THE EVOLUTION OF THE WEYL TENSOR UNDER THE RICCI FLOW

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ABSTRACT. We compute the evolution equation of the Weyl tensor under the Ricci flow of a Riemannian manifold and we discuss some consequences for the classification of locally conformally flat Ricci solitons.

CONTENTS

1. THE EVOLUTION EQUATION OF THE WEYL TENSOR

The Riemann curvature operator of a Riemannian manifold (M^n, g) is defined as in [14] by

$$\mathrm{Riem}(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z\,.$$

In a local coordinate system the components of the $(3,1)$–Riemann curvature tensor are given by $\mathrm{R}^l_{ijk}\frac{\partial}{\partial x^l} = \mathrm{Riem}\big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\big)\frac{\partial}{\partial x^k}$ and we denote by $\mathrm{R}_{ijkl} = g_{lm}\mathrm{R}^m_{ijk}$ its $(4,0)$–version.

In all the paper the Einstein convention of summing over the repeated indices will be adopted.

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v,w,v,w) = \mathrm{R}_{ijkl}v^iw^jv^kw^l > 0$.

The Ricci tensor is obtained by the contraction $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$ and $\mathrm{R} = g^{ik}\mathrm{R}_{ik}$ will denote the scalar curvature.

The so called Weyl tensor is then defined by the following decomposition formula (see [14, Chapter 3, Section K]) in dimension $n \geq 3$,

$$\begin{aligned}\mathrm{W}_{ijkl} =& \mathrm{R}_{ijkl} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) - \frac{1}{n-2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\\ =& \mathrm{R}_{ijkl} + \mathrm{A}_{ijkl} + \mathrm{B}_{ijkl}\,,\end{aligned}$$

where we introduced the tensors

$$\mathrm{A}_{ijkl} = \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})$$

and

$$\mathrm{B}_{ijkl} = -\frac{1}{n-2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\,.$$

The Weyl tensor satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero, as it can be easily seen by the above formula.

In dimension three W is identically zero for every Riemannian manifold (M^3, g), it becomes

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relevant instead when $n \geq 4$ since its nullity is a condition equivalent for (M^n, g) to be *locally conformally flat*, that is, around every point $p \in M^n$ there is a conformal deformation $\widetilde{g}_{ij} = e^f g_{ij}$ of the original metric g, such that the new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p).

We suppose now that $(M^n, g(t))$ is a Ricci flow in some time interval, that is, the time–dependent metric $g(t)$ satisfies

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{R}_{ij} \,.$$

We have then the following evolution equations for the curvature (see for instance [15]),

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R} &= \Delta\mathrm{R} + 2|\mathrm{Ric}|^2 \\
\frac{\partial}{\partial t}\mathrm{R}_{ij} &= \Delta\mathrm{R}_{ij} + 2\mathrm{R}^{kl}\mathrm{R}_{kilj} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} \,, \\
\frac{\partial}{\partial t}\mathrm{R}_{ijkl} &= \Delta\mathrm{R}_{ijkl} + 2(\mathrm{C}_{ijkl} - \mathrm{C}_{ijlk} + \mathrm{C}_{ikjl} - \mathrm{C}_{iljk}) \\
&\qquad - g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{qjkl} + \mathrm{R}_{jp}\mathrm{R}_{iqkl} + \mathrm{R}_{kp}\mathrm{R}_{ijql} + \mathrm{R}_{lp}\mathrm{R}_{ijkq}) \,,
\end{aligned} \tag{1.1}$$

where $\mathrm{C}_{ijkl} = g^{pq}g^{rs}\mathrm{R}_{pijr}\mathrm{R}_{slkq}$.

All the computations which follow will be done in a fixed local frame, not in a moving frame.

The goal of this section is to work out the evolution equation under the Ricci flow of the Weyl tensor W_{ijkl}. In the next sections we will see the geometric consequences of the assumption that a manifold evolving by the Ricci flow is locally conformally flat at every time. In particular, we will be able to classify the so called Ricci solitons under the hypothesis of locally conformally flatness.

Since $\mathrm{W}_{ijkl} = \mathrm{R}_{ijkl} + \mathrm{A}_{ijkl} + \mathrm{B}_{ijkl}$ and we already have the evolution equation (1.1) for R_{ijkl}, we start differentiating in time the tensors A_{ijkl} and B_{ijkl}

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{A}_{ijkl} &= \frac{\Delta\mathrm{R} + 2|\mathrm{Ric}|^2}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\
&\quad + \frac{\mathrm{R}}{(n-1)(n-2)}(-2\mathrm{R}_{ik}g_{jl} - 2\mathrm{R}_{jl}g_{ik} + 2\mathrm{R}_{il}g_{jk} + 2\mathrm{R}_{jk}g_{il}) \\
&= \Delta\mathrm{A}_{ijkl} + \frac{2|\mathrm{Ric}|^2}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) + \frac{2\mathrm{R}}{n-1}\mathrm{B}_{ijkl}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{B}_{ijkl} = &- \frac{1}{n-2}\Big((\Delta\mathrm{R}_{ik} + 2\mathrm{R}^{pq}\mathrm{R}_{piqk} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq})g_{jl} \\
&- (\Delta\mathrm{R}_{il} + 2\mathrm{R}^{pq}\mathrm{R}_{piql} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq})g_{jk} \\
&+ (\Delta\mathrm{R}_{jl} + 2\mathrm{R}^{pq}\mathrm{R}_{pjql} - 2g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq})g_{ik} \\
&- (\Delta\mathrm{R}_{jk} + 2\mathrm{R}^{pq}\mathrm{R}_{pjqk} - 2g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq})g_{il} \\
&+ 4\mathrm{R}_{jk}\mathrm{R}_{il} - 4\mathrm{R}_{ik}\mathrm{R}_{jl}\Big) \\
= \Delta\mathrm{B}_{ijkl} &- \frac{2}{n-2}\Big((\mathrm{R}^{pq}\mathrm{R}_{piqk} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq})g_{jl} - (\mathrm{R}^{pq}\mathrm{R}_{piql} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq})g_{jk} \\
&+ (\mathrm{R}^{pq}\mathrm{R}_{pjql} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq})g_{ik} - (\mathrm{R}^{pq}\mathrm{R}_{pjqk} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq})g_{il}\Big) \\
&+ \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il}) \,.
\end{aligned}$$

Now we deal with the terms like $\mathrm{R}^{pq}\mathrm{R}_{piqk}$.
We have by definition $\mathrm{R}^{pq}\mathrm{R}_{piqk} = \mathrm{R}^{pq}\mathrm{W}_{piqk} - \mathrm{R}^{pq}\mathrm{A}_{piqk} - \mathrm{R}^{pq}\mathrm{B}_{piqk}$ and

$$\begin{aligned}
\mathrm{R}^{pq}\mathrm{A}_{piqk} &= \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}^{pq}g_{pq}g_{ik} - \mathrm{R}^{pq}g_{pk}g_{iq}) \\
&= \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}g_{ik} - \mathrm{R}_{ik})\,, \\
\mathrm{R}^{pq}\mathrm{B}_{piqk} &= -\frac{1}{n-2}(\mathrm{R}^{pq}\mathrm{R}_{pq}g_{ik} - \mathrm{R}^{pq}\mathrm{R}_{pk}g_{iq} + \mathrm{R}^{pq}\mathrm{R}_{ik}g_{pq} - \mathrm{R}^{pq}\mathrm{R}_{iq}g_{pk}) \\
&= -\frac{1}{n-2}(|\mathrm{Ric}|^2 g_{ik} + \mathrm{R}\mathrm{R}_{ik} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq})\,,
\end{aligned}$$

hence, we get

$$\begin{aligned}
\mathrm{R}^{pq}\mathrm{R}_{piqk} =& \,\mathrm{R}^{pq}\mathrm{W}_{piqk} - \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}g_{ik} - \mathrm{R}_{ik}) \\
&+ \frac{1}{n-2}(|\mathrm{Ric}|^2 g_{ik} + \mathrm{R}\mathrm{R}_{ik} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}) \\
=& \,\mathrm{R}^{pq}\mathrm{W}_{piqk} + \frac{1}{n-2}(|\mathrm{Ric}|^2 g_{ik} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}) + \frac{\mathrm{R}}{(n-1)(n-2)}(n\mathrm{R}_{ik} - \mathrm{R}g_{ik})\,.
\end{aligned}$$

Substituting these terms in the formula for $\frac{\partial}{\partial t}\mathrm{B}_{ijkl}$ we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{B}_{ijkl} =& \Delta \mathrm{B}_{ijkl} - \frac{2}{n-2}(\mathrm{R}^{pq}\mathrm{W}_{piqk}g_{jl} - \mathrm{R}^{pq}\mathrm{W}_{piql}g_{jk} + \mathrm{R}^{pq}\mathrm{W}_{pjql}g_{ik} - \mathrm{R}^{pq}\mathrm{W}_{pjqk}g_{il})\\
&- \frac{2|\mathrm{Ric}|^2}{(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk} + g_{jl}g_{ik} - g_{jk}g_{il})\\
&+ \frac{4}{(n-2)^2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk} + g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&- \frac{2n\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\\
&+ \frac{2\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk} + g_{jl}g_{ik} - g_{jk}g_{il})\\
&+ \frac{2}{n-2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk} + g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&+ \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il})\\
=& \Delta \mathrm{B}_{ijkl} - \frac{2}{n-2}(\mathrm{R}^{pq}\mathrm{W}_{piqk}g_{jl} - \mathrm{R}^{pq}\mathrm{W}_{piql}g_{jk} + \mathrm{R}^{pq}\mathrm{W}_{pjql}g_{ik} - \mathrm{R}^{pq}\mathrm{W}_{pjqk}g_{il})\\
&+ \frac{2n}{(n-2)^2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk} + g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&- \frac{2n\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\\
&+ \frac{2\mathrm{R}^2 - 2(n-1)|\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk} + g_{jl}g_{ik} - g_{jk}g_{il})\\
&+ \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il})\\
=& \Delta \mathrm{B}_{ijkl} - \frac{2}{n-2}(\mathrm{R}^{pq}\mathrm{W}_{piqk}g_{jl} - \mathrm{R}^{pq}\mathrm{W}_{piql}g_{jk} + \mathrm{R}^{pq}\mathrm{W}_{pjql}g_{ik} - \mathrm{R}^{pq}\mathrm{W}_{pjqk}g_{il})\\
&+ \frac{2n}{(n-2)^2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl} - g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk} + g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik} - g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&+ \frac{2n\mathrm{R}}{(n-1)(n-2)}\mathrm{B}_{ijkl} + \frac{4\mathrm{R}}{n-2}\mathrm{A}_{ijkl} - \frac{4|\mathrm{Ric}|^2}{(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk})\\
&+ \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il})\,.
\end{aligned}
$$

Hence,
(1.2)

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{W}_{ijkl} =&\Big(\frac{\partial}{\partial t}-\Delta\Big)(\mathrm{R}_{ijkl}+\mathrm{A}_{ijkl}+\mathrm{B}_{ijkl})\\
=&\,2(\mathrm{C}_{ijkl}-\mathrm{C}_{ijlk}+\mathrm{C}_{ikjl}-\mathrm{C}_{iljk})\\
&-g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{qjkl}+\mathrm{R}_{jp}\mathrm{R}_{iqkl}+\mathrm{R}_{kp}\mathrm{R}_{ijql}+\mathrm{R}_{lp}\mathrm{R}_{ijkq})\\
&+\frac{2|\mathrm{Ric}|^2}{(n-1)(n-2)}(g_{ik}g_{jl}-g_{il}g_{jk})+\frac{2\mathrm{R}}{n-1}\mathrm{B}_{ijkl}\\
&-\frac{2}{n-2}(\mathrm{R}^{pq}\mathrm{W}_{piqk}g_{jl}-\mathrm{R}^{pq}\mathrm{W}_{piql}g_{jk}+\mathrm{R}^{pq}\mathrm{W}_{pjql}g_{ik}-\mathrm{R}^{pq}\mathrm{W}_{pjqk}g_{il})\\
&+\frac{2n}{(n-2)^2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl}-g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk}+g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik}-g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&+\frac{2n\mathrm{R}}{(n-1)(n-2)}\mathrm{B}_{ijkl}+\frac{4\mathrm{R}}{n-2}\mathrm{A}_{ijkl}-\frac{4|\mathrm{Ric}|^2}{(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\\
&+\frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il})\\
=&\,2(\mathrm{C}_{ijkl}-\mathrm{C}_{ijlk}+\mathrm{C}_{ikjl}-\mathrm{C}_{iljk})\\
&-g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{qjkl}+\mathrm{R}_{jp}\mathrm{R}_{iqkl}+\mathrm{R}_{kp}\mathrm{R}_{ijql}+\mathrm{R}_{lp}\mathrm{R}_{ijkq})\\
&-\frac{2}{n-2}(\mathrm{R}^{pq}\mathrm{W}_{piqk}g_{jl}-\mathrm{R}^{pq}\mathrm{W}_{piql}g_{jk}+\mathrm{R}^{pq}\mathrm{W}_{pjql}g_{ik}-\mathrm{R}^{pq}\mathrm{W}_{pjqk}g_{il})\\
&+\frac{2n}{(n-2)^2}(g^{pq}\mathrm{R}_{ip}\mathrm{R}_{kq}g_{jl}-g^{pq}\mathrm{R}_{ip}\mathrm{R}_{lq}g_{jk}+g^{pq}\mathrm{R}_{jp}\mathrm{R}_{lq}g_{ik}-g^{pq}\mathrm{R}_{jp}\mathrm{R}_{kq}g_{il})\\
&-\frac{4\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{jl}-\mathrm{R}_{il}g_{jk}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il})\\
&+\frac{4\mathrm{R}^2-2n|\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})+\frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il})\,.
\end{aligned}
$$

Now, in order to simplify the formulas, we assume to be in an orthonormal basis, then $\mathrm{C}_{ijkl}=\mathrm{R}_{pijq}\mathrm{R}_{qlkp}$ and we have

$$
\begin{aligned}
\mathrm{C}_{ijkl}=&\,\mathrm{R}_{pijq}\mathrm{R}_{qlkp}\\
=&\,\mathrm{W}_{pijq}\mathrm{W}_{qlkp}+\mathrm{A}_{pijq}\mathrm{A}_{qlkp}+\mathrm{B}_{pijq}\mathrm{B}_{qlkp}+\mathrm{A}_{pijq}\mathrm{B}_{qlkp}+\mathrm{B}_{pijq}\mathrm{A}_{qlkp}\\
&-\mathrm{W}_{pijq}\mathrm{A}_{qlkp}-\mathrm{W}_{pijq}\mathrm{B}_{qlkp}-\mathrm{A}_{pijq}\mathrm{W}_{qlkp}-\mathrm{B}_{pijq}\mathrm{W}_{qlkp}\,.
\end{aligned}
$$

Substituting the expressions for the tensors A and B in the above terms and simplifying, we obtain the following identities.

$$
\mathrm{A}_{pijq}\mathrm{A}_{qlkp}=\frac{\mathrm{R}^2}{(n-1)^2(n-2)^2}(g_{ik}g_{jl}+(n-2)g_{ij}g_{lk})\,,
$$

$$
\begin{aligned}
\mathrm{B}_{pijq}\mathrm{B}_{qlkp}=&\,\frac{1}{(n-2)^2}(\mathrm{R}_{pj}g_{iq}+\mathrm{R}_{iq}g_{pj}-\mathrm{R}_{pq}g_{ij}-\mathrm{R}_{ij}g_{pq})(\mathrm{R}_{qk}g_{lp}+\mathrm{R}_{lp}g_{qk}-\mathrm{R}_{pq}g_{lk}-\mathrm{R}_{lk}g_{pq})\\
=&\,\frac{1}{(n-2)^2}\big(2\mathrm{R}_{ik}\mathrm{R}_{lj}+(n-4)\mathrm{R}_{ij}\mathrm{R}_{lk}+\mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik}+\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}-2\mathrm{R}_{pj}\mathrm{R}_{pi}g_{lk}-2\mathrm{R}_{pl}\mathrm{R}_{pk}g_{ij}\\
&+\mathrm{R}\mathrm{R}_{ij}g_{lk}+\mathrm{R}\mathrm{R}_{lk}g_{ij}+|\mathrm{Ric}|^2g_{ij}g_{lk}\big)\,,
\end{aligned}
$$

$$
\mathrm{A}_{pijq}\mathrm{B}_{qlkp}=-\frac{\mathrm{R}}{(n-1)(n-2)^2}\big(\mathrm{R}_{ik}g_{lj}+\mathrm{R}_{lj}g_{ik}-\mathrm{R}_{ij}g_{lk}+(n-3)\mathrm{R}_{lk}g_{ij}+\mathrm{R}g_{ij}g_{lk}\big)\,,
$$

$$
\mathrm{B}_{pijq}\mathrm{A}_{qlkp}=-\frac{\mathrm{R}}{(n-1)(n-2)^2}\big(\mathrm{R}_{lj}g_{ik}+\mathrm{R}_{ik}g_{lj}-\mathrm{R}_{lk}g_{ij}+(n-3)\mathrm{R}_{ij}g_{lk}+\mathrm{R}g_{ij}g_{lk}\big)\,,
$$

$$
\mathrm{W}_{pijq}\mathrm{A}_{qlkp}=\frac{\mathrm{R}}{(n-1)(n-2)}\mathrm{W}_{lijk}\,,
$$

$$\mathrm{A}_{pijq}\mathrm{W}_{qlkp} = \frac{\mathrm{R}}{(n-1)(n-2)}\mathrm{W}_{ilkj}\,,$$

$$\mathrm{W}_{pijq}\mathrm{B}_{qlkp} = -\frac{1}{n-2}(\mathrm{W}_{lijp}\mathrm{R}_{pk} + \mathrm{W}_{pijk}\mathrm{R}_{lp} - \mathrm{W}_{pijq}\mathrm{R}_{pq}g_{lk})\,,$$

$$\mathrm{B}_{pijq}\mathrm{W}_{qlkp} = -\frac{1}{n-2}(\mathrm{W}_{ilkp}\mathrm{R}_{pj} + \mathrm{W}_{plkj}\mathrm{R}_{pi} - \mathrm{W}_{qlkp}\mathrm{R}_{pq}g_{ij})$$

where in these last four computations we used the fact that every trace of the Weyl tensor is null.

Interchanging the indexes and summing we get

$$\begin{aligned}
\mathrm{A}_{pijq}\mathrm{A}_{qlkp} - \mathrm{A}_{pijq}\mathrm{A}_{qklp} + \mathrm{A}_{pikq}\mathrm{A}_{qljp} \ - \mathrm{A}_{pilq}\mathrm{A}_{qkjp}&\\
= \frac{\mathrm{R}^2}{(n-1)^2(n-2)^2}\Big(&g_{ik}g_{jl} + (n-2)g_{ij}g_{lk} - g_{il}g_{jk} - (n-2)g_{ij}g_{lk}\\
&+ g_{ij}g_{kl} + (n-2)g_{ik}g_{lj} - g_{ij}g_{kl} - (n-2)g_{il}g_{jk}\Big)\\
= \frac{\mathrm{R}^2}{(n-1)(n-2)^2}\,&(g_{ik}g_{jl} - g_{il}g_{jk})\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{B}_{pijq}\mathrm{B}_{qlkp} - \mathrm{B}_{pijq}\mathrm{B}_{qklp} + \mathrm{B}_{pikq}\mathrm{B}_{qljp} - \mathrm{B}_{pilq}\mathrm{B}_{qkjp}&\\
= \frac{1}{(n-2)^2}\Big(&2\mathrm{R}_{ik}\mathrm{R}_{lj} + (n-4)\mathrm{R}_{ij}\mathrm{R}_{lk} + \mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik} + \mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}\\
&- 2\mathrm{R}_{pj}\mathrm{R}_{pi}g_{lk} - 2\mathrm{R}_{pl}\mathrm{R}_{pk}g_{ij} + \mathrm{R}\mathrm{R}_{ij}g_{lk} + \mathrm{R}\mathrm{R}_{lk}g_{ij} + |\mathrm{Ric}|^2g_{ij}g_{lk}\\
&- 2\mathrm{R}_{il}\mathrm{R}_{kj} - (n-4)\mathrm{R}_{ij}\mathrm{R}_{lk} - \mathrm{R}_{pj}\mathrm{R}_{pk}g_{il} - \mathrm{R}_{pl}\mathrm{R}_{pi}g_{kj}\\
&+ 2\mathrm{R}_{pj}\mathrm{R}_{pi}g_{lk} + 2\mathrm{R}_{pk}\mathrm{R}_{pl}g_{ij} - \mathrm{R}\mathrm{R}_{ij}g_{lk} - \mathrm{R}\mathrm{R}_{lk}g_{ij} - |\mathrm{Ric}|^2g_{ij}g_{lk}\\
&+ 2\mathrm{R}_{ij}\mathrm{R}_{lk} + (n-4)\mathrm{R}_{ik}\mathrm{R}_{lj} + \mathrm{R}_{pk}\mathrm{R}_{pl}g_{ij} + \mathrm{R}_{pj}\mathrm{R}_{pi}g_{lk}\\
&- 2\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj} - 2\mathrm{R}_{pl}\mathrm{R}_{pj}g_{ik} + \mathrm{R}\mathrm{R}_{ik}g_{lj} + \mathrm{R}\mathrm{R}_{lj}g_{ik} + |\mathrm{Ric}|^2g_{ik}g_{lj}\\
&- 2\mathrm{R}_{ij}\mathrm{R}_{kl} - (n-4)\mathrm{R}_{il}\mathrm{R}_{jk} - \mathrm{R}_{pl}\mathrm{R}_{pk}g_{ij} - \mathrm{R}_{pj}\mathrm{R}_{pi}g_{kl}\\
&+ 2\mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk} + 2\mathrm{R}_{pk}\mathrm{R}_{pj}g_{il} - \mathrm{R}\mathrm{R}_{il}g_{jk} - \mathrm{R}\mathrm{R}_{jk}g_{il} - |\mathrm{Ric}|^2g_{il}g_{jk}\Big)\\
= \frac{1}{(n-2)^2}\Big(&(n-2)(\mathrm{R}_{ik}\mathrm{R}_{lj} - \mathrm{R}_{il}\mathrm{R}_{jk})\\
&- \mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik} - \mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj} + \mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk} + \mathrm{R}_{pk}\mathrm{R}_{pj}g_{il}\\
&+ \mathrm{R}(\mathrm{R}_{ik}g_{lj} + \mathrm{R}_{lj}g_{ik} - \mathrm{R}_{il}g_{jk} - \mathrm{R}_{jk}g_{il})\\
&+ |\mathrm{Ric}|^2(g_{ik}g_{lj} - g_{il}g_{jk})\Big)\,,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{A}_{pijq}\mathrm{B}_{qlkp} + \mathrm{B}_{pijq}\mathrm{A}_{qlkp} - &\mathrm{A}_{pijq}\mathrm{B}_{qklp} - \mathrm{B}_{pijq}\mathrm{A}_{qklp} \\
&+ \mathrm{A}_{pikq}\mathrm{B}_{qljp} + \mathrm{B}_{pikq}\mathrm{A}_{qljp} - \mathrm{A}_{pilq}\mathrm{B}_{qkjp} - \mathrm{B}_{pilq}\mathrm{A}_{qkjp} \\
= -\frac{\mathrm{R}}{(n-1)(n-2)^2} &\Big(\mathrm{R}_{ik}g_{lj} + \mathrm{R}_{lj}g_{ik} - \mathrm{R}_{ij}g_{lk} + (n-3)\mathrm{R}_{lk}g_{ij} + \mathrm{R}g_{ij}g_{lk} \\
&+ \mathrm{R}_{lj}g_{ik} + \mathrm{R}_{ik}g_{lj} - \mathrm{R}_{lk}g_{ij} + (n-3)\mathrm{R}_{ij}g_{lk} + \mathrm{R}g_{lk}g_{ij} \\
&- \mathrm{R}_{il}g_{kj} - \mathrm{R}_{jk}g_{il} + \mathrm{R}_{ij}g_{kl} - (n-3)\mathrm{R}_{kl}g_{ij} - \mathrm{R}g_{ij}g_{kl} \\
&- \mathrm{R}_{kj}g_{il} - \mathrm{R}_{il}g_{kj} + \mathrm{R}_{kl}g_{ij} - (n-3)\mathrm{R}_{ij}g_{kl} - \mathrm{R}g_{kl}g_{ij} \\
&+ \mathrm{R}_{ij}g_{lk} + \mathrm{R}_{lk}g_{ij} - \mathrm{R}_{ik}g_{lj} + (n-3)\mathrm{R}_{lj}g_{ik} + \mathrm{R}g_{ik}g_{lj} \\
&+ \mathrm{R}_{lk}g_{ij} + \mathrm{R}_{ij}g_{lk} - \mathrm{R}_{lj}g_{ik} + (n-3)\mathrm{R}_{ik}g_{lj} + \mathrm{R}g_{lj}g_{ik} \\
&- \mathrm{R}_{ij}g_{kl} - \mathrm{R}_{lk}g_{ij} + \mathrm{R}_{il}g_{kj} - (n-3)\mathrm{R}_{kj}g_{il} - \mathrm{R}g_{il}g_{kj} \\
&- \mathrm{R}_{kl}g_{ij} - \mathrm{R}_{ij}g_{kl} + \mathrm{R}_{kj}g_{il} - (n-3)\mathrm{R}_{il}g_{kj} - \mathrm{R}g_{kj}g_{il}\Big) \\
= -\frac{\mathrm{R}}{(n-1)(n-2)} &\Big(\mathrm{R}_{ik}g_{jl} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il} - \mathrm{R}_{il}g_{jk}\Big) \\
-\frac{2\mathrm{R}^2}{(n-1)(n-2)^2} &(g_{ik}g_{jl} - g_{il}g_{jk})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{W}_{pijq}\mathrm{A}_{qlkp} - \mathrm{W}_{pijq}\mathrm{A}_{qklp} + \mathrm{W}_{pikq}\mathrm{A}_{qljp} - \mathrm{W}_{pilq}\mathrm{A}_{qkjp}& \\
= \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{W}_{lijk} - \mathrm{W}_{kijl} + \mathrm{W}_{likj} - \mathrm{W}_{kilj}) = 0\,,&
\end{aligned}
$$

since the Weyl tensor, sharing the same symmetries of the Riemann tensor, is skew–symmetric in the third–fourth indexes.
The same result holds for the other sum as

$$
\mathrm{A}_{pijq}\mathrm{W}_{qlkp} = \frac{\mathrm{R}}{(n-1)(n-2)}\mathrm{W}_{ilkj} = \frac{\mathrm{R}}{(n-1)(n-2)}\mathrm{W}_{lijk} = \mathrm{W}_{pijq}\mathrm{A}_{qlkp}
$$

hence,

$$
\mathrm{A}_{pijq}\mathrm{W}_{qlkp} - \mathrm{A}_{pijq}\mathrm{W}_{qklp} + \mathrm{A}_{pikq}\mathrm{W}_{qljp} - \mathrm{A}_{pilq}\mathrm{W}_{qkjp} = 0\,.
$$

Finally, for the remaining two terms we have

$$
\begin{aligned}
-\mathrm{W}_{pijq}\mathrm{B}_{qlkp} - \mathrm{B}_{pijq}\mathrm{W}_{qlkp} &+ \mathrm{W}_{pijq}\mathrm{B}_{qklp} + \mathrm{B}_{pijq}\mathrm{W}_{qklp} \\
&- \mathrm{W}_{pikq}\mathrm{B}_{qljp} - \mathrm{B}_{pikq}\mathrm{W}_{qljp} + \mathrm{W}_{pilq}\mathrm{B}_{qkjp} + \mathrm{B}_{pilq}\mathrm{W}_{qkjp} \\
= \frac{1}{n-2} &\Big(\mathrm{W}_{lijp}\mathrm{R}_{pk} + \mathrm{W}_{pijk}\mathrm{R}_{lp} - \mathrm{W}_{pijq}\mathrm{R}_{pq}g_{lk} \\
&+ \mathrm{W}_{ilkp}\mathrm{R}_{pj} + \mathrm{W}_{plkj}\mathrm{R}_{pi} - \mathrm{W}_{qlkp}\mathrm{R}_{pq}g_{ij} \\
&- \mathrm{W}_{kijp}\mathrm{R}_{pl} - \mathrm{W}_{pijl}\mathrm{R}_{kp} + \mathrm{W}_{pijq}\mathrm{R}_{pq}g_{kl} \\
&- \mathrm{W}_{iklp}\mathrm{R}_{pj} - \mathrm{W}_{pklj}\mathrm{R}_{pi} + \mathrm{W}_{qklp}\mathrm{R}_{pq}g_{ij} \\
&+ \mathrm{W}_{likp}\mathrm{R}_{pj} + \mathrm{W}_{pikj}\mathrm{R}_{lp} - \mathrm{W}_{pikq}\mathrm{R}_{pq}g_{jl} \\
&+ \mathrm{W}_{iljp}\mathrm{R}_{pk} + \mathrm{W}_{pljk}\mathrm{R}_{pi} - \mathrm{W}_{qljp}\mathrm{R}_{pq}g_{ik} \\
&- \mathrm{W}_{kilp}\mathrm{R}_{pj} - \mathrm{W}_{pilj}\mathrm{R}_{kp} + \mathrm{W}_{pilq}\mathrm{R}_{pq}g_{kj} \\
&- \mathrm{W}_{ikjp}\mathrm{R}_{pl} - \mathrm{W}_{pkjl}\mathrm{R}_{pi} + \mathrm{W}_{qkjp}\mathrm{R}_{pq}g_{il}\Big) \\
= \frac{1}{n-2} &\Big(\mathrm{W}_{pilq}\mathrm{R}_{pq}g_{kj} + \mathrm{W}_{qkjp}\mathrm{R}_{pq}g_{il} - \mathrm{W}_{pikq}\mathrm{R}_{pq}g_{jl} - \mathrm{W}_{qljp}\mathrm{R}_{pq}g_{ik}\Big)
\end{aligned}
$$

where we used repeatedly the symmetries of the Weyl and the Ricci tensors.
Hence, summing all these terms we conclude

$$
\begin{aligned}
2(\mathrm{C}_{ijkl}-\mathrm{C}_{ijlk}+\mathrm{C}_{ikjl}-\mathrm{C}_{iljk}) &= 2(\mathrm{D}_{ijkl}-\mathrm{D}_{ijlk}+\mathrm{D}_{ikjl}-\mathrm{D}_{iljk})\\
&\quad+\frac{2\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\\
&\quad+\frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{lj}-\mathrm{R}_{il}\mathrm{R}_{jk})\\
&\quad+\frac{2}{(n-2)^2}(-\mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik}-\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}+\mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk}+\mathrm{R}_{pk}\mathrm{R}_{pj}g_{il})\\
&\quad+\frac{2\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{lj}+\mathrm{R}_{lj}g_{ik}-\mathrm{R}_{il}g_{jk}-\mathrm{R}_{jk}g_{il})+\frac{2|\mathrm{Ric}|^2}{(n-2)^2}(g_{ik}g_{lj}-g_{il}g_{jk})\\
&\quad-\frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}-\mathrm{R}_{il}g_{jk})\\
&\quad-\frac{4\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\\
&\quad+\frac{2}{n-2}(\mathrm{W}_{pilq}\mathrm{R}_{pq}g_{kj}+\mathrm{W}_{qkjp}\mathrm{R}_{pq}g_{il}-\mathrm{W}_{pikq}\mathrm{R}_{pq}g_{jl}-\mathrm{W}_{qljp}\mathrm{R}_{pq}g_{ik})\\
&= 2(\mathrm{D}_{ijkl}-\mathrm{D}_{ijlk}+\mathrm{D}_{ikjl}-\mathrm{D}_{iljk})\\
&\quad+\frac{2(n-1)|\mathrm{Ric}|^2-2\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\\
&\quad+\frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{lj}-\mathrm{R}_{il}\mathrm{R}_{jk})\\
&\quad-\frac{2}{(n-2)^2}(\mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik}+\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}-\mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk}-\mathrm{R}_{pk}\mathrm{R}_{pj}g_{il})\\
&\quad+\frac{2\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{R}_{ik}g_{jl}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}-\mathrm{R}_{il}g_{jk})\\
&\quad+\frac{2}{n-2}(\mathrm{W}_{pilq}\mathrm{R}_{pq}g_{kj}+\mathrm{W}_{qkjp}\mathrm{R}_{pq}g_{il}-\mathrm{W}_{pikq}\mathrm{R}_{pq}g_{jl}-\mathrm{W}_{qljp}\mathrm{R}_{pq}g_{ik})\,,
\end{aligned}
\tag{1.3}
$$

where $\mathrm{D}_{ijkl}=\mathrm{W}_{pijq}\mathrm{W}_{qlkp}$.

Then we deal with the following term appearing in equation (1.2),

$$
\begin{aligned}
\mathrm{R}_{ip}\mathrm{R}_{pjkl} + \mathrm{R}_{jp}\mathrm{R}_{ipkl} &+ \mathrm{R}_{kp}\mathrm{R}_{ijpl} + \mathrm{R}_{lp}\mathrm{R}_{ijkp}\\
&= \mathrm{R}_{ip}\mathrm{W}_{pjkl} + \mathrm{R}_{jp}\mathrm{W}_{ipkl} + \mathrm{R}_{kp}\mathrm{W}_{ijpl} + \mathrm{R}_{lp}\mathrm{W}_{ijkp}\\
&\quad - \frac{\mathrm{R}}{(n-1)(n-2)}\Big(\mathrm{R}_{ip}(g_{pk}g_{jl} - g_{pl}g_{jk}) + \mathrm{R}_{jp}(g_{ik}g_{pl} - g_{il}g_{pk})\Big)\\
&\quad - \frac{\mathrm{R}}{(n-1)(n-2)}\Big(\mathrm{R}_{kp}(g_{ip}g_{jl} - g_{il}g_{jp}) + \mathrm{R}_{lp}(g_{ik}g_{jp} - g_{ip}g_{jk})\Big)\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{ip}(\mathrm{R}_{pk}g_{jl} - \mathrm{R}_{pl}g_{jk} + \mathrm{R}_{jl}g_{pk} - \mathrm{R}_{jk}g_{pl}))\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{jp}(\mathrm{R}_{ik}g_{pl} - \mathrm{R}_{il}g_{pk} + \mathrm{R}_{pl}g_{ik} - \mathrm{R}_{pk}g_{il}))\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{kp}(\mathrm{R}_{ip}g_{jl} - \mathrm{R}_{il}g_{jp} + \mathrm{R}_{jl}g_{ip} - \mathrm{R}_{jp}g_{il}))\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{lp}(\mathrm{R}_{ik}g_{jp} - \mathrm{R}_{ip}g_{jk} + \mathrm{R}_{jp}g_{ik} - \mathrm{R}_{jk}g_{ip}))\\
&= \mathrm{R}_{ip}\mathrm{W}_{pjkl} + \mathrm{R}_{jp}\mathrm{W}_{ipkl} + \mathrm{R}_{kp}\mathrm{W}_{ijpl} + \mathrm{R}_{lp}\mathrm{W}_{ijkp}\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{ip}\mathrm{R}_{pk}g_{jl} - \mathrm{R}_{ip}\mathrm{R}_{pl}g_{jk} + \mathrm{R}_{jl}\mathrm{R}_{ik} - \mathrm{R}_{il}\mathrm{R}_{jk})\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{jl}\mathrm{R}_{ik} - \mathrm{R}_{jk}\mathrm{R}_{il} + \mathrm{R}_{jp}\mathrm{R}_{pl}g_{ik} - \mathrm{R}_{jp}\mathrm{R}_{pk}g_{il})\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{kp}\mathrm{R}_{ip}g_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il} + \mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{kp}\mathrm{R}_{jp}g_{il})\\
&\quad + \frac{1}{n-2}(\mathrm{R}_{jl}\mathrm{R}_{ik} - \mathrm{R}_{lp}\mathrm{R}_{ip}g_{jk} + \mathrm{R}_{lp}\mathrm{R}_{jp}g_{ik} - \mathrm{R}_{il}\mathrm{R}_{jk})\\
&\quad - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\\
&= \mathrm{R}_{ip}\mathrm{W}_{pjkl} + \mathrm{R}_{jp}\mathrm{W}_{ipkl} + \mathrm{R}_{kp}\mathrm{W}_{ijpl} + \mathrm{R}_{lp}\mathrm{W}_{ijkp}\\
&\quad + \frac{2}{n-2}(\mathrm{R}_{ip}\mathrm{R}_{kp}g_{jl} - \mathrm{R}_{ip}\mathrm{R}_{lp}g_{jk} + \mathrm{R}_{jp}\mathrm{R}_{lp}g_{ik} - \mathrm{R}_{jp}\mathrm{R}_{kp}g_{il})\\
&\quad + \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il})\\
&\quad - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il})\,.
\end{aligned}
$$

Inserting expression (1.3) and this last quantity in equation (1.2) we obtain

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{W}_{ijkl} =\, & 2(\mathrm{D}_{ijkl}-\mathrm{D}_{ijlk}+\mathrm{D}_{ikjl}-\mathrm{D}_{iljk}) \\
& + \frac{2(n-1)|\mathrm{Ric}|^2-2\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk}) \\
& + \frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{lj}-\mathrm{R}_{il}\mathrm{R}_{jk}) \\
& - \frac{2}{(n-2)^2}(\mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik}+\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}-\mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk}-\mathrm{R}_{pk}\mathrm{R}_{pj}g_{il}) \\
& + \frac{2\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{R}_{ik}g_{jl}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}-\mathrm{R}_{il}g_{jk}) \\
& + \frac{2}{n-2}(\mathrm{W}_{pilq}\mathrm{R}_{pq}g_{kj}+\mathrm{W}_{qkjp}\mathrm{R}_{pq}g_{il}-\mathrm{W}_{pikq}\mathrm{R}_{pq}g_{jl}-\mathrm{W}_{qljp}\mathrm{R}_{pq}g_{ik}) \\
& - \mathrm{R}_{ip}\mathrm{W}_{pjkl}-\mathrm{R}_{jp}\mathrm{W}_{ipkl}-\mathrm{R}_{kp}\mathrm{W}_{ijpl}-\mathrm{R}_{lp}\mathrm{W}_{ijkp} \\
& - \frac{2}{n-2}(\mathrm{R}_{ip}\mathrm{R}_{kp}g_{jl}-\mathrm{R}_{ip}\mathrm{R}_{lp}g_{jk}+\mathrm{R}_{jp}\mathrm{R}_{lp}g_{ik}-\mathrm{R}_{jp}\mathrm{R}_{kp}g_{il}) \\
& - \frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il}) \\
& + \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl}-\mathrm{R}_{il}g_{jk}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}) \\
& - \frac{2}{n-2}(\mathrm{R}_{pq}\mathrm{W}_{piqk}g_{jl}-\mathrm{R}_{pq}\mathrm{W}_{piql}g_{jk}+\mathrm{R}_{pq}\mathrm{W}_{pjql}g_{ik}-\mathrm{R}_{pq}\mathrm{W}_{pjqk}g_{il}) \\
& + \frac{2n}{(n-2)^2}(\mathrm{R}_{ip}\mathrm{R}_{kp}g_{jl}-\mathrm{R}_{ip}\mathrm{R}_{lp}g_{jk}+\mathrm{R}_{jp}\mathrm{R}_{lp}g_{ik}-\mathrm{R}_{jp}\mathrm{R}_{kp}g_{il}) \\
& - \frac{4\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{jl}-\mathrm{R}_{il}g_{jk}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}) \\
& + \frac{4\mathrm{R}^2-2n|\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})+\frac{4}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il}) \\
=\, & 2(\mathrm{D}_{ijkl}-\mathrm{D}_{ijlk}+\mathrm{D}_{ikjl}-\mathrm{D}_{iljk}) \\
& - (\mathrm{R}_{ip}\mathrm{W}_{pjkl}+\mathrm{R}_{jp}\mathrm{W}_{ipkl}+\mathrm{R}_{kp}\mathrm{W}_{ijpl}+\mathrm{R}_{lp}\mathrm{W}_{ijkp}) \\
& + \frac{2(\mathrm{R}^2-|\mathrm{Ric}|^2)}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk}) \\
& + \frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{lj}-\mathrm{R}_{il}\mathrm{R}_{jk}) \\
& + \frac{2}{(n-2)^2}(\mathrm{R}_{pj}\mathrm{R}_{pl}g_{ik}+\mathrm{R}_{pk}\mathrm{R}_{pi}g_{lj}-\mathrm{R}_{pl}\mathrm{R}_{pi}g_{jk}-\mathrm{R}_{pk}\mathrm{R}_{pj}g_{il}) \\
& - \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{jl}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il}-\mathrm{R}_{il}g_{jk})\,.
\end{aligned}
$$

Hence, we resume this long computation in the following proposition, getting back to a standard coordinate basis.

Proposition 1.1. *During the Ricci flow of an n–dimensional Riemannian manifold (M^n, g), the Weyl tensor satisfies the following evolution equation*

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{W}_{ijkl} =\; & 2\,(\mathrm{D}_{ijkl}-\mathrm{D}_{ijlk}+\mathrm{D}_{ikjl}-\mathrm{D}_{iljk})\\ & - g^{pq}(\mathrm{R}_{ip}\mathrm{W}_{qjkl}+\mathrm{R}_{jp}\mathrm{W}_{iqkl}+\mathrm{R}_{kp}\mathrm{W}_{ijql}+\mathrm{R}_{lp}\mathrm{W}_{ijkq})\\ & + \frac{2}{(n-2)^2}g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{qk}g_{jl}-\mathrm{R}_{ip}\mathrm{R}_{ql}g_{jk}+\mathrm{R}_{jp}\mathrm{R}_{ql}g_{ik}-\mathrm{R}_{jp}\mathrm{R}_{qk}g_{il})\\ & - \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{jl}-\mathrm{R}_{il}g_{jk}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il})\\ & + \frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il})+\frac{2(\mathrm{R}^2-|\mathrm{Ric}|^2)}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\,,\end{aligned}$$

where $\mathrm{D}_{ijkl} = g^{pq}g^{rs}\mathrm{W}_{pijr}\mathrm{W}_{slkq}$.

From this formula we immediately get the following rigidity result on the eigenvalues of the Ricci tensor.

Corollary 1.2. *Suppose that under the Ricci flow of (M^n, g) of dimension $n \geq 4$, the Weyl tensor remains identically zero. Then, at every point, either the Ricci tensor is proportional to the metric or it has an eigenvalue of multiplicity $(n-1)$ and another of multiplicity 1.*

Proof. By the above proposition, as every term containing the Weyl tensor is zero, the following relation holds at every point in space and time

$$\begin{aligned}0 =\; & \frac{2}{(n-2)^2}g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{qk}g_{jl}-\mathrm{R}_{ip}\mathrm{R}_{ql}g_{jk}+\mathrm{R}_{jp}\mathrm{R}_{ql}g_{ik}-\mathrm{R}_{jp}\mathrm{R}_{qk}g_{il})\\ & + \frac{2\mathrm{R}^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})-\frac{2|\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g_{ik}g_{jl}-g_{il}g_{jk})\\ & - \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{R}_{ik}g_{jl}-\mathrm{R}_{il}g_{jk}+\mathrm{R}_{jl}g_{ik}-\mathrm{R}_{jk}g_{il})+\frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl}-\mathrm{R}_{jk}\mathrm{R}_{il})\,.\end{aligned}$$

In normal coordinates such that the Ricci tensor is diagonal we get, for every couple of different eigenvectors v_i with relative eigenvalues λ_i,

$$(n-1)[\lambda_i^2+\lambda_j^2]-(n-1)\mathrm{R}(\lambda_i+\lambda_j)+(n-1)(n-2)\lambda_i\lambda_j+\mathrm{R}^2-|\mathrm{Ric}|^2=0\,. \tag{1.4}$$

As $n \geq 4$, fixing i, then the equation above is a second order polynomial in λ_j, hence it can only have at most 2 solutions, hence, we can conclude that there are at most three possible values for the eigenvalues of the Ricci tensor.
Since the dimension is at least four, at least one eigenvalues must have multiplicity two, let us say λ_i, hence the equation (1.4) holds also for $i = j$, and it remains at most only *one* possible value for the other eigenvalues λ_l with $l \neq i$. In conclusion, either the eigenvalues are all equal or they divide in only two possible values, λ with multiplicity larger than one, say k and $\mu \neq \lambda$. Suppose that μ also has multiplicity larger than one, that is, $k < n-1$, then we have

$$\begin{aligned}n\lambda^2-2\mathrm{R}\lambda &= \frac{|\mathrm{Ric}|^2-\mathrm{R}^2}{n-1}\\ n\mu^2-2\mathrm{R}\mu &= \frac{|\mathrm{Ric}|^2-\mathrm{R}^2}{n-1}\end{aligned} \tag{1.5}$$

taking the difference and dividing by $(\lambda-\mu)$ we get

$$n(\lambda+\mu)=2\mathrm{R}=2[k\lambda+(n-k)\mu]$$

then,

$$(n-2k)\lambda=(n-2k)\mu$$

hence, $n = 2k$, but then getting back to equation (1.5), $\mathrm{R} = n(\mu+\lambda)/2$ and

$$n\lambda^2-n(\mu+\lambda)\lambda=\frac{n(\lambda^2+\mu^2)/2-n^2(\mu^2+\lambda^2+2\lambda\mu)/4}{n-1}$$

which implies

$$-4n\lambda\mu = -\frac{n(n-2)}{n-1}(\lambda^2+\mu^2) - \frac{2n^2}{n-1}\mu\lambda$$

that is, after some computation,

$$\frac{2n(n-2)}{n-1}\mu\lambda = \frac{n(n-2)}{n-1}(\lambda^2+\mu^2)\,,$$

which implies $\lambda = \mu$.

At the end we conclude that at every point of M^n, either $\mathrm{Ric} = \lambda g$ or there is an eigenvalue λ of multiplicity $(n-1)$ and another μ of multiplicity 1. □

Remark 1.3. Notice that in dimension three equation (1.4) becomes

$$\begin{aligned}2[\lambda_i^2+\lambda_j^2] &- 2\mathrm{R}(\lambda_i+\lambda_j) + 2\lambda_i\lambda_j + \mathrm{R}^2 - |\mathrm{Ric}|^2\\ &= 2(\lambda_i+\lambda_j)^2 - 2\mathrm{R}(\lambda_i+\lambda_j) - 2\lambda_i\lambda_j + \mathrm{R}^2 - |\mathrm{Ric}|^2\\ &= -2\lambda_l(\lambda_i+\lambda_j) - 2\lambda_i\lambda_j + \mathrm{R}^2 - |\mathrm{Ric}|^2\\ &= 0\,,\end{aligned}$$

where λ_i, λ_j and λ_l are the three eigenvalues of the Ricci tensor.
Hence, the condition is void and our argument does not work. This is clearly not unexpected as the Weyl tensor is identically zero for every three–dimensional Riemannian manifold.

2. Locally Conformally Flat Ricci Solitons

Let (M^n, g), for $n \geq 4$, be a connected, complete, Ricci soliton, that is, there exists a smooth 1–form ω and a constant $\alpha \in \mathbb{R}$ such that

$$\mathrm{R}_{ij} + \frac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i) = \frac{\alpha}{n}g_{ij}\,.$$

If $\alpha > 0$ we say that the soliton is *shrinking*, if $\alpha = 0$ *steady*, if $\alpha < 0$ *expanding*.
If there exists a smooth function $f : M^n \to \mathbb{R}$ such that $df = \omega$ we say that the soliton is a *gradient* Ricci soliton and f its *potential function*, then we have

$$\mathrm{R}_{ij} + \nabla^2_{ij}f = \frac{\alpha}{n}g_{ij}\,.$$

If the metric dual field to the form ω is complete, then a Ricci soliton generates a self–similar solution to the Ricci flow (if the soliton is a gradient *soliton this condition is automatically satisfied* [33]*).*
In all this section we will assume to be in this case.

In this section we discuss the classification of Ricci solitons (M^n, g), for $n \geq 4$, which are locally conformally flat (LCF). As a consequence of Corollary 1.2 we have the following fact.

Proposition 2.1. *Let (M^n, g) be a complete, LCF Ricci soliton of dimension $n \geq 4$. Then, at every point, either the Ricci tensor is proportional to the metric or it has an eigenvalue of multiplicity $(n-1)$ and another of multiplicity 1.*

If a manifold (M^n, g) is LCF, it follows that

$$\begin{aligned}
0 =& \nabla^l \mathrm{W}_{ijkl} \\
=& \nabla^l \Big(\mathrm{R}_{ijkl} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) - \frac{1}{n-2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il}) \Big) \\
=& - \nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} + \frac{\nabla_j \mathrm{R}}{(n-1)(n-2)} g_{ik} - \frac{\nabla_i \mathrm{R}}{(n-1)(n-2)} g_{jk} \\
&- \frac{1}{n-2}(\nabla_j \mathrm{R}_{ik} - \nabla^l \mathrm{R}_{il} g_{jk} + \nabla^l \mathrm{R}_{jl} g_{ik} - \nabla_i \mathrm{R}_{jk} g_{il}) \\
=& - \frac{n-3}{n-2}(\nabla_i \mathrm{R}_{jk} - \nabla_j \mathrm{R}_{ik}) + \frac{\nabla_j \mathrm{R}}{(n-1)(n-2)} g_{ik} - \frac{\nabla_i \mathrm{R}}{(n-1)(n-2)} g_{jk} \\
&+ \frac{1}{2(n-2)}(\nabla_i \mathrm{R} g_{jk}/2 - \nabla_j \mathrm{R} g_{ik}/2) \\
=& - \frac{n-3}{n-2}\Big[\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \frac{(\nabla_i \mathrm{R} g_{jk} - \nabla_j \mathrm{R} g_{ik})}{2(n-1)} \Big] \\
=& \frac{n-3}{n-2}\Big[\nabla_j \Big(\mathrm{R}_{ik} - \frac{1}{2(n-1)} \mathrm{R} g_{ik} \Big) - \nabla_i \Big(\mathrm{R}_{jk} - \frac{1}{2(n-1)} \mathrm{R} g_{jk} \Big) \Big] ,
\end{aligned}$$

where we used the second Bianchi identity and Schur's Lemma $\nabla \mathrm{R} = 2 \operatorname{div} \mathrm{Ric}$.
Hence, since we assumed that the dimension n is at least four, the Schouten tensor defined by $\mathrm{S} = \mathrm{Ric} - \frac{1}{2(n-1)} \mathrm{R} g$ satisfies the equation

$$(\nabla_X \mathrm{S})\, Y = (\nabla_Y \mathrm{S})\, X, \quad X, Y \in TM\,.$$

Any symmetric two tensor satisfying this condition is called a Codazzi tensor (see [2, Chapter 16] for a general overview of Codazzi tensors).
Suppose that we have a local orthonormal frame $\{E_1, \dots, E_n\}$ in an open subset Ω of M^n such that $\mathrm{Ric}(E_1) = \lambda E_1$ and $\mathrm{Ric}(E_i) = \mu E_i$ for $i = 2, \dots, n$ and $\lambda \neq \mu$. For every point in Ω also the Schouten tensor S has two distinct eigenvalues σ_1 of multiplicity one and σ_2 of multiplicity $(n-1)$, with the same eigenspaces of λ and μ respectively, and

$$\sigma_1 = \frac{2n-3}{2(n-1)}\lambda - \frac{1}{2}\mu \quad \text{and} \quad \sigma_2 = \frac{1}{2}\mu - \frac{1}{2(n-1)}\lambda\,.$$

Splitting results for Riemannian manifolds admitting a Codazzi tensor with only two distinct eigenvalues were obtained by Derdzinski [11] and Hiepko–Reckziegel [20, 21] (see again [2, Chapter 16] for further discussion). In particular, it can be proved that, if the two distinct eigenvalues σ_1 and σ_2 are both "constant along the eigenspace $span\{E_2, \dots, E_n\}$" then the manifold is locally a warped product on an interval of $\mathbb{R}$ of a $(n-1)$–dimensional Riemannian manifold (see [2, Chapter 16] and [31]).
Since σ_2 has multiplicity $(n-1)$, larger than 2, we have for any two distinct indexes $i, j \geq 2$,

$$\begin{aligned}
\partial_i \sigma_2 =& \partial_i \mathrm{S}(E_j, E_j) \\
=& \nabla_i \mathrm{S}_{jj} + 2\mathrm{S}(\nabla_{E_i} E_j, E_j) \\
=& \nabla_j \mathrm{S}_{ij} + 2\sigma_2 g(\nabla_{E_i} E_j, E_j) \\
=& \partial_j \mathrm{S}(E_i, E_j) - \mathrm{S}(\nabla_{E_j} E_i, E_j) - \mathrm{S}(E_i, \nabla_{E_j} E_j) \\
=& - \sigma_2 g(\nabla_{E_j} E_i, E_j) - \sigma_2 g(E_i, \nabla_{E_j} E_j) \\
=& 0\,,
\end{aligned}$$

hence, σ_2 is always constant along the eigenspace $span\{E_2, \dots, E_n\}$. The eigenvalue σ_1 instead, for a general LCF manifold, can vary, for example $\mathbb{R}^n$ endowed with the metric

$$g = \frac{dx^2}{[1 + (x_1^2 + x_2^2 + \dots + x_{n-1}^2)]^2}$$

is LCF and

$$\mathrm{R}^g_{ij} = -(n-2)(\nabla^2_{ij}\log A - \nabla_i \log A \nabla_j \log A) + (\Delta \log A - (n-2)|\nabla \log A|^2)\delta_{ij}$$

where the derivatives are the standard ones of $\mathbb{R}^n$ and $A(x) = 1 + (x_1^2 + x_2^2 + \cdots + x_{n-1}^2)$ (see [2, Theorem 1.159]). Hence, this Ricci tensor "factorizes" on the eigenspaces $\langle e_1, \dots, e_{n-1}\rangle$ and $\langle e_n \rangle$ but the eigenvalue σ_1 of the Schouten tensor, which is given by

$$\begin{aligned}\sigma_1 = g^{nn}\mathrm{R}^g_{nn} &= (\Delta \log A - (n-2)|\nabla \log A|^2)A^2\\ &= A\Delta A - (n-1)|\nabla A|^2\\ &= 2(n-1)A - 4(n-1)(A-1)\\ &= -2(n-1)(A-2)\,,\end{aligned}$$

is clearly not constant along the directions $e_1, \dots, e_{n-1}$.

The best we can say in general is that the metric of (M^n, g) locally around every point can be written as $I \times N$ and

$$g(t,p) = \frac{dt^2 + \sigma^K(p)}{[\alpha(t) + \beta(p)]^2}$$

where σ^K is a metric on N of constant curvature K, $\alpha : I \to \mathbb{R}$ and $\beta : N \to \mathbb{R}$ are smooth functions such that $\mathrm{Hess}^K\beta = f\sigma^K$, for some function $f : N \to \mathbb{R}$ and where Hess^K is the Hessian of (N, σ^K).

2.1. **Compact LCF Ricci Solitons.** A compact Ricci soliton is actually a gradient soliton (by the work of Perelman [27]).
In general (even if they are not LCF), steady and expanding compact Ricci solitons are Einstein, hence, when also LCF, they are of constant curvature (respectively zero and negative).
In [7, 12] it is proved that also shrinking, compact, LCF Ricci solitons are of constant positive curvature, hence quotients of spheres.

Any compact, n–dimensional, LCF Ricci soliton is a quotient of $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ with their canonical metrics, for every $n \in \mathbb{N}$.

2.2. **LCF Ricci Solitons with Constant Scalar Curvature.** Getting back to the Schouten tensor, if the scalar curvature R of an LCF Ricci soliton (M^n, g) is constant, we have that also the other eigenvalue σ_1 of the Schouten tensor is constant along the eigenspace $span\{E_2, \dots, E_n\}$, that is, $\partial_i\sigma_1 = 0$, by simply differentiating the equality $\mathrm{R} = \frac{2(n-1)}{n-2}(\sigma_1 + (n-1)\sigma_2)$.

Hence, by the above discussion, we can conclude that around every point of M^n in the open set $\Omega \subset M^n$ where the two eigenvalues of the Ricci tensor are distinct the manifold (M^n, g) is locally a warped product $I \times N$ with $g(t,p) = dt^2 + h^2(t)\sigma(p)$ (this argument is due to Derdzinski [11]).
Then the LCF hypothesis implies that the warp factor (N, σ) is actually a space of constant curvature K (see for instance [4]).

As the scalar curvature R is constant, by the evolution equation $\partial_t \mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Ric}|^2$ we see that also $|\mathrm{Ric}|^2$ is constant, that is, locally $\mathrm{R} = \lambda + (n-1)\mu = C_1$ and $|\mathrm{Ric}|^2 = \lambda^2 + (n-1)\mu^2 = C_2$. Putting together these two equations it is easy to see that then both the eigenvalues μ and λ are locally constant in Ω. Hence, by connectedness, either (M^n, g) is Einstein, so a constant curvature space, or the Ricci tensor has two distinct constant eigenvalues everywhere. Using now the local warped product representation, the Ricci tensor is expressed by (see [2, Proposition 9.106] or [10, p. 65] or [5, p. 168])

$$\mathrm{Ric} = -(n-1)\,\frac{h''}{h}dt^2 + \big((n-2)K - h\,h'' - (n-2)(h')^2\big)\sigma^K\,. \tag{2.1}$$

hence, h''/h and $((n-2)K - h\,h'' - (n-2)(h')^2)/h^2$ are constant in t. This implies that $(K-(h')^2)/h^2$ is also constant and $h'' = Ch$, then locally either the manifold (M^n, g) is of constant curvature or it is the Riemannian product of a constant curvature space with an interval of $\mathbb{R}$.

By a maximality argument, passing to the universal covering of the manifold, we get the following conclusion.

If $n \geq 4$, any n–dimensional, LCF Ricci soliton with constant scalar curvature is either a quotient of $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ with their canonical metrics or a quotient of the Riemannian products $\mathbb{R}\times\mathbb{S}^{n-1}$ and $\mathbb{R}\times\mathbb{H}^{n-1}$ (see also [29]*).*

2.3. **Gradient LCF Ricci Solitons with Nonnegative Ricci Tensor.** Getting back again to the Codazzi property of the Schouten tensor S, for every index $i>1$, we have locally

$$0=\nabla_1\mathrm{R}_{i1}-\nabla_i\mathrm{R}_{11}-\frac{\partial_1\mathrm{R}}{2(n-1)}g_{i1}+\frac{\partial_i\mathrm{R}}{2(n-1)}g_{11}=\nabla_1\mathrm{R}_{i1}-\nabla_i\mathrm{R}_{11}+\frac{\partial_i\mathrm{R}}{2(n-1)}\,.$$

If the soliton is a gradient LCF Ricci soliton, that is, $\mathrm{Ric}=-\nabla^2 f+\frac{\alpha}{n}g$, we have $\mathrm{R}=-\Delta f+\alpha$ and taking the divergence of both sides

$$\begin{aligned}
\partial_i\mathrm{R}/2 &= \mathrm{div}\,\mathrm{Ric}_i\\
&= g^{jk}\nabla_k\mathrm{R}_{ij}\\
&= -\,g^{jk}\nabla_k\nabla_i\nabla_j f\\
&= -\,g^{jk}\nabla_i\nabla_k\nabla_j f-g^{jk}\mathrm{R}_{kijl}\nabla^l f\\
&= -\,\nabla_i\Delta f-\mathrm{R}_{il}\nabla^l f\\
&= \partial_i\mathrm{R}-\mathrm{R}_{il}\nabla^l f\,,
\end{aligned}$$

where we used Schur's Lemma $\partial_i\mathrm{R}=2\,\mathrm{div}\,\mathrm{Ric}_i$ and the formula for the interchange of covariant derivatives.
Hence, the relation $\partial_i\mathrm{R}=2\mathrm{R}_{il}\nabla^l f$ holds and

$$\nabla_1\nabla^2_{i1}f-\nabla_i\nabla^2_{11}f=\frac{\mathrm{R}_{ij}\nabla^j f}{n-1}\,.$$

By means of the fact that $\mathrm{W}=0$, we compute now for $i>1$ (this is a special case of the computation in Lemma 3.1 of [6]),

$$\begin{aligned}
\frac{\mu}{n-1}\nabla_i f &= \frac{\mathrm{R}_{ij}\nabla^j f}{n-1}\\
&= \nabla_1\nabla^2_{i1}f-\nabla_i\nabla^2_{11}f\\
&= \mathrm{R}_{1i1j}\nabla^j f\\
&= \left[\frac{1}{n-2}(\mathrm{R}_{11}g_{ij}-\mathrm{R}_{1j}g_{i1}+\mathrm{R}_{ij}g_{11}-\mathrm{R}_{i1}g_{1j})-\frac{\mathrm{R}}{(n-1)(n-2)}(g_{11}g_{ij}-g_{1j}g_{i1})\right]\nabla^j f\\
&= \left[\frac{1}{n-2}(\lambda g_{ij}+\mu g_{ij})-\frac{\mathrm{R}}{(n-1)(n-2)}g_{ij}\right]\nabla^j f\\
&= \left[\frac{\lambda+\mu}{n-2}-\frac{\mathrm{R}}{(n-1)(n-2)}\right]\nabla_i f\\
&= \frac{(n-1)\lambda+(n-1)\mu-\lambda-(n-1)\mu}{(n-1)(n-2)}\nabla_i f\\
&= \frac{\lambda}{n-1}\nabla_i f\,.
\end{aligned}$$

Then, in the open set $\Omega\subset M^n$ where the two eigenvalues of the Ricci tensor are distinct, the vector field ∇f is parallel to E_1, hence it is an eigenvector of the Ricci tensor and $\partial_i\mathrm{R}=2\mathrm{R}_{il}\nabla^l f=0$, for every index $i>1$.
As $\sigma_1=\frac{n-2}{2(n-1)}\mathrm{R}-(n-1)\sigma_2$ we get that also $\partial_i\sigma_1=0$ for every index $i>1$.

The set Ω is dense, otherwise its complement where $\mathrm{Ric}-Rg/n=0$ has interior points and, by Schur's Lemma, the scalar curvature would be constant in some open set of M^n. Then, strong maximum principle applied to the equation $\partial_t \mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Ric}|^2$ implies that R is constant everywhere on M^n, and we are in the previous case.

So we can conclude also in this case by the previous argument that the manifold, locally around every point in Ω, is a warped product on an interval of $\mathbb{R}$ of a constant curvature space $\mathbb{L}^K$. Moreover, Ω is obviously invariant by "translation" in the $\mathbb{L}^K$–direction.

We consider a point $p \in \Omega$ and the maximal geodesic curve $\gamma(t)$ passing from p orthogonal to $\mathbb{L}^K$, contained in Ω. It is easy to see that for every compact, connected segment of such geodesic we have a neighborhood U and a representation of the metric in g as

$$g = dt^2 + h^2(t)\sigma^K ,$$

covering the segment with the local charts and possibly shrinking them in the orthogonal directions.

Assuming from now on that the the Ricci tensor is nonnegative, by the local warped representation formula (2.1) we see that $h'' \leq 0$ along such geodesic, as $\mathrm{R}_{tt} \geq 0$.

If such geodesic has no "endpoints", being concave the function h must be constant and we have either a flat quotient of $\mathbb{R}^n$ or the Riemannian product of $\mathbb{R}$ with a quotient of $\mathbb{S}^{n-1}$. The same holds if the function h is constant in some interval, indeed, the manifold would be locally a Riemannian product and the scalar curvature would be locally constant (hence we are in the case above).

If there is at least one endpoint, one of the following two situations happens:

- the function h goes to zero at such endpoint,
- the geodesic hits the boundary of Ω.

If h goes to zero at an endpoint, by concavity $(h')^2$ must converge to some positive limit and by the smoothness of the manifold, considering again formula (2.1), the quantity $K - (h')^2$ must go to zero as h goes to zero, hence $K > 0$ and the constant curvature space $\mathbb{L}^K$ must be a quotient of the sphere $\mathbb{S}^{n-1}$ (if the same happens also at the other endpoint, the manifold is compact). Then, by topological reasons we conclude that actually the only possibility for $\mathbb{L}^K$ is the sphere $\mathbb{S}^{n-1}$ itself.

Assuming instead that h does not go to zero at any endpoint, where the geodesic hits the boundary of Ω the Ricci tensor is proportional to the metric, hence, again by the representation formula (2.1), the quantity $K - (h')^2$ is going to zero and either $K = 0$ or $K > 0$.

The case $K = 0$ is impossible, indeed h' would tend to zero at such endpoint, then by the concavity of h the function h' has a sign, otherwise h is constant in an interval, implying that in some open set (M^n, g) is flat, which cannot happens since we are in Ω. Thus, being $h' \neq 0$, h concave and we assumed that h does not go to zero, there must be another endpoint where the geodesic hits the boundary of Ω, which is in contradiction with $K = 0$ since also in this point $K - (h')^2$ must go to zero but instead h' tends to some nonzero value. Hence, K must be positive and also in this case we are dealing with a warped product of a quotient of $\mathbb{S}^{n-1}$ on an interval of $\mathbb{R}$.

Resuming, in the non–product situation, every connected piece of Ω is a warped product of a quotient of the sphere $\mathbb{S}^{n-1}$ on some intervals of $\mathbb{R}$. Then, we can conclude that the universal cover $(\widetilde{M}, \widetilde{g})$ can be recovered "gluing together", along constant curvature spheres, warped product pieces that can be topological "caps" (when h goes to zero at an endpoint) and "cylinders". Nontrivial quotients (M, g) of $(\widetilde{M}, \widetilde{g})$ are actually possible only when there are no "caps" in this gluing procedure. In such case, by its concavity, the function h must be constant along every piece of geodesic and the manifold $(\widetilde{M}, \widetilde{g})$ is a Riemannian product. If there is at least one "cap", the whole manifold is a warped product of $\mathbb{S}^{n-1}$ on an interval of $\mathbb{R}$.

Remark 2.2. We do not know if the condition on (M^n, g) to be a *gradient* LCF Ricci soliton is actually necessary to have locally a warped product. We conjecture that such conclusion should hold also for *nongradient* LCF Ricci solitons.

If $n \geq 4$, any n–dimensional, LCF gradient Ricci soliton with nonnegative Ricci tensor is either a quotient of $\mathbb{R}^n$ and $\mathbb{S}^n$ with their canonical metrics, or a quotient of $\mathbb{R} \times \mathbb{S}^{n-1}$ or it is a warped product of $\mathbb{S}^{n-1}$ on a proper interval of $\mathbb{R}$.

2.4. **The Classification of Steady and Shrinking Gradient LCF Ricci Solitons.** The class of solitons with nonnegative Ricci tensor is particularly interesting as it includes all the shrinking and steady Ricci solitons.
Indeed, by the same arguments of [32] (keeping in mind, in following the proof of the main Proposition 3.2 in such paper, that the nonnegativity of the scalar curvature for every complete, ancient Ricci was proved in [8, Corollary 2.5]), where the author generalizes the well–known Hamilton–Ivey curvature estimate to locally conformally flat, gradient, shrinking Ricci solitons (Corollary 3.3 in the same paper [32]), it follows that actually *every* complete ancient solution $g(t)$ to the Ricci flow whose Weyl tensor is identically zero for all times, is forced to have nonnegative curvature operator for every time t.
In particular, this holds for any complete, steady or shrinking Ricci soliton (even if not gradient) as they generate self–similar ancient solutions of Ricci flow.

By the previous discussion and the analysis of Bryant in the steady case [5] (see also [9, Chapter 1, Section 4]) showing that there exists a unique (up to dilation of the metric) nonflat, steady, gradient Ricci soliton which is a warped product of $\mathbb{S}^{n-1}$ on a halfline of $\mathbb{R}$, called *Bryant soliton*, we get the following classification.

Proposition 2.3. *The steady, gradient, LCF Ricci solitons of dimension $n \geq 4$ are given by the quotients of $\mathbb{R}^n$ and the Bryant soliton.*

This classification result, including also the three–dimensional LCF case, was first obtained recently by H.-D. Cao and Q. Chen [6].

In the shrinking case, the analysis of Kotschwar [22] of rotationally invariant shrinking, gradient Ricci solitons gives the following classification where the *Gaussian soliton* is defined as the flat $\mathbb{R}^n$ with a potential function $f = \alpha|x|^2/2n$, for a constant $\alpha \in \mathbb{R}$.

Proposition 2.4. *The shrinking, gradient, LCF Ricci solitons of dimension $n \geq 4$ are given by the quotients of $\mathbb{S}^n$, the Gaussian solitons with $\alpha > 0$ and quotients of $\mathbb{R} \times \mathbb{S}^{n-1}$.*

This classification of shrinking, gradient, LCF Ricci solitons follows by the works of L. Ni and N. Wallach [26], P. Petersen and W. Wylie [29] and Z.-H. Zhang [32].
Several other authors contributed to the subject, including X. Cao, B. Wang and Z. Zhang [7], B.-L. Chen [8], M. Fernández–López and E. García–Río [13], M. Eminenti, G. La Nave and C. Mantegazza [12], O. Munteanu and N. Sesum [24] and again P. Petersen and W. Wilye [28].

We show now that every complete, warped, LCF Ricci soliton with nonnegative Ricci tensor is actually a gradient soliton.
Proving our conjecture in Remark 2.2 that every Ricci soliton is locally a warped product would then lead to have a general classification of also nongradient Ricci solitons, in the steady and shrinking cases.

Remark 2.5. In the compact case, the fact that every Ricci soliton is actually a gradient is a consequence of the work of Perelman [27]. Naber [25] showed that it is true also for shrinking Ricci solitons with bounded curvature.
For examples of nongradient Ricci solitons see Baird and Danielo [1].

Proposition 2.6. *Let (M^n, g) be a complete, warped, LCF Ricci soliton with nonnegative Ricci tensor, then it is a gradient Ricci soliton with a potential function $f : M^n \to \mathbb{R}$ depending only on the t variable of the warping interval.*

Proof. We assume that (M^n, g) is globally described by $M^n = I \times \mathbb{L}^K$ and

$$g = dt^2 + h^2(t)\sigma^K \,,$$

where I is an interval of $\mathbb{R}$ or $\mathbb{S}^1$ and $(\mathbb{L}^K, \sigma^K)$ is a complete space of constant curvature K.
In the case h is constant, which clearly follows if $I = \mathbb{S}^1$, as $h'' \leq 0$ the conclusion is trivial.
We deal then with the case where $h : I \to \mathbb{R}$ is zero at some point, let us say $h(0) = 0$ and $I = [0, +\infty)$, (if the interval I is bounded the manifold M^n is compact and we are done). Then,

$\mathbb{L}^K = \mathbb{S}^{n-1}$ with its constant curvature metric σ^K. As a consequence, we have $M^n = \mathbb{R}^n$, simply connected. We consider the form ω satisfying the structural equation

$$\mathrm{R}_{\gamma\beta} + \frac{1}{2}(\nabla_\gamma\omega_\beta + \nabla_\beta\omega_\gamma) = \frac{\alpha}{n}g_{\gamma\beta}\,,$$

If $\varphi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is an isometry of the standard sphere, the associated map $\phi : M^n \to M^n$ given by $\phi(t,p) = (t, \varphi(p))$ is also an isometry, moreover, by the warped structure of M^n we have that the 1–form $\phi^*\omega$ also satisfies

$$\mathrm{R}_{\gamma\beta} + \frac{1}{2}\big[(\nabla\phi^*\omega)_{\gamma\beta} + (\nabla\phi^*\omega)_{\beta\gamma}\big] = \frac{\alpha}{n}g_{\gamma\beta}\,,$$

Calling $\mathcal{I}$ the Lie group of isometries of $\mathbb{S}^{n-1}$ and ξ the Haar unit measure associated to it, we define the following 1–form

$$\theta = \int_{\mathcal{I}} \phi^*\omega\, d\xi(\varphi)\,.$$

By the linearity of the structural equation, we have

$$\mathrm{R}_{\gamma\beta} + \frac{1}{2}(\nabla_\gamma\theta_\beta + \nabla_\beta\theta_\gamma) = \frac{\alpha}{n}g_{\gamma\beta}\,,$$

moreover, by construction, we have $L_X\theta = 0$ for every vector field X on M^n which is a generator of an isometry ϕ of M^n as above (in other words, θ depends only on t). Computing in normal coordinates on $\mathbb{S}^{n-1}$, we get

$$\begin{aligned}
\nabla_i\theta_j &= -\,\theta(\nabla_j\partial_i) = -\Gamma^t_{ij}\theta_t = hh'\sigma^K_{ij}\theta_t\,,\\
\nabla_i\theta_t &= -\,\theta(\nabla_t\partial_i) = -\Gamma^j_{ti}\theta_j = -\frac{h'}{h}\theta_i\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\alpha}{n} &= \mathrm{R}_{tt} + \nabla_t\theta_t = -(n-1)\frac{h''}{h} + \partial_t\theta_t\,,\\
0 &= \nabla_i\theta_t + \nabla_t\theta_i = \partial_t\theta_i - 2\frac{h'}{h}\theta_i\,,\\
\frac{\alpha}{n}g^K_{ij} &= \mathrm{R}_{ij} + \frac{1}{2}(\nabla_i\theta_j + \nabla_j\theta_i) = \big((n-2)(K-(h')^2) - hh'' + hh'\theta_t\big)g^K_{ij}\,.
\end{aligned}$$

It is possible to see that, by construction, actually $\theta_i = 0$ for every i at every point, but it is easier to consider directly the 1–form $\sigma = \theta_t dt$ on M^n and checking that it also satisfies these three equations as θ, hence the structural equation

$$\mathrm{R}_{\gamma\beta} + \frac{1}{2}(\nabla_\gamma\sigma_\beta + \nabla_\beta\sigma_\gamma) = \frac{\alpha}{n}g_{\gamma\beta}\,.$$

It is now immediate to see that, $d\sigma_{it} = \nabla_i\sigma_t - \nabla_t\sigma_i = 0$ and $d\sigma_{ij} = \nabla_i\sigma_j - \nabla_j\sigma_i = 0$, so the form σ is closed and being M^n simply connected, there exists a smooth function $f : M \to \mathbb{R}$ such that $df = \sigma$, thus

$$\mathrm{R}_{\gamma\beta} + \nabla^2_{\gamma\beta}f = \frac{\alpha}{n}g_{\gamma\beta}\,,$$

that is, the soliton is a gradient soliton.
It is also immediate to see that the function f depends only on $t \in I$. □

In the expanding, noncompact case (in the compact case the soliton can be only a quotient of the hyperbolic space $\mathbb{H}^n$), if the Ricci tensor is nonnegative and (M^n, g) is a gradient soliton, then either it is a warped product of $\mathbb{S}^{n-1}$ (and $M^n = \mathbb{R}^n$) or it is the product of $\mathbb{R}$ with a constant curvature space, but this last case is possible only if the soliton is the Gaussian expanding Ricci soliton, $\alpha < 0$, on the flat $\mathbb{R}^n$.

For a discussion of the expanding Ricci solitons which are warped products of $\mathbb{S}^{n-1}$ see [9, Chapter 1, Section 5], where the authors compute, for instance, an example with positive Ricci tensor (analogous to the Bryant soliton).

To our knowledge, the complete classification of complete, expanding, gradient, LCF Ricci solitons is an open problem, even if they are rotationally symmetric.

3. Singularities of Ricci Flow with Bounded Weyl Tensor

Let $(M^n, g(t))$ be a Ricci flow with M^n compact on the maximal interval $[0, T)$, with $T < +\infty$. Hamilton proved that

$$\max_M |\mathrm{Rm}|(\cdot, t) \to \infty$$

as $t \to T$.

We say that the solution has a Type I singularity if

$$\max_{M\times[0,T)} (T-t)|\mathrm{Rm}|(p,t) < +\infty\,,$$

otherwise we say that the solution develops a Type IIa singularity.

By Hamilton's procedure in [19], one can choose a sequence of points $p_i \in M^n$ and times $t_i \uparrow T$ such that, dilating the flow around these points in space and time, such sequence of rescaled Ricci flows (using Hamilton–Cheeger–Gromov compactness theorem in [18] and Perelman's injectivity radius estimate in [27]) converges to a complete maximal Ricci flow $(M_\infty, g_\infty(t))$ in an interval $t \in (-\infty, b)$ where $0 < b \leq +\infty$.
Moreover, in the case of a Type I singularity, we have $0 < b < +\infty$, $|\mathrm{Rm}_\infty|(p_\infty, 0) = 1$ for some point $p_\infty \in M_\infty$ and $|\mathrm{Rm}_\infty|(p,t) \leq 1$ for every $t \leq 0$ and $p \in M_\infty$.
In the case of a Type IIa singularity, $b = +\infty$, $|\mathrm{Rm}_\infty|(p_\infty, 0) = 1$ for some point $p_\infty \in M_\infty$ and $|\mathrm{Rm}_\infty|(p,t) \leq 1$ for every $t \in \mathbb{R}$ and $p \in M_\infty$.

These ancient limit flows were called by Hamilton *singularity models*. We want now to discuss them in the special case of a Ricci flow with uniformly bounded Weyl tensor (or with a blow up rate of the Weyl tensor which is of lower order than the one of the Ricci tensor). The Ricci flow under this condition is investigated also in [23].
Clearly, any limit flow consists of LCF manifolds, hence, by Corollary 1.2 and the cited results of Chen [8] and Zhang [32] at every time and every point the manifold has nonnegative curvature operator and either the Ricci tensor is proportional to the metric or it has an eigenvalue of multiplicity $(n-1)$ and another of multiplicity 1.

We follow now the argument in the proof of Theorem 1.1 in [6].
We recall the following splitting result (see [10, Chapter 7, Section 3]) which is a consequence of Hamilton's strong maximum principle for systems in [16].

Theorem 3.1. *Let $(M^n, g(t))$, $t \in (0,T)$ be a simply connected complete Ricci flow with nonnegative curvature operator. Then, for every $t \in (0,T)$ we have that $(M^n, g(t))$ is isometric to the product of the following factors,*

1. *the Euclidean space,*
2. *an irreducible nonflat compact Einstein symmetric space with nonnegative curvature operator and positive scalar curvature,*
3. *a complete Riemannian manifold with positive curvature operator,*
4. *a complete Kähler manifold with positive curvature operator on real $(1,1)$–forms.*

Since we are in the LCF case, every Einstein factor above must be a sphere (the scalar curvature is positive). The Kähler factors can be excluded as the following relation holds for Kähler manifolds of complex dimension $m > 1$ at every point (see [2, Proposition 2.68])

$$|\mathrm{W}|^2 \geq \frac{3(m-1)}{m(m+1)(2m-1)}\mathrm{R}^2\,.$$

Thus, any Kähler factor would have zero scalar curvature, hence would be flat. Finally, by the structure of the Ricci tensor and the fact that these limit flows are nonflat, it is easy to see that only a single Euclidean factor of dimension one is admissible, moreover, in this case there is only another factor $\mathbb{S}^{n-1}$.

In conclusion, passing to the universal cover, the possible limit flows are quotients of $\mathbb{R} \times \mathbb{S}^{n-1}$ or have a positive curvature operator.

Proposition 3.2 (LCF Type I singularity models)**.** *Let $(M^n, g(t))$, for $t \in [0, T)$, be a compact smooth solution to the Ricci flow with uniformly bounded Weyl tensor.*
If $g(t)$ develops a Type I singularity, then there are two possibilities:

(1) *M^n is diffeomorphic to a quotient of $\mathbb{S}^n$ and the solution to the normalized Ricci flow converges to a constant positive curvature metric.*
In this case the singularity model must be a shrinking compact Ricci soliton by a result of Sesum [30]*, hence by the analysis in the previous section, a quotient of $\mathbb{S}^n$ (this also follows by the work of Böhm and Wilking* [3]*).*
(2) *There exists a sequence of rescalings which converges to the flow of a quotient of $\mathbb{R} \times \mathbb{S}^{n-1}$.*

Proof. By the previous discussion, either the curvature operator is positive at every time or the limit flow is a quotient of $\mathbb{R} \times \mathbb{S}^{n-1}$.
Hence, we assume that every manifold in the limit flow has positive curvature operator. The family of metrics $g_\infty(t)$ is a complete, nonflat, LCF, ancient solution with uniformly bounded positive curvature operator which is k–non collapsed at all scales (hence a k–solution in the sense of [27]). By a result of Perelman in [27], we can find a sequence of times $t_i \searrow -\infty$ such that a sequence of suitable dilations of $g_\infty(t_i)$ converges to a nonflat, gradient, shrinking, LCF Ricci soliton. Hence, we can find an analogous sequence for the original flow. By the classification in the previous section, the thesis of the proposition follows. □

Remark 3.3. Notice that in case (2) we are not claiming that every Type I singularity model is a gradient shrinking Ricci soliton.
This problem is open also in the LCF situation.

Proposition 3.4 (LCF Type IIa singularity models)**.** *Let $(M^n, g(t))$, for $t \in [0, T)$, be a compact smooth solution to the Ricci flow with uniformly bounded Weyl tensor. If the flow develops a Type IIa singularity, then there exists a sequence of dilations which converges to the Bryant soliton.*

Proof. As we said, if the curvature operator gets some zero eigenvalue, the limit flow is a quotient of $\mathbb{R} \times \mathbb{S}^{n-1}$ which cannot be a steady soliton as it is not eternal. Hence, the curvature operator is positive.
By Hamilton's work [17], any Type IIa singularity model with nonnegative curvature operator and positive Ricci tensor is a steady, nonflat, gradient Ricci soliton. Since in our case such soliton is also LCF, by the analysis of the previous section, it must be the Bryant soliton. □

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