DISCRETE-TO-CONTINUOUS CRYSTALLINE CURVATURE FLOWS

ANTONIN CHAMBOLLE, DANIELE DE GENNARO, AND MASSIMILIANO MORINI

Abstract. We consider here a fully discrete variant of the implicit variational scheme for mean curvature flow [2, 36], in a setting where the flow is governed by a crystalline surface tension defined by the limit of pairwise interactions energy on the discrete grid. The algorithm is based on a new discrete distance from the evolving sets, which prevents the occurrence of the spatial drift and pinning phenomena identified in [11] in a similar discrete framework. We provide the first rigorous convergence result holding in any dimension, for any initial set and for a large class of purely crystalline anisotropies, in which the spatial discretization mesh can be of the same order or coarser than the time step.

1. Introduction

In this paper we analyse a space- and time-discrete approximation of crystalline mean curvature flows of the form

$$V(x,t) = -\phi(\nu_{E(t)}(x))\kappa^{\phi}_{E(t)}(x), \qquad x \in \partial E(t), \quad t \geq 0, \tag{1.1}$$

for a class of crystalline norms ϕ. We recall that an anisotropy ϕ is said to be crystalline if and only if $\{\phi \leq 1\}$ is a polytope (or equivalently, ϕ is the support function of a polytope). Moreover in the current paper we restrict ourselves to the case where $\{\phi \leq 1\}$ is a zonotope with rational generators [38, 9]. Here $V(x,t)$ stands for the (outer) normal velocity of the boundary $\partial E(t)$ at x, ϕ is a crystalline norm on $\mathbb{R}^N$ representing the surface tension, $\kappa^{\phi}_{E(t)}$ is the crystalline mean curvature of $\partial E(t)$ associated to ϕ, and $\nu_{E(t)}$ is the outer unit normal to $\partial E(t)$. The evolution law (1.1) has been considered to describe some phenomena in materials science and crystal growth; see e.g. [32, 44]. Our main result is a convergence result of the discrete approximation to the continuous evolution, as the time and space steps go to zero, even in the somewhat surprising case where the space-step is greater or equal to the time-step.

From the mathematical point of view, the lack of regularity of the differential operator involved in the definition of the crystalline curvature (see [6, 7]) is the main reason why the well-posedness of the crystalline mean curvature flow in every dimension has been a long-standing open problem. After some partial results (see for instance [1, 4, 5, 15, 25, 26, 27]), important breakthroughs have been obtained simultaneously in [28, 29, 31], where a suitable crystalline theory of viscosity

---

Key words and phrases. Crystalline flows, dicrete-to-continuous approximation, minimizing movements, distributional solutions.

Aknowledgements. Antonin Chambolle is partially funded by a "France 2030" support managed by the Agence Nationale de la Recherche, under the reference ANR-23-PEIA-0004. Daniele De Gennaro has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 94532. Massimiliano Morini is a member of the Italian National Rsearch Group GNAMPA (INDAM) and is partially supported by PRIN 2022 Project "Geometric Evolution Problems and Shape Optimization (GEPSO)", PNRR Italia Domani, financed by European Union via the Program NextGenerationEU, CUP_D53D23005820006. He wishes to warmly thank the hospitality of CEREMADE, where part of this reserach was conducted.

solutions was developed, and with a different approach in [22, 21, 20], where a new notion of distributional solutions was proposed.

Let us focus on the definition of distributional solutions, referring to the nice review [30] for further information on viscosity solutions to (1.1) (we just note that the two notions are equivalent in the setting of this paper [20, Remark 6.1]). The exact definition of distributional solutions will be recalled in Definition 2.1, but when ϕ is smooth it can be motivated as follows: It is known (see for instance [43] for the isotropic case) that $E(t)$ evolves according to (1.1) if and only if the signed distance function $d(\cdot,t) := \mathrm{sd}^{\phi^\circ}_{E(t)}$ to $\partial E(t)$ induced by the polar norm[1] ϕ°, satisfies

$$\partial_t d \geq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d>0\}, \tag{1.2}$$

$$\partial_t d \leq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d<0\} \tag{1.3}$$

in the viscosity sense. The idea of the new definition introduced in [22] is to reinterpret the equations above in the distributional sense. In particular, note that replacing $\nabla\phi(\nabla u)$ by a vector field $z \in L^\infty(\{d>0\};\mathbb{R}^N)$ such that $z(x) \in \partial\phi(\nabla d)$ for a.e. x, where $\partial\phi$ denotes the subdifferential of ϕ, the equations (1.2), (1.3) make sense even when ϕ is crystalline. The corresponding notion of super- and sub-solutions bears a comparison principle, which yields uniqueness of the motion up to fattening. Existence is obtained either by a variant of the minimizing movements scheme of [2, 36] in the spirit of [16], which consists in building a discrete-in-time evolution obtained by a recursive minimization procedure (see [22, 20]), or by approximation with smooth anisotropies [21]. We observe that the convergence of such time discrete approaches to a motion characterized by (1.2)-(1.3) in the *viscosity sense* was shown in [35], including in the 2D crystalline setting, while convergence in a distributional sense was established in [15] in the convex case only. Briefly, given a time-step $h>0$ and an initial closed set $E_0 =: E^{h,0}$, one defines $E^{h,k+1} = \{u^{h,k+1} \leq 0\}$, where $u^{h,k+1}$ is defined as the minimizer of a so-called "Rudin-Osher-Fatemi" [41] problem:

$$u^{h,k+1} \in \operatorname{argmin}\left\{\int_{\mathbb{R}^N}\phi(Du) + \frac{1}{2h}\int_{\mathbb{R}^N}|u - \mathrm{sd}^{\phi^\circ}_{E^{h,k}}|^2\right\}. \tag{1.4}$$

The idea of the present work is to combine this discretization in time with a simultaneous discretization in space for the particular class of purely crystalline anisotropies ϕ of the following form

$$\phi(v) = \sum_{i\in\mathcal{E}} \beta(i)|i\cdot v|, \tag{1.5}$$

where $\beta(i)>0$ and $\mathcal{E} \subseteq \mathbb{Z}^N \setminus \{0\}$ is a finite set of generators such that $\operatorname{Span}\mathcal{E} = \mathbb{R}^N$.

We now specify the discrete setting we are interested in, referring the reader to [13] for a more thorough introduction to related topics. We consider an ε-spaced square lattice $\varepsilon\mathbb{Z}^N$ and discrete functions $u : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, and denote $u_i := u(i)$. We observe that we could also consider a general finite-dimensional Bravais lattice, at the expense of more tedious notation. A natural discrete version of total variation-like energies are those appearing in Ising systems, namely energies of the form

$$TV^\varepsilon_\beta(v) := \varepsilon^{N-1}\sum_{i,j\in\varepsilon\mathbb{Z}^N}\beta(i/\varepsilon - j/\varepsilon)|v_i - v_j|, \tag{1.6}$$

[1]defined by $\phi^\circ(x) = \sup_{\phi(\nu)\leq 1}\nu\cdot x$ and which satisfies $\phi(x) = \sup_{\phi^\circ(x)\leq 1}\nu\cdot x$.

where β is as in (1.5) and extended to 0 in $\mathbb{Z}^N\setminus\mathcal{E}$. Under the hypotheses above on β, the functionals TV^ε_β are shown to Γ-converge[2] as $\varepsilon\to 0$ to the total variation functional

$$TV_\phi(v)=\int_{\mathbb{R}^N}\phi(Dv)$$

where ϕ is as in (1.5), see e.g. [19]. It is thus natural to define a minimizing movements scheme based on TV^ε_β which is the discrete counterpart of the minimizing procedure (1.4), as follows: given $E_0\subseteq\mathbb{R}^N$, we define $E^0_{\varepsilon,h}=\{i\in\varepsilon\mathbb{Z}^N:(i+[0,\varepsilon)^N)\cap E_0\neq\emptyset\}$ and for every $k\in\mathbb{N}$ we let $u^{k+1}_{\varepsilon,h}$ be such that

$$u^{k+1}_{\varepsilon,h}\in\operatorname{argmin}\left\{TV^\varepsilon_\beta(v)+\frac{1}{2h}\sum_{i\in\varepsilon\mathbb{Z}^N}|v_i-(\,\mathrm{sd}^k_{\varepsilon,h})_i|^2\ :\ v:\varepsilon\mathbb{Z}^N\to\mathbb{R}\right\},\tag{1.7}$$

where $\mathrm{sd}^k_{\varepsilon,h}$ denotes a suitable signed ϕ°-distance function to $E^k_{\varepsilon,h}$ defined on $\varepsilon\mathbb{Z}^N$. (Actually, the energy in (1.7) might be infinite and we rather consider the Euler-Lagrange equation of the problem.) Then, one sets $E^{k+1}_{\varepsilon,h}:=\{u^{k+1}_{\varepsilon,h}\le 0\}$.

The idea is to study the asymptotic behaviour of the discrete evolutions $E^k_{\varepsilon,h}$ as both $\varepsilon,h\to 0$. Notice that a similar analysis has been performed in [11] in the planar case, for $\phi=\|\cdot\|_1$ and $\mathrm{sd}^k_{\varepsilon,h}$ the continuous signed distance function from the discrete sets $E^k_{\varepsilon,h}$ restricted to the lattice $\varepsilon\mathbb{Z}^N$, see also [10, 12, 14, 37, 42] for further related results. With this choice, if $\varepsilon\gg h$ it is easy to see that the dissipation-like term in (1.7)

$$\frac{1}{2h}\sum_{i\in\varepsilon\mathbb{Z}^N}|v_i-(\,\mathrm{sd}^{k+1}_{\varepsilon,h})_i|^2$$

forces the functions $u^k_{\varepsilon,h}$ to be constant as k varies, therefore producing *pinning* on the moving interfaces. Moreover, when the two scales ε,h are going to zero at the same speed it is shown in [11] that a direct implementation of the standard scheme with the choice above for the distance, introduces a systematic error of order $\varepsilon=h$ at each step, which accumulates and produces a drift in the limiting evolution. As a result, low curvature shapes remain pinned, while sets with higher curvature evolve with a law which is a nonlinear modification of the crystalline curvature flow (1.1). Thus, the evolution law (1.1) can be approximated with the scheme of [11] only if $\varepsilon\ll h$.

We show in our main result, Theorem 5.2, that with a new appropriate definition of the distance $\mathrm{sd}^k_{\varepsilon,h}$, we can recover in the limit $\varepsilon,h\to 0$ the actual distributional solution to (1.1) for every initial set $E_0\subseteq\mathbb{R}^N$, for every purely crystalline anisotropy ϕ of the form (1.5) with rational coefficients, in any dimension and irrespective of relative size of the space- and time-steps. In fact, the assumption of the rational character of β can be removed in the regime $\varepsilon\le O(h)$. To the best of our knowledge this is the first general rigorous convergence result for a fully discrete scheme without restrictions on the dimension, on the initial sets and in which the spatial mesh is allowed to be of the same order or even coarser than the time step.

Let us further comment on the analysis carried out in [11] in the planar case (see also [13] for many more references on the topic). One important change between these older results and ours is that we consider distributional solutions to the crystalline mean curvature flow (1.1), instead of relying on the characterization of the motion via ODEs, which dates back to [1, 4]. The latter notion of solution is indeed suited only for planar evolutions, thus the limitation $N=2$ in the

[2]Note that we do not need to assume that the lattice generated by $\{e_k\}_{k=1,\dots,m}$ is $\mathbb{Z}^N$, which is necessary to ensure the equi-coercivity of the discrete functionals.

past works. With the ODE definition and for $\phi = \|\cdot\|_1$, the authors of [11] precisely prove the following results. If $\varepsilon \ll h$ then the limiting motion is consistent with (1.1), while if $h \ll \varepsilon$ pinning happens for any nonempty initial data. As already mentioned, in the critical case $\varepsilon = h$, the limit planar motion is not driven by (1.1), but instead by a slightly modified nonlinear crystalline mean curvature flow, and pinning may happen for some particular (low curvature) initial data. This striking difference with our result may be (vaguely) justified by the following remark. While in [11], the focus is on discrete sets, we rather evolve, in accordance with the definition of distributional solutions, the *signed distance functions* to the boundaries. In this way we can effectively achieve a sub-pixel precision in our approximation, as $u_{\varepsilon,h}$ and the signed distance function carry more information than the evolving level set $\{u_{\varepsilon,h}(t) \leq 0\}$. Our new definition of the interpolated signed distance is detailed in Section 4.

The consistency result in this paper validates the numerical experiments which we carry on in Section 6 to illustrate our results. These experiments are derived from previous experiments in [18], which however were using a different redistancing operation for which no consistency was proven. Numerical schemes based on the variational approach [2, 36] have been introduced for crystal growth [3]. Since then, there have been many attempts to implement implicit schemes based on this approach for isotropic and anisotropic curvature flows in various settings [16, 24, 39, 40, 23], all relying on the consistency of the spatial discretization with respect to the time-discrete scheme (hence assuming $\varepsilon \ll h$).

Let us conclude this introduction with two comments. The first one concerns the hypothesis that ϕ is purely crystalline. It seems quite technical, as it implies that the associated interaction function β (in the sense of (1.5)) has finite range. While this is not necessary to carry our the existence part for the discrete minimizing movements scheme, it is essential for building a calibration which yields a bound on the speed of Wulff shapes, see Appendix A. In practice, being the Wulff shape $\mathscr{W} := \{\phi^\circ \leq 1\}$ a finite Minkowski sum of (rational) segments (which is called a *zonotope*), we can effectively handcraft a calibration along the directions identified by these segments. It is a remarkable difference between this discrete setting and the continuous one, where instead the vector field $x/\phi^\circ(x)$ in $\mathbb{R}^N$ is the right calibration *for any* anisotropy ϕ.

The second one is on possible generalizations of the present analysis to more general evolution laws than (1.1). The more general evolution law which is shown to admit a unique distributional solution is

$$V(x,t) = \psi(\nu_{E(t)}(x))\left(-\kappa^\phi_{E(t)}(x) + f(x,t)\right), \qquad x \in \partial E(t), t \geq 0, \tag{1.8}$$

where ψ is a norm (usually referred to as the *mobility*), and f is a forcing term, see [22, 20] . We expect most of the present analysis to be valid even if $\psi \neq \phi$, under suitable compatibility assumptions on ψ (see [22, 20] for details), and it should not be difficult to consider a driving force f as long as it is Lipschitz in space and globally bounded, see [20] again.

The paper is organized as follows: in the next Section 2, we recall the definition of distributional crystalline curvature flows from [22, 20]. Then, we study the discrete "Rudin-Osher-Fatemi" problem and its Euler-Lagrange equation in Section 3. In Section 4, we introduce the discrete minimizing movement scheme, with our particular definition of the signed distance function. We study in detail the properties of these distances, then in Section 4.2 we analyse the particular case of an initial Wulff shape. In the continuous setting, it is well known that under the law (1.1), it decreases in a self-similar way with a speed proportional to the inverse of its radius. We show an estimate bounding the decay of the discrete Wulff shapes, it relies on the delicate construction of a calibration z for the Rudin-Osher-Fatemi problem with datum ϕ°, detailed in Appendix A.

Our main result, which is that in the limit $\varepsilon, h \to 0$, the motion defined in Section 4 converges to a crystalline flow, is stated, and proved, in Section 5. We implemented the discrete scheme in 2D and show some numerical simulations in Section 6. Some technical results are collected in the Appendix.

2. Distributional crystalline curvature flows

We recall the distributional formulation for the crystalline mean curvature motion of sets evolving with normal velocity (1.1)

introduced in [22] (see also [20]). Here and in what follows ϕ is any norm, ϕ° denotes the polar (or dual) norm of ϕ and given a closed set $F \subseteq \mathbb{R}^N$, $\mathrm{dist}^{\phi^\circ}(\cdot, F)$ stands for the ϕ°-distance function from F defined by

$$\mathrm{dist}^{\phi^\circ}(x, F) := \min\{\phi^\circ(x-y) : \, y \in F\}\,.$$

Analogously, for any E, F closed we set

$$\mathrm{dist}^{\phi^\circ}(E, F) := \min\{\phi^\circ(x-y) : \, x \in E\,,\, y \in F\}\,.$$

We recall that a sequence of closed sets $(E_k)_{k\geq 1}$ in $\mathbb{R}^N$ converges to a closed set E in the *Kuratowski sense*: if the following conditions are satisfied

(i) if $x_k \in E_k$ for each k, any limit point of $\{x_k\}$ belongs to E;
(ii) for all $x \in E$ there exists a sequence $\{x_k\}$ such that $x_k \in E_k$ for each k and $x_k \to x$.

We will write in this case:

$$E_k \xrightarrow{\mathscr{K}} E\,.$$

One can easily verify that $E_k \xrightarrow{\mathscr{K}} E$ if and only if (for any norm ψ) $\mathrm{dist}^\psi(\cdot, E_k) \to \mathrm{dist}^\psi(\cdot, E)$ locally uniformly in $\mathbb{R}^N$. Hence, by Ascoli-Arzelà Theorem we have that any sequence of closed sets admits a converging subsequence in the Kuratowski sense (possibly to $\emptyset$, when $\mathrm{dist}^\psi(\cdot, E_k) \to +\infty$).

Definition 2.1. Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ denote $E(t) := \{x \in \mathbb{R}^N \, : \, (x,t) \in E\}$. We say that E is a *superflow* for (1.1) with initial datum E_0 if

(a) $E(0) \subseteq E_0$;
(b) $E(s) \xrightarrow{\mathscr{K}} E(t)$ as $s \nearrow t$ for all $t > 0$;
(c) If $E(t) = \emptyset$ for some $t \geq 0$, then $E(s) = \emptyset$ for all $s > t$.
(d) Set $T^* := \inf\{t > 0 \, : \, E(s) = \emptyset \text{ for } s \geq t\}$, and

$$d(x,t) := \mathrm{dist}^{\phi^\circ}(x, E(t)) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then,

$$\partial_t d \geq \mathrm{div} z \tag{2.1}$$

holds in the distributional sense in $\mathbb{R}^N \times (0,T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0,T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\mathrm{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\mathrm{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) : \, d(x,t) \geq \delta\})$ for every $\delta \in (0,1)$.

We say that A, open set in $\mathbb{R}^N \times [0,+\infty)$, is a *subflow* for (1.1) with initial datum E_0 if $\mathbb{R}^N \times [0,+\infty) \setminus A$ is a superflow for (1.1) with initial datum $\mathbb{R}^N \setminus \mathrm{int}\big(E_0\big)$.

Finally, we say that E, closed set in $\mathbb{R}^N \times [0,+\infty)$, is a *weak flow* for (1.1) with initial datum E_0 if it is a superflow and if $\mathrm{int}(E)$[3] is a subflow, both with initial datum E_0.

In [22] the following crucial inclusion principle between sub- and superflows is proven.

Theorem 2.2. *Let E be a superflow with initial datum E_0 and F be a subflow with initial datum F_0 in the sense of Definition 2.1. Assume that $\mathrm{dist}^{\phi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. Then,*

$$\mathrm{dist}^{\phi^\circ}\big(E(t), \mathbb{R}^N \setminus F(t)\big) \geq \Delta \qquad \textit{for all } t \geq 0$$

(with the convention that $\mathrm{dist}^{\phi^\circ}(G, \emptyset) = \mathrm{dist}^{\phi^\circ}(\emptyset, G) = +\infty$ for any G).

We also recall the corresponding notion of sub- and supersolution to the level set flow associated with (1.1). In what follows $\mathrm{UC}(\mathbb{R}^N)$ stands for the space of uniformly continuous functions on $\mathbb{R}^N$.

Definition 2.3 (Level set subsolutions and supersolutions)**.** Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. A lower semicontinuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is called a *level set superflow* for (1.1), with initial datum u_0, if $u(\cdot, 0) \geq u_0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{u(\cdot, t) \leq \lambda\}$ is a superflow for (1.1) in the sense of Definition 2.1, with initial datum $\{u_0 \leq \lambda\}$.

An upper-semicontinuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is called a *level set subflow* for (1.1), with initial datum u_0, if $-u$ is level set superflow in the previous sense, with initial datum $-u_0$.

Finally, a continuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is called a *level set flow* for (1.1) if it is both a level set sub- and superflow.

Using Theorem 2.2, it is not difficult to deduce the following parabolic comparison principle between level set sub- and superflows, which yields in particular the uniqueness of level set flows (in the sense of Definition 2.3), see [20].

Theorem 2.4. *Let u_0, $v_0 \in \mathrm{UC}(\mathbb{R}^N)$ and let u, v be respectively a level set subflow starting from u_0 and a level set superflow starting from v_0. If $u_0 \leq v_0$, then $u \leq v$.*

We finally recall that in [22] (see also [20]) the existence of level set flows is established by implementing a level-by-level minimizing movements scheme. This in turn yields existence and uniqueness (up to fattening) for weak flows. This is made precise in the following statement, see [22, Corollary 4.6] and [20, Theorem 4.8].

Theorem 2.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. Then the following holds:*

(i) *There exists a unique level set flow u in the sense of Definition 2.3 starting u_0.*

(ii) *For all $\lambda \in \mathbb{R}$ the sets $\{(x,t) : u(x,t) \leq \lambda\}$ and $\{(x,t) : u(x,t) < \lambda\}$ are respectively the maximal superflow and minimal sublow with initial datum $\{u_0 \leq \lambda\}$.*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, the fattening phenomenon does not occur; that is,*

$$\begin{aligned} \{(x,t) : u(x,t) < \lambda\} &= \mathrm{int}\,\big(\{(x,t) : u(x,t) \leq \lambda\}\big)\,, \\ \mathrm{cl}\,\big(\{(x,t) : u(x,t) < \lambda\}\big) &= \{(x,t) : u(x,t) \leq \lambda\}\,, \end{aligned} \tag{2.2}$$

where interior and closure are relative to space-time.

[3]Here we are taking the interior with respect to $\mathbb{R}^N \times [0,+\infty)$

For all such λ, $\{(x,t) : u(x,t) \le \lambda\}$ *is the unique weak flow in the sense of Definition 2.1, starting from* $\{u_0 \le \lambda\}$.

The aim of this paper is to show that the convergence to the continuum level set flow holds true also when the Euler implicit time discretisation is combined with a suitable spatial discretisation procedure.

3. The discrete "Rudin-Osher-Fatemi" problem

In this part, we describe our discrete setting, we then introduce and analyse the discrete variant (1.7) of Problem (1.4).

3.1. **Discrete functions spaces and operators.** For $\varepsilon > 0$, we define the function spaces $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N}$. Given a function $u \in X_\varepsilon$ and a discrete "vector field" $z \in Y_\varepsilon$, with a slight abuse of notation we will denote $u_i = u(i)$ and $z_{ij} = z(i,j)$, $i, j \in \varepsilon\mathbb{Z}^N$. The discrete gradient $D_\varepsilon : X_\varepsilon \to Y_\varepsilon$ is defined, for $u \in X_\varepsilon$ as

$$(D_\varepsilon u)_{ij} = \frac{u_i - u_j}{\varepsilon}.$$

We denote its adjoint operator by $D_\varepsilon^* : Y_\varepsilon \to X_\varepsilon$, namely the operator such that, for $\eta \in X_\varepsilon$ compactly supported and for $z \in Y_\varepsilon$, is defined as

$$\sum_i (D_\varepsilon^* z)_i \eta_i := \sum_{ij} z_{ij}(D_\varepsilon \eta)_{ij} = \sum_{ij} z_{ij}(\eta_i - \eta_j),$$

where the indexes, here and throughout the paper, range over $\varepsilon\mathbb{Z}^N$ if not otherwise stated. In particular, taking $\eta = \chi_{\{i\}}$, one finds that

$$(D_\varepsilon^* z)_i = \sum_j \frac{z_{ij} - z_{ji}}{\varepsilon}, \tag{3.1}$$

which can be seen as a discrete divergence operator.

3.2. **Discrete ROF problem.** In this section we consider the discrete anisotropic ROF problem associated with the discrete total variation functional. Without loss of generality, we consider $\varepsilon = 1$ in this section, and denote $X := X_1$, $Y := Y_1$ and $D := D_1$. Given a nonnegative $\beta \in X$, which will be called the *interaction function*, satisfying

$$\sum_{i \in \mathbb{Z}^N} \beta(i) =: c_\beta < +\infty, \tag{3.2}$$

we set $\alpha_{ij} = \beta(i-j)$ and, for any $u \in X$ we define

$$TV(u) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij}|u_i - u_j| = \sum_{i,j} \alpha_{ij}|(Du)_{i,j}|. \tag{3.3}$$

We also consider the discrete perimeter $\mathscr{P}$ defined for every $E \subseteq \mathbb{Z}^N$ as

$$\mathscr{P}(E) := TV(\chi^E) = \sum_{i,j \in \mathbb{Z}^N} \alpha_{ij}|\chi_i^E - \chi_j^E|.$$

We also consider a suitable localization of the perimeter: namely, for any set $A \subseteq \mathbb{R}^N$ we define

$$\mathscr{P}(E; A) = \sum_{i \in A \cap \mathbb{Z}^N \text{ or } j \in A \cap \mathbb{Z}^N} \alpha_{ij}|\chi_i^E - \chi_j^E|.$$

Note that the quantities above may well be infinite.

Then, given $g \in X$, we consider the following problem: find a pair $(u,z) \in X \times Y$ such that

$$\begin{cases} D^* z + u = g \\ z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|, \quad |z_{ij}| \le \alpha_{ij} \quad \forall i,j \in \mathbb{Z}^N. \end{cases} \tag{3.4}$$

Note that the equation above is the Euler-Lagrange equation of the ROF discrete functional

$$ROF_g(v) = TV(v) + \frac{1}{2}\sum_{i\in\mathbb{Z}^N}(v_i - g_i)^2. \tag{3.5}$$

However, (3.4) makes sense also for those g such that $ROF_g \equiv +\infty$.

We will also consider the following geometric minimization problem. Given $g \in X$, find

$$\min_{F\subseteq\mathbb{Z}^N} \mathscr{P}(F) + \sum_{i\in\mathbb{Z}^N} \chi_i^F g_i. \tag{3.6}$$

In order to deal with unbounded sets, possibly with infinite perimeter, we will consider the following notion of global minimality with respect to compactly supported perturbations.

Definition 3.1. A set $E \subseteq \mathbb{Z}^N$ is a global minimizer for the problem (3.6) if for every $R > 0$

$$\mathscr{P}(E; B_R) + \sum_{|i|<R} \chi_i^E g_i \le \mathscr{P}(F; B_R) + \sum_{|i|<R} \chi_i^F g_i \tag{3.7}$$

for every $F \subseteq \mathbb{Z}^N$ such that $F \triangle E \subseteq B_R$. Here $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ is the open ball of radius R centered in the origin.

Proposition 3.2. *Let $g, g \in X$ such that $g' - g \ge \delta > 0$. Let E, E' be two global minimizers of problem (3.7), in the sense of Definition 3.1, corresponding to g, g' respectively. Then, $E' \subseteq E$.*

Proof. Let us denote in the following $\chi := \chi^{E_s}, \chi' := \chi^{E'_s}$. For a given $R > 0$ we define the competitor sets $F = (E_s \setminus B_R) \cup ((E'_s \cup E_s) \cap B_R)$ and $F' = (E'_s \setminus B_R) \cup ((E'_s \cap E_s) \cap B_R)$. By minimality of E_s, E'_s in B_R one has

$$\begin{aligned} \sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi'_i - \chi'_j| + \sum_{|i|<R} g'_i(\chi'_i - \chi'_i \wedge \chi_i) \le &\sum_{\substack{|i|<R \\ |j|<R}} \alpha_{ij}|\chi'_i \wedge \chi_i - \chi'_j \wedge \chi_j| \\ &+ \sum_{\substack{|i|<R \\ |j|\ge R}} (\alpha_{ij} + \alpha_{ji})|\chi'_i \wedge \chi_i - \chi'_j| \end{aligned} \tag{3.8}$$

$$\begin{aligned} \sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi_i - \chi_j| + \sum_{|i|<R} g_i(\chi_i - \chi'_i \vee \chi_i) \le &\sum_{\substack{|i|<R \\ |j|<R}} \alpha_{ij}|\chi'_i \vee \chi_i - \chi'_j \vee \chi_j| \\ &+ \sum_{\substack{|i|<R \\ |j|\ge R}} (\alpha_{ij} + \alpha_{ji})|\chi'_i \vee \chi_i - \chi_j|. \end{aligned} \tag{3.9}$$

Using the inequality[4] $|a\wedge b - c\wedge d| + |a\vee b - c\vee d| \le |a-c| + |b-d|$ and summing together (3.8) and (3.9) we obtain

$$\begin{aligned}&\sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})\left(|\chi_i-\chi_j|+|\chi_i'-\chi_j'|\right)+2\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+\\&\le\sum_{\substack{|i|<R\\|j|\ge R}}(\alpha_{ij}+\alpha_{ji})\left(|\chi_i'\wedge\chi_i-\chi_j'|+|\chi_i'\vee\chi_i-\chi_j|\right).\end{aligned}\tag{3.10}$$

We then remark that $|\chi_i'\wedge\chi_i-\chi_j'|\le|\chi_i'\wedge\chi_i-\chi_i'|+|\chi_i'-\chi_j'|=(\chi_i'-\chi_i)^++|\chi_i'-\chi_j'|$ and analogously $|\chi_i'\vee\chi_i-\chi_j|\le(\chi_i'-\chi_i)^++|\chi_i-\chi_j|$. Therefore, (3.10) entails

$$\sum_{|i|<R}(g_i'-g_i)(\chi_i'-\chi_i)^+\le\sum_{|i|<R}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji}).\tag{3.11}$$

Fix now $R_\delta>0$ such that

$$\sum_{|k|\ge R_\delta}\beta(k)\le\frac{\delta}{4}$$

and define $V_R:=\sum_{|i|<R}(\chi_i'-\chi_i)^+$. Assuming $R>R_\delta$, for every $\ell<R$ we use (3.11) and $g+\delta\le g'$ to get

$$\begin{aligned}\delta V_R&\le\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji})+2c_\beta\sum_{\ell\le|i|<R}(\chi_i'-\chi_i)^+\\&\le 2\sum_{|i|<\ell}(\chi_i'-\chi_i)^+\sum_{|k|\ge R-\ell}\beta(k)+2c_\beta(V_R-V_\ell).\end{aligned}\tag{3.12}$$

Therefore, choosing $\ell=R-R_\delta$ in (3.12) we obtain

$$\frac{\delta}{2}V_R\le 2c_\beta(V_R-V_{R-R_\delta}),\tag{3.13}$$

which implies that for every $k,\ell\in\mathbb{N}$ it holds

$$V_{kR_\delta}\le\left(1-\frac{\delta}{4c_\beta}\right)^\ell V_{(k+\ell)R_\delta}.\tag{3.14}$$

Letting $\ell\to+\infty$, since $V_{(k+\ell)R_\delta}=O(\ell^N)$, we infer that $V_{kR_\delta}=0$ for every $k\in\mathbb{N}$. In particular, this implies that $(\chi'-\chi)^+=0$ i.e. $\chi'\le\chi$. □

We will prove the following theorem.

Theorem 3.3. *Given $g\in X$ there exists a unique function $u^g\in X$ and there exists a discrete vector field $z\in Y$ such that (u^g,z) is a solution of (3.4). Moreover, the following comparison principle holds: if $g\le g'$ then $u^g\le u^{g'}$. Finally, for any $R>0$ and $s\in\mathbb{R}$ the sublevel set $E_s:=\{i\in\mathbb{Z}^N: u_i^g\le s\}$ is a global minimizer (in the sense of Definition 3.1) for (3.6) with g replaced by $g-s$.*

Proof. **Step 1.** (Existence) For every $n\in\mathbb{N}$ set $g^n:=g\chi^{B_n}$ and note that $g^n\in\ell^2(\mathbb{Z}^N)$. Therefore, by standard methods and by strict convexity the functional (3.5), with g replaced by g^n admits a unique minimizer u^n and, as previously observed, the optimality condition is the existence of a

[4] Indeed, if $a\ge b$ and $c\ge d$, this is an equality, while if $a>b$ and $c<d$, one deduces that $b-d<a-d<a-c$, $b-d<b-c<a-c$ so that there exists $t\in(0,1)$ with $a-d=t(b-d)+(1-t)(a-c)$, $b-c=(1-t)(b-d)+t(a-c)$: the conclusion follow by convexity of $|\cdot|$.

discrete field z^n such that (u^n, z^n) solves (3.4) (with g^n in place of g). Note that, for any $k \in \mathbb{Z}^N$, by equation (3.4) it holds

$$|u^n_k| \le |g^n_k| + |(D^*z)_k| \le |g_k| + c_\beta \quad \text{for every } n \in \mathbb{N}, \tag{3.15}$$

where the last inequality follows from the definition (3.1) and from $|z_{ij}| \le \alpha_{ij}$ and $|g^n| \le |g|$. Now, it is clear that we can extract a subsequence n_k and find (u, z) such that $u_i^{n_k} \to u_i$ and $z_{ij}^{n_k} \to z_{ij}$ as $k \to +\infty$. Clearly we have that $|z_{ij}| \le \alpha_{ij}$ and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$ and it is immediate to check that (u, z) satisfies equation (3.4).

Step 2. (Minimality of the sublevelsets) Let $R > 0, s \in \mathbb{R}$ and let $F \subseteq \mathbb{Z}^N$ such that $E_s \triangle F \subset\subset B_R$. We first remark that $\alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| = -z_{ij}(\chi_i^{E_s} - \chi_j^{E_s})$, which follows easily from the definition of E_s and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$.

We set $I_R := \{(i, j) \in \mathbb{Z}^N \times \mathbb{Z}^N : |i| < R \text{ or } |j| < R\}$ and compute

$$\begin{aligned}
\mathscr{P}(F; B_R) - \mathscr{P}(E_s; B_R) &= \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^F - \chi_j^F| - \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| \\
&\ge - \sum_{(i,j)\in I_R} z_{ij}(\chi_i^F - \chi_j^F) + \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_j^{E_s}) \\
&= \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)),
\end{aligned} \tag{3.16}$$

where in the last equality we used the fact that $\chi_i^{E_s} = \chi_i^F$ if $|i| \ge R$. Noting that the function $\chi^{E_s} - \chi^F$ is compactly supported, we may use it as a test function for (3.4). Therefore, from (3.16) we deduce

$$\begin{aligned}
\mathscr{P}(F; B_R) - \mathscr{P}(E_s; B_R) &\ge \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_i (\chi_i^{E_s} - \chi_i^F)(g_i - u_i) \ge \sum_{i\in E_s\setminus F} (g_i - s) - \sum_{i\in F\setminus E_s} (g_i - s),
\end{aligned}$$

which shows the minimality of E_s.

Step 3. (Comparison and uniqueness for (3.4)) Assume $g \le g'$ and let $(u, z), (u', z')$ two corresponding solutions for (3.4). Let $s > s'$ and recall that by Step 2 $\{u' \le s'\}$ and $\{u \le s\}$ are global minimizers for (3.6) according to Definition 3.1, with g replaced by $g' - s'$ and $g - s$ respectively. Since $g' - s' - (g - s) \ge s - s' > 0$, from Proposition 3.2 we obtain $\{u' \le s'\} \subseteq \{u \le s\}$. By the arbitrariness of s, s' we conclude that $u \le u'$.

□

Remark 3.4. We remark that, given $g \in X$ it clearly holds that $u^{-g} = -u^g$.

4. The minimizing movements scheme

In this section we provide a combined spatial and time discretisation of the flow (1.1) for a particular class of norms ϕ and show the convergence of the scheme to the continuum flow. In what follows, we consider $\{e_1, \ldots, e_m\} \subseteq \mathbb{Z}^N$ a finite number of integer vectors spanning the whole $\mathbb{R}^N$, and set $\mathcal{E} = \{\pm e_k\}_{k=1}^m$. We let $\beta \in X$ be a non-negative function such that

$$\beta(-i) = \beta(i) \quad \text{and} \quad \beta(i) > 0 \text{ if and only if } i \in \mathcal{E}.$$

One can naturally associate an anisotropy ϕ with the function β setting

$$\phi(v) = \sum_{i\in\mathcal{E}} \beta(i)|i\cdot v| = \sum_{k=1}^{m} 2\beta(e_k)|v\cdot e_k|. \tag{4.1}$$

Note that, in particular, it holds

$$\#\{k\in\mathbb{Z}^N : \beta(k)\neq 0\} < +\infty. \tag{4.2}$$

We recall that the ϕ-perimeter associated with (4.1)

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu_E)\, d\mathcal{H}^{N-1}$$

(defined for every $E\subseteq\mathbb{R}^N$ of finite perimeter) is the Γ-limit (in a suitable sense) as $\varepsilon\to 0$ of the following scaled discrete perimeters

$$\mathcal{P}^\varepsilon(E) := \varepsilon^{N-1}\sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{ij}|\chi^E_i - \chi^E_j| = \varepsilon^N \sum_{i,j\in\varepsilon\mathbb{Z}^N} \alpha^\varepsilon_{i,j}|(D_\varepsilon\chi^E)_{i,j}|$$

defined for all $E\subseteq\varepsilon\mathbb{Z}^N$, see for instance [9]. Here we have set

$$\alpha^\varepsilon_{ij} := \beta\left(\frac{i}{\varepsilon} - \frac{j}{\varepsilon}\right). \tag{4.3}$$

4.1. **The discrete scheme.** In this section we describe our minimizing movements scheme, discretized in both time and space.

Given ϕ a norm on $\mathbb{R}^N$ and a closed set $E\notin\{\emptyset,\mathbb{R}^N\}$, we denote with $\mathrm{sd}^{\phi^\circ}_E$ the signed ϕ°−distance function from E, which is defined as

$$\mathrm{sd}^{\phi^\circ}_E(x) := \min_{y\in E}\phi^\circ(x-y) - \min_{y\notin E}\phi^\circ(x-y).$$

We also set $\mathrm{sd}^{\phi^\circ}_\emptyset \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N}\equiv -\infty$. We denote

$$C_\phi = \min_{i\in\mathbb{Z}^N\setminus\{0\}} \phi^\circ(i) > 0 \tag{4.4}$$

and define the ϕ-Wulff shape $\mathcal{W}_R(x)$ of radius $R>0$ and center $x\in\mathbb{R}^N$ as $\mathcal{W}_R(x) = \{y\in\mathbb{R}^N : \phi^\circ(x-y) < R\}$.

Recalling (4.3), we rescale equation (3.4) on the lattice $\varepsilon\mathbb{Z}^N$ in the following way. We recall that $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N}$. Given $g\in X_\varepsilon$ the problem (3.4) now becomes to find $(u,z)\in X_\varepsilon\times Y_\varepsilon$ satisfying

$$\begin{cases} hD^*_\varepsilon z + u = g \quad \text{on } \varepsilon\mathbb{Z}^N \\ z_{ij}(u_i-u_j) = \alpha^\varepsilon_{ij}|u_i-u_j|, \quad |z_{ij}|\le\alpha^\varepsilon_{ij}, \end{cases} \tag{4.5}$$

where $D^*_\varepsilon z$ is defined in (3.1).

Given $u \in X_\varepsilon$ we define the operators $d_{\pm}^{\varepsilon,\phi^\circ}$, $\mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}$, $\mathrm{sd}^{\varepsilon,\phi^\circ} : X_\varepsilon \to X_\varepsilon$ in the following way: letting $E = \{i \in \varepsilon\mathbb{Z}^N : u_i \leq 0\}$, we first define

$$
\begin{aligned}
(d_{-}^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\}\,,\\
(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\leq 0\}} \Big\{(d_{-}^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\Big\}\,,\\
(d_{+}^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\leq 0\}} \{u_j + \phi^\circ(i-j)\}\,,\\
(\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\geq 0\}} \Big\{(d_{+}^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\Big\}\,,\\
(\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_i &= \frac{1}{2}(\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u))_i + \frac{1}{2}(\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u))_i.
\end{aligned}
\tag{4.6}
$$

Note that $d_{+}^{\varepsilon,\phi^\circ}(u) = -d_{-}^{\varepsilon,\phi^\circ}(-u)$ and $\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) = -\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(-u)$.

We will say that $f \in X_\varepsilon$ is (L,ϕ°)-Lipschitz if for all $i,j \in \varepsilon\mathbb{Z}^N$ it holds $|f_i - f_j| \leq L\phi^\circ(i-j)$.

Remark 4.1. We assume in what follows that u is $(1,\phi^\circ)$-Lipschitz. Then, concerning $d_{-}^{\varepsilon,\phi^\circ}$, $\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}$, we remark that

$$
d_{-}^{\varepsilon,\phi^\circ}(u) = \min\{f \in X_\varepsilon : f \geq u \text{ in } \{u \geq 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}\,, \tag{4.7}
$$

and analogously

$$
\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) = \max\Big\{f \in X_\varepsilon : f \leq d_{-}^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \leq 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\Big\}\,. \tag{4.8}
$$

Correspondingly it holds

$$
\begin{aligned}
d_{+}^{\varepsilon,\phi^\circ}(u) &= \max\{f \in X_\varepsilon : f \leq u \text{ in } \{u \leq 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}\,,\\
\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) &= \min\Big\{f \in X_\varepsilon : f \geq d_{+}^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \geq 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\Big\}\,,
\end{aligned}
\tag{4.9}
$$

In particular, the functions $d_{\pm}^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are also $(1,\phi^\circ)$-Lipschitz. Let us show (4.7) the other identities being analogous. To this aim, denote by $\hat{d}$ the function defined by the right-hand side of (4.7). Since $d_{-}^{\varepsilon,\phi^\circ}(u)$ is the pointwise supremum of $(1,\phi^\circ)$-Lipschitz functions, we clearly have that $d_{-}^{\varepsilon,\phi^\circ}(u)$ is itself $(1,\phi^\circ)$-Lipschitz. Moreover, testing with $j = i$ in the definition of $d_{-}^{\varepsilon,\phi^\circ}(u)$, we get $d_{-}^{\varepsilon,\phi^\circ}(u) \geq u$ in $\{u \geq 0\}$. Thus, we infer $\hat{d} \leq d_{-}^{\varepsilon,\phi^\circ}(u)$. For the opposite inequality, let f be any functions as in the minimisation problem on the right-hand side of (4.7). Then for any $i \in \varepsilon\mathbb{Z}^N$ and $j \in \{u \geq 0\}$ we have

$$
f_i \geq f_j - \phi^\circ(i-j) \geq u_j - \phi^\circ(i-j)\,.
$$

By maximising with respect to $j \in \{u \geq 0\}$, we get $f \geq d_{-}^{\varepsilon,\phi^\circ}(u)$ and in turn, by the arbitrariness of f, $\hat{d} \geq d_{-}^{\varepsilon,\phi^\circ}(u)$, which concludes the proof of (4.7)

Since the functions $d_{\pm}^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_{\pm}^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are $(1,\phi^\circ)$-Lipschitz, from (4.7) it follows that

$$
d_{-}^{\varepsilon,\phi^\circ}(u) \leq u \text{ in } \varepsilon\mathbb{Z}^N, \quad d_{-}^{\varepsilon,\phi^\circ}(u) = u \text{ in } \{u \geq 0\}, \tag{4.10}
$$

while (4.8) implies that

$$
\mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) \geq d_{-}^{\varepsilon,\phi^\circ}(u) \text{ in } \varepsilon\mathbb{Z}^N, \quad \mathrm{sd}_{-}^{\varepsilon,\phi^\circ}(u) = d_{-}^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \leq 0\}. \tag{4.11}
$$

Reasoning in the same way, we see that

$$
\begin{aligned}
d_{+}^{\varepsilon,\phi^\circ}(u) &\geq u \text{ in } \varepsilon\mathbb{Z}^N, \quad d_{+}^{\varepsilon,\phi^\circ}(u) = u \text{ in } \{u \leq 0\},\\
\mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) &\leq d_{+}^{\varepsilon,\phi^\circ}(u) \text{ in } \varepsilon\mathbb{Z}^N, \quad \mathrm{sd}_{+}^{\varepsilon,\phi^\circ}(u) = d_{+}^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \geq 0\}.
\end{aligned}
\tag{4.12}
$$

In particular we conclude

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \geq u \text{ in } \{u \geq 0\}, \quad \mathrm{sd}^{\varepsilon,\phi^\circ}(u) \leq u \text{ in } \{u \leq 0\}. \tag{4.13}$$

We remark that, always for u a $(1,\phi^\circ)$-Lipschitz function, $\{u \leq 0\} = \{\,\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \leq 0\}$ and $\{u \geq 0\} = \{\,\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \geq 0\}$. In particular, if the level set 0 of u is "fat", then this is preserved by these discrete "signed distance functions". Further properties of these discrete signed distance functions are presented in Lemma 4.5 below and in Remark 4.9

Moreover, it follows directly from the definition of $d^{\varepsilon,\phi^\circ}_{\pm}(u)$, $\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u)$ that the function $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ is invariant under integer translations, meaning that for any $i, \tau \in \varepsilon\mathbb{Z}^N$ it follows

$$\left(\mathrm{sd}^{\varepsilon,\phi^\circ}(u(\cdot+\tau))\right)_i = \left(\mathrm{sd}^{\varepsilon,\phi^\circ}(u)\right)_{i+\tau}. \tag{4.14}$$

Given a set $E \subseteq \varepsilon\mathbb{Z}^N$, we will denote with $\widehat{E} \subseteq \mathbb{R}^N$ the closed set defined by

$$\widehat{E} := E + [0,\varepsilon]^N.$$

We now define the discrete evolution scheme. A particularity of our scheme is that in practice, it evolves the distance function to a set rather than the set itself. In particular, at the discrete level, it may depend on the initialization (even if in the limit the flow is geometric and only depends on the initial set). For ease of notation we assume $\varepsilon = \varepsilon(h)$, with $\varepsilon \to 0$ as $h \to 0$ and we will specify the dependence on h only.

Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. We define $E^{h,0} := \{i \in \varepsilon\mathbb{Z}^N : (i + [0,\varepsilon)^N) \cap E_0 \neq \emptyset\}$. We note that

$$\widehat{E}^{h,0} \to E_0, \quad E^{h,0} \to E_0 \tag{4.15}$$

as $h \to 0$ in the Kuratowski sense, where with a slight abuse of notation we write $\widehat{E}^{h,0}$ to denote the set $\widehat{E^{h,0}} = E^{h,0} + [0,\varepsilon]^N$.

Given a closed set $E_0 \subseteq \mathbb{R}^N$ with $E_0 \notin \{\emptyset, \mathbb{R}^N\}$, we consider $u^{h,0}$ a $(1,\phi^\circ)$-Lipschitz function on $\varepsilon\mathbb{Z}^N$ which is negative inside $E^{h,0}$ and positive outside. For instance, we set

$$u^{h,0} := \frac{1}{2}C_\phi\varepsilon(1-\chi_{E^{h,0}}) - \frac{1}{2}C_\phi\varepsilon\chi_{E^{h,0}},$$

where C_ϕ is defined in (4.4), so that $u^{h,0}$ is $(1,\phi^\circ)$-Lipschitz. Let us set $(z^{h,0})_{ij} = 0$ for all $i,j \in \varepsilon\mathbb{Z}^N$. Then, as long as $E^{h,k} \notin \{\emptyset, \mathbb{R}^N\}$, we can iteratively define $u^{h,k+1}, z^{h,k+1}$ for $k \in \mathbb{N}$ by solving (4.5) with $g = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k})$; i.e.,

$$\begin{cases} hD^*_\varepsilon z^{h,k+1} + u^{h,k+1} = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k}) \quad \text{on } \varepsilon\mathbb{Z}^N \\ z^{h,k+1}_{ij}(u^{h,k+1}_i - u^{h,k+1}_j) = \alpha^\varepsilon_{ij}|u^{h,k+1}_i - u^{h,k+1}_j|, \quad |z^{h,k+1}_{ij}| \leq \alpha^\varepsilon_{ij}. \end{cases} \tag{4.16}$$

We then set

$$E^{h,k+1} = \{i \in \varepsilon\mathbb{Z}^N : u^{h,k+1}_i \leq 0\}.$$

If either $E^{h,k} = \emptyset$ or $E^{h,k} = \mathbb{R}^N$, we define $E^{h,k+1} = E^{h,k}$. We denote by T^*_h the first discrete time hk such that $E^{h,k} = \emptyset$, if any; otherwise we let $T^*_h = +\infty$. Analogously, we set ${T'}^*_h$ first discrete time hk such that $E^{h,k} = \mathbb{R}^N$, if any; otherwise we let $T'^*_h = +\infty$.

For ease of notation we will set

$$
\begin{aligned}
&E^h(t) := E^{h,[t/h]} \subseteq \varepsilon\mathbb{Z}^N \\
&d^h(t) := \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,[t/h]}) \in X_\varepsilon \\
&u^h(t) := u^{h,[t/h]} \in X_\varepsilon \\
&z^h(t) := z^{h,[t/h]} \in Y_\varepsilon \\
&\widehat{d}^h(\cdot,t) := \mathrm{sd}^{\phi^\circ}_{\widehat{E}^h(t)} \in \mathrm{Lip}(\mathbb{R}^N),
\end{aligned}
\tag{4.17}
$$

where again, with a slight abuse of notation, $\widehat{E}^h(t)$ stands for $\widehat{E^h(t)}$. Note that in the definition of $\widehat{d}^h(\cdot,t)$ we are possibly using the convention $\mathrm{sd}^{\phi^\circ}_{\emptyset} \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. Note also that $z^h(t)$ is well defined only for $0 \le t < \min\{T^*_h, T'^*_h\}$; however, if needed, we can set $z^h(t) = 0$ for $t \ge \min\{T^*_h, T'^*_h\}$.

Remark 4.2. If u is the solution of (4.5) with datum (L,ϕ°)-Lipschitz datum g, by standard arguments, based on the comparison principle and translation invariance, one can show that u satisfies the same Lipschitz bound of g. Indeed, given $j \in \varepsilon\mathbb{Z}^N$, the function $u(\cdot - j) \pm L\phi^\circ(j)$ solves (4.5) with datum $g(\cdot - j) \pm L\phi^\circ(j)$. By comparison one concludes as $g(\cdot - j) - L\phi^\circ(j) \le g(\cdot) \le g(\cdot - j) + L\phi^\circ(j)$.

Lemma 4.3. *Let u^h, E^h, d^h be defined as in (4.17). Then, d^h is $(1,\phi^\circ)$-Lipschitz and satisfies for every $t \ge 0$*

$$
\begin{cases}
u^h(t) \le d^h(t) & \text{in } \varepsilon\mathbb{Z}^N \setminus E^h(t) \\
u^h(t) \ge d^h(t) & \text{in } E^h(t).
\end{cases}
\tag{4.18}
$$

Proof. It follows from Remarks 4.1 and 4.2. □

Lemma 4.4. *Given a $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, one has that*

$$
\sup_{\varepsilon\mathbb{Z}^N\setminus E} |\,\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) - \mathrm{sd}^{\phi^\circ}_{\widehat{E}}| \le c_\phi \varepsilon,
\tag{4.19}
$$

for a suitable positive constant c_ϕ, where $E = \{i \in \varepsilon\mathbb{Z}^N : u_i \le 0\}$. Moreover,

$$
\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \ge \mathrm{sd}^{\phi^\circ}_{\widehat{E}} - c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.
\tag{4.20}
$$

Proof. In this proof we let c_ϕ denote a positive constant which depends on ϕ and that may change from line to line and also within the same line.

We start introducing a slightly modified definition of the discrete signed distance $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. Namely, setting

$$
\begin{aligned}
&\partial^+_\varepsilon E := \{i \in \varepsilon\mathbb{Z}^N \setminus E \;:\; \exists j \in E \text{ with } \|i-j\|_\infty = \varepsilon\} \\
&\partial^-_\varepsilon E := \{i \in E \;:\; \exists j \in \varepsilon\mathbb{Z}^N \setminus E \text{ with } \|i-j\|_\infty = \varepsilon\}
\end{aligned},
\tag{4.21}
$$

we define

$$
\tilde{d}_i = \begin{cases}
\inf\{u_j + \phi^\circ(i-j) \;:\; j \in \partial^-_\varepsilon E\}, & \text{for } i \in \varepsilon\mathbb{Z}^N \setminus E \\
\sup\{u_j - \phi^\circ(i-j) \;:\; j \in \partial^+_\varepsilon E\} & \text{for } i \in E
\end{cases}.
\tag{4.22}
$$

We start by showing that

$$
\begin{aligned}
&\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \ge \tilde{d} \quad \text{in } E, \\
&\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \le \tilde{d} \quad \text{in } \varepsilon\mathbb{Z}^N \setminus E.
\end{aligned}
\tag{4.23}
$$

Indeed, we note that for every $i \in E$ we have

$$(\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u))_i = (d_-^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}}\{u_j - \phi^\circ(i-j)\} \geq \sup_{j\in\partial_\varepsilon^+ E}\{u_j - \phi^\circ(i-j)\} = \tilde{d}_i.$$

On the other hand, recalling that $d_-^{\varepsilon,\phi^\circ}(u) \leq u$ in E, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we see

$$(\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\leq 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \leq \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} = \tilde{d}_i.$$

Reasoning analogously we show the same inequalities between $\operatorname{sd}_+^{\varepsilon,\phi^\circ}$ and $\tilde{d}$ and thus prove (4.23).

Next, we prove

$$\sup_{\varepsilon\mathbb{Z}^N}|\tilde{d} - \operatorname{sd}_{\widehat{E}}^{\phi^\circ}| \leq c_\phi\varepsilon. \tag{4.24}$$

Recall that by definition (4.21), since $u \leq 0$ in E and $u > 0$ in $\varepsilon\mathbb{Z}^N \setminus E$ and since u is $(1,\phi^\circ)$-Lipschitz, it holds

$$|u_j| \leq c_\phi\varepsilon \quad \text{for } j \in \partial_\varepsilon^\pm E.$$

Then, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we have

$$\tilde{d}_i = \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} \geq \inf_{j\in\partial_\varepsilon^- E}\phi^\circ(i-j) - c_\phi\varepsilon \geq \operatorname{sd}_{\widehat{E}}^{\phi^\circ}(i) - c_\phi\varepsilon. \tag{4.25}$$

On the other hand, by definition of $\operatorname{sd}_{\widehat{E}}^{\phi^\circ}$ there exists $x \in \partial\widehat{E}$ such that $\operatorname{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x)$. Let $k \in \varepsilon\mathbb{Z}^N$ be the closest point from x in $\partial_\varepsilon^- E$. We have

$$\begin{aligned}\operatorname{sd}_{\widehat{E}}^{\phi^\circ}(i) &= \phi^\circ(i-x) \geq \phi^\circ(i-k) - c_\phi\varepsilon \\ &\geq \phi^\circ(i-k) + u_k - c_\phi\varepsilon \geq \tilde{d}_i - c_\phi\varepsilon.\end{aligned} \tag{4.26}$$

Finally, equation (4.25) and (4.26) imply (4.24) outside E. The other case is analogous.

We now finally prove (4.19) outside E. From (4.23) and (4.24) it holds

$$d_-^{\varepsilon,\phi^\circ}(u) = \operatorname{sd}_-^{\varepsilon,\phi^\circ}(u) \geq \tilde{d} \geq \operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{ in } E.$$

In particular, $\operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is an admissible competitor in (4.8), thus $\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u) \geq \operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ in $\varepsilon\mathbb{Z}^N$. On the other hand, in $\varepsilon\mathbb{Z}^N \setminus E$ it holds (4.23), thus we conclude (4.19) for $\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u)$. Concerning $\operatorname{sd}_+^{\varepsilon,\phi^\circ}(u)$, we note that by Remark 4.1 and the equation above it holds

$$u \geq \operatorname{sd}_-^{\varepsilon,\phi^\circ}(u) \geq \operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

The function $\operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is therefore admissible in (4.9), thus by maximality

$$d_+^{\varepsilon,\phi^\circ}(u) \geq \operatorname{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon.$$

Since $\operatorname{sd}_+^{\varepsilon,\phi^\circ}(u) = d_+^{\varepsilon,\phi^\circ}(u)$ in $\varepsilon\mathbb{Z}^N \setminus E$ we conclude (4.19), taking also into account again (4.23) and (4.24). Finally, (4.20) follows by combining (4.19), (4.23) and (4.24). □

Lemma 4.5. *Given $u \in X_\varepsilon$ and $(1,\phi^\circ)$-Lipschitz, it holds*

$$\operatorname{sd}^{\varepsilon,\phi^\circ}(-u) = -\operatorname{sd}^{\varepsilon,\phi^\circ}(u). \tag{4.27}$$

Furthermore, if $u_1, u_2 \in X_\varepsilon$ are $(1,\phi^\circ)$-Lipschitz and $u_1 \leq u_2$ then

$$\operatorname{sd}^{\varepsilon,\phi^\circ}(u_1) \leq \operatorname{sd}^{\varepsilon,\phi^\circ}(u_2). \tag{4.28}$$

Finally, for any $s > 0$ and $u \in X_\varepsilon$ and $(1,\phi^\circ)$-Lipschitz, it holds

$$\operatorname{sd}^{\varepsilon,\phi^\circ}(u-s) \leq \operatorname{sd}^{\varepsilon,\phi^\circ}(u) - s. \tag{4.29}$$

Proof. For every $i \in \varepsilon\mathbb{Z}^N$ it holds

$$(d_-^{\varepsilon,\phi^\circ}(-u))_i = \max_{j\in\{(-u)\geq 0\}} \{-u_j - \phi^\circ(i-j)\} = -\min_{j\in\{u\leq 0\}} \{u_j + \phi^\circ(i-j)\} = -(d_+^{\varepsilon,\phi^\circ}(u))_i.$$

In turn,

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u))_i &= \min_{j\in\{(-u)\leq 0\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(-u))_j + \phi^\circ(i-j)\Big\} \\ &= -\max_{j\in\{u\geq 0\}} \Big\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\Big\} = -(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.\end{aligned}$$

Reasoning in the same way for $d_+^{\varepsilon,\phi^\circ}$, $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$ we arrive at

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(-u) = -\,\mathrm{sd}_\mp^{\varepsilon,\phi^\circ}(u) \tag{4.30}$$

and thus $\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\,\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. The monotonicity property (4.28) follows easily from Definition (4.6). The proofs of the other results also follow from Definition (4.6), we present only the one concerning (4.29). Fix $s > 0$ and $u \in X_\varepsilon$ be a $(1, \phi^\circ)$-Lipschitz function. By definition of $d_-^{\varepsilon,\phi^\circ}(u)$ we have

$$(d_-^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\} \geq s + \sup_{j\in\{u\geq s\}} \{(u_j - s) - \phi^\circ(i-j)\} = (d_-^{\varepsilon,\phi^\circ}(u-s))_i + s.$$

Analogously

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\leq 0\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\Big\} \\ &\geq s + \inf_{j\in\{u\leq s\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(u-s))_j + \phi^\circ(i-j)\Big\} = s + (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u-s))_i.\end{aligned}$$

Since the proofs for $d_+^{\varepsilon,\phi^\circ}(u)$, $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$ are analogous, we conclude. □

Remark 4.6. (Evolution of the complement) Let $E^h(t), u^h(t)$ be as in (4.17). We note that, if $F_0 \subseteq \mathbb{R}^N$ is a closed set such that $F^{h,0} = \varepsilon\mathbb{Z}^N \setminus E^{h,0}$, then the discrete evolution starting from F_0 coincides with $\{u^h(t) \geq 0\}$ for every $t \geq 0$. Indeed, denoting v^h the discrete evolution starting from F_0, it holds by definition $v^{h,0} = -u^{h,0}$, thus recalling (4.27) we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) = -\,\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0})$$

and, by uniqueness for (4.5) it follows that $v^h(h) = -u^h(h)$. Then we can iterate to conclude.

Remark 4.7 (Comparison principle)**.** Let E_0, F_0 be closed sets in $\mathbb{R}^N$ such that $E^{h,0} \subseteq F^{h,0}$ (note that this condition is satisfied if $E_0 \subseteq F_0$). Let $E^h(t), F^h(t)$ be the corresponding discrete evolutions and let $u^h(t), v^h(t)$ be the associated functions as in (4.17). Then, for every $t \geq 0$ it holds $E^h(t) \subseteq F^h(t)$. This follows easily by iteration from the monotonicity property (4.28) and from the comparison principle for (4.5). One in fact could also consider the "open" discrete evolution given by $\mathring{E}^h(t) := \{u^h(t) < 0\}$ and $\mathring{F}^h(t) := \{v^h(t) < 0\}$. Then, by the same argument one also have that $\mathring{E}^h(t) \subseteq \mathring{F}^h(t)$.

Remark 4.8 (Avoidance principle)**.** Let $E_0, F_0 \subseteq \mathbb{R}^N$ be closed sets such that $E^{h,0} \cap F^{h,0} = \emptyset$ (which is, for example, implied by $\mathrm{dist}(E_0, F_0) > c_\phi \varepsilon$ for a suitable $c_\phi > 0$). Let E^h, u^h and $\mathring{F}^h(t), v^h$ be the closed and open discrete evolutions starting from E_0, F_0 respectively (where the open discrete evolution has been defined in Remark 4.7). Then,

$$\mathring{F}^h(t) \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Indeed, $F^{h,0} \subseteq \varepsilon\mathbb{Z}^N \setminus E^{h,0}$ implies that $-u^{h,0} \le v^{h,0}$ and thus by (4.27) and (4.28)

$$-\operatorname{sd}^{\varepsilon,\phi^\circ}(u^{h,0}) = \operatorname{sd}^{\varepsilon,\phi^\circ}(-u^{h,0}) \le \operatorname{sd}^{\varepsilon,\phi^\circ}(v^{h,0}).$$

By the comparison principle for (4.5) and iterating one sees that $-u^h(t) \le v^h(t)$ for all $t \ge 0$, which implies

$$\mathring{F}^h(t) = \{v^h(t) < 0\} \subseteq \{u^h(t) > 0\} = \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Remark 4.9. We conclude this section by observing that we could have made different choices of the distance function, without affecting the final convergence result. In definition (4.6) we could have set

$$\begin{aligned}
(d^{<}(u))_i &= \inf_{j\in\{u<0\}} \{u_j + \phi^\circ(i-j)\}, \\
(\operatorname{sd}^{<}(u))_i &= \sup_{j\in\{u\ge0\}} \left\{(d^{<}(u))_j - \phi^\circ(i-j)\right\}, \\
(d^{\le}(u))_i &= \inf_{j\in\{u\le0\}} \{u_j + \phi^\circ(i-j)\}, \\
(\operatorname{sd}^{\le}(u))_i &= \sup_{j\in\{u>0\}} \left\{(d^{<}(u))_j - \phi^\circ(i-j)\right\}.
\end{aligned} \tag{4.31}$$

One can see that $\operatorname{sd}^{\le}(u)$ mimics the signed distance function to the boundary of $\{u \le 0\}$ while $\operatorname{sd}^{<}(u)$ mimics the signed distance function to the boundary of $\{u < 0\}$. Defining the algorithm as in (4.16) but with $\operatorname{sd}^{<}$, $\operatorname{sd}^{\le}$ replacing $\operatorname{sd}^{\varepsilon,\phi^\circ}$, adapting our proof one can conclude the same convergence result. Let us further comment on the relation between $\operatorname{sd}^{\varepsilon,\phi^\circ}$, $\operatorname{sd}^{\le}$, $\operatorname{sd}^{<}$. One can prove that for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, then

$$\operatorname{sd}^{\le}(u) \le \operatorname{sd}_-^{\varepsilon,\phi^\circ}(u) \le \operatorname{sd}_+^{\varepsilon,\phi^\circ}(u) \le \operatorname{sd}^{<}(u). \tag{4.32}$$

Thus, between the many possible choices we could have performed in (4.6), it turns out that $\operatorname{sd}^{<}$ is the "maximal" one, while $\operatorname{sd}^{\le}$ is the "minimal". Indeed, let us show that $\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u) \le \operatorname{sd}_+^{\varepsilon,\phi^\circ}(u)$. By definition (4.6) and (4.10), (4.12) for every $i \in \{u \ge 0\}$ it holds

$$(\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\le0\}} \left\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\right\} = \inf_{j\in\{u\le0\}} \{u_j + \phi^\circ(i-j)\} = (\operatorname{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning analogously, for every $i \in \{u \le 0\}$ it holds

$$(\operatorname{sd}_+^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\ge0\}} \left\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\right\} = \sup_{j\in\{u\ge0\}} \{u_j - \phi^\circ(i-j)\} = (\operatorname{sd}_-^{\varepsilon,\phi^\circ}(u))_i.$$

Furthermore, for any two $(1,\phi^\circ)$-Lipschitz functions $u, u' \in X_\varepsilon$, if $u \le u' - s$ for $s > 0$ then

$$\operatorname{sd}^{<}(u) \le \operatorname{sd}^{\le}(u') - s.$$

In particular, this implies that for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $s' > s$ then

$$\operatorname{sd}^{\varepsilon,\phi^\circ}(u-s) \le \operatorname{sd}^{\varepsilon,\phi^\circ}(u-s') + s' - s.$$

Fix $u_0 \in X_\varepsilon$ is a $(1,\phi^\circ)$-Lipschitz function. Using the properties above and standard arguments, one can see that for all but countably many $s \in \mathbb{R}$ the discrete evolutions starting from $\{u_0 \le s\}$ and corresponding to the three possible choices of distances in (4.32) coincide.

4.2. **Discrete evolution of Wulff shapes.** In this section we provide some control on the evolution speed of discrete Wulff shapes. The first result estimates the solution of (4.5) for the distance to the Wulff shape.

Lemma 4.10. *There exists a constant $C = C(\phi) > 0$ with the following property. If u is the solution of (4.5) with $g = \phi^\circ$, then $u \le \phi^h$, where $\phi^h \in X_\varepsilon$ is defined as*

$$\phi^h_i := \begin{cases} \phi^\circ(i) + \frac{Ch}{\phi^\circ(i)} & \text{if } \phi^\circ(i) \ge C(\sqrt{h} \vee \varepsilon) \\ C(\sqrt{h} \vee \varepsilon) + \frac{Ch}{\sqrt{h} \vee \varepsilon} & \text{otherwise.} \end{cases} \tag{4.33}$$

The proof of Lemma 4.10, based on the construction of a calibration, is postponed to Appendix A. We now prove a useful lemma used to estimate the redistancing step in our algorithm for functions of the form of (4.33).

Lemma 4.11. *Let $R \ge \delta > 0$ and set*

$$u := (\phi^\circ - R) \vee (\delta/2 - R).$$

Then, for ε, h small enough depending on δ it holds

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R + \hat{c}\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N, \tag{4.34}$$

for a suitable positive constant $\hat{c}$, depending on ϕ. Furthermore, if we assume (B.1), it holds

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \phi^\circ - R \quad \text{in } \varepsilon\mathbb{Z}^N. \tag{4.35}$$

Proof. By (4.32), it is sufficient to prove the claim for $\mathrm{sd}^{\varepsilon,\phi^\circ}_+$. We start showing that $d^{\varepsilon,\phi^\circ}_+(u) = u$, noting that by (4.12) it suffices to prove $d^{\varepsilon,\phi^\circ}_+(u) \le u$ in $\{u \ge 0\} = \{\phi^\circ \ge R\}$. Assuming (B.1), given $i \in \{u \ge 0\}$ we note that $\phi^\circ(i) \ge R$ thus by Lemma B.1 there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ satisfying

$$\phi^\circ(j) + \phi^\circ(i-j) = \phi^\circ(i).$$

Taking $\varepsilon = \varepsilon(\delta)$ we can ensure that $R - 2\varepsilon\ell_1 \ge \delta/2$, so that $j \in (\mathcal{W}_R \setminus \mathcal{W}_{\delta/2}) \cap \varepsilon\mathbb{Z}^N$. By definition (4.31) and the equation above we conclude that

$$d^{\varepsilon,\phi^\circ}_+(u) \le u_j + \phi^\circ(i-j) = \phi^\circ(j) - R + \phi^\circ(i-j) = \phi^\circ(i) - R,$$

hence we have shown that $d^{\varepsilon,\phi^\circ}_+(u) = u$. Finally, from the definition (4.6) and since $d^{\varepsilon,\phi^\circ}_+(u) = u = \phi^\circ - R$ on $\{u \ge 0\}$, we conclude by the triangular inequality that $\mathrm{sd}^{\varepsilon,\phi^\circ}_+(u) \le \phi^\circ - R$. All in all, we have obtained (4.35).

If instead (B.1) does not hold, using the first part of Lemma B.1 and reasoning as above, one concludes that

$$\mathrm{sd}^{\varepsilon,\phi^\circ}_+(u) \le \phi^\circ - R + \hat{c}\varepsilon,$$

for a positive constant $\hat{c}$, and then the conclusion follows. □

Combining the two results above we can provide a bound on the evolution speed of Wulff shapes in the algorithm (4.16).

Proposition 4.12. *Assume either $\varepsilon \le O(h)$ or that (B.1) holds. For every $\delta > 0$ there exist ε_0, h_0, c_0 positive constants depending on δ with the following property. If $R \ge \delta$, $\varepsilon \le \varepsilon_0$ and $h \le h_0$, then the discrete evolution of $\mathcal{W}_R$ defined in (4.16), denoted $\mathcal{W}^h(t)$, satisfies*

$$\mathcal{W}^h(t) \supseteq (\mathcal{W}_{R-c_0(t+\varepsilon)}) \cap \varepsilon\mathbb{Z}^N), \tag{4.36}$$

as long as $R - c_0(t+\varepsilon) \ge \delta/2$.

Proof. Let $\mathring{\mathcal{W}}^h(t)$ be the open discrete evolution (see Remark 4.7) starting from the closure of $\mathcal{W}_R$, for some $R > 0$ and let $v^h(t)$ be the associated function as in (4.17). Using the definition of $v^{h,0}$, (4.11) and the first definition in (4.6), it is easy to see that

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 = (d_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le -R + c_\phi\varepsilon. \tag{4.37}$$

On the other hand, consider $i \in \{v^{h,0} \ge 0\}$ and let $x' \in \partial\mathcal{W}_R$ be such that

$$\phi^\circ(i - x') = \phi^\circ(i) - \phi^\circ(x') = \phi^\circ(i) - R.$$

Since there exists $j' \in \{v^{h,0} \le 0\}$ such that $\phi^\circ(j' - x') \le c_\phi\varepsilon$, then by triangular inequality

$$\phi^\circ(i - j') \le \phi^\circ(i) - R + c_\phi\varepsilon.$$

Thus, using again definition (4.6), we get

$$(d_+^{\varepsilon,\phi^\circ}(v^{h,0}))_i \le \inf_{j\in\{v^{h,0}\le 0\}} \phi^\circ(i - j) \le \phi^\circ(i) - R + c_\phi\varepsilon,$$

which implies

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \le \sup_{j\in\{v^{h,0}\ge 0\}} (d_+^{h,0}(v^{h,0}))_j - \phi^\circ(j) \le -R + c_\phi\varepsilon. \tag{4.38}$$

Therefore, since $\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0})$ is a $(1,\phi^\circ)$-Lipschitz function, from (4.37), (4.38) we get that

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) \le \phi^\circ - R + c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.$$

By comparison and Lemma 4.10 we obtain

$$v^h(h) \le \phi^h - R + c_\phi\varepsilon, \tag{4.39}$$

where $\phi^h \in X_\varepsilon$ is defined in (4.33). Considering $R \ge \delta$ and $h = h(\delta), \varepsilon = \varepsilon(\delta)$ small enough, the equation above implies that

$$v^h(h) \le (\phi^\circ - R + c_0 h + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right) \tag{4.40}$$

where $c_0 = 4C/\delta$, with C the same as in (4.33). Assume first (B.1). From Lemma 4.11, with R replaced by $R - c_0 h - c_\phi\varepsilon$, we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + c_\phi\varepsilon, \tag{4.41}$$

therefore by comparison and Lemma 4.10 we get

$$v^h(2h) \le \phi^h - R + c_0 h + c_\phi\varepsilon,$$

which, reasoning as above, implies for $\varepsilon(\delta), h(\delta)$ small

$$v^h(2h) \le (\phi^\circ - R + 2c_0 h + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right).$$

Hence, we can iterate the argument to conclude that

$$v^h(t) \le (\phi^\circ - R + c_0 t + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right), \tag{4.42}$$

as long as $R - c_0 t - c_\phi\varepsilon \ge \delta/2$ and ε, h are sufficiently small. In particular, this implies (4.36) (possibly changing the value of c_0).

If instead (B.1) does not hold and $\varepsilon \le O(h)$, we obtain (4.39), (4.40) in the same way. Then, using the first part of Lemma 4.11 we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \le \phi^\circ - R + c_0 h + \hat{c}\varepsilon + c_\phi\varepsilon, \tag{4.43}$$

then iterating we get

$$v^h(kh) \le (\phi^\circ - R + kc_0 h + k\hat{c}\varepsilon + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right),$$

hence, recalling that $\varepsilon \le O(h)$ we conclude (4.42) and (4.36), as long as $R - c_0 t - c_\phi \varepsilon \ge \delta/2$, with ε, h sufficiently small and possibly changing the value of c_0. □

As a corollary of the previous result, we deduce an estimate of the evolution of the distance function $\widehat{d}^h$ at distance from the evolving boundary, which we show next.

Corollary 4.13. *Let $E_0 \subseteq \mathbb{R}^N$ be a closed set and consider the discrete evolution defined in (4.17). Assume either that $\varepsilon \le O(h)$ or that (B.1) holds. Then, for every $\delta > 0$ there exist $c_0 = c_0(\delta) > 0$, $h_0 = h_0(\delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta)$ such that the following holds. If $\widehat{d}^h(x,t) \ge \delta$, then for $s \ge t$,*

$$\widehat{d}^h(x,s) \ge \widehat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \tag{4.44}$$

provided $0 < h \le h_0$, $0 < \varepsilon < \varepsilon_0$ and as long as $\widehat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \ge \delta/2$. Similarly, if $\widehat{d}^h(x,t) \le -\delta$, then for $s \ge t$,

$$\widehat{d}^h(x,s) \le \widehat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \tag{4.45}$$

provided $0 < h \le h_0$ and as long as $\widehat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \le -\delta/2$.

Proof. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

Assume $\widehat{d}^h(x,t) \ge \delta$. Without loss of generality we may assume $t \in [0, T_h^*)$ so that $\widehat{d}^h(x,t)$ is finite. Denote by $x_\varepsilon \in \varepsilon\mathbb{Z}^N$ such that $x \in x_\varepsilon + [0,\varepsilon)^N$. Note that there exists a constant $c_\phi > 0$ such that, setting $R := \widehat{d}^h(x,t) - c_\phi\varepsilon$, one has $(\mathcal{W}_R(x_\varepsilon))^{h,0} \cap E^h(t) = \emptyset$ and $R > \delta/2$ (if ε, h are sufficiently small, depending on δ). By the avoidance principle stated in Remark 4.8, we deduce that the open discrete evolution of $\mathcal{W}_R(x_\varepsilon)$, which we denote by $F(\tau)$, lies outside $E^h([\frac{t}{h}]h + \tau)$ for all $\tau \ge 0$. By Proposition 4.12 we deduce

$$F(\tau) \supseteq \mathcal{W}_{R - c_0(\tau+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N, \tag{4.46}$$

provided that $R - c_0(\tau + \varepsilon) \ge \delta/2$. Note that in particular

$$(\mathcal{W}_{R - c_0(\tau+h+\varepsilon)}(x_h) \cap \varepsilon\mathbb{Z}^N) \subseteq (\varepsilon\mathbb{Z}^N \setminus E^h(t+\tau)),$$

as long as $R - c_0(\tau + h + \varepsilon) \ge \delta/2$. In turn, we get

$$\widehat{d}^h(x_\varepsilon, t+\tau) \ge R - c_0(\tau + h + \varepsilon), \tag{4.47}$$

provided $R - c_0(\tau + h + \varepsilon) \ge \delta/2$ (for a possibly larger value of c_0). Recalling the definition of R and x_ε and possibly increasing the value of c_0, we infer

$$\widehat{d}^h(x, t+\tau) \ge \widehat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \tag{4.48}$$

as long as $\widehat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \ge \delta$. The case $\widehat{d}^h(x,t) \le -\delta$ is analogous. □

5. Convergence of the scheme

We now are ready to study the convergence of the scheme as $\varepsilon \to 0, h \to 0$. Recall that we assumed that $\varepsilon = \varepsilon(h)$ goes to 0 as $h \to 0$. In this section we assume that either $\varepsilon \le O(h)$ or that (B.1) holds. Let $E^h(\cdot)$ be the discrete evolution defined in (4.17) and recall that $\widehat{E}^h(\cdot) = E^h(\cdot) + [0,\varepsilon]^N$. We introduce the closed space-time tubes

$$\overline{E}^h := \mathrm{cl}\big(\{(x,t) \in \mathbb{R}^N \times [0,+\infty) : x \in \widehat{E}^h(t)\}\big) \tag{5.1}$$

where the closure is in space-time. Then, there exist A, E open and closed (respectively) subsets of $\mathbb{R}^N \times [0,+\infty)$, with $A \subseteq E$, and a subsequence $h_k \to 0$ such that

$$\overline{E}^{h_k} \xrightarrow{\mathcal{K}} E \qquad \text{and} \qquad \mathbb{R}^N \times [0,+\infty) \setminus \mathrm{int}\big(\overline{E}^{h_k}\big) \xrightarrow{\mathcal{K}} \mathbb{R}^N \times [0,+\infty) \setminus A,$$

where interior, and Kuratowski convergence are meant in space-time. Let $E(t)$ and $A(t)$ be the t-time slice of E and A, respectively..

Note that if $E(t) = \emptyset$ for some $t \ge 0$, then (4.44) implies $E(s) = \emptyset$ for all $s \ge t$ so that we can define, as in Definition 2.1, the extinction time T^* of E. In the same fashion one can define the extinction time T'^* of $\mathbb{R}^N \times [0,+\infty) \setminus A$ (notice that at least one between T^* and T'^* is $+\infty$). Possibly extracting a further (not relabelled) subsequence and arguing exactly as in [22, Proof of Proposition 4.4] (and relying on the bounds (4.44) and (4.45)), one can in fact show the following result.

Proposition 5.1. *There exists a countable set $\mathcal{N} \subseteq (0,+\infty)$ such that $\widehat{d}^{h_k}(\cdot,t)^+ \to \mathrm{dist}^{\phi^\circ}(\cdot, E(t))$ and $\widehat{d}^{h_k}(\cdot,t)^- \to \mathrm{dist}^{\phi^\circ}(\cdot, \mathbb{R}^N \setminus A(t))$ locally uniformly for all $t \in (0,+\infty) \setminus \mathcal{N}$. Moreover, E and $\mathbb{R}^N \times [0,+\infty) \setminus A$ satisfy the continuity properties (b) and (c) of Definition 2.1. In addition, if $T^* > 0$, then $\{\widehat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0,T^*) \setminus E$ and analogously $\{\widehat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0,T'^*) \cap A$ if $T'^* > 0$. Finally, $E(0) = E_0$ and $A(0) = \mathrm{int}(E_0)$.*

Theorem 5.2. *The set E is a superflow in the sense of Definition 2.1 with initial datum E_0, while A is a subflow with initial datum E_0.*

The proof of this result follows the main lines of the proof of [22, Theorem 4.5]. One important difference with respect to the local, continuous setting is that the variable z^{h_k} is defined on the *edges* (i,j) between the vertices $i \in \varepsilon\mathbb{Z}^N$ and it is therefore unclear how to pass to the limit in this variable to obtain the limiting vector field $z(x,t)$. In order to do so, we associate with the discrete vector field $z^h_{ij}(t) \in Y_\varepsilon$ a vector field $\mathbf{z}^h(\cdot,t)$ in $\mathbb{R}^N$ defined as follows:

$$\mathbf{z}^h(x,t) := \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i-j), \tag{5.2}$$

where $i \in \varepsilon\mathbb{Z}^N$ is such that $x \in i + [0,\varepsilon)^N$. Recall that we can take $z^h_{ij}(t)$ and thus $\mathbf{z}^h(\cdot,t)$ identically zero for $t \ge \min\{T^*_h, T'^*_h\}$. First, we show the following:

Lemma 5.3. *The vector field $\mathbf{z}^h$ satisfies*

$$\phi^\circ(\mathbf{z}^h) \le 1. \tag{5.3}$$

Proof. Take $v \ne 0$ in $\mathbb{R}^N$. Recalling that $\phi(v) = \sum_{\ell \in \mathbb{Z}^N} \beta(\ell)|v \cdot \ell|$, one has for any $x \in \mathbb{R}^N$ and $i \in \varepsilon\mathbb{Z}^N$ such that $x \in i + [0,\varepsilon)^N$

$$\mathbf{z}^h(x,t) \cdot v = \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i-j) \cdot v = \sum_{\ell \in \mathbb{Z}^N} z^h_{i,i+\varepsilon\ell}(t)\ell \cdot v \le \phi(v), \tag{5.4}$$

where we used that $|z^h_{i,i+\varepsilon\ell}(t)| \le \beta(\ell)$. ∎

Hence, being globally bounded, this vector field is weakly-$*$ compact in $L^\infty(\mathbb{R}^N\times(0,T);\mathbb{R}^N)$ for any $T>0$. The following lemma establishes a relationship between the divergence of its limits and the limits of the discrete divergences of z^h.

Lemma 5.4. *Assume that $\mathbf{z}^{h_k} \overset{*}{\rightharpoonup} z$ in $L^\infty(\mathbb{R}^N\times(0,T);\mathbb{R}^N)$ along a subsequence $h_k\to 0$. Then, for every $\varphi\in C^\infty(\mathbb{R}^N\times(0,T))$ and $\eta\in C^\infty_c(\mathbb{R}^N\times(0,T))$ it holds*

$$\lim_{k\to\infty}\left(\varepsilon_k^N\int\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\,\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k}\,\mathrm{d}t\right)=\iint \eta\, z\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $\varphi\in C^\infty(\mathbb{R}^N)$ and $\eta\in C^\infty_c(\mathbb{R}^N)$ and denote $S(t)=\operatorname{supp}(\eta(t))$ and $Q_k:=[0,\varepsilon_k)^N$. We have

$$\varepsilon_k^N\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\,\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k}=\varepsilon_k^N\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)}{\varepsilon_k}\eta(i,t)\,\nabla\varphi(x_{ij})\cdot(i-j), \tag{5.5}$$

where x_{ij} belongs to the segment between i and j. Furthermore we have

$$\begin{aligned}
&\left|\varepsilon_k^N\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\,\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k}-\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)}{\varepsilon_k}\int_{i+Q_k}\eta\nabla\varphi\cdot(i-j)\,\mathrm{d}x\right|\\
&\le\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}|\eta(i,t)|\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x+O(\varepsilon_k^N) &(5.6)\\
&\le 2\|\eta\|_\infty\sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x+O(\varepsilon_k^N)\\
&\le c\varepsilon_k^N\sum_{i\in S\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}+\alpha_{ji}^{\varepsilon_k}}{\varepsilon_k}|i-j|^2+O(\varepsilon_k^N) &(5.7)\\
&=c\varepsilon_k^{N+1}\sum_{\substack{i\in\mathbb{Z}^N\\ \varepsilon_k i\in S(t)}}\sum_{j\in\mathbb{Z}^N}\alpha_{ij}|i-j|^2+O(\varepsilon_k^N)\\
&\le c\varepsilon_k^{N+1}\left(\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2\right)(\#S(t)\cap\varepsilon_k\mathbb{Z}^N)+O(\varepsilon_k^N)\\
&\le c\varepsilon_k\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2+O(\varepsilon_k^N) &(5.8)
\end{aligned}$$

where in (5.6) we used the Lipschitz property of η and (4.2), while in (5.7) we used the Lipschitz property of $\nabla\varphi$ and $|x_{ij}-x|\le(1+\sqrt{N})|i-j|$ for $i\neq j$ and $x\in i+Q_k$, and finally in (5.8) we used that $\#(S(t)\cap\varepsilon\mathbb{Z}^N)=O(\varepsilon_k^{-N})$, which holds locally uniformly in time. Moreover, note that the the estimate provided above is uniform as t varies in compact subsets of $(0,T)$. Recalling (4.2), we conclude integrating in time and sending $k\to\infty$. □

At this point, we may proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

We only show that E is a superflow, as the subflow property of A can be proven analogously. Points (a), (b) and (c) of Definition 2.1 follow from Proposition 5.1. We are left with showing (d). Without loss of generality we may assume $T^*>0$ (which follows from Corollary 4.13 if the initial set is not trivial). Note also that by Proposition 5.1 we have $\liminf_k T^*_{h_k}\ge T^*$.

Step 1: (Proof of (2.1)). For $(x,t) \in \mathbb{R}^N \times (0,T^*) \setminus E$ we set $d(x,t) := \mathrm{dist}^{\phi^\circ}(\cdot, E(t))$. By Lemma 4.4 and Proposition 5.1 we have

$$\sup_{\varepsilon_k \mathbb{Z}^N \cap K} |d^{h_k}(t) - d(\cdot,t)| \to 0 \text{ as } k \to \infty \text{ for } t \in (0,T^*) \setminus \mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N \setminus E(t). \tag{5.9}$$

Moreover, d^{h_k} and d are locally uniformly bounded in $\mathbb{R}^N \times (0,T^*) \setminus E$. Set $\mathbf{z}^{h_k}(\cdot,t) := 0$ for $t > T^*_{h_k}$ if $T^*_{h_k} < T^*$. Extracting a further subsequence, if needed, and recalling Lemma 5.3, we may assume that $\mathbf{z}^{h_k}$ converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*); \mathbb{R}^N)$ to some vector-field z satisfying

$$\phi^\circ(z) \le 1 \tag{5.10}$$

almost everywhere. Recall that by (4.18) we have $u^{h_k}(t) \le d^{h_k}(t)$ in $\varepsilon_k \mathbb{Z}^N \setminus E^{h_k}(t)$; i.e., in the region where $d^{h_k}(t)$ is nonnegative. Combining with (4.16) (and recalling (4.17)) we infer that for $t < T^*_{h_k}$ it holds

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \le \frac{d^{h_k}(t+h_k) - d^{h_k}(t)}{h_k} \qquad \text{in } \varepsilon_k \mathbb{Z}^N \setminus E^{h_k}(t). \tag{5.11}$$

Consider a nonnegative test function $\varphi \in C^\infty_c((\mathbb{R}^N \times (0,T^*)) \setminus E)$. If k is large enough, then the distance of the support of φ from $\overline{E}^{h_k}$ is bounded away from zero. In particular, d^{h_k} is finite and positive on $\operatorname{supp} \varphi$. We deduce from (5.11) that

$$\begin{aligned}
&\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \varphi(i,t) \left(\frac{d_i^{h_k}(t+h_k) - d_i^{h_k}(t)}{h_k} + (D^*_{\varepsilon_k} z^{h_k}(t+h_k))_i \right) \mathrm{d}t \\
&= -\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \frac{\varphi(i,t) - \varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\, \mathrm{d}t + \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} \frac{z_{ij}^{h_k}(t+h_k) - z_{ji}^{h_k}(t+h_k)}{h_k} \varphi(i,t)\, \mathrm{d}t \\
&= -\varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} \frac{\varphi(i,t) - \varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\, \mathrm{d}t + \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} z_{ij}^{h_k}(t+h_k) \frac{\varphi(i,t) - \varphi(j,t)}{h_k}\, \mathrm{d}t \ge 0.
\end{aligned} \tag{5.12}$$

It is easy to check that the first integral in (5.12) converges to $-\iint d\, \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t$ as $k \to \infty$ thanks to (5.9) and since d^{h_k}, d are uniformly bounded. Recalling that $\mathbf{z}^{h_k}$ converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*))$ to z, we use Lemma 5.4 to conclude that the second integral in (5.12) converges to $\iint z \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t$. We thus conclude (2.1).

Step 2: (Convergence of u^{h_k} to d). Firstly, we establish an upper bound for $-D^*_{\varepsilon_k} z_{h_k}$ away from E^{h_k}. We start by noting that definition (4.6) implies

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \le \frac{1}{2} \Big((d_-^{\varepsilon,\phi^\circ}(u))_j + u_\ell + \phi^\circ(\cdot - j) + \phi^\circ(\cdot - \ell) \Big) \quad \text{in } \varepsilon \mathbb{Z}^N \setminus \{u \le 0\}, \tag{5.13}$$

for every $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $j, \ell \in \{u \le 0\}$. Therefore, specifying the inequality above for $u^{h_k}(t)$, by the comparison principle and Lemma 4.10 we conclude

$$u_i^{h_k}(t+h_k) \le \frac{1}{2} \Big(\phi^{h_k}_{i-j} + \phi^{h_k}_{i-\ell} + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t)))_j + u_\ell^{h_k}(t) \Big), \quad \forall i \in \varepsilon_k \mathbb{Z}^N \setminus E^{h_k}(t), \tag{5.14}$$

where $j, \ell \in E^{h_k}(t)$. If $\widehat{d}^{h_k}(i,t) \ge R > 0$, recalling the definition of ϕ^h, we get

$$u_i^{h_k}(t+h_k) \le \frac{1}{2} \Big(\phi^\circ(i-j) + \phi^\circ(i-\ell) + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t)))_j + u_\ell^{h_k}(t) \Big) + \frac{Ch_k}{R - c_\phi \varepsilon}, \tag{5.15}$$

for all $i \in \varepsilon_k\mathbb{Z}^N \setminus E^{h_k}(t)$. Infimizing in j, ℓ over $E^{h_k}(t)$ in (5.15) and using again (4.6) and (4.12), we conclude

$$u_i^{h_k}(t+h_k) \le d_i^{h_k}(t) + h_k \frac{C}{R - c_\phi \varepsilon_k} \le d_i^{h_k}(t) + h_k \frac{C}{R}. \tag{5.16}$$

provided h_k, ε_k are small enough depending on R, and for a possibly larger value of C. As a consequence of (5.16), we obtain

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \le \frac{C}{R} \qquad \text{in } \{\widehat{d}^{h_k}(\cdot,t) \ge R\} \cap \varepsilon_k\mathbb{Z}^N. \tag{5.17}$$

Using again Lemma 5.4 and the convergences of E_{h_k} and d_{h_k} it follows that

$$\mathrm{div} z \le \frac{C}{R} \qquad \text{in } \{(x,t) \in \mathbb{R}^N \times (0,T^*) :\, d(x,t) > R\}$$

in the sense of distributions. Hence $\mathrm{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\mathrm{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) :\, d(x,t) \ge \delta\})$ for every $\delta > 0$.

On the other hand, note that for every $i \in \varepsilon_k\mathbb{Z}^N$ it holds

$$d^{h_k}(t) \ge d_i^{h_k}(t) - \phi^\circ(\cdot - i).$$

Thus, by Lemma 4.10 and by comparison as before, we get

$$u_i^{h_k}(t+h_k) \ge d_i^{h_k}(t) - \phi_0^{h_k} = d_i^{h_k}(t) - (C+1)\sqrt{h_k}.$$

Combining the above inequality with (5.16), we deduce for all $t \in (0,T^*) \setminus \mathcal{N}$ and any $\delta > 0$ that

$$\sup_{\{\widehat{d}_{h_k}(\cdot,t) \ge \delta\} \cap \varepsilon_k\mathbb{Z}^N} |u^{h_k}(t+h_k) - d^{h_k}(t)| \le \sqrt{h_k}(C+2),$$

provided that k is large enough. In particular, recalling also (5.9), we deduce that

$$\sup_{\varepsilon_k\mathbb{Z}^N \cap K} |u^{h_k}(t) - d(\cdot,t)| \to 0 \text{ as } k \to \infty \text{ for } t \in (0,T^*) \setminus \mathcal{N} \text{ and for any compact } K \subseteq \mathbb{R}^N \setminus E(t), \tag{5.18}$$

also with the sequence $\{u^{h_k}\}$ locally (in space and time) uniformly bounded.

Step 3: (The subdifferential inclusion). It remains to show that

$$z \in \partial\phi(\nabla d) \quad \text{a.e. in } \mathbb{R}^N \times (0,T^*) \setminus E. \tag{5.19}$$

Recall that $\xi \in \partial\phi(\eta)$ if and only if $\xi \in \{v : \phi^\circ(v) \le 1,\ v \cdot \eta \ge \phi(\eta)\}$. Since one inequality has been proved in (5.10), we show the other one. Consider a test function $\eta \ge 0$, $\eta \in C_c^\infty((\mathbb{R}^N \times (0,T^*)) \setminus E)$. Let $\sigma > 0$ and set $d_\sigma \in C^\infty(\mathbb{R}^N \times (0,T^*))$ as $d_\sigma = d * \rho_\sigma$, where ρ_σ are space-time mollifiers. Obviously

$$\begin{aligned}\sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t) - u_j^{h_k}(t)) &= \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t))(d_\sigma(i,t) - d_\sigma(j,t)) \\ &\quad + \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\left(u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) + d_\sigma(j,t)\right).\end{aligned} \tag{5.20}$$

In turn, Lemma 5.4 implies that

$$\lim_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t) \frac{d_\sigma(i,t) - d_\sigma(j,t)}{\varepsilon_k} \right) \mathrm{d}t = \iint z \cdot \nabla d_\sigma\, \eta \,\mathrm{d}x\,\mathrm{d}t. \tag{5.21}$$

Let us thus show that

$$\lim_{\sigma\to 0}\lim_{k\to\infty} \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} \left(z_{ij}^{h_k}(t)\eta(i,t) \frac{u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) - d_\sigma(j,t)}{\varepsilon_k} \right) \mathrm{d}t = 0, \tag{5.22}$$

We set for every $t \in (0, T_h^*)$ and $\sigma > 0$

$$m_{k,\sigma}(t) := \min_{i \in \mathrm{supp}(\eta) \cap \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)),$$
$$M_{k,\sigma}(t) := \max_{i \in \mathrm{supp}(\eta) \cap \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)).$$

The convergence (5.18) implies that these quantities are uniformly bounded and

$$\lim_{\sigma \to 0} \lim_{k \to +\infty} m_{k,\sigma}(t) = 0, \qquad \lim_{\sigma \to 0} \lim_{k \to +\infty} M_{k,\sigma}(t) = 0, \tag{5.23}$$

uniformly for all $t \notin \mathcal{N}$. For all times $t \in (0, T^*) \setminus \mathcal{N}$ it holds

$$\varepsilon_k^N \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} z_{ij}^{h_k}(t) \eta(i,t) \frac{u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) + d_\sigma(j,t)}{\varepsilon_k}$$
$$= \varepsilon_k^N \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} z_{ij}^{h_k}(t) \eta(i,t) \frac{(u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) - (u_j^{h_k}(t) - d_\sigma(j,t) - m_{k,\sigma}(t))}{\varepsilon_k}$$
(5.24)
$$= \varepsilon_k^N \sum_{i \in \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) \sum_{j \in \varepsilon_k \mathbb{Z}^N} \left(\frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \eta(i,t) + z_{ji}^{h_k}(t) \frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k} \right).$$

For k large enough, since the support of η is at positive distance from E, by the bound (5.17) one has $D_{\varepsilon_k}^* z^{h_k}(t) \geq -c(\delta)$ on the support for h_k small enough. Thus it holds

$$\varepsilon_k^N \sum_{i \in \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) \eta(i,t) \sum_{j \in \varepsilon_k \mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k}$$
$$\geq -c(\delta) \varepsilon_k^N \sum_{i \in \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{k,\sigma}(t)) \eta(i,t).$$

Recalling that $\#(\mathrm{supp}(\eta) \cap \varepsilon_k \mathbb{Z}^N) = O(h_k^{-N})$ uniformly in time, by uniform convergence and (5.18) we conclude that

$$\lim_{\sigma \to 0} \liminf_{k \to \infty} \varepsilon_k^N \int \sum_{i \in \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t)) \eta(i,t) \sum_{j \in \varepsilon_k \mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \,\mathrm{d}t \geq 0. \tag{5.25}$$

The other term in (5.24) can be estimated using the Lipschitz constant of η:

$$\left| \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} \varepsilon_k^N (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k} \,\mathrm{d}t \right|$$
$$\leq \|\nabla \eta\|_\infty \varepsilon_k^N \int \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t) - m_{\varepsilon,k}(t)) \alpha_{ji}^{h_k} \frac{|i-j|}{\varepsilon_k} \,\mathrm{d}t \to 0$$

letting first $k \to +\infty$ and then $\sigma \to 0$, thanks to (5.18) and (5.23). Note now that adding and subtracting $M_{\varepsilon,k}(t)$ to (5.22) instead of $m_{\varepsilon,k}(t)$ and reasoning as above, one proves that

(5.26)
$$\lim_{\sigma \to 0} \limsup_{k \to \infty} \varepsilon_k^N \int \left(\sum_{i \in \varepsilon_k \mathbb{Z}^N} ((u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t)) \eta(i,t) \sum_{j \in \varepsilon_k \mathbb{Z}^N} \frac{z_{ij}^{h_k}(t) - z_{ji}^{h_k}(t)}{\varepsilon_k} \right) \mathrm{d}t \leq 0,$$
$$\lim_{\varepsilon \to 0} \lim_{k \to \infty} \varepsilon_k^N \int \left| \sum_{i,j \in \varepsilon_k \mathbb{Z}^N} ((u_i^{h_k}(t) - d_\sigma(i,t) - M_{\varepsilon,k}(t)) z_{ji}^{h_k}(t) \frac{\eta(i,t) - \eta(j,t)}{\varepsilon_k} \right| \mathrm{d}t = 0.$$

Combining (5.24), (5.25) and (5.26), we conclude (5.22).

Integrating in time (5.20) and combining (5.21) and (5.22), since $\nabla d_\sigma = \rho_\sigma * \nabla d \to \nabla d$ pointwise a.e. and are uniformly bounded in $L^\infty(\mathbb{R}^N \times (0, T^*); \mathbb{R}^N)$, it holds

$$\lim_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\, z_{ij}^{h_k}(t) \frac{u_i^{h_k}(t) - u_j^{h_k}(t)}{\varepsilon_k} \right) \mathrm{d}t = \iint z \cdot \nabla d\, \eta \,\mathrm{d}x\,\mathrm{d}t.$$

The convergence above can be paired with the lower semicontinuity of the Γ-convergence of the discrete total variations (which follows from an adaptation of classical arguments, see e.g. [19]) and $z_{ij}^\varepsilon(u_i^\varepsilon - u_j^\varepsilon) = \alpha_{ij}^\varepsilon |u_i^\varepsilon - u_j^\varepsilon|$ to obtain

$$\begin{aligned} \iint \phi(\nabla d)\eta &\le \liminf_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\, \alpha_{ij}^{h_k} \frac{|u_i^{h_k}(t) - u_j^{h_k}(t)|}{\varepsilon_k} \right) \mathrm{d}t \\ &= \liminf_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\, z_{ij}^{h_k}(t) \frac{u_i^{h_k}(t) - u_j^{h_k}(t)}{\varepsilon_k} \right) \mathrm{d}t = \iint z \cdot \nabla d\, \eta, \end{aligned}$$

which shows that $\phi(\nabla d) = z \cdot \nabla d$ a.e. on the support of η, from which we deduce (5.19). □

We conclude this section by observing that the discrete scheme converges to the unique weak flow (in the sense of Definition 2.1) starting from E_0 for "generic" initial data E_0, i.e. whenever fattening does not occur. More precisely, we have the following Corollary.

Corollary 5.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$ and for every $\lambda \in \mathbb{R}$ let $\overline{E}_\lambda^h$ be the closed space-time tube of the h-discrete evolution starting from $\{u_0 \le \lambda\}$; i.e., as in (5.1) with $E_0 = \{u_0 \le \lambda\}$. Then, there exists a countable set $\mathcal{N}$ such that for all $\lambda \in \mathbb{R}^N \setminus \mathcal{N}$*

$$\overline{E}_\lambda^h \xrightarrow{\mathscr{K}} E_\lambda \quad \textit{in } \mathbb{R}^N \times [0, +\infty)$$

as $h \to 0$, where E_λ is the unique weak flow in the sense of Definition 2.1 starting from $\{u_0 \le \lambda\}$.

Proof. It follows by combining Theorems 5.2 and 2.5. □

6. Numerical experiments

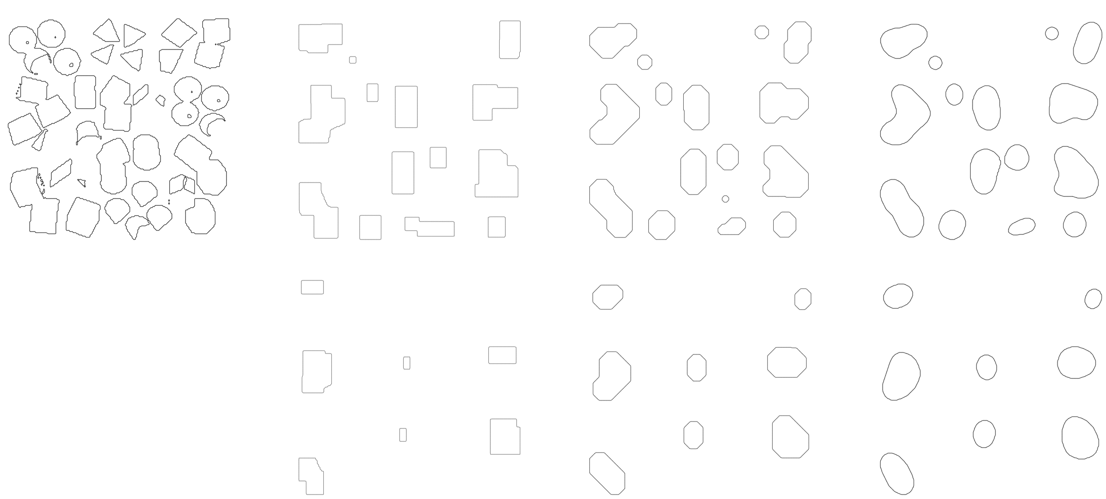

Figure 1. An initial datum and evolutions for square, octagonal and "almost isotropic" anisotropies, at two different times.

We show some numerical experiments to illustrate our results, in dimension 2 (an implementation in 3D is currently being developed). We follow the implementation described in [18] (see also [17]), except that now the distance is properly computed using using the inf/sup-convolution formulas (4.6). The (exact) numerical resolution of the discrete ROF functional is computed using Hochbaum's parametric maximum flow algorithm [33, 34], implemented upon the maxflow/mincut implementation of Boykov and Kolmogorov [8]. Other implementations of the algorithm yielding approximate minimizers have been considered for instance in [16, 39], of course they work in practice and allow to address more (an)isotropies than the current work, yet the joint convergence as $\varepsilon = h \to 0$ is not clear in these contexts. For numerical speedup, the infimum and supremum of definition (4.6) are computed only in a neighborhood of fixed size and not on the whole grid. Similarly, the ROF minimization is only performed in a neighborhood of the boundary. We observe that Corollary B.2 in Appendix B justifies this restriction in some particular case, notably the case $\phi = \|\cdot\|_{\ell^1}, \phi^\circ = \|\cdot\|_{\ell^\infty}$ (where $\ell_1 = 1$ can be chosen in Lemma B.1), which is particularly relevant. The code is available at `https://plmlab.math.cnrs.fr/chambolle/discretecrystals/` (implemented in `C/C++` and running on GNU/linux with `gcc`).

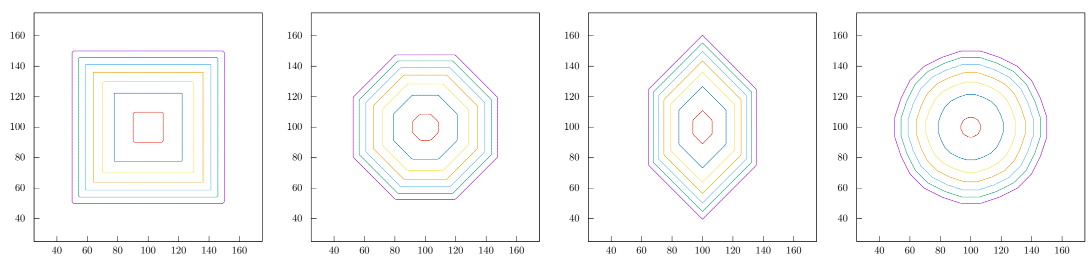

FIGURE 2. Wulff shapes of initial radius $R_0 = 50$ evolved at times $t = 0$, 200, $400, \ldots, 1200$ for four different anisotropies (square, octagon, diamond and "almost isotropic").

Figure 1 shows three examples of flows from the same starting set, composed of random shapes. The anisotropies are square (nearest neighbours interactions), octagonal (next nearest neighbours, weighted so that the corresponding Wulff shape is a regular octagon), and "almost isotropic", which is generated by the interactions in the directions $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 2)$, $(\pm 1, \pm 3)$ weighted so that the Wulff shape is a polygon with 24 facets of equal lengths.

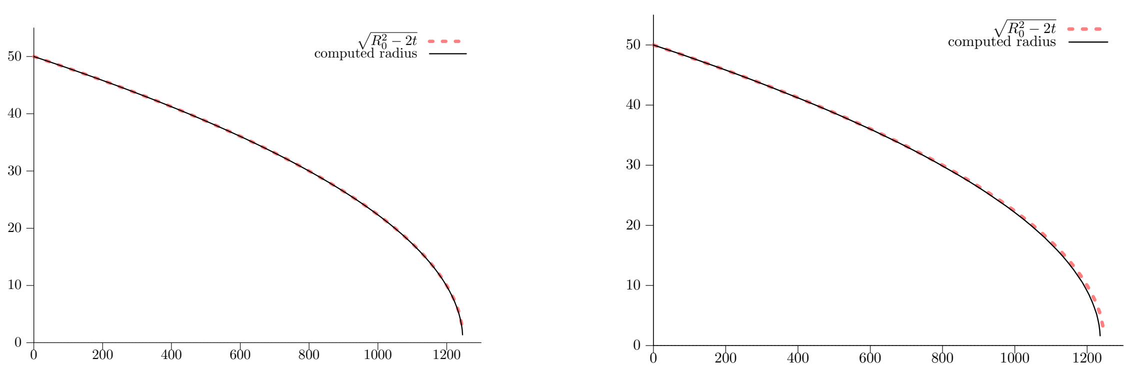

FIGURE 3. Evolution of the radius for the square (left) and octogonal (right) anisotropies.

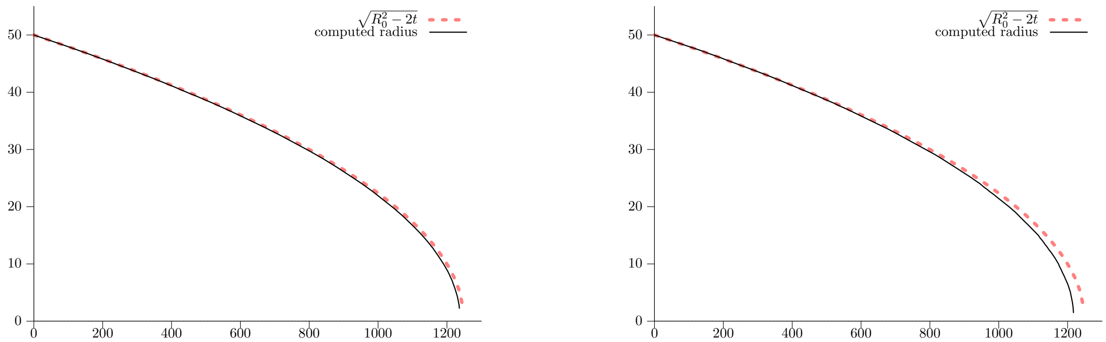

FIGURE 4. Evolution of the radius for the diamond (left) and "almost isotropic" (right) anisotropies.

Then, we estimate the decay of the radius of an initial Wulff shape $\mathcal{W}_{R_0} = \{\phi \le R_0\}$ along the evolution, up to extinction. In our experiment, $R_0 = 50$. It is well known that the solution is the Wulff shape of radius $R(t) = \sqrt{R_0^2 - 2(N-1)t}$ (where here $N = 2$). The evolutions are depicted in Figure 2. We use the same anisotropies as in figure 1, with additionally a "diamond" Wulff shape generated by the directions $(0, \pm 1)$, $(\pm 1, \pm 2)$ and with sides of equal lengths. In all cases, the weights have been calibrated so that the perimeters of the Wulff shapes are $6.28 \approx 2\pi$.

The plots in Figure 4 show that the decay of the radii is remarkably close to the theoretical prediction, even if this is less precise when more directions of interactions are involved, near extinction. This might be due in part to the fact that the computation of the distance through truncated variants of (4.6) become less precise.

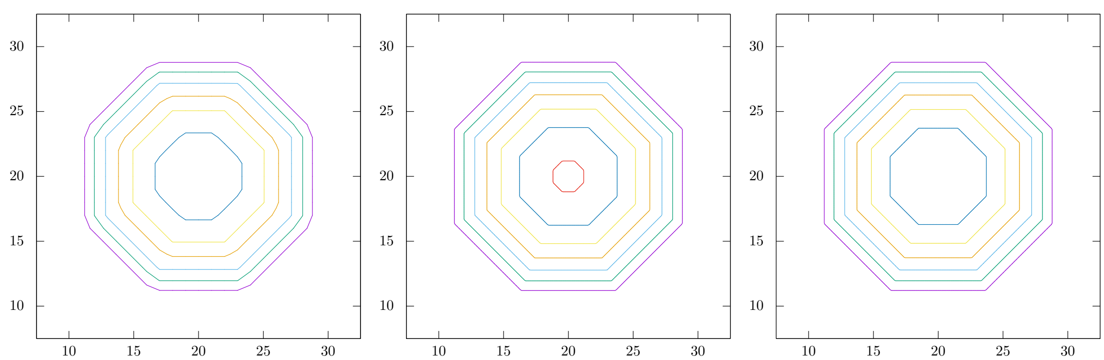

FIGURE 5. Evolution of an initial octagon with $R_0 = 10$ at times $0, 7, 14, \ldots$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

Finally, we perform the same experiment with varying ε and h. We observe that the results look remarkably close even if, at low resolution, the error becomes huge when the size of the Wulff shape is of the order of the discretization. Figure 5 shows the shapes. Observe that the shape at time $t = 49$ is only computed for $\varepsilon = 0.1$ and $h = 0.1$ (the shape vanishes before for the two other experiments). On the other hand, this computation took more than one hour, while the case $\varepsilon = 1$ took less than a minute and the case $\varepsilon = 0.1$, $h = 0.5$ a bit less than an hour. Figure 6 shows the decay of the radii, which should be $\sqrt{R_0^2 - 2t}$ for $R_0 = 10$ and $t \in [0, 50]$.

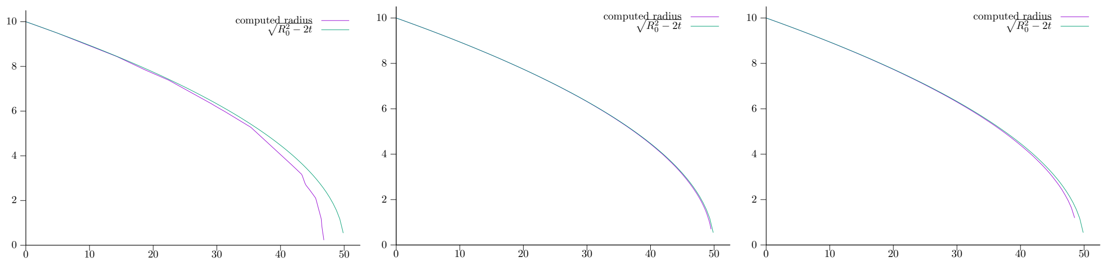

FIGURE 6. Evolution of the radius for an initial octagon with $R_0 = 10$ until the vanishing time $t = 50$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

APPENDIX A. PROOF OF LEMMA 4.10

We build here a supersolution to Problem (4.5) when $g = \phi^\circ$. Let us first recall some notation and results concerning zonotopes (see e.g. [38]). Recall that $\mathcal{E} = \{\pm e_k\}_{k=1}^m \subseteq \mathbb{Z}^N$ where, without loss of generality, the vectors $e_1, \dots, e_m$ span the whole $\mathbb{R}^N$. Given a non-negative interaction function $\beta \in X$, we assume that $\beta = 0$ on $\mathbb{Z}^N \setminus \mathcal{E}$ and that $\beta(-i) = \beta(i)$ for every $i \in \mathbb{Z}^N$. The anisotropy ϕ associated to β, as defined in (1.5), is such that its 1-Wulff shape $\mathcal{W}_1 \subseteq \mathbb{R}^N$ is a zonotope, which can be expressed as the Minkowski sum

$$\mathcal{W}_1 = \sum_{e \in \mathcal{E}} \beta(e)(-e, e) = \sum_{k=1}^m 2\beta(e_k)(-e_k, e_k).$$

Alternatively, one can define the zonotope $\mathcal{W}_1$ as the image of a cube under an affine map. Indeed, it holds

$$\mathcal{W}_1 = V(Q^{(m)}) \tag{A.1}$$

where $V = (2\beta(e_1)e_1, \dots, 2\beta(e_m)e_m) \in \mathbb{R}^{N \times m}$ and $Q^{(m)} = (-1, 1)^m$. Since the set $\mathcal{E}$ is uniquely defined up to sign changes, the matrix V is also uniquely detemined up to permutations of columns or sign changes.

Note that by definition of zonotope any element $x \in \overline{\mathcal{W}_\ell}$ for $\ell > 0$ can be written as

$$x = \ell \sum_{k=1}^m 2\beta(e_k)\lambda_k e_k,$$

for suitable coefficients $|\lambda_k| \le 1$. We note that (the closure of) a facet F (of non-zero dimension) of the zonotope $\mathcal{W}_\ell$ can be described in the following form:

$$F = \ell \sum_{j=1}^r 2\beta(e_{\sigma(j)})[-e_{\sigma(j)}, e_{\sigma(j)}] + \ell \sum_{j=r+1}^m 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A.2}$$

where σ is a permutation of $\{1, \dots, m\}$, $1 \le r \le m$ and $|\varepsilon_j| = 1$. Moreover (see [38, page 206] for details) the vectors $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ uniquely identify

$$\{e \in \mathcal{E} : e \parallel F\},$$

and r is uniquely defined as the number of vectors in the family $\mathcal{E}$ which are parallel to the facet F. Analogously, any vertex v of the zonotope $\mathcal{W}_\ell$ is of the form

$$v = \ell \sum_{j=1}^m 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{A.3}$$

where $\varepsilon_j \in \{\pm 1\}$ for every $j = 1, \dots, m$ and σ is a permutation of $\{1, \dots, m\}$. Note however that not every point of this form is a vertex of the zonotope.

Lemma A.1. *There exists $\ell_0 > 0$ such that for every $\varepsilon > 0$ and every $\ell \geq \ell_0$, if $i \in \varepsilon\mathbb{Z}^N$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$, then for each $k \in \{1, \dots, m\}$ either one of the following holds:*

i) *neither $i + \varepsilon e_k$ nor $i - \varepsilon e_k$ belong to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case it holds either $\phi^\circ(i+\varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i - \varepsilon e_k)$ or $\phi^\circ(i - \varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i + \varepsilon e_k)$;*

ii) *one between $i \pm \varepsilon e_k$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case $\phi^\circ(i \pm \varepsilon e_k) \geq \ell$ and it holds*

$$\#((i + \varepsilon\mathbb{Z}e_k) \cap \partial\mathcal{W}_{\varepsilon\ell}) \geq 2[\ell/\ell_0]. \tag{A.4}$$

Proof. By scaling, it suffices to prove the result in the case $\varepsilon = 1$. We take ℓ_0 such that

$$\ell_0 \geq \max_{k=1,\dots,m} \frac{1}{2\beta(e_k)} \tag{A.5}$$

and remark that $\ell_0 \in (0, +\infty)$. Note that the choice (A.5) implies for every $j = 1, \dots, m$ that

$$|(-2\ell\beta(e_j)e_j, 2\ell\beta(e_j)e_j)| = 4\ell\beta(e_j)|e_j| \geq 2\frac{\ell}{\ell_0}|e_j|.$$

We then fix $i \in \partial\mathcal{W}_\ell \cap \mathbb{Z}^N$ and $e_k \in \mathcal{E}$. We have to distinguish two cases.

Case 1. There exists a facet $F \ni i$ of $\mathcal{W}_\ell$ such that $e_k \parallel F$. By (A.2) we then see that

$$i \in 2\ell\beta(e_k)[-e_k, e_k] + j,$$

where $j \in F$. This implies in particular that $\{n \in \mathbb{Z} : i + ne_k \in F\}$ is an interval of $\mathbb{Z}$ containing 0. Furthermore, by the assumption (A.5), it contains at least $[2\ell|e_k|/\ell_0]$ points and we conclude (A.4). Since i and one between $i \pm e_k$ belong to $\partial\mathcal{W}_\ell$, then $\phi^\circ(i \pm e_k) \geq \ell$ by convexity.

Case 2. For every facet $F \ni i$ of $\mathcal{W}_\ell$ it holds $e_k \nparallel F$. Let us fix a facet $F \ni i$ and note that by (A.2) and up to relabelling the indexes, it holds

$$i \in \ell\sum_{j=1}^{r} 2\beta(e_j)[-e_j, e_j] + \ell\sum_{j=r+1}^{m} 2\beta(e_j)\varepsilon_j e_j,$$

with $k > r$ and $|\varepsilon_j| = 1$ for $j = r+1, \dots, m$. Recalling (A.1), we see that

$$i - \varepsilon_k e_k = \ell V(y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k),$$

where $\tilde{e}_1, \dots \tilde{e}_m$ denotes the canonical base of $\mathbb{R}^m$ and $y \in \sum_{j=1}^r[-\tilde{e}_j, \tilde{e}_j] + \sum_{j=r+1}^m \varepsilon_j\tilde{e}_j \subseteq \partial Q^{(m)}$. By the choice (A.5) and since $k > r$, one deduces that $y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k \in Q^{(m)}$, thus $i - \varepsilon_k e_k \in \overline{\mathcal{W}}_\ell$. Since then $e_k \nparallel F$ for any facet containing i, it must hold $\phi^\circ(i - \varepsilon_k e_k) < \ell$. By convexity one easily concludes that $\phi^\circ(i + \varepsilon_k e_k) > \ell$, which shows i). □

We now define a calibration z_{ij} for every $(i,j) \in (\{\phi^\circ > \varepsilon\ell_0\} \cap \varepsilon\mathbb{Z}^N) \times \varepsilon\mathbb{Z}^N$. Fix $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) > \varepsilon\ell_0$. In the following we write $i \sim j$ if $\frac{i-j}{\varepsilon} \in \mathcal{E}$. We start defining

$$z_{ij} = \begin{cases} 0 & \text{if } j \not\sim i \\ -\beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) > \phi^\circ(i) \\ \beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) < \phi^\circ(i). \end{cases} \tag{A.6}$$

In particular, this definition covers case i) in Lemma A.1. Assume then that there exists $j \sim i$ with $\phi^\circ(j) = \phi^\circ(i)$ and $\frac{j-i}{\varepsilon} = e_k \in \mathcal{E}$. Since $i \in \varepsilon\mathbb{Z}^N$ and $e_k \in \mathcal{E}$ fall in case ii) of Lemma A.1, there exists an interval $[-\underline{n}, \bar{n}] \cap \mathbb{Z}$ for $\underline{n}, \bar{n} \in \mathbb{N}$ such that

$$(i + \varepsilon\mathbb{Z}e_k) \cap \partial W^{\phi^\circ}_{\phi^\circ(i)} = i + ([-\underline{n}, \bar{n}] \cap \mathbb{Z})\varepsilon e_k$$

and moreover

$$\#([-\underline{n},\bar{n}]\cap\mathbb{Z}) \geq 2[\phi^\circ(i)/(\varepsilon\ell_0)]. \tag{A.7}$$

Thus, we define z_{ij} as a linear interpolation of the values assumed at the extremal points of $i+[-\underline{n},\bar{n}]\varepsilon e_k$ as

$$\begin{aligned} z_{i+t\varepsilon e_k, i+(t+1)\varepsilon e_k} &:= \beta(e_k)\left(1-2\frac{t+\underline{n}+1}{\underline{n}+\bar{n}+1}\right) \quad \forall t\in[-\underline{n}-1,\bar{n}]\cap\mathbb{Z}, \\ z_{i+t\varepsilon e_k, i+(t-1)\varepsilon e_k} &:= \beta(e_k)\left(1-2\frac{-t+\underline{n}+1}{\underline{n}+\bar{n}+1}\right) \quad \forall t\in[-\underline{n},\bar{n}+1]\cap\mathbb{Z}. \end{aligned} \tag{A.8}$$

By definition one easily sees that

$$|z_{ij}|\leq \alpha^\varepsilon_{ij}, \qquad z_{ij}(\phi^\circ(i)-\phi^\circ(j)) = \alpha^\varepsilon_{ij}|\phi^\circ(i)-\phi^\circ(j)|. \tag{A.9}$$

We now show how to bound the divergence $(D^*_\varepsilon z)_i$. Assume that $\phi^\circ(i+\varepsilon e_k)=\phi^\circ(i)$ or that $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. Then by definition (A.8) and by (A.7) one deduces

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}-z_{i+\varepsilon e_k,i}-z_{i-\varepsilon e_k,i} = -\frac{4\beta(e_k)}{\underline{n}+\bar{n}+1} \geq -\frac{2\beta(e_k)}{[\phi^\circ(i)/(\varepsilon\ell_0)]} \geq -\frac{C\varepsilon}{\phi^\circ(i)}, \tag{A.10}$$

and similarly if $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. If instead $\phi^\circ(i\pm\varepsilon e_k)\neq\phi^\circ(i)$ and $\phi^\circ(i\pm\varepsilon e_k)\geq\varepsilon\ell_0$, one sees that

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}=0 \text{ and } z_{i+\varepsilon e_k,i}+z_{i-\varepsilon e_k,i}=0 \tag{A.11}$$

Combining (A.10) and (A.11) and recalling (4.2) we conclude that if $\phi^\circ(i)\geq\ell_1\varepsilon$ then

$$h(D^*_\varepsilon z)_i \geq -\frac{c_\phi h}{\phi^\circ(i)} \tag{A.12}$$

for a suitable positive constant c_ϕ depending on ϕ.

We now illustrate a procedure that allows to extend the calibration above to $\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N$. We set $C>1$ a sufficiently big constant and define a function $v\in X_\varepsilon$ setting

$$v := \begin{cases} \phi^\circ + \dfrac{Ch}{\phi^\circ} & \text{on } \{\phi^\circ \geq C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N \\ C(\sqrt{h}\vee\varepsilon)+\frac{h}{\sqrt{h}\vee\varepsilon} & \text{on } \{\phi^\circ < C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N \end{cases}. \tag{A.13}$$

A calibration $w\in Y_\varepsilon$ can be defined setting for $i,j\in\varepsilon\mathbb{Z}^N$

$$w_{ij} := \begin{cases} z_{ij} & \text{if } \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) \\ -\alpha^\varepsilon_{ij} & \text{if } \phi^\circ(i)< 2\sqrt{C}(\sqrt{h}\vee\varepsilon) \end{cases}. \tag{A.14}$$

Since $x\mapsto x+Chx^{-1}$ is strictly monotone in the region $\{x\geq\sqrt{Ch}\}$, we can employ (A.9) to prove that, for every $i,j\in\varepsilon\mathbb{Z}^N$ with $\phi^\circ(i)\geq C(\sqrt{h}\vee\varepsilon)$, it holds

$$w_{ij}(v_i-v_j)=\alpha^\varepsilon_{ij}|v_i-v_j|, \quad |w_{ij}|\leq\alpha^\varepsilon_{ij}. \tag{A.15}$$

Moreover, taking C large enough ensures that whenever $j\sim i$, then

$$\begin{aligned} \phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\leq C(\sqrt{h}\vee\varepsilon) \\ \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\geq \sqrt{C}(\sqrt{h}\vee\varepsilon) \end{aligned}. \tag{A.16}$$

Thus, equation (A.15) can be directly checked in the case $\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ using the definition (A.14).

Note now that definition (A.14) implies $D^*_\varepsilon w = 0$ in the region $\{\phi^\circ < 2\sqrt{C}(\sqrt{h} \vee \varepsilon)\}$ thus we assume $\phi^\circ(i) \geq 2\sqrt{C}(\sqrt{h} \vee \varepsilon)$ and estimate $(D^*_\varepsilon w)_i$. If $\phi^\circ(i - \varepsilon e_k) < 2\sqrt{C}(\sqrt{h} \vee \varepsilon)$ by convexity $\phi^\circ(i + \varepsilon e_k) > 2\sqrt{C}(\sqrt{h} \vee \varepsilon)$, thus by definition (A.14) we get

$$z_{i,i+\varepsilon e_k} - z_{i+\varepsilon e_k,i} + z_{i,i-\varepsilon e_k} - z_{i-\varepsilon e_k,i} = -\beta(e_k) - \beta(e_k) + \beta(e_k) - (-\beta(e_k)) = 0.$$

The symmetric case is analogous. On the other hand, if every $j \sim i$ is in $\{\phi^\circ \geq 2\sqrt{C}(\sqrt{h} \vee \varepsilon)\}$ equation (A.12) holds. Therefore, we have shown

$$hD^*_\varepsilon w \geq -\frac{c_\phi h}{\phi^\circ} \chi_{\{\phi^\circ \geq \sqrt{C}(\sqrt{h} \vee \varepsilon)\}}. \tag{A.17}$$

By a direct computation, using (A.17) and assuming the $C > c_\phi$, we see that the pair (v, w) defined above satisfies

$$\begin{cases} hD^*_\varepsilon w + v \geq \phi^\circ \\ w_{ij}(v_i - v_j) = \alpha^\varepsilon_{ij}|v_i - v_j|, \quad |w_{ij}| \leq \alpha^\varepsilon_{ij}. \end{cases}$$

Recalling the comparison result in Theorem 3.3, we conclude that the solution u to (3.4) satisfies $u \leq v$ in $\varepsilon\mathbb{Z}^N$.

Appendix B. A remark on the inf/sup-convolution formulas (4.6)

In this section we show that in some particular cases, the $\inf, \sup$ in the definition (4.6) can be replaced by $\min, \max$ and that this minimization/maximization procedure can be made in a fixed neighborhood of the point considered. Yet, our proof also shows that this neighborhood can become very large, depending on the weights of the interaction, and it seems that we cannot expect in general cases that the $\min, \max$ are actually reached.

We assume here that ϕ satisfies the following assumption. There exists $\ell_\phi > 0$ such that for every $\varepsilon_k \in \{0, \pm 1\}$ for $k = 1, \dots, m$, there exists $\ell \leq \ell_\phi$ such that

$$\ell \sum_{k=1}^m 2\beta(e_k)\varepsilon_k e_k \in \mathbb{Z}^N. \tag{B.1}$$

Note that this condition is satisfied if and only if $\beta(e_k) \in \mathbb{Q}$ for all $k = 1, \dots, m$.

Lemma B.1. *There exists $\ell_1 > 0$ with the following property. For any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq \varepsilon\ell_1$ there exists $j \in \varepsilon\mathbb{Z}^N \setminus \{0\}$ with $\phi^\circ(j) < \phi^\circ(i)$ and satisfying*

$$\phi^\circ(i) \geq \phi^\circ(j) + \phi^\circ(i - j) - c_\phi\varepsilon. \tag{B.2}$$

If (B.1) holds, for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq 2\varepsilon\ell_1$ there exists $j \in (\mathcal{W}_{\varepsilon\ell_1} \setminus \{0\}) \cap \varepsilon\mathbb{Z}^N$ such that

$$\phi^\circ(i) = \phi^\circ(j) + \phi^\circ(i - j). \tag{B.3}$$

Moreover, for every $R \in (2\varepsilon\ell_1, \phi^\circ(i))$ there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ such that (B.3) holds.

Proof. By scaling we prove the result in the case $\varepsilon = 1$. Given $i \in \mathbb{Z}^N \setminus \{0\}$, inequality (B.2) follows easily choosing $\ell_1 \geq 2$, considering $\sigma i \in \mathbb{R}^N \setminus \{0\}$ for an appropriate $\sigma \in (0, 1)$ and $j \in \mathbb{Z}^N$ so that $\sigma i \in (j + [0, 1]^N)$.

We now assume (B.1) and denote by ℓ_ϕ the radius associated to ϕ. We then choose $\ell_1 = \ell_\phi$. Let us fix $i \in \mathbb{Z}^N$ with $\phi^\circ(i) = \ell \geq 2\ell_1$. By (A.2) there exist $r > 0$, ε_k, λ_k with $|\varepsilon_k| = 1$ and $|\lambda_k| < 1$ such that

$$i = \ell \left(\sum_{k=1}^r 2\beta(e_k)\varepsilon_k e_k + \sum_{k=r+1}^m \lambda_k 2\beta(e_k) e_k \right).$$

Let us denote the point

$$v = \sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k \in \partial \mathcal{W}_1,$$

and define the function sign by $\mathrm{sign}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. For any $\ell' \leq \ell_\phi$ we rewrite i as follows

$$i = \ell' \left(v + \sum_{k=r+1}^{m} 2\beta(e_k)\mathrm{sign}(\lambda_k) e_k \right) + (\ell - \ell') \left(v + \sum_{k=r+1}^{m} 2\beta(e_k) \left(\frac{\ell}{\ell - \ell'}\lambda_k - \frac{\ell'}{\ell - \ell'}\mathrm{sign}(\lambda_k) \right) e_k \right)$$

$$=: \ell' w + (\ell - \ell') \left(v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda_k' e_k \right).$$

Notice that, since $\ell \geq 2\ell'$ and $|\lambda_k| \leq 1$ it holds $|\lambda_k'| \leq 1$, thus by formula (A.2) we get

$$v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda_k' e_k \in \partial \mathcal{W}_1$$

and therefore $\phi^\circ(i - \ell' w) = \ell - \ell'$. We conclude noting that by the hypothesis (B.1) we can choose $\ell' \leq \ell_1$ so that $\ell' w \in \mathbb{Z}^N$, which implies (B.3) since $\phi^\circ(\ell' w) = \ell'$.

We now prove the last assertion. Since $\phi^\circ(i) \geq 2\ell_1$, by the previous result there exists $j_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that $\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(i - j_0)$. Now, if $R - 2\ell_1 \leq \phi^\circ(j_0)$ we conclude. If not, then $\phi^\circ(i - j_0) \geq 2\ell_1$ by (B.3), and thus we can find $k_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that

$$\phi^\circ(i - j_0) = \phi^\circ(k_0) + \phi^\circ(i - j_0 - k_0). \tag{B.4}$$

Denoting $j_1 = j_0 + k_0$, on one hand (B.4) implies

$$\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(j_1 - j_0) + \phi^\circ(i - j_1) \geq \phi^\circ(j_1) + \phi^\circ(i - j_1) \tag{B.5}$$

thus equality holds instead. If $\phi^\circ(j_1) \geq R - 2\ell_1$ we conclude, if not (B.5) yields $\phi^\circ(i - j_1) \geq 2\ell_1$ and we can iterate. Recalling that $\phi^\circ \geq c_\phi > 0$ on $\varepsilon\mathbb{Z}^N \setminus \{0\}$, it is clear that after a finite number of iterations the process stops, and one can check that the required properties are satisfied. $\square$

By the previous lemma it is easy to prove the following result.

Corollary B.2. *Let $u \in X$ be a $(1, \phi)$-Lipschitz function and ℓ_1 as in Lemma B.1. Then, for all $i \in \varepsilon\mathbb{Z}^N$ it holds*

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \max_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\}.$$

In addition, if $i \in \{u \leq 0\}$, the maximum is reached in a point in $(\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \varepsilon\mathbb{Z}^N$.

Proof. It is enough to consider $i \in \{u < 0\} \cap \varepsilon\mathbb{Z}^N$. Let us denote $F = (\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \{u > 0\}$. Firstly, by a variant of the argument by iteration employed in the proof of Lemma B.1, one can prove that

$$\sup_{j \in \{u \geq 0\}} \{u_j - \phi^\circ(i - j)\} = \sup_{j \in F} \{u_j - \phi^\circ(i - j)\}. \tag{B.6}$$

On the other hand, take a point $j_0 \in \{u > 0\}$. If $j \in F$ satisfies $u_j - \phi^\circ(i - j) \geq u_{j_0} - \phi^\circ(i - j_0)$, since $u \leq 2\varepsilon\ell_1$ in F (as u is $(1, \phi^\circ)$-Lipschitz) we obtain

$$2\varepsilon\ell_1 + \phi^\circ(i - j_0) \geq \phi^\circ(i - j),$$

which implies that the sup in (B.6) is indeed a max. $\square$

References

[1] Fred Almgren and Jean E. Taylor. "Flat flow is motion by crystalline curvature for curves with crystalline energies". In: *J. Differential Geom.* 42.1 (1995), pp. 1–22. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214457030.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. "Curvature-driven flows: a variational approach". In: *SIAM J. Control Optim.* 31.2 (1993), pp. 387–438. ISSN: 0363-0129. DOI: 10.1137/0331020. URL: https://doi.org/10.1137/0331020.

[3] Robert Almgren. "Variational algorithms and pattern formation in dendritic solidification". In: *J. Comput. Phys.* 106.2 (1993), pp. 337–354. ISSN: 0021-9991.

[4] Sigurd Angenent and Morton E. Gurtin. "Multiphase thermomechanics with interfacial structure. II. Evolution of an isothermal interface". In: *Arch. Rational Mech. Anal.* 108.4 (1989), pp. 323–391. ISSN: 0003-9527. DOI: 10.1007/BF01041068. URL: http://dx.doi.org/10.1007/BF01041068.

[5] Giovanni Bellettini, Vicent Caselles, Antonin Chambolle, and Matteo Novaga. "Crystalline mean curvature flow of convex sets". In: *Arch. Ration. Mech. Anal.* 179.1 (2006), pp. 109–152. ISSN: 0003-9527. DOI: 10.1007/s00205-005-0387-0. URL: http://dx.doi.org/10.1007/s00205-005-0387-0.

[6] Giovanni Bellettini, Matteo Novaga, and Maurizio Paolini. "On a crystalline variational problem. I. First variation and global L^∞ regularity". In: *Arch. Ration. Mech. Anal.* 157.3 (2001), pp. 165–191. ISSN: 0003-9527. DOI: 10.1007/s002050010127. URL: https://doi.org/10.1007/s002050010127.

[7] Giovanni Bellettini and Maurizio Paolini. "Anisotropic motion by mean curvature in the context of Finsler geometry". In: *Hokkaido Math. J.* 25.3 (1996), pp. 537–566. ISSN: 0385-4035. DOI: 10.14492/hokmj/1351516749. URL: http://dx.doi.org/10.14492/hokmj/1351516749.

[8] Yuri Boykov and Vladimir Kolmogorov. "An experimental comparison of min-cut/max- flow algorithms for energy minimization in vision". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 26.9 (2004), pp. 1124–1137. DOI: 10.1109/TPAMI.2004.60.

[9] Andrea Braides and Antonin Chambolle. "Ising systems, measures on the sphere, and zonoids". In: (2023). arXiv: 2305.11054 [math.AP].

[10] Andrea Braides, Marco Cicalese, and Nung Kwan Yip. "Crystalline motion of interfaces between patterns". In: *J. Stat. Phys.* 165.2 (2016), pp. 274–319. ISSN: 0022-4715. DOI: 10.1007/s10955-016-1609-6. URL: https://doi.org/10.1007/s10955-016-1609-6.

[11] Andrea Braides, Maria Stella Gelli, and Matteo Novaga. "Motion and pinning of discrete interfaces". In: *Arch. Ration. Mech. Anal.* 195.2 (2010), pp. 469–498. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-009-0215-z. URL: https://doi.org/10.1007/s00205-009-0215-z.

[12] Andrea Braides and Giovanni Scilla. "Motion of discrete interfaces in periodic media". In: *Interfaces Free Bound.* 15.4 (2013), pp. 451–476. ISSN: 1463-9963. DOI: 10.4171/IFB/310.

[13] Andrea Braides and Margherita Solci. *Geometric flows on planar lattices.* Pathways in Mathematics. Birkhäuser/Springer, Cham, 2021, pp. x+134. DOI: 10.1007/978-3-030-69917-8. URL: https://doi.org/10.1007/978-3-030-69917-8.

[14] Andrea Braides and Margherita Solci. "Motion of discrete interfaces through mushy layers". In: *J. Nonlinear Sci.* 26.4 (2016), pp. 1031–1053. ISSN: 0938-8974. DOI: 10.1007/s00332-016-9297-6.

[15] Vicent Caselles and Antonin Chambolle. "Anisotropic curvature-driven flow of convex sets". In: *Nonlinear Anal.* 65.8 (2006), pp. 1547–1577. ISSN: 0362-546X. DOI: 10.1016/j.na.2005.10.029.

[16] Antonin Chambolle. "An algorithm for mean curvature motion". In: *Interfaces Free Bound.* 6.2 (2004), pp. 195–218. ISSN: 1463-9963. DOI: 10.4171/IFB/97. URL: http://dx.doi.org/10.4171/IFB/97.

[17] Antonin Chambolle and Jérôme Darbon. "Image Processing and Analysis with Graphs: Theory and Practice". In: Digital Imaging and Computer Vision. Editors: Olivier Lézoray, University of Caen, France; Leo Grady, Siemens Corporate Research, Princeton, New Jersey, USA. CRC Press, 2012. Chap. 4: "A Parametric Maximum Flow Approach for Discrete Total Variation Regularization".

[18] Antonin Chambolle and Jérôme Darbon. "On total variation minimization and surface evolution using parametric maximum flows". In: *International journal of computer vision* 84.3 (2009), pp. 288–307.

[19] Antonin Chambolle and Leonard Kreutz. "Crystallinity of the homogenized energy density of periodic lattice systems". In: *Multiscale Model. Simul.* 21.1 (2023), pp. 34–79. ISSN: 1540-3459. DOI: 10.1137/21M1442073. URL: https://doi.org/10.1137/21M1442073.

[20] Antonin Chambolle, Massimiliano Morini, Matteo Novaga, and Marcello Ponsiglione. "Existence and uniqueness for anisotropic and crystalline mean curvature flows". In: *J. Amer. Math. Soc.* 32.3 (2019), pp. 779–824. ISSN: 0894-0347. DOI: 10.1090/jams/919.

[21] Antonin Chambolle, Massimiliano Morini, Matteo Novaga, and Marcello Ponsiglione. "Generalized crystalline evolutions as limits of flows with smooth anisotropies". In: *Anal. PDE* 12.3 (2019), pp. 789–813. ISSN: 2157-5045. DOI: 10.2140/apde.2019.12.789. URL: https://doi.org/10.2140/apde.2019.12.789.

[22] Antonin Chambolle, Massimiliano Morini, and Marcello Ponsiglione. "Existence and uniqueness for a crystalline mean curvature flow". In: *Comm. Pure Appl. Math.* 70.6 (2017), pp. 1084–1114. ISSN: 0010-3640. DOI: 10.1002/cpa.21668. URL: https://doi.org/10.1002/cpa.21668.

[23] Tokuhiro Eto and Yoshikazu Giga. "On a minimizing movement scheme for mean curvature flow with prescribed contact angle in a curved domain and its computation". In: *Annali di Matematica Pura ed Applicata (1923 -)* (2023). DOI: 10.1007/s10231-023-01398-9. URL: https://doi.org/10.1007/s10231-023-01398-9.

[24] Tokuhiro Eto, Yoshikazu Giga, and Katsuyuki Ishii. "An area-minimizing scheme for anisotropic mean-curvature flow". In: *Adv. Differential Equations* 17.11-12 (2012), pp. 1031–1084. ISSN: 1079-9389.

[25] Mi-Ho Giga and Yoshikazu Giga. "Generalized motion by nonlocal curvature in the plane". In: *Arch. Ration. Mech. Anal.* 159.4 (2001), pp. 295–333. ISSN: 0003-9527. DOI: 10.1007/s002050100154.

[26] Mi-Ho Giga, Yoshikazu Giga, and Norbert Požár. "Periodic total variation flow of non-divergence type in $\mathbb{R}^n$". In: *J. Math. Pures Appl. (9)* 102.1 (2014), pp. 203–233. ISSN: 0021-7824. DOI: 10.1016/j.matpur.2013.11.007. URL: https://doi.org/10.1016/j.matpur.2013.11.007.

[27] Yoshikazu Giga, Morton E. Gurtin, and José Matias. "On the dynamics of crystalline motions". In: *Japan J. Indust. Appl. Math.* 15.1 (1998), pp. 7–50. ISSN: 0916-7005. DOI: 10.1007/BF03167395. URL: http://dx.doi.org/10.1007/BF03167395.

[28] Yoshikazu Giga and Norbert Požár. "A level set crystalline mean curvature flow of surfaces". In: *Adv. Differential Equations* 21.7-8 (2016), pp. 631–698. ISSN: 1079-9389. URL: http://projecteuclid.org/euclid.ade/1462298654.

[29] Yoshikazu Giga and Norbert Požár. "Approximation of general facets by regular facets with respect to anisotropic total variation energies and its application to crystalline mean curvature flow". In: *Comm. Pure Appl. Math.* 71.7 (2018), pp. 1461–1491. ISSN: 0010-3640. DOI: 10.1002/cpa.21752. URL: https://doi.org/10.1002/cpa.21752.

[30] Yoshikazu Giga and Norbert Požár. "Motion by crystalline-like mean curvature: a survey". In: *Bull. Math. Sci.* 12.2 (2022), Paper No. 2230004, 68. ISSN: 1664-3607. DOI: 10.1142/S1664360722300043.

[31] Yoshikazu Giga and Norbert Požár. "Viscosity solutions for the crystalline mean curvature flow with a nonuniform driving force term". In: *Partial Differ. Equ. Appl.* 1.6 (2020), Paper No. 39, 26. ISSN: 2662-2963. DOI: 10.1007/s42985-020-00040-0. URL: https://doi.org/10.1007/s42985-020-00040-0.

[32] Morton E. Gurtin. *Thermomechanics of evolving phase boundaries in the plane.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993, pp. xi+148. ISBN: 0-19-853694-1.

[33] Dorit S. Hochbaum. "An efficient algorithm for image segmentation, Markov random fields and related problems". In: *J. ACM* 48.4 (2001), pp. 686–701. ISSN: 0004-5411,1557-735X. DOI: 10.1145/502090.502093. URL: https://doi.org/10.1145/502090.502093.

[34] Dorit S. Hochbaum. "Multi-label Markov random fields as an efficient and effective tool for image segmentation, total variations and regularization". In: *Numer. Math. Theory Methods Appl.* 6.1 (2013), pp. 169–198. ISSN: 1004-8979,2079-7338.

[35] Katsuyuki Ishii. "An approximation scheme for the anisotropic and nonlocal mean curvature flow". In: *NoDEA Nonlinear Differential Equations Appl.* 21.2 (2014), pp. 219–252. ISSN: 1021-9722,1420-9004. DOI: 10.1007/s00030-013-0244-z. URL: https://doi.org/10.1007/s00030-013-0244-z.

[36] Stephan Luckhaus and Thomas Sturzenhecker. "Implicit time discretization for the mean curvature flow equation". In: *Calc. Var. Partial Differential Equations* 3.2 (1995), pp. 253–271. ISSN: 0944-2669. DOI: 10.1007/BF01205007.

[37] Annalisa Malusa and Matteo Novaga. "Crystalline evolutions in chessboard-like microstructures". In: *Netw. Heterog. Media* 13.3 (2018), pp. 493–513. ISSN: 1556-1801. DOI: 10.3934/nhm.2018022.

[38] Peter McMullen. "On zonotopes". In: *Trans. Amer. Math. Soc.* 159 (1971), pp. 91–109. ISSN: 0002-9947. DOI: 10.2307/1996000. URL: https://doi.org/10.2307/1996000.

[39] Adam Oberman, Stanley Osher, Ryo Takei, and Richard Tsai. "Numerical methods for anisotropic mean curvature flow based on a discrete time variational formulation". In: *Commun. Math. Sci.* 9.3 (2011), pp. 637–662. ISSN: 1539-6746,1945-0796. DOI: 10.4310/CMS.2011.v9.n3.a1. URL: https://doi.org/10.4310/CMS.2011.v9.n3.a1.

[40] Norbert Požár. *On the self-similar solutions of the crystalline mean curvature flow in three dimensions.* 2018. arXiv: 1806.02482.

[41] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. "Nonlinear total variation based noise removal algorithms". In: *Phys. D* 60.1-4 (1992). Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991), pp. 259–268. ISSN: 0167-2789. DOI: 10.1016/0167-2789(92)90242-F.

[42] Giovanni Scilla. "Motion of discrete interfaces on the triangular lattice". In: *Milan J. Math.* 88.2 (2020), pp. 315–346. ISSN: 1424-9286. DOI: 10.1007/s00032-020-00316-5.

[43] Halil Mete Soner. "Motion of a set by the curvature of its boundary". In: *J. Differential Equations* 101.2 (1993), pp. 313–372. ISSN: 0022-0396. DOI: 10.1006/jdeq.1993.1015. URL: http://dx.doi.org/10.1006/jdeq.1993.1015.

[44] Jean E. Taylor. "Crystalline variational problems". In: *Bull. Amer. Math. Soc.* 84.4 (1978), pp. 568–588. ISSN: 0002-9904.

(Antonin Chambolle) CEREMADE, CNRS, Université Paris-Dauphine, PSL, and Mokaplan, INRIA Paris, France.

Email address, A. Chambolle: antonin.chambolle@ceremade.dauphine.fr

(Daniele De Gennaro) CEREMADE, CNRS, Université Paris-Dauphine, PSL, France, and Dipartimento di Scienze Matematiche Fisiche e Informatiche, Università di Parma, Italy.

Email address, D. De Gennaro: degennaro@ceremade.dauphine.fr

(Massimiliano Morini) Dipartimento di Scienze Matematiche Fisiche e Informatiche, Università di Parma, Italy

Email address, M. Morini: massimiliano.morini@unipr.it