this chapter consisting of always including the substrate regions inside the admissible region A (with the film region then being represented by $\overline{A} \setminus S^{(1)}$). We also notice that the hypotheses on the surface tensions in this chapter coincide with the ones in [58] up to the observation that on the right hand-side of (3.2.16) one can disregard the absolute value for the setting with a fixed substrate).

We conclude the section by outlining the consequences of the results obtained in this chapter for the two-phase setting in the multiple-phase setting of film multilayers that is the object of investigation in Chapter 4.

Remark 3.2.14 (Relation to the setting of film multilayers)**.** In Chapter 4, it is considered the setting in which also the film region is subject to a graph constraint, by introducing a family of admissible regions for the film and the substrate of the form $\mathcal{B}^1 := \{(S_{h^1,K^1}, S_{h^0,K^0}) \in \mathrm{AS}(\Omega) \times \mathrm{AS}(\Omega) : h^0 \leq h^1 \text{ and } \partial S_{h^1,K^1} \cap \mathrm{Int}(S_{h^0,K^0}) = \emptyset\} \subset \mathcal{B}$ and the related family $\mathcal{C}^1 := \{(S_{h^1,K^1}, S_{h^0,K^0}, u) : (S_{h^1,K^1}, S_{h^0,K^0}) \in \mathcal{B}^1 \text{ and } u \in H^1_{\mathrm{loc}}(S_{h^1,K^1}; \mathbb{R}^2)\} \subset \mathcal{C}$ of admissible configurations. By implementing in the compactness for both the substrate and the film the arguments employed in this chapter for the substrate, a similar result to Theorem 3.2.10 is established, thus providing an existence result for the problem of films resting on deformable substrates in the presence of delaminations, which can be seen as an extension of [34, 35, 46]. Furthermore, in Chapter 4 by then performing also an iteration procedure an existence result is provided also for the setting of finitely-many film multilayers.

3.3. Compactness

In this section, we fix $\mathbf{m} := (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ and we prove that the families $\mathcal{B}_{\mathbf{m}}$ and $\mathcal{C}_{\mathbf{m}}$ are compact with respect to $\tau_{\mathcal{B}}$ and $\tau_{\mathcal{C}}$ topologies, respectively.

Proposition 3.3.1. *The following assertions hold:*

(i) For every sequence of closed sets $\{E_k\}$, there exists $E \subset \mathbb{R}^2$ such that $E_k \xrightarrow{\mathcal{K}} E$.

(ii) For every sequence $\{E_k\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^2$ there exists a subsequence $\{E_{k_l}\}_{l\in\mathbb{N}}$ and $E \subset \mathbb{R}^2$ such that $\mathrm{sdist}(\cdot, \partial E_{k_l}) \to \mathrm{sdist}(\cdot, \partial E)$ *locally uniformly in $\mathbb{R}^2$.*

The proof of Proposition 3.3.1-(i) and -(ii) can be found in [4, Theorem 6.1] and [58, Theorem 6.1], respectively (see also [4, Theorem 6.1] for the version of Item (ii) with the signed-distance convergence replaced by the *Hausdorff-metric convergence*).

Theorem 3.3.2 (Compactness of $\mathcal{B}_{\mathbf{m}}$)**.** *Let* $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\mathbf{m}}$ *such that*

$$\sup_{k\in\mathbb{N}} \mathcal{S}(A_k, S_{h_k,K_k}) < \infty$$

for $(h_k, K_k) \in \mathrm{AHK}(\Omega)$*. Then, there exist a not relabeled subsequence* $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\mathbf{m}}$ *and* $(A, S_{h,K}) \in \mathcal{B}_{\mathbf{m}}$ *such that* $(A_k, S_{h_k,K_k}) \xrightarrow{\tau_{\mathcal{B}}} (A, S_{h,K})$*.*

Proof. For simplicity we denote $S_k := S_{h_k,K_k}$. We begin by observing that by Proposition 3.3.1-(ii) there exist $A \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2$ such that $\mathrm{sdist}(\cdot, \partial A_{k_l}) \to \mathrm{sdist}(\cdot, \partial A)$ and $\mathrm{sdist}(\cdot, \partial S_{k_l}) \to \mathrm{sdist}(\cdot, \partial S)$ locally uniformly in $\mathbb{R}^2$. Let $R := \sup_{k\in\mathbb{N}} \mathcal{S}(A_k, S_{h_k,K_k})$. In view of Remark 3.2.4 and by (2.2.7) we have the following decomposition of ∂A_k,

$$\partial A_k = \partial^* A_k \cup (\partial A_k \cap (A_k^{(0)} \cup A_k^{(1)})) \cup N_k,$$

where N_k is a $\mathcal{H}^1$-negligible set for every $k \in \mathbb{N}$. Thus, for every $k \in \mathbb{N}$ we observe that

$$\partial A_k \setminus \partial S_k = \partial^* A_k \setminus \partial S_k \cup \Big((\partial A_k \setminus \partial S_k) \cap \Big(A_k^{(0)} \cup A_k^{(1)}\Big)\Big) \cup N'_k, \tag{3.3.1}$$

where $N'_k := N_k \setminus \partial S_k$ is a $\mathcal{H}^1$-negligible set. Since for any $k \in \mathbb{N}$, S_k is a set of finite perimeter, by (2.2.7) we have that

$$\partial S_k = \partial^* S_k \cup (\partial S_k \cap (S_k^{(0)} \cup S_k^{(1)})) \cup \widetilde{N}_k,$$

where $\widetilde{N}_k$ is a $\mathcal{H}^1$-negligible set for every $k \in \mathbb{N}$. Reasoning similarly to (3.3.1) we have that

$$\partial S_k \setminus \partial A_k = \partial^* S_k \setminus \partial A_k \cup (\partial S_k \setminus \partial A_k) \cap \Big(S_k^{(0)} \cup S_k^{(1)}\Big) \cup \widetilde{N}'_k, \tag{3.3.2}$$

$$(\partial S_k \setminus \partial A_k) \cap A_k^{(1)} = \Big(\partial^* S_k \setminus \partial A_k \cup \Big(\partial S_k \setminus \partial A_k \cap \Big(S_k^{(1)} \cup S_k^{(0)}\Big)\Big)\Big) \cap A_k^{(1)} \cup \widetilde{N}''_k \tag{3.3.3}$$

and

$$\partial S_k \cap \partial A_k \cap A_k^{(1)} = \Big((\partial^* S_k \cap \partial A_k) \cup \Big((\partial S_k \cap \partial A_k) \cap (S_k^{(1)} \cup S_k^{(0)})\Big)\Big) \cap A_k^{(1)} \cup \widetilde{N}'''_k, \tag{3.3.4}$$

where $\widetilde{N}'_k$, $\widetilde{N}''_k$ and $\widetilde{N}'''_k$ are $\mathcal{H}^1$-negligible sets for every $k \in \mathbb{N}$. Furthermore, we can deduce that

$$\begin{aligned}
\partial S_k \cap \partial A_k &= \Big(\partial^* S_k \cup (\partial S_k \cap (S_k^{(0)} \cup S_k^{(1)}))\Big) \cap \Big(\partial^* A_k \cup (\partial A_k \cap (A_k^{(0)} \cup A_k^{(1)}))\Big) \cup \widehat{N}_k \\
&= (\partial S_k^* \cap \partial^* A_k) \cup \Big(\partial S_k \cap \partial^* A_k \cap (S_k^{(0)} \cup S_k^{(1)})\Big) \cup \Big(\partial^* S_k \cap \partial A_k \cap (A_k^{(0)} \cup A_k^{(1)})\Big) \\
&\quad \Big(\partial S_k \cap \partial A_k \cap ((A_k^{(0)} \cup A_k^{(1)}) \cup (S_k^{(0)} \cup S_k^{(1)}))\Big) \cup \widehat{N}_k,
\end{aligned} \tag{3.3.5}$$

where $\widehat{N}_k$ is a $\mathcal{H}^1$-negligible set, for every $k \in \mathbb{N}$. By (H1),(H3), (3.3.1)-(3.3.5) and thanks to the fact that for every $k \in \mathbb{N}$, $S_k \subset \overline{A_k}$ we have that

$$\begin{aligned}
&c_1 \Big(\mathcal{H}^1(\partial A_k) + \mathcal{H}^1(\partial S_k \setminus \partial A_k) \Big) \\
&\quad \leq c_1 \Bigg(\int_{\partial^* A_k \setminus \partial S_k} d\mathcal{H}^1 + \int_{\partial^* S_k \cap \partial^* A_k} d\mathcal{H}^1 + \int_{(\partial A_k \setminus \partial S_k) \cap ({A_k}^{(0)} \cup {A_k}^{(1)})} d\mathcal{H}^1 \\
&\qquad + \int_{(\partial^* S_k \setminus \partial A_k) \cap {A_k}^{(1)}} d\mathcal{H}^1 + \int_{\partial S_k \cap \partial A_k \cap {S_k}^{(1)}} d\mathcal{H}^1 + \int_{\partial S_k \cap \partial^* A_k \cap {S_k}^{(0)}} d\mathcal{H}^1 \\
&\qquad + \int_{(\partial S_k \setminus \partial A_k) \cap ({S_k}^{(1)} \cup {S_k}^{(0)}) \cap {A_k}^{(1)}} d\mathcal{H}^1 + \int_{\partial^* S_k \cap \partial A_k \cap {A_k}^{(1)}} d\mathcal{H}^1 \\
&\qquad + \int_{(\partial S_k \cap \partial A_k) \cap {S_k}^{(0)} \cap {A_k}^{(1)}} d\mathcal{H}^1 \Bigg) \\
&\quad \leq 2(\mathcal{S}(A_k, S_k)) \leq 2R,
\end{aligned} \tag{3.3.6}$$

for every $k \in \mathbb{N}$, where in the first inequality we used Lemma 3.2.1 and in the second inequality we used (3.2.15). It follows that

$$\mathcal{H}^1(\partial A_k) \leq \frac{2R}{c_1} \tag{3.3.7}$$

and

$$\mathcal{H}^1(\partial S_k) = \mathcal{H}^1(\partial S_k \cap \partial A_k) + \mathcal{H}^1(\partial S_k \setminus \partial A_k) \leq \frac{2R}{c_1}, \tag{3.3.8}$$

for any $k \in \mathbb{N}$.

In view of (3.3.7) and (3.3.8) we conclude by Lemma 3.2.9 that $A \subset \overline{\Omega}$ is $\mathcal{L}^2$-measurable, ∂A is $\mathcal{H}^1$-finite, $\mathcal{H}^1$-rectifiable, and with at most m_1-connected components, and that there exists $(h, K) \in \mathrm{AHK}(\Omega)$ such that $S = S_{h,K} \in \mathrm{A}S(\Omega)$ and $\partial S_{h,K}$ has at most m_0-connected components. Furthermore, in view of Remark 3.2.8-(i), since $\overline{S_k} \subset \overline{A_k}$ for any $k \in \mathbb{N}$, we have that $S_{h,K} \subset \overline{A}$.

In order to prove that $(A, S_{h,K}) \in \mathcal{B}_{\mathbf{m}}$ it remains to check that $\partial A \cap \mathrm{Int}(S_{h,k}) = \emptyset$, to which the rest of the proof is devoted. Assume by contradiction that

$$\partial A \cap \mathrm{Int}(S_{h,k}) \neq \emptyset. \tag{3.3.9}$$

Then, there exists $x \in \partial A \cap \mathrm{Int}(S_{h,k})$. By Remark 3.2.8-(i), there exists $x_k \in \partial A_k$ such that $x_k \to x$ and hence, by $\tau_{\mathcal{B}}$-convergence we observe that

$$\mathrm{sdist}(x, \partial S_{h_k,K_k}) \to \mathrm{sdist}(x, \partial S_{h,K}) \quad \text{as } k \to \infty. \tag{3.3.10}$$

Since by (3.3.9) there exists $\varepsilon > 0$ such that $\mathrm{sdist}(x, \partial S_{h,K}) = -\varepsilon$, we can find $k_0 := k_0(x)$ for which $\mathrm{sdist}(x, \partial S_{h_{k_0},K_{k_0}})$ is negative. Then, $x \in \mathrm{Int}(S_{h_{k_0},K_{k_0}})$ and so, there exists $\delta \leq \varepsilon/2$ such that

$$x_{k_0} \in B_\delta(x) \subset \mathrm{Int}(S_{h_{k_0},K_{k_0}}),$$

which is an absurd since $\partial A_k \cap \mathrm{Int}(S_{h_k,K_k}) = \emptyset$ for every $k \in \mathbb{N}$. □

We are now in the position to prove Theorem 3.2.11. To this end, we implement the arguments used in [58, Theorem 2.7] to the situation with free-boundary substrates and in particular the ones contained in Step 1 of the proof of [58, Theorem 2.7]. In fact, the original setting introduced in [66] with respect to [58] to model the delaminated interface regions allows to avoid the further modification of the film admissible regions that was performed in Steps 2 and 3 of [58, Theorem 2.7].

Proof of Theorem 3.2.11. Denote $R := \sup_{k\in\mathbb{N}} \mathcal{F}(A_k, S_{h_k,K_k}, u_k)$. Without loss of generality (by passing, if necessary, to a not relabeled subsequence), we assume that

$$\liminf_{k\to\infty} \mathcal{F}(A_k, S_{h_k,K_k}, u_k) = \lim_{k\to\infty} \mathcal{F}(A_k, S_{h_k,K_k}, u_k) \leq R. \tag{3.3.11}$$

Since $\mathcal{W}$ is non-negative, by Theorem 3.3.2 there exist a subsequence $\{(A_{k_n}, S_{h_{k_n},K_{k_n}})\} \subset \mathcal{B}_{\mathbf{m}}$ and $(A, S) \in \mathcal{B}_{\mathbf{m}}$ such that $(A_{k_n}, S_{h_{k_n},K_{k_n}}) \xrightarrow{\tau_{\mathcal{B}}} (A, S)$. As a consequence of Theorem 3.3.2, there exists $(h, K) \in \text{AHK}$ such that $S = S_{h,K}$.

The rest of the proof is devoted to the construction of a sequence $(\widetilde{A}_n, \widetilde{S}_n) \subset \mathcal{B}_{\mathbf{m}}$ to which we can apply [58, Corollary 3.8] (with $P = \text{Int}(A)$ and $P_n = \text{Int}(\widetilde{A}_n)$, respectively) in order to obtain $u \in H^1_{\text{loc}}(\text{Int}(A); \mathbb{R}^2)$ such that $(A, S, u) \in \mathcal{C}_{\mathbf{m}}$ has finite energy, and a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise rigid displacements such that $\left(\widetilde{A}_n, \widetilde{S}_n, u_{k_n} + b_n\right) \xrightarrow{\tau_{\mathcal{C}}} (A, S, u)$. Furthermore, we observe that also Equation (3.2.21) will be a consequence of such construction and hence, the assertion will directly follow.

By [58, Proposition 3.6] applied to A_{k_n} and A there exist a not relabeled subsequence $\{A_{k_n}\}$ and a sequence $\{\widetilde{A}_n\}$ with $\mathcal{H}^1$-rectifiable boundary $\partial\widetilde{A}_n$ of at most m_1-connected components such that

$$\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial\widetilde{A}_n) < \infty, \tag{3.3.12}$$

that satisfy the following properties:

(a1) $\partial A_{k_n} \subset \partial\widetilde{A}_n$ and $\lim_{n\to\infty} \mathcal{H}^1(\partial\widetilde{A}_n \setminus \partial A_{k_n}) = 0$,

(a2) $\text{sdist}(\cdot, \partial\widetilde{A}_n) \to \text{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ as $n \to \infty$,

(a3) if $\{E_i\}_{i\in I}$ is the family of all connected components of $\text{Int}(A)$, there exist connected components of $\text{Int}(\widetilde{A}_n)$, which we enumerate as $\{E_i^n\}_{i\in I}$, such that for every i and $G \subset\subset E_i$ one has that $G \subset\subset E_i^n$ for all n large (depending only on i and G),

(a4) $\mathcal{L}^2(\widetilde{A}_n) = \mathcal{L}^2(A_{k_n})$.

Furthermore, from the construction of $\widetilde{A}_n$ (namely from the fact that $\widetilde{A}_n$ is constructed by adding extra "internal" topological boundary to the selected subsequence A_{k_n}, see [58, Propositions 3.4 and 3.6]) it follows that

$$\widetilde{A}_n = A_{k_n} \setminus (\partial\widetilde{A}_n \setminus \partial A_{k_n}) \tag{3.3.13}$$

with $\partial\widetilde{A}_n \setminus \partial A_{k_n}$ given by a finite union of closed $\mathcal{H}^1$-rectifiable sets connected to ∂A_{k_n}. More precisely, there exist a finite index set J and a family $\{\Gamma_j\}_{j\in J}$ of closed $\mathcal{H}^1$-rectifiable sets of Ω connected to ∂A_{k_n} such that

$$\partial\widetilde{A}_n \setminus \partial A_{k_n} = \bigcup_{j\in J} \Gamma_j.$$

We define

$$\widetilde{K}_n := K_{k_n} \cup ((\partial\widetilde{A}_n \setminus \partial A_{k_n}) \cap \overline{\text{Int}(S_{h_{k_n}})}) \subset \overline{\text{Int}(S_{h_{k_n}})}$$

and we observe that $\widetilde{K}_n$ is closed and $\mathcal{H}^1$-rectifiable in view of the fact that $\partial\widetilde{A}_n \setminus \partial A_{k_n}$ is a closed set in Ω and is $\mathcal{H}^1$-rectifiable, since $\partial\widetilde{A}_n$ is $\mathcal{H}^1$-rectifiable. Therefore, $(h_{k_n}, \widetilde{K}_n) \in \text{AHK}(\Omega)$ and $\widetilde{S}_n := S_{h_{k_n},\widetilde{K}_n} \subset \overline{\widetilde{A}_n}$. We claim that $\partial\widetilde{S}_n$ has at most m_0-connected components, so that $(\widetilde{A}_n, \widetilde{S}_n) \in \mathcal{B}_{\mathbf{m}}$. Indeed, if for every $j \in J$, $S_{h_{k_n},K_{k_n}} \cap \Gamma_j$ is empty there is nothing to prove, so we assume that there exists $j \in J$ such that $S_{h_{k_n},K_{k_n}} \cap \Gamma_j \neq \emptyset$. On one hand if $\Gamma_j \subset S_{h_{k_n},K_{k_n}}$, thanks to the facts that Γ_j is connected to ∂A_{k_n} and $S_{h_{k_n},K_{k_n}} \subset \overline{A_{k_n}}$, we deduce that Γ_j needs to be connected to $\partial S_{h_{k_n},K_{k_n}}$. On the other hand, if $\Gamma_j \cap (A_{k_n} \setminus S_{h_{k_n}}) \neq \emptyset$, then we can find $x_1 \in \Gamma_j \cap S_{h_{k_n},K_{k_n}}$ and $x_2 \in \Gamma_j \cap (A_{k_n} \setminus S_{h_{k_n}})$. Since Γ_j is closed and connected, by [44, Lemma

3.12] there exists a parametrization $r:[0,1]\to\mathbb{R}^2$ whose support $\gamma\subset\Gamma_j$ joins the point x_1 with x_2. Thus, γ crosses $\partial S_{h_{k_n},K_{k_n}}$ and we conclude that Γ_j is connected to $\partial S_{h_{k_n},K_{k_n}}$.

We claim that $(\widetilde{A}_n,\widetilde{S}_n)\xrightarrow{\tau_{\mathcal{B}}}(A,S_{h,K})$ as $n\to\infty$. In view of (3.3.12), (a2) and the fact that by (3.2.7) and the previous construction of $\widetilde{K}_n$,

$$\sup_{n\in\mathbb{N}}\mathcal{H}^1(\partial\widetilde{S}_n)=\sup_{n\in\mathbb{N}}\mathcal{H}^1(\partial S_{h_{k_n},\widetilde{K}_n})<\infty,$$

it remains to prove that

$$\operatorname{sdist}(\cdot,\partial S_{h_{k_n},\widetilde{K}_n})\to\operatorname{sdist}(\cdot,\partial S_{h,K}) \tag{3.3.14}$$

locally uniformly in $\mathbb{R}^2$ as $n\to\infty$. Indeed, by Remark 3.2.8-(i), it suffices to prove that $S_{h_{k_n},\widetilde{K}_n}\xrightarrow{\mathcal{K}}S_h$ and that $\Omega\setminus S_{h_{k_n},\widetilde{K}_n}\xrightarrow{\mathcal{K}}\Omega\setminus\operatorname{Int}(S_{h,K})$. On one hand, by the $\tau_{\mathcal{B}}$-convergence of $\{(A_{k_n},S_{k_n})\}$, the fact that $\widetilde{S}_n:=\overline{S_{h_{k_n},\widetilde{K}_n}}=S_{h_{k_n}}$, and the properties of Kuratowski convergence, it follows that $\widetilde{S}_n:=S_{h_{k_n},\widetilde{K}_n}\xrightarrow{\mathcal{K}}S_h$. On the other hand, let $x\in\Omega\setminus\operatorname{Int}(S_{h,K})$, since

$$\operatorname{Int}(S_{h_{k_n},\widetilde{K}_n})=\operatorname{Int}(S_{h_{k_n}})\setminus\widetilde{K}_n\subset\operatorname{Int}(S_{h_{k_n}})\setminus K_{k_n}=\operatorname{Int}(S_{h_{k_n},K_{k_n}})$$

and by the fact that $\Omega\setminus\operatorname{Int}(S_{h_{k_n},K_{k_n}})\xrightarrow{\mathcal{K}}\Omega\setminus\operatorname{Int}(S_{h,K})$, there exists

$$x_n\in\Omega\setminus\operatorname{Int}(S_{h_{k_n},K_{k_n}})\subset\Omega\setminus\operatorname{Int}(S_{h_{k_n},\widetilde{K}_n})$$

such that $x_n\to x$. Now, we consider a sequence $x_n\in\Omega\setminus\operatorname{Int}(S_{h_{k_n},\widetilde{K}_n})$ converging to a point $x\in\Omega$. We proceed by contradiction, namely we assume that $x\in\operatorname{Int}(S_{h,K})$. Therefore, there exists $\epsilon>0$ such that $\operatorname{sdist}(x,\partial S_{h,K})=-\epsilon$, which implies that $\operatorname{sdist}(x,\partial S_{h_{k_n},K_{k_n}})\to-\epsilon$ as $n\to\infty$. Thus, there exists $n_\epsilon\in\mathbb{N}$, such that $x_n\in B_{\epsilon/2}(x)\subset\operatorname{Int}(S_{h_{k_n},K_{k_n}})$, for every $n\geq n_\epsilon$. However, notice that

$$\begin{aligned}x_n\in\Omega\setminus\operatorname{Int}\left(S_{h_{k_n},\widetilde{K}_n}\right)&=\Omega\setminus\left(\operatorname{Int}(S_{h_{k_n}})\setminus\widetilde{K}_n\right)\\&=\left(\Omega\setminus\operatorname{Int}\left(S_{h_{k_n},K_{k_n}}\right)\right)\cup\left(\left(\partial\widetilde{A}_n\setminus\partial A_{k_n}\right)\cap\overline{\operatorname{Int}(S_{h_{k_n}})}\right),\end{aligned} \tag{3.3.15}$$

where in the last equality we used the definition of $\widetilde{K}_n:=K_{k_n}\cup((\partial\widetilde{A}_n\setminus\partial A_{k_n})\cap\overline{\operatorname{Int}(S_{h_{k_n}})})$ and the fact that $\operatorname{Int}(S_{h_{k_n},K_{k_n}})=\operatorname{Int}(S_{h_{k_n}})\setminus K_{k_n}$. Therefore, by (3.3.15) we deduce that $x_n\in\partial\widetilde{A}_n\setminus\partial A_{k_n}$ for every $n\geq n_\epsilon$ and hence, $x\in\partial A$ by (a2) and Remark 3.2.8-(i). We reached an absurd as it follows that $x\in\operatorname{Int}(S_{h,K})\cap\partial A=\emptyset$. This conclude the proof of (3.3.14) and hence, of the claim.

By (3.2.15) and by conditions (a1), (a4) and (3.3.13), we observe that

$$\lim_{n\to\infty}\left|\mathcal{S}(A_{k_n},S_{k_n})-\mathcal{S}(\widetilde{A}_n,\widetilde{S}_n)\right|=\lim_{n\to\infty}\left|\mathcal{S}(A_{k_n},S_{h_{k_n},K_{k_n}})-\mathcal{S}(\widetilde{A}_n,S_{h_{k_n},\widetilde{K}_n})\right|=0, \tag{3.3.16}$$

and

$$\mathcal{W}(A_{k_n},u_{k_n})=\mathcal{W}(\widetilde{A}_n,u_{k_n}). \tag{3.3.17}$$

By (3.2.17), (3.3.11), (3.3.13), (3.3.17), (a3) and thanks to the fact that $\mathcal{S}$ is non-negative, we obtain that

$$\int_{E_i^n}|e(u_{k_n})|^2dx\leq\int_{\widetilde{A}_n}|e(u_{k_n})|^2dx\leq\frac{R}{2c_3},$$

for every $i\in I$ and for n large enough. Therefore, by a diagonal argument and by [58, Corollary 3.8] (applied to, with the notation of [58], $P=E_i$ and $P_n=E_i^n$) up to extracting not relabeled subsequences both for $\{u_{k_n}\}\subset H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ and $\{E_i^n\}_n$ there exist $w_i\in H^1_{\mathrm{loc}}(E_i,\mathbb{R}^2)$, and a sequence of rigid displacements $\{b_n^i\}$ such that $(u_{k_n}+b_n^i)\mathbb{1}_{E_i^n}\to w_i$ a.e. in E_i. Let $\{D_i^n\}_{i\in\widetilde{I}}$ for an index set $\widetilde{I}$ be the family of open and connected components of $\widetilde{A}_n \setminus \bigcup_{i\in I} E_i^n$ such that by (a3) $\mathrm{Int}(D_i^n)$ converges to the empty set for every $i \in \widetilde{I}$. In D_i^n we consider the null rigid displacement, and we define

$$b_n := \sum_{i\in I} b_n^i \mathbb{1}_{E_i^n} \quad \text{and} \quad u := \sum_{i\in I} w_i \mathbb{1}_{E_i}.$$

We have that $u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A);\mathbb{R}^2)$, b_n is a rigid displacement associated to $\widetilde{A}_n$, $u_{k_n} + b_n \to u$ a.e. in $\mathrm{Int}(A)$ and hence, $(A, S, u) =(A, S_{h,K}, u) \in \mathcal{C}_{\mathbf{m}}$ and $(\widetilde{A}_n, \widetilde{S}_n, u_{k_n} + b_n) :=(\widetilde{A}_n, S_{h_{k_n},\widetilde{K}_n}, u_{k_n} + b_n) \xrightarrow{\tau_{\mathcal{C}}} (A, S_{h,K}, u)$. Furthermore, as $e(u_{k_n} + b_n) = e(u_{k_n})$, from (3.3.16) and (3.3.17) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \left|\mathcal{F}(A_{k_n}, S_{h_{k_n},K_{k_n}}, u_{k_n}) - \mathcal{F}(\widetilde{A}_n, \widetilde{S}_n, u_{k_n} + b_n)\right| \\
&= \lim_{n\to\infty} \left|\mathcal{F}(A_{k_n}, S_{h_{k_n},K_{k_n}}, u_{k_n}) - \mathcal{F}(\widetilde{A}_n, S_{h_{k_n},\widetilde{K}_n}, u_{k_n} + b_n)\right| = 0,
\end{aligned}$$

which implies (3.2.21) and completes the proof. □

3.4. Lower semicontinuity

In this section we prove that for any fixed $\mathbf{m} := (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ the energy $\mathcal{F}$ is lower semicontinuous in the family of configurations $\mathcal{C}_{\mathbf{m}}$ with respect to the topology $\tau_{\mathcal{C}}$. Since $\mathcal{F}$ is given as the sum of the surface energy $\mathcal{S}$ and the elastic energy $\mathcal{W}$, we proceed by proving that both $\mathcal{S}$ and $\mathcal{W}$ are independently lower semicontinuous with respect to $\tau_{\mathcal{C}}$.

We begin with $\mathcal{S}$ and we adopt *Fonseca-Müller blow-up technique* [48], for which we make use of a localized version $\mathcal{S}_L$ of the surface energy, which we can consider with surface tensions constant with respect to the variable in $\overline{\Omega}$.

Definition 3.4.1. *Let $\phi_F, \phi_S, \phi_{FS}$ be three functions and let $\phi := \min\{\phi_{\mathrm{S}}, \phi_{\mathrm{F}} + \phi_{\mathrm{FS}}\}$, $\phi' := \min\{\phi_{\mathrm{F}}, \phi_{\mathrm{S}}\}$ be such that ϕ_F, ϕ_{FS}, ϕ , $\phi' \in C(\mathbb{R}^2; [0,\infty])$ are Finsler norms and the hypotheses (H1) and (H2) are satisfied by the functions $\varphi_\alpha, \varphi, \varphi' \in C(\overline{\Omega} \times \mathbb{R}^2)$ given for $\alpha = S, FS, F$ by $\varphi_\alpha(x,\cdot) := \phi_\alpha(\cdot)$, $\varphi(x,\cdot) := \phi(\cdot)$ and $\varphi'(x,\cdot) = \phi'(\cdot)$ for every $x \in \overline{\Omega}$. We define the localized surface energy $\mathcal{S}_L : \mathcal{B}_L \to [0, +\infty]$ by*

$$\begin{aligned}
\mathcal{S}_L\,(A, S, O) := &\int_{O\cap(\partial^* A\setminus\partial S)} \phi_{\mathrm{F}}(\nu_A)\, d\mathcal{H}^1 + \int_{O\cap\partial^* S\cap\partial^* A} \phi(\nu_A)\, d\mathcal{H}^1 \\
&+ \int_{O\cap(\partial^* S\setminus\partial A)\cap A^{(1)}} \phi_{\mathrm{FS}}(\nu_S)\, d\mathcal{H}^1 + \int_{O\cap\partial^* S_{h,K}\cap\partial A\cap A^{(1)}} (\phi_{\mathrm{F}} + \phi)(\nu_A)\, d\mathcal{H}^1 \\
&+ \int_{O\cap\partial A\cap A^{(1)}\cap S^{(0)}} 2\phi_{\mathrm{F}}(\nu_A)\, d\mathcal{H}^1 + \int_{O\cap\partial A\cap A^{(0)}} 2\phi'(\nu_A)\, d\mathcal{H}^1 \\
&+ \int_{O\cap(\partial S\setminus\partial A)\cap\left(S^{(1)}\cup S^{(0)}\right)\cap A^{(1)}} 2\phi_{\mathrm{FS}}(\nu_S)\, d\mathcal{H}^1 + \int_{O\cap\partial S\cap\partial A\cap S^{(1)}} 2\phi(\nu_S)\, d\mathcal{H}^1
\end{aligned} \tag{3.4.1}$$

for every $(A, S, O) \in \mathcal{B}_L := \{(A, S, O) \,:\, (A, S) \in \mathcal{B},\ O$ open and contained in $\Omega\}$.

We start with some technical results needed in the blow-up argument used in Theorem 3.4.13.

Lemma 3.4.2. *Let Q be any open square, $K \subset \overline{Q}$ be a nonempty closed set and $E_k \subset \overline{Q}$ be such that sdist$(\cdot, \partial E_k) \to$ dist$(\cdot, K)$ uniformly in $\overline{Q}$ as $k \to \infty$. Then $E_k \xrightarrow{\mathcal{K}} K$ as $k \to \infty$. Analogously, if sdist$(\cdot, \partial E_k) \to -$dist$(\cdot, K)$ uniformly in $\overline{Q}$ as $k \to \infty$, then $\overline{Q} \setminus E_k \xrightarrow{\mathcal{K}} K$ as $k \to \infty$.*

The proof of the previous lemma follows from the same arguments of [58, Lemma 4.2].

Proposition 3.4.3. *Let $m \in \mathbb{N}$ and let $E \subset \mathbb{R}^2$ be a set such that ∂E has at most m-connected components, is $\mathcal{H}^1$-rectifiable, and satisfies $\mathcal{H}^1(\partial E) < \infty$. Let $x \in \partial E$ be such that the measure-theoretic unit normal $\nu_E(x)$ of ∂E at x and there exists $R > 0$ such that $\overline{Q_{R,\nu_E(x)}(x)} \cap \partial\sigma_{\rho,x}(E) \xrightarrow{\mathcal{K}} \overline{Q_{R,\nu_E(x)}(x)} \cap T_x$ as $\rho \to 0^+$, where $T_x := T_{x,\nu_E(x)}$. Then, the following assertions hold true:*

(a) If $x \in E^{(1)} \cap \partial E$, then $sdist(\cdot, \partial\sigma_{\rho,x}(E)) \to -dist(\cdot, T_x)$ uniformly in $\overline{Q_{R,\nu_E(x)}}$ as $\rho \to 0^+$;

(b) If $x \in E^{(0)} \cap \partial E$, then $sdist(\cdot, \partial\sigma_{\rho,x}(E)) \to dist(\cdot, T_x)$ uniformly in $\overline{Q_{R,\nu_E(x)}}$ as $\rho \to 0^+$;

(c) If $x \in \partial^* E$ then $sdist(\cdot, \partial\sigma_{\rho,x}(E)) \to sdist(\cdot, \partial H_x)$ uniformly in $\overline{Q_{R,\nu_E(x)}}$ as $\rho \to 0^+$.

Proof. The cases (a) and (b) follow directly from [58, Proposition A.5]. It remains to prove the case (c) to which the remaining of the proof is devoted. Let $x \in \partial^* E$ and $R > 0$ be such that

$$\overline{Q_{R,\nu_E(x)}(x)} \cap \partial\sigma_{\rho,x}(E) \xrightarrow{\mathcal{K}} \overline{Q_{R,\nu_E(x)}(x)} \cap T_x \text{ as } \rho \to 0^+. \tag{3.4.2}$$

Without loss of generality, we assume that $x = 0, \nu_E(0) = \mathbf{e_2}$, $H_x = H_0$ and $T_x = T_0 = \partial H_0$.

Let $\{\rho_k\} \subset (0,1)$ be such that $\rho_k \to 0$ and let $f_k := \text{sdist}(\cdot, \sigma_{\rho_k}(\partial E))\big|_{Q_R} \in W^{1,\infty}(Q_R)$. We see that for any $k > 0$, f_k is 1-Lipschitz continuous, moreover by the fact that $f_k(0) = \text{sdist}(0, \sigma_{\rho_k}(\partial E)) = 0$, we deduce that $\{f_k\}$ is uniformly bounded. By applying Ascoli-Arzelà Theorem, there exists $f \in W^{1,\infty}(Q_R)$ and a non-relabeled subsequence $\{f_k\}$ such that $f_k \to f$ uniformly in Q_R. In view of [58, Proposition A.1], by (3.4.2) we obtain that $|f_k| = \text{dist}(\cdot, \sigma_{\rho_k}(\partial E)) \to \text{dist}(\cdot, T_0)$ uniformly in Q_R and thus, $|f(x)| = \text{dist}(x, \partial H_0)$ for any $x \in Q_R$.

It remains to prove that $f(\cdot) = \text{sdist}(\cdot, \partial H_0)$ in Q_R. We proceed by absurd. Assume by contradiction that $f \neq \text{sdist}(\cdot, \partial H_0)$ then, either $f \equiv \text{sdist}(\cdot, \partial(Q_R \setminus H_0))$ or $f \equiv \text{dist}(x, T_0)$ or $f \equiv -\text{dist}(x, T_0)$. Let us first consider the case in which $f \equiv \text{sdist}(\cdot, \partial(Q_R \setminus H_0))$. In view of Remark 3.2.8-(i), it follows that $\sigma_{\rho_k}(E) \xrightarrow{\mathcal{K}} Q_R \setminus \text{Int}(H_0)$ and so, as a consequence we have that $\mathcal{L}^2(\sigma_{\rho_k}(E) \triangle (Q_R \setminus \text{Int}(H_0))) \to 0$ as $k \to \infty$, which is in contradiction with the De Giorgi's structure theorem of sets of finite perimeter (see [43, Theorem 5.13] or [68, Theorem 15.5]).

In the case $f(\cdot) \equiv \text{dist}(\cdot, T_0)$, thanks to the fact that $f_k \to f$ uniformly in $\overline{Q_R}$ and by Lemma 3.4.2, we have that $\sigma_{\rho_k}(E) \cap \overline{Q_R} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q_R}$, and thus, $\mathbb{1}_{\sigma_{\rho_k}(E)} \to \mathbb{1}_{T_0}$ in $L^1(\overline{Q_R})$, which is a contradiction with the fact that $\mathbb{1}_{\sigma_{\rho_k}(E)} \to \mathbb{1}_{H_0}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$ by [43, Theorem 5.13].

In the last case in which $f(\cdot) \equiv -\text{dist}(\cdot, T_0)$, we proceed analogously, and by Lemma 3.4.2 we obtain that $\overline{Q_R} \setminus \sigma_{\rho_k}(E) \xrightarrow{\mathcal{K}} T_0$ and hence, $\mathbb{1}_{(\overline{Q_1} \setminus \sigma_{\rho_k}(E))} \to \mathbb{1}_{T_0}$ in $L^1(\overline{Q_R})$. Therefore, we reach a contradiction again by applying [43, Theorem 5.13] since $x \in \partial^*(\mathbb{R}^2 \setminus E)$ and so, $\mathbb{1}_{(\overline{Q_R} \setminus \sigma_{\rho_k}(E))} \to \mathbb{1}_{\mathbb{R}^2 \setminus \text{Int}(H_0)}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$. ☐

We now introduce the notions of *film free boundary*, *substrate free boundary*, and *film-substrate adhesion interface* for triples $(A, h, K) \in \mathcal{B}$, and of *triple junctions* at the points where they "meet".

Definition 3.4.4. For any admissible pair $(A, S) \in \widetilde{\mathcal{B}}$ we denote:

- the *film free boundary*, the *substrate free* boundary and the *film-substrate adhesion interface* by

$$\begin{aligned}
\Gamma_{\text{F}}(A, S) &:= \Big((\partial A \setminus \partial S) \cup (\partial S \cap \partial A \cap A^{(1)}) \cup (\partial S \cap \partial^* A \cap S^{(0)})\Big) \cap \Omega, \\
\Gamma_{\text{S}}(A, S) &:= \Big((\partial S \cap \partial A) \setminus (\partial S \cap \partial^* A \cap S^{(0)})\Big) \cap \Omega, \\
\Gamma^{\text{A}}_{\text{FS}}(A, S) &:= \Big(\big((\partial S \setminus \partial A) \cap A^{(1)}\big)\Big) \cap \Omega,
\end{aligned}$$

respectively. Notice that the *film-substrate delamination interface*, that we define by

$$\Gamma^{\mathrm{D}}_{\mathrm{FS}}(A,S) := \Big((\partial S\cap \partial A)\cap A^{(1)}\Big)\cap\Omega,$$

is contained both in $\Gamma_{\mathrm{F}}(A,S)$ and in $\Gamma_{\mathrm{S}}(A,S)$.

- *triple junction* (by including for simplicity also the "double" junctions at the boundary) any point

$$\begin{aligned} p\in &\Big(Cl(\Gamma_{\mathrm{F}}(A,S))\cap Cl(\Gamma_{\mathrm{S}}(A,S))\cap Cl(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A,S))\cap\Omega\Big)\\ &\cup\Big(Cl(\Gamma_{\mathrm{F}}(A,S))\cap Cl(\Gamma_{\mathrm{S}}(A,S))\cap\partial\mathrm{Int}(\overline{S})\cap\partial\Omega)\Big)\\ &\cup\Big(Cl(\Gamma_{\mathrm{F}}(A,S))\cap Cl(\Gamma^{A}_{\mathrm{FS}}(A,S))\cap\partial\Omega)\Big), \end{aligned}$$

where the closures are considered with respect to the relative topology of $\partial A\cup\partial S$.

The next result allows us to assume that the adhesion interface of any admissible pair $(A,S)\in\mathcal{B}_{\mathbf{m}}$ (without the substrate internal cracks) can be considered, up to an error and up to passing to the family $\widetilde{\mathcal{B}}$, to be given by a finite number (depending on the initial pair (A,S)) of connected components.

Lemma 3.4.5. *Let R be an open rectangle with two sides, that are denoted by T_1 and T_2, perpendicular to $\boldsymbol{e_1}$. Let $(A,S_{h,K})\in\mathcal{B}_{\boldsymbol{m}}$ for $(h,K)\in$ AHK be such that $\mathcal{S}_L(A,S_{h,K},R)<\infty$, where $\mathcal{S}_L$ is the localized surface energy defined in* (3.4.1). *If $\mathcal{H}^1\left(\left(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A,S_{h,K})\setminus Int(S_h)\right)\cap R\right)>0$, for every $\eta\in(0,1)$ small enough there exist $M:=M(A,S_{h,K},\eta)\in\mathbb{N}\cup\{0\}$ and $(\widetilde{A},\widetilde{S})\in\widetilde{\mathcal{B}}$ such that $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widetilde{A},\widetilde{S})\setminus Int(\overline{\widetilde{S}}))\cap R$ has at most M connected components and*

$$\mathcal{S}_L(A,S_{h,K},R)\geq\mathcal{S}_L(\widetilde{A},\widetilde{S},R)-\eta. \tag{3.4.3}$$

Furthermore, $(\widetilde{A},\widetilde{S})$ satisfies the following properties:

(i) If $\overline{\partial S_h\cap R}\cap T_\ell\neq\emptyset$ for $\ell=1,2$, then also $\overline{\partial\widetilde{S}\cap R}\cap T_\ell\neq\emptyset$ for $\ell=1,2$;

(ii) If there exists a closed connected set $\Lambda\subset\overline{\partial A\cap R}$ such that $\Lambda\cap T_\ell\neq\emptyset$ for $\ell=1,2$, then there exists a curve with support $\widetilde{\Lambda}\subset\overline{\partial\widetilde{A}\cap R}$ such that $\widetilde{\Lambda}\cap T_\ell\neq\emptyset$ for $\ell=1,2$.

Proof. Notice that we cannot a priori exclude that $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A,S_{h,K})\setminus\mathrm{Int}(S_h))\cap R$ is a totally disconnected set with positive $\mathcal{H}^1$-measure (see, for instance, the *Smith-Volterra-Cantor set* in [76, Chapter 3]). We denote by $L(h)$ the set of substrate filaments of the substrate $S_{h,K}$, namely,

$$L(h):=\{(x_1,x_2)\in\overline{\Omega}:x_1\in(-l,l)\text{ and }h^+(x_1)<x_2\leq h(x_1)\}\subset\partial S_h. \tag{3.4.4}$$

Since h is upper semicontinuous, there exist an index set J_1 and a countable family of disjoint points $\{x_1^j\}_{j\in J_1}\subset(-l,l)$ such that

$$L(h)=\bigcup_{j\in J_1}L_j(h), \tag{3.4.5}$$

where $L_j(h):=\{(x_1^j,x_2)\in\overline{\Omega}:h^+(x_1^j)<x_2\leq h(x_1^j)\}$ for every $j\in J_1$. In the following three steps, in view of the outer regularity of Borel measures, we construct an admissible pair $(\widetilde{A},\widetilde{S})\subset\widetilde{\mathcal{B}}$ by modifying some portions of ∂S_h and ∂A. More precisely, in the first step, we construct an admissible height h^1 by eliminating a family of "small" filaments of $L(h)$ so that $L(h^1)$ consists of only a finite number of filaments, we accordingly modify A in an admissible region A^1 containing $L(h)\setminus L(h^1)$, and we define $K^1:=K$. In the second step, we construct S^2 by modifying S_{h^1,K^1} and we introduce an admissible region A^2 in such a way that $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2}))\cap R$ is a countable

union of connected components, and (i) and (ii) hold true. In the third step, by eliminating some components of $\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^2,S^2)\setminus \mathrm{Int}(\overline{S^2}))\cap R$ we define an admissible pair $(\widetilde{A},\widetilde{S})\in\widetilde{\mathcal{B}}$ for which $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widetilde{A},\widetilde{S})\setminus \mathrm{Int}(\overline{\widetilde{S}}))\cap R$ has at most M-connected components, (i) and (ii) are preserved, and (3.4.3) holds true.

Step 1 (Modification of substrate filaments). We modify $(A,S_{h,K})$ in (A^0,S_{h^1,K^1}) to have a finite number of substrate filaments. We denote by $J_2\subset J_1$ the set of indexes $j\in J_1$ such that $\mathcal{H}^1(L_j\setminus \mathrm{Int}(A))=0$ and L_j is not connected to ∂A, and we denote by $F_{J_2}(h)$ the set of the x_1-coordinates corresponding to the points in each vertical segment L_j for $j\in J_2$, i.e., $F_{J_2}(h):=\{x_1^j\}_{j\in J_2}$, where $x_1^j\in[-l,l]$ is such that $(x_1^j,h(x_1^j))\in L_j(h)$ and $h^+(x_1^j)<h(x_1^j)$ for every $j\in J_2$. We define as h^0 the modification of h, given by

$$\begin{aligned} h^0:\ [-l,l]\ &\to\ [0,L]\\ x\ &\mapsto\ h^0(x)\ :=\begin{cases} h(x) & \text{if } x\in[-l,l]\setminus F_{J_2}(h),\\ h^+(x) & \text{if } x\in F_{J_2}(h),\end{cases}\end{aligned}$$

and observe that by construction $h^0\in\mathrm{AH}$ and

$$\mathcal{H}^1(\partial S_{h^0})\le\mathcal{H}^1(\partial S_h),\tag{3.4.6}$$

where S_{h^0} is defined as in (3.2.2). We notice that the triple $(A,S_{h^0,K^0})\in\mathcal{B}_{\mathbf{m}}$, where $K^0:=K$. As a consequence of the construction and of the non-negativity of ϕ_{FS}, it follows that

$$\mathcal{S}_L(A,S_{h,K},R)\ge\mathcal{S}_L(A,S_{h^0,K^0},R).\tag{3.4.7}$$

We notice that by (3.4.5) we have that

$$\mathcal{H}^1(L(h^0))=\mathcal{H}^1(L(h))-\sum_{j\in J_2}\mathcal{H}^1(L_j(h))=\sum_{j\in J_3}\mathcal{H}^1(L_j(h)),$$

where $J_3:=J_1\setminus J_2$ and $L(h^0)$ is the set of substrate filaments of S_{h^0,K^0} defined as in (3.4.4). Therefore, for a fix $\widetilde{\eta}>0$ there is $j_1':=j_1'(\widetilde{\eta})\in J_3$ such that

$$\sum_{j=j_1'+1,j\in J_3}^{\infty}\mathcal{H}^1(L_j(h))\le\widetilde{\eta},\tag{3.4.8}$$

and we define

$$A^1:=\left(A\setminus\left(\bigcup_{j=j_1'+1,j\in J_3}^{\infty}(L_j(h)\cap\mathrm{Int}(A))\right)\right)\cup\bigcup_{j=j_1'+1,j\in J_3}^{\infty}(L_j(h)\cap(\Omega\setminus\overline{\mathrm{Int}(A)})).$$

Furthermore, we denote by h^1 the modification of h^0, defined by

$$\begin{aligned} h^1:\ [-l,l]\ &\to\ [0,L]\\ x\ &\mapsto\ h^1(x)\ :=\begin{cases} h^0(x) & \text{if } x\in[-l,l]\setminus F_{J_4}(h^0),\\ h^{0+}(x) & \text{if } x\in F_{J_4}(h^0),\end{cases}\end{aligned}$$

where $J_4:=\{j\in J_3: j\ge j_1'+1\}$ and $F_{J_4}(h^0):=\{x_1^j\}_{j\in J_4}$ such that $(x_1^j,h(x_1^j))\in L_j(h)$ for every $j\in J_4$. We define $K^1:=K^0$, we notice that $(h^1,K^1)\in\mathrm{AHK}$ and $S_{h^1,K^1}\subset\overline{A^1}$, thus $(A^1,S_{h^1,K^1})\in\mathcal{B}_{\mathbf{m}}$ and since $L_j(h)$ is connected to ∂A for every $j\in J_4\subset J_3$, we deduce that ∂A^1 has at most m_1-connected components. Finally, we observe that

$$\mathcal{S}_L(A,S_{h,K},R)-\mathcal{S}_L(A^1,S_{h^1,K^1},R)\ge-\sum_{j\in J_4}2\int_{L_j(h)}\phi_{\mathrm{F}}(\nu_{L_j(h)})d\mathcal{H}^1\ge-2c_2\widetilde{\eta},\tag{3.4.9}$$

where we used the non-negativeness of ϕ, ϕ_{FS} and (3.2.15), and we observe that $(A^1, S_{h^1,K^1}) \in \mathcal{B}_{\mathbf{m}}$ has a finite number of substrate filaments, more precisely, we denote by $j' \in \mathbb{N}$ the cardinality of the index set $J := J_3 \setminus J_4 = \{j \in J_3 : j \leq j_1'\}$ and we have that $L(h^1)$ is the union of j' filaments, i.e.,

$$L(h^1) = \bigcup_{j \in J} L_j(h), \tag{3.4.10}$$

where $L(h^1)$ is defined as in (3.4.4) with respect to the substrate $S_{h^1,K^1} \in \mathrm{AS}(\Omega)$.

Step 2 (Modification of the substrate free boundary). Without loss of generality in the following we assume that $\partial\mathrm{Int}(S_{h^1}) \cap \overline{L(h^1)} \subset R$. Since $\mathcal{H}^1 \llcorner \partial S_{h^1,K^1}$ is a finite Borel measure and

$$\Gamma^{\mathrm{A}} := \left(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^1, S_{h^1,K^1}) \setminus \mathrm{Int}(S_{h^1})\right) \cap R$$

is a Borel set, by the outer regularity of measures (see [68, Theorem 2.10]), there exists an open set $O = O(\widetilde{\eta}) \subset R$ such that $\Gamma^{\mathrm{A}} \subset O \cap \partial S_{h^1,K^1}$ and

$$\mathcal{H}^1\left(\widehat{\Lambda}\right) = \mathcal{H}^1\left((O \cap \partial^* S_{h^1,K^1}) \setminus \Gamma^{\mathrm{A}}\right) \leq \mathcal{H}^1\left((O \cap \partial S_{h^1,K^1}) \setminus \Gamma^{\mathrm{A}}\right) < \frac{2^{-5/2}}{j'+1}\widetilde{\eta}, \tag{3.4.11}$$

where $\widehat{\Lambda} := (O \cap \partial\mathrm{Int}(\overline{S_{h^1,K^1}})) \setminus \Gamma^{\mathrm{A}}$ and j' is defined in the Step 1 as the number of filaments of S_{h^1}. Moreover, by using the notation introduced in (3.2.2) and the fact that $h^1 \in \mathrm{AH}(\Omega)$ we conclude that

$$\partial\mathrm{Int}\left(\overline{S_{h^1,K^1}}\right) = \partial\mathrm{Int}(S_{h^1}) = \partial S_{{h^1}^+}, \tag{3.4.12}$$

and hence, since $\partial\mathrm{Int}(\overline{S_{h^1,K^1}})$ is a connected and compact set in $\mathbb{R}^2$ with finite $\mathcal{H}^1$-measure, by [44, Lemma 3.12] there exists a parametrization $r : [0,1] \to \mathbb{R}^2$ of $\partial\mathrm{Int}(\overline{S_{h^1,K^1}})$ whose support γ joins the points $(-l, h^1(-l^+))$ with $(l, h^1(l^-))$.

Notice that by Step 1, $\partial\mathrm{Int}(S_{h^1}) \cap \overline{L(h^1)}$ is the union of j'-points that we can order by labeling them with $p_1, \dots, p_{j'}$. Furthermore, we denote with $p_0 := r(t_0)$ and $p_{j'+1} := r(t_1)$, where $t_0 := \inf\{t \in [0,1] : r(t) \in \partial R\}$ and $t_1 := \sup\{t \in [0,1] : r(t) \in \partial R\}$, and we consider the family $\{R^i\}_{i=1}^{j'+1}$ of the strips R^i defined as the open regions of R contained between the vertical lines passing though the points p_{i-1} and p_i.

Since $\widehat{\Lambda}$ is a Borel measurable set, by (2.2.3) and (3.4.11) we have that

$$\mathcal{N}^1(\widehat{\Lambda}) \leq 2^{\frac{5}{2}}\mathcal{H}^1(\widehat{\Lambda}) < \frac{\widetilde{\eta}}{j'+1}, \tag{3.4.13}$$

where $\mathcal{N}^1$ is the net measure defined in (2.2.1). Therefore, there exists $\delta > 0$ such that we can find a family of disjoint dyadic squares $\{U_n\}_{n \in \mathbb{N}} \subset \mathcal{Q}$ such that $\widehat{\Lambda} \subset \bigcup_{n \in \mathbb{N}} U_n$, $\mathrm{diam}(U_n) \leq \delta$ for any $n \in \mathbb{N}$ and

$$\sum_{n \in \mathbb{N}} \mathrm{diam}(U_n) \leq \mathcal{N}^1(\widehat{\Lambda}) + \frac{\widetilde{\eta}}{j'+1} < \frac{2}{j'+1}\widetilde{\eta}. \tag{3.4.14}$$

Without loss of generality we assume that ($\widehat{\Lambda} \neq \emptyset$ and that) $U_n \cap \widehat{\Lambda}$ is non-empty for every $n \in \mathbb{N}$. Let $\{U_n^i\} \subset \{U_n\}$ be the subfamily of dyadic squares such that $U_n^i \cap R^i \cap \gamma \neq \emptyset$ for every $n \in \mathbb{N}$. Furthermore, we assume for simplicity that $\mathrm{Int}(U_n^i) \subset R^i$. We begin by modifying the pair (A^1, S_{h^1,K^1}) in the strip R^1, by denoting the modification by (A_1^2, S_1^2). We characterize A_1^2 and S_1^2 as

$$A_1^2 := \left(A^1 \cup \bigcup_{n \in \mathbb{N}} U_n^1\right) \setminus \bigcup_{n \in \mathbb{N}} (\partial U_n^1 \setminus U_n^1) \tag{3.4.15}$$

and

$$S_1^2 := \left(S_{h^1,K^1} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{Int}(U_n^1) \right) \cup \bigcup_{n\in\mathbb{N}} (\partial U_n^1 \cap (R^1 \setminus S_{h^1})), \tag{3.4.16}$$

respectively. By construction, it follows that $S_1^2 \subset \overline{A_1^2}$, ∂A_1^2 and ∂S_1^1 have finite $\mathcal{H}^1$ measure and are $\mathcal{H}^1$-rectifiable, and $\partial A_1^2 \cap \mathrm{Int}(\overline{S_1^2}) = \emptyset$ and hence, $(A_1^2, S_1^2) \in \widetilde{\mathcal{B}}$. Furthermore, we have that

$$\begin{aligned} &\mathcal{S}_L\left(A^1, S_{h^1,K^1}, R^1\right) - \mathcal{S}_L\left(A_1^2, S_1^2, R^1\right) \\ &\geq -2\int_{\bigcup_{n\in\mathbb{N}} \partial U_n^1} \phi_{\mathrm{F}}(\nu_{U_n^1}) + \phi(\nu_{U_n^1}) + \phi_{\mathrm{FS}}(\nu_{U_n^1})\, d\mathcal{H}^1 \\ &\geq -\sum_{n\in\mathbb{N}} 2\int_{\partial U_n^1} \phi_{\mathrm{F}}(\nu_{U_n^1}) + \phi(\nu_{U_n^1}) + \phi_{\mathrm{FS}}(\nu_{U_n^1})\, d\mathcal{H}^1 \\ &\geq -\sum_{n\in\mathbb{N}} 24\, c_2 \mathrm{diam}(U_n^1) \geq -\frac{48}{j'+1} c_2 \widetilde{\eta}, \end{aligned} \tag{3.4.17}$$

where in the first inequality we used the non-negativeness of ϕ_{F}, ϕ_{FS} and ϕ, in the second inequality we used the subadditivity of measures, in the third inequality we used (3.2.15), and in the last inequality we used (3.4.14). We notice that $\Gamma_{\mathrm{FS}}^{\mathrm{A}}(A_1^2, S_1^2) \cap \partial\mathrm{Int}(\overline{S_1^2}) \cap R^1$ is a countable union of connected sets because by construction every connected component of $\Gamma_{\mathrm{FS}}^{\mathrm{A}}(A_1^2, S_1^2) \cap \partial\mathrm{Int}(\overline{S_1^2}) \cap R^1$ is connected to an element in the family of sets $\{\partial U_n^1 \cap U_n^1\}_{n\in\mathbb{N}}$.

Now, we modify (A_1^2, S_1^2) in a new configuration (A_1^3, S_1^3) in order to prove Assertion (ii). To this end let $\Lambda \subset \overline{\partial A \cap R}$ be a closed connected set such that $\Lambda \cap T_\ell \neq \emptyset$ for $\ell = 1, 2$. By [44, Lemma 3.12] there exists a parametrization $r_1 : [0,1] \to \mathbb{R}^2$ whose support $\gamma^1 \subset \Lambda$ joins T_1 with T_2. We define $\gamma_1^1 := \gamma^1 \cap \overline{R^1}$ and we observe that $(\gamma_1^1 \setminus \mathrm{Int}(U_n^1)) \cup \partial U_n^1$ is a connected set. Let $Z_1 \subset \mathbb{N}$ be the set of indexes n such that $\gamma_1 \cap (\partial U_n^1 \cap U_n^1) \neq \emptyset$. If $Z_1 = \emptyset$, then we define $A_1^3 := A_1^2$ and $S_1^3 := S_1^2$. If $Z_1 \neq \emptyset$, then we modify γ_1^1 in $\bigcup_{n\in Z_1} U_n^1$ by defining a new set Λ_1. More precisely, by using the fact that dyadic squares by definition do not intersect each other, we fix $n \in Z_1$ and we replace with a set Λ_n^1 (see (3.4.24) below) the portion of γ^1 passing through U_n^1. To this end, let us denote the closures of the left and bottom sides of U_n^1 by L_n^1 and L_n^2, respectively, and proceed by defining Λ_n^1 in different way with respect to following three cases (see Figure 3.2):

Case 1 $\boldsymbol{\gamma_1^1 \cap L_n^1 \neq \emptyset}$ **and** $\boldsymbol{\gamma_1 \cap L_n^2 = \emptyset}$**.** Since L_n^1 is closed, we deduce that $\gamma_1^1 \cap L_n^1$ is closed. Therefore, there exist $a_n^2 := \max\{a \in \mathbb{R}; (l_n^1, a) \in \gamma_1^1 \cap L_n^1\}$ and $b_n^2 := \min\{b \in \mathbb{R}; (l_n^1, b) \in \gamma_1^1 \cap L_n^1\}$, where l_n^1 is the element in the singleton $\pi_1(L_n^1)$. Since γ_1 is parametrized by r_1, there exist $t_n^1, t_n^2 \in [0,1]$ such that $p_{n,1}^1 := (l_n^1, a_n^2) = r_1(t_n^1)$ and $p_{n,1}^2 := (l_n^1, b_n^2) = r_1(t_n^2)$, and by the continuity of r_1 there exists $q_{n,1}^1 \in \gamma_1^1 \setminus U_n^1$ such that $\mathrm{dist}(p_{n,1}^1, q_{n,1}^1) \leq \frac{\mathrm{diam}(U_n^1)}{2}$. If $\pi_1(q_{n,1}^1) = l_n^1$, then we define $\widetilde{q}_{n,1}^1 := (l_n^1 - \varepsilon, \pi_2(q_{n,1}^1) - \varepsilon)$ for a $\varepsilon > 0$ small enough such that $\mathrm{dist}(q_{n,1}^1, \widetilde{q}_{n,1}^1) \leq \frac{\mathrm{diam}(U_n^1)}{2}$, otherwise we let $\widetilde{q}_{n,1}^1 := q_{n,1}^1$. We denote by $\widetilde{L}_{n,1}^1$ the segment connecting $q_{n,1}^1$ with $\widetilde{q}_{n,1}^1$, we denote by $\widetilde{L}_{n,1}^2$ the segment connecting $\widetilde{q}_{n,1}^1$ with $p_{n,1}^2$, and we denote by $\widetilde{L}_{n,1}^3$ the segment connecting $p_{n,1}^1$ with the vertex of U_n^1 in $\partial U_n^1 \setminus U_n^1$. Let $\Lambda_n^1 := \widetilde{L}_{n,1}^1 \cup \widetilde{L}_{n,1}^2 \cup \widetilde{L}_{n,1}^3$ and observe that by construction it follows that

$$\mathcal{H}^1(\Lambda_n^1) = \mathcal{H}^1(\widetilde{L}_{n,1}^1 \cup \widetilde{L}_{n,1}^2 \cup \widetilde{L}_{n,1}^3) \leq 2\,\mathrm{diam}(U_n^1). \tag{3.4.18}$$

Case 2 $\boldsymbol{\gamma_1^1 \cap L_n^1 = \emptyset}$ **and** $\boldsymbol{\gamma_1 \cap L_n^2 \neq \emptyset}$**.** By arguing analogously to Case 1 there exist $t_n^1, t_n^2 \in [0,1]$ such that $p_{n,2}^1 := (a_n^1, l_n^2) = r_1(t_n^1)$ and $p_{n,2}^2 := (b_n^1, l_n^2) = r_1(t_n^2)$, where $\pi_2(L_n^2) = \{l_n^2\}$, $a_n^1 := \max\{a \in \mathbb{R}; (a, l_n^2) \in \gamma_1^1 \cap L_n^2\}$, and $b_n^1 := \min\{b \in \mathbb{R}; (b, l_n^2) \in \gamma_1^1 \cap L_n^2\}$. By the continuity of r_1 there exists $q_{n,2}^1 \in \gamma_1^1 \setminus U_n^1$ such that $\mathrm{dist}(p_{n,2}^1, q_{n,2}^1) \leq \frac{\mathrm{diam}(U_n^1)}{2}$. If $\pi_2(q_{n,2}^1) = l_n^2$, then we define $\widetilde{q}_{n,2}^1 := (\pi_1(q_{n,2}^1) - \varepsilon, l_n^2 - \varepsilon)$ for $\varepsilon > 0$ small enough such that

$\operatorname{dist}(q^1_{n,2}, \widetilde{q}^1_{n,2}) \leq \frac{\operatorname{diam}(U^1_n)}{2}$, otherwise we let $\widetilde{q}^1_{n,2} := q^1_{n,2}$. We denote by $\widetilde{L}^1_{n,2}$ the segment connecting $\widetilde{q}^1_{n,2}$ with $p^1_{n,2}$, we denote by $\widetilde{L}^2_{n,2}$ the segment connecting $\widetilde{q}^1_{n,2}$ with $p^2_{n,2}$ and we denote by $\widetilde{L}^3_{n,2}$ the segment connecting $p^1_{n,2}$ with with the vertex of U^1_n in $\partial U^1_n \setminus U^1_n$. Let $\Lambda^1_n := \widetilde{L}^1_{n,2} \cup \widetilde{L}^2_{n,2} \cup \widetilde{L}^3_{n,2}$ and observe that by construction it follows that

$$\mathcal{H}^1(\Lambda^1_n) = \mathcal{H}^1(\widetilde{L}^1_{n,2} \cup \widetilde{L}^2_{n,2} \cup \widetilde{L}^3_{n,2}) \leq 2 \operatorname{diam}(U^1_n). \tag{3.4.19}$$

Case 3 $\boldsymbol{\gamma^1_1 \cap L^1_n \neq \emptyset}$ **and** $\boldsymbol{\gamma_1 \cap L^2_n \neq \emptyset}$**.** We define $p^k_{n,\ell}, q^1_{n,\ell}, \widetilde{q}^1_{n,\ell}, \widetilde{L}^\alpha_{n,\ell}$ for $k = 1, 2$, $\alpha = 1, 2, 3$ as in Case 1 for $\ell = 1$ and as in Case 2 for $\ell = 2$. Furthermore, we denote by $\widetilde{L}^4_n$ the segment connecting $p^2_{n,1}$ with $p^2_{n,2}$. Let $\Lambda^1_n := \widetilde{L}^1_{n,1} \cup \widetilde{L}^2_{n,1} \cup \widetilde{L}^1_{n,2} \cup \widetilde{L}^2_{n,2} \cup \widetilde{L}^3_{n,1} \cup \widetilde{L}^3_{n,2} \cup \widetilde{L}^4_n$ and observe that by construction it follows that

$$\begin{aligned} \mathcal{H}^1(\Lambda^1_n) &\leq \mathcal{H}^1(\widetilde{L}^1_{n,1} \cup \widetilde{L}^2_{n,1} \cup \widetilde{L}^3_{n,1}) + \mathcal{H}^1(\widetilde{L}^1_{n,2} \cup \widetilde{L}^2_{n,2} \cup \widetilde{L}^3_{n,2}) + \mathcal{H}^1(\widetilde{L}^4_n) \\ &\leq 5 \operatorname{diam}(U^1_n). \end{aligned} \tag{3.4.20}$$

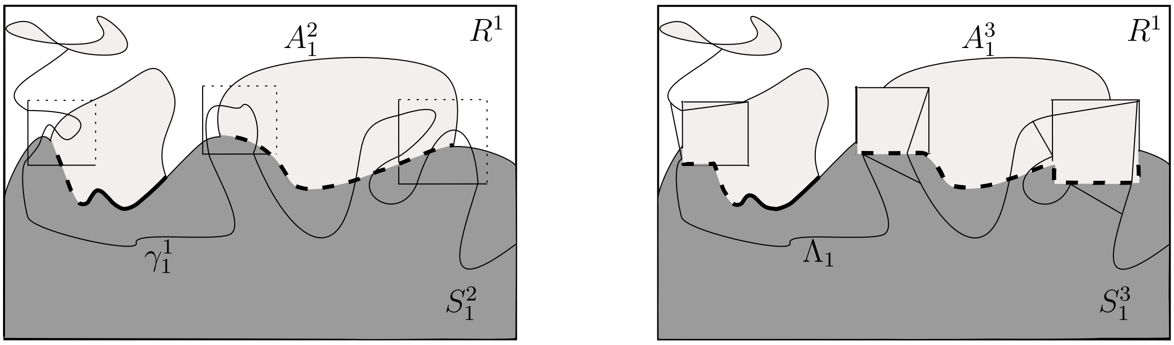

Figure (3.2): The three squares appearing in the illustration represent dyadic squares U^1_n of the first strip R^1 in the three different cases, namely, by moving from the left to the right, Cases 1, 2, and 3, that are considered in Step 2 of the proof of Lemma 3.4.5. On the left the initial pair (A^2_1, S^2_1) is represented, while on the right the pair (A^3_1, S^3_1) that is obtained after the modification described in such step is depicted.

Let

$$\Gamma_1 := \overline{\Big(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\overline{A^2_1}, S^2_1) \cap \partial \operatorname{Int}(\overline{S^2_1}) \cap R^1\Big) \setminus \overline{\bigcup_{n \in Z_1} (U^1_n)}}.$$

We now observe that the previous construction of $\bigcup_{n \in Z_1} \Lambda^1_n$ does not divide Γ_1 in an uncountable number of connected components. More precisely, we claim that for every given a connected component $\widehat{\Gamma}$ of Γ_1, $\widehat{\Gamma} \setminus \overline{\bigcup_{n \in Z_1} \Lambda^1_n}$ is a countable union of disjoint connected sets. To prove this claim, we notice that, since $\overline{\widehat{\Gamma}} \subset \gamma$ and γ is parameterized by r, also $\overline{\widehat{\Gamma}}$ is parametrizable and hence, there exists a continuous injective map $\hat{r} : [0, 1] \to \mathbb{R}^2$ whose support coincides with $\overline{\widehat{\Gamma}}$. This in particular proves that $\hat{r}$ is a homeomorphism between $[0, 1]$ and $\overline{\widehat{\Gamma}}$. The claim then follows from the fact that $\overline{\widehat{\Gamma}} \setminus \overline{\bigcup_{n \in Z_1} \Lambda^1_n}$ is open with respect to the relative topology of $\overline{\widehat{\Gamma}}$ and [76, Proposition 8 in Part 1].

We are now in the position to define A^3_1 and S^3_1 as follows

$$\begin{aligned} A^3_1 := \overline{A^2_1} \setminus \Bigg(\Bigg(\gamma^1 \setminus \bigcup_{n \in Z_1} U^1_n \Bigg) \cup \bigcup_{n \in \mathbb{N}} (\partial U^1_n \setminus U^1_n) \cup \bigcup_{n \in Z_1} (\overline{A^2_1} \cap \Lambda^1_n) \Bigg) \\ \cup \bigcup_{n \in Z_1} ((R^1 \setminus \operatorname{Int}(A^2_1)) \cap \Lambda^1_n) \end{aligned} \tag{3.4.21}$$

and

$$S_1^3 := S_1^2 \setminus \left(\bigcup_{n\in Z_1} (\overline{A_1^2} \cap \Lambda_n^1) \right). \tag{3.4.22}$$

Therefore, we have that

$$\begin{aligned} \mathcal{S}_L\left(A_1^2, S_1^2, R^1\right) - \mathcal{S}_L\left(A_1^3, S_1^3, R^1\right) &\geq -2 \int_{\bigcup_{n\in Z_1} \Lambda_n^1} \phi_{\mathrm{F}}(\nu_{A_3^2}) + \phi(\nu_{A_3^2})\, d\mathcal{H}^1 \\ &\geq - \sum_{n\in Z_1} 2 \int_{\Lambda_n^1} \phi_{\mathrm{F}}(\nu_{U_n^1}) + \phi(\nu_{U_n^1})\, d\mathcal{H}^1 \\ &\geq - \sum_{n\in Z_1} 20\, c_2 \mathrm{diam}(U_n^1) \geq -\frac{40}{j'+1} c_2 \widetilde{\eta}, \end{aligned} \tag{3.4.23}$$

where in the first inequality we used the non-negativeness of $\phi_{\mathrm{F}}, \phi_{\mathrm{FS}}$ and ϕ, in the second inequality we used the subadditivity of measures, in the third inequality we used (3.2.15) and (3.4.18)–(3.4.20), and in the last inequality we used (3.4.14).

Moreover, by construction we have that

$$\Lambda_1 := \left(\gamma^1 \setminus \left(\bigcup_{n\in Z_1} U_n^1 \right) \right) \cup \bigcup_{n\in Z_1} \Lambda_n^1 \subset \partial A_1^3 \tag{3.4.24}$$

is closed, connected, and joins T_1 with T_2.

We now modify the pair (A_1^3, S_1^3) and Λ_1 in the strip R^2, first by employing the same construction of (3.4.15) and (3.4.16) to obtain a configuration $(A_2^2, S_2^2) \in \widetilde{\mathcal{B}}$, and then by employing the same construction of (3.4.21) and (3.4.22) to modify the pair (A_2^2, S_2^2), by denoting the final modified pair with $(A_2^3, S_2^3) \in \widetilde{\mathcal{B}}$. We then define

$$\Lambda_2 := \left(\Lambda_1 \setminus \left(\bigcup_{n\in Z_2} U_n^2 \right) \right) \cup \bigcup_{n\in Z_2} \Lambda_n^2 \subset \partial A_2^3. \tag{3.4.25}$$

By iterating the same procedure on the strips R^i for $i = 3, \dots, j'+1$ we obtain the pair $(A^2, S^2) := (A_{j'+1}^3, S_{j'+1}^3) \in \widetilde{\mathcal{B}}$ and we define $\Lambda_{j'+1}$ as in (3.4.25) by replacing all the index 1 and 2 with j' and $j'+1$, respectively. We observe that by [44, Lemma 3.12] there exists a support $\widetilde{\Lambda} \subset \Lambda_{j'+1}$ of a curve joining T_1 with T_2.

Furthermore, as done in (3.4.23) for i=1,

$$\mathcal{S}_L\left(A_i^2, S_i^2, R^i\right) - \mathcal{S}_L\left(A_i^3, S_i^3, R^i\right) \geq -\frac{40}{j'+1} c_2 \widetilde{\eta} \tag{3.4.26}$$

for every $i = 1, \dots, j'+1$. Therefore, by iteration it follows that

$$\begin{aligned} \mathcal{S}_L\left(A^1, S_{h^1,K^1}, R\right) - \mathcal{S}_L\left(A^2, S^2, R\right) &\geq \sum_{i=1}^{j'+1} \Big(\mathcal{S}_L\left(A^1, S_{h^1,K^1}, R^i\right) - \mathcal{S}_L\left(A_i^2, S_i^2, R^i\right) \\ &\qquad + \mathcal{S}_L\left(A_i^2, S_i^2, R^i\right) - \mathcal{S}_L\left(A_i^3, S_i^3, R^i\right) \Big) \\ &\geq - \sum_{i=1}^{j'+1} \frac{88}{j'+1} c_2 \widetilde{\eta} \\ &\geq -88 c_2 \widetilde{\eta}, \end{aligned} \tag{3.4.27}$$

where in the second inequality we used (3.4.17) and (3.4.26). We notice that $\Gamma_{\mathrm{FS}}^{\mathrm{A}}(A^2, S^2) \cap \partial \mathrm{Int}(\overline{S^2}) \cap R$ is a countable union of connected sets because by construction every connected component of $\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^2,S^2)\cap\partial\mathrm{Int}(\overline{S^2})\cap R$ is connected to an element in the family of sets $\{\partial U_n\cap U_n\}_{n\in\mathbb{N}}$. Therefore, $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2}))\cap R$ is equal to a countable union of connected sets. More precisely, there exists a family of connected and disjoint sets $\{\widetilde{\Gamma}_i\}_{i\in\mathbb{N}}$ such that

$$(\Gamma_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2}))\cap R=\bigcup_{i\in\mathbb{N}}\widetilde{\Gamma}_i, \tag{3.4.28}$$

and hence,

$$\mathcal{H}^1\left((\Gamma_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2})\cap R\right)=\sum_{i\in\mathbb{N}}\mathcal{H}^1\left(\widetilde{\Gamma}_i\right). \tag{3.4.29}$$

We conclude this step by observing that Assertion (i) follows by the construction of S^2, while Assertion (ii) holds with the defined set $\widetilde{\Lambda}$.

Step 3 (From countable to a finite number of components). Since $\mathcal{H}^1((\Gamma_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2}))\cap R)\leq\mathcal{H}^1(\partial S^2)<\infty$, by (3.4.29) there exists $i_0:=i_0(\widetilde{\eta})\in\mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty}\mathcal{H}^1(\widetilde{\Gamma}_i)\leq\widetilde{\eta}. \tag{3.4.30}$$

Notice that $(\widetilde{A},\widetilde{S})\in\widetilde{\mathcal{B}}$, where $\widetilde{S}:=S^2$ and $\widetilde{A}:=A^2\setminus\bigcup_{i\in\widetilde{I}}\widetilde{\Gamma}_i$ with $\widetilde{I}:=\{i\in\mathbb{N}:i\geq i_0+1\}$. Furthermore, it follows from Steps 1 and 2 that

$$\begin{aligned}\mathcal{S}_L(A,S_{h,K},R)-\mathcal{S}_L(\widetilde{A},\widetilde{S},R)&=\mathcal{S}_L(A,S_{h,K},R)\pm\mathcal{S}_L(A^2,S^2,R)-\mathcal{S}_L(\widetilde{A},\widetilde{S},R)\\&\geq-88c_2\widetilde{\eta}-2\sum_{i=i_0+1}^{\infty}\int_{\widetilde{\Gamma}_i}\phi_{\mathrm{F}}(\nu_{\widetilde{\Gamma}_i})+\phi(\nu_{\widetilde{\Gamma}_i})\,d\mathcal{H}^1\\&\geq-92c_2\widetilde{\eta},\end{aligned} \tag{3.4.31}$$

where in the first inequality we used (3.4.9) and (3.4.27), and the definition of $\widetilde{A}$ and the non-negativeness of ϕ_{FS} and, in the second inequality we used (3.2.15) and (3.4.30). We conclude this step by defining $M\in\mathbb{N}\cup\{0\}$ as the cardinality of $\mathbb{N}\setminus\widetilde{I}$, and we notice by construction that $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widetilde{A},\widetilde{S})\setminus\mathrm{Int}(\overline{\widetilde{S}}))\cap R$ has at most M-connected components.

Finally, we observe that Assertion (i) is a direct consequence of the construction in Steps 1 and 2, while Assertion (ii) follows from the definition of $\widetilde{\Lambda}$ in Step 2 (which is not modified in Step 3 since $\widetilde{\Lambda}\cap(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A^2,S^2)\setminus\mathrm{Int}(\overline{S^2}))=\emptyset$). The proof of this lemma is concluded by taking $\widetilde{\eta}:=\frac{\eta}{92c_2}$ in (3.4.31) with $\eta\in(0,\min\{1,92c_2\})$. □

We formalize below the notions of *film islands*, *composite voids*, and *substrate grains* for any admissible pair $(A,S_{h,K})\in\mathcal{B}$.

Definition 3.4.6. Let $R\subset\Omega$ be an open rectangle and let $(A,S)\in\mathcal{B}$. We refer to:

- any closed component $V\subset\overline{R}$ of $\Omega\setminus\mathrm{Int}(A)$ such that $\partial V\cap(\Gamma_{\mathrm{S}}(A,S)\cup\Gamma^{\mathrm{D}}_{\mathrm{FS}}(A,S))$ is not empty and it consists in one and only one connected component as an *extended* (as we also include "connected delamination regions") *composite void* of the configuration (A,S) (or sometime for simplicity of the film region or of A).
- any open connected component $P\subset R$ of $\mathrm{Int}(A\setminus\overline{S})$ such that $\partial P\cap\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A,S)$ is not empty and it consists in one and only one connected component as a *film island* of the configuration (A,S) (or sometime for simplicity of the film region or of A), and to a film island $P=\mathrm{Int}(A\setminus\overline{S})\cap R$ of (A,S) as the *full island* of A.
- any open connected component $G\subset R$ of $\mathrm{Int}(S)$ such that $\partial G\cap\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A,S)$ is not empty and it consists in one and only one connected component as a *substrate grain* of the configuration (A,S) (or sometime for simplicity of the substrate region or of S), and to a substrate grain $G=\mathrm{Int}(S)\cap R$ of (A,S) as the *full grain* of S.

The following results can be seen as analogous of [58, Lemmas 4.4 and 4.5] though in our more involved setting of three free interfaces (see Table 3.1), where we have to distinguish among the blow-ups at:

- the substrate free boundary (Lemma 3.4.7),
- the film-substrate incoherent (delaminated) interface (Lemma 3.4.8),
- the substrate cracks in the film-substrate incoherent interface (Lemma 3.4.9),
- the filaments of both the substrate and the film (Lemma 3.4.10),
- the substrate filaments on the film free boundary (Lemma 3.4.11),
- the delaminated substrate filaments in the film (Lemma 3.4.12).

The strategy employed in these proofs is based on reducing to the situation of a finite number of connected components for the film-substrate coherent interfaces by Lemma 3.4.5 and then on designing induction arguments (with respect to the index of such components) in which we “shrink" islands, “fill" voids, and modify “grains” in new voids (see Figures 3.3, 3.4, and 3.6, respectively) by means of the *minimality of segments* (see [68, Remark 20.3]).

We begin by addressing the setting of the substrate free boundary.

Lemma 3.4.7. *Let $R \subset \mathbb{R}^2$ be an open rectangle with a side parallel to $\boldsymbol{e_1}$ and let $T \subset \mathbb{R}^2$ be a line such that $T \cap R \neq \emptyset$. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $R \subset \sigma_{\rho_1}(\Omega)$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $\partial S_{h_k,K_k} \cap \overline{R} \xrightarrow{\mathcal{K}} T \cap \overline{R}$ in $\mathbb{R}^2$ and $(\overline{A_k} \setminus Int(S_{h_k,K_k})) \cap \overline{R} \xrightarrow{\mathcal{K}} T \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, then for every $\delta \in (0,1)$ small enough, there exists $k_\delta \in \mathbb{N}$ such that*

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \geq \int_{T\cap\overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - \delta \tag{3.4.32}$$

for any $k \geq k_\delta$.

Proof. We prove (3.4.32) in three steps. In the first step, we prove (3.4.32) for every $k \in \mathbb{N}$ such that $\Gamma_k^{\mathrm{A}} := (\Gamma_{\mathrm{FS}}^{\mathrm{A}}(A_k, S_{h_k,K_k}) \setminus \mathrm{Int}(S_{h_k,K_k})) \cap R$ is $\mathcal{H}^1$-negligible by following the program of [58, Lemma 4.4]. In the second step, we consider those $k \in \mathbb{N}$ such that Γ_k^{A} has $\mathcal{H}^1$-positive measure and observe that in view of Lemma 3.4.5 we can pass, up to a small error in the energy, to a triple $(\widetilde{A}_k, \widetilde{S}_k) \in \mathcal{B}_{\mathbf{m}}$ such that $\widetilde{\Gamma}_k^{\mathrm{A}} := (\Gamma_{\mathrm{FS}}^{\mathrm{A}}(\widetilde{A}_k, \widetilde{S}_k) \setminus \mathrm{Int}(\overline{\widetilde{S}_k})) \cap R$ has M_k connected components, and which then is shown to always admit either an island or a void. Finally, in the third step, we apply the anisotropic minimality of segments to prove (3.4.32) by means of an induction argument based on shrinking the islands and/or filling the voids of the triple $(\widetilde{A}_k, \widetilde{S}_k)$.

If T is a vertical segment we define $c_\theta := 1$, otherwise we define $c_\theta := (1/\sin\theta) + (1/\cos\theta)$, where $\theta < \pi/2$ is the smallest angle formed by the direction of T with $\mathbf{e_1}$. Since $\overline{A_k} \setminus \mathrm{Int}(S_{h_k,K_k}) \cap \overline{R} \xrightarrow{\mathcal{K}} T \cap \overline{R}$, for every $\delta' \in (0,1)$ there exits $k_{\delta'} \in \mathbb{N}$ such that

$$\emptyset \neq \overline{A_k} \setminus \mathrm{Int}(S_{h_k,K_k}) \cap \overline{R} \subset T^{\delta'} \tag{3.4.33}$$

for any $k \geq k_{\delta'}$, where $T^{\delta'} := \{x \in R : \mathrm{dist}(x,T) < \delta'/(2c_\theta)\}$ is the tubular neighborhood of T in R.

Step 1. Assume that $\mathcal{H}^1\left(\Gamma_k^{\mathrm{A}}\right) = 0$ for a fix $k \geq k_{\delta'}$. Since

$$\partial \mathrm{Int}\left(\overline{S_{h_k,K_k}}\right) = \partial \mathrm{Int}(S_{h_k}) = \partial S_{h_k^+}, \tag{3.4.34}$$

by [44, Lemma 3.12] there exists a parametrization $r_k : [0,1] \to \mathbb{R}^2$ of $\partial \mathrm{Int}\left(\overline{S_{h_k,K_k}}\right)$, whose support we denote by γ_k. Notice that by (3.4.33), there exists $p_0 := r_k(t_0)$ and $p_1 := r_k(t_1)$, where $t_0 := \inf\{t \in [0,1] : r_k(t) \in \partial R\}$ and $t_1 := \sup\{t \in [0,1] : r_k(t) \in \partial R\}$, and also by (3.4.34) $t_0 < t_1$. Let $\widetilde{T}_i \subset \partial R \cap \overline{T^{\delta'}}$ for $i = 0, 1$ be the closed and connected set with minimal length connecting p_i with $T \cap \overline{R}$. Therefore, by trigonometric identities we obtain that $\mathcal{H}^1(\widetilde{T}_i) \le \frac{\delta'}{2c_\theta} c_\theta = \frac{\delta'}{2}$. We define $\Lambda_k := (\partial R \cap \overline{T^{\delta'}}) \cap (\sigma_{\rho_k}(\Omega) \setminus \mathrm{Int}(S_{h_k,K_k}))$ and we observe that

$$\begin{aligned}\mathcal{S}_L(A_k, S_{h_k,K_k}, R) + &\int_{\Lambda_k} \phi(\nu_{\Lambda_k})\, d\mathcal{H}^1 \\ \ge &\int_{\gamma_k \cap R \cap \partial^* S_{h_k,K_k} \cap \partial^* A_k} \phi(\nu_{A_k})\, d\mathcal{H}^1 + \int_{\gamma_k \cap R \cap \partial^* S_{h_k,K_k} \cap \partial A_k \cap A_k^{(1)}} \phi(\nu_{A_k})\, d\mathcal{H}^1 \\ &\quad + \int_{\Lambda_k} \phi(\nu_{\Lambda_k})\, d\mathcal{H}^1 \\ = &\int_{\Gamma_k} \phi(\nu_{\Gamma_k})\, d\mathcal{H}^1 - \sum_{i=0}^{1} \int_{\widetilde{T}_i} \phi(\mathbf{e_1})\, d\mathcal{H}^1,\end{aligned} \tag{3.4.35}$$

where $\Gamma_k := \widetilde{T}_0 \cup (\gamma_k \cap \overline{R}) \cup \Lambda_k \cup \widetilde{T}_1$. By the anisotropic minimality of segments (see [68, Remark 20.3]), it yields that

$$\int_{\Gamma_k} \phi(\nu_{\Gamma_k})\, d\mathcal{H}^1 \ge \int_{T \cap \overline{R}} \phi(\nu_T)\, d\mathcal{H}^1, \tag{3.4.36}$$

and so, thanks to the facts that $\mathcal{H}^1(\widetilde{T}_0 \cup \widetilde{T}_1) \le \delta'$ and $\mathcal{H}^1(\Lambda_k) \le \delta'$, and by (3.2.15), (3.4.33)-(3.4.36), we deduce that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \ge \int_{T \cap \overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta'. \tag{3.4.37}$$

Step 2. Assume that $\mathcal{H}^1(\Gamma_k^{\mathrm{A}}) > 0$ for a fixed $k \ge k_{\delta'}$. By applying Lemma 3.4.5, with $\Omega = \mathcal{B}_{\mathbf{m}}(\sigma_{\rho_k}(\Omega))$, there exist $M_k := M_k(\delta', A_k, h_k, K_k) \in \mathbb{N}$ and an admissible triple $(\widetilde{A}_k^1, \widetilde{S}_k) \in \mathcal{B}(\sigma_{\rho_k}(\Omega))$ such that $\widetilde{\Gamma}_k^{\mathrm{A}}$ has M_k connected components and

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \ge \mathcal{S}_L(\widetilde{A}_k^1, \widetilde{S}_k, R) - c_2\delta'. \tag{3.4.38}$$

We consider $\widetilde{A}_k := \widetilde{A}_k^1 \cup ((\partial \widetilde{A}_k^1 \setminus \partial \widetilde{S}_k) \cap (\widetilde{A}_k^1)^{(1)})$. By (3.4.38) and by the non-negativeness of ϕ_{F} we deduce that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \ge \mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, R) - c_2\delta'. \tag{3.4.39}$$

We denote by $\{\Gamma^j\}_{j=1}^{M_k}$ the family of open connected components of $\widetilde{\Gamma}_k^{\mathrm{A}}$. In this step we prove that there exists at least an island or an extended void (see definition in (3.4.6)) in $\widetilde{A}_k$. More precisely, by arguing by contradiction we prove that one of the following two cases always applies: $M_k \ge 1$ and there exists at least an island in $\widetilde{A}_k$, or $M_k \ge 2$ and there exists at least an extended void in $\widetilde{A}_k$.

To this end, assume that the admissible pair $(\widetilde{A}_k, \widetilde{S}_k)$ does not present any island or extended void. We begin by observing that, since $\mathcal{H}^1(\widetilde{\Gamma}_k^{\mathrm{A}}) > 0$, there exist a $j_0 \in \{1, \dots, M_k\}$ and an open connected component F_1 of $\mathrm{Int}(\overline{\widetilde{A}_k} \setminus \overline{\widetilde{S}_k}) \cap R$ such that $\Gamma^{j_0} \subset \partial F_1$. By the contradiction hypothesis, since F_1 cannot be an island of $\widetilde{A}_k$, there exists $j_1 \in \{1, \dots, M_k\} \setminus \{j_0\}$ such that $\Gamma^{j_1} \subset \partial F_1$ and $\Gamma^{j_0} \cap \Gamma^{j_1} = \emptyset$. Furthermore, as by the contradiction hypothesis there cannot be also an extended void between Γ^{j_0} and Γ^{j_1}, and by using the fact that $\partial \overline{\widetilde{S}_k}$ contains Γ^{j_0} and Γ^{j_1}, and consists of only one component, we conclude that there exist an open connected component F_2 of $\mathrm{Int}(\overline{\widetilde{A}_k} \setminus \overline{\widetilde{S}_k}) \cap R$, which could coincide or not with F_1, and at least an extra $j_2 \in \{1, \dots, M_k\} \setminus \{j_0, j_1\}$ such that $\Gamma^{j_2} \subset \partial F_2$.

We now claim that there exist an open connected component F_3 of $\mathrm{Int}(\overline{\widetilde{A}_k} \setminus \overline{\widetilde{S}_k}) \cap R$, which could or not coincide with F_2, and at least an extra $j_3 \in \{1, \dots, M\} \setminus \{j_0, j_1, j_2\}$ such that $\Gamma^{j_3} \subset \partial F_3$.

Indeed, if $F_1 \neq F_2$, the claim is a direct consequence of applying to F_2 the same argument applied to F_1 to find Γ^{j_1}, and we define $F_3 = F_2$, while, if $F_2 = F_1$, the claim is a direct consequence of applying to F_2 with the pair of components consisting, e.g., of Γ^{j_0} and Γ^{j_2}, the same argument applied to F_1 with the pair of components $(\Gamma^{j_0}, \Gamma^{j_1})$ to find Γ^{j_2}, with F_3 being possibly, but not necessary, equal to F_2.

Moreover, the same reasoning applied on F_2, can be implement also on F_3, yielding an open connected component F_4 of $\mathrm{Int}(\overline{\widetilde{A}_k} \setminus \widetilde{S}_k) \cap R$ and at least an extra $j_4 \notin \{j_n\}_{n=0,\dots,3}$ such that $\Gamma^{j_4} \subset \partial F_4$. As such, by keeping on iterating this reasoning we reach a contradiction with the fact that the family of connected components of $\widetilde{\Gamma}^{\mathrm{A}}_k$ consists of at most $M_k < \infty$ elements.

Step 3. In this step we prove (3.4.32) for those $k \geq k_{\delta'}$ such that $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k) > 0$, which together with Step 1 concludes the proof of (3.4.32). More precisely, we prove that

$$\mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \int_{T \cap \overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 6c_2\delta', \tag{3.4.40}$$

which, in view of (3.4.39), yields Assertion (i) by taking $\delta' := \frac{\delta}{7c_2}$ and $k_\delta := k_{\delta'}$ for any $\delta \in (0, \min\{7c_2, 1\})$.

In order to prove (3.4.40) we consider an auxiliary energy $\mathcal{S}^1_L$ in $\mathcal{B}$ by defining

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) := \mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, R) + \sum_{i=1}^{2} \int_{T^k_i} \phi_{\mathrm{F}}(\nu_{\partial R}) + \phi(\nu_{\partial R})\, d\mathcal{H}^1 \tag{3.4.41}$$

for every $(\widetilde{A}_k, \widetilde{S}_k) \in \mathcal{B}_{\mathbf{m}}$, where T^k_1 and T^k_2 are the closed connected components of $\partial R \cap \overline{T^{\delta'}}$, we recall that $\widetilde{\Gamma}^{\mathrm{A}}_k$ has M_k connected components, and we prove that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \int_{T \cap \overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta', \tag{3.4.42}$$

by proceeding by induction on the number $M_k \in \mathbb{N}$ of connected components of $\widetilde{\Gamma}^{\mathrm{A}}_k$ in three steps. Notice that (3.4.40) directly follows from (3.4.42), since

$$\mathcal{H}^1(T^k_i) \leq \delta'$$

by (3.4.33) and the definition of T^k_i and hence,

$$\mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) - 4c_2\delta', \tag{3.4.43}$$

by (3.2.15).

In Substeps 3.1 and 3.2 we prove the basis of the induction by proving it in both the two cases provided by Step 2, i.e., if $M_k = 1$ and $\widetilde{A}_k$ presents an island, and if $M_k = 2$ and $\widetilde{A}_k$ presents an extended void, respectively. Finally in Substep 3.3 we prove the induction and obtain (3.4.42).

Step 3.1. We consider the basis of the induction in the case in which $M_k = 1$ and there is an island $P_1 \subset R$ of $\widetilde{A}_k$ such that $\Gamma^1 \subset \partial P_1$. Let $p_1, p_2 \in \partial P_1$ be two different *triple junctions* of P_1 (see Definition 3.4.4) such that p_1 and p_2 belong to the relative boundary of Γ^1 in $\partial \overline{\widetilde{S}_k}$ and let L_1 by the closed segment connecting p_1 with p_2. It follows that

$$\widehat{\Gamma} := (\partial \overline{\widetilde{S}_k} \setminus \Gamma^1) \cup L_1$$

is connected and closed. We consider by $\{P^1_n\}_{n \in \mathbb{N}}$ the family of open connected components enclosed by L_1 and Γ^1. We are now in the position to characterize the modification $\widehat{S}_k$ of $\widetilde{S}_k$ by

$$\widehat{S}_k := \left(\widetilde{S}_k \setminus \bigcup_{n \in \mathbb{N}, P^1_n \subset \overline{\widetilde{S}_k}} \overline{P^1_n} \right) \cup \bigcup_{n \in \mathbb{N}, P^1_n \subset R \setminus \mathrm{Int}(\widetilde{S}_k)} \overline{P^1_n}.$$

Furthermore, we consider by $\{P_n^2\}_{n\in\mathbb{N}}$ the family of open connected components enclosed by L_1 and $\partial P_1 \setminus \Gamma^1$ such that for every $n \in \mathbb{N}$, $\partial P_n^2 \cap \Gamma^1$ and $\partial P_n^2 \setminus \Gamma^1$ have one non-empty connected component. We define

$$\widehat{A}_k := \left(\widetilde{A}_k \setminus \bigcup_{n\in\mathbb{N}, P_n^2\subset \overline{\widetilde{A}_k}} \overline{P_n^2}\right) \cup \bigcup_{n\in\mathbb{N}, P_n^2\subset R\setminus \mathrm{Int}(\widetilde{A}_k)} \overline{P_n^2}$$

(see Figure 3.3). By applying the anisotropic minimality of segments (see [68, Remark 20.3]), it yields that

$$\begin{aligned}
&\int_{(\partial P_1\setminus\Gamma^1)\cap\partial^*\widetilde{A}_k\setminus\partial\widetilde{S}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 + \int_{(\partial P_1\setminus\Gamma^1)\cap\left(\partial\widetilde{A}_k\setminus\partial\widetilde{S}_k\right)\cap\left(\widetilde{A}_k^{(0)}\cup\widetilde{A}_k^{(1)}\right)} 2\phi_{\mathrm{F}}(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 \\
&\quad + \int_{\Gamma^1\cap\left(\partial^*\widetilde{S}_k\setminus\partial\widetilde{A}_k\right)\cap\widetilde{A}_k^{(1)}} \phi_{\mathrm{FS}}(\nu_{\widetilde{S}_k})\, d\mathcal{H}^1 + \int_{\Gamma^1\cap\left(\partial\widetilde{S}_k\setminus\partial\widetilde{A}_k\right)\cap\widetilde{S}_k^{(0)}\cap\widetilde{A}_k^{(1)}} 2\phi_{\mathrm{FS}}(\nu_{\widetilde{S}_k})\, d\mathcal{H}^1 \\
&\quad + \int_{(\partial P_1\setminus\Gamma^1)\cap\partial\widetilde{S}_k\cap\partial^*\widetilde{A}_k\cap\widetilde{S}_k^{(0)}} \phi_{\mathrm{F}}(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 + \int_{(\partial P_1\setminus\Gamma^1)\cap(T_1^k\cup T_2^k)} \phi_{\mathrm{F}}(\nu_{\partial P_1})\, d\mathcal{H}^1 \\
&\geq \int_{L_1} \phi_{\mathrm{F}}(\nu_{L_1}) + \phi_{\mathrm{FS}}(\nu_{L_1})\ d\mathcal{H}^1 \geq \int_{L_1} \phi(\nu_{L_1})\ d\mathcal{H}^1,
\end{aligned} \tag{3.4.44}$$

where in the last inequality we used the definition of ϕ. We notice that the last term in the left side of the previous inequality is needed to include in the analysis the situation in which $\overline{P_1} \cap \partial R \neq \emptyset$. From (3.4.44), the inequality (3.4.42) directly follows as

$$\mathcal{S}_L^1(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}_L^1(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{T\cap\overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta', \tag{3.4.45}$$

where in the last inequality we used the non-negativeness of ϕ_{F} and we proceeded by applying Step 1 to the configuration $\left(\widehat{A}_k, \widehat{S}_k\right)$ for which by construction it holds that $(\Gamma_{\mathrm{FS}}^{\mathrm{A}}\left(\widehat{A}_k, \widehat{S}_k\right)\setminus\mathrm{Int}(\overline{\widehat{S}_k}))\cap R$ is $\mathcal{H}^1$-negligible.

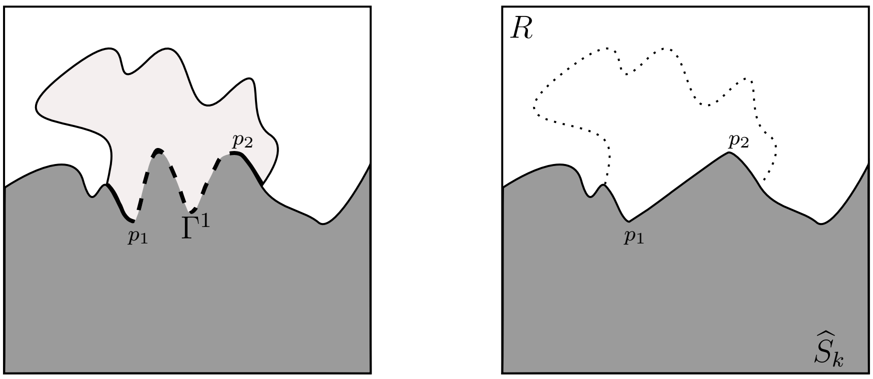

Figure (3.3): The two illustrations above represent, passing from the left to the right, the construction that consists in “shrinking” a film island, which is contained in Step 3.1 of the proof of Lemma 3.4.7 for the basis of the induction in the case with $M = 1$ and with $\widetilde{A}_k$ presenting an island P_1.

Step 3.2. To conclude the basis of the induction, we consider the case with $M = 2$ and the presence of an extended void $V_1 \subset \overline{R}$ of $\widetilde{A}_k$. Let p_1 and p_2 be the two triple junctions such that $p_1, p_2 \in \partial V_1$ and $p_i \in \Gamma^i$, for $i = 1, 2$. By [44, Lemma 3.12] there exists a curve with support $\gamma^1 \subset \partial V_1 \cap \partial\overline{\widetilde{S}_k}$ connecting p_1 with p_2. Furthermore, by [58, Lemma 4.3], since ∂V_1 is connected, $\mathcal{H}^1$-finite and V_1 is bounded, there exists a curve with support $\gamma^2 \subset \overline{\partial V_1 \setminus (\gamma^1 \cap \partial\mathrm{Int}(\overline{V_1})}$. Notice that γ^1 and γ^2 can intersect only in the delamination area, or more precisely $\gamma^1 \cap \partial^*\overline{\widetilde{S}_k} \cap \partial^*\widetilde{A}_k$

and $\gamma^2 \cap \partial^* \widetilde{A}_k \setminus \partial^* \overline{\widetilde{S}_k}$ are disjoint up to a $\mathcal{H}^1$-negligible set. We denote by L_1 the closed segment connecting p_1 with p_2, notice that

$$\widehat{\Gamma} := \overline{(\partial \overline{\widetilde{S}_k} \setminus \gamma^1)} \cup L_1.$$

is connected and closed. We consider by $\{V_n^1\}_{n\in\mathbb{N}}$ the family of open connected components enclosed by L_1 and $\partial V_1 \cap \partial \overline{\widetilde{S}_k}$ such that for every $n \in \mathbb{N}$, $\partial V_n^1 \cap \partial \overline{\widetilde{S}_k}$ and $\partial V_n^1 \cap L_1$ have one non-empty connected component. We characterize the modification $\widehat{S}_k$ of $\widetilde{S}_k$ as

$$\widehat{S}_k := \left(\widetilde{S}_k \setminus \bigcup_{n\in\mathbb{N}, V_n^1 \subset \overline{\widetilde{S}_k}} \overline{V_n^1} \right) \cup \bigcup_{n\in\mathbb{N}, V_n^1 \subset R \setminus \mathrm{Int}(\widetilde{S}_k)} V_n^1 .$$

Furthermore, we define

$$\widehat{A}_k := \widehat{A}_k \cup (\mathrm{Int}(\widehat{S}_k) \setminus \overline{\widetilde{S}_k}).$$

We notice that by construction $(\widehat{A}_k, \widetilde{S}_k) \in \mathcal{B}(\Omega_k)$ and $\Gamma^{\mathrm{A}}_{\mathrm{FS}}\left(\widehat{A}_k, \widetilde{S}_k\right) \setminus \mathrm{Int}(\overline{\widetilde{S}_k}) = \Gamma_1 \cup L_1 \cup \Gamma_2$ is a connected set (see Figure (3.4)). Moreover, by the anisotropic minimality of segments (see [68, Remark 20.3]), it follows that

$$\begin{aligned}
&\int_{\gamma^2\cap\partial^*\widetilde{A}_k\setminus\partial\widetilde{S}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 + \int_{\gamma^1\cap\partial^*\widetilde{S}_k\cap\partial^*\widetilde{A}_k} \phi(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 + \int_{\gamma^2\cap\partial^*\widetilde{A}_k\cap\widetilde{S}_k^{(0)}} \phi_{\mathrm{F}}(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 \\
&\quad + \int_{(\gamma^1\cup\gamma^2)\cap\partial^*\widetilde{S}_k\cap\partial\widetilde{A}_k\cap\widetilde{A}_k^{(1)}} (\phi_{\mathrm{F}} + \phi)(\nu_{\widetilde{A}_k})\, d\mathcal{H}^1 \\
&\geq \int_{L_1} \phi_{\mathrm{F}}(\nu_{L_1}) + \phi(\nu_{L_1})\, d\mathcal{H}^1 \geq \int_{L_1} \phi_{\mathrm{FS}}(\nu_{L_1})\, d\mathcal{H}^1,
\end{aligned} \tag{3.4.46}$$

where in the last inequality we used (3.2.16). We now obtain (3.4.42) by observing that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{T\cap\overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta', \tag{3.4.47}$$

where in the first inequality we used (3.4.46) and in the second inequality we proceed by applying Step 3.1 to the configuration $\left(\widehat{A}_k, \widehat{S}_k\right)$, which by construction presents a island and is such that $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}\left(\widehat{A}_k, \widehat{S}_k\right) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap R$ consists of one and only component.

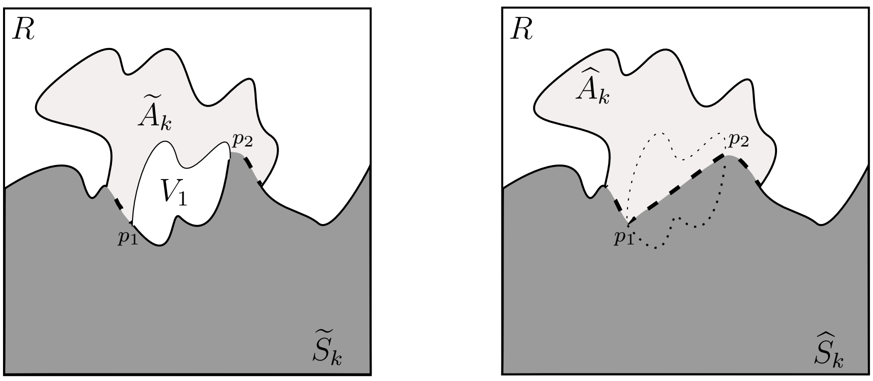

Figure (3.4): The two illustrations above represent, passing from the left to the right, the construction that consists in "filling" a void, which is contained in Step 3.2 of the proof of the Lemma 3.4.7 for the basis of the induction in the case with $M = 2$ and with $\widetilde{A}_k$ presenting a void V_1.

Step 3.3. Now we make the inductive hypothesis that (3.4.42) holds true if $\widetilde{\Gamma}^{\mathrm{A}}_k$ has $M_k = j - 1$ connected components, and we prove that (3.4.42) holds also if $\widetilde{\Gamma}^{\mathrm{A}}_k$ has $M_k = j$ connected components. We observe that by Step 2 we have the two cases:

(a) $j \geq 1$ and there exists at least an island $P \subset R$ in $\widetilde{A}_k$;

(b) $j \geq 2$ and there exists at least an extended void $V \subset \overline{R}$ in $\widetilde{A}_k$.

In the case (a) we proceed by applying the same construction done in Step 3.1 with respect to the island P instead of P_1 obtaining the configuration $\left(\widehat{A}_k, \widehat{S}_k\right) \in \mathcal{B}(\Omega_k)$. We observe that by construction $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}\left(\widehat{A}_k, \widehat{S}_k\right) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap R$ presents $j-1$ connected components (since a component is canceled) and hence, we obtain that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{T\cap\overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta', \tag{3.4.48}$$

where we used (3.4.45) in the first inequality and we applied the induction hypothesis on $(\widehat{A}_k, \widehat{S}_k)$ in the second.

In the case (b) we proceed by applying the same construction done in Step 3.2 with respect to the extended void V instead of V_1 obtaining the configuration $\left(\widehat{A}_k, \widehat{S}_k\right) \in \mathcal{B}_{\mathbf{m}}$. We observe that by construction $\left(\widehat{A}_k, \widehat{S}_k\right) \in \mathcal{B}(\Omega_k)$ and $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}\left(\widehat{A}_k, \widehat{S}_k\right) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap R$ presents $j-1$ connected components (since two components are connected in one). Finally, we have that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{T\cap\overline{R}} \phi(\nu_T)\, d\mathcal{H}^1 - 2c_2\delta' \tag{3.4.49}$$

where we used (3.4.47) in the first inequality and we applied the induction hypothesis on $(\widehat{A}_k, \widehat{S}_k)$ in the second. □

We continue with the setting of the film-substrate incoherent delaminated interface.

Lemma 3.4.8. *Let $R \subset \mathbb{R}^2$ be an open rectangle with a side parallel to $\boldsymbol{e_1}$ and let $T \subset \mathbb{R}^2$ be a line such that $T \cap R \neq \emptyset$ and let $x \in T$. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $R \subset \sigma_{\rho_1}(\Omega)$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $S_{h_k,K_k} \cap \overline{R} \xrightarrow{\mathcal{K}} H_{x,\nu_T} \cap \overline{R}$ in $\mathbb{R}^2$ and $\overline{R} \setminus A_k \xrightarrow{\mathcal{K}} T \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, then for every $\delta \in (0,1)$ small enough, there exists $k_\delta \in \mathbb{N}$ such that*

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \geq \int_{T\cap\overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - \delta. \tag{3.4.50}$$

for any $k \geq k_\delta$. The same statement remains true if we replace H_{x,ν_T} with $H_{x,-\nu_T}$.

Proof. Without loss of generality, we assume that $\sup_{k\in\mathbb{N}} \mathcal{S}_L(A_k, S_{h_k,K_k}, R) < \infty$. We prove (3.4.50) in two steps. In the first step, we prove (3.4.50) for every $k \in \mathbb{N}$ such that $\Gamma^{\mathrm{A}}_k := (\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A_k, S_{h_k,K_k}) \setminus \mathrm{Int}(S_{h_k,K_k})) \cap R$ is $\mathcal{H}^1$-negligible by repeating the same arguments of Step 1 in the proof of Lemma 3.4.7. In the second step, by arguing as in [58, Lemma 4.4] we prove (3.4.50) for those $k \in \mathbb{N}$ such that $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k)$ is positive.

If T is a vertical segment we define $c_\theta := 1$, otherwise we define $c_\theta := (1/\sin\theta) + (1/\cos\theta)$, where $\theta < \pi/2$ is the smallest angle formed by the direction of T with $\mathbf{e_1}$. Since $\partial S_{h_k,K_k} \cap \overline{R} \xrightarrow{\mathcal{K}} T \cap \overline{R}$ and $\overline{R} \setminus A_k \xrightarrow{\mathcal{K}} T \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, for every $\delta' \in (0,1)$ there exits $k_{\delta'} \in \mathbb{N}$ such that

$$\partial S_{h_k,K_k} \cap \overline{R},\, \overline{R} \setminus A_k \subset T^{\delta'} \tag{3.4.51}$$

for every $k \geq k_{\delta'}$, where $T^{\delta'} := \{x \in R : \mathrm{dist}(x,T) < \delta'/(2c_\theta)\}$ is a tubular neighborhood of T in R. By arguing as in (3.4.34) there exists a parametrization $r_k : [0,1] \to \mathbb{R}^2$ of $\partial\mathrm{Int}\left(\overline{S_{h_k,K_k}}\right)$, whose support we denote by γ_k. Finally, let $T^{\delta'}_1$ and $T^{\delta'}_2$ be the connected components of $\partial R \cap \overline{T^{\delta'}}$.

Step 1. Assume that $\mathcal{H}^1\left(\Gamma_k^{\mathrm{A}}\right) = 0$ for a fix $k \geq k_{\delta'}$. Notice that by (3.4.51), there exists $p_1 := r_k(t_1)$ and $p_2 := r_k(t_2)$, where $t_1 := \inf\{t \in [0,1] : r_k(t) \in \partial R\}$ and $t_2 := \sup\{t \in [0,1] : r_k(t) \in \partial R\}$, and without loss of generality we assume that $t_1 < t_2$. Let $\widetilde{T}_i^1 \subset \partial R \cap T_i^{\delta'}$ for $i = 1,2$ be the closed and connected set with minimal length connecting p_i with $T \cap \overline{R}$. Therefore, by trigonometric identities we obtain that $\mathcal{H}^1(\widetilde{T}_i) \leq \frac{\delta'}{2c_\theta} c_\theta = \frac{\delta'}{2}$. Since $\mathcal{H}^1\left(\Gamma_k^{\mathrm{A}}\right) = 0$ and by [58, Lemma 4.3], there exists a curve with support $\widetilde{\gamma}_k \subset \overline{\partial A_k \setminus (\gamma^1 \cap \partial \mathrm{Int}(\overline{A_k})}$ such that γ_k and $\widetilde{\gamma}_k$ can intersect only in the delamination area, more precisely, $\gamma_k \cap \partial^* S_{h_k} \cap \partial^* A_k$ and $\widetilde{\gamma}_k \cap \partial^* A_k \setminus \partial S_{h_k}$ are disjoint up to a $\mathcal{H}^1$-negligible set. We define $\Lambda_k := (\partial R \cap \overline{T^{\delta'}}) \cap (\Omega_k \setminus \mathrm{Int}(S_{h_k,K_k}))$ and we denote by $\widetilde{T}_i^2 \subset \partial R \cap T_i^{\delta'}$ for $i = 1,2$ be the closed and connected set with minimal length connecting $\widetilde{\gamma}_k$ with each point of $T \cap T_i^{\delta'}$. It yields that

$$\begin{aligned}
\mathcal{S}_L(A_k, S_{h_k,K_k}, R) &+ \int_{\Lambda_k} \phi(\nu_{\Lambda_k})\, d\mathcal{H}^1 \\
&\geq \int_{\widetilde{\gamma}_k \cap \partial^* A_k \setminus \partial S_{h_k,K_k}} \phi_{\mathrm{F}}(\nu_{A_k})\, d\mathcal{H}^1 + \int_{\gamma_k \cap \partial^* S_{h_k,K_k} \cap \partial^* A_k} \phi(\nu_{A_k})\, d\mathcal{H}^1 \\
&\quad + \int_{(\gamma_k \cup \widetilde{\gamma}_k) \cap \partial^* S_{h_k,K_k} \cap \partial A_k \cap A_k^{(1)}} (\phi_{\mathrm{F}} + \phi)(\nu_{A_k})\, d\mathcal{H}^1 + \int_{\Lambda_k} \phi(\nu_{\Lambda_k})\, d\mathcal{H}^1 \\
&= \int_{\widetilde{\Gamma}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{\Gamma}_k})\, d\mathcal{H}^1 + \int_{\Gamma_k} \phi(\nu_{\Gamma_k})\, d\mathcal{H}^1 - \sum_{i=1}^2 \int_{\widetilde{T}_i^1} \phi(\nu_{\widetilde{T}_i^j})\, d\mathcal{H}^1 - \sum_{i=1}^2 \int_{\widetilde{T}_i^2} \phi_{\mathrm{F}}(\nu_{\widetilde{T}_i^j})\, d\mathcal{H}^1,
\end{aligned} \tag{3.4.52}$$

where $\Gamma_k := \widetilde{T}_1^1 \cup (\gamma_k \cap \overline{R}) \cup \Lambda_k \cup \widetilde{T}_2^1$ and $\widetilde{\Gamma}_k := \widetilde{T}_1^2 \cup \widetilde{\gamma}_k \cup \widetilde{T}_2^2$. By the anisotropic minimality of segments (see [68, Remark 20.3]), it yields that

$$\int_{\widetilde{\Gamma}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{\Gamma}_k})\, d\mathcal{H}^1 + \int_{\Gamma_k} \phi(\nu_{\Gamma_k})\, d\mathcal{H}^1 \geq \int_{T \cap \overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1, \tag{3.4.53}$$

and so, thanks to the facts that $\mathcal{H}^1(\widetilde{T}_0^j \cup \widetilde{T}_1^j) \leq \delta'$ and $\mathcal{H}^1(\Lambda_k) \leq \delta'$ for $j = 1,2$, and by (3.2.15), (3.4.51)-(3.4.53), we deduce that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \geq \int_{T \cap \overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 3c_2\delta'. \tag{3.4.54}$$

Step 2. Since $(A_k, S_{h_k,K_k}) \in \mathcal{B}_{\mathbf{m}}(\sigma_k(\Omega))$ we can find an enumeration $\{\Lambda_k^n\}_{n=1,\dots,m_k^1}$ of the connected components Λ_k^n of ∂A_k lying strictly inside of R such that $m_k^1 \leq m_1$. Moreover, thanks to the fact that $\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$ for each $k \in \mathbb{N}$, the family $\{\Lambda_k^\alpha\}_{\alpha \in \mathbb{N}}$ of connected components Λ_k^α that intersect $T_1^{\delta'}$ or $T_2^{\delta'}$ of $\partial A_k \cap \overline{Q_1}$, respectively, are at most countable. Furthermore, we define Λ^{m_k+i} for $i = 1,2$ by

$$\Lambda^{m_k+i} := \left(\bigcup_{\alpha \in \mathbb{N},\, \Lambda^\alpha \cap T_i^{\delta'} \neq \emptyset} \Lambda^\alpha \right) \cup T_i^{\delta'}$$

We denote by $\pi_T : \mathbb{R}^2 \to \mathbb{R}$ the orthogonal projection of $\mathbb{R}^2$ onto T and since for every $n = 1, \dots, m_k + 2$, Λ^n is a connected set we have that $\pi_T(\Lambda^n)$ is a homeomorphic to a closed interval in $\mathbb{R}$. More precisely, $\bigcup_{n=1}^{m_k+2} \pi_T(\Lambda^j)$ is equal to a finite family of closed segments. Thanks to the facts that $\overline{R} \setminus A_k \xrightarrow{\mathcal{K}} T \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, and $m_k \leq m_1$ for every $k \in \mathbb{N}$, we have that

$$\lim_{k \to \infty} \mathcal{H}^1 \left((T \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k+2} \pi_T(\Lambda^n) \right) = 0$$

and hence, there exists $k_{\delta'}^1 > k_{\delta'}$ such that

$$\mathcal{H}^1 \left((T \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k+2} \pi_T(\Lambda^n) \right) < (m_1 + 2)\delta' \tag{3.4.55}$$

for every $k \geq k^1_{\delta'}$. We denote by $a_n, b_n \in T \cap \overline{R}$ for every $n = 1, \dots, m_k+2$ the initial and final point of each $\pi_T(\Lambda^j)$, respectively (notice that $a_{m_k+1} \in T \cap T_1^{\delta'}$ and $b_{m_k+2} \in T \cap T_2^{\delta'}$). We decompose $\bigcup_{n=1}^{m_k+2} \pi_T(\Lambda^n)$ as the finite union of disjoint open connected sets $\{C_j\}_{j \in J}$, where the endpoints of C_j are denoted by $a'_j, b'_j \in \bigcup_{j=1}^{m_k+2}\{a_j, b_j\}$ for every $j \in J$, and $\bigcup_{j \in J}\{a'_j, b'_j\} = \bigcup_{n=1}^{m_k+2}\{a_n, b_n\}$. Therefore, by definition the cardinality of J is bounded by $2m_k + 3$,

$$\bigcup_{n=1}^{m_k+2} \pi_T(\Lambda^n) = \bigcup_{j \in J} C_j,$$

and also by (3.4.55) we have that

$$\mathcal{H}^1\left((T \cap \overline{R}) \setminus \bigcup_{j \in J} C_j\right) < (m_1 + 2)\delta' \tag{3.4.56}$$

for every $k \geq k^1_{\delta'}$. Let $T_{a'_j}$ and $T_{b'_j}$ be the lines parallel to ν_T and passing through a'_j and b'_j, respectively. Finally, we denote by S^j the intersection of the strip between $T_{a'_j}$ and $T_{b'_j}$ and $R \cap T^{\delta'}$ for every $j \in J$. If $j \in J$ is such that $C_j \cap T_1^{\delta}$, we define $T'_{a'_j} := T_1^{\delta'}$ and $T'_{b'_j} = T_{b'_j} \cap \overline{S^j}$, analogously, if $j \in J$ is such that $C_j \cap T_2^{\delta}$ we define $T'_{a'_j} = T_{a'_j} \cap \overline{S^j}$ and $T'_{b'_j} := T_2^{\delta'}$, otherwise, if $C_j \cap \bigcup_{i=1}^2 T_i^{\delta} = \emptyset$, $T'_{a'_j} = T_{a'_j} \cap \overline{S^j}$ and $T'_{b'_j} = T_{b'_j} \cap \overline{S^j}$. It follows that

$$\mathcal{H}^1(T'_{a'_j} \cup T'_{b'_j}) \leq 2\delta'. \tag{3.4.57}$$

From now on we fix $j \in J$ and we consider a fixed $k \geq k^1_{\delta'}$ such that $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k \cap S^j) > 0$. For simplicity in the following part of this step we denote $\Lambda^n := \Lambda^n_k$ for every $n = 1, \dots, m_k + 2$. By applying Lemma 3.4.5 (with R, as from the notation of Lemma 3.4.5, coinciding with S^j) there exist $M_k := M_k(A_k, S_{h_k,K_k}, \delta') \in \mathbb{N} \cup \{0\}$ and $(\widetilde{A}_k, \widetilde{S}_k) \in \mathcal{B}_{\mathbf{m}}$ such that $\widetilde{\Gamma}^{\mathrm{A}}_k := (\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widetilde{A}_k, \widetilde{S}_k) \setminus \mathrm{Int}(\overline{\widetilde{S}_k})) \cap S^j$ has M_k connected components and

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, S^j) - c_2\delta', \tag{3.4.58}$$

and there exists a path $\widetilde{\Lambda}^j \subset \partial\widetilde{A}_k$ such that $\widetilde{\Lambda}^j \cap T'_{c_j} \neq \emptyset$ and $\partial\widetilde{S}_k \cap \overline{S^j} \cap T'_{c_j} \neq \emptyset$ for $c_j \in \{a'_j, b'_j\}$. Let $\{\Gamma_\ell\}_{\ell=1}^{M_k}$ be the family of connected components of $\widetilde{\Gamma}^{\mathrm{A}}_k$. Without loss of generality, we assume that $\widetilde{\Lambda}^j$ intersects all islands and voids of $\widetilde{A}_k$ that are not full ones (see Definition 3.4.6), because otherwise we can always reduce to this situation by repeating Steps 3.1 and 3.2 of Lemma 3.4.7. If $M_k = 0$, by repeating the same arguments of Step 1 we obtain that

$$\mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 3c_2\delta'. \tag{3.4.59}$$

Therefore, by (3.4.58) and (3.4.59), we deduce that

$$\mathcal{S}_L(A_k, S_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 4c_2\delta'. \tag{3.4.60}$$

In the remaining of this step we assume that $M_k \geq 1$ by proving the following claim: If $M_k \geq 1$, then there exists an island of $\widetilde{A}_k$ or a substrate grain of $\widetilde{S}_k$ (see Definition 3.4.6). In order to prove this claim we proceed by contradiction. Therefore, let us assume that $(\widetilde{A}_k, \widetilde{S}_k)$ does not contain any island and substrate grain. Since $M_k \geq 1$, there exists $\ell_1 \in \{1, \dots, M_k\}$ and, since the endpoints of Γ_{ℓ_1} must be connected through $\widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k})$, then there exists an open connected set $D_1 \subset \mathrm{Int}(\widetilde{A}_k)$ enclosed by $\widetilde{\Gamma}^{\mathrm{A}}_k$ and $\widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k})$ such that $\Gamma_{\ell_1} \subset \partial D_1$. By assumption, since D_1 cannot be an island and cannot be a grain of $(\widetilde{A}_k, \widetilde{S}_k)$, there must exist $\ell_2 \in \{1, \dots, M_k\} \setminus \{\ell_1\}$

such that $\Gamma_{\ell_2} \subset \overline{D_1}$. Since the endpoints of Γ_{ℓ_2} must be connected through $\widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k})$, then there exists an open connected set $D_2 \subset \mathrm{Int}(\widetilde{A}_k)$ enclosed by $\widetilde{\Gamma}_k^{\mathrm{A}}$ and $\widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k})$ such that $\Gamma_{\ell_2} \subset \partial D_2$, and we notice that D_2 cannot coincide with D_1 since $\partial D_1 \cap (\widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k}))$ is not connecting the endpoints of Γ_{ℓ_2} (besides not connecting also the endpoints of Γ_{ℓ_1}). By keeping on iterating this reasoning we reach a contradiction with the fact that the family $M_k < \infty$, and hence the claim holds true.

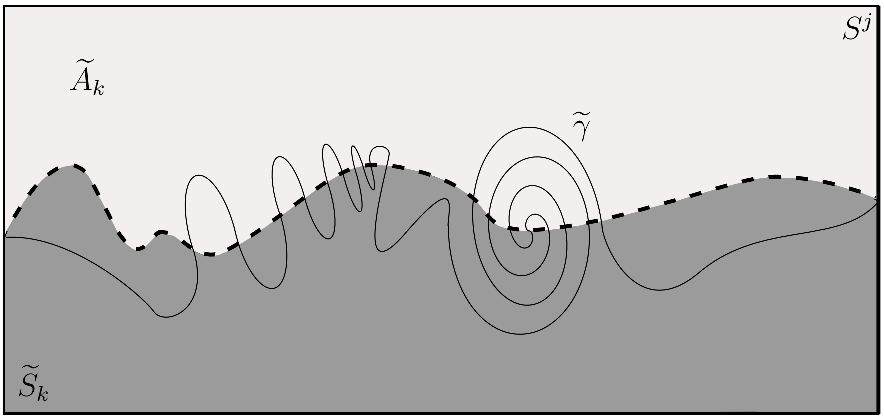

Figure (3.5): The illustration represents the configuration of the admissible pair $(\widetilde{A}_k, \widetilde{S}_k)$ that is modified in Step 2 of Lemma 3.4.8 to obtain the pair $(\breve{A}_k, \breve{S}_k)$ for which in the same step is proved the existence of at least a film island or a substrate grain in accordance with Definition 3.4.4.

Step 3. In this step we assume that $M_k \geq 1$ (for the case $M_k = 0$ see Step 2) and we prove that

$$\mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 8c_2\delta', \tag{3.4.61}$$

from which it easily follows from (3.2.15), (3.4.57) and (3.4.58), that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.62}$$

In order to prove (3.4.61) we consider an auxiliary energy $\mathcal{S}_L^1$ given by $\mathcal{S}_L$ with an extra term, namely defined by

$$\mathcal{S}_L^1(\widetilde{A}_k, \widetilde{S}_k, R) := \mathcal{S}_L(\widetilde{A}_k, \widetilde{S}_k, R) + \sum_{c_j \in \{a'_j, b'_j\}} 2\int_{T'_{c_j}} (\phi_{\mathrm{F}} + \phi)\, d\mathcal{H}^1 \tag{3.4.63}$$

for every $(\widetilde{A}_k, \widetilde{S}_k) \in \mathcal{B}$, and we prove that

$$\mathcal{S}_L^1(\widetilde{A}_k, \widetilde{S}_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1, \tag{3.4.64}$$

since (3.4.61) directly follows from (3.4.64) by (3.2.15) and (3.4.57). To prove (3.4.64) we proceed by induction on the number $M_k \in \mathbb{N}$ of connected components of $\widetilde{\Gamma}_k^{\mathrm{A}}$ in three steps. In Steps 3.1 and 3.2 we show the basis of the induction for $M_k = 1$, by considering the two cases provided by Step 2, i.e., the case in which $\widetilde{A}_k$ presents an island in Step 3.1 and the case in which $\widetilde{S}_k$ presents a substrate grain in Step 3.2. We conclude then the induction in Step 3.3.

Step 3.1 Assume that $M_k = 1$ and that there exists an island $P_1 \subset \overline{\widetilde{A}_k} \setminus \mathrm{Int}(\widetilde{S}_k)$ of $\widetilde{A}_k$ such that P_1 is enclosed by $\Gamma_1 \cup \widetilde{\gamma} \cup \partial\mathrm{Int}(\overline{\widetilde{A}_k}))$ with $\Gamma_1 \subset \partial P_1$. Let p_1 and p_2 be the endpoints of Γ_1, and let L_1 be the segment connecting p_1 with p_2. We denote by P_1^1 the open set enclosed by L_1 and $\partial P_1 \cap \Gamma_1$ and we denote by P_1^2 the open set enclosed by L_1 and $\partial P_1 \setminus \Gamma_1$.

We define a modification of $\widetilde{S}_k$, denoted by $\widehat{S}_k$, and a modification of $\widetilde{A}_k$, denoted by $\widehat{A}_k$, as

$$\widehat{S}_k := (\widetilde{S}_k \setminus (P_1^1 \cap \mathrm{Int}(\widetilde{S}_k))) \cup (P_1^1 \cap (S^j \setminus \overline{\widetilde{S}_k})),$$

and

$$\widehat{A}_k := \widetilde{A}_k \setminus \left(P_1^2 \cap (\mathrm{Int}(\widetilde{A}_k) \setminus \overline{\widetilde{S}_k})\right) \cup \left(P_1^2 \cap (S^j \setminus \overline{\widetilde{A}_k})\right),$$

respectively. Notice that $(\widehat{A}_k, \widehat{S}_k) \in \mathcal{B}(\sigma_k(\Omega))$. By the anisotropic minimality of segments, it follows that

$$\begin{aligned}\int_{\partial P_1 \setminus \widetilde{C}_1} \phi_{\mathrm{F}}(\nu_{\partial P_1})\, d\mathcal{H}^1 + \int_{\partial P_1 \cap \widetilde{C}_1} \phi_{\mathrm{FS}}(\nu_{\partial P_1})\, d\mathcal{H}^1 &\geq \int_{L_1} \phi_{\mathrm{F}}(\nu_{L_1}) + \phi_{\mathrm{FS}}(\nu_{L_1})\, d\mathcal{H}^1 \\ &\geq \int_{L_1} \phi(\nu_{L_n})\, d\mathcal{H}^1,\end{aligned} \tag{3.4.65}$$

where in the second inequality we used the definition of ϕ, and hence, by (3.4.65) we obtain that

$$\mathcal{S}_L^1(\widetilde{A}_k, \widetilde{S}_k, S^j) \geq \mathcal{S}_L^1(\widehat{A}_k, \widehat{S}_k, S^j). \tag{3.4.66}$$

Moreover, we observe that by construction $\widehat{\gamma} := (\widetilde{\gamma} \setminus (\widetilde{\gamma} \cap \partial \mathrm{Int}(\overline{\widetilde{A}_k}))) \cup L_1$ is path connected and it joins $T \cap T'_{a'_j}$ with $T \cap T'_{b'_j}$, and $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widehat{A}_k, \widehat{S}_k) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap S^j$ is $\mathcal{H}^1$-negligible. Thus, by repeating the same arguments of Step 1, we deduce that

$$\mathcal{S}_L^1(\widehat{A}_k, \widehat{S}_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1. \tag{3.4.67}$$

By (3.4.57), (3.4.58), (3.4.63), (3.4.66), and (3.4.67) we obtain (3.4.61).

Step 3.2 Assume that $M_k = 1$ and that there exists a substrate grain $G_1 \subset \overline{\widetilde{S}_k}$ of $\widetilde{S}_k$ such that G_1 is enclosed by $\Gamma_1 \cup \widetilde{\gamma} \cup \partial \mathrm{Int}(\overline{\widetilde{A}_k}))$ with $\Gamma_1 \subset \partial G_1$. Let p_1 and p_2 be the endpoints of Γ_1, and let L_1 be the segment connecting p_1 with p_2. We denote by G_1^1 the open set enclosed by L_1 and $\partial G_1 \cap \Gamma_1$ and we denote by G_1^2 the open set enclosed by L_1 and $\partial G_1 \setminus \Gamma_1$.

We define a modification of $\widetilde{S}_k$, denoted by $\widehat{S}_k$, and a modification of $\widetilde{A}_k$, denoted by $\widehat{A}_k$, as

$$\widehat{S}_k := \widetilde{S}_k \setminus ((G_1^2 \cap \mathrm{Int}(\widetilde{S}_k)) \cup (G_1^1 \cap \mathrm{Int}(\widetilde{S}_k))),$$

and

$$\widehat{A}_k := \widetilde{A}_k \setminus \left(G_1^2 \cap \mathrm{Int}(\widetilde{A}_k)\right) \cup \left(G_1^1 \cap \mathrm{Int}(\widetilde{A}_k)\right),$$

respectively (see Figure 3.6). Notice that $(\widehat{A}_k, \widehat{S}_k) \in \mathcal{B}(\sigma_k(\Omega))$.

By the anisotropic minimality of segments, it follows that

$$\begin{aligned}\int_{\partial G_1 \setminus \Gamma_1} \phi(\nu_{\partial G_1})\, d\mathcal{H}^1 + \int_{\partial G_1 \cap \Gamma_1} \phi_{\mathrm{FS}}(\nu_{\partial G_1})\, d\mathcal{H}^1 &\geq \int_{L_1} \phi(\nu_{L_1}) + \phi_{\mathrm{FS}}(\nu_{L_1})\, d\mathcal{H}^1 \\ &\geq \int_{L_1} \phi_{\mathrm{F}}(\nu_{L_1})\, d\mathcal{H}^1,\end{aligned} \tag{3.4.68}$$

where in the second inequality we used (3.2.16) and hence, by (3.4.68) we obtain that

$$\mathcal{S}_L^1(\widetilde{A}_k, \widetilde{S}_k, S^j) \geq \mathcal{S}_L^1(\widehat{A}_k, \widehat{S}_k, S^j). \tag{3.4.69}$$

Moreover, we observe that by construction $\widehat{\gamma} := (\widetilde{\gamma} \setminus (\widetilde{\gamma} \cap \partial \mathrm{Int}(\overline{\breve{A}_k}))) \cup L_1$ is path connected and it joins $T \cap T'_{a'_j}$ with $T \cap T'_{b'_j}$, and $(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widehat{A}_k, \widehat{S}_k) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap S^j$ is $\mathcal{H}^1$-negligible. Thus, by repeating the same arguments of Step 1, we deduce that

$$\mathcal{S}_L^1(\widehat{A}_k, \widehat{S}_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1. \tag{3.4.70}$$

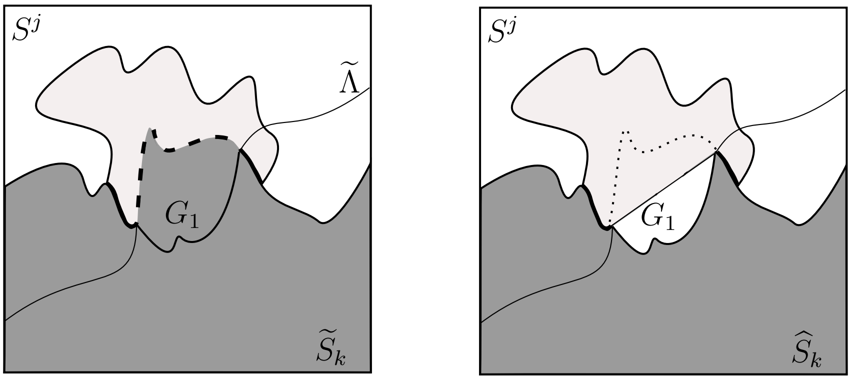

Figure (3.6): The two illustrations above represent, passing from the left to the right, the construction that consists in "modifying grains in new voids", which is contained in Step 3.2 of the proof of the Lemma 3.4.8 for the modification of the grain in a new void.

By (3.2.15), (3.4.57), (3.4.58), (3.4.63), (3.4.69), and (3.4.70) we obtain (3.4.61).

Step 3.3. Assume that (3.4.61) holds true if $M_k = i - 1$. We need to show that (3.4.61) holds also if $M_k = i$. By Step 2 we have two cases:

(a) $i \geq 1$ and there exists at least an island $P \subset S^j$ of $\widetilde{A}_k$;

(b) $i \geq 1$ and there exists at least a grain substrate $G \subset S^j$ of $\widetilde{S}_k$.

In the case (a) we proceed by applying the same construction done in Step 3.1 with respect to the island P instead of P_1 obtaining the configuration $\left(\widehat{A}_k, \widehat{S}_k\right) \in \mathcal{B}(\Omega_k)$. We observe that by construction the set $\widehat{\Gamma}^{\mathrm{A}}_k := (\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widehat{A}_k, \widehat{S}_k) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap S^j$ has cardinality $i-1$ (since the island P of $\breve{A}_k$ is shrunk in $\widehat{A}_k$) and hence, we obtain that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 \tag{3.4.71}$$

where we used the inductive hypothesis in the last inequality.

In the case (b) we proceed by applying the same construction done in Step 3.2 with respect to the substrate grain G instead of G_1 obtaining the configuration $\left(\widehat{A}_k, \widehat{S}_k\right) \in \mathcal{B}$. We observe that by construction the set $\widehat{\Gamma}^{\mathrm{A}}_k := (\Gamma^{\mathrm{A}}_{\mathrm{FS}}(\widehat{A}_k, \widehat{S}_k) \setminus \mathrm{Int}(\overline{\widehat{S}_k})) \cap S^j$ has cardinality $i-1$ (since the grain G of $\breve{S}_k$ is opened in a void) and hence, we obtain that

$$\mathcal{S}^1_L(\widetilde{A}_k, \widetilde{S}_k, R) \geq \mathcal{S}^1_L(\widehat{A}_k, \widehat{S}_k, R) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 \tag{3.4.72}$$

where we used the inductive hypothesis in the last inequality.

Step 4. Let $k \geq k^1_{\delta'}$, which was defined in Step 2. If $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k) > 0$, we consider $j \in J$. By repeating the arguments of Step 1 in S^j if $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k \cap S^j) = 0$ (see (3.4.54)), and by Steps 2 and 3 (see (3.4.60) and (3.4.61)) we obtain that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.73}$$

Therefore, in view of the fact that the cardinality of J is bounded by $2m_k + 3$, by (3.4.56) and

(3.4.73) it follows that

$$\begin{aligned} \mathcal{S}_L(A_k, S_{h_k,K_k}, R) &\geq \sum_{j\in J} \mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \\ &\geq \int_{\bigcup_{j\in J} C_j} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 9(2m_k+3)c_2\delta' \\ &\geq \int_{T\cap\overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - 2(m_1+2)c_2\delta' - 9(2m_k+3)c_2\delta' \\ &= \int_{T\cap\overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - (20m_1+31)c_2\delta', \end{aligned} \tag{3.4.74}$$

where in the last inequality we used that $m_k \leq m_1$. By recalling once again (3.4.54) of Step 1, we observe that by (3.4.74) we obtain that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, R) \geq \int_{T\cap\overline{R}} \phi_{\mathrm{F}}(\nu_T) + \phi(\nu_T)\, d\mathcal{H}^1 - (20m_1+31)c_2\delta', \tag{3.4.75}$$

for both the case with $\mathcal{H}^1(\Gamma_k^{\mathrm{A}}) > 0$ and the case with $\mathcal{H}^1(\Gamma_k^{\mathrm{A}}) = 0$. Finally, follows (3.4.50) from choosing $k_\delta := k^1_{\delta'}$ and $\delta' = \frac{\delta}{(20m_1+31)c_2}$ for $\delta \in (0, \min\{(20m_1+31)c_2, 1\})$ in (3.4.75). This completes the proof. □

We continue with the situation of the substrate cracks in the film-substrate incoherent interface.

Lemma 3.4.9. *Let T_0 be the x_2-axis. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $Q_1 \subset \sigma_{\rho_1}(\Omega)$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $\overline{Q_1}\backslash A_k \xrightarrow{\mathcal{K}} T_0\cap\overline{Q_1}$ and $\overline{Q_1}\backslash S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0\cap\overline{Q_1}$ in $\mathbb{R}^2$ as $k\to\infty$, then for every $\delta\in(0,1)$ small enough, there exists $k_\delta\in\mathbb{N}$ such that*

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, Q_1) \geq 2\int_{T_0\cap\overline{Q_1}} \phi(\boldsymbol{e_1})\, d\mathcal{H}^1 - \delta \tag{3.4.76}$$

for any $k \geq k_\delta$.

Proof. Without loss of generality we assume that $\sup_{k\in\mathbb{N}} \mathcal{S}_L(A_k, S_{h_k,K_k}, Q_1) < \infty$. Since $\overline{Q_1}\setminus A_k \xrightarrow{\mathcal{K}} T_0\cap\overline{Q_1}$ and $\overline{Q_1}\setminus S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0\cap\overline{Q_1}$ in $\mathbb{R}^2$ as $k\to\infty$, for every $\delta'\in(0,1)$ there exits $k_{\delta'}\in\mathbb{N}$ such that

$$\overline{Q_1}\setminus A_k,\ \overline{Q_1}\setminus S_{h_k,K_k} \subset T^{\delta'} \tag{3.4.77}$$

for every $k\geq k_{\delta'}$, where $T^{\delta'} := \{x\in Q_1 : \operatorname{dist}(x,T_0) < \delta'/2\}$ is the tubular neighborhood with thickness δ' of T_0 in Q_1. Let $T_1^{\delta'}$ and $T_2^{\delta'}$ be the connected components of $\partial Q_1 \cap \overline{T^{\delta'}}$. Since $(A_k, S_{h_k,K_k}) \in \mathcal{B}_{\mathbf{m}}(\sigma_k(\Omega))$ we can find an enumeration $\{\Gamma_k^n\}_{1,\dots,m_k^0}$ of connected components Γ_k^n of $\partial S_{h_k,K_k}$ lying strictly inside of Q_1, and an enumeration $\{\Lambda_k^n\}_{1,\dots,m_k^1}$ of the connected components Λ_k^n of ∂A_k lying strictly inside of Q_1, such that $m_k^\ell \leq m_\ell$ for $\ell = 0,1$. Moreover, thanks to the fact that $\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$ for each $k\in\mathbb{N}$, the families $\{\Gamma_k^\alpha\}_{\alpha\in\mathbb{N}}$ and $\{\Lambda_k^\alpha\}_{\alpha\in\mathbb{N}}$ of connected components Γ_k^α and Λ_k^α that intersect $T_1^{\delta'}$ or $T_2^{\delta'}$ of $\partial S_{h_k,K_k}\cap\overline{Q_1}$ and of $\partial A_k\cap\overline{Q_1}$, respectively, are at most countable. Furthermore, we define Γ^{m_k+i} and Λ^{m_k+i} for $i=1,2$ by

$$\Gamma^{m_k+i} := \left(\bigcup_{\alpha\in\mathbb{N},\,\Gamma^\alpha\cap T_i^{\delta'}\neq\emptyset} \Gamma^\alpha\right)\cup T_i^{\delta'} \quad\text{and}\quad \Lambda^{m_k+i} := \left(\bigcup_{\alpha\in\mathbb{N},\,\Lambda^\alpha\cap T_i^{\delta'}\neq\emptyset} \Lambda^\alpha\right)\cup T_i^{\delta'}.$$

Thanks to the Kuratowski convergences of $\overline{Q_1}\setminus A_k$ and $\overline{Q_1}\setminus S_{h_k,K_k}$ to $T_0\cap\overline{Q_1}$ in $\mathbb{R}^2$ as $k\to\infty$, the fact that $m_k^\ell \leq m_\ell$ for $\ell=0,1$ and for every $k\in\mathbb{N}$, we have that

$$\lim_{k\to\infty}\mathcal{H}^1\left((T_0\cap\overline{Q_1})\setminus\bigcup_{n=1}^{m_k^0+2}\pi_2(\Gamma^n)\right) = 0 \quad\text{and}\quad \lim_{k\to\infty}\mathcal{H}^1\left((T_0\cap\overline{Q_1})\setminus\bigcup_{n=1}^{m_k^1+2}\pi_2(\Lambda^n)\right) = 0$$

Hence, there exists $k^1_{\delta'} \geq k_{\delta'}$ such that

$$\mathcal{H}^1\left((T_0 \cap \overline{Q_1}) \setminus \bigcup_{n=1}^{m^0_k+2} \pi_2(\Gamma^n)\right) < (m_0+2)\delta' \tag{3.4.78}$$

for every $k \geq k^1_{\delta'}$, and there exists $k^2_{\delta'} \geq k_{\delta'}$ such that

$$\mathcal{H}^1\left((T_0 \cap \overline{Q_1}) \setminus \bigcup_{n=1}^{m^1_k+2} \pi_2(\Lambda^n)\right) < (m_1+2)\delta' \tag{3.4.79}$$

for every $k \geq k^2_{\delta'}$. Let $k^3_{\delta'} := \max\{k^1_{\delta'}, k^2_{\delta'}\}$. Similarly to Step 2 of Lemma 3.4.8, we can decompose $\bigcup_{n=1}^{m_k+2} \pi_2(\Gamma^n)$ and $\bigcup_{n=1}^{m_k+2} \pi_2(\Lambda^n)$ as the finite union of disjoint open connected sets $\mathscr{C}^0 := \{C^0_j\}_{j\in J_0}$ and $\mathscr{C}^1 := \{C^1_j\}_{j\in J_1}$, respectively. Notice that the cardinality of J_ℓ is bounded by $2m^\ell_k + 3$ for $\ell = 0, 1$. Therefore

$$\bigcup_{n=1}^{m^0_k+2} \pi_2(\Gamma^n) = \bigcup_{j\in J_0} C^0_j, \quad \text{and} \quad \bigcup_{n=1}^{m^1_k+2} \pi_2(\Lambda^n) = \bigcup_{j\in J_1} C^1_j,$$

and also by (3.4.78) and (3.4.79) we have that

$$\mathcal{H}^1\left((T_0 \cap \overline{Q_1}) \setminus \bigcup_{j\in J_\ell} C^\ell_j\right) < (m_\ell+2)\delta' \tag{3.4.80}$$

for every $k \geq k^3_{\delta'}$ and $\ell = 0, 1$. We observe that

$$\mathscr{C} := \{C : \emptyset \neq C = C^0 \cap C^1 \text{ with } C^\ell \in \mathscr{C}^\ell \text{ for } \ell = 0, 1\}$$

is a family of pairwise disjoint sets and has cardinality m^k bounded by $(2m^0_k + 3)(2m^1_k + 3)$, i.e., $\mathscr{C} := \{C_j\}_{j\in J}$ for an index set J with cardinality $m^k \leq (2m^0_k + 3)(2m^1_k + 3)$. We observe that

$$\mathcal{H}^1\left((T_0 \cap \overline{Q_1}) \setminus \bigcup_{j\in J} C_j\right) \leq \sum_{\ell\in\{0,1\}} \mathcal{H}^1\left((T_0 \cap \overline{Q_1}) \setminus \bigcup_{j\in J_\ell} C^\ell_j\right) < (m_0+m_1+4)\delta', \tag{3.4.81}$$

where we used (3.4.80) in the last inequality. Finally, let $S^j := ([-1,1] \times_{\mathbb{R}^2} C_j) \cap T^{\delta'}$ and let T^j_1 and T^j_2 the portions of the boundary of S^j parallels to the x_1-axis for every $j \in J$.

We now prove (3.4.76) in three steps. In the first step, we prove (3.4.76) for $k \geq k^3_{\delta'}$ such that $\Gamma^{\mathrm{A}}_k := \Gamma^{\mathrm{A}}_{\mathrm{FS}}(A_k, S_{h_k,K_k}) \cap S^j$ is $\mathcal{H}^1$-negligible by repeating the same arguments of Step 1 in the proof of Lemma 3.4.8. In the second step, by arguing as in Steps 2 and 3 in the proof of 3.4.8 we prove (3.4.50) for those $k \geq k^3_{\delta'}$ such that $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k)$ is positive. In the last step, by arguing as in Step 4 in the proof of Lemma 3.4.8 we obtain (3.4.76).

Step 1. Assume that $\mathcal{H}^1\left(\Gamma^{\mathrm{A}}_k\right) = 0$ for a fix $k \geq k^3_{\delta'}$. By construction and by [44, Lemma 3.12] there exists a curve with support $\gamma_1 \subset \partial S_{h_k,K_k} \cap (S^j \setminus \mathrm{Int}(S_{h_k,K_k}))$ connecting T^j_1 with T^j_2, and hence, by [58, Lemma 4.3], there exists also a curve with support $\gamma_2 \subset \overline{\partial S_{h_k,K_k} \setminus (\gamma_1 \cap \partial \mathrm{Int}(\overline{S_{h_k,K_k}})}$ such that $\gamma_1 \cap \partial^* S_{h_k,K_k}$, $\gamma_2 \cap \partial^* S_{h_k,K_k}$ are disjoint up to an $\mathcal{H}^1$-negligible set and γ_2 joins T^j_1

with T_2^j. Since $\mathcal{H}^1\left(\Gamma_k^{\mathrm{A}}\right) = 0$, it yields that

$$\begin{aligned}
\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) + \sum_{i=1}^{2} 2\int_{T_i^j} \phi(\nu_{T_i^j})\, d\mathcal{H}^1 \\
\geq \int_{(\gamma_1\cup\gamma_2)\cap\partial^* S_{h_k,K_k}\cap\partial^* A_k} \phi(\nu_{A_k})\, d\mathcal{H}^1 \\
+ \int_{(\gamma_1\cup\gamma_2)\cap(\partial S_{h_k,K_k}\cap\partial A_k)\cap(S^{(1)}_{h_k,K_k}\cup A_k^{(0)})} 2\phi(\nu_{A_k})\, d\mathcal{H}^1 \\
+ \int_{(\gamma_1\cup\gamma_2)\cap\partial^* S_{h_k,K_k}\cap\partial A_k\cap {A_k}^{(1)}} (\phi_{\mathrm{F}} + \phi)(\nu_{A_k})\, d\mathcal{H}^1 \\
+ \sum_{i=1}^{2} 2\int_{T_i^j} \phi(\nu_{T_i^j})\, d\mathcal{H}^1 \\
= \int_{\Gamma_1} \phi(\nu_{\Gamma_1})\, d\mathcal{H}^1 + \int_{\Gamma_2} \phi(\nu_{\Gamma_2})\, d\mathcal{H}^1,
\end{aligned} \tag{3.4.82}$$

where $\Gamma_1 := T_1^j \cup \gamma_1 \cup T_2^j$ and $\Gamma_2 := T_1^j \cup \gamma_2 \cup T_2^j$. From the anisotropic minimality of segments (see [68, Remark 20.3]) it follows that

$$\int_{\Gamma_1} \phi(\nu_{\Gamma_1})\, d\mathcal{H}^1 + \int_{\Gamma_2} \phi(\nu_{\Gamma_2})\, d\mathcal{H}^1 \geq 2\int_{C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1, \tag{3.4.83}$$

and so, thanks to the facts that $\mathcal{H}^1(T_1^j \cup T_2^j) \leq 2\delta'$, and by (3.2.15), (3.4.77), (3.4.82) and (3.4.83), we deduce that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - 4c_2\delta'. \tag{3.4.84}$$

Step 2. Assume that $\mathcal{H}^1\left(\Gamma_k^{\mathrm{A}}\right) > 0$ for a fix $k \geq k_{\delta'}$. By construction and by [44, Lemma 3.12], there exists a curve with support $\widetilde{\gamma}_1 \subset \partial A_k$ connecting $T_1^{j_1}$ with $T_2^{j_1}$. If $\mathcal{H}^1(\widetilde{\gamma}_1 \cap (\partial A_k \setminus \partial S_{h_k})) = 0$, by repeating the same arguments of Step 1, we can deduce that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - 4c_2\delta'. \tag{3.4.85}$$

If $\mathcal{H}^1(\widetilde{\gamma}_1 \cap (\partial A_k \setminus \partial S_{h_k})) > 0$, then by reasoning as it was done in the proof of Lemma 3.4.8 to reach (3.4.62) we obtain

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.86}$$

More precisely, we reach (3.4.86) by noticing that $\nu_{T_0} = \mathbf{e_1}$ and by following the proof of Step 2 (from after equation (3.4.57)) and of Step 3 in Lemma 3.4.8 with the only difference that we replace the reference to Step 1 of Lemma 3.4.8 with the reference to Step 1 of the current lemma and the extra term appearing in (3.4.63) with

$$\sum_{i=1,2} \int_{T_i^j} (\phi_{\mathrm{F}} + 3\phi)(\boldsymbol{e_2})\, d\mathcal{H}^1.$$

.

Step 3. Let $k \geq k^3_{\delta'}$ and let $j \in J$. By applying (3.4.84) if $\mathcal{H}^1(\Gamma_k^{\mathrm{A}} \cap S^j) = 0$ and (3.4.86) if $\mathcal{H}^1(\Gamma_k^{\mathrm{A}} \cap S^j) > 0$ we obtain that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, \widetilde{S}^j) \geq 2\int_{C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.87}$$

Therefore, since the cardinality of J is bounded by $(2m_k^0+3)\cdot(2m_k^1+3)$, (3.4.87) yields

$$\begin{aligned} \mathcal{S}_L(A_k, S_{h_k,K_k}, Q_1) &\geq \sum_{j\in J} \mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \\ &\geq 2\int_{\bigcup_{j\in J} C_j} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - 9(2m_k^0+3)\cdot(2m_k^1+3)c_2\delta' \\ &\geq 2\int_{T_0\cap\overline{Q_1}} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - (m_0+m_1+4)c_2\delta' - 9(2m_k^0+3)\cdot(2m_k^1+3)c_2\delta' \\ &= 2\int_{T\cap\overline{R}} \phi(\mathbf{e_1})\, d\mathcal{H}^1 - \alpha\delta', \end{aligned} \tag{3.4.88}$$

where $\alpha := (m_0+m_1+4)c_2+9(2m_0+3)\cdot(2m_1+3)$ and in the last inequality we used that $m_k^\ell \leq m_\ell$ for $\ell = 0,1$. Finally, (3.4.76) follows from choosing $k_\delta := k_{\delta'}^3$ and $\delta' = \frac{\delta}{\alpha}$ for $\delta \in (0, \min\{\alpha, 1\})$ in (3.4.88). This completes the proof. □

We continue by treating the situation related to the blow up at a point in the filaments of both the substrate and the film.

Lemma 3.4.10. *Let T_0 be the x_2-axis. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $Q_1 \subset \sigma_{\rho_1}(\Omega)$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $Q_1 \cap A_k \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q_1}$ in $\mathbb{R}^2$ and $Q_1 \cap S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q_1}$ in $\mathbb{R}^2$ as $k \to \infty$, then for every $\delta \in (0,1)$ small enough, there exists $k_\delta \in \mathbb{N}$ such that*

$$\mathcal{S}_L(A_k, h_k, K_k, Q_1) \geq 2\int_{T_0} \phi'(\boldsymbol{e_1})\, d\mathcal{H}^1 - \delta \tag{3.4.89}$$

for any $k \geq k_\delta$.

Proof. The situation is symmetric to the situation of Lemma 3.4.9 and the proof is the same since $\phi' \leq \phi$. □

We continue with the situation in the blow up of the substrate filaments on the film free boundary.

Lemma 3.4.11. *Let T_0 be the x_2-axis. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $Q \subset \sigma_{\rho_1}(\Omega) \cap H_{0,-\boldsymbol{e_2}}$ be an open square whose sides are either parallel or perpendicular to $\boldsymbol{e_1}$ and $T_0 \cap Q \neq \emptyset$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $\overline{Q} \cap S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q}$ and $\overline{Q} \cap A_{k_n} \xrightarrow{\mathcal{K}} H_{0,\boldsymbol{e_1}} \cap \overline{Q}$, then for every $\delta \in (0,1)$, there exists $k_\delta \in \mathbb{N}$ such that for any $k \geq k_\delta$,*

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) \geq \int_{T_0\cap\overline{Q}} \phi_{\mathrm{F}}(\boldsymbol{e_1})\, d\mathcal{H}^1 - \delta. \tag{3.4.90}$$

Proof. Without loss of generality we assume that $\sup_{k\in\mathbb{N}} \mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$. Since $\overline{Q} \cap S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q_1}$ and $\overline{Q} \cap A_{k_n} \xrightarrow{\mathcal{K}} H_{0,\mathbf{e_1}} \cap \overline{Q}$ in $\mathbb{R}^2$ as $k \to \infty$ for every $\delta' \in (0,1)$ there exists $k_{\delta'} \in \mathbb{N}$ such that

$$\overline{Q} \cap S_{h_k,K_k},\ Q \cap \partial A_k \subset T^{\delta'}, \tag{3.4.91}$$

where $T^{\delta'} := \{x \in Q : \operatorname{dist}(x, T_0) < \frac{\delta'}{2}\}$. Let T_1 be the upper side of Q and let $T_1^{\delta'} := \{x \in Q : \operatorname{dist}(x, T_1) < \delta'/2\}$. By the Kuratowski convergence of $\overline{Q} \cap S_{h_k,K_k}$, there exists $k_{\delta'}^1 \geq k_{\delta'}$ such that $S_{h_k} \cap \overline{T_1^{\delta'}} \neq \emptyset$ for every $k \geq k_{\delta'}^1$. Let $R := (T^{\delta'} \setminus \overline{T_1^{\delta'}})$ and let $T_1', T_2' \subset \partial R$ be the upper and lower side of the rectangle R, respectively.

Since $(A_k, S_{h_k,K_k}) \in \mathcal{B}_{\mathbf{m}}(\sigma_k(\Omega))$ we can find an enumeration $\{\Lambda_k^n\}_{n=1,\dots,m_k^1}$ of the connected components Λ_k^n of ∂A_k lying strictly inside of R, such that $m_k^1 \leq m_1$. Moreover, thanks to the fact that $\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$ for each $k \in \mathbb{N}$, the family $\{\Lambda_k^\alpha\}_{\alpha\in\mathbb{N}}$ of connected components

Λ_k^α of $\partial A_k \cap \overline{R}$ that intersect T_1' or T_2', respectively, are at most countable. Furthermore, we define Λ^{m_k+i} for $i = 1, 2$ by

$$\Lambda_k^{m_k+i} := \left(\bigcup_{\alpha\in\mathbb{N},\, \Lambda_k^\alpha \cap T_i^{\delta'} \neq \emptyset} \Lambda_k^\alpha \right) \cup T_i'.$$

Thanks to the Kuratowski convergences of $\overline{R} \cap \partial A_k$ to $T_0 \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, the fact that $m_k^1 \leq m_1$ for every $k \in \mathbb{N}$, we have that

$$\lim_{k\to\infty} \mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) \right) = 0$$

Hence, there exists $k_{\delta'}^2 \geq k_{\delta'}^1$ such that

$$\mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) \right) < (m_1 + 2)\delta' \tag{3.4.92}$$

for every $k \geq k_{\delta'}^2$. Similarly to Step 2 of Lemma 3.4.8, we can decompose $\bigcup_{n=1}^{m_k+2} \pi_2(\Lambda_k^n)$ as the finite union of disjoint open connected sets $\mathscr{C} := \{C_j\}_{j\in J}$. Notice that the cardinality of J is bounded by $2m_k^1 + 3$. Therefore

$$\bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) = \bigcup_{j\in J} C_j,$$

and also by (3.4.104) we have that

$$\mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{j\in J} C_j \right) < (m_1 + 2)c_2\delta' \tag{3.4.93}$$

for every $k \geq k_{\delta'}^2$. Finally, let $S^j := (\pi_1(R) \times_{\mathbb{R}^2} C_j) \cap T^{\delta'}$ and let T_1^j and T_2^j be the upper and lower sides of the boundary of each rectangle S^j, respectively.

Step 1. Let $k \geq k_{\delta'}^2$ and $j \in J$ be such that $\mathcal{H}^1(\Gamma_{\mathrm{FS}}^{\mathrm{A}}(A_k, S_{h_k,K_k}) \cap S^j) = 0$. In view of the construction of S^j, by [44, Lemma 3.12] there exists a curve with support $\gamma_k \subset \partial(A_k \setminus \mathrm{Int}(S_{h_k}))$ joining T_1^j with T_2^j. It follows that

$$\begin{aligned}
\mathcal{S}_L(A_k, S_{h_k,K_k}, R) &+ \sum_{i=1}^{2} \int_{T_i^j} \phi_{\mathrm{F}}(\mathbf{e_2})\, d\mathcal{H}^1 \\
&\geq \int_{\gamma_k \cap \partial^* A_k \setminus \partial S_{h_k,K_k}} \phi_{\mathrm{F}}(\nu_{A_k})\, d\mathcal{H}^1 \\
&\quad + \int_{\gamma_k \cap (\partial A_k \setminus \partial S_{h_k,K_k}) \cap (A_k^{(0)} \cup A_k^{(1)})} 2\phi_{\mathrm{F}}(\nu_{A_k})\, d\mathcal{H}^1 \\
&\quad + \int_{\gamma_k \cap \partial^* S_{h_k,K_k} \cap \partial A_k \cap A_k^{(1)}} (\phi_{\mathrm{F}} + \phi)(\nu_{A_k})\, d\mathcal{H}^1 \\
&\quad + \int_{\gamma_k \cap \partial S_{h_k,K_k} \cap \partial^* A_k \cap S_{h_k,K_k}^{(0)}} \phi_{\mathrm{F}}(\nu_{A_k})\, d\mathcal{H}^1 + \sum_{i=1}^{2} \int_{T_i^j} \phi_{\mathrm{F}}(\mathbf{e_2})\, d\mathcal{H}^1 \\
&\geq \int_{\widetilde{\gamma}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{\gamma}_k})\, d\mathcal{H}^1,
\end{aligned} \tag{3.4.94}$$

where $\widehat{\gamma}_k := T_1^j \cup \gamma_k \cup T_2^j$. By the anisotropic minimality of segments, we deduce that

$$\int_{\widehat{\gamma}_k} \phi_{\mathrm{F}}(\nu_{\widetilde{\gamma}_k})\, d\mathcal{H}^1 \geq \int_{C_j} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1. \tag{3.4.95}$$

By (3.2.15), (3.4.94) and (3.4.95), we conclude that

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - 2c_2\delta'. \tag{3.4.96}$$

Step 2. Let $k \geq k^2_{\delta'}$ and $j \in J$ be such that $\mathcal{H}^1(\Gamma^{\mathrm{A}}_{\mathrm{FS}}(A_k, S_{h_k,K_k}) \cap S^j) > 0$. By reasoning as in the proof of Lemma 3.4.8 to reach (3.4.62) we obtain

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - 7c_2\delta'. \tag{3.4.97}$$

More precisely, we reach (3.4.97) by noticing that $\nu_{T_0} = \mathbf{e_1}$ and by following the proof of Step 2 (from after equation (3.4.57)) and of Step 3 of Lemma 3.4.8 with the only difference that we replace the reference to Step 1 of Lemma 3.4.8 with the reference to Step 1 of the current lemma, and the extra term appearing in (3.4.63) with

$$\sum_{i=1,2} \int_{T_i^j} (2\phi_{\mathrm{F}} + \phi)(\boldsymbol{e_2})\, d\mathcal{H}^1. \tag{3.4.98}$$

Notice that the difference of two units more in the error term of (3.4.97) with respect to (3.4.62) is due to the fact that the extra term (3.4.98) contributes to the final error with exactly two units more.

Step 3. Fix $k \geq k^2_{\delta'}$ and let $j \in J$. By (3.4.96) if $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k \cap S^j) = 0$ and by (3.4.97) if $\mathcal{H}^1(\Gamma^{\mathrm{A}}_k \cap S^j) > 0$ we obtain that

$$\mathcal{S}_L(A_k, S_k, S^j) \geq \int_{C_j} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - 7c_2\delta'. \tag{3.4.99}$$

Therefore, since the cardinality of J is bounded by $2(m_1 + 3)$, (3.4.99) yields

$$\begin{aligned} \mathcal{S}_L(A_k, S_{h_k,K_k}, R) &\geq \sum_{j\in J} \mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq \int_{\bigcup_{j\in J} C_j} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - 7(2m_1 + 3)c_2\delta' \\ &\geq \int_{T_0 \cap \overline{R}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (m_1 + 2)c_2\delta' - 7(2m_1 + 3)c_2\delta' \\ &= \int_{T_0 \cap \overline{R}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (15m_1 + 24)c_2\delta'. \end{aligned} \tag{3.4.100}$$

By the non-negativeness of ϕ_{F}, ϕ and ϕ_{FS} we have that

$$\begin{aligned} \mathcal{S}_L(A_k, S_{h_k,K_k}, Q) \geq \mathcal{S}_L(A_k, S_{h_k,K_k}, R) &\geq \int_{T_0 \cap \overline{R}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (15m_1 + 24)c_2\delta' \\ &\geq \int_{T_0 \cap \overline{Q}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (15m_1 + 25)c_2\delta', \end{aligned} \tag{3.4.101}$$

where in the second inequality we used (3.4.101) and in the last inequality we added and subtracted $\int_{T_0 \cap \overline{T_1^{\delta'}}} \phi_{\mathrm{F}}(\boldsymbol{e_1})\, d\mathcal{H}^1$, and we used (3.2.15) and the fact that $\mathcal{H}^1(T_0 \cap \overline{T_1^{\delta'}}) \leq \delta'/2$. Finally, (3.4.90) follows from choosing $k_\delta := k^2_{\delta'}$ and $\delta' = \frac{\delta}{(15m_1+25)c_2}$ for $\delta \in (0, \min\{(15m_1 + 25)c_2, 1\})$ in (3.4.111). This completes the proof. □

We conclude these list of estimates by addressing the setting of the delaminated substrate filaments in the film.

Lemma 3.4.12. *Let T_0 be the x_2-axis. Let $\{\rho_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\rho_k \searrow 0$ and $Q \subset \sigma_{\rho_1}(\Omega) \cap H_{0,-\boldsymbol{e_2}}$ be an open square whose sides are either parallel or perpendicular to $\boldsymbol{e_1}$ and $T_0 \cap Q \neq \emptyset$. If $\{(A_k, S_{h_k,K_k})\} \subset \mathcal{B}_{\boldsymbol{m}}(\sigma_{\rho_1}(\Omega))$ is a sequence such that $\overline{Q} \cap S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q}$ and $\overline{Q} \setminus A_k \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q}$, then for every $\delta \in (0,1)$, there exists $k_\delta \in \mathbb{N}$ such that for any $k \geq k_\delta$,*

$$\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) \geq \int_{T_0\cap\overline{Q}} 2\phi_{\mathrm{F}}(\boldsymbol{e_1})\, d\mathcal{H}^1 - \delta. \tag{3.4.102}$$

Proof. Without loss of generality we assume that $\sup_{k\in\mathbb{N}} \mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$. Since $\overline{Q} \cap S_{h_k,K_k} \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q}$ and $\overline{Q} \setminus A_k \xrightarrow{\mathcal{K}} T_0 \cap \overline{Q}$ in $\mathbb{R}^2$ as $k \to \infty$ for every $\delta' \in (0,1)$ there exists $k_{\delta'} \in \mathbb{N}$ such that

$$\overline{Q} \cap S_{h_k,K_k},\ \overline{Q} \setminus A_k \subset T^{\delta'}, \tag{3.4.103}$$

where $T^{\delta'} := \{x \in Q : \mathrm{dist}(x, T_0) < \frac{\delta'}{2}\}$. Let T_1 be the upper side of Q and let $T_1^{\delta'} := \{x \in Q : \mathrm{dist}(x, T_1) < \delta'/2\}$. By the Kuratowski convergence of $\overline{Q} \cap S_{h_k,K_k}$, there exists $k^1_{\delta'} \geq k_{\delta'}$ such that $S_{h_k} \cap \overline{T_1^{\delta'}} \neq \emptyset$ for every $k \geq k^1_{\delta'}$. Let $R := (T^{\delta'} \setminus \overline{T_1^{\delta'}})$ and let $T_1', T_2' \subset \partial R$ be the upper and lower side of the rectangle R, respectively.

Since $(A_k, S_{h_k,K_k}) \in \mathcal{B}_{\mathbf{m}}(\sigma_k(\Omega))$ we can find an enumeration $\{\Lambda_k^n\}_{n=1,\dots,m_k^1}$ of the connected components Λ_k^n of ∂A_k lying strictly inside of R, such that $m_k^1 \leq m_1$. Moreover, thanks to the fact that $\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) < \infty$ for each $k \in \mathbb{N}$, the family $\{\Lambda_k^\alpha\}_{\alpha\in\mathbb{N}}$ of connected components Λ_k^α of $\partial A_k \cap \overline{R}$ that intersect T_1' or T_2', respectively, are at most countable. Furthermore, we define Λ^{m_k+i} for $i = 1, 2$ by

$$\Lambda_k^{m_k+i} := \left(\bigcup_{\alpha\in\mathbb{N},\, \Lambda_k^\alpha \cap T_i^{\delta'} \neq \emptyset} \Lambda_k^\alpha \right) \cup T_i'.$$

Thanks to the Kuratowski convergences of $\overline{Q} \setminus A_k$ to $T_0 \cap \overline{R}$ in $\mathbb{R}^2$ as $k \to \infty$, the fact that $m_k^1 \leq m_1$ for every $k \in \mathbb{N}$, we have that

$$\lim_{k\to\infty} \mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) \right) = 0$$

Hence, there exists $k^2_{\delta'} \geq k^1_{\delta'}$ such that

$$\mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) \right) < (m_1 + 2)\delta' \tag{3.4.104}$$

for every $k \geq k^2_{\delta'}$. Similarly to Step 2 of Lemma 3.4.8, we can decompose $\bigcup_{n=1}^{m_k+2} \pi_2(\Lambda_k^n)$ as the finite union of disjoint open connected sets $\mathscr{C} := \{C_j\}_{j\in J}$. Notice that the cardinality of J is bounded by $2m_k^1 + 3$. Therefore

$$\bigcup_{n=1}^{m_k^1+2} \pi_2(\Lambda_k^n) = \bigcup_{j\in J} C_j,$$

and also by (3.4.104) we have that

$$\mathcal{H}^1 \left((T_0 \cap \overline{R}) \setminus \bigcup_{j\in J} C_j \right) < (m_1 + 2)c_2\delta' \tag{3.4.105}$$

for every $k \geq k^2_{\delta'}$. Finally, let $S^j := (\pi_1(R) \times_{\mathbb{R}^2} C_j) \cap T^{\delta'}$ and let T_1^j and T_2^j be the upper and lower sides of the boundary of each rectangle S^j, respectively.

Step 1. Let $k \geq k^2_{\delta'}$ and $j \in J$ be such that $\mathcal{H}^1(\Gamma^{\rm A}_{\rm FS}(A_k, S_{h_k,K_k}) \cap S^j) = 0$. In view of the construction of S^j, by [44, Lemma 3.12] we can find a curve with support $\gamma^1_k \subset \partial(A_k \setminus \mathrm{Int}(S_{h_k}))$ joining T^j_1 with T^j_2, and there exists only one connected component of $\partial(A_k \setminus \mathrm{Int}(S_{h_k}))$ in S^j. Therefore, by applying [58, Lemma 4.3] there exists a curve connecting T^j_1 and T^j_2 with support $\gamma^2_k \subset S^j \cap \partial(A_k \setminus \mathrm{Int}(S_{h_k})) \setminus (\gamma^1_k \cap \partial\mathrm{Int}(\overline{A_k \setminus \mathrm{Int}(S_{h_k})}))$. It follows that

$$
\begin{aligned}
&\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) + 2\sum_{i=1}^{2} \int_{T'_i} \phi_{\rm F}(\mathbf{e_2})\, d\mathcal{H}^1 \\
&\qquad \geq \int_{(\gamma^1_k \cup \gamma^2_k) \cap (\partial^* A_k \setminus \partial S_{h_k,K_k})} \phi_{\rm F}(\nu_{A_k})\, d\mathcal{H}^1 \\
&\qquad\quad + \int_{(\gamma^1_k \cup \gamma^2_k) \cap (\partial A_k \setminus \partial S_{h_k,K_k}) \cap A^{(0)}_k} 2\phi_{\rm F}(\nu_{A_k})\, d\mathcal{H}^1 \\
&\qquad\quad + \int_{(\gamma^1_k \cup \gamma^2_k) \cap \partial^* S_{h_k,K_k} \cap \partial A_k \cap A^{(1)}_k} \phi_{\rm F}(\nu_{A_k})\, d\mathcal{H}^1 \\
&\qquad\quad + \int_{(\gamma^1_k \cup \gamma^2_k) \cap \partial S_{h_k,K_k} \cap \partial^* A_k \cap S^{(0)}_{h_k,K_k}} \phi_{\rm F}(\nu_{A_k})\, d\mathcal{H}^1 + 2\sum_{i=1}^{2} \int_{T'_i} \phi_{\rm F}(\mathbf{e_2})\, d\mathcal{H}^1 \\
&\qquad \geq \sum_{i=1}^{2} \int_{\widetilde{\gamma}^i_k} \phi_{\rm F}(\nu_{\widetilde{\gamma}^i_k})\, d\mathcal{H}^1,
\end{aligned}
\tag{3.4.106}
$$

where $\widetilde{\gamma}^i_k := T^j_1 \cup \gamma^i_k \cup T^j_2$ for $i = 1, 2$. By the anisotropic minimality of segments, we deduce that

$$
\sum_{i=1}^{2} \int_{\widetilde{\gamma}^i_k} \phi_{\rm F}(\nu_{\widetilde{\gamma}^i_k})\, d\mathcal{H}^1 \geq 2\int_{C_j} \phi_{\rm F}(\mathbf{e_1})\, d\mathcal{H}^1. \tag{3.4.107}
$$

By (3.2.15), (3.4.106) and (3.4.107), we conclude that

$$
\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi_{\rm F}(\mathbf{e_1})\, d\mathcal{H}^1 - 4c_2\delta'. \tag{3.4.108}
$$

Step 2. Let $k \geq k^2_{\delta'}$ and $j \in J$ be such that $\mathcal{H}^1(\Gamma^{\rm A}_{\rm FS}(A_k, S_{h_k,K_k}) \cap S^j) > 0$. By reasoning as in the proof of Lemma 3.4.8 to reach (3.4.62) we obtain

$$
\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi_{\rm F}(\mathbf{e_1})\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.109}
$$

More precisely, we reach (3.4.109) by noticing that $\nu_{T_0} = \mathbf{e_1}$ and by following the proof of Step 2 (from after equation (3.4.57)) and of Step 3 of Lemma 3.4.8 with the only difference that we replace the reference to Step 1 of Lemma 3.4.8 with the reference to Step 1 of the current lemma, and the extra term appearing in (3.4.63) with

$$
\sum_{i=1,2} \int_{T^j_i} (3\phi_{\rm F} + \phi)(\boldsymbol{e_2})\, d\mathcal{H}^1.
$$

Step 3. Fix $k \geq k^2_{\delta'}$ and let $j \in J$. By (3.4.108) if $\mathcal{H}^1(\Gamma^{\rm A}_k \cap S^j) = 0$ and by (3.4.109) if $\mathcal{H}^1(\Gamma^{\rm A}_k \cap S^j) > 0$ we obtain that

$$
\mathcal{S}_L(A_k, S_{h_k,K_k}, S^j) \geq 2\int_{C_j} \phi_{\rm F}(\mathbf{e_1})\, d\mathcal{H}^1 - 9c_2\delta'. \tag{3.4.110}
$$

Therefore, the same reasoning of Step 3 of Lemma 3.4.11 yields that

$$\begin{aligned}\mathcal{S}_L(A_k, S_{h_k,K_k}, Q) &\geq \mathcal{S}_L(A_k, S_{h_k,K_k}, R) \geq 2\int_{T\cap\overline{R}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (35m_1+56)c_2\delta' \\ &\geq 2\int_{T\cap\overline{Q}} \phi_{\mathrm{F}}(\mathbf{e_1})\, d\mathcal{H}^1 - (35m_1+57)c_2\delta',\end{aligned} \tag{3.4.111}$$

where as a difference with Step 3 of Lemma 3.4.11 we added and subtracted $2\int_{T_0\cap\overline{T_1^{\delta'}}} \phi_{\mathrm{F}}(\boldsymbol{e_1})\, d\mathcal{H}^1$ in the last inequality. Finally, (3.4.102) follows from choosing $k_\delta := k^2_{\delta'}$ and $\delta' = \frac{\delta}{(35m_1+57)c_2}$ for $\delta \in (0, \min\{(35m_1+57)c_2, 1\})$ in (3.4.111). This completes the proof. ☐

We are now in the position to prove that the surface energy $\mathcal{S}$ is lower semicontinuous in $\mathcal{B}_{\mathbf{m}}$ with respect to the $\tau_{\mathcal{B}}$-convergence.

Theorem 3.4.13 (Lower semicontinuity of $\mathcal{S}$)**.** *Let $(A_k, S_{h_k,K_k})_{k\in\mathbb{N}} \subset \mathcal{B}_{\mathbf{m}}$ and $(A, S_{h,K}) \in \mathcal{B}_{\mathbf{m}}$ such that $(A_k, S_{h_k,K_k}) \xrightarrow{\tau_{\mathcal{B}}} (A, S_{h,K})$ as $k \to \infty$. Then,*

$$\mathcal{S}(A, S_{h,K}) \leq \liminf_{k\to\infty} \mathcal{S}(A_k, S_{h_k,K_k}). \tag{3.4.112}$$

Proof. Without loss of generality, we assume that the *liminf* in the right side of (3.4.112) is reached and finite in $\mathbb{R}$. For every $k \in \mathbb{N}$ we denote $S_k := S_{h_k,K_k} \in \mathrm{A}S(\Omega)$ for simplicity and we define μ_k as the Radon measure associated to $\mathcal{S}(A_k, S_k)$, i.e., the measure μ_k given by

$$\mu_k(B) := \int_{B\cap\Omega\cap(\partial A_k\cup\partial S_k)} \psi_k(x, \nu_k\,(x))\; d\mathcal{H}^1$$

for every Borel set $B \subset \mathbb{R}^2$, where $\nu_k := \nu_{\partial A_k \cup \partial S_k}$ and the surface tension ψ_k is defined by

$$\psi_k(x,\nu_k(x)) := \begin{cases} \varphi_{\mathrm{F}}(x,\nu_{A_k}(x)) & \text{if } x \in \partial^* A_k \setminus \partial S_k \\ \varphi(x,\nu_{A_k}(x)) & \text{if } x \in \partial^* A_k \cap \partial^* S_k, \\ 2\varphi_{\mathrm{F}}(x,\nu_{A_k}(x)) & \text{if } x \in (\partial A_k \setminus \partial S_k)\cap A^{(1)}, \\ 2\varphi'(x,\nu_{A_k}(x)) & \text{if } x \in (\partial A_k \setminus \partial S_k)\cap A^{(0)}, \\ \varphi_{\mathrm{FS}}(x,\nu_{S_k}(x)) & \text{if } x \in (\partial^* S_k \setminus \partial A_k)\cap A_k^{(1)}, \\ 2\varphi(x,\nu_{A_k}(x)) & \text{if } x \in (\partial S_k \cap \partial A_k)\cap S_k^{(1)}, \\ 2\varphi'(x,\nu_{A_k}(x)) & \text{if } x \in (\partial S_k \cap \partial A_k)\cap A^{(0)}, \\ \phi(x,\nu_{A_k}(x)) & \text{if } x \in \partial S_k \cap \partial^* A_k \cap S_k^{(0)}, \\ 2\varphi_{\mathrm{FS}}(x,\nu_{S_k}(x)) & \text{if } x \in (\partial S_k \setminus \partial A_k)\cap (S_k^{(1)} \cup S_k^{(0)})\cap A_k^{(1)}, \\ (\varphi_{\mathrm{F}}+\varphi)(x,\nu_{A_k}(x)) & \text{if } x \in \partial A_k \cap \partial^* S_k \cap A^{(1)}, \\ 2\varphi_{\mathrm{F}}(x,\nu_{S_k}(x)) & \text{if } x \in (\partial S_k \cap \partial A_k)\cap S_k^{(0)} \cap A_k^{(1)}. \end{cases} \tag{3.4.113}$$

Furthermore, we denote by μ the Radon measure associated to $\mathcal{S}(A, h, K)$, i.e., the measure μ given by

$$\mu(B) := \int_{B\cap\Omega\cap(\partial A\cup\partial S_{h,K})} \psi(x, \nu_{\partial A\cup\partial S_{h,K}}\,(x))\; d\mathcal{H}^1$$

for every Borel set $B \subset \mathbb{R}^2$, where ψ is defined analogously to ψ_k in (3.4.113), but with the sets A_k and S_k replaced with A and $S_{h,K}$, respectively.

We observe that by (H1) there exists $c := c(c_2) > 0$ such that

$$\mu_k(\mathbb{R}^2) = \mathcal{S}(A_k, S_k) \leq c\left(\mathcal{H}^1(\partial A_k) + \mathcal{H}^1(\partial S_k)\right)$$

and since $(A_k, S_k) \xrightarrow{\tau_{\mathcal{B}}} (A, S_{h,K})$, we obtain that $\sup_k \mu_k(\mathbb{R}^2) < +\infty$. It follows that $\{\mu_k\}$ is a sequence of bounded Radon measures and hence, owing to the weak* compactness of Radon measures (see [68, Theorem 4.33]), there exist a not relabeled subsequence $\{\mu_k\}$ and a Radon measure μ_0 such that $\mu_k \overset{*}{\rightharpoonup} \mu_0$ as $k \to \infty$. The purpose of this proof is to show the following inequality in the sense of measures

$$\mu_0 \geq \mu. \tag{3.4.114}$$

Since μ_0 and μ are non-negative measures and $\mu << \mathcal{H}^1 \llcorner (\partial A \cup \partial S_{h,K})$, to obtain (3.4.114), it is enough to prove that the surface tension ψ of μ on each subset of $\partial A \cup \partial S_{h,K}$ on which it is uniquely defined, is bounded from above by the Radon-Nikodym derivative of μ_0 with respect to the $\mathcal{H}^1$-measure of the corresponding subset, namely the following 12 inequalities:

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* A \setminus \partial S_{h,K})}(x) \geq \varphi_{\mathrm{F}}(x, \nu_A(x)) \text{ for } \mathcal{H}^1\text{-a.e. } x \in \partial^* A \setminus \partial S_{h,K}, \tag{3.4.115}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial^* A)}(x) \geq \varphi(x, \nu_A(x)) \text{ for } \mathcal{H}^1\text{-a.e. } x \in \partial^* S_{h,K} \cap \partial^* A, \tag{3.4.116}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A \setminus \partial S_{h,K}) \cap A^{(1)})}(x) \geq 2\varphi_{\mathrm{F}}(x, \nu_A(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in (\partial A \setminus \partial S_{h,K}) \cap A^{(1)}, \tag{3.4.117}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A \setminus \partial S_{h,K}) \cap A^{(0)})}(x) \geq 2\varphi'(x, \nu_A(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in (\partial A \setminus \partial S_{h,K}) \cap A^{(0)}, \tag{3.4.118}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial^* S_{h,K} \setminus \partial A) \cap A^{(1)})}(x) \geq \varphi_{\mathrm{FS}}(x, \nu_{S_{h,K}}(x)) \quad \text{for } \mathcal{H}^1\text{- a.e. } x \in (\partial^* S_{h,K} \setminus \partial A) \cap A^{(1)}, \tag{3.4.119}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(1)})}(x) \geq 2\varphi(x, \nu_A(x)) \text{ for } \mathcal{H}^1\text{-a.e. } x \in (\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(1)}, \tag{3.4.120}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap A^{(0)})}(x) \geq 2\varphi'(x, \nu_A(x)) \text{ for } \mathcal{H}^1\text{-a.e. } x \in (\partial S_{h,K} \cap \partial A) \cap A^{(0)}, \tag{3.4.121}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial S_{h,K} \cap \partial^* A \cap S_{h,K}^{(0)})}(x) \geq \varphi_{\mathrm{F}}(x, \nu_A(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial S_{h,K} \cap \partial^* A \cap S_{h,K}^{(0)}, \tag{3.4.122}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(1)} \cap A^{(1)})}(x) \geq 2\varphi_{\mathrm{FS}}(x, \nu_{S_{h,K}}(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in (\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(1)} \cap A^{(1)}, \tag{3.4.123}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)})}(x) \geq 2\varphi_{\mathrm{FS}}(x, \nu_{S_{h,K}}(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in (\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)}, \tag{3.4.124}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial A \cap A^{(1)})}(x) \geq (\varphi_{\mathrm{F}} + \varphi)(x, \nu_A(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial^* S_{h,K} \cap \partial A \cap A^{(1)} \tag{3.4.125}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)})}(x) \geq 2\varphi_{\mathrm{F}}(x, \nu_{S_{h,K}}(x)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in (\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)}. \tag{3.4.126}$$

The rest of the proof is devoted to establish the previous 12 lower-bound estimates. In order to do that we fix $\varepsilon > 0$ small enough and we recall that from the uniform continuity of $\varphi_{\mathrm{F}}, \varphi_{\mathrm{S}}, \varphi_{\mathrm{FS}}$ it follows that there exists a $\delta_\varepsilon > 0$ such that

$$\varphi_{\mathrm{F}}(y,\xi) \geq \varphi_{\mathrm{F}}(x_0,\xi) - \varepsilon, \quad \varphi_{\mathrm{S}}(y,\xi) \geq \varphi_{\mathrm{S}}(x_0,\xi) - \varepsilon, \text{ and } \varphi_{\mathrm{FS}}(y,\xi) \geq \varphi_{\mathrm{FS}}(x_0,\xi) - \varepsilon, \tag{3.4.127}$$

for every $y \in Q_{\delta_\varepsilon}(x_0) \subset \Omega$, $x_0 \in \Omega$ and $|\xi| = 1$. The proofs of (3.4.115) and (3.4.119) are based on [68, Theorem 20.1], the proofs of (3.4.116), (3.4.120), (3.4.121), (3.4.122), (3.4.125) and (3.4.126) are based on Lemmas 3.4.7, 3.4.8, 3.4.9, 3.4.10, 3.4.11 and 3.4.12, respectively. Finally, the proofs of (3.4.118) and (3.4.124) are based on [58, Lemma 4.4], and the proofs of (3.4.117) and (3.4.123) are based on [58, Lemma 4.5] (See Table 3.1).

Sets	**Conditions**	**Surf. t.**	**Assertions**
$\partial^* A \setminus \partial S_{h,K}$	$\nu_{A_k}\mathcal{H}^1(\partial^* A_k) \xrightharpoonup{*} \nu_A \mathcal{H}^1(\partial^* A)$	φ_{F}	[68, Theorem 20.1]
$\partial^* S_{h,K} \cap \partial^* A$	$\overline{R_{\nu_A}} \cap (\overline{A_{k_n}} \setminus \mathrm{Int}(S_{k_n})) \xrightarrow{\mathcal{K}} \overline{R_{\nu_A}} \cap T_{0,\nu_A}$	φ	Lemma 3.4.7
$(\partial A \setminus \partial S_{h,K}) \cap A^{(1)}$	$\overline{Q_1} \setminus A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi_{\mathrm{F}}$	[58, Lemma 4.5]
$(\partial A \setminus \partial S_{h,K}) \cap A^{(0)}$	$\overline{Q_1} \cap A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi'$	[58, Lemma 4.4]
$\partial^* S_{h,K} \setminus \partial A$	$\nu_{S_k}\mathcal{H}^1(\partial^* S_k) \xrightharpoonup{*} \nu_{S_{h,K}} \mathcal{H}^1(\partial^* S_{h,K})$	φ_{FS}	[68, Theorem 20.1]
$(\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(1)}$	$\overline{Q_1} \setminus A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$, $\overline{Q_1} \setminus S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	2φ	Lemma 3.4.9
$(\partial S_{h,K} \cap \partial A) \cap A^{(0)}$	$\overline{Q_1} \cap A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$, $\overline{Q_1} \cap S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi'$	Lemma 3.4.10
$\partial S_{h,K} \cap \partial^* A \cap S_{h,K}^{(0)}$	$\overline{Q_1} \cap A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap H_{0,\mathbf{e_1}}$, $\overline{Q_1} \cap S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	φ_{F}	Lemma 3.4.11
$(\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(1)} \cap A^{(1)}$	$\overline{Q_1} \setminus S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi_{\mathrm{FS}}$	[58, Lemma 4.5]
$(\partial S_{h,K} \setminus \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)}$	$\overline{Q_1} \cap S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi_{\mathrm{FS}}$	[58, Lemma 4.4]
$\partial^* S_{h,K} \cap \partial A \cap A^{(1)}$	$\overline{R_{\nu_{S_{h,K}}}} \cap S_{k_n} \xrightarrow{\mathcal{K}} \overline{R_{\nu_{S_{h,K}}}} \cap H_{0,\nu_{S_{h,K}}}$, $\overline{R_{\nu_{S_{h,K}}}} \setminus A_{k_n} \xrightarrow{\mathcal{K}} \overline{R_{\nu_{S_{h,K}}}} \cap T_{0,\nu_{S_{h,K}}}$	$\varphi_{\mathrm{F}} + \varphi$	Lemma 3.4.8
$(\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(0)} \cap A^{(1)}$	$\overline{Q_1} \cap S_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$, $\overline{Q_1} \setminus A_{k_n} \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_{0,\mathbf{e_1}}$	$2\varphi_{\mathrm{F}}$	Lemma 3.4.12

Table (3.1): Sketch of the proof of (3.4.115)–(3.4.126) for Theorem 3.4.13: in the blow-ups centered at a point of the sets listed in the first column, the corresponding conditions listed in the second column are proven to hold and the lower bounds of the localized surface energy are reached with surface tensions given in the third column by means of the assertions listed in the fourth column. Note that R_{ν_U} for $U = A, S_{h,K}$ is defined as $R_{\nu_U} := Q_1$ if $\nu_U = \mathbf{e}_i$ for $i = \mathbf{1}, \mathbf{2}$, or, otherwise, $R_{\nu_U} := (-\cos\theta_{\nu_U}, \cos\theta_{\nu_U}) \times_{\mathbb{R}^2} (-\sin\theta_{\nu_U}, \sin\theta_{\nu_U})$, where θ_{ν_U} is the angle formed between T_{0,ν_U} and the x_1-axis.

Proof of (3.4.115). We begin by observing that by the definition of $\partial^* A$, the continuity of φ, the Borel regularity of $y \in \partial^* A \mapsto \varphi_{\mathrm{F}}(y, \nu_A(y))$ and the Besicovitch Derivation Theorem (see [47, Theorem 1.153]), the set of points in $\partial^* A \setminus \partial S_{h,K}$ not satisfying the following 3 conditions:

(a1) $\nu_A(x)$ exists,

(a2) x is a Lebesgue point of $y \in \partial^* A \setminus \partial S_{h,K} \mapsto \varphi_{\mathrm{F}}(y, \nu_A(y))$, i.e.,

$$\lim_{r\to 0} \frac{1}{2r} \int_{Q_r \cap \partial^* A \setminus \partial S_{h,K}} |\varphi_{\mathrm{F}}(y, \nu_A(y)) - \varphi_{\mathrm{F}}(x, \nu_A(x))| \, d\mathcal{H}^1(y) = 0,$$

(a3) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* A \setminus \partial S_{h,K})}(x)$ exists and it is finite,

is $\mathcal{H}^1$-negligible. Therefore, we prove (3.4.115) for a fixed $x \in \partial^* A \setminus \partial S_{h,K}$ satisfying (a1)-(a3). Without loss of generality, we consider $x = 0$ and $\nu_A(0) = \mathbf{e_1}$, where we used (a1).

By [58, Lemma 3.2-(b)] and $\tau_{\mathcal{B}}$-convergence we have that $A_k \to A$ and $S_k \to S$ in $L^1(\mathbb{R}^2)$ and hence, $D\mathbb{1}_{A_k} \xrightharpoonup{*} D\mathbb{1}_A$ and $D\mathbb{1}_{S_k} \xrightharpoonup{*} D\mathbb{1}_S$, hence by the Structure Theorem for sets of finite perimeter (see [43, Theorem 5.15]), it holds that

$$\nu_{A_k} \mathcal{H}^1(\partial^* A_k) \xrightharpoonup{*} \nu_A \mathcal{H}^1(\partial^* A). \tag{3.4.128}$$

Furthermore, by Remark (3.2.8)-(i) and again the the $\tau_{\mathcal{B}}$-convergence we obtain that $\partial S_k \xrightarrow{\mathcal{K}} \partial S_{h,K}$, from which it follows that for any $\eta > 0$ there exists $k_\eta \in \mathbb{N}$ such that $\partial S_k \subset S^\eta$ for every $k \geq k_\eta$, where $S^\eta := \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial S_{h,K}) \leq \eta\}$.

We observe that from the properties of Radon measures there exists a sequence $\rho_n \searrow 0$ such that $Q_{\rho_n} \subset\subset \Omega$, $\mu_0(\partial Q_{\rho_n}) = 0$,

$$\mu_0(Q_{\rho_n}) = \lim_{k\to+\infty} \mu_k(\overline{Q_{\rho_n}}) \tag{3.4.129}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* A \setminus \partial S_{h,K})}(0) = \lim_{n\to\infty} \frac{\mu_0(Q_{\rho_n})}{2\rho_n}, \tag{3.4.130}$$

where we also used (a3) and Besicovitch Derivation Theorem (see, e.g., [47, Theorem 1.153]). Therefore, by (3.4.129) we deduce that

$$\begin{aligned} \mu_0(Q_{\rho_n}) = \lim_{k\to+\infty} \mu_k(\overline{Q_{\rho_n}}) &\geq \liminf_{k\to\infty} \int_{Q_{\rho_n} \cap \partial^* A_k \setminus \partial S_k} \varphi_{\mathrm{F}}(y, \nu_{A_k}) \, d\mathcal{H}^1 \\ &\geq \liminf_{k\to\infty} \int_{Q_{\rho_n} \cap \partial^* A_k \setminus S^\eta} \varphi_{\mathrm{F}}(y, \nu_{A_k}) \, d\mathcal{H}^1 \geq \int_{Q_{\rho_n} \cap \partial^* A \setminus S^\eta} \varphi_{\mathrm{F}}(y, \nu_A) \, d\mathcal{H}^1, \end{aligned} \tag{3.4.131}$$

where in the first inequality we used the non-negativeness of ψ_k, in the second inequality we used the fact that $\partial S_k \subset S^\eta$ for every $k \geq k_\eta$ and in the last inequality, by using the fact that $A_k \to A$ in $L^1(\mathbb{R}^2)$ and (3.4.128), we apply (see, e.g., [68, Theorem 20.1]). Moreover, taking $\eta \to 0$ in (3.4.131) we obtain that

$$\mu_0(Q_{\rho_n}) \geq \lim_{\eta\to 0} \int_{Q_{\rho_n} \cap \partial^* A \setminus S^\eta} \varphi_{\mathrm{F}}(y, \nu_A) \, d\mathcal{H}^1 = \int_{Q_{\rho_n} \cap \partial^* A \setminus \partial S_{h,K}} \varphi_{\mathrm{F}}(y, \nu_A) \, d\mathcal{H}^1 \tag{3.4.132}$$

by Lebesgue monotone convergence theorem [47, Theorem 1.79]. Finally, by (3.4.130) and (a2) we conclude that

$$\begin{aligned} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* A \setminus \partial S_{h,K})}(0) &= \lim_{n\to\infty} \frac{\mu_0(Q_{\rho_n})}{2\rho_n} \geq \liminf_{n\to\infty} \frac{1}{2\rho_n} \int_{Q_{\rho_n} \cap \partial^* A \setminus \partial S_{h,K}} \varphi_{\mathrm{F}}(y, \nu_A) \, d\mathcal{H}^1 \\ &= \varphi_{\mathrm{F}}(0, \mathbf{e_1}). \end{aligned}$$

□

Proof of (3.4.116). By the definition of $\partial^* A$ and $\partial^* S_{h,K}$, by [58, Proposition A.4] (applied with K taken as first ∂A and then $\partial S_{h,K}$), and by the Besicovitch Derivation Theorem the set of points $x \in \partial^* A \cap \partial^* S_{h,K}$ not satisfying the following 3 conditions:

(b1) $\nu_A(x)$, $\nu_{S_{h,K}}(x)$ exist and, either $\nu_A(x) = \nu_{S_{h,K}}(x)$ or $\nu_A(x) = -\nu_{S_{h,K}}(x)$,

(b2) for every open rectangle R containing x with sides parallel or perpendicular to $\mathbf{e_1}$ we have that $\overline{R} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{R} \cap T_{x,\nu_A(x)}$ and $\overline{R} \cap \partial\sigma_{\rho,x}(S_{h,K}) \xrightarrow{\mathcal{K}} \overline{R} \cap T_{x,\nu_A(x)}$ as $\rho \to 0$, where $T_{x,\nu_A(x)}$ is the approximate tangent line at x of ∂A (or of $\partial S_{h,K}$),

(b3) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial^* A)}(x)$ exists and it is finite,

is $\mathcal{H}^1$-negligible. Therefore, we prove (3.4.116) for any fixed $x \in \partial^* A \cap \partial^* S_{h,K}$ satisfying (b1)-(b3). Without loss of generality we assume that $x = 0$ and we denote $T_0 = T_{0,\nu_A(0)}$. Furthermore, by using (b1) we choose in (b2) the rectangle $R_{\nu_A} := Q_1$ if $\nu_A(0) = \mathbf{e}_i$ for $i = \mathbf{1}, \mathbf{2}$, or $R_{\nu_A} := (-\cos\theta_{\nu_A}, \cos\theta_{\nu_A}) \times_{\mathbb{R}^2} (-\sin\theta_{\nu_A}, \sin\theta_{\nu_A})$, where θ_{ν_A} is the angle formed between the tangent line T_0 and the x_1-axis, otherwise. For any $\rho > 0$, we write $R_\rho := \rho R_{\nu_A}$.

In view of the definition of R_{ν_A} and again by using also the Besicovitch Derivation Theorem (see [47, Theorem 1.153]) there exists a subsequence $\rho_n \searrow 0$ such that

$$\mu_0(\partial R_{\rho_n}) = 0, \qquad \lim_{k \to +\infty} \mu_k(\overline{R_{\rho_n}}) = \mu_0(R_{\rho_n}) \tag{3.4.133}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial^* A)}(0) = \lim_{n \to \infty} \frac{\mu_0(R_{\rho_n})}{2\rho_n}. \tag{3.4.134}$$

We now claim that

$$\operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to \operatorname{sdist}(\cdot, \partial H_0) \quad \text{and} \quad \operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{h,K})) \to \operatorname{sdist}(\cdot, \partial H_0) \tag{3.4.135}$$

uniformly in $\overline{R_{\nu_A}}$ as $n \to \infty$, where H_0 is the half space centered in 0 with respect to the vector ν_A. To prove the claim we can for example observe that by [58, Proposition A.4] we have (not only (b2), but also) that $\overline{Q_r} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{Q_r} \cap T_x$ and $\overline{Q_r} \cap \partial\sigma_{\rho,x}(S_{h,K}) \xrightarrow{\mathcal{K}} \overline{Q_r} \cap T_x$ as $\rho \to 0$ for any square Q_r such that $R_{\nu_A} \subset Q_r$ and hence, by Proposition 3.4.3-(c) applied to Q_r, $\operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to \operatorname{sdist}(\cdot, \partial H_0)$ and $\operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{h,K})) \to \operatorname{sdist}(\cdot, \partial H_0)$ uniformly in $\overline{Q_r} \supset R_{\nu_A}$.

Furthermore, from the $\tau_{\mathcal{B}}$-convergence it follows that

$$\operatorname{sdist}(\cdot, \partial A_k) \to \operatorname{sdist}(\cdot, \partial A) \quad \text{and} \quad \operatorname{sdist}(\cdot, \partial S_k) \to \operatorname{sdist}(\cdot, \partial S_{h,K}) \tag{3.4.136}$$

uniformly in $\overline{R_{\nu_A}}$ as $k \to \infty$ and hence, by (3.4.135) and (3.4.136), a standard diagonalization argument yields that there exists a subsequence $\{(A_{k_n}, h_{k_n}, K_{k_n})\}$ such that

$$\operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(A_{k_n})) \to \operatorname{sdist}(\cdot, \partial H_0), \quad \operatorname{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{k_n})) \to \operatorname{sdist}(\cdot, \partial H_0) \tag{3.4.137}$$

uniformly in $\overline{R_{\nu_A}}$ as $n \to \infty$ and by (3.4.133) such that

$$\mu_{k_n}(\overline{R_{\rho_n}}) \le \mu_0(R_{\rho_n}) + \rho_n^2, \tag{3.4.138}$$

for every $n \in \mathbb{N}$. We also observe that

$$\begin{aligned} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial^* A)}(0) &\ge \limsup_{n \to \infty} \frac{\mu_{k_n}(\overline{R_{\rho_n}})}{2\rho_n} \\ &\ge c_1 \limsup_{n \to \infty} \frac{\mathcal{H}^1(\overline{R_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{R_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})}{2\rho_n}, \end{aligned} \tag{3.4.139}$$

where in the first inequality we used (3.3.1)-(3.3.5), as A_{k_n} and S_{k_n} are sets of finite perimeter, and (3.2.15) while in the second inequality we used (3.4.134) and (3.4.138).

Now, we claim that

$$\overline{R_{\nu_A}} \cap (\overline{\sigma_{\rho_n}(A_{k_n})} \setminus \mathrm{Int}(\sigma_{\rho_n}(S_{k_n}))) \xrightarrow{\mathcal{K}} \overline{R_{\nu_A}} \cap T_0. \tag{3.4.140}$$

We proceed by contradiction, let $x_n \in \overline{R_{\nu_A}} \cap \overline{\sigma_{\rho_n}(A_{k_n})} \setminus \mathrm{Int}(\sigma_{\rho_n}(S_{k_n}))$ such that $x_n \to x$ and assume that $x \in \mathrm{Int}(\overline{R_{\nu_A}} \cap H_0)$ or $x \in R_{\nu_A} \setminus H_0$. In the first case, there exists $\varepsilon > 0$ such that $\mathrm{sdist}(x, \partial H_0) = -\varepsilon$. By (3.4.137) we observe that $\mathrm{sdist}(x, \partial\sigma_{\rho_n}(S_{k_n})) \to -\varepsilon$ uniformly in R_{ν_A} as $n \to \infty$. Furthermore, for n large enough, $x_n \in B_{\varepsilon/2}(x)$ and it follows that $\mathrm{sdist}(x_n, \partial\sigma_{\rho_n}(S_{k_n}))$ is negative. Therefore, for n large enough, $x_n \in \mathrm{Int}(\sigma_{\rho_n}(S_{k_n})))$ which is an absurd. Analogously if $x \in R_{\nu_A} \setminus H_0$ we have that $\mathrm{sdist}(x, \partial H_0) = \varepsilon$ and by (3.4.137), $\mathrm{sdist}(x, \partial\sigma_{\rho_n}(A_{k_n})) \to \varepsilon$ uniformly in R_{ν_A} as $n \to \infty$, similarly as before, we can conclude, for n large enough, that $x_n \in R_{\nu_A} \setminus \overline{\sigma_{\rho_n}(A_{k_n})}$, which is an absurd.

Now, let $x \in \overline{R_{\nu_A}} \cap T_0$. By Kuratowski convergence there exists $\{x_n\} \subset R_{\nu_A} \cap \partial\sigma_{\rho_n}(A_{k_n})$ such that $x_n \to x$. We see that for all $n \in \mathbb{N}$, $x_n \in R_{\nu_A} \cap \overline{\sigma_{\rho_n}(A_{k_n})}$ and $x_n \notin R_{\nu_A} \setminus \mathrm{Int}(\sigma_{\rho_n}(S_{k_n}))$, if not, there exists $n' \in \mathbb{N}$ such that $x_{n'} \in R_{\nu_A} \cap \mathrm{Int}(\sigma_{\rho_{n'}}(S_{k_{n'}})) \subset\subset R_{\nu_A} \cap \mathrm{Int}(\sigma_{\rho_{n'}}(A_{k_{n'}}))$, which is an absurd.

Since $\left\{(A_{k_n}, S_{h_{k_n},K_{k_n}})\right\} \subset \mathcal{B}_{\mathbf{m}}$, we know that $(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})}) \in \mathcal{B}_{\mathbf{m}}(\sigma_{\rho_n}(\Omega))$. In view of (3.4.137) and (3.4.140) by applying Lemma 3.4.7 to $(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})})$ and R_{ν_A}, with $\phi_\alpha(\cdot) = \varphi_\alpha(0,\cdot)$ for $\alpha =$ F,S,FS, and by fixing $\varepsilon \in (0,1)$, there exists $n^1_\varepsilon \in \mathbb{N}$ such that for every $n \geq n^1_\varepsilon$,

$$\mathcal{S}_L(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})}, R_{\nu_A}) \geq \int_{T_0\cap\overline{R_{\nu_A}}} \varphi(0,\nu_{T_0})\, d\mathcal{H}^1 - \varepsilon \geq 2\varphi(0,\nu_{T_0}) - \varepsilon. \tag{3.4.141}$$

Moreover, by the uniform continuity of the Finsler norm φ_α for $\alpha =$ F,S,FS there exists $n^2_\varepsilon \geq n^1_\epsilon$

such that

$$
\begin{aligned}
\mu_{k_n}(\overline{R_{\rho_n}}) \geq \mu_{k_n}&(R_{\rho_n}) \\
\geq &\int_{R_{\rho_n}\cap\partial^* A_{k_n}\setminus\partial S_{k_n}} \varphi_{\mathrm{F}}(0,\nu_{A_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap(\partial A_{k_n}\setminus\partial S_{k_n})\cap\left(A^{(0)}_{k_n}\cup A^{(1)}_{k_n}\right)} 2\varphi_{\mathrm{F}}(0,\nu_{A_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap\partial^* S_{k_n}\cap\partial^* A_{k_n}} \varphi(0,\nu_{A_{k_n}(0)})\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap(\partial^* S_{k_n}\setminus\partial A_{k_n})\cap A^{(1)}_{k_n}} \varphi_{\mathrm{FS}}(0,\nu_{S_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap(\partial S_{k_n}\cap\partial A_{k_n})\cap\left(S^{(1)}_{k_n}\cup A^{(0)}_{k_n}\right)} 2\varphi(0,\nu_{A_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap\partial S_{k_n}\cap\partial^* A_{k_n}\cap S^{(0)}_{k_n}} (\varphi+\varphi_{\mathrm{FS}})(0,\nu_{A_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap(\partial S_{k_n}\setminus\partial A_{k_n})\cap\left(S^{(1)}_{k_n}\cup S^{(0)}_{k_n}\right)\cap A^{(1)}_{k_n}} 2\varphi_{\mathrm{FS}}(0,\nu_{S_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap\partial^* S_{k_n}\cap\partial A_{k_n}\cap A^{(1)}_{k_n}} (\varphi_{\mathrm{F}}+\varphi)(0,\nu_{A_{k_n}}(0))\, d\mathcal{H}^1 \\
&+\int_{R_{\rho_n}\cap(\partial S_{k_n}\cap\partial A_{k_n})\cap S^{(0)}_{k_n}\cap A^{(1)}_{k_n}} (\varphi_{\mathrm{F}}+\varphi+\varphi_{\mathrm{FS}})(0,\nu_{S_{k_n}}(0))\, d\mathcal{H}^1 \\
&-\varepsilon\left(\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right), \\
=:&\, \mathcal{S}_L(A_{k_n},S_{h_{k_n},K_{k_n}},R_{\rho_n})-\varepsilon\left(\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right)
\end{aligned}
\tag{3.4.142}
$$

for every $n \geq n^2_\varepsilon$, where in the first inequality we used the definition of μ_{k_n} and in the second inequality (3.4.127), and hence,

$$
\begin{aligned}
\mu_{k_n}(\overline{R_{\rho_n}}) &\geq \rho_n\left(\mathcal{S}_L(\sigma_{\rho_n}(A_{k_n}),S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})},R_{\nu_A})\right) \\
&\quad-\varepsilon\left(\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right) \\
&\geq 2\rho_n\varphi(0,\nu_{T_0})-\varepsilon\rho_n-\varepsilon\left(\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right),
\end{aligned}
\tag{3.4.143}
$$

where in the first inequality we used (3.4.142) and the properties of the blow up map, and in the last inequality we used (3.4.141). Finally, we conclude that

$$
\begin{aligned}
\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K}\cap\partial^* A)}(0) &\geq \liminf_{n\to\infty}\frac{\mu_{k_n}(\overline{R_{\rho_n}})}{2\rho_n} \\
&\geq \varphi(0,\nu_{T_0})-\frac{\varepsilon}{2} \\
&\qquad-\varepsilon\limsup_{n\to\infty}\frac{\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{R_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})}{2\rho_n} \\
&\geq \varphi(0,\nu_{T_0})-\frac{\varepsilon}{2}-\frac{\varepsilon}{c_1}\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K}\cap\partial^* A)}(0),
\end{aligned}
\tag{3.4.144}
$$

where in the first inequality we used (3.4.138), in the second inequality we used (3.4.143) and in the last inequality we used (3.4.139). By (b3) and taking $\varepsilon \to 0^+$, in the inequality above, we deduce (3.4.116). □

Proof of (3.4.117). By the $\mathcal{H}^1$-rectifiability of ∂A, by [58, Proposition A.4] (applied with K taken as ∂A), and by the Besicovitch Derivation theorem, the set of points $x \in (\partial A \setminus \partial S_{h,K}) \cap A^{(1)}$ not satisfying the following 4 conditions:

(c1) $\theta^*(\partial A, x) = \theta_*(\partial A, x) = 1$,

(c2) $\nu_A(x)$ exists,

(c3) $\overline{Q_{1,\nu_A(x)}} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{Q_{1,\nu_A(x)}} \cap T_{x,\nu_A(x)}$ as $\rho \to 0$,

(c4) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A \setminus \partial S_{h,K}) \cap A^{(1)})}(x)$ exists and it is finite,

is $\mathcal{H}^1$-negligible. Therefore, we prove (3.4.117) for any fixed $x \in (\partial A \setminus \partial S_{h,K}) \cap A^{(1)}$ satisfying (c1)-(c4). Without loss of generality we assume that $x = 0$ and $\nu_A(0) = \mathbf{e_1}$, and we use the notation $T_0 := T_{0,\nu_A(0)}$. Again by the Besicovitch Derivation Theorem there exists a subsequence $\rho_n \searrow 0$ such that

$$\mu_0(\partial Q_{\rho_n}) = 0, \qquad \lim_{k\to+\infty} \mu_k(\overline{Q_{\rho_n}}) = \mu_0(Q_{\rho_n}) \tag{3.4.145}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A \setminus \partial S_{h,K}) \cap A^{(1)})}(0) = \lim_{n\to\infty} \frac{\mu_0(Q_{\rho_n})}{2\rho_n}. \tag{3.4.146}$$

By (c3) and applying Proposition 3.4.3-(a) to A we have that

$$\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to -\mathrm{dist}(\cdot, T_0) \tag{3.4.147}$$

uniformly in $\overline{Q_1}$ as $n \to \infty$. Furthermore, from the $\tau_{\mathcal{B}}$-convergence it follows that

$$\mathrm{sdist}(\cdot, \partial A_k) \to \mathrm{sdist}(\cdot, \partial A) \tag{3.4.148}$$

uniformly in $\overline{Q_1}$ as $k \to \infty$ and hence, by (3.4.147) and (3.4.148), a standard diagonalization argument yields that there exists a subsequence $\left\{(A_{k_n}, S_{h_{k_n},K_{k_n}})\right\}$ such that

$$\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A_{k_n})) \to -\mathrm{dist}(\cdot, T_0). \tag{3.4.149}$$

uniformly in $\overline{Q_1}$ as $n \to \infty$ and, by also using (3.4.145), such that

$$\mu_{k_n}(\overline{Q_{\rho_n}}) \le \mu_0(Q_{\rho_n}) + \rho_n^2, \tag{3.4.150}$$

for every $n \in \mathbb{N}$. By arguing as in (3.4.139), we infer that

$$\limsup_{n\to\infty} \frac{\mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})}{2\rho_n} \le c_1^{-1} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner (((\partial A \setminus \partial S_{h,K}) \cap A^{(1)})}(0). \tag{3.4.151}$$

By (3.4.149) and by applying Lemma 3.4.2, we have that $\overline{Q_1} \setminus \sigma_{\rho_n}(A_{k_n}) \xrightarrow{\mathcal{K}} \overline{Q_1} \cap T_0$ as $n \to \infty$. Since the number of connected components of $\partial\sigma_{\rho_n}(A_{k_n})$ lying inside of Q_1 does not exceed m_1, by [58, Lemma 4.5] (applied by taking m_0, δ, ϕ in the notation of [58, Lemma 4.5] as m_1, ε, and $\varphi_{\mathrm{F}}(0,\cdot)$, respectively) implies that there exists $n_\varepsilon^1 \ge n_\varepsilon$ such that for every $n \ge n_\varepsilon^1$,

$$\begin{aligned}\int_{Q_1 \cap \partial^*\sigma_{\rho_n}(A_{k_n})} &\varphi_{\mathrm{F}}(0, \nu_{\sigma_{\rho_n}(A_{k_n})}(0))\, d\mathcal{H}^1 \\ &+ \int_{Q_1 \cap \partial\sigma_{\rho_n}(A_{k_n}) \cap ((\sigma_{\rho_n}(A_{k_n}))^{(0)} \cup (\sigma_{\rho_n}(A_{k_n}))^{(1)})} 2\varphi_{\mathrm{F}}(0, \nu_{\sigma_{\rho_n}(A_{k_n})}(0))\, d\mathcal{H}^1 \\ \ge 2 \int_{\overline{Q_1} \cap T_0} &\varphi_{\mathrm{F}}(0, \mathbf{e_1})\, d\mathcal{H}^1 - \varepsilon = 4\varphi_{\mathrm{F}}(0, \mathbf{e_1}) - \varepsilon.\end{aligned} \tag{3.4.152}$$

In view of Remark 3.2.8-(iii) and by $\tau_{\mathcal{B}}$-convergence, there exists a ball $B_{r(0)}(0)$ and $n_\varepsilon^2 \geq n_\varepsilon^1$ such that $Q_{\rho_n} \cap \partial S_{k_n} \subset B_{r(0)}(0) \cap \partial S_{k_n} = \emptyset$ for any $n \geq n_\varepsilon^2$, and thus,

$$\emptyset = Q_{\rho_n} \cap \partial S_{k_n} = \rho_n(Q_1 \cap \partial\sigma_{\rho_n}(S_{k_n})). \tag{3.4.153}$$

Therefore, by (3.4.152) and (3.4.153), we obtain that

$$\begin{aligned}
&\int_{Q_1\cap(\partial^*\sigma_{\rho_n}(A_{k_n})\setminus\partial\sigma_{\rho_n}(S_{k_n}))} \varphi_{\mathrm{F}}(0,\nu_{\sigma_{\rho_n}(A_{k_n})})\,d\mathcal{H}^1\\
&\quad+\int_{Q_1\cap(\partial\sigma_{\rho_n}(A_{k_n})\setminus\partial\sigma_{\rho_n}(S_{k_n}))\cap((\sigma_{\rho_n}(A_{k_n}))^{(0)}\cup(\sigma_{\rho_n}(A_{k_n}))^{(1)})} 2\varphi_{\mathrm{F}}(0,\nu_{\sigma_{\rho_n}(A_{k_n})}(0))\,d\mathcal{H}^1\\
&\geq 4\varphi_{\mathrm{F}}(0,\mathbf{e}_2)-\varepsilon.
\end{aligned}\tag{3.4.154}$$

Furthermore, by (3.4.127) it follows that there exists $n_\varepsilon^3 \geq n_\varepsilon^2$ such that

$$\begin{aligned}
\mu_{k_n}(\overline{Q_{\rho_n}}) &\geq \int_{Q_{\rho_n}\cap\partial^*A_{k_n}\setminus\partial S_{k_n}} \varphi_{\mathrm{F}}(0,\nu_{A_{k_n}})\,d\mathcal{H}^1\\
&\quad+\int_{Q_{\rho_n}\cap(\partial A_{k_n}\setminus\partial S_{k_n})\cap(A_{k_n}^{(0)}\cup A_{k_n}^{(1)})} 2\varphi_{\mathrm{F}}(0,\nu_{A_{k_n}})\,d\mathcal{H}^1\\
&\quad-\varepsilon\left(\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right)\\
&=\rho_n\Bigg(\int_{Q_1\cap(\partial^*\sigma_{\rho_n}(A_{k_n})\setminus\partial\sigma_{\rho_n}(S_{k_n}))} \varphi_{\mathrm{F}}(0,\nu_{\sigma_{\rho_n}(A_{k_n})})\,d\mathcal{H}^1\\
&\quad+\int_{Q_1\cap(\partial\sigma_{\rho_n}(A_{k_n})\setminus\partial\sigma_{\rho_n}(S_{k_n}))\cap((\sigma_{\rho_n}(A_{k_n}))^{(0)}\cup(\sigma_{\rho_n}(A_{k_n}))^{(1)})} 2\varphi_{\mathrm{F}}(0,\nu_{\sigma_{\rho_n}(A_{k_n})})\,d\mathcal{H}^1\Bigg)\\
&\quad-\varepsilon\left(\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right)\\
&\geq 4\rho_n\varphi_{\mathrm{F}}(0,\mathbf{e_1})-\varepsilon\rho_n-\varepsilon\left(\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial A_{k_n})+\mathcal{H}^1(\overline{Q_{\rho_n}}\cap\partial S_{k_n}\setminus\partial A_{k_n})\right),
\end{aligned}\tag{3.4.155}$$

for every $n \geq n_\varepsilon^3$, where in the first inequality we argued as in (3.4.142) (with Q_{ρ_n} instead of R_{ρ_n}) and we used the non-negativeness of ψ_{k_n}, in the equality we used properties of the blow up map, and in the second inequality we used (3.4.154). Finally, by (3.4.146), (3.4.150) and (3.4.155) and by repeating the same arguments of (3.4.144), we deduce that

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A\setminus\partial S_{h,K})\cap A^{(1)})}(0) \geq 2\varphi_{\mathrm{F}}(0,\mathbf{e_1}) - \frac{\varepsilon}{2} - \frac{\varepsilon}{c_1}\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial A\setminus\partial S_{h,K})\cap A^{(1)})}(0).$$

By (c4) and taking $\varepsilon \to 0^+$, in the inequality above, we deduce (3.4.117). □

Proof of (3.4.118). Since $\varphi' \leq \varphi_{\mathrm{F}}$, we repeat the same arguments of the proof of (3.4.117) by using Proposition 3.4.3-(b) and [58, Lemma 4.4] instead of Proposition 3.4.3-(a) and [58, Lemma 4.5], respectively. □

Proof of (3.4.119). We observe that (3.4.119) follows from the same arguments used in (3.4.115), which are based on [68, Theorem 20.1], by "interchanging the roles" of A_k, A with $S_k, S_{h,K}$. □

Proof of (3.4.120). By the $\mathcal{H}^1$-rectifiability of ∂A and $\partial S_{h,K}$, by [58, Proposition A.4] (applied with K taken as first ∂A and then $\partial S_{h,K}$), and by the Besicovitch Derivation Theorem the set of points $x \in (\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(1)}$ not satisfying the following 4 conditions:

(f1) $\theta^*(\partial A, x) = \theta_*(\partial A, x) = \theta^*(\partial S_{h,K}, x) = \theta_*(\partial S_{h,K}, x) = 1$,

(f2) $\nu_A(x)$, $\nu_{S_{h,K}}(x)$ exist and either $\nu_A(x) = \nu_{S_{h,K}}(x)$ nor $\nu_A(x) = -\nu_{S_{h,K}}(x)$, (f3) $\overline{Q_{1,\nu_A(x)}(x)} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{Q_{1,\nu_A(x)}(x)} \cap T_{x,\nu_A(x)}$ and $\overline{Q_{1,\nu_A(x)}(x)} \cap \partial\sigma_{\rho,x}(S_{h,K}) \xrightarrow{\mathcal{K}} \overline{Q_{1,\nu_A(x)}(x)} \cap T_{x,\nu_A(x)}$,

(f4) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K})}(x)$ exists and it is finite.

is $\mathcal{H}^1$-negligible. Therefore we prove (3.4.120) for any fixed $x \in (\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K}$ satisfying (f1)-(f4). By (f2) and without loss of generality we assume that $x = 0$ and $\nu_A(0) = \mathbf{e_1}$, and we denote $T_0 := T_{0,\nu_A(0)}$. Again by the Besicovitch Derivation Theorem there exists a subsequence $\rho_n \searrow 0$ such that

$$\mu_0(\partial Q_{\rho_n}) = 0, \quad \lim_{k\to+\infty} \mu_k(\overline{Q_{\rho_n}}) = \mu_0(Q_{\rho_n}) \tag{3.4.156}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K})}(0) = \lim_{n\to\infty} \frac{\mu_0(Q_{\rho_n})}{2\rho_n}. \tag{3.4.157}$$

By (f3) and applying Proposition 3.4.3-(a) to A and $S_{h,K}$ we have that

$$\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to -\mathrm{dist}(\cdot, T_0) \quad \text{and} \quad \mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{h,K})) \to -\mathrm{sdist}(\cdot, T_0) \tag{3.4.158}$$

uniformly in $\overline{Q_1}$ as $n \to \infty$. Furthermore, from the $\tau_{\mathcal{B}}$-convergence it follows that

$$\mathrm{sdist}(\cdot, \partial A_k) \to \mathrm{sdist}(\cdot, \partial A) \quad \text{and} \quad \mathrm{sdist}(\cdot, \partial S_k) \to \mathrm{sdist}(\cdot, \partial S_{h,K}) \tag{3.4.159}$$

uniformly in $\overline{Q_1}$ as $k \to \infty$ and hence, by (3.4.158) and (3.4.159), a standard diagonalization argument yields that there exists a subsequence $\{(A_{k_n}, h_{k_n}, K_{k_n})\}$ such that

$$\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A_{k_n})) \to -\mathrm{dist}(\cdot, T_0) \quad \text{and} \quad \mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{k_n})) \to -\mathrm{dist}(\cdot, T_0) \tag{3.4.160}$$

uniformly in $\overline{Q_1}$ as $n \to \infty$ and by (3.4.156) such that

$$\mu_{k_n}(\overline{Q_{\rho_n}}) \le \mu_0(Q_{\rho_n}) + \rho_n^2, \tag{3.4.161}$$

for every $n \in \mathbb{N}$. By arguing as in (3.4.139) we deduce that

$$\limsup_{n\to\infty} \frac{\mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})}{2\rho_n} \le c_1^{-1} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K})}(0). \tag{3.4.162}$$

By (3.4.160) and by applying Lemma 3.4.2 we deduce that

$$\overline{Q_1} \setminus \sigma_{\rho_n}(A_{k_n}) \xrightarrow{\mathcal{K}} \overline{Q}_1 \cap T_0 \quad \text{and} \quad \overline{Q_1} \setminus \sigma_{\rho_n}(S_{k_n}) \xrightarrow{\mathcal{K}} \overline{Q}_1 \cap T_0. \tag{3.4.163}$$

Since $\left\{(A_{k_n}, S_{h_{k_n},K_{k_n}})\right\} \subset \mathcal{B}_{\mathbf{m}}$, we know that $(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})}) \in \mathcal{B}_{\mathbf{m}}(\sigma_{\rho_n}(\Omega))$. By (3.4.163) and by applying Lemma 3.4.9 to $(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})})$ and Q_1, with $\phi_\alpha(\cdot) = \varphi_\alpha(0,\cdot)$ for $\alpha =$ F,S,FS, there exists $n^1_\varepsilon \in \mathbb{N}$ such that for every $n \ge n^1_\varepsilon$,

$$\mathcal{S}_L(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})}, Q_1) \ge 2\int_{\overline{Q_1}\cap T_0} \varphi(0,\mathbf{e_1})\, d\mathcal{H}^1 - \varepsilon \ge 4\varphi(0,\mathbf{e_1}) - \varepsilon. \tag{3.4.164}$$

By (3.4.164), by the uniform continuity in (3.4.127) and by repeating the same arguments of (3.4.143) we obtain that there exists $n^2_\varepsilon \ge n^1_\varepsilon$ such that

$$\mu_{k_n}(\overline{Q_{\rho_n}}) \ge 4\varphi(0,\mathbf{e_1}) - \varepsilon\rho_n - \varepsilon\left(\mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})\right), \tag{3.4.165}$$

for every $n \geq n_\varepsilon^2$. By (3.4.157), (3.4.161), (3.4.165) and by arguing as in (3.4.144) we have that

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K})}(0) \geq 2\varphi(0, \mathbf{e_1}) - \frac{\varepsilon}{2} - \frac{\varepsilon}{c_1} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S^{(1)}_{h,K})}(0).$$

Finally, by (f4) and taking $\varepsilon \to 0^+$, in the inequality above, we reach (3.4.120). □

Proof of (3.4.121). Since $\varphi' \leq \varphi$, we repeat the same arguments of the proof of (3.4.120) in view of the fact that $\partial S_{h,K} \cap \partial A \cap S^{(0)}_{h,K} \cap A^{(0)} = \partial S_{h,K} \cap \partial A \cap A^{(0)}$ by employing Proposition 3.4.3-(b) in place of Proposition 3.4.3-(a) and Lemma 3.4.10 in place of Lemma 3.4.9. □

Proof of (3.4.122). In order to obtain (3.4.122) we combine the arguments of the proof of (3.4.116) and the proof of (3.4.120), by using the argumentations of the former with Proposition 3.4.3-(c) with $\nu_A = \mathbf{e_1}$ for the sets A and A_k and their convergence, and the argumentations of the latter with Proposition 3.4.3-(b) for the sets $S_{h,K}$ and S_k and their convergence, but employing Lemma 3.4.11 in place of Lemmas 3.4.7 and 3.4.9, which were used in such previous proofs. □

Proof of (3.4.123). We repeat the same arguments of the proof of (3.4.117) which are based on [58, Lemma 4.5], by "interchanging the roles" of A_k, A with $S_k, S_{h,K}$. □

Proof of (3.4.124). We repeat the same arguments of the proof of (3.4.118) which are based on [58, Lemma 4.4], by "interchanging the roles" of A_k, A with $S_k, S_{h,K}$. □

Proof of (3.4.125). By the definition of $\partial^* S_{h,K}$, by the $\mathcal{H}^1$-rectifiability of ∂A, by [58, Proposition A.4] (applied with K taken as first ∂A and then $\partial S_{h,K}$), and the Besicovitch Derivation Theorem the set of points $x \in \partial^* S_{h,K} \cap \partial A \cap A^{(1)}$ not satisfying the following 4 conditions:

(h1) $\theta^*(\partial A, x) = \theta_*(\partial A, x) = 1$,

(h2) $\nu_A(x)$, $\nu_{S_{h,K}}(x)$ exist and, either $\nu_A(x) = \nu_{S_{h,K}}(x)$ or $\nu_A(x) = -\nu_{S_{h,K}}(x)$,

(h3) for every open rectangle R containing x with sides parallel or perpendicular to $\mathbf{e_1}$ we have that $\overline{R} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{R} \cap T_{x,\nu_A(x)}$ and $\overline{R} \cap \partial\sigma_{\rho,x}(S_{h,K}) \xrightarrow{\mathcal{K}} \overline{R} \cap T_{x,\nu_A(x)}$ as $\rho \to 0$, where $T_{x,\nu_A(x)}$ is the approximate tangent line at x of ∂A (or of $\partial S_{h,K}$),

(h4) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial A \cap A^{(1)})}(x)$ exists and it is finite,

is $\mathcal{H}^1$-negligible. Therefore, we prove (3.4.125) for any fixed $x \in \partial^* S_{h,K} \cap \partial A \cap A^{(1)}$ satisfying (h1)-(h4). Without loss of generality we assume that $x = 0$ and we denote $T_0 = T_{0,\nu_A(0)}$. Furthermore, by using (h2) we choose in (h3) the rectangle $R_{\nu_A} := Q_1$ if $\nu_A(0) = \mathbf{e}_i$ for $i = \mathbf{1}, \mathbf{2}$, or $R_{\nu_A} := (-\cos\theta_{\nu_A}, \cos\theta_{\nu_A}) \times_{\mathbb{R}^2} (-\sin\theta_{\nu_A}, \sin\theta_{\nu_A})$, where θ_{ν_A} is the angle formed between the tangent line T_0 and the x_1-axis, otherwise. For any $\rho > 0$, we write $R_\rho := \rho R_{\nu_A}$.

In view of the definition of R_{ν_A} and again by using also the Besicovitch Derivation Theorem (see [47, Theorem 1.153]) there exists a subsequence $\rho_n \searrow 0$ such that

$$\mu_0(\partial R_{\rho_n}) = 0, \quad \lim_{k\to+\infty} \mu_k(\overline{R_{\rho_n}}) = \mu_0(R_{\rho_n}) \tag{3.4.166}$$

and

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial A \cap A^{(1)})}(0) = \lim_{n\to\infty} \frac{\mu_0(R_{\rho_n})}{2\rho_n}. \tag{3.4.167}$$

We now claim that

$$\text{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to -\text{dist}(\cdot, T_0) \quad \text{and} \quad \text{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{h,K})) \to \text{sdist}(\cdot, \partial H_0) \tag{3.4.168}$$

uniformly in $\overline{R_{\nu_A}}$ as $n \to \infty$. To prove the claim we can for example observe that by [58, Proposition A.4] we have (not only (h3), but also) that $\overline{Q_r} \cap \partial\sigma_{\rho,x}(A) \xrightarrow{\mathcal{K}} \overline{Q_r} \cap T_x$ and $\overline{Q_r} \cap \partial\sigma_{\rho,x}(S_{h,K}) \xrightarrow{\mathcal{K}} \overline{Q_r} \cap T_x$ as $\rho \to 0$ for any square Q_r such that $R_{\nu_A} \subset Q_r$ and hence, by Proposition 3.4.3 Items (a) and (c), applied to Q_r, $\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A)) \to -\mathrm{dist}(\cdot, T_0)$ and $\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{h,K})) \to \mathrm{sdist}(\cdot, \partial H_0)$ uniformly in $\overline{Q_r} \supset R_{\nu_A}$.

Furthermore, from the $\tau_{\mathcal{B}}$-convergence it follows that

$$\mathrm{sdist}(\cdot, \partial A_k) \to \mathrm{sdist}(\cdot, \partial A) \quad \text{and} \quad \mathrm{sdist}(\cdot, \partial S_k) \to \mathrm{sdist}(\cdot, \partial S_{h,K}) \tag{3.4.169}$$

uniformly in $\overline{R_{\nu_A}}$ as $k \to \infty$ and hence, by (3.4.168) and (3.4.169), a standard diagonalization argument yields that there exists a subsequence $\{(A_{k_n}, h_{k_n}, K_{k_n})\}$ such that

$$\mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(A_{k_n})) \to -\mathrm{dist}(\cdot, T_0) \quad \text{and} \quad \mathrm{sdist}(\cdot, \partial\sigma_{\rho_n}(S_{k_n})) \to \mathrm{sdist}(\cdot, \partial H_0) \tag{3.4.170}$$

uniformly in $\overline{R_{\nu_A}}$ as $n \to \infty$ and by (3.4.166) such that

$$\mu_{k_n}(\overline{R_{\rho_n}}) \leq \mu_0(R_{\rho_n}) + \rho_n^2, \tag{3.4.171}$$

for any $n \in \mathbb{N}$. By (3.4.166), (3.4.171) and arguing as in (3.4.140) we infer that

$$\limsup_{n\to\infty} \frac{\mathcal{H}^1(\overline{R_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{R_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})}{2\rho_n} \leq c_1^{-1} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial^* S_{h,K} \cap \partial A \cap A^{(1)})}(0). \tag{3.4.172}$$

By applying Lemma 3.4.2, we have that $\overline{R_{\nu_A}} \setminus \sigma_{\rho_n}(A_{k_n}) \xrightarrow{\mathcal{K}} T_0$ and in view of Remark 3.2.8-(i) we have that $\overline{R_{\nu_A}} \cap \partial\sigma_{\rho_n}(S_{k_n}) \xrightarrow{\mathcal{K}} T_0$. Since $\left\{(A_{k_n}, S_{h_{k_n},K_{k_n}})\right\} \subset \mathcal{B}_{\mathbf{m}}$, we know that $(\sigma_{\rho_n}(A_{k_n}), S_{(1/\rho_n)h_{k_n}(\rho_n\cdot),\sigma_{\rho_n}(K_{k_n})}) \in \mathcal{B}_{\mathbf{m}}(\sigma_{\rho_n}(\Omega))$. By applying Lemma 3.4.8 with $\phi_\alpha(\cdot) = \varphi_\alpha(0,\cdot)$ for $\alpha =$ F,S,FS and $\delta = \varepsilon$, there exists $n_\varepsilon^1 \in \mathbb{N}$ such that for every $n \geq n_\varepsilon^1$,

$$\begin{aligned} \mathcal{S}_L(\sigma_{\rho_n}(A_{k_n}), S_{\sigma_{\rho_n}(S_{k_n}),\sigma_{\rho_n}(K_{k_n})}, R_{\nu_A}) &\geq \int_{\overline{R_{\nu_A}} \cap T_0} \varphi_{\mathrm{F}}(0, \nu_{T_0})) + \varphi(0, \nu_{T_0})\, d\mathcal{H}^1 - \varepsilon \\ &= 2(\varphi_{\mathrm{F}}(0, \nu_{T_0}) + \varphi(0, \nu_{T_0})) - \varepsilon. \end{aligned} \tag{3.4.173}$$

By definition of μ_k, the non-negativeness of $\varphi_{\mathrm{F}}, \varphi$ and φ_{FS}, (3.4.127), (3.4.173) and arguing as in (3.4.143) we deduce that

$$\mu_{k_n}(\overline{Q_{\rho_n}}) \geq 2(\varphi_{\mathrm{F}}(0, \nu_{T_0}) + \varphi(0, \nu_{T_0})) - \varepsilon\rho_n - \varepsilon\left(\mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial A_{k_n}) + \mathcal{H}^1(\overline{Q_{\rho_n}} \cap \partial S_{k_n} \setminus \partial A_{k_n})\right), \tag{3.4.174}$$

for every $n > n'_\varepsilon$. By (3.4.167), (3.4.171), (3.4.174) and by arguing as in (3.4.144) we have that

$$\begin{aligned} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\partial A \cap \partial^* S \cap A^{(1)})}(0) &\geq \varphi_{\mathrm{F}}(0, \nu_{T_0}) + \varphi(0, \nu_{T_0}) - \frac{\varepsilon}{2} \\ &\quad - \varepsilon c_1^{-1} \frac{d\mu_0}{d\mathcal{H}^1 \llcorner ((\partial S_{h,K} \cap \partial A) \cap S_{h,K}^{(1)})}(0). \end{aligned}$$

Finally, by (h4) and by taking $\varepsilon \to 0^+$ in the inequality above, we deduce (3.4.123). □

Proof of (3.4.126). We repeat the arguments of (3.4.120) by using Proposition 3.4.3-(a) for the sets A and A_k, and Proposition 3.4.3-(b) for the sets $S_{h,K}$ and S_k, and by replacing Lemma 3.4.7 with Lemma 3.4.12. □

□

We are finally in position to prove Theorem 3.2.12.

Proof of Theorem 3.2.12. Without loss of generality, we assume that the *liminf* in the right side of (3.2.22) is reached and finite in $\mathbb{R}$. From Theorem 3.4.13 it follows that

$$\mathcal{S}(A, S_{h,K}) \leq \liminf_{k\to\infty} \mathcal{S}(A_k, S_{h_k,K_k}). \tag{3.4.175}$$

In view of the definition of $\mathcal{F}$, in order to reach the assertion, it suffices to establish the lower semicontinuity of $\mathcal{W}$, to which the rest of the proof is devoted.

Let $D \subset\subset \mathrm{Int}(A)$, by the fact that $\mathrm{Int}(A_k) \xrightarrow{\mathcal{K}} \mathrm{Int}(A)$, we deduce that $D \subset\subset \mathrm{Int}(A_k)$ for k large enough. As $u_k \to u$ a.e. in $\mathrm{Int}(A)$, then, $u_k \to u$ a.e. in D. Furthermore, since $e(u_k)$ are bounded in the $L^2(D)$ norm, we have that $e(u_k) \rightharpoonup e(u)$ in $L^2(D)$. By convexity of $\mathcal{W}(D, \cdot)$ we obtain that

$$\mathcal{W}(D, u) \leq \liminf_{k\to+\infty} \mathcal{W}(D, u_k) \leq \liminf_{k\to+\infty} \mathcal{W}(A_k, u_k).$$

To conclude it is now enough to let $D \nearrow \mathrm{Int}(A)$. □

3.5. Existence

In view of Theorems 3.2.11 and 3.2.12 we are in position to prove Theorem 3.2.10 by employing the *direct method of the calculus of variations.*

Proof of Theorem 3.2.10. Fix $m \in \mathbb{N}$ and let $\{(A_k, S_{h_k,K_k}, u_k)\} \subset \mathcal{C}_{\mathbf{m}}$ be a minimizing sequence of $\mathcal{F}$ such that $\mathcal{L}^2(A_k) = \mathbb{v}_1$, $\mathcal{L}^2(S_{h_k,K_k}) = \mathbb{v}_0$, and

$$\sup_{k\in\mathbb{N}} \mathcal{F}(A_k, S_{h_k,K_k}, u_k) < \infty.$$

By Theorem 3.2.11 there exist a subsequence $\{(A_{k_l}, S_{h_{k_l},K_{k_l}}, u_{k_l})\}$, a sequence $\{(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l)\} \subset \mathcal{C}_{\mathbf{m}}$, and $(A, S_{h,K}, u) \in \mathcal{C}_{\mathbf{m}}$ such that $(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l) \xrightarrow{\tau_{\mathcal{C}}} (A, S_{h,K}, u)$ as $l \to \infty$ and

$$\liminf_{l\to\infty} \mathcal{F}(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l) = \liminf_{l\to\infty} \mathcal{F}(A_{k_l}, S_{h_{k_l},K_{k_l}}, u_{k_l}). \tag{3.5.1}$$

By Theorem 3.2.12, we have that

$$\mathcal{F}(A, S_{h,K}, u) \leq \liminf_{l\to\infty} \mathcal{F}(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l). \tag{3.5.2}$$

We claim that $\{(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l)\}$ and $(A, S_{h,K}, u)$ satisfy the volume constraints of (3.2.18). Indeed, by Theorem 3.2.11, for any $l \geq 1$, $\mathbb{v}_1 = \mathcal{L}^2(A_{k_l}) = \mathcal{L}^2(\widetilde{A}_l)$ and $\mathbb{v}_0 = \mathcal{L}^2(S_{h_{k_l},K_{k_l}}) = \mathcal{L}^2\left(S_{h_l,\widetilde{K}_l}\right)$, where $S_{h_l,\widetilde{K}_l} \in \mathrm{A}S(\Omega)$. Thanks to the fact that $(\widetilde{A}_l, S_{h_{k_l},K_{k_l}}) \xrightarrow{\tau_{\mathcal{B}}} (A, S_{h,K})$ as $l \to \infty$, by applying [58, Lemma 3.2] we infer that $\widetilde{A}_l \to A$ in $L^1(\mathbb{R}^2)$ as $l \to \infty$, and thus $\mathcal{L}^2(A) = \mathbb{v}_1$, and, similarly, we deduce that $\mathcal{L}^2(S_{h,K}) = \mathbb{v}_0$. Finally, from (3.5.1) and (3.5.2) we deduce that

$$\begin{aligned}\inf_{(A,S_{h,K},u)\in\mathcal{C}_{\mathbf{m}},\, \mathcal{L}^2(A)=\mathbb{v}_1,\, \mathcal{L}^2(S_{h,K})=\mathbb{v}_0} \mathcal{F}(A, S_{h,K}, u) &= \lim_{k\to\infty} \mathcal{F}(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l) \\ &\geq \liminf_{l\to\infty} \mathcal{F}\left(\widetilde{A}_l, S_{h_{k_l},\widetilde{K}_{k_l}}, v_l\right) \\ &\geq \mathcal{F}(A, S_{h,K}, u)\end{aligned}$$

and hence, $(A, S_{h,K}, u)$ is a solution of the minimum problem (3.2.18). By observing that (3.5.1) and (3.5.2) hold true also by replacing $\mathcal{F}$ with $\mathcal{F}^\lambda$ for $\lambda := (\lambda_0, \lambda_1)$, with $\lambda_0, \lambda_1 > 0$, we deduce that also the unconstrained minimum problem (3.2.19) can be solved by employing the same method. This concludes the proof. □

4. Film multilayers

In this chapter, the results contained in the following preprint are presented:

- R. Llerena, P. Piovano: *Solutions for a free-boundary problem modeling film multilayers with coherent and incoherent interfaces*, preprint (2023).

4.1. Introduction

In this chapter, it is addressed the problem of modeling the morphology of multilayered film composites consisting of different crystalline materials deposited on a substrate. The goal is to advance the literature on the variational modeling of single-layered films deposited on a fixed substrate [25, 34, 35, 46] in a twofold direction: on the one hand, by letting the substrate surface free and by addressing the presence of multiple layers of various materials, and, on the other hand, by including into the analysis the possibility of a failure of the film coatings, since, as described in [82] for the case of some oxide films, the compressive stresses generated during film growth can lead to the delamination (and the buckling) between different layers.

Nowadays film-based nanostructures find several applications, in particular for the manufacturing of electronic and photonic devices, such as for the creation of their semiconductor components, and of solar and photovoltaic cells. The great interest that films and, in particular multilayer films [81, 87, 90], created by vapor deposition of different material constituents, continue attracting is due to the fact that, as they are self-assembled heterostructures, their employment represents one of the nanostructure design methods with most feasibility potential; therefore, any advancement in the mathematical modeling of film and multilayer film materials can have an important practical impact for their design control. Examples of multilayer films that are used for optoelectronic applications, are multiple quantum well structures with alternating compressive and tensile strained layers, and short-period *quantum-dot superlattices*. Also for the latter, as described in [87, 90], it is really the superposition of various layers of materials that allows to reach the highest degree order needed for the applications with respect to the size, the density and the distribution of the quantum dots.

The adopted strategy consists in combining the implementation to the multi-phase setting of the film models considered in [25, 34, 35, 46], in which delamination is not taken into account, with the recent results for a two-phase setting of [66], in which the interfaces between phases are instead allowed to present both coherent and incoherent portions. Coherency is interpreted as a microscopic organization of atoms that can be regarded as a (possibly deformed) uniform lattice that is homogeneous through the interface, while incoherency refers to the presence of debonding and delamination at the interface. In this way the extension of the single-layer literature to the multilayer setting (with possible delamination at each layer interface) is performed within the theory of stress driven rearrangement instabilities (SDRI) [9, 33, 53, 77], which was also at the basis of the variational single-layer models introduced in [79, 80] and analytical validated in [25, 34, 35, 46]. In fact, as in [79, 80] for thin films, and more generally for free crystals in [58, 59, 60], it is considered the mismatch between the free-standing equilibrium lattices of the materials of each pair of film layers and of the first layer with the substrate by means of the so-called *mismatch strain* in the elastic energy. As described by the SDRI theory the lattice mismatch is responsible for the migration of the atoms of each phase from their crystalline order, since the lattice mismatch induces large stresses in the bulk material and, in order to release the related elastic energy, the atoms move forming corrugations, cracks, and other interface instabilities [9, 53, 33, 77].

In regard to the literature results for settings with phase interfaces exclusively assumed to be coherent, the authors of [65] refer to the literature on optimal shape of partitions in the absence of elastic effects, which was initiated by Almgren in [1], who formulated the problem in $\mathbb{R}^d$, for $d > 1$, for surface tensions proportional at each interface. By working in the framework of *integral currents* of geometric measure theory he singled out a condition referred to as “partitioning regularity", that ensures the lower semicontinuity of the overall surface energy with respect to the L^1-convergence of the sets in the partition. Then, Ambrosio and Braides expanded the scope in [2, 3] by including also non-proportional surface tensions and by introducing an integral condition called *BV-ellipticity*, which they proved to be both sufficient and necessary for the L^1-lower semicontinuity. Afterwards, various other conditions have been introduced and studied, such as *B-convexity* and *joint convexity*, in the attempt of finding a more practical condition than *BV-ellipticity*, as the latter can be challenging to be verified as it represents the analogous of Morrey’s *quasi-convexity* condition in the setting of *Caccioppoli partitions*. BV-ellipticity though remains the only known condition characterizing the the L^1-lower semicontinuity apart from specific contexts (see [23, 71] for more details), and the fact that it coincides with the *triangle-inequality condition*, which is simpler to check, for the case with 3 phases [2, 3]. Finally, in [52] the analogous version of the BV-ellipticity condition in the framework of *BD-spaces* has been studied.

Instead, in regard to the settings with only incoherent interfaces, the authors of [65] refer to the results obtained with respect to the related *Mumford-Shah* problem for also the application to *image segmentation*, which was actually originally introduced in [72] as a multi-phase formulation. In this context interfaces represent the contours of the image color areas that can be characterized as the discontinuity set of an auxiliary state function. The reference for existence and *Ahlfors-type regularity* results is made to [4, 30], which has been then extended also to the *Griffith model* in fracture mechanics in the context of linear elasticity with respect to vectorial state functions representing the bulk displacement of crystalline materials [26, 50]. Finally, Bucur, Fragalà, and Giacomini addressed the original multi-phase setting of [72] in [20] and [21] by providing a rigorous mathematical formulation with incoherent interfaces (see also [27] for a related multi-phase boundary problem for reaction-diffusion systems).

In [20] they recover Ahlfors-type regularity results for an *ad hoc* nonstandard notion of multi-phase local almost-quasi minimizers for an energy accounting for the incoherent portions of each interface and disregarding the contribution of the remaining coherent portions. Afterwards, in [21] the same Authors introduced what they refer to as the *multi-phase Mumford-Shah problem*, that is characterized by the sum of possibly different Mumford-Shah-type energy contributions, each related to a different phase, to which an extra term (justified on statistical reasons) is added. Such extra term is needed as otherwise minimizing configurations would present a single phase. However, in [21] coherent interfaces are not counted in the energy as “no-jump interface portions” along the reduced phase boundary are weighted in each phase energy in the same way as the jump portions.

To include the interplay between coherency and incoherency in the model considered in this chapter, by allowing each phase interface to present also both coherent and incoherent portions, it is adopted the strategy initiated in [66] for the setting with a film phase deposited on a substrate. Since the results in [66] regards $d = 2$ and were achieved under a so-called *exterior graph constraint* on the substrate surface, in order to implement those results to multiple film phases, it is also restricted to $d = 2$ and the exterior graph constraint is assumed on both the substrate surface and the film profiles. It is noticed that even in the presence of such condition internal cracks in each film layer and in the substrate are allowed to be also of non-graph type.

In this chapter, it is denoted by $\Omega := (-l, l) \times (-L, +\infty)$ for positive parameters $l, L \in \mathbb{R}$ the region where the multilayer films and the substrate are located, and, given $\alpha \in \mathbb{N}$, it is denoted a multilayered film composite with α layers on top of the substrate phase S_0, which is also denoted in the following as the *0th layer*, by S_α. Furthermore, for each $j \in \{0, \ldots, \alpha\}$ it is assumed that the profile of each *jth layer* is parametrizable by a *height function* $h^j : [-l, l] \mapsto [-L, +\infty)$ measuring the thickness of the profile of *jth composite* S_j, i.e., the j-layered film composite including all ith layers for $i \in \{0, \ldots, j\}$, by assuming that $h^{j-1} \leq h^j$ for $j \in \{1, \ldots, \alpha\}$. It is also denoted by $K^j \subset \overline{\mathrm{Int}(S_{h^j})}$, where S_{h^j} is the subgraph of h^j, the *cracks of the jth composite*, which are assumed such that $K^j \cap \mathrm{Int}(S_{h^{j-1}}) \subset K^{j-1}$, so that then for $j \in \{1, \ldots, \alpha\}$ the *j-composite* S_j coincides with

$$S_{h^j, K^j} := S_{h^j} \setminus K^j,$$

and the jth film layer coincides with $S_{h^j,K^j} \setminus S^{(1)}_{h^{j-1},K^{j-1}}$ (see Figure 4.1). It follows that there is no formal distinction in the hypotheses taken on the substrate phase S_0 and the one taken on each jth film composite S_{h^j,K^j} (apart from the fact of being contained in all of them).

In particular, by writing that $(h^j, K^j) \in \mathrm{AHK}(\Omega)$ it is assumed that each jth layer height function h^j is an upper semicontinuous function with bounded pointwise variation and each jth composite crack set K^j is a closed $\mathcal{H}^1$-rectifiable set with finite $\mathcal{H}^1$ measure. More precisely, it is denoted the family $\mathcal{B}^\alpha$ of admissible multilayered film composites in Ω with α layers (on the substrate layer), as a $(\alpha+1)$-tuple of all the jth composites S_{h^j,K^j} for $j \in \{0, \ldots, \alpha\}$, namely

$$\begin{aligned}\mathcal{B}^\alpha := \{(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}) : (h^j, K^j) \in \mathrm{AHK}(\Omega),\, h^{j-1} \leq h^j,\, K^j \subset \overline{\mathrm{Int}(S_{h^j})},\\ K^j \cap \mathrm{Int}(S_{h^{j-1}}) \subset K^{j-1} \text{ for } j \in \{1, \ldots, \alpha\}\}.\end{aligned}$$

Furthermore, by following the SDRI theory [9, 33, 53, 79, 77] and in the analogy with the single-layer film setting [25, 34, 35, 46, 66], the family of *admissible configurations* $\mathcal{C}^\alpha$ is defined by

$$\mathcal{C}^\alpha := \{(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}, u) : (S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}) \in \mathcal{B}^\alpha,\, u \in \mathrm{H}^1_{\mathrm{loc}}(\mathrm{Int}(S_{h^\alpha,K^\alpha}))\},$$

where the functions u represent the *bulk displacement* in the multilayered film composites, and the total configurational energy $\mathcal{F}^\alpha : \mathcal{C} \to [-\infty, \infty]$ as given by the sum of an elastic energy $\mathcal{W}$ and a surface energy $\mathcal{S}^\alpha$, i.e.,

$$\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}, u) := \mathcal{W}(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}, u) + \mathcal{S}^\alpha(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0})$$

for any $(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha$.

The elastic energy $\mathcal{W}$ is defined in $\mathcal{C}^\alpha$ by

$$\mathcal{W}(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0}, u) := \int_{S_{h^\alpha,K^\alpha}} W(x, Eu(x) - E_0^\alpha(x))\, dx,$$

where the elastic density W denotes the quadratic form

$$W(x, M) := \mathbb{C}(x) M : M,$$

defined for the fourth-order tensor $\mathbb{C} : \Omega \to \mathbb{M}^2_{sym}$, E denotes the symmetric part of the gradient, i.e., $E(v) := \frac{\nabla v + \nabla^T v}{2}$ for any $v \in H^1_{\mathrm{loc}}(\mathrm{Int}(A); \mathbb{R}^2)$ for a set A, and represents the *strain*, and E_0^α is the mismatch strain $x \in \Omega \mapsto E_0^\alpha(x) \in \mathbb{M}^2_{\mathrm{sym}}$ defined as

$$E_0^\alpha := \begin{cases} E(u_0^\alpha) & \text{in } \Omega \setminus S_{h^{\alpha-1}}, \\ E(u_0^i) & \text{in } \mathrm{Int}(S_{h^i}) \setminus S_{h^{1-1}} \text{ for } i = 1, \ldots, \alpha - 1 \\ 0 & \text{in } \mathrm{Int}(S_{h^0,K^0}), \end{cases}$$

with respect to fixed α functions $u_0^i \in H^1(\Omega;\mathbb{R}^2)$ for $i \in \{1,\dots,\alpha\}$.

The surface energy $\mathcal{S}^\alpha$ is defined as the sum of all the pairwise contributions $\mathcal{S}^{(i,j)} : \mathcal{B}^1 \to [0,\infty]$ for $0 \le i < j \le \alpha$ given by

$$\mathcal{S}^{(i,j)}(S_{h^j,K^j}, S_{h^i,K^i}) := \int_{\partial S_{h^i,K^i} \cup \partial S_{h^j,K^j}} \psi_{i,j}(z,\nu)\, d\mathcal{H}^1,$$

for every admissible multilayered composite $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0}) \in \mathcal{B}^\alpha$, where $\psi_{i,j}$ denotes the anisotropic surface tension that takes different definition with respect to the various portions of $\partial S_{h^i,K^i} \cup \partial S_{h^j,K^j}$. More precisely, in order to properly define $\psi_{i,j}$ it is considered the three surface tensions φ_i, φ_j, $\varphi_{ij} : \overline{\Omega} \times \mathbb{R}^2 \to [0,\infty]$ characterizing the vapor-ith layer interface, the vapor-j layer interface, and the ith layer-j layer interface. Moreover, in order to address both the wetting and dewetting regimes with respect to the materials of each pair of film layers, it is introduced two additional surface tensions for each pair $(S_{h^j,K^j}, S_{h^i,K^i})$, denoted as the i,j *regime surface tensions*, which are defined as follows:

$$\varphi_{ij}^1 := \min\{\varphi_i, \varphi_j + \varphi_{ij}\} \qquad \text{and} \qquad \varphi_{ij}^2 := \min\{\varphi_i, \varphi_j\},$$

in analogy to the definitions given in [66] for two-phase setting. It follows that

$$\psi_{i,j}(x,\nu(x)) := \begin{cases} \varphi_j(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap (\partial^* S_{h^j,K^j} \setminus \partial^* S_{h^i,K^i}) \\ \varphi_{ij}^1(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap \partial^* S_{h^i,K^i} \cap \partial^* S_{h^j,K^j}, \\ \varphi_{ij}(x,\nu_{S_{h^i,K^i}}(x)) & x \in \Omega \cap (\partial^* S_{h^i,K^i} \setminus \partial S_{h^j,K^j}) \\ (\varphi_j + \varphi_i^1)(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap \partial^* S_{h^i,K^i} \cap \partial S_{h^j,K^j} \cap S_{h^j,K^j}^{(1)}, \\ 2\varphi_j(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap \partial S_{h^j,K^j} \cap S_{h^j,K^j}^{(1)} \cap S_{h^i,K^i}^{(0)}, \\ 2\varphi_{ij}^2(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap \partial S_{h^j,K^j} \cap {S_{h^j,K^j}}^{(0)}, \\ 2\varphi_{ij}(x,\nu_{S_{h^i,K^i}}(x)) & \begin{aligned} x \in \Omega &\cap (\partial S_{h^i,K^i} \setminus \partial S_{h^j,K^j}) \\ &\cap \left(S_{h^i,K^i}^{(1)} \cup S_{h^i,K^i}^{(0)}\right) \cap S_{h^j,K^j}^{(1)}, \end{aligned} \\ \varphi_{ij}^1(x,\nu_{S_{h^j,K^j}}(x)) & x \in \Omega \cap \partial S_{h^i,K^i} \cap \partial S_{h^j,K^j} \cap S_{h^i,K^i}^{(1)}, \end{cases}$$

where, given a set $U \subset \mathbb{R}^2$, ν_U, $\partial^* U$, and $U^{(\alpha)}$ denote, when well defined, the outward pointing normal to ∂U, the *reduced boundary*, and the set of points of density $\alpha \in [0,1]$, respectively. Notice that if $\alpha = 1$ the energy $\mathcal{S}^{(0,1)}$ coincides with the surface energy defined in [66] as, by following the notation of [66] it follows that $\varphi_0 := \varphi_{\mathrm{S}}, \varphi_1 := \varphi_{\mathrm{F}}, \varphi_{01} := \varphi_{\mathrm{FS}}$ and as a consequence $\varphi_{01}^1 = \varphi$ and $\varphi_{0,1}^2 = \varphi'$. Finally, the α-surface energy $\mathcal{S}^\alpha : \mathcal{B}^\alpha \to [-\infty,\infty]$ is given by

$$\mathcal{S}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0}) := \sum_{j=1}^{\alpha} \sum_{i=0}^{j-1} \mathcal{S}^{(i,j)}(S_{h^j,K^j}, S_{h^i,K^i}).$$

Observe that are choices for the definition of $\mathcal{S}^\alpha$ are possible, which will be evaluated in the future.

It was observed in [58, 66] that the family $\mathcal{B}^1$ lacks compactness with respect to the signed distance convergence. In order to overcome this issue and being able to apply Gołąb's's Theorem [53] to recover compactness, it is imposed a constraint $m_j \in \mathbb{N}$ on the number of connected components of the cracks of the jth composite that are not connected to the jth layer for $j = 0,\dots,\alpha$. Therefore, it is restricted to the family of configurations $\mathcal{C}_{\boldsymbol{m}}^\alpha \subset \mathcal{C}^\alpha$ for which such constraints hold, where $\mathbf{m} := (m_0,\dots,m_\alpha) \in \mathbb{N}^{\alpha+1}$.

The main goal of [65] is to prove that the minimum problem

$$\inf_{\substack{(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u) \in \mathcal{C}_{\mathbf{m}}^\alpha, \\ \mathcal{L}^2(S_{h^i,K^i}) = \mathtt{v}_i, \text{ for } i=0,\dots,\alpha}} \mathcal{F}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u). \tag{4.1.1}$$

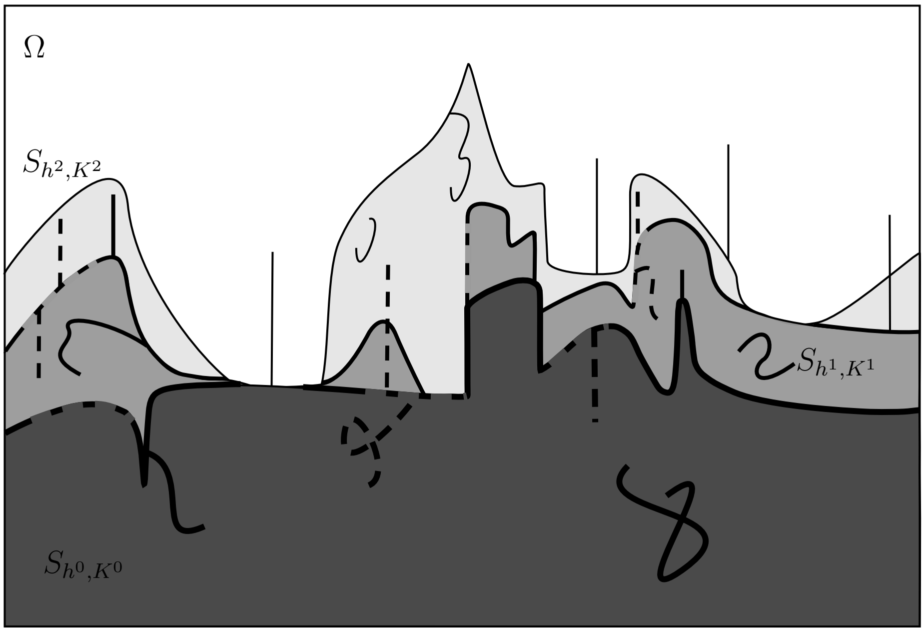

Figure (4.1): A multilayered film composite with 2 layers (on the substrate 0th layer S_{h^0,K^0}) associated to an admissible configuration $(S_{h^2,K^2},\dots,S_{h^0,K^0},u)\in\mathcal{C}^2_{\mathbf{m}}$ (see Definition 4.2.1) is represented by indicating each jth layer with a gray color with decreasing value with respect to the increasing order of the index $j=0,1,2$. Furthermore, the jth layer is indicated with a thinner line with respect to the increasing order of the index $j=0,1,2$, and for the 0th and 1st layer it is distinguished between their coherent and incoherent portions by using a dashed or a continuous line, respectively.

admits a solution for every family of area constraints $\{\mathbb{v}_i\}_{i=0}^{\alpha}\subset[\mathcal{L}^2(\Omega)/2,\mathcal{L}^2(\Omega)]$.

To do that the *Direct Method* of Calculus of Variations is employed, which consists in in finding a proper topology $\tau_{\mathcal{C}^\alpha}$ weak enough to prove compactness in $\mathcal{C}^\alpha_{\mathbf{m}}\subset\mathcal{C}^\alpha$ and strong enough to have lower semicontinuity of $\mathcal{F}^\alpha$ in $\mathcal{C}^\alpha_{\mathbf{m}}$. The topology $\tau_{\mathcal{C}^\alpha}$ considered is the one for which the convergence

$$(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k)\xrightarrow[k\to\infty]{\tau_{\mathcal{C}^\alpha}}(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)$$

is equivalent to

$$\begin{cases}\text{for every } i=0,\dots,\alpha,\ \sup_{k\in\mathbb{N}}\mathcal{H}^1\left(\partial S_{h^i_k,K^i_k}\right)<\infty,\\ \operatorname{sdist}\left(\cdot,\partial S_{h^i_k,K^i_k}\right)\xrightarrow[k\to\infty]{}\operatorname{sdist}\left(\cdot,\partial S_{h^i,K^i}\right)\text{ locally uniformly in }\mathbb{R}^2\text{ and}\\ u_k\xrightarrow[k\to\infty]{}u\quad\text{a.e. in Int}\,(\mathrm{S}_{\mathrm{h}^\alpha,\mathrm{K}^\alpha}),\end{cases}$$

where the *signed distance function* is defined for any $E\subset\mathbb{R}^2$ as follows

$$\operatorname{sdist}(x,\partial E):=\begin{cases}\operatorname{dist}(x,E) & \text{if } x\in\mathbb{R}^2\setminus E,\\ -\operatorname{dist}(x,E) & \text{if } x\in E.\end{cases}$$

For the compactness it is implemented in the multilayer setting the compactness results proven for the substrate in [66], which were based on [25, 34, 35, 46] (with the difference that instead of lower semicontinuous graph it is assumed an upper semicontinuity property). Notice that in order to include incoherency in the setting of [25, 34, 35, 46] it is implemented for multilayers the setting of [66], where in the elements in the $(\alpha+1)$-tuple are not each film layers, but the jth composites. In particular, this allows to include in the model also the possible presence of a countable island of one material onto the other layers. In order to establish the lower semicontinuity property, the mathematical induction proceeds by directly using the lower semicontinuity result of [66] for the basis of the induction.

This chapter is organized as follows: in Section 4.2, the model and the main results of the chapter are introduced. In Section 4.3 it is proved the existence of minimizers for single-layer films with delamination. Finally, in Section 4.4 it is proved the existence result for the minimum problem (4.1.1) with a finite number α of layers over the substrate 0th layer.

4.2. Mathematical setting and main results

4.2.1. Multilayer model

In this section, we introduce the family of admissible regions with finite number of composite layers and the respective family of admissible configurations. Let $\Omega := (-l, l) \times (-L, \infty) \subset \mathbb{R}^2$ for positive parameters $l, L \in \mathbb{R}$.

Analogously to [66], we assume a *graph-crack* constraint of the composite of layers, in other words, we consider a *graph constraint* on the strict epigraph of the composite of layers, while inside of it, we consider *cracks* as closed and $\mathcal{H}^1$-rectifiable sets of Ω, roughly speaking, the profile of the composite is given by a function representing its thickness, plus a countable number of external vertical filaments and internal cracks. More precisely, we consider the *family of admissible heights* $\mathrm{AH}(\Omega)$ defined by

$$\mathrm{AH}(\Omega) := \{h : [-l, l] \to [0, L] : h \text{ is upper semicontinuous and } \operatorname{Var} h < \infty\} \tag{4.2.1}$$

and let S_h denote the closed subgraph with height $h \in \mathrm{AH}(\Omega)$, i.e.,

$$S_h := \{(x, y) : -l < x < l, y \leq h(x)\}. \tag{4.2.2}$$

Furthermore, we define the *family of admissible cracks* $\mathrm{AK}(\Omega)$ by

$$\mathrm{AK}(\Omega) := \{K \subset \Omega : K \text{ is a closed set in } \mathbb{R}^2,\ \mathcal{H}^1\text{-rectifiable and } \mathcal{H}^1(K) < \infty\} \tag{4.2.3}$$

and the *family of pairs of admissible heights and cracks* $\mathrm{AHK}(\Omega)$ by

$$\mathrm{AHK}(\Omega) := \{(h, K) \in \mathrm{AH}(\Omega) \times \mathrm{AK}(\Omega) : K \subset \overline{\mathrm{Int}(S_h)}\}. \tag{4.2.4}$$

Finally, given $(h, K) \in \mathrm{AHK}(\Omega)$ we refer to the region characterized as the subgraph of the height function h without the internal cracks of K, namely,

$$S_{h,K} := (S_h \setminus K) \cap \Omega, \tag{4.2.5}$$

as the *(generalized) subgraph with height h and cracks K*, and we define the *family of admissible subgraphs* as

$$\mathrm{AS}(\Omega) := \{S \subset \Omega : S = S_{h,K} \text{ for a pair } (h, K) \in \mathrm{AHK}(\Omega)\}. \tag{4.2.6}$$

We observe that for every $(h, K) \in \mathrm{AHK}(\Omega)$

$$\overline{S_{h,K}} = S_h, \quad \mathrm{Int}(S_{h,K}) = \mathrm{Int}(S_h) \setminus K \quad \text{and} \quad \partial S_{h,K} = \partial S_h \cup K. \tag{4.2.7}$$

We have that ∂S_h is connected and, ∂S_h and $\partial S_{h,K}$ have finite $\mathcal{H}^1$-measure. By [44, Lemma 3.12 and Lemma 3.13], for any $h \in \mathrm{AH}(\Omega)$, ∂S_h is rectifiable and applying the Besicovitch-Marstrand-Mattila Theorem (see [4, Theorem 2.63]), ∂S_h is $\mathcal{H}^1$-rectifiable, and hence, $\partial S_{h,K}$ is $\mathcal{H}^1$-rectifiable. Furthermore, by applying [59, Proposition A.1] S_h and $S_{h,K}$ are sets of finite perimeter.

Definition 4.2.1 (Admissible multilayers and admissible configurations)**.** We define the family of two layers $\mathcal{B}^1$ by

$$\mathcal{B}^1 := \{(S_{h^1,K^1}, S_{h^0,K^0}) :\ \text{for } i = 0,1 \text{ there exists } (h^i, K^i) \in \mathrm{AHK}(\Omega),\ S_{h^i,K^i} \in \mathrm{AS}(\Omega),\\ h^0 \le h^1 \text{ and } \partial S_{h^1,K^1} \cap \mathrm{Int}(S_{h^0,K^0}) = \emptyset\} \subset \mathrm{AS}(\Omega) \times \mathrm{AS}(\Omega).$$

Let $\alpha \in \mathbb{N}$, we define the family of admissible $(\alpha+1)$-layers $\mathcal{B}^\alpha$ by

$$\mathcal{B}^\alpha := \{(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in [\mathrm{AS}(\Omega)]^{\alpha+1} :\ (S_{h^j,K^j}, S_{h^i,K^i}) \in \mathcal{B}^1\\ \text{for every } 0 \le i \le j \le \alpha\},$$

where $[\mathrm{AS}(\Omega)]^{\alpha+1} := \mathrm{AS}(\Omega) \times \ldots \times \mathrm{AS}(\Omega)$ represents the $(\alpha+1)$ cartesian product of $\mathrm{AS}(\Omega)$. We define the family of admissible configurations by

$$\mathcal{C}^\alpha := \{(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) : (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in \mathcal{B}^\alpha,\ u \in \mathrm{H}^1_{\mathrm{loc}}(\mathrm{Int}(S_{h^\alpha,K^\alpha}))\}.$$

For any $\alpha \in \mathbb{N}$, motivated in [58, 59, 66] we introduce a notion of convergence for the families $\mathcal{B}^\alpha$ and $\mathcal{C}^\alpha$.

Definition 4.2.2 ($\tau_{\mathcal{B}^\alpha}$-Convergence)**.** A sequence $\{(S_{h^\alpha_n,K^\alpha_n}, \dots, S_{h^0_n,K^0_n})\} \subset \mathcal{B}^\alpha$ $\tau^\alpha_{\mathcal{B}}$-converges to $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in \mathcal{B}^\alpha$ if

- $\sup_{k\in\mathbb{N}} \mathcal{H}^1(\partial S_{h^i_k,K^i_k}) < \infty$, for every $i = 0, \dots, \lambda$
- $\mathrm{sdist}\left(\cdot, \partial S_{h^i_k,K^i_k}\right) \to \mathrm{sdist}\left(\cdot, \partial S_{h^i,K^i}\right)$ locally uniformly in $\mathbb{R}^2$ as $k \to \infty$ for every $i = 0, \dots, \lambda$,

where

$$S_{h^i_k,K^i_k} := S_{h^i_k} \setminus K^i_k \quad \text{and} \quad S_{h^i,K^i} := S_{h^i} \setminus K^i,$$

for every $k \in \mathbb{N}$ and $i = 0, \dots, \alpha$.

Definition 4.2.3 ($\tau_{\mathcal{C}^\alpha}$-Convergence)**.** A sequence $\{(S_{h^\alpha_n,K^\alpha_n}, \dots, S_{h^0_n,K^0_n}, u_n)\} \subset \mathcal{C}^\alpha$ $\tau_{\mathcal{C}^\alpha}$-converges to $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha$ if

- $(S_{h^\alpha_n,K^\alpha_n}, \dots, S_{h^0_n,K^0_n}) \xrightarrow{\tau_{\mathcal{B}^\alpha}} (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0})$,
- $u_n \to u$ a.e. in $\mathrm{Int}(S_{h^\alpha,K^\alpha})$.

Analogously to [58, 59, 66], we introduce a subfamily of $\mathcal{B}^\alpha$ with a restriction on the number of connected components of the boundary of each composite layer.

Definition 4.2.4. Let $\alpha \in \mathbb{N}$. For any $\boldsymbol{m} := (m_0, \dots, m_\alpha) \in \mathbb{N}^{\alpha+1}$ the family $\mathcal{B}^\alpha_{\boldsymbol{m}}$ is given by all elements $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in \mathcal{B}$ such that $\partial S_{h^i,K^i}$ has at most m_i-connected components for $i = 0, \dots, \alpha$. We define

$$\mathcal{C}^\alpha_{\boldsymbol{m}} := \{(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C} :\ (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in \mathcal{B}^\alpha\}.$$

In the sequel we fix $\alpha \in \mathbb{N}$. Motivated in the model introduced in [66], we consider the surface tension between two layers $\mathcal{S}^{(i,j)} : \mathcal{B}^\alpha \to [-\infty, \infty]$ in the family of admissible layers $\mathcal{B}^\alpha$ by

$$\mathcal{S}^{(i,j)}(S_{h^j,K^j}, S_{h^i,K^i}) := \int_{\partial S_{h^i,K^i} \cup \partial S_{h^j,K^j}} \psi_{i,j}(z,\nu)\, d\mathcal{H}^1,$$

where $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0})\in\mathcal{B}^\alpha$ for $0\le i<j\le\alpha$, and the surface tension $\psi_{i,j}$ is defined in different portions of $\partial S_{h^i,K^i}\cup\partial S_{h^j,K^j}$, more precisely,

$$\psi(x,\nu(x)):=\begin{cases}\varphi_j(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap(\partial^*S_{h^j,K^j}\setminus\partial^*S_{h^i,K^i})\\ \varphi^1_{ij}(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap\partial^*S_{h^i,K^i}\cap\partial^*S_{h^j,K^j},\\ \varphi_{ij}(x,\nu_{S_{h^i,K^i}}(x)) & x\in\Omega\cap(\partial^*S_{h^i,K^i}\setminus\partial S_{h^j,K^j})\\ (\varphi_j+\varphi^1_i)(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap\partial^*S_{h^i,K^i}\cap\partial S_{h^j,K^j}\cap S^{(1)}_{h^j,K^j},\\ 2\varphi_j(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap\partial S_{h^j,K^j}\cap S^{(1)}_{h^j,K^j}\cap S^{(0)}_{h^i,K^i},\\ 2\varphi^2_{ij}(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap\partial S_{h^j,K^j}\cap S_{h^j,K^j}{}^{(0)},\\ 2\varphi_{ij}(x,\nu_{S_{h^i,K^i}}(x)) & \begin{array}{l}x\in\Omega\cap(\partial S_{h^i,K^i}\setminus\partial S_{h^j,K^j})\\ \qquad\cap\left(S^{(1)}_{h^i,K^i}\cup S^{(0)}_{h^i,K^i}\right)\cap S^{(1)}_{h^j,K^j},\end{array}\\ \varphi^1_{ij}(x,\nu_{S_{h^j,K^j}}(x)) & x\in\Omega\cap\partial S_{h^i,K^i}\cap\partial S_{h^j,K^j}\cap S^{(1)}_{h^i,K^i},\end{cases}\tag{4.2.8}$$

where φ_j, $\varphi_{ij}:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$ and, given also the function $\varphi_i:\overline{\Omega}\times\mathbb{R}^2\to[0,\infty]$, we define the functions φ^1_{ij} and φ^2_{ij} in $C(\overline{\Omega}\times\mathbb{R}^2;[0,\infty])$ by

$$\varphi^1_{ij}:=\min\{\varphi_i,\varphi_j+\varphi_{ij}\}\qquad\text{and}\qquad\varphi^2_{ij}:=\min\{\varphi_j,\varphi_i\}.$$

In view of [66], for every $0\le i\le j\le\alpha$, $\varphi_j,\varphi_i,\varphi_{ij}$ represent the anisotropic surface tensions of the film/vapor, the substrate/vapor and the substrate/film interfaces, respectively, while φ^1_{ij} and φ^2_{ij} are referred to as the anisotropic *regime surface tensions* and are introduced to include into the analysis the wetting and dewetting regimes.

Remark 4.2.5. If $\alpha=1$, we can observe that the surface tension $\mathcal{S}$ considered in [66] coincides with $\mathcal{S}^{(0,1)}$ by considering, with respect to the notation of [66], $\varphi_0:=\varphi_{\mathrm{S}},\varphi_1:=\varphi_{\mathrm{F}},\varphi_{01}:=\varphi_{\mathrm{FS}}$ and as a consequence $\varphi^1_{01}=\varphi$ and $\varphi^2_{01}=\varphi'$.

Now, we are in position to define the α-surface energy $\mathcal{S}^\alpha:\mathcal{B}^\alpha\to[-\infty,\infty]$ by

$$\mathcal{S}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0}):=\sum_{j=1}^{\alpha}\sum_{i=0}^{j-1}\mathcal{S}^{(i,j)}(S_{h^j,K^j},S_{h^i,K^i}).$$

Let $\alpha\in\mathbb{N}$, the total energy $\mathcal{F}^\alpha:\mathcal{C}^\alpha\to[-\infty,\infty]$ is defined by

$$\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u):=\mathcal{S}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0})+\mathcal{W}(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)$$

for any $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)\in\mathcal{C}^\alpha$, where $\mathcal{W}$ stands for the elastic energy, more precisely,

$$\mathcal{W}(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u):=\int_{S_{h^\alpha,K^\alpha}}W(x,Eu(x)-E^\alpha_0(x))\,dx,$$

and W is determined by the quadratic form

$$W(x,M):=\mathbb{C}(x)\,M:M,$$

for a fourth-order tensor $\mathbb{C}:\Omega\to\mathbb{M}^2_{sym}$, E denotes the symmetric gradient, i.e., $E(v):=\frac{\nabla v+\nabla^T v}{2}$ for any $v\in\mathrm{H}^1_{\mathrm{loc}}(\Omega)$ and E^α_0 is the mismatch strain $x\in\Omega\mapsto E^\alpha_0(x)\in\mathbb{M}^2_{\mathrm{sym}}$ defined as

$$E^\alpha_0:=\begin{cases}E(u^\alpha_0) & \text{in }\Omega\setminus S_{h^{\alpha-1}},\\ E(u^i_0) & \text{in }\mathrm{Int}(S_{h^i})\setminus S_{h^{1-1}}\text{ for }i=1,\dots,\alpha-1\\ 0 & \text{in }\mathrm{Int}(S_{h^0,K^0}),\end{cases}$$

for a fixed sequence $\{u^i_0\}_{i=1}^{\alpha-1}\subset H^1(\Omega;\mathbb{R}^2)$.

4.2.2. Main results

We state here the main results of the paper [65]. Let $\alpha \in \mathbb{N}$. We fix $l, L > 0$ and we consider $\Omega := (-l, l) \times (-L, \infty)$. For every pair of integers $0 \le i \le j \le \alpha$ we consider $\varphi^1_{ij} := \min\{\varphi_i, \varphi_j + \varphi_{ij}\}$ and $\varphi^2_{ij} := \min\{\varphi_j, \varphi_i\}$ and we assume throughout this manuscript that:

(H1) $\varphi_j, \varphi_{ij}, \varphi^1_{ij}, \varphi^2_{ij} \in C(\overline{\Omega} \times \mathbb{R}^2)$ are Finsler norms such that there exists $c_2 \ge c_1 > 0$ such that

$$c_1|\xi| \le \varphi_j(x,\xi), \varphi^1_{ij}(x,\xi), \varphi_{ij}(x,\xi) \le c_2|\xi| \quad \text{for every } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^2, \tag{4.2.9}$$

(H2) We have

$$\varphi^1_{ij}(x,\xi) \ge |\varphi_{ij}(x,\xi) - \varphi_j(x,\xi)| \quad \text{for every } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^2. \tag{4.2.10}$$

(H3) $\mathbb{C} \in L^\infty(\Omega; \mathbb{M}^2_{sym})$ and there exists $c_3 > 0$ such that

$$\mathbb{C}\,(x)\,M : M \ge 2c_3 M : M \tag{4.2.11}$$

for every $M \in \mathbb{M}^2_{sym}$.

We notice that under assumptions (H1)-(H3), the energy $\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in [0,\infty]$ for every $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha$.

Theorem 4.2.6 (Existence of minimizers). *Let $\alpha \in \mathbb{N}$ and let $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_\alpha) \in \mathbb{R}^{\alpha+1}$ such that $\lambda_i > 0$ for every $i = 0, \dots, \alpha$. Assume (H1)-(H3) and let $\{\mathbb{v}_i\}_{i=0}^\alpha \subset [\mathcal{L}^2(\Omega)/2, \mathcal{L}^2(\Omega)]$ such that for every $0 \le i_1 \le i_2 \le \alpha$, $\mathbb{v}_{i_1} \le \mathbb{v}_{i_2}$. Then for every $\mathbf{m} = (m_0, \dots, m_\alpha) \in \mathbb{N}^{\alpha+1}$ the volume constrained minimum problem*

$$\inf_{\substack{(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha_{\mathbf{m}}, \\ \mathcal{L}^2(S_{h^i,K^i}) = \mathbb{v}_i,\ i = 0,\dots,\alpha}} \mathcal{F}^\alpha(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \tag{4.2.12}$$

and the unconstrained minimum problem

$$\inf_{(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u) \in \mathcal{C}^\alpha_{\mathbf{m}}} \mathcal{F}^{\alpha,\boldsymbol{\lambda}}(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha_{\mathbf{m}} \tag{4.2.13}$$

have solution, where $\mathcal{F}^{\alpha,\boldsymbol{\lambda}} : \mathcal{C}^\alpha_{\mathbf{m}} \to \mathbb{R}$ is defined as

$$\mathcal{F}^{\alpha,\boldsymbol{\lambda}}(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) := \mathcal{F}^\alpha(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) + \sum_{i=0}^\alpha \lambda_i \Big|\mathcal{L}^2(S_{h^i,K^i}) - \mathbb{v}_i\Big|.$$

In [65] we employ the Direct Method of Calculus of Variations to prove Theorem 4.2.6. In order to apply this method we prove that any energy equi-bounded sequence $\{(A_k, S_k, u_k)\} \subset \mathcal{C}_{\mathbf{m}}$ satisfy the following compactness property.

Theorem 4.2.7. *Let $\{(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}^\alpha_{\mathbf{m}}$ be such that*

$$\sup_{k\in\mathbb{N}} \left(\mathcal{F}^\alpha(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k) + \mathcal{L}^2\left(S_{h^\alpha_k,K^\alpha_k}\right)\right) < \infty. \tag{4.2.14}$$

Then, there exist an admissible configuration $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha_{\mathbf{m}}$ of finite energy, a subsequence $\{(S_{h^\alpha_{k_n},K^\alpha_{k_n}}, \dots, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n})\}_{n\in\mathbb{N}}$, a sequence $\{(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n}, \dots, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n})\}_{n\in\mathbb{N}} \subset \mathcal{C}^\alpha_{\mathbf{m}}$ and a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise rigid displacements associated to $S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n}$ such that

$$(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n}, \dots, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n} + b_n) \xrightarrow{\tau_{\mathcal{C}^\alpha}} (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u)$$

and

$$\liminf_{n\to\infty} \mathcal{F}^\alpha(S_{h^\alpha_{k_n},K^\alpha_{k_n}}, \dots, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n}) = \liminf_{n\to\infty} \mathcal{F}^\alpha(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n}, \dots, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n} + b_n). \tag{4.2.15}$$

Furthermore, we show that $\mathcal{F}^\alpha$ is lower semicontinuous in $\mathcal{C}^\alpha_{\mathbf{m}}$ with respect to the topology $\tau_{\mathcal{C}^\alpha}$ for any $\alpha \in \mathbb{N}$.

Theorem 4.2.8 (Lower semicontinuity of $\mathcal{F}^\alpha$)**.** *Assume (H1)-(H3). Let* $\{(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}^\alpha_{\mathbf{m}}$ *and* $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u) \in \mathcal{C}^\alpha_{\mathbf{m}}$ *be such that*

$$(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k) \xrightarrow{\tau_{\mathcal{C}^\alpha}} (S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u).$$

Then

$$\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u) \le \liminf_{k\to\infty} \mathcal{F}^\alpha(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k). \tag{4.2.16}$$

4.3. Single-layer films with delamination

In order to establish Theorem 4.2.6 in [65] we use an induction argument, and in this section, we prove the basis of the induction. More precisely, we prove Theorem 4.2.6 by assuming that $\alpha = 1$ and hence, in the following, we consider $\boldsymbol{m} := (m_1, m_0) \in \mathbb{N}^2$. We begin by observing that the double-layered film setting of $\alpha = 1$ is a particular case of the two-phase setting considered in [66], with the only difference that the "exterior graph condition" is assumed not only on the substrate region but also on the film phase. The analogy comes also from the fact, that as proved below, for energy equibounded admissible configurations in $\mathcal{C}^1$, we can easily reduced to bounded rectangular containers $\widetilde{\Omega} := (-l,l)\times(-L,\widetilde{L})$ for a properly chosen constant $\widetilde{L} > 0$.

We recall that in [66] the families of admissible regions $\mathcal{B}(\widetilde{\Omega})$ and $\mathcal{B}_{\mathbf{m}}(\widetilde{\Omega})$ are defined as

$$\begin{aligned}\mathcal{B}(\widetilde{\Omega}) := \{(A, S_{h,K}) :\quad & (h,K)\in \mathrm{AHK}(\widetilde{\Omega}),\ A \text{ is } \mathcal{L}^2\text{-measurable set with } S_{h,K}\subset \overline{A}\subset \overline{\widetilde{\Omega}}\\ & \text{such that } \partial A\cap \mathrm{Int}(S_{h,K}) = \emptyset,\ \partial A \text{ is } \mathcal{H}^1\text{-rectifiable},\\ & \mathcal{H}^1(\partial A)+\mathcal{H}^1(\partial S)<\infty\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{\mathbf{m}}(\widetilde{\Omega}) := \{(A,S_{h,K})\in\mathcal{B}(\widetilde{\Omega}) :\ & \partial A \text{ and } \partial S_{h,K} \text{ have at most}\\ & m_1 \text{ and } m_0 \text{ connected components, respectively}\}.\end{aligned}$$

Notice that $\mathcal{B}^1(\widetilde{\Omega}) \subset \mathcal{B}(\widetilde{\Omega})$ and $\mathcal{B}^1_{\mathbf{m}}(\widetilde{\Omega}) \subset \mathcal{B}_{\mathbf{m}}(\widetilde{\Omega})$. Furthermore, the families of admissible configurations in [66] are $\mathcal{C}(\widetilde{\Omega})$ and $\mathcal{C}_{\mathbf{m}}(\widetilde{\Omega})$ defined by

$$\mathcal{C}(\widetilde{\Omega}) := \{(A,S_{h,K},u) :\ (A,S_{h,K})\in\mathcal{B}(\widetilde{\Omega}) \text{ and } u\in H^1_{\mathrm{loc}}(\mathrm{Int}(A);\mathbb{R}^2)\}$$

and

$$\mathcal{C}_{\mathbf{m}}(\widetilde{\Omega}) := \{(A,S_{h,K},u)\in\mathcal{C}(\widetilde{\Omega}) :\ (A,S_{h,K})\in\mathcal{B}_{\mathbf{m}}(\widetilde{\Omega})\},$$

so that $\mathcal{C}^1(\widetilde{\Omega}) \subset \mathcal{C}(\widetilde{\Omega})$ and $\mathcal{C}^1_{\mathbf{m}}(\widetilde{\Omega}) \subset \mathcal{C}_{\mathbf{m}}(\widetilde{\Omega})$. Therefore, since the elastic energy $\mathcal{W}$ and the surface energy $\mathcal{S}$ considered in [66] coincide with the energies $\mathcal{W}$ and $\mathcal{S}^1$ of this manuscript (by also observing that, following the notation of [66], $\varphi_0 = \varphi_{\mathrm{S}}, \varphi_1 = \varphi_{\mathrm{F}}, \varphi_{01} = \varphi_{\mathrm{FS}}$, $\varphi^1_{01} = \varphi$ and $\varphi^2_{01} = \varphi'$), we have that

$$\mathcal{S}\equiv\mathcal{S}^1 \quad \text{and} \quad \mathcal{F}\equiv\mathcal{F}^1, \tag{4.3.1}$$

in $\mathcal{C}^1(\widetilde{\Omega})$ and $\mathcal{C}^1_{\mathbf{m}}(\widetilde{\Omega})$. Finally, we also observe have that the topologies $\tau_{\mathcal{B}}$ and $\tau_{\mathcal{C}}$ defined in [66] coincide with the topologies $\tau_{\mathcal{B}^1}$ and $\tau_{\mathcal{C}^1}$, respectively.

On the basis of this observations and by using the results for the two-phase setting of [66], we now prove that energy-equibounded sequences in $\mathcal{C}^1_{\boldsymbol{m}}$ are compact and that $\mathcal{F}^1$ is lower semicontinuous with respect to the topology $\tau_{\mathcal{C}^1}$.

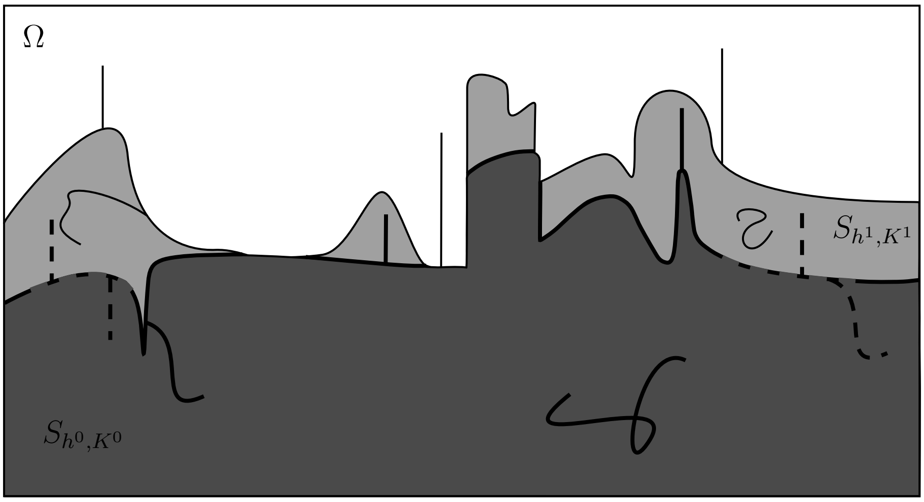

Figure (4.2): A single-layer film (on the substrate 0th layer S_{h^0,K^0}) associated to an admissible configuration $(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}}$ (see Definition 4.2.1) is represented by indicating each jth layer with a gray color with decreasing value with respect to the increasing order of the index $j = 0, 1$ and each jth layer with a thinner line with respect to the increasing order of the index $j = 0, 1$. Furthermore, in the 0th layer we distinguish between its coherent and incoherent portions by using a dashed or a continuous line, respectively.

Proposition 4.3.1. *Let* $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k})\} \subset \mathcal{B}^1_{\mathbf{m}}$ *be such that*

$$\sup_{k\in\mathbb{N}} \Big(\mathcal{S}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) + \mathcal{L}^2(S_{h^1_k,K^1_k})\Big) < \infty. \tag{4.3.2}$$

Then, there exist a not relabeled subsequence $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k})\} \subset \mathcal{B}^1_{\mathbf{m}}$ *and* $(S_{h^1,K^1}, S_{h^0,K^0}) \in \mathcal{B}^1_{\mathbf{m}}$ *such that* $(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}^1}} (S_{h^1,K^1}, S_{h^0,K^0})$.

Proof. We begin by observing that in view of [4, Theorem 3.47] from (4.3.2) it follows that there exists $\widetilde{L} > 0$ such that for every $k \in \mathbb{N}$, $S_{h^1_k,K^1_k} \subset (-l,l) \times (-L, \widetilde{L}) =: \widetilde{\Omega}$. Since $(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) \in \mathcal{B}^1_{\mathbf{m}}$ for every $k \in \mathbb{N}$, by (4.3.1), where in [66] we consider $\varphi_0 = \varphi_{\mathrm{S}}, \varphi_1 = \varphi_{\mathrm{F}}, \varphi_{01} = \varphi_{\mathrm{FS}}$, $\varphi^1_{01} = \varphi$ and $\varphi^2_{01} = \varphi'$, we have that

$$\sup_{k\in\mathbb{N}} \mathcal{S}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) = \sup_{k\in\mathbb{N}} \mathcal{S}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) < \infty. \tag{4.3.3}$$

By applying [66, Theorem 4.2] with respect to the region $\widetilde{\Omega}$, there exist a not relabeled subsequence $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k})\} \subset \mathcal{B}_{\mathbf{m}}$ and $(S, S_{h^0,K^0}) \in \mathcal{B}_{\mathbf{m}}$ such that

$$(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}}} (S, S_{h^0,K^0}). \tag{4.3.4}$$

By definition of $\tau_{\mathcal{B}}$-convergence and by the second statement of [66, Lemma 3.8] there exists $(h^1, K^1) \in$ AHK(Ω) such that

$$S = S_{h^1,K^1}. \tag{4.3.5}$$

In view of the definition of h^0 and h^1 that comes from [66, Lemma 3.8], we see that

$$h^0(x_1) := \sup\{\limsup_{k\to\infty} h^0_k(x^k_1) : x^k_1 \to x_1\} \le \sup\{\limsup_{k\to\infty} h^1_k(x^k_1) : x^k_1 \to x_1\} =: h^1(x_1)$$

for every $x_1 \in [-l,l]$. Thus, $(S_{h^1,K^1}, S_{h^0,K^0}) \in \mathcal{B}^1_{\mathbf{m}}$. Finally, from (4.3.4) and (4.3.5), and by the fact that the $\tau_{\mathcal{B}^1}$-convergence is similar to the $\tau_{\mathcal{B}}$-convergence of [66] we obtain that

$$(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}^1}} (S_{h^1,K^1}, S_{h^0,K^0})$$

which concludes the proof. □

We are in the position to prove that $\mathcal{C}^1_{\mathbf{m}}$ is compact with respect to the topology $\tau_{\mathcal{C}^1}$.

Theorem 4.3.2 (Compactness of $\mathcal{C}^1_{\mathbf{m}}$)**.** *Let $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}^1_{\mathbf{m}}$ be such that*

$$\sup_{k\in\mathbb{N}} \Big(\mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)+\mathcal{L}^2(S_{h^1_k,K^1_k})\Big) < \infty. \tag{4.3.6}$$

Then, there exist an admissible configuration $(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}}$ of finite $\mathcal{F}^1$ energy, a subsequence $\{(S_{h^1_{k_n},K^1_{k_n}}, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n})\}_{n\in\mathbb{N}}$, a sequence $\{(S_{h^1_{k_n},\widetilde{K}^1_n}, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n})\}_{n\in\mathbb{N}} \subset \mathcal{C}^1_{\mathbf{m}}$ and a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise rigid displacements associated to $S_{h^1_{k_n},\widetilde{K}^1_n}$ such that

$$(S_{h^1_{k_n},\widetilde{K}^1_n}, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n}+b_n) \xrightarrow{\tau_{\mathcal{C}^1}} (S_{h^1,K^1}, S_{h^0,K^0}, u)$$

and

$$\liminf_{n\to\infty} \mathcal{F}^1(S_{h^1_{k_n},K^1_{k_n}}, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n}) = \liminf_{n\to\infty} \mathcal{F}^1(S_{h^1_{k_n},\widetilde{K}^1_n}, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n}+b_n). \tag{4.3.7}$$

Proof. We begin by observing that in view of [4, Theorem 3.47] from (4.3.2) it follows that there exists $\widetilde{L} > 0$ such that for every $k \in \mathbb{N}$, $S_{h^1_k,K^1_k} \subset (-l,l)\times(-L,\widetilde{L}) =: \widetilde{\Omega}$. In view of the observations at the beginning of the section by (4.3.1) and (4.3.6) we have that

$$\sup_{k\in\mathbb{N}} \mathcal{F}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) = \sup_{k\in\mathbb{N}} \mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) < \infty,$$

where $\mathcal{F}: \mathcal{C}(\widetilde{\Omega}) \to [0,\infty]$ is the total energy considered in [66]. Hence, by applying [66, Theorem 4.3] with respect to the region $\widetilde{\Omega}$, we deduce that there exist a triple $(S, S_{h^0,K^0}, u) \in \mathcal{C}_{\mathbf{m}}(\widetilde{\Omega})$ of finite $\mathcal{F}$-energy, a subsequence $\{(S_{h^1_{k_n},K^1_{k_n}}, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n})\}_{n\in\mathbb{N}}$, a sequence $\{(\widetilde{S}_n, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n})\}_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathbf{m}}$ and a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise rigid displacements associated to $\widetilde{S}_n$ such that

$$(\widetilde{S}_n, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n}+b_n) \xrightarrow{\tau_{\mathcal{C}}} (S, S_{h^0,K^0}, u) \tag{4.3.8}$$

and

$$\liminf_{n\to\infty} \mathcal{F}(S_{h^1_{k_n},K^1_{k_n}}, S_{h^0_{k_n},K^0_{k_n}}, u_{k_n}) = \liminf_{n\to\infty} \mathcal{F}(\widetilde{S}_n, S_{h^0_{k_n},\widetilde{K}^0_n}, u_{k_n}+b_n). \tag{4.3.9}$$

In view of the proof of [66, Theorem 4.3] we have that

$$\widetilde{S}_n := S_{h^1_{k_n},K^1_{k_n}} \setminus \Big(\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}\Big) \tag{4.3.10}$$

In analogy to the definition of $\widetilde{K^0_n}$ in the proof of [66, Theorem 4.3], we define

$$\widetilde{K^1_n} := K^1_n \cup \Big(\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}\Big) \tag{4.3.11}$$

and we claim that $\widetilde{S}_n = S_{h^1_{k_n},\widetilde{K}^1_n}$. Indeed, we have that

$$\begin{aligned} S_{h^1_{k_n},\widetilde{K}^1_n} &:= \partial S_{h^1_{k_n}} \cup \Big(S_{h^1_{k_n}} \setminus \widetilde{K^1_n}\Big) = \partial S_{h^1_{k_n}} \cup \Big(S_{h^1_{k_n}} \setminus \Big(K^1_n \cup \Big(\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}\Big)\Big)\Big) \\ &= \partial S_{h^1_{k_n}} \cup \Big(S_{h^1_{k_n}} \cap \Big((K^1_n)^c \cap \Big(\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}\Big)^c\Big)\Big) \\ &= \Big(\partial S_{h^1_{k_n}} \cup \Big(S_{h^1_{k_n}} \setminus K^1_n\Big)\Big) \cap \Big(\partial S_{h^1_{k_n}} \cup \Big(\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}\Big)^c\Big) \\ &=: S_{h^1_{k_n},K^1_{k_n}} \cap \Big(\partial S_{h^1_{k_n}} \cup (\partial\widetilde{S}_n)^c \cup \partial S_{h^1_{k_n},K^1_{k_n}}\Big) = S_{h^1_{k_n},K^1_{k_n}} \setminus (\partial\widetilde{S}_n \setminus \partial S_{h^1_{k_n},K^1_{k_n}}) \\ &=: \widetilde{S}_n, \end{aligned} \tag{4.3.12}$$

where we used (4.2.5) in the first and fifth equalities, (4.3.11) in the second equality, De Morgan's laws in the third, forth and sixth equalities, and (4.3.10) in the last equality, and hence $(S_{h^1_{k_n},\widetilde{K}^1_{k_n}}, S_{h^0_{k_n},\widetilde{K}^0_{k_n}}, u_{k_n}) \in \mathcal{C}^1_{\mathbf{m}}$.

Furthermore, by (4.3.6) and the non-negativeness of $\mathcal{F}$ it follows from Proposition 4.3.1 that $S = S_{h^1,K^1}$ for a proper pair $(h^1, K^1) \in \mathrm{AHK}(\Omega)$ and, in particular, we have that $(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}}$. Finally, in view of the definition of $\tau_{\mathcal{C}^1}$-convergence, by (4.3.8) we obtain that

$$(S_{h^1_{k_n},\widetilde{K}^1_{k_n}}, S_{h^0_{k_n},\widetilde{K}^0_{k_n}}, u_{k_n} + b_n) \xrightarrow{\tau_{\mathcal{C}^1}} (S_{h^1,K^1}, S_{h^0,K^0}, u),$$

and, by (4.3.1), (4.3.9) we obtain (4.3.7), which concludes the proof. □

Now, by applying [66, Theorem 5.14] we prove that $\mathcal{F}^1$ is lower semicontinuous with respect to the $\tau_{\mathcal{C}^1}$-topology.

Theorem 4.3.3 (Lower semicontinuity of $\mathcal{F}^1$)**.** *Assume (H1)-(H3). Let* $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{C}^1_{\mathbf{m}}$ *and* $(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}}$ *be such that* $(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) \xrightarrow{\tau_{\mathcal{C}^1}} (S_{h^1,K^1}, S_{h^0,K^0}, u)$. *Then*

$$\mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) \le \liminf_{k\to\infty} \mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k). \tag{4.3.13}$$

Proof. Without loss of generality, we assume that the right side of (4.3.13) is finite. In view of [4, Theorem 3.47] and from the fact that and since $(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) \xrightarrow{\tau_{\mathcal{C}^1}} (S_{h^1,K^1}, S_{h^0,K^0}, u)$it follows that there exists $\widetilde{L} > 0$ such that for every $k \in \mathbb{N}$, $S_{h^1_k,K^1_k} \subset (-l, l) \times (-L, \widetilde{L}) =: \widetilde{\Omega}$. Since the topology $\tau_{\mathcal{B}}$ considered in [66] coincide with the topology $\tau_{\mathcal{B}^1}$, by (4.3.1) and by applying [66, Theorem 5.13] applied in the regions $\Omega = \widetilde{\Omega}$, we have that

$$\mathcal{S}^1\left(S_{h^1,K^1}, S_{h^0,K^0}\right) \le \liminf_{k\to\infty} \mathcal{S}^1\left(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}\right). \tag{4.3.14}$$

Now, we are going to prove that the elastic energy is lower semicontinuous. Indeed, let $D \subset\subset \mathrm{Int}(S_{h^1,K^1})$, by properties of the signed distance convergence we have that $D \subset\subset \mathrm{Int}(S_{h^1_k,K^1_k})$ for k large enough. By definition of $\tau_{\mathcal{C}^1}$ convergence we have that $u_k \to u$ a.e. in D. Furthermore, since $E\,u_k$ are bounded in the $L^2(D)$ norm, we have that $E\,u_k \rightharpoonup E\,u$ in $L^2(D)$. By convexity of W we obtain that

$$\int_D W(x, E\,u - E^1_0)\,dx \le \liminf_{k\to\infty} \int_D W(x, E\,u_k - E^1_0)\,dx \le \liminf_{k\to+\infty} \mathcal{W}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)$$

By taking $D \nearrow \mathrm{Int}(S_{h^1,K^1})$ we conclude that

$$\mathcal{W}(S_{h^1,K^1}, S_{h^0,K^0}, u) \le \liminf_{k\to+\infty} \mathcal{W}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k). \tag{4.3.15}$$

By (4.3.14) and (4.3.15) and thanks to the superadditivity of the *liminf*, we get that

$$\begin{aligned}
\mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) &:= \mathcal{W}(S_{h^1,K^1}, S_{h^0,K^0}, u) + \mathcal{S}^1\left(S_{h^1,K^1}, S_{h^0,K^0}\right) \\
&\le \liminf_{k\to+\infty} \mathcal{W}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) + \liminf_{k\to\infty} \mathcal{S}^1\left(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}\right) \\
&\le \liminf_{k\to+\infty} \mathcal{W}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) + \mathcal{S}^1\left(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}\right) \\
&=: \liminf_{k\to\infty} \mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k),
\end{aligned}$$

which concludes the proof. □

Finally, we state the main result of this section. The following result is the analogous result of Theorem 4.2.6, more precisely, we prove the existence of minimizers for a volume constrained problem and for an unconstrained problem, respectively, with respect to the admissible family of deformable film and substrate $\mathcal{C}^1_{\mathbf{m}}$ for every $\mathbf{m} := (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$.

Theorem 4.3.4 (Existence of minimizers)**.** *Assume (H1)-(H3) and let* $\mathbb{v}_0, \mathbb{v}_1 \in [\mathcal{L}^2(\Omega/2), \mathcal{L}^2(\Omega)]$ *such that* $\mathbb{v}_0 \le \mathbb{v}_1$*. Then for every* $\mathbf{m} = (m_0, m_1) \in \mathbb{N}^2$*, the volume constrained minimum problem*

$$\inf_{(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}},\, \mathcal{L}^2(S_{h^1,K^1}) = \mathbb{v}_1,\, \mathcal{L}^2(S_{h^0,K^0}) = \mathbb{v}_0} \mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) \tag{4.3.16}$$

and the unconstrained minimum problem

$$\inf_{(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{m}} \mathcal{F}^{1,\boldsymbol{\lambda}}(S_{h^1,K^1}, S_{h^0,K^0}, u) \tag{4.3.17}$$

have solution, where $\mathcal{F}^{1,\boldsymbol{\lambda}} : \mathcal{C}^1_{\mathbf{m}} \to \mathbb{R}$ *is defined as*

$$\mathcal{F}^{1,\boldsymbol{\lambda}}(S_{h^1,K^1}, S_{h^0,K^0}, u) := \mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) + \sum_{i=0}^{1} \lambda_i \Big| \mathcal{L}^2(S_{h^i,K^i}) - \mathbb{v}_i \Big|.$$

for any $\boldsymbol{\lambda} = (\lambda_0, \lambda_1) \in \mathbb{R}^2$ *such that* $\lambda_0, \lambda_1 > 0$.

Proof. We follow the *Direct Method of the Calculus of Variations.* Fix $\mathbf{m} := (m_1, m_0) \in \mathbb{N}^2$. Let $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)\} \subset \mathcal{C}^1_{\mathbf{m}}$ be a minimizing sequence of $\mathcal{F}^1$ such that $\mathcal{L}^2(S_{h^i_k,K^i_k}) = \mathbb{v}_i$ for $i = 0, 1$, and

$$\sup_{k\in\mathbb{N}} \mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) < \infty.$$

Since $\mathcal{L}^2(S_{h^1_k,K^1_k}) = \mathbb{v}_1$ for every $k \in \mathbb{N}$, by Theorem 4.3.2, there exist a subsequence $\{(S_{h^1_{k_l},K^1_{k_l}}, S_{h^0_{k_l},K^0_{k_l}}, u_{k_l})\}$, a sequence $\{(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}, v_l)\}_{n\in\mathbb{N}} \subset \mathcal{C}^1_{\mathbf{m}}$ and $(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}}$ such that

$$(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}, v_l) \xrightarrow{\tau_{\mathcal{C}^1}} (S_{h^1,K^1}, S_{h^0,K^0}, u)$$

as $l \to \infty$ and

$$\liminf_{l\to\infty} \mathcal{F}^1(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}, v_l) = \liminf_{l\to\infty} \mathcal{F}^1(S_{h^1_{k_l},K^1_{k_l}}, S_{h^0_{k_l},K^0_{k_l}}, u_{k_l}). \tag{4.3.18}$$

According to Theorem 4.3.3, we have that

$$\mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) \le \liminf_{l\to\infty} \mathcal{F}^1(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}, v_l). \tag{4.3.19}$$

We claim that $\{(S_{h^1,K^1}, S_{h^0,K^0})\}$ and $(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l})$ satisfy the volume constraints of (4.3.16). Indeed, fix $i = 0, 1$, by [66, Theorem 4.3], for any $l \ge 1$, $\mathbb{v}_i = \mathcal{L}^2(S_{h^i_{k_l},K^i_{k_l}}) = \mathcal{L}^2(S_{h^i_{k_l},\widetilde{K}^i_l})$. Thanks to the fact that

$$(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}) \xrightarrow{\tau_{\mathcal{B}^1}} (S_{h^1,K^1}, S_{h^0,K^0}),$$

applying [58, Lemma 3.2] we infer that $S_{h^i_{k_l},\widetilde{K}^i_l} \to S_{h^i,K^i}$ in $L^1(\mathbb{R}^2)$ as $l \to \infty$, and thus $\mathcal{L}^2(S_{h^i,K^i}) = \mathbb{v}_i$. From (4.3.18) and (4.3.19), we deduce that

$$\begin{aligned}
&\inf_{(S_{h^1,K^1}, S_{h^0,K^0}, u) \in \mathcal{C}^1_{\mathbf{m}},\, \mathcal{L}^2(S_{h^1,K^1}) = \mathbb{v}_1,\, \mathcal{L}^2(S_{h^0,K^0}) = \mathbb{v}_0} \mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u) \\
&\quad = \lim_{k\to\infty} \mathcal{F}^1(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) \ge \liminf_{l\to\infty} \mathcal{F}^1(S_{h^1_{k_l},\widetilde{K}^1_l}, S_{h^0_{k_l},\widetilde{K}^0_l}, v_l) \\
&\quad \ge \mathcal{F}^1(S_{h^1,K^1}, S_{h^0,K^0}, u).
\end{aligned}$$

We conclude from the previous inequality that (A, h, K, u) is a minimum of (4.3.16). The same arguments are used to solve the unconstrained problem (4.3.17) by noticing that for a minimizing sequence $\{(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k)\} \subset \mathcal{C}^1_{\mathbf{m}}$ of $\mathcal{F}^{1,\boldsymbol{\lambda}}$ such that

$$\sup_{k\in\mathbb{N}} \mathcal{F}^{1,\boldsymbol{\lambda}}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) < \infty$$

we have that

$$\mathcal{L}^2(S_{h^1_k,K^1_k}) \le \left|\mathcal{L}^2(S_{h^1_k,K^1_k}) - \mathbb{v}_1\right| + \mathbb{v}_1 \le \frac{1}{\lambda_1}\mathcal{F}^{1,\boldsymbol{\lambda}}(S_{h^1_k,K^1_k}, S_{h^0_k,K^0_k}, u_k) + \mathbb{v}_1,$$

and thus $\sup_{k\in\mathbb{N}} \mathcal{L}^2(S_{h^1_k,K^1_k}) < \infty$. □

4.4. Multilayered films

In this section, we consider $\alpha > 1$ and we denote $\mathbf{m} := (m_0, \ldots, m_\alpha) \in \mathbb{N}^{\alpha+1}$. The main goal of this section is to prove Theorem 4.2.6. In order to do this, first we prove that $\mathcal{C}^\alpha_{\boldsymbol{m}}$ is compact and by induction, with respect to $\alpha \in \mathbb{N}$ we show that $\mathcal{F}^\alpha$ is lower semicontinuous with respect to the topology of $\tau_{\mathcal{C}^\alpha}$. Notice that in the previous section, we proved the basis of the induction for the lower semicontinuity property. We start by proving that $\mathcal{B}^\alpha_{\boldsymbol{m}}$ and $\mathcal{C}^\alpha_{\boldsymbol{m}}$ are compact.

Proposition 4.4.1. *Let* $\{(S_{h^\alpha_k,K^\alpha_k}, \ldots, S_{h^0_k,K^0_k})\} \subset \mathcal{B}^\alpha_{\mathbf{m}}$ *such that*

$$\sup_{k\in\mathbb{N}} \Big(\mathcal{S}^\alpha(S_{h^\alpha_k,K^\alpha_k}, \ldots, S_{h^0_k,K^0_k}) + \mathcal{L}^2(S_{h^\alpha_k,K^\alpha_k})\Big) < \infty \tag{4.4.1}$$

Then, there exist a not relabeled subsequence $\{(S_{h^\alpha_k,K^\alpha_k}, \ldots, S_{h^0_k,K^0_k})\} \subset \mathcal{B}_{\mathbf{m}}$ *and* $\{(S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0})\} \in \mathcal{B}^\alpha_{\mathbf{m}}$ *such that* $(S_{h^\alpha_k,K^\alpha_k}, \ldots, S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}^\alpha}} (S_{h^\alpha,K^\alpha}, \ldots, S_{h^0,K^0})$.

Proof. We proceed by induction on $\alpha \in \mathbb{N}$. If $\alpha = 1$ by Proposition 4.3.1 the assertion holds. Assume now that for $\alpha = n$ the thesis of the proposition is true. We prove that the assertion holds if $\alpha = n + 1$. First, we observe that

$$\begin{aligned}\mathcal{S}^{n+1}(S_{h^{n+1}_k,K^{n+1}_k}, \ldots, S_{h^0_k,K^0_k}) &:= \sum_{j=1}^{n+1}\sum_{i=0}^{j-1} \mathcal{S}^{(i,j)}(S_{h^j,K^j}, S_{h^i,K^i}) \\ &= \sum_{j=1}^{n}\sum_{i=0}^{j-1} \mathcal{S}^{(i,j)}(S_{h^j,K^j}, S_{h^i,K^i}) \\ &\quad + \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^i,K^i}) \\ &=: \mathcal{S}^n(S_{h^n_k,K^n_k}, \ldots, S_{h^0_k,K^0_k}) \\ &\quad + \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^i,K^i})\end{aligned} \tag{4.4.2}$$

for every $k \in \mathbb{N}$. By (4.4.1), (4.4.2), the non-negativeness of the second term in the right side of (4.4.2) and thanks to the induction hypothesis we obtain that there exists $(S_{h^n,K^n}, \ldots, S_{h^0,K^0}) \in \mathcal{B}^n_{\mathbf{m_n}}$ such that $(S_{h^n_k,K^n_k}, \ldots, S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}^n}} (S_{h^n,K^n}, \ldots, S_{h^0,K^0})$, where $\mathbf{m_n} := (m_0, \ldots, m_n)$. Furthermore, by non-negativeness of $\mathcal{S}^{(n+1,j)}$ for every $j = 0, \ldots, n$, by (4.4.1) and (4.4.2) we see that

$$\mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^0,K^0}) \le \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^j,K^j}), \tag{4.4.3}$$

for every $j = 0, \dots, n$. Now we observed that since $\sup_{k\in\mathbb{N}} \mathcal{L}^2(S_{h_k^{n+1},K_k^{n+1}}) < \infty$ by (4.4.2) and (4.4.3) Proposition 4.3.1 applied for every $j = 0, \dots, n$ yields that there exist a subsequence $(S_{h_k^{n+1},K_k^{n+1}}, S_{h_k^j,K_k^j})$ and a region $S_{h^{n+1},K^{n+1}}$ with $(h^{n+1}, K^{n+1}) \in \mathrm{AHK}(\Omega)$ (and $(S_{h^{n+1},K^{n+1}}, S_{h^j,K^j}) \in \mathcal{B}^1_{\mathbf{m_j}}$ for $\mathbf{m_j} := (m_j, m_{n+1}) \in \mathbb{N}^2$) such that $(S_{h_k^{n+1},K_k^{n+1}}, S_{h_k^j,K_k^j}) \xrightarrow{\tau_{\mathcal{B}^1}} (S_{h^{n+1},K^{n+1}}, S_{h^j,K^j})$ for every $j = 0, \dots, n$, where we used the uniqueness of the sign-distance convergence.

It remains to prove that $(S_{h^{n+1},K^{n+1}}, \dots, S_{h^0,K^0}) \in \mathcal{B}^{n+1}_{\mathbf{m}}$. Since $\partial S_{h^i,K^i}$ has at most m_i connected components for $i = 0, \dots, n+1$, it remains to check that $\partial S_{h^{n+1},K^{n+1}} \cap \mathrm{Int}(S_{h^i,K^i}) = \emptyset$ for every $i = 0, \dots, n+1$, to which the rest of the proof is devoted. Let us fix $i = 0, \dots, n+1$ and assume by contradiction that

$$\partial S_{h^{n+1},K^{n+1}} \cap \mathrm{Int}(S_{h^i,K^i}) \neq \emptyset. \tag{4.4.4}$$

Then, there exists $x \in \partial S_{h^{n+1},K^{n+1}} \cap \mathrm{Int}(S_{h^i,K^i})$. By properties of the signed distance convergence (see [66, Remark 3.8]) there exists $x_k \in \partial S_{h_k^{n+1},K_k^{n+1}}$ such that $x_k \to x$, and by the $\tau_{\mathcal{B}^1}$-convergence it follows that

$$\mathrm{sdist}(x, \partial S_{h_k^i,K_k^i}) \to \mathrm{sdist}(x, \partial S_{h^i,K^i}) \quad \text{as } k \to \infty. \tag{4.4.5}$$

By (4.4.4) there exists $\varepsilon > 0$ such that $\mathrm{sdist}(x, \partial S_{h^i,K^i}) = -\varepsilon$, we can find $k_0 := k_0(x)$ for which $\mathrm{sdist}(x, \partial S_{h_{k_0}^i,K_{k_0}^i})$ is negative. Then, $x \in \mathrm{Int}(S_{h_{k_0}^i,K_{k_0}^i})$ and so, there exists $\delta \le \varepsilon/2$ such that

$$x_{k_0} \in B_\delta(x) \subset \mathrm{Int}(S_{h_{k_0}^i,K_{k_0}^i}),$$

which is an absurd since $\partial S_{h_k^{n+1},K_k^{n+1}} \cap \mathrm{Int}(S_{h_k^i,K_k^i}) = \emptyset$. Finally, we conclude the proof by observing that there exists $(S_{h^{n+1},K^{n+1}}, \dots, S_{h^0,K^0}) \in \mathcal{B}^{n+1}_{\mathbf{m}}$ such that $(S_{h_k^{n+1},K_k^{n+1}}, \dots, S_{h_k^0,K_k^0}) \xrightarrow{\tau_{\mathcal{B}^{n+1}}} (S_{h^{n+1},K^{n+1}}, \dots, S_{h^0,K^0})$. □

Now, we prove that $\mathcal{C}^\alpha_{\boldsymbol{m}}$ is compact with respect to the topology $\tau_{\mathcal{C}^\alpha}$.

Proof of Theorem 4.2.7. Denote $R := \sup_{k\in\mathbb{N}} \Big(\mathcal{F}^\alpha(S_{h_k^\alpha,K_k^\alpha}, \dots, S_{h_k^0,K_k^0}, u_k) + \mathcal{L}^2(S_{h_k^\alpha,K_k^\alpha})\Big)$. Without loss of generality (by passing, if necessary, to a not relabeled subsequence), we assume that

$$\liminf_{k\to\infty} \mathcal{F}^\alpha(S_{h_k^\alpha,K_k^\alpha}, \dots, S_{h_k^0,K_k^0}, u_k) = \lim_{k\to\infty} \mathcal{F}^\alpha(S_{h_k^\alpha,K_k^\alpha}, \dots, S_{h_k^0,K_k^0}, u_k) \le R. \tag{4.4.6}$$

Since $\mathcal{W}$ is a non-negative energy, by Proposition 4.4.1 there exist a subsequence $\{(S_{h_k^\alpha,K_k^\alpha}, \dots, S_{h_k^0,K_k^0})\} \subset \mathcal{B}^\alpha_{\mathbf{m}}$ and $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) \in \mathcal{B}^\alpha_{\mathbf{m}}$ such that

$$(S_{h_k^\alpha,K_k^\alpha}, \dots, S_{h_k^0,K_k^0}) \xrightarrow{\tau_{\mathcal{B}^\alpha}} (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}).$$

The rest of the proof is devoted to the construction of a sequence $(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha}, \dots, S_{h_{k_n}^0,\widetilde{K}_n^0}, u_{k_n}) \subset \mathcal{B}^\alpha_{\mathbf{m}}$ to which we can apply [58, Corollary 3.8] (with $P = \mathrm{Int}(S_{h^\alpha,K^\alpha})$ and $P_n = \mathrm{Int}(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha})$, respectively) in order to obtain $u \in H^1_{\mathrm{loc}}(\mathrm{Int}(S_{h^\alpha,K^\alpha}); \mathbb{R}^2)$ such that $(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) \in \mathcal{C}^\alpha_{\mathbf{m}}$ has finite energy, and a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise rigid displacements such that

$$(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha}, \dots, S_{h_{k_n}^0,\widetilde{K}_n^0}, u_{k_n} + b_n) \xrightarrow{\tau_{\mathcal{C}^\alpha}} (S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u).$$

Furthermore, we observe that also Equation (4.2.15) will be a consequence of such construction and hence, the assertion of the theorem will directly follow.

By [58, Proposition 3.6] applied to $S_{h^\alpha_{k_n},K^\alpha_{k_n}}$ and S_{h^α,K^α} there exist a not relabeled subsequence $\{S_{h^\alpha_{k_n},K^\alpha_{k_n}}\}$ and a sequence $\{\widetilde{A}_n\}$ with $\mathcal{H}^1$-rectifiable boundary $\partial\widetilde{A}_n$ of at most m_α-connected components such that

$$\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial\widetilde{A}_n) < \infty, \tag{4.4.7}$$

that satisfy the following properties:

(a1) $\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}} \subset \partial\widetilde{A}_n$ and $\lim_{n\to\infty} \mathcal{H}^1(\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}) = 0$,

(a2) $\operatorname{sdist}(\cdot,\partial\widetilde{A}_n) \to \operatorname{sdist}(\cdot,\partial S_{h^\alpha,K^\alpha})$ locally uniformly in $\mathbb{R}^2$ as $n\to\infty$,

(a3) If $\{E_i\}_{i\in I}$ is the family of all connected components of $\operatorname{Int}(S_{h^\alpha,K^\alpha})$, we can find the connected components of $\operatorname{Int}(\widetilde{A}_n)$, which we enumerate as $\{E^n_i\}_{i\in I}$, such that for any i and $G \subset\subset E_i$ one has $G \subset\subset E^n_i$ for all n large (depending only on i and G),

(a4) $\mathcal{L}^2(\widetilde{A}_n) = \mathcal{L}^2(S_{h^\alpha_{k_n},K^\alpha_{k_n}})$.

Furthermore, from the construction of $\widetilde{A}_n$ (namely from the fact that $\widetilde{A}_n$ is constructed by adding extra “internal” topological boundary to the selected subsequence $S_{h^\alpha_{k_n},K^\alpha_{k_n}}$, see [58, Propositions 3.4 and 3.6]) it follows that

$$\widetilde{A}_n = S_{h^\alpha_{k_n},K^\alpha_{k_n}} \setminus (\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}) \tag{4.4.8}$$

with $\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$ given by a finite union of closed $\mathcal{H}^1$-rectifiable sets connected to $\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$. More precisely, there exist a finite index set J and a family $\{\Gamma_j\}_{j\in J}$ of closed $\mathcal{H}^1$-rectifiable sets of Ω connected to ∂S_{k_n} such that

$$\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}} = \bigcup_{j\in J} \Gamma_j.$$

We define

$$\widetilde{K^i_n} := K^i_{k_n} \cup ((\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}) \cap S_{h^i_{k_n}}) \subset S_{h^i_{k_n}},$$

for every $i = 0,\dots,\alpha$, and we observe that $\widetilde{K^i_n}$ is closed and $\mathcal{H}^1$-rectifiable in view of the fact that $\partial\widetilde{A}_n \setminus \partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$ is a closed set in Ω and is $\mathcal{H}^1$-rectifiable, since $\partial\widetilde{A}_n$ is $\mathcal{H}^1$-rectifiable. Therefore, $(h^i_{k_n},\widetilde{K^i_n}) \in \mathrm{AHK}(\Omega)$ for every $i = 0,\dots,\alpha$. Furthermore, we have that

$$S_{h^i_{k_n},\widetilde{K^i_n}} \subset S_{h^i_{k_n}} \subset S_{h^j_{k_n}} = \overline{S_{h^j_{k_n},\widetilde{K^j_n}}},$$

for every $0 \le i \le j \le \alpha$. We claim that $\partial S_{h^i_{k_n},\widetilde{K^i_n}}$ has at most m_i-connected components for $i = 0,\dots,\lambda$. Indeed, let $i \in \{0,\dots,\alpha\}$, if for every $j \in J$, $S_{h^i_{k_n},K^i_{k_n}} \cap \Gamma_j$ is empty there is nothing to prove, so we assume that there exists $j \in J$ such that $S_{h^i_{k_n},K^i_{k_n}} \cap \Gamma_j \neq \emptyset$. On one hand if $\Gamma_j \subset S_{h^i_{k_n},K^i_{k_n}}$, thanks to the facts that Γ_j is connected to $\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$ and $S_{h^i_{k_n},K^i_{k_n}} \subset S_{h^\lambda_{k_n}}$, we deduce that Γ_j needs to be connected to $\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$. On the other hand, if $\Gamma_j \cap (S_{h^\alpha_{k_n},K^\alpha_{k_n}} \setminus S_{h^i_{k_n}}) \neq \emptyset$, then we can find $x_1 \in \Gamma_j \cap S_{h^i_{k_n},K^i_{k_n}}$ and $x_2 \in \Gamma_j \cap (S_{h^\alpha_{k_n},K^\alpha_{k_n}} \setminus S_{h_{k_n}})$. Since Γ_j is closed and connected, by [44, Lemma 3.12] there exists a parametrization $r : [0,1] \to \mathbb{R}^2$ whose support $\gamma \subset \Gamma_j$ joins the point x_1 with x_2. Thus, γ crosses $\partial S_{h^i_{k_n},K^i_{k_n}}$ and we conclude that Γ_j is connected to $\partial S_{h^i_{k_n},K^i_{k_n}}$. Finally, by repeating the same arguments of (4.3.12), we obtain that

$$\widetilde{A}_n = S_{h^\lambda_{k_n},\widetilde{K}^\lambda_{k_n}}$$

and thus, $(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n}) \in \mathcal{B}^\alpha_{\mathbf{m}}$.

We claim that $(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n}) \xrightarrow{\tau_{\mathcal{B}^\alpha}} (S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0})$ as $n\to\infty$. In view of (4.4.7), (a2), by (4.2.7) and the previous construction of $\widetilde{K}^i_n$,

$$\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial S_{h^i_{k_n},\widetilde{K}^i_n}) < \infty,$$

for every $i=0,\dots,\lambda$. It remains to prove that

$$\operatorname{sdist}(\cdot,\partial S^i_{h^i_{k_n},\widetilde{K}_n}) \to \operatorname{sdist}(\cdot,\partial S_{h^i,K^1}) \tag{4.4.9}$$

locally uniformly in $\mathbb{R}^2$ as $n\to\infty$ for every $i=0,\dots,\alpha$. Let us fix $i=0,\dots,\alpha$, by properties of the signed distance convergence, it suffices to prove that $S_{h^i_{k_n},\widetilde{K}^i_n} \xrightarrow{\mathcal{K}} S_{h^i}$ and that $\Omega\setminus S_{h^i_{k_n},\widetilde{K}^i_n} \xrightarrow{\mathcal{K}} \Omega\setminus \operatorname{Int}(S_{h^i,K^i})$. On one hand, by the $\tau_{\mathcal{B}^\alpha}$-convergence of $\{(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n})\}$, the fact that $\overline{S_{h^i_{k_n},\widetilde{K}^i_n}} = S_{h^i_{k_n}}$, and the properties of Kuratowski convergence, it follows that $S_{h^i_{k_n},\widetilde{K}^i_n} \xrightarrow{\mathcal{K}} S_{h^i}$. On the other hand, let $x\in\Omega\setminus\operatorname{Int}(S_{h^i,K^i})$, since

$$\operatorname{Int}(S_{h^i_{k_n},\widetilde{K}^i_n}) = \operatorname{Int}(S_{h^i_{k_n}})\setminus \widetilde{K}^i_n \subset \operatorname{Int}(S_{h^i_{k_n}})\setminus K^i_{k_n} = \operatorname{Int}(S_{h^i_{k_n},K_{k_n}})$$

and by the fact that $\Omega\setminus\operatorname{Int}(S_{h^i_{k_n},K^i_{k_n}}) \xrightarrow{\mathcal{K}} \Omega\setminus\operatorname{Int}(S_{h^i,K^i})$, there exists

$$x_n \in \Omega\setminus\operatorname{Int}(S_{h^i_{k_n},K^i_{k_n}}) \subset \Omega\setminus\operatorname{Int}(S_{h^i_{k_n},\widetilde{K}^i_n})$$

such that $x_n\to x$. Now, we consider a sequence $x_n\in\Omega\setminus\operatorname{Int}(S_{h^i_{k_n},\widetilde{K}^i_n})$ converging to a point $x\in\Omega$. We proceed by contradiction, namely we assume that $x\in\operatorname{Int}(S_{h^i,K^i})$. Therefore, there exists $\epsilon>0$ such that $\operatorname{sdist}(x,\partial S_{h^i,K^i}) = -\epsilon$, which implies that $\operatorname{sdist}(x,\partial S_{h^i_{k_n},K^i_{k_n}}) \to -\epsilon$ as $n\to\infty$. Thus, there exists $n_\epsilon\in\mathbb{N}$, such that $x_n\in B_{\epsilon/2}(x)\subset \operatorname{Int}(S_{h^i_{k_n},K^i_{k_n}})$, for every $n\geq n_\epsilon$. However, notice that

$$\begin{aligned} x_n \in \Omega\setminus\operatorname{Int}\left(S_{h^i_{k_n},\widetilde{K}^i_n}\right) &= \Omega\setminus\left(\operatorname{Int}(S_{h^i_{k_n}})\setminus\widetilde{K}^i_n\right)\\ &= \left(\Omega\setminus\operatorname{Int}\left(S_{h^i_{k_n},K^i_{k_n}}\right)\right)\cup\left(\left(\partial S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}}\setminus\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}\right)\cap S_{h^i_{k_n}}\right), \end{aligned} \tag{4.4.10}$$

where in the last equality we used the definition of $\widetilde{K}^i_n := K^i_{k_n}\cup((\partial S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}}\setminus\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}})\cap S_{h^i_{k_n}})$ and the fact that $\operatorname{Int}(S_{h^i_{k_n},K^i_{k_n}}) = \operatorname{Int}(S_{h^i_{k_n}})\setminus K^i_{k_n}$. Therefore, by (4.4.10) we deduce that $x_n\in\partial S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}}\setminus\partial S_{h^\alpha_{k_n},K^\alpha_{k_n}}$ for every $n\geq n_\epsilon$ and hence, $x\in\partial S_{h^\alpha,K^\alpha}$ by (a2) and by [66, Remark 3.7]. We reached an absurd as it follows that $x\in\operatorname{Int}(S_{h^i,K^i})\cap\partial S_{h^\alpha,K^\alpha}=\emptyset$. This conclude the proof of (4.4.9) and hence, of the claim.

By (4.2.9) and by conditions (a1), (a4) and (4.4.8), we observe that

$$\lim_{n\to\infty}\left|\mathcal{S}^\alpha(S_{h^\alpha_{k_n},K^\alpha_{k_n}},\dots,S_{h^0_{k_n},K^0_{k_n}}) - \mathcal{S}^\alpha(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n})\right| = 0, \tag{4.4.11}$$

and

$$\mathcal{W}(S_{h^\alpha_{k_n},K^\alpha_{k_n}},\dots,S_{h^0_{k_n},K^0_{k_n}},u_{k_n}) = \mathcal{W}(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n},u_{k_n}). \tag{4.4.12}$$

By (4.2.11), (4.4.6), (4.4.8), (4.4.12), (a3) and thanks to the fact that $\mathcal{S}^\alpha$ is non-negative, we obtain that

$$\int_{E^n_i} |e(u_{k_n})|^2dx \leq \int_{S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}}} |e(u_{k_n})|^2dx \leq C\frac{R}{2c_3},$$

for every $i \in I$, for n large enough and for a constant $C := C(u_0^1, \dots, u_0^\alpha) > 0$. Therefore, by a diagonal argument and by [58, Corollary 3.8] (applied to, with the notation of [58], $P = E_i$ and $P_n = E_i^n$) up to extracting not relabeled subsequences both for $\{u_{k_n}\} \subset H^1_{\text{loc}}(\Omega;\mathbb{R}^2)$ and $\{E_i^n\}_n$ there exist $w_i \in H^1_{\text{loc}}(E_i,\mathbb{R}^2)$, and a sequence of rigid displacements $\{b_n^i\}$ such that $(u_{k_n} + b_n^i)\mathbb{1}_{E_i^n} \to w_i$ a.e. in E_i. Let $\{D_i^n\}_{i\in\widetilde{I}}$ for an index set $\widetilde{I}$ be the family of open and connected components of $S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}} \setminus \bigcup_{i\in I} E_i^n$ such that by (a3) $\text{Int}(D_i^n)$ converges to the empty set for every $i \in \widetilde{I}$. In D_i^n we consider the null rigid displacement, and we define

$$b_n := \sum_{i\in I} b_n^i \mathbb{1}_{E_i^n} \quad \text{and} \quad u := \sum_{i\in I} w_i \mathbb{1}_{E_i}.$$

We have that $u \in H^1_{\text{loc}}(\text{Int}(S_{h^\alpha,K^\alpha});\mathbb{R}^2)$, b_n is a rigid displacement associated to $S_{h^\alpha_{k_n},\widetilde{K}^\alpha_{k_n}}$, $u_{k_n} + b_n \to u$ a.e. in $\text{Int}(S_{h^\alpha,\widetilde{K}^\alpha})$ and hence, $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u) \in \mathcal{C}_{\mathbf{m}}$ and $(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n},u_{k_n}+b_n) \xrightarrow{\tau_{\mathcal{C}}} (S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)$. Furthermore, as $E(u_{k_n}+b_n) = Eu_{k_n}$, from (4.4.11) and (4.4.12) it follows that

$$\lim_{n\to\infty}\left|\mathcal{F}^\alpha(S_{h^\alpha_{k_n},K^\alpha_{k_n}},\dots,S_{h^0_{k_n},K^0_{k_n}},u_{k_n}) - \mathcal{F}^\alpha(S_{h^\alpha_{k_n},\widetilde{K}^\alpha_n},\dots,S_{h^0_{k_n},\widetilde{K}^0_n},u_{k_n}+b_n)\right| = 0, \qquad (4.4.13)$$

which implies (4.2.15) and completes the proof. □

In the following proof, we show by induction that $\mathcal{F}^\alpha$ is lower semicontinuous.

Proof of Theorem 4.2.8. Since

$$\mathcal{F}^\alpha(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k) := \mathcal{S}^\alpha(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k}) + \mathcal{W}(S_{h^\alpha_k,K^\alpha_k},\dots,S_{h^0_k,K^0_k},u_k),$$

and by non-negativeness of $\mathcal{S}^\alpha$ and $\mathcal{W}$ we prove first that $\mathcal{S}^\alpha$ is lower semicontinuous with respect to the convergence in $\tau_{\mathcal{B}^\alpha}$, and then we prove that $\mathcal{W}$ is lower semicontinuous with respect to the convergence in $\tau_{\mathcal{C}^\alpha}$.

To prove that $\mathcal{S}^\alpha$ is lower semicontinuous we proceed by induction on $\alpha \in \mathbb{N}$. Notice that if $\alpha = 1$, by Theorem 4.3.3 the assertion holds. Assume now that for $\alpha = n$ the assertion of the theorem holds. We are going to prove that the assertion of theorem holds if $\alpha = n+1$. By definition of the energy $\mathcal{S}^{n+1}$ we see that

$$\begin{aligned}\mathcal{S}^{n+1}(S_{h^{n+1}_k,K^{n+1}_k},\dots,S_{h^0_k,K^0_k}) &= \mathcal{S}^n(S_{h^n_k,K^n_k},\dots,S_{h^0_k,K^0_k}) \\ &\quad + \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k},S_{h^j_k,K^j_k})\end{aligned} \qquad (4.4.14)$$

for every $k \in \mathbb{N}$. By the induction hypothesis and the fact that

$$(S_{h^n_k,K^n_k},\dots,S_{h^0_k,K^0_k}) \xrightarrow{\tau_{\mathcal{B}^n}} (S_{h^n,K^n},\dots,S_{h^0,K^0})$$

we have that

$$\mathcal{S}^n(S_{h^n,K^n},\dots,S_{h^0,K^0}) \le \liminf_{k\to\infty} \mathcal{S}^n(S_{h^n_k,K^n_k},\dots,S_{h^0_k,K^0_k}). \qquad (4.4.15)$$

Furthermore, by definition of $\tau_{\mathcal{B}^{n+1}}$-convergence it follows that

$$(S_{h^{n+1}_k,K^{n+1}_k},S_{h^j_k,K^j_k}) \xrightarrow{\tau_{\mathcal{B}^1}} (S_{h^{n+1},K^{n+1}},S_{h^j_k,K^j_k}) \qquad (4.4.16)$$

for every $j = 0,\dots,n$. Thus, by (4.4.16) and by the induction hypothesis we deduce that

$$\mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}},S_{h^j_k,K^j_k}) \le \liminf_{k\to\infty} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k},S_{h^j_k,K^j_k}) \qquad (4.4.17)$$

for every $j = 0, \dots, n$. It follows from the superadditivity of the *liminf* that

$$\begin{aligned}\sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^j_k,K^j_k}) &\leq \sum_{j=0}^{n} \liminf_{k\to\infty} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k}, S_{h^j_k,K^j_k}) \\ &\leq \liminf_{k\to\infty} \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k}, S_{h^j_k,K^j_k}),\end{aligned} \tag{4.4.18}$$

where in the first inequality we used (4.4.17). Therefore, we have that

$$\begin{aligned}\mathcal{S}^{n+1}(S_{h^{n+1},K^{n+1}}, \dots, S_{h^0,K^0}) &= \mathcal{S}^{n}(S_{h^{n},K^{n}}, \dots, S_{h^0,K^0}) \\ &\quad + \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1},K^{n+1}}, S_{h^j,K^j}) \\ &\leq \liminf_{k\to\infty} \mathcal{S}^{n}(S_{h^n_k,K^n_k}, \dots, S_{h^0_k,K^0_k}) \\ &\quad + \liminf_{k\to\infty} \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k}, S_{h^j_k,K^j_k}) \\ &\leq \liminf_{k\to\infty} \Big(\mathcal{S}^{n}(S_{h^n_k,K^n_k}, \dots, S_{h^0_k,K^0_k}) \\ &\quad + \sum_{j=0}^{n} \mathcal{S}^{(n+1,j)}(S_{h^{n+1}_k,K^{n+1}_k}, S_{h^j_k,K^j_k})\Big) \\ &= \liminf_{k\to\infty} \mathcal{S}^{n+1}(S_{h^{n+1}_k,K^{n+1}_k}, \dots, S_{h^0_k,K^0_k}),\end{aligned} \tag{4.4.19}$$

where in first and second equality we used (4.4.14) the definition of $\mathcal{S}^{n+1}$, in the first inequality we used (4.4.15) and (4.4.18) and in the second inequality we used the superadditivity of the *liminf*, and thus, $\mathcal{S}^{\alpha}$ is lower semicontinuous with respect to the topology $\tau_{\mathcal{B}^{\alpha}}$.

By repeating the same arguments of the proof of lower semicontinuity of $\mathcal{W}$ in the proof of Theorem 4.3.3 we can deduce that

$$\mathcal{W}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u) \leq \liminf_{k\to\infty} \mathcal{W}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k) \tag{4.4.20}$$

Finally, we conclude the proof by observing that by the superadditivity of the *liminf* it follows that

$$\begin{aligned}\mathcal{F}^{\alpha}(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}, u) &:= \mathcal{S}^{\alpha}(S_{h^\alpha,K^\alpha}, \dots, S_{h^0,K^0}) + \mathcal{W}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u) \\ &\leq \liminf_{k\to\infty} \mathcal{S}^{\alpha}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}) \\ &\quad + \liminf_{k\to\infty} \mathcal{W}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k) \\ &\leq \liminf_{k\to\infty} \Big(\mathcal{S}^{\alpha}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}) + \mathcal{W}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k)\Big) \\ &=: \liminf_{k\to\infty} \mathcal{F}^{\alpha}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k),\end{aligned}$$

where in the first inequality we used the lower semicontinuity of $\mathcal{S}^{\alpha}$ and $\mathcal{W}$. □

Finally, we are now in a position to prove the main result of this chapter.

Proof of Theorem 4.2.6. We follow the *Direct Method of the Calculus of Variations*. Fix $\mathbf{m} = (m_0, \dots, m_\alpha) \in \mathbb{N}^{\alpha+1}$ and let $\{(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k)\} \subset \mathcal{C}^{\alpha}_{\mathbf{m}}$ be a minimizing sequence of $\mathcal{F}^{\alpha}$ such that $\mathcal{L}^2(S_{h^i_k,K^i_k}) = \mathbb{v}_i$ for $i = 0, \dots, \alpha$, and

$$\sup_{k\in\mathbb{N}} \mathcal{F}^{\alpha}(S_{h^\alpha_k,K^\alpha_k}, \dots, S_{h^0_k,K^0_k}, u_k) < \infty.$$

Since $\mathcal{L}^2(S_{h_k^\alpha,K_k^\alpha}) = \mathbb{v}_\alpha$, by Theorem 4.2.7 there exist a subsequence $\{(S_{h_{k_n}^\alpha,K_{k_n}^\alpha},\dots,S_{h_{k_n}^0,K_{k_n}^0},u_{k_n})\}$, a sequence $\{(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},v_n)\}_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathbf{m}}^\alpha$ and $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)\in\mathcal{C}_{\mathbf{m}}^\alpha$ such that

$$(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},v_n)\xrightarrow{\tau_{\mathcal{C}^\alpha}}(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)$$

as $n\to\infty$ and

$$\liminf_{n\to\infty}\mathcal{F}^\alpha(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},v_n)=\liminf_{n\to\infty}\mathcal{F}^\alpha(S_{h_{k_n}^\alpha,K_{k_n}^\alpha},\dots,S_{h_{k_n}^0,K_{k_n}^0},u_{k_n}). \tag{4.4.21}$$

According to Theorem 4.2.8, we have that

$$\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)\le\liminf_{n\to\infty}\mathcal{F}^\alpha(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},v_n). \tag{4.4.22}$$

We claim that for every $i=0,\dots,\alpha$, $S_{h_{k_n}^i,\widetilde{K}_n^i}$ and $S_{h^i,\widetilde{K}^i}$ satisfy the volume constraints of (4.2.12). Indeed, by Theorem 4.2.7, for any $n\in\mathbb{N}$, $\mathbb{v}_i=\mathcal{L}^2(S_{h_{k_n}^i,K_{k_n}^i})=\mathcal{L}^2(S_{h_{k_n}^i,\widetilde{K}_n^i})$ for every $i=0,\dots,\alpha$. Fix $i=0,\dots,\alpha$. By definition of $\tau_{\mathcal{B}^\alpha}$-convergence and by applying [58, Lemma 3.2] we infer that $S_{h_{k_n}^i,\widetilde{K}_n^i}\to S_{h^i,K^i}$ in $L^1(\mathbb{R}^2)$ as $n\to\infty$, and thus $\mathcal{L}^2(S_{h^i,K^i})=\mathbb{v}_i$. From (4.4.21) and (4.4.22), we deduce that

$$\begin{aligned}&\inf_{\substack{(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)\in\mathcal{C}_{\mathbf{m}}^\alpha,\\ \mathcal{L}^2(S_{h^i,K^i})=\mathbb{v}_i,\ i=0,\dots,\alpha}}\mathcal{F}^\alpha(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)\\ &\qquad=\lim_{n\to\infty}\mathcal{F}^\lambda(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},u_{k_n})\\ &\qquad\ge\liminf_{n\to\infty}\mathcal{F}^\lambda(S_{h_{k_n}^\alpha,\widetilde{K}_n^\alpha},\dots,S_{h_{k_n}^0,\widetilde{K}_n^0},u_{k_n})\ge\mathcal{F}^\lambda(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u).\end{aligned}$$

We conclude from the previous inequality that $(S_{h^\alpha,K^\alpha},\dots,S_{h^0,K^0},u)$ is a minimum of (4.2.12).

The same strategy is used to solve the unconstrained problem (4.2.13) thanks to the extra observation that for any minimizing sequence $\{(S_{h_k^\alpha,K_k^\alpha},\dots,S_{h_k^0,K_k^0},u_k)\}\subset\mathcal{C}_{\mathbf{m}}^\alpha$ of $\mathcal{F}^{\alpha,\boldsymbol{\lambda}}$ such that

$$\sup_{k\in\mathbb{N}}\mathcal{F}^{\alpha,\boldsymbol{\lambda}}(S_{h_k^\alpha,K_k^\alpha},\dots,S_{h_k^0,K_k^0},u_k)<\infty$$

we have that

$$\mathcal{L}^2(S_{h_k^\alpha,K_k^\alpha})\le\left|\mathcal{L}^2(S_{h_k^\alpha,K_k^\alpha})-\mathbb{v}_\alpha\right|+\mathbb{v}_\alpha\le\frac{1}{\lambda_\alpha}\mathcal{F}^{\alpha,\boldsymbol{\lambda}}(S_{h_k^\alpha,K_k^\alpha},\dots,S_{h_k^0,K_k^0},u_k)+\mathbb{v}_\alpha$$

for every $k\in\mathbb{N}$.

This conclude the proof. □

Part B.

Perfect Dynamical Elasto-Plasticity with dissipative boundary conditions

5. Dissipative boundary conditions

In this chapter, we present the results published in the paper

- J.-F. Babadjian, R. Llerena: *Mixed boundary conditions as limits of dissipative boundary conditions in dynamic perfect plasticity*, Journal of Convex Analysis **30** (2023), no. 1, 81–110.

5.1. Introduction

Elasto-plasticity is a classical theory of continuum mechanics [57, 67] that predicts the appearance of permanent deformations in materials when an internal critical stress is reached. At the atomistic level, these plastic deformations occur when the crystal lattice of the atoms are misaligned due to the accumulation of slips defects, called dislocations. These dislocations determine the change of behavior of a body from an elastic and reversible state to a plastic and irreversible one.

At the continuum level, and in the context of small deformations, the theory involves the *displacement field* $u : \Omega \times (0,T) \to \mathbb{R}^n$ and the *Cauchy stress tensor* $\sigma : \Omega \times (0,T) \to \mathbb{M}^n_{sym}$, both defined on the reference configuration Ω of the body, a bounded open subset of $\mathbb{R}^n$ $(n = 2,3)$. They first satisfy the *equation of motion*

$$\ddot{u} - \mathrm{d}iv\sigma = f \quad \text{ in } \Omega \times (0,T), \tag{5.1.1}$$

for some (given) external body load $f : \Omega \times (0,T) \to \mathbb{R}^n$. In the previous expression, and in the sequel, the dot stands for the partial derivative with respect to time. One particular feature of perfect plasticity is that the stress tensor is constrained to take its values into a fixed closed and convex set $\mathbf{K}$ of the space $\mathbb{M}^n_{sym}$ of symmetric $n \times n$ matrices, also called *elasticity set*:

$$\sigma \in \mathbf{K}. \tag{5.1.2}$$

In classical elasticity, the linearized strain is purely elastic and it is represented by the symmetric part of the gradient of displacement, i.e. $Eu := (Du + Du^T)/2$. In perfect elasto-plasticity, the elastic strain $e : \Omega \times (0,T) \to \mathbb{M}^n_{sym}$ only represents a part of the linearized strain Eu. It stands for the reversible part of the total deformation and it is related to σ by means of *Hooke's law*, which we assume to be isotropic:

$$\sigma = \mathbb{C}e = \lambda(\mathrm{tr}\, e)\mathrm{I}d + 2\mu e, \tag{5.1.3}$$

for some constants $(\lambda, \mu) \in \mathbb{R}^2$, called Lamé coefficients, which satisfy the ellipticity conditions $\mu > 0$ and $n\lambda + 2\mu > 0$. The remaining part of the strain,

$$p := Eu - e \tag{5.1.4}$$

stands for the plastic strain leading to irreversible deformations. It is a new unknown of the problem whose evolution is described by means of a *flow rule*. Assuming that $\mathbf{K}$ has nonempty interior, it stipulates that if σ belongs to the interior of $\mathbf{K}$, then the material behaves elastically and no additional inelastic strains are created, i.e. $\dot{p} = 0$. On the other hand, if σ reaches the boundary of $\mathbf{K}$, then $\dot{p}$ may develop in such a way that a non–trivial permanent plastic strain p may remain after unloading. The evolution of p is described by the *Prandtl-Reuss law*

$$\dot{p} \in N_{\mathbf{K}}(\sigma),$$

where $N_{\mathbf{K}}(\sigma)$ stands for the normal cone to $\mathbf{K}$ at σ, or equivalently, thanks to convex analysis, by *Hill's principle of maximum plastic work*

$$H(\dot{p}) = \sigma : \dot{p}, \tag{5.1.5}$$

where $H(q) := \sup_{\tau \in \mathbf{K}} \tau : q$ is the support function $\mathbf{K}$. The system (5.1.1)–(5.1.5) has to be supplemented by initial conditions

$$(u(0), \dot{u}(0), e(0), p(0)) = (u_0, v_0, e_0, p_0) \tag{5.1.6}$$

as well as suitable boundary conditions to be discussed later, and which will be one of the main focus of this work.

For most of metals and alloys, standard models of perfect plasticity involve elasticity sets $\mathbf{K}$ which are invariant in the direction of hydrostatic matrices (multiples of the identity) and bounded in the direction of deviatoric (trace free) ones. This is for example the case of the Von Mises and Tresca models (see e.g. [5, 7, 89] in the static case, [6, 49, 85, 32] in the quasi-static case and [8, 36] in the dynamic one). In other situations like in the context of soils mechanics, it is of importance to consider elasticity sets $\mathbf{K}$ that are not necessarily invariant with respect to hydrostatic matrices. So called Drucker-Prager or Mohr-Coulomb models fall within this framework (see [11, 14, 15]). In this chapter, we treat as utmost as possible the case of a general elasticity set $\mathbf{K}$.

Let us now discuss the boundary conditions. Having in mind that the system of dynamic elasto-plasticity described so far has a hyperbolic nature, one has to consider boundary conditions compatible with this hyperbolic structure, in particular, with the finite speed propagation of the initial data along the characteristic lines. A general approach to this type of initial–boundary value constrained hyperbolic systems has been studied in [41] (see also [40]) where a class of so-called admissible dissipative boundary conditions has been introduced. This problem has subsequently been specified to the case of plasticity, first in [13] for a simplified scalar model, and then in [11] for the general vectorial model as described before. In this context, all admissible (homogeneous) *dissipative boundary conditions* take the form (see [11, Section 3])

$$S\dot{u} + \sigma\nu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{5.1.7}$$

where ν denotes the outer unit normal to Ω, and $S : \partial\Omega \to \mathbb{M}^n_{sym}$ is a spatially dependent positive definite boundary matrix. The well posedness of the initial–boundary value system (5.1.1)–(5.1.7) has been carried out in [11]. It has been established existence and uniqueness of two equivalent notions of relaxed solutions (variational and entropic solutions). The relaxation phenomena is a simple consequence of the fact that, formally, the stress constraint (5.1.2) might not be compatible with the boundary condition (5.1.7). Indeed, if $\sigma(t) \in \mathbf{K}$ in Ω, we would expect that $\sigma(t)\nu \in \mathbf{K}\nu$ on $\partial\Omega$ while $\sigma(t)\nu = -S\dot{u}(t)$ is free on the boundary. Thus, the boundary condition and the stress constraint have to accommodate to each other and the dissipative boundary condition (5.1.7) has to be relaxed into

$$P_{-\mathbf{K}\nu}(S\dot{u}) + \sigma\nu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{5.1.8}$$

where, for $x \in \partial\Omega$, $P_{-\mathbf{K}\nu(x)}$ stands for the orthogonal projection in $\mathbb{R}^n$ onto the convex set $-\mathbf{K}\nu(x)$ with respect to a suitable scalar product. This is indeed a relaxation in the sense of the Calculus of Variations, because the energy balance involves a term of the form

$$\int_\Omega H(\dot{p})\, dx + \frac{1}{2}\int_{\partial\Omega} S\dot{u} \cdot \dot{u}\, d\mathcal{H}^{n-1} + \frac{1}{2}\int_{\partial\Omega} S^{-1}(\sigma\nu) \cdot (\sigma\nu)\, d\mathcal{H}^{n-1}.$$

The previous energy functional turns out of not being lower semicontinuous with respect to weak convergence in the energy space, and its relaxation with respect to this topology is explicitly given by

$$\int_\Omega H(\dot p)\,dx + \int_{\partial\Omega} \psi(x,\dot u)\,d\mathcal{H}^{n-1} + \frac{1}{2}\int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,d\mathcal{H}^{n-1},$$

where $\psi(x,\cdot)$ is the inf-convolution of the functions $z \mapsto \frac{1}{2}S(x)z\cdot z$ and $z \mapsto H(-z \odot \nu(x))$. The connection between the relaxed energy and the modified boundary condition (5.1.8) comes from a first order minimality condition and the following formula (see [11, Section 4])

$$D_z\psi(x,\dot u(t,x)) = P_{-\mathbf{K}\nu(x)}(S(x)\dot u(t,x)).$$

Unfortunately, Dirichlet, Neumann and mixed boundary conditions are not admissible because the matrix S is not allowed to vanish nor to take the value ∞. It is the main focus of the present work to show that these type of natural boundary conditions can actually be obtained by means of an asymptotic analysis letting $S \to \infty$ in a portion of the boundary where we want to recover a Dirichlet condition, and letting $S \to 0$ on the complementary part where one wishes to formulate a Neumann condition. This type of analysis has already been performed in [13] in the simplified case of antiplane scalar plasticity where pure Dirichlet and pure Neumann boundary conditions have been derived. We extend here this analysis to the general vectorial case where additional issues arise, and to the case of mixed boundary conditions.

To be more precise, in the spirit of [49, 61, 32], we partition $\partial\Omega$ into the disjoint union of Γ_D, Γ_N and Σ, where Γ_D and Γ_N stand for the Dirichlet and Neumann parts of the boundary, respectively, and Σ is the interface between Γ_D and Γ_N which is $\mathcal{H}^{n-1}$-negligible. We consider a boundary matrix of the form

$$S_\lambda(x) := \left(\lambda \mathbf{1}_{\Gamma_D} + \frac{1}{\lambda}\mathbf{1}_{\Gamma_N}\right)\mathrm{Id} \tag{5.1.9}$$

for some parameter $\lambda > 0$ which will be sent to ∞. Denoting by $(u_\lambda, e_\lambda, p_\lambda, \sigma_\lambda)$ the unique weak solutions of the system (5.1.1)–(5.1.6) with the relaxed dissipative boundary condition (5.1.8) associated to the boundary matrix S_λ, using the results of [11], we easily derive bounds in the energy space for this quadruple, which allow one to get weak limits (u, e, p, σ) and pass to the limit into the equation of motion (5.1.1), the stress constraint (5.1.2), Hooke's law (5.1.3), the additive decomposition (5.1.4) and the initial condition (5.1.6). This is the object of Lemma 5.3.10. As usual in plasticity, the main difficulty consists in passing to the limit in the flow rule expressed by (5.1.5) and in the relaxed boundary condition (5.1.8). In accordance with [11, 13, 14], the idea consists in taking the limit as $\lambda \to \infty$ into the energy balance. The main difficulty is concerned with the term

$$\int_\Omega H(\dot p_\lambda)\,dx + \int_{\partial\Omega} \psi_\lambda(x,\dot u_\lambda)\,d\mathcal{H}^{n-1} + \frac{1}{2}\int_{\partial\Omega} S_\lambda^{-1}(\sigma_\lambda\nu)\cdot(\sigma_\lambda\nu)\,d\mathcal{H}^{n-1},$$

where

$$\psi_\lambda(x,z) := \inf_{w\in\mathbb{R}^n}\left\{\frac{1}{2}\left(\lambda\mathbf{1}_{\Gamma_D} + \frac{1}{\lambda}\mathbf{1}_{\Gamma_N}\right)|w|^2 + H((w-z)\odot\nu(x))\right\}.$$

A uniform bound on the previous energy easily shows that

$$\int_{\Gamma_N} |\sigma_\lambda\nu|^2\,d\mathcal{H}^{n-1} \le \frac{C}{\lambda} \to 0, \qquad \text{as } \lambda\to\infty,$$

which leads to the Neumann boundary condition $\sigma\nu = 0$ on Γ_N. The obtention of the Dirichlet boundary condition on Γ_D is more involved because, as usual in perfect plasticity, concentration phenomena might occur. A convex analysis argument based on the Moreau-Yosida approximation of H yields the following lower bound on the energy (see Lemma 5.3.11)

$$\int_\Omega H(\dot p)\,dx + \int_{\Gamma_D} H(-\dot u \odot \nu)\,d\mathcal{H}^{n-1} \le \liminf_{\lambda\to\infty}\left(\int_\Omega H(\dot p_\lambda)\,dx + \int_{\partial\Omega} \psi_\lambda(x,\dot u_\lambda)\,d\mathcal{H}^{n-1}\right).$$

Proving that this lower bound is also an upper bound is formally a consequence the convexity inequality

$$H(\dot p) \ge \sigma : \dot p$$

(because $\sigma \in \mathbf{K}$), and integrations by parts in space and time. Unfortunately, this formal convexity inequality is difficult to justify in the context of perfect plasticity because the Cauchy stress σ and the plastic strain rate $\dot p$ are not in duality. Indeed, the natural energy space gives $\sigma(t) \in H(\mathrm{d}iv,\Omega)$ while $\dot p((t) \in \mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^n_{sym})$ since the support function H grows linearly with respect to its argument. In particular, the plastic dissipation

$$\int_\Omega H(\dot p(t))\,dx$$

has to be understood as a convex function of a measure (see [38, 39, 54]). Whenever the quadruple (u,e,p,σ) belongs to the energy space, it follows that $(\dot u(t),\dot e(t),\dot p(t))$ belongs to the space of all kinematically admissible triples

$$\Big\{(v,\eta,q) \in [BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)] \times L^2(\Omega;\mathbb{M}^n_{sym}) \times \mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^n_{sym}) : \\ Ev = \eta + q \text{ in } \Omega, \quad q = -v\odot\nu\mathcal{H}^{n-1} \text{ on } \Gamma_D\Big\},$$

and $\sigma(t)$ belongs to the space of all statically and plastically admissible stresses

$$\{\tau \in H(\mathrm{d}iv,\Omega) :\ \tau\nu = 0 \text{ on } \Gamma_N,\ \tau(x)\in\mathbf{K} \text{ a.e. in } \Omega\}.$$

In the spirit of [49, 61, 32], it allows one to consider a generalized stress/strain duality (see Definition 5.2.2) as the first order distribution $[\sigma(t)\colon \dot p(t)] \in \mathcal{D}'(\mathbb{R}^n)$, compactly supported in $\overline{\Omega}$, defined as

$$\langle[\sigma(t)\colon \dot p(t)],\varphi\rangle = -\int_\Omega \varphi\sigma(t) : \dot e(t)\,dx - \int_\Omega \dot u(t)\cdot\mathrm{div}\sigma(t)\,\varphi\,dx - \int_\Omega \sigma(t)\colon\,(u(t)\odot\nabla\varphi)\,dx \quad (5.1.10)$$

or any $\varphi \in C^\infty_c(\mathbb{R}^n)$. The question now reduces to prove that

$$H(\dot p(t)) \ge [\sigma(t)\colon \dot p(t)] \quad \text{in } \mathcal{M}(\mathbb{R}^n), \qquad (5.1.11)$$

and this is the object of Section 5.2 . In Propositions 5.2.4 we show that this generalized convexity inequality is always satisfied in the pure Dirichlet ($\Gamma_D = \partial\Omega$) and pure Neumann ($\Gamma_N = \partial\Omega$) cases. In the case of mixed boundary conditions, there might be some concentration effects at the interface Σ between the Dirichlet and the Neumann parts, and the previous convexity inequality is shown to hold only in $\mathcal{M}(\mathbb{R}^n\setminus\Sigma)$ in Proposition 5.2.5. Unfortunately, this weaker result is not enough to conclude the energy upper bound because, although Σ is $\mathcal{H}^{n-1}$-negligible, some undesirable energy concentration might accumulate on that set. We further exhibit special cases in dimensions $n = 2$ and $n = 3$ which guarantee the validity of (5.1.11) also in the case of mixed boundary conditions (see Propositions 5.2.6 and 5.2.7). In dimension $n = 2$, it is enough to assume that Σ is a finite set (as in [49]) while in dimension $n = 3$, we suppose that the convex set $\mathbf{K}$ is invariant in the direction of hydrostatic matrices and bounded in the direction of deviatoric ones, as well as additional regularity assumptions on the reference configuration Ω (as in [61]).

To conclude this introduction, let us mention that our method only allows one to derive homogeneous mixed boundary conditions. Indeed, at a formal level, even starting from a non-homogeneous dissipative boundary condition of the form $S\dot{u} + \sigma\nu = g$ on $\partial\Omega \times (0,T)$, for some non trivial source term g, (or its relaxed counterpart $P_{-\mathbf{K}\nu}(S\dot{u} - g) + \sigma\nu = 0$ on $\partial\Omega \times (0,T)$ given by an adaptation of [11]), we obtain an energy balance involving the following additional term

$$\int_0^T \int_{\partial\Omega} S^{-1} g \cdot g \, d\mathcal{H}^{n-1} \, dt.$$

Specializing the problem to a boundary matrix $S = S_\lambda$ of the form (5.1.9) and some λ-dependent source term $g_\lambda \in L^2(\partial\Omega \times (0,T); \mathbb{R}^n)$, the previous discussion shows that a uniform bound on the solution $(u_\lambda, e_\lambda, p_\lambda, \sigma_\lambda)$ in the energy space would require that

$$\sup_{\lambda>0} \left\{ \frac{1}{\lambda} \int_{\Gamma_D} |g_\lambda|^2 \, d\mathcal{H}^{n-1} + \lambda \int_{\Gamma_N} |g_\lambda|^2 \, d\mathcal{H}^{n-1} \right\} < \infty.$$

It would imply that

$$\sigma_\lambda \nu = g_\lambda - \lambda^{-1} \dot{u}_\lambda \to 0 \quad \text{in } \Gamma_N \times (0,T)$$

in a weak sense as $\lambda \to \infty$ (because the trace of $\dot{u}_\lambda$ is bounded in $L^1(\partial\Omega \times (0,T); \mathbb{R}^n)$), leading to a homogeneous Neumann condition in Γ_N. Concerning the Dirichlet part, formally reporting this information in the dissipative boundary condition restricted to Γ_D would lead to

$$\dot{u}_\lambda = \lambda^{-1} g_\lambda - \lambda^{-1} \sigma_\lambda \nu \to 0 \quad \text{in } \Gamma_D \times (0,T),$$

in some weak sense as $\lambda \to \infty$ (because $\sigma_\lambda \nu$ is bounded in $L^2(0,T; H^{-1/2}(\partial\Omega; \mathbb{R}^n))$), leading to a homogeneous Dirichlet boundary condition. Strictly speaking one should rather consider the relaxed boundary condition which would lead to a strain concentration on Γ_D associated to a homogeneous Dirichlet boundary condition.

This chapter is organized as follows. In Section 2, we discuss the notion duality between plastic strains and Cauchy stresses, and we prove generalized convexity inequalities of the form (5.1.11) involving these two arguments which are not in duality in the energy space. Finally, in Section 3, we state and prove our main result, Theorem 5.3.9, about the convergence of the solutions obtained in [11] to the (unique) solution of a dynamical elasto-plastic model with homogeneous mixed boundary conditions.

5.2. Duality between stress and plastic strain

In the spirit of [14, 49, 61], we define a generalized notion of stress/strain duality.

(H_1) **The reference configuration.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We assume that $\partial\Omega$ is decomposed as the following disjoint union

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Sigma,$$

where Γ_D and Γ_N are open sets in the relative topology of $\partial\Omega$, and $\Sigma = \partial_{|\partial\Omega}\Gamma_D = \partial_{|\partial\Omega}\Gamma_N$ is $\mathcal{H}^{n-1}$-negligible.

On the Neumann part Γ_N, we will prescribe a surface load given by a function $g \in L^\infty(\Gamma_N; \mathbb{R}^n)$. The space of *statically admissible stresses* is defined by

$$\mathcal{S}_g := \{\sigma \in H(\mathrm{div}, \Omega) : \ \sigma\nu = g \text{ on } \Gamma_N\}.$$

In the sequel we will also be interested in stresses σ taking values in a given set.

(H_2) **Plastic properties.** Let $\mathbf{K} \subset \mathbb{M}^n_{sym}$ be a closed convex set such that 0 belongs to the interior point of $\mathbf{K}$. In particular, there exists $r > 0$ such that

$$\left\{\tau \in \mathbb{M}^n_{sym} : |\tau| \le r\right\} \subset \mathbf{K}. \tag{5.2.1}$$

The support function $H : \mathbb{M}^n_{sym} \to [0, +\infty]$ of $\mathbf{K}$ is defined by

$$H(q) := \sup_{\sigma \in \mathbf{K}} \sigma : q \quad \text{for all } q \in \mathbb{M}^n_{sym}.$$

We can deduce from (5.2.1) that

$$H(q) \geq r|q| \quad \text{for all } q \in \mathbb{M}^n_{sym}. \tag{5.2.2}$$

If $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^n_{sym})$, we denote the convex function of a measure $H(p)$ by

$$H(p) := H\left(\frac{dp}{d|p|}\right)|p|,$$

and the plastic dissipation is defined by

$$\mathcal{H}(p) := \int_{\Omega \cup \Gamma_D} H\left(\frac{dp}{d|p|}\right) d|p|.$$

We define the set of all *plastically admissible stresses* by

$$\mathcal{K} := \{\sigma \in H(\mathrm{d}iv, \Omega) : \ \sigma(x) \in \mathbf{K} \text{ for a.e. } x \in \Omega\}$$

which defines a closed and convex subset of $H(\mathrm{d}iv, \Omega)$.

The portion Γ_D of $\partial\Omega$ stands for the Dirichlet part of the boundary where a given displacement w will be prescribed. We assume that it extends into a function $w \in H^1(\Omega; \mathbb{R}^n)$ (so that $w_{|\Gamma_D} \in H^{1/2}(\Gamma_D; \mathbb{R}^n)$). We define the space of *kinematically admissible triples* by

$$\begin{aligned} \mathcal{A}_w \ := \ \Big\{ & (u, e, p) \in [BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)] \times L^2(\Omega; \mathbb{M}^n_{sym}) \times \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^n_{sym}) : \\ & Eu = e + p \text{ in } \Omega, \quad p = (w - u) \odot \nu \mathcal{H}^{n-1} \text{ on } \Gamma_D \Big\}, \end{aligned}$$

where ν is the outer unit normal to Ω. The function u stands for the displacement, e is the elastic strain and p is the plastic strain. The following result provides an approximation for triples $(u, e, p) \in \mathcal{A}_w$ and its proof follows the line of Step 1 in [49, Theorem 6.2].

Lemma 5.2.1. *Let $(u, e, p) \in [BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)] \times L^2(\Omega; \mathbb{M}^n_{sym}) \times \mathcal{M}(\Omega; \mathbb{M}^n_{sym})$ be such that $Eu = e + p$ in Ω. Then, there exists a sequence $\{(u_k, e_k, p_k)\}_{k \in \mathbb{N}}$ in $C^\infty(\overline{\Omega}; \mathbb{R}^n \times \mathbb{M}^n_{sym} \times \mathbb{M}^n_{sym})$ such that*

$$Eu_k = e_k + p_k \quad \text{in } \Omega,$$

$$\begin{cases} u_k \to u & \text{strongly in } L^2(\Omega; \mathbb{R}^n), \\ e_k \to e & \text{strongly in } L^2(\Omega; \mathbb{M}^n_{sym}), \\ p_k \rightharpoonup p & \text{weakly* in } \mathcal{M}(\Omega; \mathbb{M}^n_{sym}), \\ |p_k|(\Omega) \to |p|(\Omega), \\ |Eu_k|(\Omega) \to |Eu|(\Omega), \\ u_k \to u & \text{strongly in } L^1(\partial\Omega; \mathbb{R}^n). \end{cases} \tag{5.2.3}$$

and for all $\varphi \in C^\infty_c(\mathbb{R}^n)$ with $\varphi \geq 0$,

$$\limsup_{k \to \infty} \int_\Omega \varphi \, dH(p_k) \leq \int_\Omega \varphi \, dH(p). \tag{5.2.4}$$

Proof. The construction of a sequence $\{(u_k, e_k, p_k)\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega}; \mathbb{R}^n \times \mathbb{M}^n_{sym} \times \mathbb{M}^n_{sym})$ such that $Eu_k = e_k + p_k$ in Ω together with the four first convergences of (5.2.3) result from Step 1 in [49, Theorem 6.2]. Moreover, a careful inspection of that proof also shows that $|Eu_k|(\Omega) \to |Eu|(\Omega)$. The strong convergence of the trace in $L^1(\partial\Omega; \mathbb{R}^n)$ is a consequence of [10, Proposition 3.4]. The last condition (5.2.4) follows as well from the proof of [49, Theorem 6.2] using the subadditivity and the positive one-homogeneity of H. Note that (5.2.4) cannot be directly obtained from the strict convergence of $\{p_k\}_{k\in\mathbb{N}}$ and Reshetnyak continuity Theorem (see [4, Theorem 2.39] or [78]) because H is just lower semicontinuous and it can take infinite values. □

We now define a distributional duality pairing between statically admissible stresses and plastic strains.

Definition 5.2.2. *Let $\sigma \in \mathcal{S}_g$ and $(u, e, p) \in \mathcal{A}_w$. We define the first order distribution $[\sigma\colon p] \in \mathcal{D}'(\mathbb{R}^n)$ by*

$$\langle[\sigma\colon p], \varphi\rangle := \int_\Omega \varphi\sigma : (Ew - e)\, dx + \int_\Omega (w-u)\cdot \operatorname{div}\sigma\, \varphi\, dx + \int_\Omega \sigma\colon ((w-u)\odot\nabla\varphi)\, dx + \int_{\Gamma_N} \varphi g\cdot(u-w)\, d\mathcal{H}^{n-1}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n)$.

Remark 5.2.3. If $\varphi \in C^\infty_c(\Omega)$, thanks to the integration by parts formula in $H^1(\Omega;\mathbb{R}^n)$, the expression of the stress/strain duality becomes independent of w and g, and it reduces to

$$\langle[\sigma\colon p], \varphi\rangle = -\int_\Omega \varphi\sigma\colon e\, dx - \int_\Omega u\cdot\operatorname{div}\sigma\,\varphi\, dx - \int_\Omega \sigma\colon (u\odot\nabla\varphi)\, dx\,. \tag{5.2.5}$$

As already observed in [14], contrary to [49, 61], we are not able to show in general that $[\sigma : p]$ extends into a bounded Radon measure. This is due to the fact that, in our context, σ_D fails to belong to $L^\infty(\Omega;\mathbb{M}^n_D)$. However, provided $\sigma \in \mathcal{K}$ and under suitable assumption on Ω and $\mathbf{K}$, we are going to show a convexity inequality which will ensure that $H(p) - [\sigma : p]$ is a nonnegative distribution, hence that $[\sigma : p]$ actually defines a bounded Radon measure supported in $\overline{\Omega}$.

5.2.1. Pure Dirichlet or pure Neumann boundary conditions

As the following result shows, the distribution $[\sigma : p]$ always extends into a bounded Radon measure in the pure Dirichlet or pure Neumann cases.

Proposition 5.2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Assume that either $\partial\Omega = \Gamma_D$ or $\partial\Omega = \Gamma_N$. Then, for every $\sigma \in \mathcal{S}_g \cap \mathcal{K}$ and $(u,e,p) \in \mathcal{A}_w$ with $H(p) \in \mathcal{M}(\Omega\cup\Gamma_D)$, the distribution $[\sigma : p]$ extends to a bounded Radon measure supported in $\overline{\Omega}$ and*

$$H(p) \geq [\sigma : p] \quad \text{in } \mathcal{M}(\mathbb{R}^n). \tag{5.2.6}$$

Proof. In the case of pure Dirichlet boundary conditions, $\partial\Omega = \Gamma_D$, we first note that $\mathcal{S}_g = H(\operatorname{div}, \Omega)$. The duality pairing is then independent of g and reduces to

$$\langle[\sigma\colon p], \varphi\rangle = \int_\Omega \varphi\sigma : (Ew - e)\, dx + \int_\Omega (w-u)\cdot\operatorname{div}\sigma\,\varphi\, dx + \int_\Omega \sigma\colon ((w-u)\odot\nabla\varphi)\, dx$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n)$. This case has already been addressed in [14, Section 2]. The result is a direct consequence an approximation result for $\sigma \in \mathcal{K}$ by smooth functions (see e.g. [32, Lemma 2.3]) as well as the integration by parts formula in $BD(\Omega)$ (see [10, Theorem 3.2]).

In the case of pure Neumann boundary conditions, $\partial\Omega = \Gamma_N$, using the integration by parts formula in $H^1(\Omega;\mathbb{R}^n)$ for the function w, the duality pairing becomes independent of w and reduces to

$$\langle[\sigma : p], \varphi\rangle := -\int_\Omega \varphi\sigma : e\, dx - \int_\Omega u \cdot \mathrm{div}\sigma\, \varphi\, dx - \int_\Omega \sigma : (u \odot \nabla\varphi)\, dx + \int_{\partial\Omega} \varphi g \cdot u\, d\mathcal{H}^{n-1}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. According to Lemma 5.2.1, there exists a sequence $\{(u_k, e_k, p_k)\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega};\mathbb{R}^n \times \mathbb{M}^n_{sym} \times \mathbb{M}^n_{sym})$ such that $Eu_k = e_k + p_k$ in Ω and (5.2.3)–(5.2.4) hold. By definition of the duality pairing $[\sigma : p_k]$, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\langle[\sigma : p_k], \varphi\rangle := -\int_\Omega \sigma : e_k\varphi\, dx - \int_\Omega \varphi u_k \cdot div\sigma\, dx - \int_\Omega \sigma : (u_k \odot \nabla\varphi)\, dx + \int_{\partial\Omega} \varphi g \cdot u_k\, d\mathcal{H}^{n-1}, \tag{5.2.7}$$

and using the integration by parts formula (2.1.1) for $\sigma \in H(\mathrm{div}, \Omega)$, we get that

$$\langle[\sigma : p_k], \varphi\rangle := \int_\Omega \sigma : p_k\varphi\, dx. \tag{5.2.8}$$

By definition of the support function H, we have that $H(p_k) \geq \sigma : p_k$ a.e. in Ω, hence if $\varphi \geq 0$, by (5.2.7), it yields

$$\begin{aligned}\int_\Omega H(p_k)\varphi\, dx &\geq \int_\Omega \sigma : p_k\varphi\, dx \\ &= -\int_\Omega \sigma : e_k\varphi\, dx - \int_\Omega \varphi u_k \cdot div\sigma\, dx - \int_\Omega \sigma : (u_k \odot \nabla\varphi)\, dx \\ &\quad + \int_{\partial\Omega} \varphi g \cdot u_k\, d\mathcal{H}^{n-1}.\end{aligned}$$

Hence, passing to the limit as $k \to \infty$ thanks to the convergences (5.2.3)–(5.2.4) yields

$$\begin{aligned}\int_\Omega \varphi\, dH(p) &\geq -\int_\Omega \sigma : e\varphi\, dx - \int_\Omega \varphi u \cdot \mathrm{div}\sigma\, dx - \int_\Omega \sigma : (u \odot \nabla\varphi)\, dx + \int_{\partial\Omega} \varphi g \cdot u\, d\mathcal{H}^{n-1} \\ &=: \langle[\sigma : p], \varphi\rangle,\end{aligned}$$

where we used once more the definition of duality $[\sigma : p]$. As a consequence, the distribution $H(p) - [\sigma : p]$ is nonnegative, hence it extends into a bounded Radon measure in $\mathbb{R}^n$. Thus, $[\sigma : p]$ extends as well into a bounded Radon measure in $\mathbb{R}^n$. Finally $[\sigma : p]$ is clearly supported in $\overline{\Omega}$ from its very definition. □

5.2.2. Mixed boundary conditions

When $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$, the situation is more delicate as in [49]. We first prove the following general result giving the required convexity inequality but only outside Σ (see [49, Theorem 6.2]) which, unfortunately, will not be enough for our purpose. We will later introduce additional assumptions in dimensions $n = 2$ and 3 which will ensure the validity of the convexity inequality in the whole $\mathbb{R}^n$.

Proposition 5.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every $\sigma \in \mathcal{S}_g \cap \mathcal{K}$ and $(u, e, p) \in \mathcal{A}_w$ with $H(p) \in \mathcal{M}(\Omega \cup \Gamma_D)$, the restriction of the distribution $[\sigma : p]$ to $\mathbb{R}^n \setminus \Sigma$ extends to a bounded Radon measure in $\mathbb{R}^n \setminus \Sigma$ and*

$$H(p) \geq [\sigma : p] \quad \text{in } \mathcal{M}(\mathbb{R}^n \setminus \Sigma). \tag{5.2.9}$$

Proof. Without loss of generality, we can assume $w = 0$ in Definition 5.2.2. Let us fix a test function $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \Sigma)$, and let $U \subset \mathbb{R}^n$ be an open set such that $\Sigma \subset U$ and $U \cap supp(\varphi) = \emptyset$.

Let us consider another open set $W \subset \mathbb{R}^n$ such that $\Gamma_N \setminus U \subset W$ and $\overline{W} \cap \partial\Omega \subset \Gamma_N$. Finally, let $W' \subset \mathbb{R}^n$ be a further open set such that $W' \subset\subset W$, $\Gamma_N \setminus U \subset W'$ and $supp(\varphi) \cap \Gamma_N \subset W'$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \psi \leq 1$, $Supp(\psi) \subset W$ and $\psi = 1$ on W'. We decompose σ as follows,

$$\sigma = \psi\sigma + (1-\psi)\sigma =: \sigma_1 + \sigma_2.$$

Note that, for $i = 1, 2$, we have that $\sigma_i \in H(\mathrm{div}, \Omega)$. Moreover,

$$\sigma_1\nu := \psi(\sigma\nu) = \psi g \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \sigma_2 = 0 \quad \text{on } W'. \tag{5.2.10}$$

Substituting σ with this decomposition in Definition 5.2.2 we get that

$$\begin{aligned}\langle[\sigma\colon p], \varphi\rangle &:= -\int_\Omega \varphi\sigma : e\,dx - \int_\Omega u \cdot \mathrm{div}\sigma\,\varphi\,dx - \int_\Omega \sigma\colon (u \odot \nabla\varphi)\,dx + \int_{\Gamma_N} \varphi g \cdot u\,d\mathcal{H}^{n-1} \\ &= -\int_\Omega \varphi\sigma_1 : e\,dx - \int_\Omega u \cdot \mathrm{div}\sigma_1\,\varphi\,dx - \int_\Omega \sigma_1\colon (u \odot \nabla\varphi)\,dx + \int_{\Gamma_N} \varphi g \cdot u\,d\mathcal{H}^{n-1} \\ &\quad - \int_\Omega \varphi\sigma_2 : e\,dx - \int_\Omega u \cdot \mathrm{div}\sigma_2\,\varphi\,dx - \int_\Omega \sigma_2\colon (u \odot \nabla\varphi)\,dx. \end{aligned} \tag{5.2.11}$$

We first approximate (u, e, p) in the expression (5.2.11) involving σ_1. Indeed, thanks to Lemma 5.2.1, there exists a sequence $\{(u_k, e_k, p_k)\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega}; \mathbb{R}^n \times \mathbb{M}^n_{sym} \times \mathbb{M}^n_{sym})$ such that $Eu_k = e_k + p_k$ in Ω and (5.2.3)–(5.2.4) hold. On the one hand, we have

$$\begin{aligned} &-\int_\Omega \varphi\sigma_1 : e_k\,dx - \int_\Omega u_k \cdot \mathrm{div}\sigma_1\,\varphi\,dx - \int_\Omega \sigma_1\colon (u_k \odot \nabla\varphi)\,dx + \int_{\Gamma_N} \varphi g \cdot u_k\,d\mathcal{H}^{n-1} \\ &\quad \to -\int_\Omega \varphi\sigma_1 : e\,dx - \int_\Omega u \cdot \mathrm{div}\sigma_1\,\varphi\,dx - \int_\Omega \sigma_1\colon (u \odot \nabla\varphi)\,dx + \int_{\Gamma_N} \varphi g \cdot u\,d\mathcal{H}^{n-1}. \end{aligned} \tag{5.2.12}$$

On the other hand, for any $k \in \mathbb{N}$, thanks to the integration by parts formula for $\sigma_1 \in H(div, \Omega)$ together with (5.2.10), we can observe that

$$\begin{aligned} &-\int_\Omega \varphi\sigma_1 : e_k\,dx - \int_\Omega u_k \cdot \mathrm{div}\sigma_1\,\varphi\,dx - \int_\Omega \sigma_1\colon (u_k \odot \nabla\varphi)\,dx + \int_{\Gamma_N} \varphi g \cdot u_k\,d\mathcal{H}^{n-1} \\ &= \int_\Omega \varphi\sigma_1 : p_k\,dx - \langle\sigma_1\nu, \varphi u_k\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n), H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)} + \int_{\Gamma_N} \varphi g \cdot u_k\,d\mathcal{H}^{n-1} \\ &= \int_\Omega \varphi\sigma_1 : p_k\,dx - \int_{\partial\Omega} \varphi\psi g \cdot u_k\,d\mathcal{H}^{n-1} + \int_{\Gamma_N} \varphi g \cdot u_k\,d\mathcal{H}^{n-1} \\ &= \int_\Omega \varphi\sigma_1 : p_k\,dx, \end{aligned} \tag{5.2.13}$$

where we used that $\psi = 1$ on $Supp(\varphi) \cap \Gamma_N$ and $\psi = 0$ in $\partial\Omega \setminus \Gamma_N$. Hence, by definition of the support function H, we have that $H(p_k) \geq \sigma : p_k$ a.e. in Ω. As a consequence, if $\varphi \geq 0$,

$$\begin{aligned} \int_\Omega H(p_k)\psi\varphi\,dx &\geq \int_\Omega \sigma_1 : p_k\varphi\,dx \\ &= -\int_\Omega \varphi u_k \cdot div\sigma_1\,dx - \int_\Omega \sigma_1 : (u_k \odot \nabla\varphi)\,dx \\ &\qquad - \int_\Omega \sigma_1 : e_k\varphi\,dx + \int_{\Gamma_N} \varphi g \cdot u_k\,d\mathcal{H}^{n-1}. \end{aligned}$$

We can pass to the limit as $k \to \infty$ owing to (5.2.4) and (5.2.12). We deduce that

$$\begin{aligned} \int_{W\cap\Omega} \varphi\,dH(p) &\geq \int_\Omega \varphi\psi\,dH(p) \\ &\geq -\int_\Omega \varphi u \cdot \mathrm{div}\sigma_1\,dx - \int_\Omega \sigma_1 : (u \odot \nabla\varphi)\,dx \\ &\qquad - \int_\Omega \sigma_1 : e\varphi\,dx + \int_{\Gamma_N} \varphi g \cdot u\,d\mathcal{H}^{n-1}, \end{aligned} \tag{5.2.14}$$

where we have used the fact that p, hence $H(p)$, does not charge Γ_N.

Coming back to (5.2.11), we now approximate the last term in the right-hand side by approximating σ_2. Arguing as in [32, Lemma 2.3] or Step 2 in [49, Theorem 6.2] and using (5.2.10), there exists a sequence $\left\{\sigma_2^k\right\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{M}^n_{sym})$ such that $\sigma_2^k(x) \in \mathbf{K}$ for all $x \in \overline{\Omega}$ and

$$\begin{cases}\sigma_2^k \to \sigma_2 & \text{strongly in } H(\mathrm{div},\Omega),\\ \sigma_2^k\nu = 0 & \text{on } W' \cap \Gamma_N.\end{cases} \tag{5.2.15}$$

Therefore, using the integration by parts formula in $BD(\Omega)$, we infer that

$$\begin{aligned}
-\int_\Omega \varphi\sigma_2^k : e\,dx - \int_\Omega u\cdot \mathrm{div}\sigma_2^k\,\varphi\,dx - \int_\Omega \sigma_2^k\colon (u\odot\nabla\varphi)\,dx\\
= \int_\Omega \varphi\sigma_2^k : dp - \int_{\partial\Omega} \varphi(\sigma_2^k\nu)\cdot u\,d\mathcal{H}^{n-1}\\
= \int_\Omega \varphi\sigma_2^k : dp - \int_{\Gamma_D} \varphi(\sigma_2^k\nu)\cdot u\,d\mathcal{H}^{n-1}\\
=: \int_{\Omega\cup\Gamma_D} \varphi\sigma_2^k : dp
\end{aligned} \tag{5.2.16}$$

where in the second equality, we have used the fact that $supp(\varphi)\cap\partial\Omega \subset \Gamma_D \cup (\Gamma_N \cap W')$ and the last condition of (5.2.15), while in the third equality we used that $p \llcorner \Gamma_D = -u\odot\nu\mathcal{H}^{n-1} \llcorner \Gamma_D$. Using that $\sigma_2^k(x) \in \mathbf{K}$ for all $x \in \overline{\Omega}$, we get that

$$\int_{\Omega\cup\Gamma_D} \varphi\,dH(p) \geq \int_{\Omega\cup\Gamma_D} \varphi\sigma_2^k : dp,$$

hence passing to the limit as $k \to \infty$ using (5.2.15) and (5.2.16) leads to

$$\int_{\Omega\cup\Gamma_D} \varphi\,dH(p) \geq -\int_\Omega \varphi\sigma_2 : e\,dx - \int_\Omega u\cdot\mathrm{div}\sigma_2\,\varphi\,dx - \int_\Omega \sigma_2\colon (u\odot\nabla\varphi)\,dx. \tag{5.2.17}$$

Combining (5.2.11), (5.2.14) and (5.2.17), we conclude that

$$\langle[\sigma : p],\varphi\rangle \leq \int_{\Omega\cup\Gamma_D} \varphi\,dH(p) + \int_{W\cap\Omega} \varphi\,dH(p).$$

Let us finally consider a decreasing sequence of open sets $\{W_j\}_{j\in\mathbb{N}}$ such that $\Gamma_N \setminus U \subset W_j$ and $W_j \cap \partial\Omega \subset \Gamma_N$ for all $j \in \mathbb{N}$, and $\bigcap_j W_j = \overline{\Gamma_N \setminus U}$. Passing to the limit in the previous expression as $j \to \infty$ owing to the monotone convergence theorem yields

$$\langle[\sigma : p],\varphi\rangle \leq \int_{\Omega\cup\Gamma_D} \varphi\,dH(p) + \int_{\overline{\Gamma_N\setminus U}} \varphi\,dH(p).$$

As $\overline{\Gamma_N \setminus U} \subset \Gamma_N \cup \Sigma$ and p is concentrated on $\Omega\cup\Gamma_D$, we deduce that

$$\langle[\sigma : p],\varphi\rangle \leq \int_{\Omega\cup\Gamma_D} \varphi\,dH(p)$$

which completes the proof of the proposition. □

In the remaining part of this section, we exhibit some particular cases where we can extend inequality (5.2.9) above into one in $\mathcal{M}(\mathbb{R}^n)$. The following result deals with the two-dimensional case where the convexity inequality holds provided Σ is a finite set.

Proposition 5.2.6. *Under the same assumptions as in Proposition 5.2.5, assume further that $n = 2$ and that Σ is a finite set. Then, for all $\sigma \in \mathcal{S}_g \cap \mathcal{K}$ and all $(u, e, p) \in \mathcal{A}_w$,*

$$H(p) \geq [\sigma : p] \quad \text{in} \;\; \mathcal{M}(\mathbb{R}^2).$$

Proof. We again reduce to the case $w = 0$. Arguing as in [49, Example 2], for all $(u, e, p) \in \mathcal{A}_0$, there exists a sequence $\{(u_k, e_k, p_k)\}_{k\in\mathbb{N}}$ in $\mathcal{A}_0$ such that, for each $k \in \mathbb{N}$, $(u_k, e_k, p_k) = 0$ in an open neighborhood U_k of Σ and

$$\begin{cases} u_k \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2), \\ e_k \to e \quad \text{strongly in } L^2(\Omega;\mathbb{M}^2_{sym}), \\ p_k \rightharpoonup p \quad \text{weakly* in } \mathcal{M}(\Omega \cup \Gamma_D;\mathbb{M}^2_{sym}), \\ |p_k|(\Omega \cup \Gamma_D) \to |p|(\Omega \cup \Gamma_D). \end{cases} \tag{5.2.18}$$

A careful inspection of the argument used in [49, Example 2] shows that $|Eu_k|(\Omega) \to |Eu|(\Omega)$. Thus, applying [10, Proposition 3.4], we deduce the convergence of the trace

$$u_k \to u \quad \text{strongly in } L^1(\partial\Omega;\mathbb{R}^n). \tag{5.2.19}$$

Moreover, for all $\varphi \in C^\infty_c(\mathbb{R}^2)$ with $\varphi \geq 0$,

$$\limsup_{k\to\infty} \int_{\Omega\cup\Gamma_D} \varphi \, dH(p_k) \leq \int_{\Omega\cup\Gamma_D} \varphi \, dH(p). \tag{5.2.20}$$

Once more, (5.2.20) does not follow from the Reshetnyak continuity Theorem because our H does not fulfill the assumptions of that result.

Let V_k be an open set satisfying $\Sigma \subset V_k \subset\subset U_k$, and let $\psi_k \in C^\infty_c(\mathbb{R}^2;[0,1])$ be a cut-off function such that $\psi_k = 1$ in V_k and $\mathrm{S}upp(\psi_k) \subset U_k$. For every $\varphi \in C^\infty_c(\mathbb{R}^2)$ with $\varphi \geq 0$, then $(1-\psi_k)\varphi \in C^\infty_c(\mathbb{R}^2 \setminus \Sigma)$ so that by Proposition 5.2.5,

$$\int_{\Omega\cup\Gamma_D} \varphi \, dH(p_k) \geq \int_{\Omega\cup\Gamma_D} \varphi(1-\psi_k) \, dH(p_k) \geq \langle [\sigma : p_k], \varphi(1-\psi_k)\rangle.$$

Since by construction $\mathrm{S}upp(u_k, e_k, p_k) \subset \mathbb{R}^2 \setminus U_k$, it is easily seen that $\mathrm{S}upp([\sigma : p_k]) \subset \mathbb{R}^2 \setminus U_k$ hence $\langle [\sigma : p_k], \varphi\psi_k\rangle = 0$. As a consequence

$$\int_{\Omega\cup\Gamma_D} \varphi \, dH(p_k) \geq \langle [\sigma : p_k], \varphi\rangle,$$

and the conclusion follows passing to the limit as $k \to \infty$ owing to the convergences (5.2.18)–(5.2.20). □

The three-dimensional case requires additional regularity assumptions for the domain Ω, and a particular geometric structure for the elasticity set $\mathbf{K}$ which has to be a cylinder whose axis is given by the set of spherical matrices. Note that these assumptions cover the physical cases of Von Mises and Tresca models.

Proposition 5.2.7. *Under the same assumptions as in Proposition 5.2.5, assume further that $n = 3$ and that:*

(i) $\Omega \subset \mathbb{R}^3$ is a bounded open set of class C^2 and Σ is 1-dimensional submanifold of class C^2;

(ii) $\mathbf{K} = K_D \oplus (\mathbb{R}\, Id) = \{\sigma \in \mathbb{M}^3_{sym} : \sigma_D \in K_D\}$ where $K_D \subset \mathbb{M}^3_D$ is a compact and convex set containing 0 in its interior.

Then, for all $\sigma \in \mathcal{S}_g \cap \mathcal{K}$ and all $(u,e,p) \in \mathcal{A}_w$,

$$H(p) \geq [\sigma : p] \quad \text{in } \mathcal{M}(\mathbb{R}^3).$$

Proof. Since $\sigma \in L^2(\Omega;\mathbb{M}^3_{sym})$ satisfies $\mathrm{d}iv\sigma \in L^2(\Omega;\mathbb{R}^3)$ and $\sigma_D \in L^\infty(\Omega;\mathbb{M}^3_{sym})$ (because $\sigma \in \mathcal{K}$ implies $\sigma_D(x) \in K_D$ a.e. in Ω), we claim that $\sigma \in L^6(\Omega;\mathbb{M}^3_{sym})$. Indeed, arguing as in [49, Proposition 6.1], using the decomposition $\sigma = \sigma_D + \frac{1}{3}(\mathrm{tr}\,\sigma)\mathrm{I}d$, we have that $\frac{1}{3}\nabla(\mathrm{tr}\,\sigma) = \mathrm{d}iv\sigma - \mathrm{d}iv\sigma_D \in L^2(\Omega;\mathbb{R}^3) + W^{-1,\infty}(\Omega;\mathbb{R}^3)$, hence by the Sobolev embedding,

$$\nabla(\mathrm{tr}\,\sigma) \in W^{-1,6}(\Omega) + W^{-1,\infty}(\Omega) \subset W^{-1,6}(\Omega).$$

Applying Nečas Lemma (see [73]), we infer that $\mathrm{tr}\,\sigma \in L^6(\Omega)$, hence $\sigma \in L^6(\Omega;\mathbb{M}^3_{sym})$.

In particular, $\sigma \in L^3(\Omega;\mathbb{M}^3_{sym})$, $\sigma_D \in L^\infty(\Omega;\mathbb{M}^3_D)$, $\mathrm{d}iv\sigma \in L^{3/2}(\Omega;\mathbb{R}^3)$ and $\sigma\nu \in L^\infty(\Gamma_N;\mathbb{R}^3)$. These conditions turn out to be sufficient to apply [61, Proposition 2.7] (with, in the notation of [61], $n = 3$, $p = 3/2$ and $p^* = 3$). Then, an immediate adaptation of the proof of [61, Lemma 3.5] (using [61, Proposition 2.7] instead of [61, Corollary 2.8]) shows the validity of the so-called Kohn-Temam condition:

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{\Sigma_\delta} |\sigma||u|\,dx = 0,$$

where $\Sigma_\delta := \Omega \cap \{x \in \mathbb{R}^3 : \mathrm{d}ist(x,\Sigma) < \delta\}$. We are thus in position to argue as in the proof of [49, Theorem 6.5] to get the conclusion. Indeed, let $\psi_\delta \in C^\infty_c(\Sigma_\delta;[0,1])$ be a cut-off function such that $\psi_\delta = 1$ in a neighborhood of Σ and $|\nabla\psi_\delta| \leq 2/\delta$. Then, for all $\varphi \in C^\infty_c(\mathbb{R}^3)$ with $\varphi \geq 0$, we have

$$\begin{aligned}\langle[\sigma : p],(1-\psi_\delta)\varphi\rangle = &-\int_\Omega (1-\psi_\delta)\varphi\sigma : e\,dx - \int_\Omega u\cdot \mathrm{div}\sigma\,(1-\psi_\delta)\varphi\,dx \\ &- \int_\Omega (1-\psi_\delta)\sigma\colon (u\odot\nabla\varphi)\,dx \\ &+ \int_\Omega \varphi\sigma\colon (u\odot\nabla\psi_\delta)\,dx + \int_{\Gamma_N}(1-\psi_\delta)\varphi g\cdot u\,d\mathcal{H}^{n-1}.\end{aligned}$$

Since $\psi_\delta \searrow 0$ pointwise, and

$$\left|\int_\Omega \varphi\sigma\colon (u\odot\nabla\psi_\delta)\,dx\right| \leq \frac{2\|\varphi\|_{L^\infty(\Omega)}}{\delta}\int_{\Sigma_\delta}|\sigma||u|\,dx \to 0,$$

the dominated convergence Theorem allows us to pass to the limit as $\delta \to 0$, and get that

$$\langle[\sigma : p],(1-\psi_\delta)\varphi\rangle \to \langle[\sigma : p],\varphi\rangle.$$

On the other hand, since $(1-\psi_\delta)\varphi \in C^\infty_c(\mathbb{R}^3 \setminus \Sigma)$, Proposition 5.2.5 ensures that

$$\int_{\Omega\cup\Gamma_D}\varphi\,dH(p) \geq \int_{\Omega\cup\Gamma_D}(1-\psi_\delta)\varphi\,dH(p) \geq \langle[\sigma : p],(1-\psi_\delta)\varphi\rangle.$$

The conclusion follows passing to the limit as $\delta \to 0$. □

5.3. Dynamic elasto-plasticity

5.3.1. The model with dissipative boundary conditions

We consider a small strain dynamical perfect plasticity problem under the following assumptions:

(H_3) **The elastic properties.** We assume that the material is isotropic, which means that the constitutive law, expressed by Hooke's tensor, is given by

$$\mathbb{C}\xi = \lambda(\mathrm{tr}\,\xi)\mathrm{I}d + 2\mu\xi \quad \text{for all } \xi \in \mathbb{M}^n_{sym},$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $2\mu + n\lambda > 0$. These conditions imply the existence of constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \beta|\xi|^2 \quad \text{for all } \xi \in \mathbb{M}^n_{sym}.$$

We define the following quadratic form

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi = \frac{\lambda}{2}(\mathrm{tr}\,\xi)^2 + \mu|\xi|^2 \quad \text{for all } \xi \in \mathbb{M}^n_{sym}.$$

If $e \in L^2(\Omega; \mathbb{M}^n_{sym})$, we further define the elastic energy by

$$\mathcal{Q}(e) := \int_\Omega Q(e)\,dx.$$

(H_4) **The dissipative boundary conditions.** Let $S \in L^\infty(\partial\Omega; \mathbb{M}^n_{sym})$ be a boundary matrix satisfying the conditions: there exists a constant $c > 0$ such that

$$S(x)z \cdot z \geq c|z|^2 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega \text{ and for all } z \in \mathbb{R}^n.$$

(H_5) **The external forces.** We assume the body is subjected to external body forces

$$f \in H^1(0, T; L^2(\Omega; \mathbb{R}^n)).$$

(H_6) **The initial conditions.** Let $u_0 \in H^1(\Omega; \mathbb{R}^n)$, $v_0 \in H^2(\Omega; \mathbb{R}^n)$, $e_0 \in L^2(\Omega; \mathbb{M}^n_{sym})$ and $p_0 \in L^2(\Omega; \mathbb{M}^n_{sym})$ be such that

$$\begin{cases} \sigma_0 := \mathbb{C}e_0 \in \mathcal{K}, & \\ Eu_0 = e_0 + p_0 & \text{in } \Omega, \\ Sv_0 + \sigma_0\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to formulate the main result of [11], we further need to introduce the function $\psi : \partial\Omega \times \mathbb{R}^n \to \mathbb{R}^+$ defined by

$$\psi(x, z) = \inf_{w \in \mathbb{R}^n} \left\{ \frac{1}{2}S(x)w \cdot w + H((w - z) \odot \nu(x)) \right\} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega \text{ and all } z \in \mathbb{R}^n, \tag{5.3.1}$$

where $\nu(x)$ is the outer normal to Ω at $x \in \partial\Omega$. We recall (see [11, Remark 4.7]) that the differential of ψ in the z-direction is given by

$$D_z\psi(x, z) = \mathrm{P}_{-\mathbf{K}\nu(x)}(S(x)z),$$

where $\mathrm{P}_{-\mathbf{K}\nu(x)}$ is the orthogonal projection in $\mathbb{R}^n$ onto the closed and convex set $-\mathbf{K}\nu(x)$ with respect to the scalar product $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle u, v\rangle_{S(x)^{-1}} := S(x)^{-1}u \cdot v$. We further denote by $\|\cdot\|_{S(x)^{-1}}$ its associated norm.

The following well posedness result with homogeneous dissipative boundary conditions has been established in [11].

Theorem 5.3.1. *Assume that assumptions* (H_1)–(H_6) *hold. Then, there exists a unique triple* (u, e, p) *such that*

$$\begin{cases} u \in W^{2,\infty}(0,T; L^2(\Omega;\mathbb{R}^n)) \cap C^{0,1}([0,T]\,; BD(\Omega)), \\ e \in W^{1,\infty}(0,T; L^2(\Omega;\mathbb{M}^n_{sym})), \\ p \in C^{0,1}([0,T]\,; \mathcal{M}(\Omega;\mathbb{M}^n_{sym})), \end{cases}$$

$$\sigma := \mathbb{C}e \in L^\infty(0,T; H(\mathrm{div},\Omega)), \quad \sigma\nu \in L^\infty(0,T; L^2(\partial\Omega;\mathbb{R}^n)),$$

and satisfying

1. *The initial conditions:*

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad e(0) = e_0, \quad p(0) = p_0;$$

2. *The additive decomposition: for all* $t \in [0,T]$,

$$Eu(t) = e(t) + p(t) \quad \textit{in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym});$$

3. *The equation of motion:*

$$\ddot{u} - \mathrm{div}\sigma = f \quad \textit{in } L^2(0,T; L^2(\Omega;\mathbb{R}^n));$$

4. *The relaxed dissipative boundary condition:*

$$P_{-\mathbf{K}\nu}(S\dot{u}) + \sigma\nu = 0 \quad \textit{in } L^2(0,T; L^2(\partial\Omega;\mathbb{R}^n));$$

5. *The stress constraint: for every* $t \in [0,T]$,

$$\sigma(t) \in \mathbf{K} \qquad \textit{a.e. in } \Omega;$$

6. *The flow rule: for a.e.* $t \in [0,T]$,

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \textit{in } \mathcal{M}(\Omega);$$

7. *The energy balance: for every* $t \in [0,T]$

$$\begin{aligned} \frac{1}{2}\int_\Omega |\dot{u}(t)|^2\,dx + \mathcal{Q}(e(t)) + \int_0^t H(\dot{p}(s))(\Omega)\,ds + \int_0^t \int_{\partial\Omega} \psi(x,\dot{u})\,d\mathcal{H}^{n-1}\,ds \\ + \frac{1}{2}\int_0^t \int_{\partial\Omega} S^{-1}(\sigma\nu)\cdot(\sigma\nu)\,d\mathcal{H}^{n-1}\,ds = \frac{1}{2}\int_\Omega |v_0|^2\,dx + \mathcal{Q}(e_0) + \int_0^t \int_\Omega f\cdot\dot{u}\,dx\,ds. \end{aligned} \tag{5.3.2}$$

Moreover, the following uniform estimate holds

$$\|\ddot{u}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{e}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} \le C_*, \tag{5.3.3}$$

for some constant $C_* > 0$ *depending on* $\|u_0\|_{H^1(\Omega;\mathbb{R}^n)}$, $\|v_0\|_{H^2(\Omega;\mathbb{R}^n)}$, $\|e_0\|_{L^2(\Omega;\mathbb{M}^n_{sym})}$, $\|\sigma_0\|_{H(\mathrm{div},\Omega)}$ *and* $\|p_0\|_{L^2(\Omega;\mathbb{M}^n_{sym})}$, *but which is independent of* S.

5.3.2. Elasto-visco-plastic model

The proof of Theorem 5.3.1 is obtained by means of an elasto-visco-plastic approximation, which is treated in this section. We follow the program of [70, Theorem 3.4.1] which treats the antiplanar case.

Theorem 5.3.2. *Assume that* (H_1)–(H_6)*. Then, for every* $\varepsilon > 0$ *there exists a unique triple* $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ *with*

$$\begin{cases} u_\varepsilon \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;H^1(\Omega;\mathbb{R}^n)),\\ e_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),\\ p_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),\end{cases}$$

which satisfies the following properties

1. *The initial conditions:*
$$u_\varepsilon(0) = u_0, \quad \dot u_\varepsilon(0) = v_0, \quad e_\varepsilon(0) = e_0 \quad \text{and} \quad p_\varepsilon(0) = p_0;$$
2. *The additive decomposition:*
$$Eu_\varepsilon = e_\varepsilon + p_\varepsilon \quad \text{a.e. in } \Omega \times (0,T);$$
3. *The constitutive law:*
$$\sigma_\varepsilon := \mathbb{C} e_\varepsilon \quad \text{a.e. in } \Omega \times (0,T);$$
4. *The equation of motion:*
$$\ddot u_\varepsilon - div(\sigma_e + \varepsilon E\dot u_\varepsilon) = f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n));$$
5. *The dissipative boundary conditions:*
$$S\dot u_\varepsilon + (\sigma_\varepsilon + \varepsilon E\dot u_\varepsilon)\nu = \varepsilon E v_0 \nu \quad \text{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n));$$
6. *The visco-plastic flow rule:*
$$\dot p_\varepsilon = \frac{\sigma_\varepsilon - P_{\boldsymbol{K}}(\sigma_\varepsilon)}{\varepsilon} \quad \text{a.e. in } \Omega \times (0,T);$$
7. *The energy balance: for every* $t \in [0,T]$
$$\begin{aligned}\frac{1}{2}\int_\Omega |\dot u_\varepsilon(t)|^2\,dx + \mathcal{Q}(e_\varepsilon(t)) + \int_0^t\int_\Omega H(\dot p_\varepsilon))\,dx\,ds + \int_0^t\int_{\partial\Omega} S\dot u_\varepsilon\cdot\dot u_\varepsilon\,d\mathcal{H}^{n-1}\,ds\\ + \varepsilon\int_0^t\int_\Omega |\dot p_\varepsilon|^2\,d\mathcal{H}^{n-1}\,ds\\ = \frac{1}{2}\int_\Omega |v_0|^2\,dx + \mathcal{Q}(e_0) + \int_0^t\int_\Omega f\cdot\dot u\,dx\,ds + \int_0^t\int_{\partial\Omega} \varepsilon E v_0\nu\cdot\dot u_\varepsilon\,d\mathcal{H}^{n-1}\,ds. \quad (5.3.4)\end{aligned}$$

Moreover, the following uniform estimate holds

$$\begin{aligned}\|\ddot u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot e_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} + \varepsilon\|E\ddot u_\varepsilon\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))}\\ + \int_0^T\int_{\partial\Omega} S\ddot u_\varepsilon\cdot\ddot u_\varepsilon\,d\mathcal{H}^{n-1}\,dt \le C, \quad (5.3.5)\end{aligned}$$

where C *is a positive constant independent of* ε *and* S*.*

First, we notice that for every $\epsilon > 0$, $\varepsilon E v_0\nu \in H^1(0,T;L^2(\partial\Omega;\mathbb{R}^n))$. The proof of Theorem 5.3.2 follows the program of [70, Section 3.4.1], which is based on a time discretization of the hyperbolic system. In the following sections, we prove the existence and uniqueness of Theorem 5.3.2.

Time discretization

For any $N \in \mathbb{N}$, we define a partition of the interval $[0,T]$ in N sub-intervals of equal length $\delta := T/N$ as follows:

$$0 =: t_0 < t_1 < \ldots < t_N := T,$$

moreover, we have that $\delta = t_i - t_{i-1}$ for every $i = 1,\ldots,N$. We define the discrete body $f_i := f(t_i)$ for every $i = 0,\ldots,N$. Consequently, we define inductively

$$(u_0, e_0, p_0) = (u_0, e_0, p_0) \quad \text{and} \quad (u_1, e_1, p_1) = (u_0, e_0, p_0) + \delta(u_0, e_0, p_0).$$

and for all $i \geq 2$, (u_i, e_i, p_i) is the unique solution to the minimum problem

$$\begin{aligned}\min_{(v,\eta,q)\in\mathcal{X}} \Big\{ Q(\eta) &+ \int_\Omega H(q - p_{i-1})\,dx + \frac{1}{2\delta^2}\int_\Omega (v - 2u_{i-1} + u_{i-2})^2\,dx \\ &+ \frac{\varepsilon}{2\delta}\int_\Omega |Ev - Eu_{i-1}|^2 + |q - p_{i-1}|^2\,dx + \frac{1}{2\delta}\int_{\partial\Omega} S(v - u_{i-1})\cdot(v - u_{i-1})\,d\mathcal{H}^{n-1} \\ &- \int_\Omega f_{i-1}\cdot v\,dx - \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot v\,d\mathcal{H}^{n-1} \Big\} \end{aligned} \tag{5.3.6}$$

where

$$\mathcal{X} = \{(v,\eta,q) \in H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_{sym}) \times L^2(\Omega;\mathbb{M}^n_{sym}) :\ Ev = \eta + q \text{ in } \Omega\}.$$

Notice that (5.3.6) admits an unique solution by strictly convexity, coercivity and sequentially weakly lower semi-continuity in $\mathcal{X}$. Let $(v,\eta,q) \in \mathcal{X}$ and $s \in (0,1)$, it follows that $(u_i, e_i, p_i) + s(v,\eta,q)$ is an admissible for the minimum problem (5.3.6). We choose $(u_i, e_i, p_i) + s(v,\eta,q)$ as a competitor in (5.3.6) and by taking $s \to 0$ we see that

$$\begin{aligned} &\int_\Omega \mathbb{C}e_i : \eta\,dx + \int_\Omega H(p_i + q - p_{i-1})\,dx - \int_\Omega H(p_i - p_{i-1})\,dx \\ &+ \frac{1}{\delta^2}\int_\Omega (u_i - 2u_i + u_{i-2})\cdot v\,dx + \frac{\varepsilon}{\delta}\int_\Omega \left((Eu_i - Eu_{i-1}) : v + (p_i - p_{i-1}) : q\right)\,dx \\ &+ \frac{1}{\delta}\int_{\partial\Omega} S(u_i - u_{i-1})\cdot v\,d\mathcal{H}^{n-1} - \int_\Omega f_{i-1}\cdot v\,dx - \int_{\partial\Omega} \varepsilon E v_0 \nu\cdot v\,d\mathcal{H}^{n-1} \geq 0. \end{aligned} \tag{5.3.7}$$

Furthermore, since $(v,\eta,q) = \pm(v,\nabla v,0) \in \mathcal{X}$ is an admissible of (5.3.6), from the inequality above we observe that

$$\begin{aligned} &\int_\Omega \mathbb{C}e_i : Ev\,dx + \frac{1}{\delta^2}\int_\Omega (u_i - 2u_{i-1} + u_{i-2})\cdot v\,dx + \frac{\varepsilon}{\delta}\int_\Omega (Eu_i - Eu_{i-1}) : Ev\,dx \\ &+ \frac{1}{\delta}\int_{\partial\Omega} S(u_i - u_{i-1})\cdot v\,d\mathcal{H}^{n-1} = \int_\Omega f_{i-1}\cdot v\,dx + \int_{\partial\Omega} \varepsilon E v_0\nu\cdot v\,d\mathcal{H}^{n-1}. \end{aligned}$$

From this and by considering $v \in C_0^\infty(\Omega)$, we have that

$$\frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2} - \operatorname{div}\left(\mathbb{C}e_i + \frac{\varepsilon}{\delta}(Eu_i - Eu_{i-1})\right) = f_{i-1} \quad \text{in } \mathcal{D}'(\Omega). \tag{5.3.8}$$

By the integration by parts, we see that

$$\begin{aligned} &\int_\Omega \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2}\cdot v\,dx - \int_\Omega \operatorname{div}\left(\mathbb{C}e_i + \frac{\varepsilon}{\delta}(Eu_i - Eu_{i-1})\right)\cdot v\,dx - \int_\Omega f_{i-1}\cdot v\,dx \\ &= \int_{\partial\Omega} \varepsilon E v_0\nu\cdot v\,d\mathcal{H}^{n-1} - \langle(\mathbb{C}e_i + \frac{\varepsilon}{\delta}(Eu_i - Eu_{i-1}))\nu\rangle|v\rangle_{H^{-\frac{1}{2}}(\partial\Omega),H^{\frac{1}{2}}(\partial\Omega)} \\ &- \frac{1}{\delta}\int_{\partial\Omega} S(u_i - u_{i-1})\cdot v\,d\mathcal{H}^{n-1} \end{aligned} \tag{5.3.9}$$

It follows from (5.3.8) and (5.3.9) that

$$\frac{1}{\delta}S(u_i - u_{i-1}) + \langle(\mathbb{C}e_i + \frac{\varepsilon}{\delta}(Eu_i - Eu_{i-1}))\nu\rangle|v\rangle_{H^{-\frac{1}{2}}(\partial\Omega),H^{\frac{1}{2}}(\partial\Omega)} = \varepsilon Ev_0\nu \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega),$$

in other words, the equation above represents the discrete boundary conditions. Let $\widetilde{q} \in L^2(\Omega;\mathbb{M}^n_{sym})$. Concerning the *visco-plastic* equation, first we consider the admissible competitor $(0, -q, \widetilde{q} - (p_i - p_{i-1})) \in \mathcal{X}$, it follows from (5.3.7) that

$$\int_\Omega H(\widetilde{q})\,dx \geq \int_\Omega H(p_i - p_{i-1})\,dx + \int_\Omega (\sigma_i - \frac{\varepsilon}{\delta}(p_i - p_{i-1})) : (\widetilde{q} - (p_i - p_{i-1}))\,dx \tag{5.3.10}$$

where $\sigma_i := \mathbb{C}e_i$. By localizing (5.3.10), we see that

$$H(\widetilde{q}) \geq H(p_i - p_{i-1}) + \left(\sigma_i - \frac{\varepsilon}{\delta}(p_i - p_{i-1})\right) : (\widetilde{q} - (p_i - p_{i-1})), \tag{5.3.11}$$

for every $\widetilde{q} \in \mathbb{M}^n_{sym} \times \mathbb{R}$ and for a.e. $x \in \Omega$. As a consequence, $\sigma_i - \frac{\varepsilon}{\delta}(p_i - p_{i-1})$ belongs to the sub-differential of $\widetilde{q} \mapsto H(\widetilde{q})$ at the point $p_i - p_{i-1}$. More precisely,

$$\sigma_i \in \partial\left(H(\cdot) + \frac{\varepsilon}{2}|\cdot|^2\right)\left(\frac{p_i - p_{i-1}}{\delta}\right).$$

By duality we have that

$$\frac{p_i - p_{i-1}}{\delta} = \frac{1}{\varepsilon}(\sigma_i - P_{\mathbf{K}}(\sigma_i)),$$

from where we deduce that

$$\frac{p_i - p_{i-1}}{\delta} = \frac{1}{\varepsilon}(\sigma_i - P_{\mathbf{K}}(\sigma_i)) \tag{5.3.12}$$

a.e. in $\Omega \times (0, T)$.

Interpolations

In this section we construct three different types of interpolations. First, we consider the piecewise constant interpolation as follows

$$\overline{u}(0) := u_0, \quad \overline{\sigma}(0) := \sigma_0, \quad \overline{p}(0) := p_0, \quad \overline{e}(0) := e_0, \quad \text{and} \quad \overline{f}(0) := f(0),$$

moreover, for all $t \in (t_{i-1}, t_i]$ and $i = 1, \dots, N$,

$$\overline{u}(t) := u_i, \quad \overline{\sigma}(t) := \sigma_i, \quad \overline{p}(t) := p_i, \quad \overline{e}(t) := e_i, \quad \text{and} \quad \overline{f}(t) := f_{i-1}.$$

Now, we consider the piecewise affine interpolation as follows

$$\widehat{u}(0) := u_0, \quad \widehat{\sigma}(0) := \sigma_0, \quad \widehat{p}(0) := p_0, \quad \text{and} \quad \widehat{e}(0) := e_0,$$

and for every $i = 1, \dots, N$ and for every $t \in (t_{i-1}, t_i]$, we define

$$\widehat{u}(t) := u_{i-1} + \frac{t - t_{i-1}}{\delta}(u_i - u_{i-1}), \qquad \widehat{\sigma}(t) := \sigma_{i-1} + \frac{t - t_{i-1}}{\delta}(\sigma_i - \sigma_{i-1}),$$
$$\widehat{p}(t) := p_{i-1} + \frac{t - t_{i-1}}{\delta}(p_i - p_{i-1}), \qquad \text{and} \quad \widehat{e}(t) := e_{i-1} + \frac{t - t_{i-1}}{\delta}(e_i - e_{i-1}).$$

Finally, we define the quadratic interpolation of u on $[t_{i-1}t_i]$, denoted as $\widetilde{u}$, as follows

$$\widetilde{u}(t) = \frac{u_i - 2u_{i-1} + u_{i-2}}{2\delta^2}(t - t_{i-1})^2 + \frac{u_i - u_{i-1}}{2\delta}(t - t_{i-1}) + u_{i-1},$$

where we define $u_{-1} := u_0$. Notice that

$$E\overline{u}(t) = \overline{e}(t) + \overline{p}(t), \quad E\widehat{u}(t) = \widehat{e}(t) + \widehat{p}(t), \quad E\widetilde{u}(t) = \widetilde{e}(t) + \widetilde{p}(t), \quad \overline{\sigma}(t) = \mathbb{C}\overline{e}(t)$$

and

$$\widehat{\sigma}(t) = \mathbb{C}\widehat{e}(t),$$

for every $t \in [0, T]$. Therefore, it follows from the discrete elastic equation (5.3.8) and from the discrete equation for the plasticity (5.3.12), that

$$\dot{\tilde{u}}(t) - \operatorname{div}(\overline{\sigma}(t) + \varepsilon E\dot{\widehat{u}}(t)) = \overline{f}(t), \quad \dot{\widehat{p}}(t) = -\frac{1}{\varepsilon}(\overline{\sigma} - P_{\mathbf{K}}(\overline{\sigma}))$$

for a.e. in Ω and for a.e. $t \in [\delta, T]$, and

$$S\dot{\widehat{u}}(t) + (\varepsilon E\dot{\widehat{u}}(t) + \overline{\sigma})\nu = \varepsilon E v_0 \nu \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega) \tag{5.3.13}$$

for a.e. $t \in [\delta, T]$.

A priori estimates

Since $u_i - u_{i-1} \in H^1(\Omega; \mathbb{R}^n)$ and by taking $v = u_i - u_{i-1}$ in (5.3.9) we deduce that

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\frac{u_i - u_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} - \left\|\frac{u_{i-1} - u_{i-2}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \left\|\frac{u_i - 2u_{i-1} + u_{i-2}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)}\right) \\
&\quad + \varepsilon\delta\left\|\frac{Eu_i - Eu_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + \int_\Omega \sigma_i : (p_i - p_{i-1})\,dx \\
&\quad + Q(e_i) - Q(e_{i-1}) + Q(e_i - e_{i-1}) + \frac{1}{\delta}\int_{\partial\Omega} S(u_i - u_{i-1}) \cdot (u_i - u_{i-1})\,d\mathcal{H}^{n-1} \\
&= \int_\Omega f_{i-1} \cdot (u_i - u_{i-1})\,dx + \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot (u_i - u_{i-1})\,d\mathcal{H}^{n-1},
\end{aligned} \tag{5.3.14}$$

where we used the discrete additive decomposition of the symmetric gradient, i.e., $Eu_i - Eu_{i-1} = e_i - e_{i-1} + p_i - p_{i-1}$. Now, if we consider $\widetilde{q} \equiv 0$ in (5.3.11), we deduce that

$$\int_\Omega \sigma_i : (p_i - p_{i-1})\,dx \geq \int_\Omega H(p_i - p_{i-1})\,dx + \varepsilon\delta\left\|\frac{p_i - p_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})}. \tag{5.3.15}$$

By inserting (5.3.15) in (5.3.14) we obtain that

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\frac{u_i - u_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} - \left\|\frac{u_{i-1} - u_{i-2}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \left\|\frac{u_i - 2u_{i-1} + u_{i-2}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)}\right) \\
&\quad + \varepsilon\delta\left\|\frac{Eu_i - Eu_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + \int_\Omega H(p_i - p_{i-1})\,dx + \varepsilon\delta\left\|\frac{p_i - p_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} \\
&\quad + Q(e_i) - Q(e_{i-1}) + Q(e_i - e_{i-1}) + \frac{1}{\delta}\int_{\partial\Omega} S(u_i - u_{i-1}) \cdot (u_i - u_{i-1})\,d\mathcal{H}^{n-1} \\
&\leq \int_\Omega f_{i-1} \cdot (u_i - u_{i-1})\,dx + \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot (u_i - u_{i-1})\,d\mathcal{H}^{n-1}.
\end{aligned} \tag{5.3.16}$$

Let $j \in \mathbb{N}$ be such that $j \le N$. By summing (5.3.16) from $i = 2$ to j yields that

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{u_j - u_{j-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{j}\left\|\frac{u_i - 2u_{i-1} + u_{i-2}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \varepsilon\delta\sum_{i=1}^{j}\left\|\frac{Eu_i - Eu_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} \\
&\quad + \sum_{i=1}^{j}\int_\Omega H(p_i - p_{i-1})\,dx + \varepsilon\delta\sum_{i=1}^{j}\left\|\frac{p_i - p_{i-1}}{\delta}\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + Q(e_j) \\
&\quad + \frac{1}{\delta}\sum_{i=1}^{j}\int_{\partial\Omega} S(u_i - u_{i-1})\cdot(u_i - u_{i-1})\,d\mathcal{H}^{n-1} \\
&\le \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + 2\varepsilon\delta\|Ev_0\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + \delta\int_\Omega H(Ev_0) + Q(e_0) + \delta\int_{\partial\Omega} Sv_0\cdot v_0\,d\mathcal{H}^{n-1} \\
&\quad + \sum_{i=2}^{j}\int_\Omega f_{i-1}\cdot(u_i - u_{i-1})\,dx + \sum_{i=2}^{j}\int_{\partial\Omega}\varepsilon Ev_0\nu\cdot(u_i - u_{i-1})\,d\mathcal{H}^{n-1}.
\end{aligned} \tag{5.3.17}$$

In view of the three different types of interpolation, it follows for all $t \in (t_{j-1}, t_j]$

$$\begin{aligned}
&\frac{1}{2}\left\|\dot{\widehat{u}}(t)\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \varepsilon\int_0^{t_j}\left\|E\dot{\widehat{u}}(t)\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} dt + \int_0^{t_j}\int_\Omega H(\dot{\widehat{p}}(t))\,dx\,dt \\
&\quad + \varepsilon\int_0^{t_j}\left\|\dot{\widehat{p}}(t)\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} dt + Q(\overline{e}(t)) + \int_0^{t_j}\int_{\partial\Omega} S\dot{\widehat{u}}(t)\cdot\dot{\widehat{u}}(t)\,d\mathcal{H}^{n-1}\,dt \\
&\le \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + 2\varepsilon\delta\|Ev_0\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + \delta\int_\Omega H(Ev_0) + Q(e_0) + \delta\int_{\partial\Omega} Sv_0\cdot v_0\,d\mathcal{H}^{n-1} \\
&\quad + \left\|\overline{f}(t)\right\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\left\|\dot{\widehat{u}}(t)\right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \\
&\quad + \|\varepsilon Ev_0\nu\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))}\left\|\dot{\widehat{u}}(t)\right\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))},
\end{aligned} \tag{5.3.18}$$

where we used the Cauchy-Schwarz inequality in the last two terms of the right side of (5.3.17). By (H4) and by applying the Young's inequality in the last two terms of the right side of (5.3.18), we deduce that

$$\begin{aligned}
&\frac{1}{2}\left\|\dot{\widehat{u}}(t)\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \varepsilon\int_0^{t_j}\left\|E\dot{\widehat{u}}(t)\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} dt + \int_0^{t_j}\int_\Omega H(\dot{\widehat{p}}(t))\,dx\,dt \\
&\quad + \varepsilon\int_0^{t_j}\left\|\dot{\widehat{p}}(t)\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} dt + \int_0^{t_j}\int_{\partial\Omega} S\dot{\widehat{u}}(t)\cdot\dot{\widehat{u}}(t)\,d\mathcal{H}^{n-1}\,dt \\
&\le \|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)} + 2\varepsilon\delta\|Ev_0\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})} + \delta\int_\Omega H(Ev_0) + Q(e_0) + \delta\int_{\partial\Omega} Sv_0\cdot v_0\,d\mathcal{H}^{n-1} \\
&\quad + \left\|\overline{f}(t)\right\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\varepsilon Ev_0\nu\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))}.
\end{aligned} \tag{5.3.19}$$

Recall that

$$\widehat{u}(t,x) = u_0(x) + \int_0^t \dot{\widehat{u}}(s,x)\,dx\,ds \quad \text{and} \quad E\widehat{u}(t,x) = Eu_0(x) + \int_0^t E\dot{\widehat{u}}(s,x)\,dx\,ds,$$

therefore, thanks to (5.3.19), there exists $c_\varepsilon > 0$, depending on ε, such that

$$\|\widehat{u}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \le c_\varepsilon. \tag{5.3.20}$$

Furthermore, by the additive decomposition of the symmetric gradient with respect to the piecewise affine interpolation, i.e., $E\dot{\widehat{u}}(t) = \dot{\widehat{\sigma}}(t) + \dot{\widehat{p}}(t)$, there exists c_ε, depending on ε, such that

$$\left\|\dot{\widehat{\sigma}}\right\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} \le c_\varepsilon. \tag{5.3.21}$$

Weak convergence

As a consequence of the inequalities at the end of the previous section, there exist not relabeled subsequences of $\widehat{u}$, $\widehat{e}$ and $\widehat{p}$ and functions

$$\begin{cases} u_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^n)), \\ e_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \\ p_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \end{cases}$$

such that

$$\widehat{u} \underset{N}{\rightharpoonup} u_\varepsilon \text{ in } H^1(0,T;H^1(\Omega;\mathbb{R}^n)), \qquad \dot{\widehat{u}} \underset{N}{\rightharpoonup} \dot{u}_\varepsilon \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$$

$$\widehat{e} \underset{N}{\rightharpoonup} e_\varepsilon \text{ in } H^1(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \quad \text{and} \quad \widehat{p} \underset{N}{\rightharpoonup} p_\varepsilon \text{ in } H^1(0,T;L^2(\Omega;\mathbb{M}^n_{sym})).$$

Since for a.e. $t \in [t_{i-1}, t_i]$

$$\|\widehat{u}(t) - \overline{u}(t)\|_{H^1(\Omega;\mathbb{R}^n)} = \left\| u_{i-1} + \frac{t - t_{i-1}}{\delta}(u_i - u_{i-1}) - u_i \right\|_{H^1(\Omega;\mathbb{R}^n)} \leq 2\delta \left\| \dot{\widehat{u}}(t) \right\|_{H^1(\Omega;\mathbb{R}^n)},$$

we deduce from (5.3.20) and the inequality above that

$$\overline{u} \underset{N}{\rightharpoonup} u_\varepsilon \text{ in } L^2(0,T;H^1(\Omega;\mathbb{R}^n)). \tag{5.3.22}$$

Thanks to the weak convergences above, by Ascoli-Arzelà theorem (also it follows from [31, Lemma 2]), we obtain that

$$\widehat{u}(t) \underset{N}{\rightharpoonup} u_\varepsilon(t) \text{ in } H^1(\Omega;\mathbb{R}^n), \quad \widehat{e}(t) \underset{N}{\rightharpoonup} e_\varepsilon(t) \text{ in } L^2(\Omega;\mathbb{M}^n_{sym}) \quad \text{and} \quad \widehat{p}(t) \underset{N}{\rightharpoonup} p_\varepsilon(t) \text{ in } L^2(\Omega;\mathbb{M}^n_{sym}),$$

for all $t \in [0,T]$. Since $(\widehat{u}(0), \widetilde{e}(0), \widetilde{p}(0)) = (u_0, e_0, p_0)$ for every N, we obtain that

$$(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u_0, e_0, p_0) \in H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_{sym}) \times L^2(\Omega;\mathbb{M}^n_{sym}),$$

moreover, since $E\widehat{u}(t) = \widehat{e}(t) + \widehat{p}(t)$, we deduce that

$$Eu_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t) \in L^2(\Omega;\mathbb{M}^n_{sym})$$

for all $t \in [0,T]$. By arguing as in (5.3.22), it follows that

$$\overline{e} \underset{N}{\rightharpoonup} e_\varepsilon \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym})) \quad \text{and} \quad \overline{p} \underset{N}{\rightharpoonup} p_\varepsilon \text{ in } L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym})).$$

Thanks to the convergences described above and by (5.3.13) we have that

$$(\sigma_\varepsilon + \varepsilon E(\dot{\hat{u}}_\varepsilon))\nu + S\dot{\hat{u}}_\varepsilon = \varepsilon E v_0 \nu \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega).$$

With respect to the quadratic approximation, we observe that

$$\sup_{t\in[0,T]} \|\widetilde{u}(t) - \widehat{u}(t)\|_{L^2(\Omega;\mathbb{R}^n)} \leq 4\delta \left\| \dot{\widehat{u}} \right\|_{L^\infty((0,T);L^2(\Omega;\mathbb{R}^n))}.$$

Therefore, $\widetilde{u} \underset{N}{\rightharpoonup} u_\varepsilon$ in $L^\infty((0,T);L^2(\Omega;\mathbb{R}^n))$.

Weak formulation of the equation of motion and initial condition for the velocity

At this stage, we do not have enough regularity to prove that $\dot{u}_\varepsilon(0) = u_0$. It is very common in hyperbolic equations that the initial condition will be obtained by giving sense to the weak formulation of the equation of motion. This is the aim of this subsection.

Proposition 5.3.3. *For all $\varphi \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ such that $\varphi(T,\cdot) = 0$, we have that*

$$\begin{aligned} -\int_0^T\int_\Omega \dot{u}_\varepsilon\dot{\varphi}\,dx\,dt + \int_0^T\int_\Omega(\sigma_\varepsilon+\varepsilon E\dot{u}_\varepsilon) : E\varphi\,dx\,dt + \int_0^T\int_{\partial\Omega} S\dot{u}_\varepsilon\cdot\varphi\,d\mathcal{H}^{n-1}\,dt \\ = \int_0^T\int_\Omega f\cdot\varphi\,dx\,dt + \int_\Omega v_0\cdot\varphi(0)\,dx + \int_0^T\int_{\partial\Omega}\varepsilon Ev_0\nu\cdot\varphi\,d\mathcal{H}^{n-1}\,dt. \end{aligned} \tag{5.3.23}$$

Proof. We define the piecewise constant and piecewise affine interpolations of $\varphi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n;\mathbb{R}^n)$ as follows

$$\overline{\varphi}(t) = \varphi(t_{i-1}) \quad\text{and}\quad \widehat{\varphi}(t) = \varphi(t_{i-1}) + \frac{t-t_i}{\delta}(\varphi(t_i - \varphi(t_{i-1})),$$

for $t \in [t_{i-1}, t_i]$. Therefore,

$$\sum_{i=2}^N \delta\int_\Omega \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2}\cdot\varphi(t_{i-1})\,dx = -\sum_{i=1}^N\delta\int_\Omega\frac{u_i-u_{i-1}}{\delta}\cdot\frac{\varphi(t_i)-\varphi(t_{i-1})}{\delta}\,dx - \int_\Omega v_0\cdot\varphi(0)\,dx$$

and by considering in (5.3.8) the test function $\varphi(t_{i-1})$ and thanks to the discrete integration by parts, we deduce that

$$\begin{aligned} -\sum_{i=1}^N\delta\int_\Omega\frac{u_i-u_{i-1}}{\delta}\cdot\frac{\varphi(t_i)-\varphi(t_{i-1})}{\delta}\,dx + \sum_{i=2}^N\delta\int_\Omega\left(\sigma_i+\varepsilon\frac{Eu_i-Eu_{i-1}}{\delta}\right) : E\varphi(t_{i-1})\,dx \\ + \sum_{i=2}^N\delta\int_{\partial\Omega}\frac{S(u_i-u_{i-1})}{\delta}\cdot\varphi(t_{i-1})\,d\mathcal{H}^{n-1} \\ = \sum_{i=2}^N\delta\int_\Omega f_{i-1}\cdot\varphi(t_{i-1})\,dx + \int_\Omega v_0\cdot\varphi(0)\,dx + \sum_{i=2}^N\delta\int_{\partial\Omega}\varepsilon Ev_0\nu\cdot\varphi(t_{i-1})\,d\mathcal{H}^{n-1}. \end{aligned}$$

This yields that

$$\begin{aligned} -\int_0^T\int_\Omega\dot{\widehat{u}}\cdot\dot{\widehat{\varphi}}\,dx\,dt + \int_\delta^T\int_\Omega\left(\overline{\sigma}+\varepsilon E\dot{\widehat{u}}\right) : E\overline{\varphi}\,dx\,dt + \int_\delta^T\int_{\partial\Omega}S\dot{\widehat{u}}\cdot\overline{\varphi}\,d\mathcal{H}^{n-1}\,dt \\ = \int_\delta^T\int_\Omega\overline{f}\cdot\overline{\varphi}\,dx\,dt + \int_\Omega v_0\cdot\varphi(0)\,dx + \int_{\partial\Omega}\varepsilon Ev_0\nu\cdot\overline{\varphi}\,d\mathcal{H}^{n-1}\,dt. \end{aligned} \tag{5.3.24}$$

Since we have the following convergences $\overline{\varphi}\to\varphi$ strongly in $L^2(0,T;H^1(\Omega;\mathbb{R}^n))$, $\dot{\widehat{\varphi}}\to\dot{\varphi}$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ and $\overline{f}\to f$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, by the absolute continuity of $t\mapsto f(t)$ in $L^2(\Omega;\mathbb{R}^n)$, by the weak convergences of the previous section and by taking $N\to\infty$ in (5.3.24) we obtain that

$$\begin{aligned} -\int_0^T\int_\Omega\dot{u}_\varepsilon\cdot\dot{\varphi}\,dx\,dt + \int_\delta^T\int_\Omega(\sigma_\varepsilon+\varepsilon E\dot{u}_\varepsilon) : E\varphi\,dx\,dt + \int_\delta^T\int_{\partial\Omega}S\dot{u}_\varepsilon\cdot\varphi\,d\mathcal{H}^{n-1}\,dt \\ = \int_\delta^T\int_\Omega f\cdot\varphi\,dx\,dt + \int_\Omega v_0\cdot\varphi(0)\,dx + \int_0^T\int_{\partial\Omega}\varepsilon Ev_0\nu\cdot\varphi\,d\mathcal{H}^{n-1}\,dt. \end{aligned} \tag{5.3.25}$$

By densities arguments, (5.3.25) also holds if $\varphi\in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ such that $\varphi(T,\cdot)=0$. $\square$

Finally, in view of Proposition 5.3.3, we obtain that $\dot{u}_\varepsilon \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ and $\ddot{u}_\varepsilon \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n))$. Thus, by [16, Theorem 1.19] we infer that $\dot{u}_\varepsilon$ is equal a.e. $t \in [0,T]$ to a function in $C^0(0,T;L^2(\Omega;\mathbb{R}^n))$, moreover the function $t \mapsto \langle \dot{u}_\varepsilon(t)|v(t)\rangle_{L^2(\Omega;\mathbb{R}^n)}$ is absolutely continuous for every $v \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ such that $\dot{v} \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n))$ and we have that

$$\frac{d}{dt}\langle \dot{u}_\varepsilon|v\rangle_{L^2(\Omega;\mathbb{R}^n)} = \langle \ddot{u}_\varepsilon|v\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)} + \langle \dot{u}_\varepsilon|\dot{v}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}.$$

Furthermore, since

$$\ddot{u}_\varepsilon - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) = f \quad \text{in } L^2(0,T;(H^1(\Omega;\mathbb{R}^n))'). \tag{5.3.26}$$

From the last equality, we can get the initial condition for the velocity. Indeed, if we consider $\varphi = \psi(t)\xi(x)$, where $\psi(t) = \frac{1}{T}(T-t)$ and $\xi \in C_0^\infty(\Omega;\mathbb{R}^n)$ as a test function in (5.3.25), we obtain that

$$-\int_0^T\int_\Omega \dot{u}_\varepsilon\cdot\dot{\varphi}\,dx\,dt + \int_\delta^T\int_\Omega(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\varphi\,dx\,dt = \int_\delta^T\int_\Omega f\cdot\varphi\,dx\,dt + \int_\Omega v_0\cdot\varphi(0)\,dx. \tag{5.3.27}$$

By (5.3.26) and (5.3.27) and thanks to [16, Theorem 1.1], we deduce that

$$\begin{aligned}
-\int_0^T\int_\Omega \dot{u}_\varepsilon\cdot\dot{\varphi}\,dx\,dt &= -\int_0^T \langle \dot{u}_\varepsilon|\dot{v}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}\,dt \\
&= \int_0^T \langle \ddot{u}_\varepsilon|v\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}\,dt - \int_0^T \frac{d}{dt}\langle \dot{u}_\varepsilon|v\rangle_{L^2(\Omega;\mathbb{R}^n)}\,dt \\
&= \int_0^T \langle \ddot{u}_\varepsilon|v\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}\,dt - \int_\Omega \dot{u}_\varepsilon(T)\cdot\varphi(T)\,dx + \int_\Omega \dot{u}_\varepsilon(0)\cdot\varphi(0)\,dx.
\end{aligned}$$

From (5.3.27), we have that

$$\begin{aligned}
&\int_0^T \langle \ddot{u}_\varepsilon|v\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}\,dt + \int_\Omega \dot{u}_\varepsilon(0)\cdot\varphi(0)\,dx + \int_\delta^T\int_\Omega(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\varphi\,dx\,dt \\
&= \int_\delta^T\int_\Omega f\cdot\varphi\,dx\,dt + \int_\Omega v_0\cdot\varphi(0)\,dx.
\end{aligned} \tag{5.3.28}$$

Finally, by the integration by parts in (5.3.28), we have that

$$\int_\Omega \dot{u}_\varepsilon(0)\cdot\xi\,dx = \int_\Omega v_0\cdot\xi\,dx.$$

Hence,

$$\dot{u}_\varepsilon(0) = v_0 \quad \text{in } H^2(\Omega;\mathbb{R}^n).$$

Strong convergence and flow rule

In order to obtain the flow rule we need to improve the weak convergences of $\overline{\sigma}$ and $\widehat{\sigma}$, more precisely, in this section we prove that $\overline{\sigma}$ and $\widehat{\sigma}$ are strongly convergent.

Lemma 5.3.4. *We have that $\overline{\sigma}$ and $\widehat{\sigma}$ converge strongly to σ_ε in $L^2(0,T,L^2(\Omega;\mathbb{R}^n))$.*

Proof. Before to start the proof, we denote by $[t]_i = t_i$ for every $t \in (0,T]$ and every $i = 2,\dots,N$. Since

$$\ddot{u}_\varepsilon - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) = f \quad \text{in } L^2(0,T;(H^1(\Omega;\mathbb{R}^n))') \text{ in } L^2(0,T;(H^1(\Omega;\mathbb{R}^n))')$$

and

$$\ddot{\widehat{u}}(t) - \operatorname{div}(\overline{\sigma}(t) + \varepsilon E\dot{\widehat{u}}(t)) = \overline{f}(t) \text{ in } (H^1(\Omega;\mathbb{R}^n))',$$

by taking the difference between the two previous equations and by considering as a test function $\mathbb{1}_{[\delta,[t]_i]}\dot{\hat{u}} \in L^2(0,T;(H^1(\Omega;\mathbb{R}^n))')$, we obtain that

$$\begin{aligned}\int_\delta^{[t]_i} \langle \ddot{\hat{u}} - \ddot{u}_\varepsilon | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds \\ - \int_\delta^{[t]_i} \langle \operatorname{div}(\overline{\sigma} + \varepsilon E\dot{\hat{u}})) - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds \\ = \int_\delta^{[t]_i} (\overline{f} - f)\cdot \dot{\hat{u}}\,ds \end{aligned} \tag{5.3.29}$$

Since $\overline{f} \to f$ strongly in $L^1(0,T;L^2(\Omega;\mathbb{R}^n))$ and $\dot{\hat{u}}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$, by dominated convergence theorem and from the equality above we infer that

$$\lim_{N\to\infty}\Bigg(\int_\delta^{[t]_i} \langle \ddot{\hat{u}} - \ddot{u}_\varepsilon | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds \\ - \int_\delta^{[t]_i} \langle \operatorname{div}(\overline{\sigma} + \varepsilon E\dot{\hat{u}})) - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\Bigg) = 0.$$

Recall that,

$$\begin{aligned} & - \int_\delta^{[t]_i} \langle \operatorname{div}(\overline{\sigma} + \varepsilon E\dot{\hat{u}})) - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds \\ & = \int_\delta^{[t]_i}\int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\hat{u}} + \varepsilon(E\dot{\hat{u}} - E\dot{u}_\varepsilon) : E\dot{\hat{u}}\,dx\,ds \\ & \quad + \int_\delta^{[t]_i}\int_{\partial\Omega} S(\dot{\hat{u}} - \dot{u}_\varepsilon)\cdot\dot{\hat{u}} + \varepsilon E v_0 \nu\cdot(\dot{\hat{u}} - \dot{u}_\varepsilon)\,d\mathcal{H}^{n-1}\,ds \end{aligned} \tag{5.3.30}$$

Furthermore, by the following convergences $E\dot{\hat{u}} \underset{N}{\rightharpoonup} E\dot{u}_\varepsilon$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ and $\dot{\hat{u}} \underset{N}{\rightharpoonup} \dot{u}_\varepsilon$ in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n))$, we deduce that

$$\int_\delta^{[t]_i}\int_\Omega \varepsilon(E\dot{\hat{u}} - E\dot{u}_\varepsilon) : E\dot{\hat{u}}\,dx\,ds + \int_\delta^{[t]_i}\int_{\partial\Omega} S(\dot{\hat{u}} - \dot{u}_\varepsilon)\cdot\dot{\hat{u}} + \varepsilon E v_0\nu\cdot(\dot{\hat{u}} - \dot{u}_\varepsilon)\,d\mathcal{H}^{n-1}\,ds \to 0 \tag{5.3.31}$$

as $N \to \infty$. By (5.3.29)–(5.3.31) we see that the expression

$$\int_0^T \Bigg(\int_\delta^{[t]_i} \langle \ddot{\hat{u}} - \ddot{u}_\varepsilon | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds + \int_\delta^{[t]_i}\int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\hat{u}} + \varepsilon(E\dot{\hat{u}} - E\dot{u}_\varepsilon) : E\dot{\hat{u}}\,dx\,ds \\ + \int_\delta^{[t]_i}\int_{\partial\Omega} S(\dot{\hat{u}} - \dot{u}_\varepsilon)\cdot\dot{\hat{u}}\,d\mathcal{H}^{n-1}\,ds\Bigg)\,dt \tag{5.3.32}$$

goes to 0 as $N \to \infty$. Thus,

$$\begin{aligned} & \liminf_{N\to\infty}\int_0^T\int_\delta^{[t]_i} \langle \ddot{\hat{u}} - \ddot{u}_\varepsilon | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt \\ & \qquad + \limsup_{N\to\infty}\int_0^T\int_\delta^{[t]_i}\int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\hat{u}}\,dx\,ds\,dt \\ & \le \limsup_{N\to\infty}\int_0^T\Bigg(\int_\delta^{[t]_i} \langle \ddot{\hat{u}} - \ddot{u}_\varepsilon | \dot{\hat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds \\ & \qquad + \int_\delta^{[t]_i}\int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\hat{u}}\,d\mathcal{H}^{n-1}\,ds\Bigg) \le 0. \end{aligned} \tag{5.3.33}$$

By the additive decomposition of the symmetric gradient with respect to the piecewise affine interpolation, we obtain that

$$\int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\widehat{u}}\, dx\, ds\, dt = \int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : (\dot{\widehat{e}} + \dot{\widehat{p}})\, dx\, ds\, dt.$$

By the fact that $\overline{\sigma} \underset{N}{\rightharpoonup} \sigma_\varepsilon$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$, there exist $\tau_\varepsilon \in L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ such that $P_{\boldsymbol{K}}(\overline{\sigma}) \underset{N}{\rightharpoonup} \tau_\varepsilon$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ and $\tau_\varepsilon \in \boldsymbol{K}$ almost everywhere. Furthermore, we have that

$$\int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : \dot{\widehat{p}}\, dx\, ds\, dt = \int_0^T \int_\delta^{[t]_i} \int_\Omega H(\dot{\widehat{p}}) + \frac{\varepsilon}{2}\left|\dot{\widehat{p}}\right|^2 + \frac{1}{2\varepsilon}|(P_{\boldsymbol{K}}(\overline{\sigma}) - \overline{\sigma}|^2 - \sigma_\varepsilon : \dot{\widehat{p}}\, dx\, ds\, dt$$

From the inequality above and by the weak convergences of $\dot{\widehat{p}}$, $\overline{\sigma}$ and $P_{\boldsymbol{K}}(\overline{\sigma})$ we obtain that

$$\liminf_{N\to\infty} \int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : \dot{\widehat{p}}\, dx\, ds\, dt \geq \int_0^T \int_\delta^{[t]_i} \int_\Omega H(\dot{p}_\varepsilon) + \frac{\varepsilon}{2}|\dot{p}_\varepsilon|^2 + \frac{1}{2\varepsilon}|\tau_\varepsilon - \sigma_\varepsilon|^2 - \sigma_\varepsilon : \dot{p}_\varepsilon\, dx\, ds\, dt \tag{5.3.34}$$

In view of the weak convergence of $P_{\boldsymbol{K}}(\overline{\sigma})$ and by the fact that

$$\dot{\widehat{p}}(t) = \frac{1}{\varepsilon}(\overline{\sigma} - P_{\mathbf{K}}(\overline{\sigma}))$$

we deduce that

$$\dot{p}_\varepsilon(t) = \frac{1}{\varepsilon}(\sigma_\varepsilon - \tau_\varepsilon).$$

From the last expression and by (5.3.34), we obtain that

$$\liminf_{N\to\infty} \int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : \dot{\widehat{p}}\, dx\, ds\, dt \geq \int_0^T \int_\delta^{[t]_i} \int_\Omega H(\dot{p}_\varepsilon) - \tau_\varepsilon : \dot{p}_\varepsilon\, dx\, ds\, dt \geq 0, \tag{5.3.35}$$

where in the last inequality we used the flow rule. Thus, by (5.3.35), for N large enough, we deduce that

$$\int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : \dot{\widehat{p}}\, dx\, ds\, dt \geq 0.$$

Thus,

$$\begin{aligned}\limsup_{N\to\infty} \int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\widehat{u}}\, dx\, ds\, dt &\geq \limsup_{N\to\infty} \int_0^T \int_\delta^{[t]_i} \int_\Omega (\widehat{\sigma} - \sigma_\varepsilon) : (\widehat{e} - e_\varepsilon)\, dx\, ds\, dt \\ &= \limsup_{N\to\infty} \int_0^T Q(\widehat{e}([t]_i) - e_\varepsilon([t]_i))\, dt,\end{aligned}$$

where we use the fact that $\widehat{e}(0) = -e_\varepsilon(0)$. Moreover, $\widehat{e}([t]_i) = \overline{e}([t]_i)$ for every $t \in [0,T]$, hence

$$\limsup_{N\to\infty} \int_0^T \int_\delta^{[t]_i} \int_\Omega (\overline{\sigma} - \sigma_\varepsilon) : E\dot{\widehat{u}}\, dx\, ds\, dt \geq \limsup_{N\to\infty} \int_0^T Q(\overline{e}([t]_i) - e_\varepsilon([t]_i))\, dt.$$

In the sequel, we are going to present some estimates for the following term

$$\begin{aligned}&\int_0^T \int_\delta^{[t]_i} \langle \ddot{\widetilde{u}} - \ddot{u}_\varepsilon | \dot{\widehat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\, ds\, dt \\ &= \int_0^T \int_\delta^{[t]_i} \langle \ddot{\widetilde{u}} | \dot{\widehat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\, ds\, dt - \int_0^T \int_\delta^{[t]_i} \langle \ddot{u}_\varepsilon | \dot{\widehat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\, ds\, dt \\ &\quad + \int_0^T \int_\delta^{[t]_i} \langle \ddot{u}_\varepsilon | \dot{u}_\varepsilon - \dot{\widehat{u}} \rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\, ds\, dt.\end{aligned}$$

First, we observe that

$$\lim_{N\to\infty}\int_0^T\int_\delta^{[t]_i}\langle \ddot{u}_\varepsilon|\dot{u}_\varepsilon-\dot{\hat{u}}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt=0.$$

Thanks to the integration by parts, we have that

$$\int_0^T\int_\delta^{[t]_i}\langle \ddot{\tilde{u}}|\dot{\hat{u}}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt\geq\frac{1}{2}\left(\left\|\dot{\hat{u}}\right\|^2_{L^2(\Omega;\mathbb{R}^n)}-\|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)}\right).$$

In view of the lower semicontinuity of $\left\|\dot{\hat{u}}\right\|^2_{L^2(\Omega;\mathbb{R}^n)}$ and from the inequality above, we obtain that

$$\liminf_{N\to\infty}\int_0^T\int_\delta^{[t]_i}\langle \ddot{\tilde{u}}|\dot{\hat{u}}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt\geq\frac{1}{2}\left(\|\dot{u}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)}-\|v_0\|^2_{L^2(\Omega;\mathbb{R}^n)}\right).$$

By [16, Theorem 1.19], we have that

$$\int_0^T\int_\delta^{[t]_i}\langle \ddot{u}_\varepsilon|\dot{u}_\varepsilon\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt=\frac{1}{2}\int_0^T\left(\|\dot{u}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)}([t]_i)-\|\dot{u}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)}(\delta)\right)\,dt$$

Since $\dot{u}_\varepsilon\in C^0([0,T];L^2(\Omega;\mathbb{R}^n))$ and by the expression above, we have that

$$\lim_{N\to\mathbb{R}^n}\int_0^T\int_\delta^{[t]_i}\langle \ddot{u}_\varepsilon|\dot{u}_\varepsilon\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt=\frac{1}{2}\int_0^T\left(\|\dot{u}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)}(t)-\|\dot{u}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)}(0)\right)\,dt.$$

As a consequence and summarizing all the inequalities above, we can deduce that

$$\liminf_{N\to\infty}\int_0^T\int_\delta^{[t]_i}\langle \ddot{\tilde{u}}-\ddot{u}_\varepsilon|\dot{\hat{u}}\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}(s)\,ds\,dt\geq 0. \tag{5.3.36}$$

Thanks to (5.3.33) and (5.3.36), and up to a relabeling, we obtain that $\overline{\sigma}$ converges strongly to σ_ε in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$. Furthermore, by the fact that $\overline{\sigma}-\widehat{\sigma}\to 0$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$, we can conclude the proof of this lemma. □

We conclude this section with the following remark: by applying Lemma 5.3.4 and the Lipschitz continuity of the projection on $\mathbf{K}$, we get that

$$\dot{\hat{p}}=-\frac{1}{\varepsilon}(P_{\mathbf{K}}(\overline{\sigma})-\overline{\sigma})\to-\frac{1}{\varepsilon}(P_{\mathbf{K}}(\sigma_\varepsilon)-\sigma_\varepsilon)=\dot{p}_\varepsilon\quad\text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym})). \tag{5.3.37}$$

Uniqueness

We proceed with the classical method to prove the uniqueness of solutions. More precisely, we consider two different solutions $(u^1_\varepsilon,e^1_\varepsilon,p^1_\varepsilon)$ and $(u^2_\varepsilon,e^2_\varepsilon,p^2_\varepsilon)$ associated to the initial conditions (u_0,e_0,p_0,v_0) and source terms f and $Ev_0\nu$. By subtracting the two equations of motion associated to both solutions, we see that

$$\ddot{u}^1_\varepsilon-\ddot{u}^2_\varepsilon-\operatorname{div}((\sigma^1_\varepsilon+\varepsilon E\dot{u}^1_\varepsilon)-(\sigma^2_\varepsilon+\varepsilon E\dot{u}^2_\varepsilon))=0\quad\text{in } L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n)). \tag{5.3.38}$$

Considering to $\dot{u}^1_\varepsilon-\dot{u}^2_\varepsilon\in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ as test function in the distribution defined in (5.3.38) we have that

$$\int_0^s\langle\ddot{u}^1_\varepsilon-\ddot{u}^2_\varepsilon|\dot{u}^1_\varepsilon-\dot{u}^2_\varepsilon\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}dt$$
$$-\int_0^s\langle\operatorname{div}((\sigma^1_\varepsilon+\varepsilon E\dot{u}^1_\varepsilon)-(\sigma^2_\varepsilon+\varepsilon E\dot{u}^2_\varepsilon))|\dot{u}^1_\varepsilon-\dot{u}^2_\varepsilon\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}dt=0.$$

By the integration by parts we infer from the previous expression that

$$
\begin{aligned}
&\int_0^s \langle \ddot u_\varepsilon^1 - \ddot u_\varepsilon^2 | \dot u_\varepsilon^1 - \dot u_\varepsilon^2 \rangle_{(H^1(\Omega;\mathbb{R}^n))', H^1(\Omega;\mathbb{R}^n)} dt + \int_0^s \int_\Omega (\sigma_\varepsilon^1 - \sigma_\varepsilon^2) : (E\dot u_\varepsilon^1 - E\dot u_\varepsilon^2)\, dx\, dt \\
&\quad + \varepsilon \int_0^s \int_\Omega \left| E\dot u_\varepsilon^1 - E\dot u_\varepsilon^2 \right|^2 dx\, dt \\
&\quad - \int_0^s \int_{\partial\Omega} (\dot u_\varepsilon^1 - \dot u_\varepsilon^2) \cdot ((\sigma_\varepsilon^1 + \varepsilon E\dot u_\varepsilon^1) - (\sigma_\varepsilon^2 + \varepsilon E\dot u_\varepsilon^2))\nu \, d\mathcal{H}^1\, dt = 0
\end{aligned}
\tag{5.3.39}
$$

Notice that the last term in the left side of (5.3.39) is equal to

$$
\int_0^s \int_{\partial\Omega} S(\dot u_\varepsilon^1 - \dot u_\varepsilon^2) : (\dot u_\varepsilon^1 - \dot u_\varepsilon^2)\, d\mathcal{H}^1\, dt \tag{5.3.40}
$$

which is greater than 0 by (H4). Moreover, we have that

$$
\int_0^s \langle \ddot u_\varepsilon^1 - \ddot u_\varepsilon^2 | \dot u_\varepsilon^1 - \dot u_\varepsilon^2 \rangle_{(H^1(\Omega;\mathbb{R}^n))', H^1(\Omega;\mathbb{R}^n)} dt = \frac{\| \dot u_\varepsilon^1(s) - \dot u_\varepsilon^2(s) \|^2_{L^2(\Omega;\mathbb{R}^n)}}{2}, \tag{5.3.41}
$$

where we used the fact that $\dot u_\varepsilon^1(0) = \dot u_\varepsilon^2(0) = 0$. Finally, by the additive decomposition of the symmetric gradient, we have that

$$
\begin{aligned}
&\int_0^s \int_\Omega (\sigma_\varepsilon^1 - \sigma_\varepsilon^2) : (E\dot u_\varepsilon^1 - E\dot u_\varepsilon^2)\, dx\, dt \\
&\quad = Q(e_\varepsilon^1(s) - e_\varepsilon^2(s)) + \int_0^s \int_\Omega (\sigma_\varepsilon^1 - \sigma_\varepsilon^2) : (\dot p_\varepsilon^1 - \dot p_\varepsilon^2)\, dx\, dt \\
&\quad = Q(e_\varepsilon^1(s) - e_\varepsilon^2(s)) + \frac{1}{\varepsilon} \int_0^s \int_\Omega (\sigma_\varepsilon^1 - \sigma_\varepsilon^2) : (\sigma_\varepsilon^1 - \sigma^2 - (P_{\mathbf{K}}(\sigma_\varepsilon^2) - P_{\mathbf{K}}(\sigma_\varepsilon^1))) dx\, dt,
\end{aligned}
\tag{5.3.42}
$$

where we used the fact that $e_\varepsilon^1(0) = e_\varepsilon^2(0) = e_0$ and (5.3.37). By (5.3.39)–(5.3.42) we conclude that $e_\varepsilon^1 = e_\varepsilon^2$ and $\dot u_\varepsilon^1 = \dot u_\varepsilon^2$. Since $u_\varepsilon^1(0) = u_\varepsilon^2(0) = u_0$ we deduce that $u_\varepsilon^1 = u_\varepsilon^2$, and by the kinetic compatibility we obtain that $p_\varepsilon^1 = p_\varepsilon^2$.

Remark 5.3.5. In view of the uniqueness of solutions, it is very well known that we do not have to subtract subsequences in the converges treated before in this section.

Energy balance

By integrating the equation of motion against the function $(s,x) \mapsto \mathbb{1}_{[0,t]}(s)\dot u_\varepsilon(s,x)$, it follows that

$$
\begin{aligned}
&\int_0^t \langle \ddot u_\varepsilon | \dot u_\varepsilon \rangle_{(H^1(\Omega;\mathbb{R}^n))', H^1(\Omega;\mathbb{R}^n)} ds + \int_0^t \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot u_\varepsilon) : E\dot u_\varepsilon\, dx\, ds + \int_0^t \int_{\partial\Omega} S\dot u_\varepsilon \cdot \dot u_\varepsilon\, d\mathcal{H}^{n-1}\, ds \\
&\quad = \int_0^t \int_\Omega f \cdot \dot u_\varepsilon\, dx\, ds + \int_0^t \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot \dot u_\varepsilon\, d\mathcal{H}^{n-1}\, ds.
\end{aligned}
\tag{5.3.43}
$$

From the additive decomposition of $E\dot u_\varepsilon$ and by (5.3.43) yields that

$$
\begin{aligned}
&\frac{1}{2}\left(\|\dot u_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} - \|\dot u_\varepsilon(0)\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) + Q(e_\varepsilon(t)) - Q(e_\varepsilon(0)) + \varepsilon \int_0^t \int_\Omega |E\dot u_\varepsilon|^2\, dx\, ds \\
&\quad + \int_0^t \int_\Omega \dot p_\varepsilon : \sigma_\varepsilon\, dx\, ds + \int_0^t \int_{\partial\Omega} S\dot u_\varepsilon \cdot \dot u_\varepsilon\, d\mathcal{H}^{n-1}\, ds \\
&= \int_0^t \int_\Omega f \cdot \dot u_\varepsilon\, dx\, ds + \int_0^t \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot \dot u_\varepsilon\, d\mathcal{H}^{n-1}\, ds.
\end{aligned}
\tag{5.3.44}
$$

Since $\dot p_\varepsilon = -\frac{1}{\varepsilon}(P_{\mathbf{K}}(\sigma_\varepsilon) - \sigma_\varepsilon)$, it follows that $\dot p_\varepsilon : \sigma_\varepsilon = \varepsilon|\dot p_\varepsilon|^2 + H(\dot p_\varepsilon)$. By inserting the previous equivalence of $\dot p_\varepsilon : \sigma_\varepsilon$ in (5.3.44), we obtain that

$$\frac{1}{2}\|\dot u_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} + Q(e_\varepsilon(t)) + \varepsilon\int_0^s\int_\Omega |E\dot u_\varepsilon|^2\,dx\,ds + \int_0^t\int_\Omega H(\dot p_\varepsilon)\,dx\,ds + \int_0^t\int_{\partial\Omega} S\dot u_\varepsilon\cdot\dot u_\varepsilon\,d\mathcal{H}^{n-1}\,ds$$
$$+\varepsilon\int_0^t\int_\Omega |\dot p_\varepsilon|^2\,dx\,ds$$
$$= \frac{1}{2}\|\dot u_\varepsilon(0)\|^2_{L^2(\Omega;\mathbb{R}^n)} + Q(e(0)) + \int_0^t\int_\Omega f\cdot\dot u_\varepsilon\,dx\,ds + \int_0^t\int_{\partial\Omega}\varepsilon E v_0\nu\cdot\dot u_\varepsilon\,d\mathcal{H}^{n-1}\,ds,$$

and thus we obtained the energy balance stated in Theorem 5.3.2 (see (5.3.4)).

A posteriori estimates

In this section, we follow the program of [14, 70]. We begin by extending, for negative times, the functions $u_\varepsilon, e_\varepsilon, p_\varepsilon, f$ and g, more precisely, we define

$$u_\varepsilon(t) := u_0 + tv_0, \quad e_\varepsilon(t) := e_0, \quad p_\varepsilon(t) := p_0 \quad \text{and} \quad f(t) = f(0),$$

for any $t < 0$. We also introduce the following notation

$$\partial_t^h\varphi := \frac{\varphi(t) - \varphi(t-h)}{h}.$$

Let $t \in [0, T]$ and left $h \in (0, t)$. We consider as a test function of the equation of motion the function $\varphi \in L^2(0, T+h; H^1(\Omega;\mathbb{R}^n))$, by integrating by parts we see that

$$\begin{aligned}\int_0^T \langle \ddot u_\varepsilon(t)|\varphi(t)\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}dt + \int_0^T\int_\Omega(\sigma_\varepsilon(t) + \varepsilon E\dot u_\varepsilon(t)) : E\varphi(t)\,dx\,dt \\ + \int_0^T\int_{\partial\Omega} S\dot u_\varepsilon(t)\cdot\varphi(t)\,d\mathcal{H}^{n-1}\,dt \\ = \int_0^T\int_\Omega f(t)\cdot\varphi(t)\,dx\,dt + \int_0^T\int_{\partial\Omega}\varepsilon E v_0\nu(t)\cdot\varphi(t)\,d\mathcal{H}^{n-1}\,dt,\end{aligned} \tag{5.3.45}$$

and by using the same arguments at the time $t-h$ we have that

$$\begin{aligned}\int_h^{T+h} \langle \ddot u_\varepsilon(t-h)|\varphi(t)\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}dt + \int_h^{T+h}\int_\Omega(\sigma_\varepsilon(t-h) + \varepsilon E\dot u_\varepsilon(t-h)) : E\varphi(t)\,dx\,dt \\ + \int_0^{T+h}\int_{\partial\Omega} S\dot u_\varepsilon(t-h)\cdot\varphi(t)\,d\mathcal{H}^{n-1}\,dt \\ = \int_0^{T+h}\int_\Omega f(t-h)\cdot\varphi(t)\,dx\,dt + \int_0^{T+h}\int_{\partial\Omega}\varepsilon E v_0\nu(t-h)\cdot\varphi(t)\,d\mathcal{H}^{n-1}\,dt.\end{aligned} \tag{5.3.46}$$

We consider, in the difference of (5.3.45) and (5.3.46), the test function

$$\varphi(t,x) := \mathbb{1}_{[0,s]}(t)\frac{\dot u_\varepsilon(t) - \dot u_\varepsilon(t-h)}{h} =: \mathbb{1}_{[0,s]}(t)\partial_h^t\dot u_\varepsilon(t)$$

and by the definition for negative times we can deduce that

$$\begin{aligned}\int_0^s \langle \partial_h^t\ddot u_\varepsilon(t)|\partial_h^t\dot u_\varepsilon(t)\rangle_{(H^1(\Omega;\mathbb{R}^n))',H^1(\Omega;\mathbb{R}^n)}dt + \int_0^s\int_\Omega \partial_h^t(\sigma_\varepsilon(t) + \varepsilon E\dot u_\varepsilon(t)) : E\partial_h^t\dot u_\varepsilon(t)\,dx\,dt \\ + \int_0^s\int_{\partial\Omega} S\partial_h^t\dot u_\varepsilon(t)\cdot\partial_h^t\dot u_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt \\ = \frac{1}{h}\int_0^h\int_\Omega(\operatorname{div}(\sigma_0 + \varepsilon E v_0) + f(0))\cdot\partial_h^t\dot u_\varepsilon(t)\,dx\,dt + \int_0^s\int_\Omega \partial_h^t f(t)\cdot\partial_h^t\dot u_\varepsilon(t)\,dx\,dt.\end{aligned} \tag{5.3.47}$$

By applying additive decomposition of $E\dot{u}_\varepsilon$ we see that

$$\begin{aligned}\int_0^s\int_\Omega E\partial_h^t\dot{u}_\varepsilon(t)):\partial_h^t\sigma_\varepsilon(t)\,dx\,dt\\ =\int_0^s\int_\Omega \partial_h^t\dot{\sigma}_\varepsilon(t)):\partial_h^t\sigma_\varepsilon(t)\,dx\,dt+\int_0^s\int_\Omega \partial_h^t\dot{p}_\varepsilon(t)):\partial_h^t\sigma_\varepsilon(t)\,dx\,dt\\ -\frac{1}{h}\int_0^h\int_\Omega Ev_0:\partial_h^t\sigma_\varepsilon(t)\,dx\,dt.\end{aligned}\tag{5.3.48}$$

Inserting (5.3.48) in the right side of (5.3.47) and by the integration by parts we see that

$$\begin{aligned}&\frac{1}{2}\left(\left\|\partial_h^t\dot{u}_\varepsilon(s)\right\|^2_{L^2(\Omega;\mathbb{R}^n)}-\left\|\partial_h^t\dot{u}_\varepsilon(0)\right\|^2_{L^2(\Omega;\mathbb{R}^n)}+Q(\partial_h^te_\varepsilon(s))-Q(\partial_h^te_\varepsilon(0))\right)\\ &\quad+\varepsilon\int_0^s\int_\Omega\left|E\partial_h^t\dot{u}_\varepsilon(t)\right|^2dx\,dt+\int_0^s\int_\Omega\partial_h^t\dot{p}_\varepsilon(t)):\partial_h^t\sigma_\varepsilon(t)\,dx\,dt-\frac{1}{h}\int_0^h\int_\Omega Ev_0:\partial_h^t\sigma_\varepsilon(t)\,dx\,dt\\ &\quad+\int_0^s\int_{\partial\Omega}S\partial_h^t\dot{u}_\varepsilon(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt\\ &\geq\frac{1}{2}\left(\left\|\partial_h^t\dot{u}_\varepsilon(s)\right\|^2_{L^2(\Omega;\mathbb{R}^n)}+\left\|\partial_h^t\sigma_\varepsilon(s)\right\|^2_{L^2(\Omega;\mathbb{M}^n_{sym})}\right)+\varepsilon\int_0^s\int_\Omega\left|E\partial_h^t\dot{u}_\varepsilon(t)\right|^2dx\,dt\\ &\quad-\frac{1}{h}\int_0^h\int_\Omega Ev_0:\partial_h^t\sigma_\varepsilon(t)\,dx\,dt+\int_0^s\int_{\partial\Omega}S\partial_h^t\dot{u}_\varepsilon(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt\end{aligned}\tag{5.3.49}$$

By (5.3.47) and (5.3.49) we obtain that

$$\begin{aligned}&\frac{1}{2}\left(\left\|\partial_h^t\dot{u}_\varepsilon(s)\right\|^2_{L^2(\Omega;\mathbb{R}^n)}+Q(\partial_h^te_\varepsilon(s))\right)+\varepsilon\int_0^s\int_\Omega\left|E\partial_h^t\dot{u}_\varepsilon(t)\right|^2dx\,dt\\ &\quad+\int_0^s\int_{\partial\Omega}S\partial_h^t\dot{u}_\varepsilon(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt\\ &\leq\frac{1}{h}\int_0^h\int_\Omega(\operatorname{div}(\sigma_0+\varepsilon Ev_0)+f(0))\cdot\partial_h^t\dot{u}_\varepsilon(t)\,dx\,dt+\frac{1}{h}\int_0^h\int_\Omega Ev_0:\partial_h^t\sigma_\varepsilon(t)\,dx\,dt\\ &\quad+\int_0^s\int_\Omega\partial_h^tf(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,dx\,dt.\end{aligned}$$

From this follows that

$$\begin{aligned}&\frac{1}{2}\left(\left\|\partial_h^t\dot{u}_\varepsilon(s)\right\|^2_{L^2(\Omega;\mathbb{R}^n)}+Q(\partial_h^te_\varepsilon(s))\right)+\varepsilon\int_0^s\int_\Omega\left|E\partial_h^t\dot{u}_\varepsilon(t)\right|^2dx\,dt\\ &\quad+\int_0^s\int_{\partial\Omega}S\partial_h^t\dot{u}_\varepsilon(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt\\ &\leq\|\operatorname{div}(\sigma_0+\varepsilon Ev_0)+f(0)\|_{L^2(\Omega;\mathbb{R}^n)}\sup_{t\in[0,T]}\left\|\partial_h^t\dot{u}_\varepsilon(t)\right\|_{L^2(\Omega;\mathbb{R}^n)}\\ &\quad+\|Ev_0\|_{L^2(\Omega;\mathbb{M}^n_{sym})}\sup_{t\in[0,T]}\left\|\partial_h^t\sigma_\varepsilon(t)\right\|_{L^2(\Omega;\mathbb{M}^n_{sym})}+\left\|\partial_h^tf(t)\right\|_{L^2(\Omega;\mathbb{R}^n)}\sup_{t\in[0,T]}\left\|\partial_h^t\dot{u}_\varepsilon(t)\right\|_{L^2(\Omega;\mathbb{R}^n)}.\end{aligned}$$

Therefore, there exists c independent of ε and S such that

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|\partial_h^t\dot{u}_\varepsilon(t)\right\|_{L^2(\Omega;\mathbb{R}^n)}+\sup_{t\in[0,T]}\left\|\partial_h^t\sigma_\varepsilon(t)\right\|_{L^2(\Omega;\mathbb{M}^n_{sym})}+\sqrt{\varepsilon}\left\|E\partial_h^t\dot{u}_\varepsilon(t)\right\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))}\\ &\quad+\left(\int_0^s\int_{\partial\Omega}S\partial_h^t\dot{u}_\varepsilon(t)\cdot\partial_h^t\dot{u}_\varepsilon(t)\,d\mathcal{H}^{n-1}\,dt\right)^{\frac{1}{2}}\\ &\leq c\left(\|\operatorname{div}(\sigma_0+\varepsilon Ev_0)+f(0)\|_{L^2(\Omega;\mathbb{R}^n)}+\|Ev_0\|_{L^2(\Omega;\mathbb{M}^n_{sym})}+\left\|\partial_h^tf\right\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\right).\end{aligned}\tag{5.3.50}$$

Letting $h \to 0$ in (5.3.50), we get that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\ddot{u}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\dot{\sigma}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^n_{sym})} + \sqrt{\varepsilon}\|E\ddot{u}_\varepsilon(t)\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} \\
&\quad + \left(\int_0^s \int_{\partial\Omega} S\partial_h^t \dot{u}_\varepsilon(t) \cdot \partial_h^t \dot{u}_\varepsilon(t)\, d\mathcal{H}^{n-1}\, dt\right)^{\frac{1}{2}} \\
&\leq c\left(\|\mathrm{div}(\sigma_0 + \varepsilon E v_0) + f(0)\|_{L^2(\Omega;\mathbb{R}^n)} + \|Ev_0\|_{L^2(\Omega;\mathbb{M}^n_{sym})} + \left\|\dot{f}\right\|_{L^2(\Omega;\mathbb{R}^n)}\right)
\end{aligned}$$

It follows from the fact that $\ddot{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$ and by the distributional equation of motion, i.e., $\ddot{u}_\varepsilon + \mathrm{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) = f$ in $\mathcal{D}'((0,T)\times\Omega)$ that

$$\mathrm{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$$

and thus

$$\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon \in L^2(0,T;H(\mathrm{div},\Omega)).$$

Hence, for every $\varphi \in C_0^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$ we see that

$$\begin{aligned}
&-\int_0^T \int_\Omega \dot{u}_\varepsilon \cdot \dot{\varphi}\, dx\, dt + \int_0^T \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\varphi\, dx\, dt + \int_0^T \int_{\partial\Omega} S\dot{u}_\varepsilon \cdot \varphi\, d\mathcal{H}^{n-1}\, dt \\
&= \int_0^T \int_\Omega f \cdot \varphi\, dx\, dt + \int_0^T \int_{\partial\Omega} \varepsilon E v_0 \nu \cdot \varphi\, d\mathcal{H}^{n-1}\, dt.
\end{aligned}$$

By the integration by parts, from the equality above, it yields that

$$\begin{aligned}
&\int_0^T \int_\Omega \ddot{u}_\varepsilon \cdot \varphi\, dx\, dt - \int_0^T \int_\Omega \mathrm{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) \cdot \varphi\, dx\, dt - \int_0^T \int_\Omega f \cdot \varphi\, dx\, dt \\
&= \int_0^T \langle \varepsilon E v_0 \nu - (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)\nu - S\dot{u}_\varepsilon | \varphi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}\, dt.
\end{aligned}$$

Thanks to the fact that $\ddot{u}_\varepsilon +$ $\mathrm{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) = f$ a.e. in $(0,T)\times\Omega$ we obtain that

$$\int_0^T \langle \varepsilon E v_0 \nu - (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)\nu - S\dot{u}_\varepsilon | \varphi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}\, dt = 0.$$

Hence, by density arguments we conclude that

$$\varepsilon E v_0 \nu - (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)\nu - S\dot{u}_\varepsilon = 0 \quad \text{in } L^2(0,T;H^{-1/2}(\partial\Omega))$$

which proves the assertion for the dissipative boundary condition in Theorem 5.3.2 and concludes the proof of Theorem 5.3.2.

Remark 5.3.6. Unfortunately, to prove Theorem 5.3.1 by means of an asymptotic analysis when $\varepsilon \to 0$ to the solutions provided in Theorem 5.3.2, it is needed that $g \equiv 0$. In other words, by assuming (H1)-(H6) we only can obtain homogeneous dissipative boundary conditions, more precisely, we have in Item 5 of Theorem 5.3.2 the following condition:

$$S\dot{u}_\varepsilon + (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)\nu = \varepsilon E v_0 \quad \text{in } L^2(0,T;L^2(\partial\Omega;\mathbb{R}^n)).$$

This is a consequence that in Theorem 5.3.2, the energy balance and the a posteriori estimates strongly depend on g. Another consequence of considering $g \neq 0$ is that we can not deduce the crucial strong convergences obtained in [11, Section 4.4] and thus Theorem 5.3.1 does not hold. That is the main reason that in Theorem 5.3.1 we consider only $(H5)$ and $(H6)$.

5.3.3. Derivation of mixed boundary condition

Our aim is to show through an asymptotic analysis how it is possible to obtain homogeneous mixed boundary conditions starting from dissipative boundary conditions. We consider a boundary matrix of the form

$$S(x) = S_\lambda(x) := \left(\lambda \mathbf{1}_{\Gamma_D}(x) + \frac{1}{\lambda}\mathbf{1}_{\Gamma_N}(x)\right) \mathrm{I}d, \quad \lambda > 0.$$

Remark 5.3.7. *Note that since*

$$\|\cdot\|_{S_\lambda(x)^{-1}} = \left(\lambda \mathbf{1}_{\Gamma_D}(x) + \frac{1}{\lambda}\mathbf{1}_{\Gamma_N}(x)\right)^{-1} |\cdot|,$$

for any $\lambda > 0$ and all $x \in \partial\Omega \setminus \Sigma$, the orthogonal projection $P_{-\mathbf{K}\nu(x)}$ onto the closed and convex set $-\mathbf{K}\nu(x)$ concerning the scalar product $\langle \cdot, \cdot \rangle_{S_\lambda(x)^{-1}}$ coincides with the orthogonal projection concerning the canonical Euclidean scalar product of $\mathbb{R}^n$. It is in particular independent of λ.

We will need to strengthen assumption (H_1) into

(H_1') **Reference configuration.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^3 boundary. We assume that $\partial\Omega$ is decomposed as the following disjoint union

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Sigma,$$

where Γ_D and Γ_N are open sets in the relative topology of $\partial\Omega$, and $\Sigma = \partial_{|\partial\Omega}\Gamma_D = \partial_{|\partial\Omega}\Gamma_N$ is a $(n-2)$-dimensional submanifold of class C^3.

Moreover, the initial condition needs to be adapted to our mixed boundary conditions.

(H_6') **The initial conditions.** Let $u_0 \in H^1(\Omega;\mathbb{R}^n)$, $v_0 \in H^2(\Omega;\mathbb{R}^n)$, $e_0 \in L^2(\Omega;\mathbb{M}^n_{sym})$, $p_0 \in L^2(\Omega;\mathbb{M}^n_{sym})$ and $\sigma_0 := \mathbf{A}e_0 \in H^2(\Omega;\mathbb{M}^n_{sym})$ be such that

$$\begin{cases} Eu_0 = e_0 + p_0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \Gamma_D, \\ \sigma_0\nu = 0 & \text{on } \Gamma_N, \\ \sigma_0 + B(0,r) \subset \mathbf{K} & \text{in } \Omega \text{ for some } r > 0. \end{cases}$$

First, we are going to construct a sequence of initial data $(u_0^\lambda, v_0^\lambda, e_0^\lambda, p_0^\lambda)$ satisfying (H_6) with $S = S_\lambda$, and approximating (u_0, v_0, e_0, p_0) as $\lambda \to \infty$. This is the object of the following result.

Lemma 5.3.8. *Let $n = 2$, 3. Under assumptions (H_1') and (H_6'), for every $\lambda > 0$, there exists $(v_0^\lambda, \sigma_0^\lambda) \in H^2(\Omega;\mathbb{R}^n) \times \mathcal{K}$ such that $(v_0^\lambda, \sigma_0^\lambda) \to (v_0, \sigma_0)$ strongly in $H^2(\Omega;\mathbb{R}^n) \times H(div,\Omega)$ as $\lambda \to \infty$ and*

$$\left(\lambda \mathbf{1}_{\Gamma_D} + \frac{1}{\lambda}\mathbf{1}_{\Gamma_N}\right) v_0^\lambda + \sigma_0^\lambda \nu = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega. \tag{5.3.51}$$

Proof. Since $\partial\Omega$ has a C^3 boundary then its normal ν belongs to $C^2(\partial\Omega;\mathbb{R}^n)$ and, thanks to the Trace Theorem in Sobolev spaces, the trace of σ_0 belongs to $H^{\frac{3}{2}}(\partial\Omega;\mathbb{M}^n_{sym})$. As a consequence, the product $\sigma_0\nu$ belongs to $H^{\frac{3}{2}}(\partial\Omega;\mathbb{R}^n)$ and there exists an extension $\hat{v}_0 \in H^2(\Omega;\mathbb{R}^n)$ whose trace on $\partial\Omega$ coincides with $-\sigma_0\nu$ with the estimate

$$\|\hat{v}_0\|_{H^2(\Omega;\mathbb{R}^n)} \leq C\|\sigma_0\nu\|_{H^{3/2}(\partial\Omega;\mathbb{R}^n)},$$

where $C > 0$ is a constant only depending on n and Ω. For each $\lambda > 0$, let us define

$$v_0^\lambda := v_0 + \lambda^{-1}\hat{v}_0 \in H^2(\Omega;\mathbb{R}^n).$$

It follows that $v_0^\lambda \to v_0$ strongly in $H^2(\Omega;\mathbb{R}^n)$ as $\lambda\to\infty$. Now, we consider $z_0 \in H^1(\Omega;\mathbb{R}^n)$ as the unique weak solution of the boundary value problem

$$\begin{cases} z_0 - \operatorname{div}(e(z_0)) = 0 & \textit{in } \Omega,\\ e(z_0)\nu = -v_0 & \textit{on } \partial\Omega. \end{cases} \tag{5.3.52}$$

According to Korn's inequality and the Lax-Milgram Lemma such a solution exists and is unique. Using that Ω has a C^3-boundary and that $v_0 \in H^{\frac{3}{2}}(\partial\Omega;\mathbb{R}^n)$, elliptic regularity ensures that $z_0 \in H^3(\Omega;\mathbb{R}^n)$. Let us define

$$\sigma_0^\lambda := \sigma_0 + \lambda^{-1} e(z_0)$$

In particular, $\sigma_0^\lambda \to \sigma_0$ strongly in $H(\operatorname{div},\Omega)$ as $\lambda\to\infty$. On Γ_D, we observe that

$$\lambda v^\lambda_{0|\Gamma_D} + \sigma_0^\lambda \nu_{|\Gamma_D} = \lambda v_{0|\Gamma_D} + \hat{v}_{0|\Gamma_D} + \sigma_0\nu_{|\Gamma_D} + \frac{1}{\lambda} e(z_0)\nu_{|\Gamma_D} = 0,$$

where we have used the fact that $e(z_0)\nu = -v_0 = 0$ and $\hat{v}_0 = -\sigma_0\nu$ on Γ_D. Similarly, on Γ_N we have

$$\frac{1}{\lambda} v^\lambda_{0|\Gamma_N} + \sigma_0^\lambda \nu_{|\Gamma_N} = \frac{1}{\lambda} v_{0|\Gamma_N} + \frac{1}{\lambda^2}\hat{v}_{0|\Gamma_N} + \sigma_0\nu_{|\Gamma_N} + \frac{1}{\lambda} e(z_0)\nu_{|\Gamma_N} = 0,$$

where we have used the fact that $\hat{v}_0 = -\sigma_0\nu = 0$ and $e(z_0)\nu = -v_0$ on Γ_N. We conclude (5.3.51) thanks to the fact that $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Sigma$ and $\mathcal{H}^{n-1}(\Sigma) = 0$.

It remains to check that $\sigma_0^\lambda \in \mathbf{K}$ a.e. in Ω. To this aim, we have by Sobolev embedding (recall that $n = 2$ or 3) that $e(z_0) \in H^2(\Omega;\mathbb{M}^n_{sym}) \subset L^\infty(\Omega;\mathbb{M}^n_{sym})$. Let $r > 0$ be the constant given by the last property of hypothesis (H_6') and $\lambda > 0$ large enough so that $\lambda^{-1}\|e(z_0)\|_{L^\infty(\Omega;\mathbb{M}^n_{sym})} < r$. It thus follows that $\sigma_0^\lambda \in \sigma_0 + B(0,r) \subset \mathbf{K}$ a.e. in Ω. □

Given the initial data $(u_0, v_0^\lambda, e_0^\lambda := \mathbf{A}^{-1}\sigma_0^\lambda, p_0^\lambda := Eu_0 - \mathbf{A}^{-1}\sigma_0^\lambda)$ satisfying (H_6), we denote by $(u_\lambda, e_\lambda, p_\lambda)$ the associated solution given by Theorem 5.3.1. Our aim is to study the asymptotic behavior of the solutions $(u_\lambda, e_\lambda, p_\lambda)$ when $\lambda\to\infty$ in order to recover Dirichlet ($\Gamma_N = \emptyset$), Neumann ($\Gamma_D = \emptyset$) and mixed boundary conditions in the other cases.

Our main result is the following:

Theorem 5.3.9. *Assume that (H_1'), (H_2), (H_3), (H_5) and (H_6') hold. For each $\lambda > 0$, let $(v_0^\lambda, \sigma_0^\lambda)$ be given by Lemma 5.3.8, and let $(u_\lambda, e_\lambda, p_\lambda)$ be the solution given by Theorem 5.3.1 associated with the boundary matrix S_λ defined in* (5.1.9) *and the initial data $(u_0, v_0^\lambda, e_0^\lambda := \mathbf{A}^{-1}\sigma_0^\lambda, p_0^\lambda := Eu_0 - \mathbf{A}^{-1}\sigma_0^\lambda)$. Then,*

$$\begin{cases} u_\lambda \rightharpoonup u & \textit{weakly* in } W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)),\\ e_\lambda \rightharpoonup e & \textit{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),\\ \sigma_\lambda \rightharpoonup \sigma & \textit{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),\\ p_\lambda(t) \rightharpoonup p(t) & \textit{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym}) \textit{ for all } t\in[0,T], \end{cases}$$

where (u,e,p) is the unique triple satisfying

$$\begin{cases} u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap C^{0,1}([0,T];BD(\Omega)),\\ e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),\\ \sigma := \mathbf{A}e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})) \cap L^\infty(0,T;H(\operatorname{div},\Omega)),\\ p \in C^{0,1}([0,T];\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^n_{sym})), \end{cases}$$

together with

5. Dissipative boundary conditions

1. *The initial conditions:*

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad e(0) = e_0, \quad p(0) = p_0;$$

2. *The kinematic compatibility: for all* $t \in [0, T]$,

$$\begin{cases} Eu(t) = e(t) + p(t) & \text{in } \Omega, \\ p(t) = -u(t) \odot \nu \mathcal{H}^{n-1} & \text{on } \Gamma_D; \end{cases}$$

3. *The equation of motion:*

$$\ddot{u} - \operatorname{div}\sigma = f \quad \text{in } L^2(0, T; L^2(\Omega; \mathbb{R}^n));$$

4. *The stress constraint: for every* $t \in [0, T]$,

$$\sigma(t) \in \mathbf{K} \quad a.e \quad in \quad \Omega;$$

5. *The boundary condition*

$$\sigma\nu = 0 \quad in \quad L^2(0, T; L^2(\Gamma_N; \mathbb{R}^n));$$

6. *The flow rule: if one of the following conditions are satisfied:*

   (i) Dirichlet case: $\Omega = \Gamma_D$,

   (ii) Neumann case: $\Omega = \Gamma_N$,

   (iii) Mixed case in dimension $n = 2$: $\Gamma_D \neq \emptyset$, $\Gamma_N \neq \emptyset$ *and* Σ *finite,*

   (iv) Mixed case in dimension $n = 3$: $\Gamma_D \neq \emptyset$, $\Gamma_N \neq \emptyset$ *and*

$$\mathbf{K} = K_D \oplus (\mathbb{R}Id) := \{\sigma \in \mathbb{M}^3_{sym} : \sigma_D \in K_D\},$$

   for some compact and convex set $K_D \subset \mathbb{M}^3_D$ *containing zero in its interior,*

   then, for a.e. $t \in [0, T]$,

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad in \quad \mathcal{M}(\Omega \cup \Gamma_D).$$

As explained before, the solution (u, e, p) to the previous boundary value problem will be obtained by means of an asymptotic analysis as $\lambda \to \infty$ of the solution $(u_\lambda, e_\lambda, p_\lambda)$ of the dissipative boundary value in the Theorem 5.3.1. This analysis is based in the spirit of [13, Theorem 5.1] in the antiplane case.

5.3.4. Weak compactness and passing to the limit into linear equations

We observe that the constant $C_* > 0$ appearing in estimate (5.3.3) of Theorem 5.3.1 depends on the various norms $\|u_0\|_{H^1(\Omega;\mathbb{R}^n)}$, $\|v_0^\lambda\|_{H^2(\Omega;\mathbb{R}^n)}$, $\|e_0^\lambda\|_{L^2(\Omega;\mathbb{M}^n_{sym})}$, $\|\sigma_0^\lambda\|_{H(\text{div},\Omega)}$ and $\|p_0^\lambda\|_{L^2(\Omega;\mathbb{M}^n_{sym})}$ of the initial data. Since, by Lemma 5.3.8, these quantities are independent of λ, it follows that the constant C_* is independent of λ as well. This is essential to get uniform bounds on the sequence $\{(u_\lambda, e_\lambda, p_\lambda)\}_{\lambda>0}$ and then weak compactness thereof.

The following compactness result follows from standard argument as, e.g., in [13, Section 5]. The weak convergences allow us to obtain, in the limit, the initial conditions, the kinetic compatibility, the equation of motion and the stress constraint,

Lemma 5.3.10. *Assume that (H_1'), (H_2), (H_3), (H_5) and (H_6') hold. There exist a subsequence (not relabeled) and*

$$\begin{cases} u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap C^{0,1}([0,T];BD(\Omega)), \\ e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \\ \sigma \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})) \cap L^\infty(0,T;H(\mathrm{div},\Omega)), \\ p \in C^{0,1}([0,T];\mathcal{M}(\Omega;\mathbb{M}^n_{sym})), \end{cases}$$

such that as $\lambda \to \infty$,

$$\begin{cases} u_\lambda \rightharpoonup u & \text{weakly* in } W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), \\ e_\lambda \rightharpoonup e & \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \\ \sigma_\lambda \rightharpoonup \sigma & \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})), \end{cases}$$

and, for every $t \in [0,T]$,

$$\begin{cases} u_\lambda(t) \rightharpoonup u(t) & \text{weakly in } L^2(\Omega;\mathbb{R}^n), \\ u_\lambda(t) \rightharpoonup u(t) & \text{weakly* in } BD(\Omega), \\ \dot u_\lambda(t) \rightharpoonup \dot u(t) & \text{weakly in } L^2(\Omega;\mathbb{R}^n), \\ e_\lambda(t) \rightharpoonup e(t) & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{sym}), \\ \sigma_\lambda(t) \rightharpoonup \sigma(t) & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{sym}), \\ p_\lambda(t) \rightharpoonup p(t) & \text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym}). \end{cases}$$

Moreover, there hold:

- *the initial conditions: $u(0) = u_0$, $\dot u(0) = v_0$, $e(0) = e_0$, $p(0) = p_0$;*
- *the additive decomposition: for all $t \in [0,T]$,*

$$Eu(t) = e(t) + p(t) \quad \text{in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym});$$

- *the equation of motion: $\ddot u - \mathrm{div}\sigma = f$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$;*
- *the stress constraint: for every $t \in [0,T]$, $\sigma(t) = \mathbf{A}e(t) \in \mathbf{K}$ a.e in Ω;*
- *the Neumann condition: $\sigma\nu = 0$ in $L^2(0,T;L^2(\Gamma_N;\acute{\mathrm{L}}\mathbb{R}^n))$.*

Proof. According to the energy balance (5.3.2) and estimate (5.3.3), we infer that

$$\begin{aligned} \|\dot u_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\sigma_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} + \|\dot p_\lambda\|_{L^1(0,T;\mathcal{M}(\Omega;\mathbb{M}^n_{sym}))} \\ + \frac{1}{\sqrt{\lambda}}\|\sigma_\lambda\nu\|_{L^2(0,T;L^2(\Gamma_D;\mathbb{R}^n))} + \sqrt{\lambda}\|\sigma_\lambda\nu\|_{L^2(0,T;L^2(\Gamma_N;\mathbb{R}^n))} \\ + \int_0^T \int_{\partial\Omega} \psi_\lambda(x,\dot u_\lambda)\, d\mathcal{H}^{n-1}\, ds \le C, \end{aligned} \tag{5.3.53}$$

where ψ_λ is given by (5.3.1) with $S = S_\lambda$, and

$$\|\ddot u_\lambda\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot e_\lambda\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} \le C_*.$$

In both previous estimates, the constants $C > 0$ and $C_* > 0$ are independent of λ. Using that $u_\lambda \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))$ and $u_0 \in L^2(\Omega;\mathbb{R}^n)$, we get

$$\sup_{\lambda>0} \|u_\lambda\|_{W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < \infty,$$

and similarly, since $e_\lambda \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ and $e_0 \in L^2(\Omega;\mathbb{M}^n_{sym})$,

$$\sup_{\lambda>0} \|e_\lambda\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))} < \infty.$$

We can thus extract a subsequence (not relabeled) and find $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))$ and $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ such that, as $\lambda \to \infty$,

$$\begin{cases} u_\lambda \rightharpoonup u & \text{weakly* in } W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), \\ e_\lambda \rightharpoonup e & \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})). \end{cases}$$

Setting $\sigma := \mathbf{A}e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$ we also have

$$\sigma_\lambda \rightharpoonup \sigma \quad \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^n_{sym})),$$

and using the equation of motion leads to

$$\mathrm{div}\sigma_\lambda = \ddot{u}_\lambda - f \rightharpoonup \ddot{u} - f \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)).$$

By uniqueness of the distributional limit, we infer that $\mathrm{d}iv\sigma = \ddot{u} - f \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$ and, thus, $\sigma \in L^\infty(0,T;H(\mathrm{div},\Omega))$.

Owing to Ascoli-Arzela Theorem, for every $t \in [0,T]$,

$$\begin{cases} u_\lambda(t) \rightharpoonup u(t) & \text{weakly in } L^2(\Omega;\mathbb{R}^n), \\ \dot{u}_\lambda(t) \rightharpoonup \dot{u}(t) & \text{weakly in } L^2(\Omega;\mathbb{R}^n), \\ e_\lambda(t) \rightharpoonup e(t) & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{sym}), \\ \sigma_\lambda(t) \rightharpoonup \sigma(t) & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{sym}). \end{cases}$$

We now derive weak compactness on the sequence $\{p_\lambda\}_{\lambda>0}$ of plastic strains. Thanks to the energy balance between two arbitrary times $0 \le t_1 \le t_2 \le T$ together with (5.2.2),

$$\begin{aligned} r\int_{t_1}^{t_2} |\dot{p}_\lambda(s)|(\Omega)\,ds \le \int_{t_1}^{t_2} H(\dot{p}_\lambda(s))(\Omega)\,ds \le \quad & \frac{1}{2}\int_\Omega (\dot{u}_\lambda(t_1) - \dot{u}_\lambda(t_2)) \cdot (\dot{u}_\lambda(t_1) + \dot{u}_\lambda(t_2))\,dx \\ & + \frac{1}{2}\int_\Omega (\sigma_\lambda(t_1) - \sigma_\lambda(t_2)) : (e_\lambda(t_1) + e_\lambda(t_2))\,dx \\ & + \int_{t_1}^{t_2}\int_\Omega f \cdot \dot{u}_\lambda\,dx\,ds. \end{aligned} \tag{5.3.54}$$

By (H_5), using that $f \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$, that $\{\dot{u}_\lambda\}_{\lambda>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$ and that $\{\sigma_\lambda\}_{\lambda>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_{sym}))$, we can find a constant $C > 0$ independent of λ such that

$$|p_\lambda(t_1) - p_\lambda(t_2)|(\Omega) \le \int_{t_1}^{t_2} |\dot{p}_\lambda(s)|(\Omega)ds \le C(t_2 - t_1).$$

Applying Ascoli-Arzela Theorem, we extract a further subsequence (independent of time) and find $p \in C^{0,1}([0,T];\mathcal{M}(\Omega;\mathbb{M}^n_{sym}))$ such that for all $t \in [0,T]$,

$$p_\lambda(t) \rightharpoonup p(t) \quad \text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym}).$$

Using the additive decomposition $Eu_\lambda = e_\lambda + p_\lambda$ in Ω, the previously established weak convergences show that $u \in C^{0,1}([0,T];BD(\Omega))$ and, for all $t \in [0,T]$,

$$u_\lambda(t) \rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega).$$

It is now possible to pass to the limit in the initial condition

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad e(0) = e_0, \quad p(0) = p_0,$$

in the additive decomposition: for all $t \in [0,T]$,

$$Eu(t) = e(t) + p(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{M}^n_{sym}),$$

and in the equation of motion

$$\ddot{u} - \text{div}\sigma = f \quad \text{ in } L^2(0,T; L^2(\Omega;\mathbb{R}^n)).$$

The stress constraint being convex, hence closed under weak $L^2(\Omega;\mathbb{M}^n_{sym})$ convergence, we further obtain that for every $t \in [0,T]$, $\sigma(t) \in \mathbf{K}$ a.e in Ω.

It remains to show the Neumann boundary condition $\sigma\nu = 0$ on Γ_N. Since $\sigma_\lambda \rightharpoonup \sigma$ weakly in $L^2(0,T; H(\text{d}iv, \Omega))$, we deduce that $\sigma_\lambda \nu \rightharpoonup \sigma\nu$ weakly in $L^2(0,T; H^{-1/2}(\partial\Omega;\mathbb{R}^n))$. On the other hand, using estimate (5.3.53), we have

$$\|\sigma_\lambda \nu\|_{L^2(0,T;L^2(\Gamma_N;\mathbb{R}^n))} \leq \frac{C}{\sqrt{\lambda}} \to 0,$$

as $\lambda \to \infty$, hence $\sigma\nu = 0$ in $L^2(0,T; L^2(\Gamma_N;\mathbb{R}^n))$. □

5.3.5. Flow rule

It remains to prove the flow rule, which will be performed by passing to the limit in the energy balance obtained in the Theorem 5.3.1, namely, for all $t \in [0,T]$,

$$\frac{1}{2}\int_\Omega |\dot{u}_\lambda(t)|^2\,dx + \int_\Omega Q(e_\lambda(t))\,dx + \int_0^t H(\dot{p}_\lambda(s))(\Omega)\,ds + \int_0^t \int_{\partial\Omega} \psi_\lambda(x, \dot{u}_\lambda)\,d\mathcal{H}^{n-1}\,ds$$
$$\leq \frac{1}{2}\int_\Omega |v_0|^2\,dx + \int_\Omega Q(e_0)\,dx + \int_0^t \int_\Omega f\cdot \dot{u}_\lambda\,dx\,ds. \quad (5.3.55)$$

The first two terms will easily pass to the lower limit by lower semicontinuity of the norm with respect to weak L^2-convergence. The main issue is to pass to the (lower) limit in both last terms in the left-hand side of the previous inequality. The following result will enable one to obtain a lower bound.

Lemma 5.3.11. *Let* $\{(\hat{u}_\lambda, \hat{e}_\lambda, \hat{p}_\lambda)\}_{\lambda>0} \subset [BD(\Omega) \cap L^2(\Omega;\mathbb{R}^n)] \times L^2(\Omega;\mathbb{M}^n_{sym}) \times \mathcal{M}(\Omega;\mathbb{M}^n_{sym})$ *be such that* $E\hat{u}_\lambda = \hat{e}_\lambda + \hat{p}_\lambda$ *in* Ω, *and*

$$\begin{cases} \hat{u}_\lambda \rightharpoonup \hat{u} & \text{weakly in } L^2(\Omega;\mathbb{R}^n), \\ \hat{u}_\lambda \rightharpoonup \hat{u} & \text{weakly* in } BD(\Omega), \\ \hat{e}_\lambda \rightharpoonup \hat{e} & \text{weakly in } L^2(\Omega;\mathbb{M}^n_{sym}), \\ \hat{p}_\lambda \rightharpoonup \hat{p} & \text{weakly* in } \mathcal{M}(\Omega;\mathbb{M}^n_{sym}), \end{cases}$$

as $\lambda \to \infty$, *for some* $(\hat{u}, \hat{e}, \hat{p}) \in [BD(\Omega) \cap L^2(\Omega;\mathbb{R}^n)] \times L^2(\Omega;\mathbb{M}^n_{sym}) \times \mathcal{M}(\Omega;\mathbb{M}^n_{sym})$. *Then,*

$$H(\hat{p})(\Omega) + \int_{\Gamma_D} H(-\hat{u}\odot\nu)\,d\mathcal{H}^{n-1} \leq \liminf_{\lambda\to\infty}\left(H(\hat{p}_\lambda)(\Omega) + \int_{\partial\Omega}\psi_\lambda(x,\hat{u}_\lambda)\,d\mathcal{H}^{n-1}\right). \quad (5.3.56)$$

Proof. Without loss of generality, we assume that the right hand side of (5.3.56) is finite. Let $(\lambda_k)_{k\in\mathbb{N}}$ be such that $\lambda_k \nearrow \infty$ and

$$\liminf_{\lambda\to\infty}\left(H(\hat{p}_\lambda)(\Omega) + \int_{\partial\Omega}\psi_\lambda(x,\hat{u}_\lambda)\,d\mathcal{H}^{n-1}\right) = \lim_{k\to\infty}\left(H(\hat{p}_{\lambda_k})(\Omega) + \int_{\partial\Omega}\psi_{\lambda_k}(x,\hat{u}_{\lambda_k})\,d\mathcal{H}^{n-1}\right).$$

As a consequence, there exists a constant $c > 0$ (independent of k) such that

$$\int_{\partial\Omega} \psi_{\lambda_k}(x, \hat{u}_{\lambda_k})\, d\mathcal{H}^{n-1} \leq c$$

for all $k \in \mathbb{N}$. By definition (5.3.1) of ψ_λ (see also [11, Lemma 4.9]), there exists a function $v_k \in L^2(\partial\Omega; \mathbb{R}^n)$ such that

$$\begin{aligned}\int_{\partial\Omega} \psi_{\lambda_k}(x, \hat{u}_{\lambda_k})\, d\mathcal{H}^{n-1} &= \frac{1}{2}\int_{\partial\Omega} S_{\lambda_k}(\hat{u}_{\lambda_k} - v_k) \cdot (\hat{u}_{\lambda_k} - v_k)\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} H(-v_k \odot \nu)\, d\mathcal{H}^{n-1} \\ &\geq \frac{\lambda_k}{2}\int_{\Gamma_D} |\hat{u}_{\lambda_k} - v_k|^2\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} H(-v_k \odot \nu)\, d\mathcal{H}^{n-1}.\end{aligned}$$

By nonnegativity of H, we infer that $\hat{u}_{\lambda_k} - v_k \to 0$ in $L^2(\Gamma_D; \mathbb{R}^n)$ as $k \to \infty$. Moreover

$$\begin{aligned}&H(\hat{p}_{\lambda_k})(\Omega) + \int_{\partial\Omega} \psi_{\lambda_k}(x, \hat{u}_{\lambda_k})\, d\mathcal{H}^{n-1} \\ &\quad\geq H(\hat{p}_{\lambda_k})(\Omega) + \int_{\Gamma_D} H(-v_k \odot \nu)\, d\mathcal{H}^{n-1} \\ &\quad\geq H_\mu(\hat{p}_{\lambda_k})(\Omega) + \int_{\Gamma_D} H_\mu(-v_k \odot \nu)\, d\mathcal{H}^{n-1}, \end{aligned} \tag{5.3.57}$$

where $H_\mu : \mathbb{M}^n_{sym} \to \mathbb{R}^+$ is the Moreau–Yosida transform of H (see [4, Lemma 1.61] or [47, Lemma 5.30]), defined by

$$H_\mu(p) := \inf_{q \in \mathbb{M}^n_{sym}} \{H(q) + \mu|p - q|\} \qquad \text{for all } p \in \mathbb{M}^n_{sym}.$$

We recall that H_μ of H enjoys the following properties:

1. For all $\mu > 0$ we have that $H_\mu \leq H$;
2. The function H_μ is μ-Lipschitz;
3. The function H_μ is convex as the inf-convolution between the proper convex functions H and $\mu|\cdot|$ (see e.g. [75, Theorem 5.4]);
4. For all $p \in \mathbb{M}^n_{sym}$, $H_\mu(p) \to H(p)$ as $\mu \to \infty$.

By the μ-Lipschitz continuity of H_μ, adding and subtracting the term $\int_{\Gamma_D} H_\mu(-\hat{u}_{\lambda_k} \odot \nu)\, d\mathcal{H}^{n-1}$ in (5.3.57) yields

$$\begin{aligned}&H(\hat{p}_{\lambda_k})(\Omega) + \int_{\partial\Omega} \psi_{\lambda_k}(x, \hat{u}_{\lambda_k})\, d\mathcal{H}^{n-1} \\ &\quad\geq H_\mu(\hat{p}_{\lambda_k})(\Omega) + \int_{\Gamma_D} H_\mu(-\hat{u}_{\lambda_k} \odot \nu)\, d\mathcal{H}^{n-1} - \mu \int_{\Gamma_D} |\hat{u}_{\lambda_k} - v_k|\, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.3.58}$$

Passing to the limit as $k \to \infty$ in (5.3.58), we obtain

$$\begin{aligned}&\lim_{k\to\infty} \left(H(\hat{p}_{\lambda_k})(\Omega) + \int_{\partial\Omega} \psi_{\lambda_k}(x, \hat{u}_{\lambda_k})\, d\mathcal{H}^{n-1} \right) \\ &\quad\geq \liminf_{k\to\infty} \left(H_\mu(\hat{p}_{\lambda_k})(\Omega) + \int_{\Gamma_D} H_\mu(-\hat{u}_{\lambda_k} \odot \nu)\, d\mathcal{H}^{n-1} \right). \end{aligned} \tag{5.3.59}$$

Let $U \subset \mathbb{R}^N$ be an open set such that $\Gamma_D = U \cap \partial\Omega$, and let $\tilde{\Omega} := \Omega \cup U$. We extend $(\hat{u}_\lambda, \hat{e}_\lambda, \hat{p}_\lambda)$ to $\tilde{\Omega}$ as

$$\tilde{u}_\lambda := \begin{cases} \hat{u}_\lambda & \text{in } \Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases} \qquad \tilde{e}_\lambda := \begin{cases} \hat{e}_\lambda & \text{in } \Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases}$$

and

$$\tilde{p}_\lambda := E\tilde{u}_\lambda - \tilde{e}_\lambda = \hat{p}_\lambda \llcorner \Omega - \hat{u}_\lambda \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D.$$

Similarly, we set

$$\tilde{u} := \begin{cases} \hat{u} & \text{in } \Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases} \qquad \tilde{e} := \begin{cases} \hat{e} & \text{in } \Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega. \end{cases}$$

Note that $\tilde{p}_\lambda \rightharpoonup \tilde{p}$ weakly* in $\mathcal{M}(\tilde{\Omega}; \mathbb{M}^n_{sym})$ with $\tilde{p} = E\tilde{u} - \tilde{e} = \hat{p} \llcorner \Omega - \hat{u} \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D$. Using that H_μ is a continuous, convex and positively one homogeneous function with $H_\mu(0) = 0$, we can apply Reshetnyak's lower semicontinuity Theorem (see [4, Theorem 2.38]) to get that

$$\begin{aligned} \liminf_{k\to\infty} \left(H_\mu(\hat{p}_{\lambda_k})(\Omega) + \int_{\Gamma_D} H_\mu(-\hat{u}_{\lambda_k} \odot \nu)\, d\mathcal{H}^{n-1} \right) \\ = \liminf_{k\to\infty} H_\mu(\tilde{p}_{\lambda_k})(\tilde{\Omega}) \geq H_\mu(\tilde{p})(\tilde{\Omega}) \\ = H_\mu(\hat{p})(\Omega) + \int_{\Gamma_D} H_\mu(-\hat{u} \odot \nu)\, d\mathcal{H}^{n-1}. \end{aligned}$$

We have thus established that for all $\mu > 0$,

$$\liminf_{\lambda\to\infty} \left(H(\hat{p}_\lambda)(\Omega) + \int_{\partial\Omega} \psi_\lambda(x, \hat{u}_\lambda)\, d\mathcal{H}^{n-1} \right) \geq H_\mu(\hat{p})(\Omega) + \int_{\Gamma_D} H_\mu(-\hat{u} \odot \nu)\, d\mathcal{H}^{n-1}.$$

We can now pass to the limit as $\mu \to \infty$ owing to the Monotone Convergence theorem to get that

$$\liminf_{\lambda\to\infty} \left(H(\hat{p}_\lambda)(\Omega) + \int_{\partial\Omega} \psi_\lambda(x, \hat{u}_\lambda)\, d\mathcal{H}^{n-1} \right) \geq H(\hat{p})(\Omega) + \int_{\Gamma_D} H(-\hat{u} \odot \nu)\, d\mathcal{H}^{n-1},$$

which leads to the desired lower bound. □

We are now in position to prove a lower bound energy inequality. Since for all $t \in [0, T]$, we have $\dot{u}_\lambda(t) \rightharpoonup \dot{u}(t)$ weakly in $L^2(\Omega; \mathbb{R}^n)$ and $e_\lambda(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^n_{sym})$, we get by weak lower semicontinuity of the norm that

$$\frac{1}{2} \int_\Omega |\dot{u}(t)|^2\, dx + \mathcal{Q}(e(t)) \leq \liminf_{\lambda\to\infty} \left\{ \frac{1}{2} \int_\Omega |\dot{u}_\lambda(t)|^2\, dx + \mathcal{Q}(e_\lambda(t)) \right\}.$$

To pass to the lower limit in the remaining terms in the left-hand side of the energy inequality (5.3.55), we consider a partition $0 = t_0 \leq t_1 \leq \ldots \leq t_N = t$ of the time interval $[0, t]$. By convexity of H and $\psi_\lambda(x, \cdot)$, we infer from Jensen's inequality that

$$\begin{aligned} \int_0^t H(\dot{p}_\lambda(s))(\Omega)\, ds + \int_0^t \int_{\partial\Omega} \psi_\lambda(x, \dot{u}_\lambda(s))\, d\mathcal{H}^{n-1}\, ds \\ \geq \sum_{i=1}^N \left\{ H(p_\lambda(t_i) - p_\lambda(t_{i-1}))(\Omega) + \int_{\partial\Omega} \psi_\lambda(x, u_\lambda(t_i) - u_\lambda(t_{i-1}))\, d\mathcal{H}^{n-1} \right\}. \end{aligned}$$

Since, for all $0 \leq i \leq N$ we have that

$$\begin{cases} u_\lambda(t_i) \rightharpoonup u(t_i) & \text{weakly in } L^2(\Omega; \mathbb{R}^n), \\ u_\lambda(t_i) \rightharpoonup u(t_i) & \text{weakly* in } BD(\Omega), \\ e_\lambda(t_i) \rightharpoonup e(t_i) & \text{weakly in } L^2(\Omega; \mathbb{M}^n_{sym}), \\ p_\lambda(t_i) \rightharpoonup p(t_i) & \text{weakly* in } \mathcal{M}(\Omega; \mathbb{M}^n_{sym}), \end{cases}$$

we can apply Proposition 5.3.11 to get that

$$\liminf_{\lambda\to\infty} \left(\int_0^t H(\dot{p}_\lambda(s))(\Omega)\, ds + \int_0^t \int_{\partial\Omega} \psi_\lambda(x, \dot{u}_\lambda)\, d\mathcal{H}^{n-1}\, ds \right) \geq \sum_{i=1}^N H(p(t_i) - p(t_{i-1}))(\Omega \cup \Gamma_D),$$

where the measure $p(t)$ is extended to Γ_D by setting

$$p(t) \llcorner \Gamma_D := -u(t) \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D.$$

Passing to the supremum with respect to all partitions, we deduce that

$$\mathcal{V}_{\mathcal{H}}(p; 0, t) := \sup \left\{ \sum_{i=1}^{N} H(p(t_i) - p(t_{i-1}))(\Omega \cup \Gamma_D) : \ 0 = t_0 \leq t_1 \leq \cdot \leq t_N = t, \, N \in \mathbb{N} \right\} < \infty.$$

Using [32, Theorem 7.1][1], we get that

$$\liminf_{\lambda \to \infty} \left(\int_0^t H(\dot{p}_\lambda(s))(\Omega) \, ds + \int_0^t \int_{\partial\Omega} \psi_\lambda(x, \dot{u}_\lambda) \, d\mathcal{H}^{n-1} \, ds \right) \geq \int_0^t H(\dot{p}(s))(\Omega \cup \Gamma_D) \, ds.$$

Passing to the lower limit in (5.3.55) as $\lambda \to \infty$ yields

$$\frac{1}{2} \int_\Omega |\dot{u}(t)|^2 \, dx + \mathcal{Q}(e(t)) + \int_0^t H(\dot{p}(s))(\Omega \cup \Gamma_D) \, ds \\ \leq \frac{1}{2} \int_\Omega |v_0|^2 \, dx + \mathcal{Q}(e_0) + \int_0^t \int_\Omega f \cdot \dot{u} \, dx \, ds. \quad (5.3.60)$$

The proof of the other energy inequality relies on the convexity inequality proved in Section 5.2. Indeed, assuming one of the following assumptions:

- $\partial\Omega = \Gamma_D$;
- $\partial\Omega = \Gamma_N$;
- $n = 2$ and Σ is a finite set;
- $n = 3$ and $\mathbf{K} = K_D \oplus (\mathbb{R}\, \mathrm{I}d)$, for some compact and convex set $K_D \subset \mathbb{M}^3_D$ containing 0 in its interior;

we can appeal Proposition 5.2.4, Proposition 5.2.6 or Proposition 5.2.7. Indeed, for a.e. $t \in [0, T]$, we have $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_0$, $\sigma(t) \in \mathcal{K} \cap \mathcal{S}_0$ and $H(\dot{p}(t))$ is a finite measure (by (5.3.60)). As a consequence, for a.e. $t \in [0, T]$, the duality pairing $[\sigma(t) \colon \dot{p}(t)] \in \mathcal{D}'(\mathbb{R}^n)$ is well defined and it extends to a bounded Radon measure supported in $\overline{\Omega}$ with

$$H(\dot{p}(t)) \geq [\sigma(t) \colon \dot{p}(t)] \quad \text{in } \mathcal{M}(\mathbb{R}^n) \,. \quad (5.3.61)$$

Since the nonnegative measure $H(\dot{p}(t)) - [\sigma(t) \colon \dot{p}(t)]$ is compactly supported in $\overline{\Omega}$, we can evaluate its mass by taking the test function $\varphi \equiv 1$ in Definition 5.2.2. We then obtain that for a.e. $t \in [0, T]$,

$$0 \leq H(\dot{p}(t))(\Omega \cup \Gamma_D) + \int_\Omega \sigma(t) : \dot{e}(t) \, dx + \int_\Omega \dot{u}(t) \cdot \mathrm{div}\sigma(t) \, dx.$$

Using the equation of motion and the regularity properties of $\dot{u}$ and e, we can integrate by parts respect to time and get that

$$0 \leq \int_0^t H(\dot{p}(s))(\Omega \cup \Gamma_D)) \, ds + \mathcal{Q}(e(t)) - \mathcal{Q}(e_0) \\ + \frac{1}{2} \int_\Omega |\dot{u}(t)|^2 \, dx - \frac{1}{2} \int_\Omega |v_0|^2 \, dx - \int_0^t \int_\Omega f \cdot u \, dx \, ds.$$

[1] Note that [32, Theorem 7.1] is stated for functions H which are bounded from above, which is not our case here because H is allowed to take the value $+\infty$. However, a careful inspection of the proof of [32, Theorem 7.1] shows the validity of this result in our case thanks to the additional property $\mathcal{V}_{\mathcal{H}}(p; 0, t) < \infty$.

Owing to the first energy inequality (5.3.60), we deduce that the last expression is zero, which implies that the nonnegative measure $H(\dot p(t)) - [\sigma(t) \colon \dot p(t)]$ has zero mass in $\overline{\Omega}$. This leads in turn that this measure vanishes in $\overline{\Omega}$, in other words the flow rule $H(\dot p(t)) = [\sigma(t) \colon \dot p(t)]$ in $\mathcal{M}(\overline{\Omega})$ is satisfied. Finally, since $H(\dot p(t))$ is concentrated on $\Omega \cup \Gamma_D$, it follows that $[\sigma(t) : \dot p(t)]$ vanishes on $\partial\Omega \setminus \Gamma_D$ and that the flow rule $H(\dot p(t)) = [\sigma(t) : \dot p(t)]$ holds in $\mathcal{M}(\Omega \cup \Gamma_D)$.

5.3.6. Uniqueness

Let (u_1, e_1, p_1) and (u_2, e_2, p_2) be two solutions given by Theorem 5.3.9. Subtracting the equations of motion of each solution, we have

$$\ddot u_1 - \ddot u_2 - \mathrm{d}iv(\sigma_1 - \sigma_2) = 0 \quad \text{in } L^2(0,T; L^2(\Omega;\mathbb{R}^n)).$$

Let us consider the test function $\varphi := \mathbf{1}_{[0,t]}(\dot u_1 - \dot u_2) \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, we deduce

$$\int_0^t \int_\Omega (\ddot u_1 - \ddot u_2) : (\dot u_1 - \dot u_2)\, dx\, ds - \int_0^t \int_\Omega (\mathrm{d}iv(\sigma_1 - \sigma_2)) \cdot (\dot u_1 - \dot u_2)\, dx\, ds = 0. \tag{5.3.62}$$

Since $\ddot u_1 - \ddot u_2 \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ and $\dot u_1(0) = \dot u_2(0) = v_0$, we infer that

$$\int_0^t \int_\Omega (\ddot u_1(s) - \ddot u_2(s)) : (\dot u_1(s) - \dot u_2(s))\, dx\, ds = \frac{\|\dot u_1(t) - \dot u_2(t)\|^2_{L^2(\Omega;\mathbb{R}^n)}}{2}. \tag{5.3.63}$$

We already know that, for a.e. $s \in [0,T]$, the distributions $[\sigma_1(s) : \dot p_1(s)]$ and $[\sigma_2(s) : \dot p_2(s)]$ belong to $\mathcal{M}(\Omega \cup \Gamma_D)$. Moreover, since $(\dot u_1(s), \dot e_1(s), \dot p_1(s))$, $(\dot u_2(s), \dot e_2(s), \dot p_2(s)) \in \mathcal{A}_0$, $\sigma_1(s)$, $\sigma_2(s) \in \mathcal{S}_0 \cap \mathcal{K}$ and $H(\dot p_1(s))$, $H(\dot p_2(s))$ are finite measures we can appeal Propositions 5.2.4, 5.2.6 and 5.2.7 which state that $[\sigma_2(t) \colon \dot p_1(s)]$ and $[\sigma_1(s) \colon \dot p_2(s)]$ extend to bounded Radon measures supported in $\overline{\Omega}$ with

$$[\sigma_1(s) : \dot p_1(s)] = H(\dot p_1(s)) \geq [\sigma_2(s) : \dot p_1(s)] \quad \text{in } \mathcal{M}(\mathbb{R}^n),$$

and

$$[\sigma_2(s) : \dot p_2(s)] = H(\dot p_2(s)) \geq [\sigma_1(s) : \dot p_2(s)] \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

As a consequence, the measure $[(\sigma_1(s) - \sigma_2(s)) : (\dot p_1(s) - \dot p_2(s))]$ is nonnegative. Furthermore, by the definition of stress duality (see Definition 5.2.2 with the test function $\varphi \equiv 1$ and $g = 0$), we infer that

$$\begin{aligned}
0 \ &\leq \ \int_0^t [(\sigma_1(s) - \sigma_2(s)) : (\dot p_1(s) - \dot p_2(s))](\overline{\Omega}) \\
&= \ -\int_0^t \int_\Omega (\sigma_1(s) - \sigma_2(s)) : (\dot e_1(s) - \dot e_2(s))\, dx\, ds \\
&\qquad\qquad - \int_0^t \int_\Omega (\mathrm{d}iv(\sigma_1(s) - \sigma_2(s))) \cdot (\dot u_1(s) - \dot u_2(s))\, dx\, ds \\
&= \ -\mathcal{Q}(e_1(t) - e_2(t)) - \int_0^t \int_\Omega (\mathrm{d}iv(\sigma_1(s) - \sigma_2(s))) \cdot (\dot u_1(s) - \dot u_2(s))\, dx\, ds,
\end{aligned} \tag{5.3.64}$$

where we have used the fact that $e_1(0) = e_2(0) = e_0$. By (5.3.62), (5.3.63) and (5.3.64), we infer that

$$\frac{\|\dot u_1(t) - \dot u_2(t)\|^2_{L^2(\Omega;\mathbb{R}^n)}}{2} + \mathcal{Q}(e_1(t) - e_2(t)) \leq 0.$$

From the expression above, we infer that $e_1 = e_2$ and $\dot u_1 = \dot u_2$. Since, $u_1(0) = u_2(0) = u_0$, we conclude that $u_1 = u_2$, and by the kinematic compatibility $p_1 = p_2$. This concludes the proof of the uniqueness. In particular, by uniqueness of the limit, there is no need of extracting subsequences when passing to the limit as $\lambda \to \infty$. The proof of Theorem 5.3.9 is now complete.

Bibliography

[1] F. J. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4-165** (1976), viii+199.

[2] L. Ambrosio and A. Braides, *Functionals defined on partitions in sets of finite perimeter I: integral*, J. Math. Pures Appl. **69** (1990), 285–305.

[3] ———, *Functionals defined on partitions in sets of finite perimeter II: semicontinuity*, J. Math. Pures Appl. **69** (1990), 307–333.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York, 2000.

[5] G. Anzellotti, *On the existence of the rates of stress and displacement for Prandtl-Reuss plasticity*, Quart. Appl. Math. **41** (1983/84), no. 2, 181–208.

[6] ———, *On the extremal stress and displacement in Hencky plasticity*, Duke Math. J. **51** (1984), no. 1, 133–147.

[7] G. Anzellotti and M. Giaquinta, *Existence of the displacement field for an elastoplastic body subject to Hencky's law and von Mises yield condition*, Manuscripta Math. **32** (1980), no. 1-2, 101–136.

[8] G. Anzellotti and S. Luckhaus, *Dynamical evolution of elasto-perfectly plastic bodies*, Appl. Math. Optim. **15** (1987), no. 2, 121–140.

[9] R. Asaro and W. Tiller, *Interface morphology development during stress corrosion cracking: Part I. Via surface diffusion*, Metall. Trans. **3** (1972), 1789–1796.

[10] J.-F. Babadjian, *Traces of functions of bounded deformation*, Indiana Univ. Math. J. **64** (2015), no. 4, 1271–1290.

[11] J.-F Babadjian and V. Crismale, *Dissipative boundary conditions and entropic solutions in dynamical perfect plasticity*, J. Math. Pures Appl. (9) **148** (2021), 75–127.

[12] J.-F. Babadjian and R. Llerena, *Mixed boundary conditions as limits of dissipative boundary conditions in dynamic perfect plasticity*, J. Convex Anal. **30** (2022), no. 1, 81–110.

[13] J.-F. Babadjian and C. Mifsud, *Hyperbolic structure for a simplified model of dynamical perfect plasticity*, Arch. Ration. Mech. Anal. **223** (2017), no. 2, 761–815.

[14] J.-F. Babadjian and M.-G. Mora, *Approximation of dynamic and quasi-static evolution problems in elasto-plasticity by cap models*, Quart. Appl. Math., **73** (2015), no. 2, 265–316.

[15] ———, *Stress regularity in quasi-static perfect plasticity with a pressure dependent yield criterion*, J. Differential Equations **264** (2018), no. 8, 5109–5151.

[16] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, Springer New York, 2010.

[17] R. Bartle, *The Elements of Integration and Lebesgue Measure*, Wiley Classics Library, Wiley, New York, 1995.

[18] M. Bonacini and R. Cristoferi, *Area quasi-minimizing partitions with a graphical constraint: Relaxation and two-dimensional partial regularity*, J Nonlinear Sci **32** (2022), no. 6, 93.

[19] B. Bourdin, G. Francfort, and J.-J Marigo, *The variational approach to fracture*, J. Elasticity **91** (2008), 5–148.

[20] D. Bucur, I. Fragalà, and A. Giacomini, *The Multiphase Mumford-Shah problem*, SIAM J. Imaging Sci. **12** (2019), 1561–1583.

[21] ———, *Multiphase free discontinuity problems: Monotonicity formula and regularity results*, Ann. Inst. Henri Poincare (C) Anal. Non Lineaire **38** (2021), 1553–1582.

[22] D. Caraballo, *The triangle inequalities and lower semi-continuity of surface energy of partitions*, Proc. R. Soc. Edinb. A: Math. **139** (2009), no. 3, 449–457.

[23] ———, *BV-Ellipticity and Lower Semicontinuity of Surface Energy of Caccioppoli Partitions of* $\mathbb{R}^n$, J Geom Anal **23** (2013), 202–220.

[24] P. Cermelli and M.E. Gurtin, *The dynamics of solid-solid phase transitions 2. Incoherent interfaces*, Arch. Rational Mech. Anal. **127** (1994), no. 1, 41–99.

[25] A. Chambolle and E. Bonnetier, *Computing the equilibrium configuration of epitaxially strained crystalline films*, SIAM J. Appl. Math. **62** (2002), 1093–1121.

[26] A. Chambolle and V. Crismale, *Existence of strong solutions to the Dirichlet problem for the Griffith energy*, Calc. Var. Partial Differential Equations **58** (2019), 136.

[27] M. Conti, S. Terracini, and G. Verzini, *A variational problem for the spatial segregation of reaction-diffusion systems*, Indiana Univ. Math. J. **54-3** (2005), 779–815.

[28] V. Crismale and M. Friedrich, *Equilibrium configurations for epitaxially strained films and material voids in three-dimensional linear elasticity*, Arch. Ration. Mech. Anal. **237** (2020), no. 2, 1041–1098.

[29] G. Dal Maso, *An introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[30] G. Dal Maso, J. Morel, and S. Solimini, *A variational method in image segmentation: existence and approximation results*, Acta Math. **168** (1992), 89–151.

[31] G. Dal Maso and R. Scala, *Quasistatic Evolution in Perfect Plasticity as Limit of Dynamic Processes*, J. Dyn. Differ. Equ. **26** (2014), no. 4, 915–954.

[32] G. Dal Maso, A. De Simone, and M.-G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal. **180** (2006), no. 2, 237–291.

[33] A Danescu, *The Asaro–Tiller–Grinfeld instability revisited*, Int. J. Solids Struct. **38** (2001), 4671–4684.

[34] E. Davoli and P. Piovano, *Analytical validation of the Young–Dupré law for epitaxially-strained thin films*, Math. Models Methods Appl. Sci **29** (2019), 2183–2223.

[35] ———, *Derivation of a heteroepitaxial thin-film model*, Interfaces Free Bound. (2020), 1–26.

[36] E. Davoli and U. Stefanelli, *Dynamic perfect plasticity as convex minimization*, SIAM J. Math. Anal. **51** (2019), no. 2, 672–730.

[37] G. De Philippis and F. Maggi, *Regularity of Free Boundaries in Anisotropic Capillarity Problems and the Validity of Young's Law*, Arch. Ration. Mech. Anal. **216** (2015), 473–568.

[38] F. Demengel and R. Temam, *Convex functions of a measure and applications*, Indiana Univ. Math. J. **33** (1984), no. 5, 673–709.

[39] ______, *Convex function of a measure: the unbounded case*, FERMAT days 85: mathematics for optimization (Toulouse, 1985), North-Holland Math. Stud., vol. 129, North-Holland, Amsterdam, 1986, pp. 103–134.

[40] B. Després, F. Lagoutière, and N. Seguin, *Weak solutions to Friedrichs systems with convex constraints*, Nonlinearity **24** (2011), no. 11, 3055–3081.

[41] B. Després, C. Mifsud, and N. Seguin, *Dissipative formulation of initial boundary value problems for Friedrichs' systems*, Comm. Partial Differential Equations **41** (2016), no. 1, 51–78.

[42] G Duvaut and Lions J.-L., *Les inéquations en mécanique et en physique*, Dunod, 1972.

[43] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Taylor & Francis, New York, 1991.

[44] K. Falconer, *The geometry of fractal sets*, no. 85, Cambridge University Press, Cambridge, 1986.

[45] I. Fonseca, N. Fusco, G. Leoni, and V. Millot, *Material voids in elastic solids with anisotropic surface energies*, J. Math. Pures Appl. **96** (2011), 591–639.

[46] I. Fonseca, N. Fusco, G. Leoni, and M. Morini, *Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results*, Arch. Ration. Mech. Anal. **186** (2007), 477–537.

[47] I. Fonseca and G. Leonni, *Modern methods in the Calculus of Variations: L^p-spaces*, Springer, New York, 2007.

[48] I. Fonseca and S. Müller, *Quasi-convex integrands and lower semicontinuity in L^1*, SIAM J. Appl. Math. **23** (1992), 1081–1098.

[49] G. Francfort and A. Giacomini, *Small strain heterogeneous elastoplasticity revisited*, Comm. Pure Appl. Math. **65** (2012), no. 9, 1185–1241.

[50] G.A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, Mech. Phys. Solids **46** (1998), 1319–1342.

[51] E. Fried and M. Gurtin, *A unified treatment of evolving interfaces accounting for small deformations and atomic transport with emphasis on grain-boundaries and epitaxy*, Adv. Appl. Mech. **40** (2004), 1–177.

[52] M. Friedrich, Perugini M., and F. Solombrino, *Lower semicontinuity for functionals defined on piecewise rigid functions and on GSBD*, Journal of Functional Analysis **280** (2021), 108929.

[53] A. Giacomini, *A generalization of Gołąb's theorem and applications to fracture mechanics*, Math. Models Methods Appl. Sci. **12** (2002), 1245–1267.

[54] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), 159–178.

[55] M.A Grinfeld, *The stress driven instability in elastic crystals: Mathematical models and physical manifestations*, J. Nonlinear Sci. **3** (1993), 35–83.

[56] M.E. Gurtin, *The dynamics of solid-solid phase transitions 1. Coherent interfaces*, Arch. Rational Mech. Anal. **123** (1993), no. 4, 305–335.

[57] R. Hill, *The mathematical theory of plasticity*, Oxford classic texts in the physical sciences, Clarendon Press, Oxford, 1950.

[58] Sh. Kholmatov and P. Piovano, *A Unified Model for Stress-Driven Rearrangement Instabilities*, Arch. Rational Mech. Anal. (2020), 415–488.

[59] ______, *Existence of minimizers for the SDRI model in 2d: wetting and dewetting regime with mismatch strain*, Adv. Calc. Var. (2023).

[60] ______, *Existence of minimizers for the SDRI model in* $\mathbb{R}^n$*: Wetting and dewetting regimes with mismatch strain*, arXiv: Analysis of PDEs (2023).

[61] R. Kohn and R. Temam, *Dual spaces of stresses and strains, with applications to Hencky plasticity*, Appl. Math. Optim. **10** (1983), no. 1, 1–35.

[62] L. Kreutz and P. Piovano, *Microscopic Validation of a Variational Model of Epitaxially Strained Crystalline Films*, SIAM J. Appl. Math. **53** (2021), 453–490.

[63] A.A. León Baldelli, J.-F. Babadjian, B. Bourdin, D. Henao, and C. Maurini, *A variational model for fracture and debonding of thin films under in-plane loadings*, J. Mech. Phys. Solids **70** (2014), 320–348.

[64] G. Leoni, *A first course in Sobolev spaces*, American Math. Soc., Providence, 2009.

[65] R. Llerena and P. Piovano, *Solutions for a free-boundary problem modeling film multilayers with coherent and incoherent interfaces*, Preprint (2022).

[66] ______, *Existence of minimizers for a two-phase free boundary problem with coherent and incoherent interfaces*, Submitted (2023).

[67] J. Lubliner, *Plasticity theory*, Macmillan Publishing Company, New York, 1970.

[68] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge University Press, 2012.

[69] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge University Press, Cambridge, 1999.

[70] C. Mifsud, *Variational and hyperbolic methods applied to constrained mechanical systems*, Theses, Université Pierre et Marie Curie, November 2016.

[71] F. Morgan, *Lowersemicontinuity of energy clusters*, Proc. R. Soc. Edinb. A: Math. **127** (1997), 819–822.

[72] D. Mumford and J. Shah, *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math. **42-5** (1989), 577–685.

[73] J. Nečas, *Sur les normes équivalentes dans* $W^k(\Omega)$ *et sur la coercivité des formes formellement positives*, Equations aux derives partielles (1966), no. 19, 102–128.

[74] F. Otto, *Initial-boundary value problem for a scalar conservation law*, CRAS **322** (1996), 729–734.

[75] R.-T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[76] H.L. Royden, *Real analysis*, Macmillan, New York, 1966.

[77] D. Spector, *Simple proofs of some results of Reshetnyak*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1681–1690.

[78] ______, *Simple proofs of some results of Reshetnyak*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1681–1690.

[79] B. J. Spencer, *Asymptotic derivation of the glued-wetting-layer model and contact-angle condition for stranski-krastanow islands*, Phys. Rev. B **59** (1999), 2011–2017.

[80] B. J. Spencer and J. Tersoff, *Equilibrium shapes and properties of epitaxially strained islands*, Phys. Rev. Lett. **79** (1997), 4858–4861.

[81] N. Sridhar, J. M. Rickman, and D. J. Srolovitz, *Multilayer film stability*, J. Appl. Phys. **82** (1997), no. 10, 4852–4859.

[82] D.J. Srolovitz and M.P. Anderson, *A criterion for compressive failure of a continuous, protective surface film*, Acta Metall. **32** (1984), no. 7, 1089–1092.

[83] P.-M. Suquet, *Un espace fonctionnel pour les équations de la plasticité*, Ann. Fac. Sci. Toulouse Math. **1** (1979), 77–87.

[84] ———, *Evolution problems for a class of dissipative materials*, Quart. Appl. Math. **38** (1981), 391–414.

[85] ———, *Sur les équations de la plasticité: existence et régularité des solutions*, J. Mécanique **20** (1981), no. 1, 3–39.

[86] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture notes of the Unione Matematica Italiana, Springer, Berlin, 2007.

[87] C. Teichert, M. G. Lagally, L. J. Peticolas, J. C. Bean, and J. Tersoff, *Stress-induced self-organization of nanoscale structures in sige/si multilayer films*, Phys. Rev. B **53** (1996), 16334–16337.

[88] R. Temam, *Navier-Stokes equations: theory and numerical analysis*, Studies in mathematics and its applications, North-Holland, Amsterdam, 1977.

[89] R. Temam, *Problèmes mathématiques en plasticité*, Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science], vol. 12, Gauthier-Villars, Montrouge, 1983.

[90] J. Tersoff, C. Teichert, and M. G. Lagally, *Self-organization in growth of quantum dot superlattices*, Phys. Rev. Lett. **76** (1996), 1675–1678.