PROPERTIES OF LIPSCHITZ SMOOTHING HEAT SEMIGROUPS

NICOLÒ DE PONTI AND GIORGIO STEFANI

Abstract. We prove several functional and geometric inequalities only assuming the linearity and a quantitative L^∞-to-Lipschitz smoothing of the heat semigroup in metric-measure spaces. Our results comprise a Buser inequality, a lower bound on the size of the nodal set of a Laplacian eigenfunction, and different estimates involving the Wasserstein distance. The approach works in large variety settings, including Riemannian manifolds with a variable Kato-type lower bound on the Ricci curvature tensor, $\mathsf{RCD}(K,\infty)$ spaces, and some sub-Riemannian structures, such as Carnot groups, the Grushin plane and the $\mathbb{SU}(2)$ group.

1. Introduction

1.1. **Framework.** In the last decades, several authors have deeply investigated the connections between fundamental functional and geometric inequalities and the properties of the *heat semigroup* $(\mathsf{H}_t)_{t\ge0}$, especially its *linearity* and *regularizing* nature. We refer the reader for instance to [4, 6, 9, 24, 32, 39, 51] and the references therein.

The *linearity* of the heat semigroup is not automatically granted by definition, as for example $(\mathsf{H}_t)_{t\ge0}$ is not additive in the so-called *Finsler structures*, see [49]. In the non-smooth framework, the heat semigroup is defined as the L^2 gradient flow of the *Cheeger energy* (see [3] for an account) and its linearity goes under the name of *infinitesimal Hilbertianity* of the ambient space. This property plays a crucial role in different fundamental aspects of the theory, including the development of a powerful non-smooth analogue of Differential Calculus [36].

Smoothing properties of the heat semigroup, such as the (generalized) *Bakry–Émery inequality* [6, 8, 9, 53],

$$|\nabla \mathsf{H}_t f|^2 \le \kappa(t)^2\, \mathsf{H}_t\left(|\nabla f|^2\right), \quad \text{for } t \ge 0, \tag{1.1}$$

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for a suitable $\kappa\colon [0,\infty) \to [0,\infty)$, usually encode curvature-type information about the ambient space. For instance, on a complete Riemannian manifold (M, g), the validity of (1.1) with $\kappa(t) = e^{-Kt}$ for some $K \in \mathbb{R}$ is equivalent to the lower bound $\mathrm{Ric}_{\mathrm{g}} \ge K$ on the Ricci curvature tensor, e.g., see [58, Th. 1.3].

In passing, we observe that the linearity does not automatically imply any smoothing property of the heat semigroup, see the example in [4, Rem. 4.12].

1.2. **Main aim and results.** An important consequence of (1.1) is the L^∞*-to-Lipschitz contraction* of $(\mathsf{H}_t)_{t\ge0}$ (L^∞-to-Lip for short), i.e.,

$$f \in \mathrm{L}^\infty \implies \mathsf{H}_t f \in \mathrm{Lip}_b(X) \text{ with } \|\nabla \mathsf{H}_t f\|_{\mathrm{L}^\infty} \le \mathsf{c}(t)\,\|f\|_{\mathrm{L}^\infty} \text{ for } t > 0, \tag{1.2}$$

for a suitable $\mathsf{c}\colon [0,\infty) \to [0,\infty)$ (see Definition 3.1 for the precise statement). Our aim is to show how several functional and geometric inequalities can be deduced uniquely from the linearity of $(\mathsf{H}_t)_{t\ge0}$ and (1.2) in a general metric-measure space $(X, \mathsf{d}, \mathfrak{m})$ (see Section 2 for a detailed description of our setting).

The novelty of our approach lies in its *minimalistic* point of view, since we do not invoke any stronger curvature-type condition. As a byproduct, all results not only come with plain and concise proofs, but also apply to a wide range of examples, including metric-measure spaces with a synthetic constant lower curvature bound, Riemannian manifolds with a variable Kato-type lower bound on the Ricci curvature tensor and several smooth sub-Riemannian structures. In view of its simplicity and flexibility, we do believe that our strategy may be revisited for other types of semigroups. We refer to Sections 4 and 5 for the comparison with the existing literature and the possible extensions to other settings.

The techniques we employ have been partly applied to some specific frameworks. However, our work provides new contributions in some contexts in which they were not previously available. Our main results include but are not limited to:

- *an indeterminacy estimate*: a lower bound on the Wasserstein distance between positive and negative parts of a function $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ in terms of its L^1 and L^∞ norms and the perimeter of its zero set;
- *the size of the nodal set*: a lower bound on the perimeter of the zero set of a Laplacian eigenfunction f_λ in terms of its eigenvalue λ and of its L^1 and L^∞ norms;
- *an indeterminacy-type estimate for eigenfunctions*: a lower bound on the Wasserstein distance between positive and negative parts of an eigenfunction f_λ in terms of its eigenvalue λ and and of its L^1 norm;
- *a Buser-type inequality*: an upper bound on the first non-trivial eigenvalue of the Laplacian in terms of the Cheeger constant of the ambient space;
- *a transport–Sobolev inequality*: an upper bound on the L^1 norm of a BV function f in terms of its total variation and of the Wasserstein distance between its positive and negative parts.

The proof of each result consists of two main steps. We first derive *implicit* inequalities depending on $t > 0$, and then we provide their *explicit* versions by optimizing with respect to the parameter t in terms of a given upper control on the function $\mathsf{c}(t)$ in (1.2). The precise form of the inequalities depends on the expression of the upper bound on $\mathsf{c}(t)$—typically, on its asymptotic behavior as $t \to 0^+$. In all the aforementioned examples, a power-logarithmic-type upper control on $\mathsf{c}(t)$ is explicitly available.

1.3. **Organization of the paper.** In Section 2, we detail the notation and several preliminary results that we use throughout the paper. In Section 3, we introduce the L^∞-to-Lip property (see Definition 3.1) and we deduce its consequences in their implicit form. In Section 4, by prescribing an upper bound on $\mathsf{c}(t)$ (see Definition 4.1), we provide explicit versions of our results. In Section 5, we discuss the settings to which our approach applies.

2. Preliminaries

2.1. **Function spaces.** We let (X, d) be a complete and separable metric space.

We let $\mathrm{C}_b(X)$ be the space of real-valued, bounded and continuous functions on X. We let $\mathrm{Lip}(X)$, $\mathrm{Lip}_b(X)$ and $\mathrm{Lip}_{bs}(X)$ be the space of Lipschitz functions which are real-valued, bounded and with bounded support, respectively, and we let $\mathrm{Lip}(f) \in [0, \infty)$ denote the Lipschitz constant of the function $f \in \mathrm{Lip}(X)$.

Given any non-negative Borel measure $\mathfrak{m}$ on X, for $p \in [1, \infty]$ we let $\mathrm{L}^p(X, \mathfrak{m})$ be the Lebesgue space of p-integrable functions. To keep the notation short, we often write $\mathrm{L}^p(X)$ or simply L^p in place of $\mathrm{L}^p(X, \mathfrak{m})$. These spaces will be endowed with the norm

$$\|f\|_{\mathrm{L}^p} = \left(\int_X |f|^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty),$$

$$\|f\|_{\mathrm{L}^\infty} = \inf\left\{C \in [0, \infty) \;:\; |f(x)| \le C \text{ for } \mathfrak{m}\text{-a.e. } x \in X\right\}.$$

Note that $\|\cdot\|_{\mathrm{L}^p}$ is well-defined (possibly equal to ∞) on $\mathfrak{m}$-measurable functions on X. As customary, we identify L^p functions up to $\mathfrak{m}$-negligible sets.

2.2. **Wasserstein distance.** We let $\mathscr{M}(X)$ be the space of finite Borel measures on X and we let $\mathscr{M}_+(X) = \{\mu \in \mathscr{M}(X) : \mu \ge 0\}$. We let

$$|\mu|(X) = \sup\left\{\int_X \varphi \,\mathrm{d}\mathfrak{m} : \varphi \in \mathrm{C}_b(X),\ \|\varphi\|_{\mathrm{L}^\infty} \le 1\right\} \in [0, \infty)$$

be the *total variation* of $\mu \in \mathscr{M}(X)$. Lastly, we define

$$\mathscr{P}_1(X) = \left\{\mu \in \mathscr{M}_+(X) : \mu(X) = 1 \text{ and } \int_X \mathsf{d}(x, x_0) \,\mathrm{d}\mu(x) < \infty \text{ for some } x_0 \in X\right\}.$$

The 1-*Wasserstein distance* W_1 between $\mu_1, \mu_2 \in \mathscr{M}_+(X)$ is given by

$$\mathsf{W}_1(\mu_1, \mu_2) = \sup\left\{\int_X f \,\mathrm{d}(\mu_1 - \mu_2) \;:\; f \in \mathrm{Lip}_b(X),\ \mathrm{Lip}(f) \le 1\right\}. \tag{2.1}$$

Whenever $\mu_1, \mu_2 \in \mathscr{M}_+(X)$, a sufficient (but not necessary) condition for $\mathsf{W}_1(\mu_1, \mu_2) < \infty$ is that $C\mu_1, C\mu_2 \in \mathscr{P}_1(X)$ for some constant $C \in (0, \infty)$.

From now on, we assume that $\mathfrak{m}$ is a non-negative Borel-regular measure which is finite on bounded sets and such that $\operatorname{supp} \mathfrak{m} = X$.

2.3. **Slope.** The *slope* of $f \in \mathrm{Lip}(X)$ is defined as

$$|\mathsf{D}f|(x) = \begin{cases} \displaystyle\limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)} & \text{if } x \in X \text{ is an accumulation point,} \\ 0 & \text{if } x \in X \text{ is isolated.} \end{cases}$$

2.4. **Relaxed gradient.** Since the set $\{f \in \mathrm{L}^2(X) : f \in \mathrm{Lip}_b(X),\ |\mathsf{D}f| \in \mathrm{L}^2(X)\}$ is dense in $\mathrm{L}^2(X)$, we can say that $G \in \mathrm{L}^2(X)$ is a *relaxed gradient* of $f \in \mathrm{L}^2(X)$ if there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathrm{L}^2(X) \cap \mathrm{Lip}(X)$ such that $f_k \to f$ in $\mathrm{L}^2(X)$ and $|\mathsf{D}f_k| \rightharpoonup \widetilde{G}$ in $\mathrm{L}^2(X)$ for some $\widetilde{G} \in \mathrm{L}^2(X)$ such that $\widetilde{G} \le G$ $\mathfrak{m}$-a.e. in X.

The set of all the relaxed gradients of $f \in \mathrm{L}^2(X)$ is a closed and convex subset of $\mathrm{L}^2(X)$. Thus, when such set is not empty, it admits an element of minimal L^2 norm, called the *minimal relaxed gradient* and denoted by $|\mathsf{D}f|_w$. Such element is minimal also in the $\mathfrak{m}$-a.e. sense, meaning that $|\mathsf{D}f|_w \le G$ $\mathfrak{m}$-a.e. for any relaxed gradient G of f. In particular, $|\mathsf{D}f|_w \le |\mathsf{D}f|$ $\mathfrak{m}$-a.e. for every $f \in \mathrm{Lip}_{bs}(X)$.

2.5. **Cheeger energy.** We let

$$\mathsf{Ch}(f) = \inf\left\{\liminf_{k\to\infty} \frac{1}{2}\int_X |\mathsf{D}f_k|^2 \,\mathrm{d}\mathfrak{m} : f_k \in \mathrm{Lip}_{bs}(X,\mathfrak{m}),\ f_k \to f \text{ in } \mathrm{L}^2(X,\mathfrak{m})\right\}$$

be the *Cheeger energy* of $f \in \mathrm{L}^2(X)$. Thanks to [4, Ths. 6.2 and 6.3], we can write

$$\mathsf{Ch}(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_X |\mathsf{D}f|_w^2 \,\mathrm{d}\mathfrak{m} & \text{if } f \text{ admits a relaxed gradient,} \\ +\infty & \text{otherwise.} \end{cases}$$

As usual, we set

$$\mathrm{W}^{1,2}(X) = \mathrm{W}^{1,2}(X,\mathsf{d},\mathfrak{m}) = \left\{f \in \mathrm{L}^2(X) : \mathsf{Ch}(f) < \infty\right\}.$$

The Cheeger energy is a 2-homogenous, convex and lower semicontinuous functional on $\mathrm{L}^2(X)$ and the set $\mathrm{W}^{1,2}(X)$, endowed with the norm

$$\|f\|_{\mathrm{W}^{1,2}}^2 = \|f\|_{\mathrm{L}^2}^2 + 2\mathsf{Ch}(f), \quad f \in \mathrm{W}^{1,2}(X),$$

is dense in $\mathrm{L}^2(X)$.

2.6. **Laplacian operator.** We let $\partial^-\mathsf{Ch}(f) \subset \mathrm{L}^2(X)$ be the *subdifferential* of Ch at $f \in \mathrm{L}^2(X)$, i.e., $\ell \in \partial^-\mathsf{Ch}(f)$ if and only if

$$\mathsf{Ch}(g) \ge \mathsf{Ch}(f) + \int_X \ell\,(g-f)\,\mathrm{d}\mathfrak{m} \quad \text{for all } g \in \mathrm{L}^2(X).$$

We write $f \in \mathrm{Dom}(\Delta)$ if $f \in \mathrm{L}^2(X)$ is such that $\partial^-\mathsf{Ch}(f) \ne \emptyset$. For $f \in \mathrm{Dom}(\Delta)$, we let Δf be the element of minimal L^2 norm in $-\partial^-\mathsf{Ch}(f)$ and we call it the *Laplacian* of f.

2.7. **Heat semigroup.** By the classical theory of gradient flow in Hilbert spaces, for every $f \in \mathrm{L}^2(X)$ there exists a unique locally Lipschitz curve $t \mapsto \mathsf{H}_t f$ from $(0,\infty)$ to $\mathrm{L}^2(X)$, called the *heat flow at time t starting from f*, such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}_t f = \Delta \mathsf{H}_t f & \text{for a.e. } t \in (0,\infty), \\ \mathsf{H}_t f \to f \text{ in } \mathrm{L}^2(X) & \text{as } t \to 0^+. \end{cases} \tag{2.2}$$

We let $\mathsf{H}_0 = \mathrm{Id}$ be the identity operator in $\mathrm{L}^2(X)$, so that $(\mathsf{H}_t)_{t\ge 0}$ is a (possibly, non-linear) semigroup, called the *heat semigroup*. Because of the 2-homogeneity of the Cheeger energy, H_t and Δ are 1-homogeneous, i.e.,

$$\mathsf{H}_t(\lambda f) = \lambda\,\mathsf{H}_t f \qquad \text{for } f \in \mathrm{L}^2(X),\ \lambda \in \mathbb{R},$$
$$\Delta(\lambda g) = \lambda\,\Delta g \qquad \text{for } g \in \mathrm{Dom}(\Delta),\ \lambda \in \mathbb{R}.$$

Moreover, for $t \geq 0$ and $p \in [1, \infty]$, the heat semigroup satisfies the *contraction property*

$$\|\mathsf{H}_t f - \mathsf{H}_t g\|_{\mathrm{L}^p} \leq \|f - g\|_{\mathrm{L}^p} \quad \forall f, g \in \mathrm{L}^2 \cap \mathrm{L}^p(X) \tag{2.3}$$

and the *maximum principle*

$$f \leq C \ \mathfrak{m}\text{-a.e. in } X \text{ for some } C \in \mathbb{R} \implies \mathsf{H}_t f \leq C \ \mathfrak{m}\text{-a.e. in } X \tag{2.4}$$

(in particular, H_t is *sign preserving*). Finally, assuming that

$$\exists\, A, B > 0 \text{ and } \bar{x} \in X \text{ such that } \mathfrak{m}(\{x \in X : \mathsf{d}(x, \bar{x}) < r\}) \leq A e^{Br^2} \text{ for all } r > 0, \tag{2.5}$$

the heat semigroup also satisfies the *mass preserving property*

$$\int_X \mathsf{H}_t f \, \mathrm{d}\mathfrak{m} = \int_X f \, \mathrm{d}\mathfrak{m} \quad \text{whenever } f \in \mathrm{L}^2 \cap \mathrm{L}^1(X) \text{ and } t > 0. \tag{2.6}$$

2.8. **Infinitesimal Hilbertianity and non-smooth Calculus.** From now on, we assume that the metric-measure space $(X, \mathsf{d}, \mathfrak{m})$ is *infinitesimally Hilbertian*, meaning that

$$2\mathsf{Ch}(f) + 2\mathsf{Ch}(g) = \mathsf{Ch}(f + g) + \mathsf{Ch}(f - g) \quad \text{for all } f, g \in \mathrm{W}^{1,2}(X). \tag{2.7}$$

In this case, the heat flow is also additive, and thus $(\mathsf{H}_t)_{t \geq 0}$ is a linear semigroup with the energy $\mathcal{E} = 2\mathsf{Ch}$ being the associated *strongly-local Dirichlet form*.

By the density of $\mathrm{L}^2 \cap \mathrm{L}^p(X)$ in $\mathrm{L}^p(X)$ and in virtue of (2.3), H_t extends to a strongly continuous linear semigroup of contractions in $\mathrm{L}^p(X)$ for any $p \in [1, \infty)$, for which we keep the same notation. By duality, H_t also extends to a linear and weakly*-continuous semigroup of contractions in $\mathrm{L}^\infty(X)$ such that

$$\int_X g \, \mathsf{H}_t f \, \mathrm{d}\mathfrak{m} = \int_X f \, \mathsf{H}_t g \, \mathrm{d}\mathfrak{m} \quad \text{for } f \in \mathrm{L}^\infty(X) \text{ and } g \in \mathrm{L}^1(X). \tag{2.8}$$

By polarization, there exists a bilinear form

$$(f, g) \mapsto \int \mathsf{D}f \cdot \mathsf{D}g \, \mathrm{d}\mathfrak{m} \leq \int_X |\mathsf{D}f|_w \, |\mathsf{D}g|_w \, \mathrm{d}\mathfrak{m} \quad \text{for } f, g \in \mathrm{W}^{1,2}(X)$$

satisfying the *integration-by-parts*

$$\int_X \mathsf{D}f \cdot \mathsf{D}g \, \mathrm{d}\mathfrak{m} = -\int_X f \, \Delta g \, \mathrm{d}\mathfrak{m} \quad \text{for } f \in \mathrm{W}^{1,2}(X) \text{ and } g \in \mathrm{Dom}(\Delta)\,. \tag{2.9}$$

In addition, the heat semigroup and the Laplacian are *self-adjoint*, i.e.,

$$\int_X f \, \Delta g \, \mathrm{d}\mathfrak{m} = \int_X g \, \Delta f \, \mathrm{d}\mathfrak{m} \quad \text{for } f, g \in \mathrm{Dom}(\Delta), \tag{2.10}$$

$$\int_X f \, \mathsf{H}_t g \, \mathrm{d}\mathfrak{m} = \int_X g \, \mathsf{H}_t f \, \mathrm{d}\mathfrak{m} \quad \text{for } f, g \in \mathrm{L}^2(X) \text{ and } t \geq 0. \tag{2.11}$$

Finally, we recall the commutation

$$\mathsf{H}_t(\Delta f) = \Delta \mathsf{H}_t f \quad \text{for } f \in \mathrm{Dom}(\Delta) \text{ and } t > 0 \tag{2.12}$$

and the *a priori* estimate

$$\|\Delta \mathsf{H}_t f\|_{\mathrm{L}^2} \leq \frac{1}{t} \, \|f\|_{\mathrm{L}^2} \quad \text{for } f \in \mathrm{L}^2(X) \text{ and } t > 0. \tag{2.13}$$

Using (2.10), (2.11) and (2.12), together with the fact that the heat flow is a semigroup with image contained in the domain of the Laplacian, we get that

$$\int_X g \, \Delta \mathsf{H}_t f \, \mathrm{d}\mathfrak{m} = \int_X f \, \Delta \mathsf{H}_t g \, \mathrm{d}\mathfrak{m} \quad \text{for } f, g \in \mathrm{L}^2(X) \text{ and } t > 0. \tag{2.14}$$

The following result is a simple consequence of the above properties. Although this result may be known to experts, we give its short proof here for the reader's convenience.

Lemma 2.1. *If $f, g \in \mathrm{W}^{1,2}(X)$, then*

$$\int_X g\,(f - \mathsf{H}_t f)\,\mathrm{d}\mathfrak{m} = \int_0^t \int_X \mathsf{D}f \cdot \mathsf{D}\mathsf{H}_s g\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \quad \textit{for all } t > 0.$$

Proof. Since Ch is quadratic and $f \in \mathrm{W}^{1,2}(X)$, we know that

$$s \mapsto \mathsf{H}_s f \in C^1((0,+\infty); \mathrm{Dom}(\Delta)) \cap C^0([0,+\infty); \mathrm{L}^2(X))$$

with

$$\lim_{h\to 0} \frac{\mathsf{H}_{s+h} f - \mathsf{H}_s f}{h} = \Delta \mathsf{H}_s f \quad \text{in } \mathrm{L}^2(X) \text{ for } s > 0.$$

As a consequence, thanks to (2.14), we can compute

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_X g\,\mathsf{H}_s f\,\mathrm{d}\mathfrak{m} = \int_X g\,\Delta\mathsf{H}_s f\,\mathrm{d}\mathfrak{m} = \int_X f\,\Delta\mathsf{H}_s g\,\mathrm{d}\mathfrak{m} \quad \text{for } s \in (0,t).$$

By (2.9), we can integrate by parts to obtain

$$\int_X f\,\Delta\mathsf{H}_s g\,\mathrm{d}\mathfrak{m} = -\int_X \mathsf{D}f \cdot \mathsf{D}\mathsf{H}_s g\,\mathrm{d}\mathfrak{m} \quad \text{for } s \in (0,t).$$

We can hence integrate in $s \in (0,t)$ to get

$$\int_X g\,(f - \mathsf{H}_t f)\,\mathrm{d}\mathfrak{m} = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \int_X g\,\mathsf{H}_s f\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \int_0^t \int_X \mathsf{D}f \cdot \mathsf{D}\mathsf{H}_s g\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s,$$

where the right-hand side is well defined since $g \in \mathrm{W}^{1,2}(X)$ (see [53, Lem. 2.1]). □

2.9. **Eigenfunctions and spectrum.** A non-zero $f_\lambda \in \mathrm{Dom}(\Delta)$ is an *eigenfunction* of the Laplacian relative to the *eigenvalue* $\lambda \in [0,\infty)$ (λ-*eigenfunction*, for short) if

$$-\Delta f_\lambda = \lambda\, f_\lambda.$$

If $\mathfrak{m}(X) < \infty$, then any non-zero constant function is a 0-eigenfunction and, moreover, every other λ-eigenfunction f_λ has zero mean, so that

$$\int_X f_\lambda^-\,\mathrm{d}\mathfrak{m} = \int_X f_\lambda^+\,\mathrm{d}\mathfrak{m}. \tag{2.15}$$

For the reader's ease, we recall the following well-known result.

Lemma 2.2. *If f_λ is a λ-eigenfunction, then $\mathsf{H}_t f_\lambda = e^{-\lambda t} f_\lambda$ for all $t \geq 0$.*

Proof. The map $(0,\infty) \ni t \mapsto G(t) = \mathsf{H}_t f_\lambda$ is locally Lipschitz in $(0,\infty)$ and continuous in $[0,\infty)$ with values in $\mathrm{L}^2(X)$. Since

$$G'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{H}_t f_\lambda = \Delta \mathsf{H}_t f_\lambda = -\lambda\,\mathsf{H}_t f_\lambda = -\lambda\, G(t) \quad \text{for a.e. } t > 0$$

due to (2.12) and the definition of eigenfunction, the function $t \mapsto e^{\lambda t} G(t)$ must be constant and thus equal to $G(0) = f_\lambda$, readily yielding the conclusion. □

The *Rayleigh quotient* of $f \in \mathrm{Dom}(\mathsf{Ch}) \setminus \{0\}$ is defined as

$$\mathcal{R}(f) = \frac{2\mathsf{Ch}(f)}{\displaystyle\int_X |f|^2 \,\mathrm{d}\mathfrak{m}}. \tag{2.16}$$

We consider

$$\lambda_0 = \inf\left\{\mathcal{R}(f) : f \in \mathrm{Dom}(\mathsf{Ch}) \setminus \{0\}\right\} \tag{2.17}$$

and

$$\lambda_1 = \inf\left\{\mathcal{R}(f) : f \in \mathrm{Dom}(\mathsf{Ch}) \setminus \{0\},\ \int_X f \,\mathrm{d}\mathfrak{m} = 0\right\} \tag{2.18}$$

Clearly, $0 \le \lambda_0 \le \lambda_1$. Moreover, if λ_k, $k = 0, 1$, is below the infimum of the essential spectrum of $-\Delta\colon \mathrm{Dom}(\Delta) \to L^2(X)$, then λ_k corresponds to the classical k-th eigenvalue of (minus) the Laplacian.

For the convenience of the reader, we recall the following simple result.

Lemma 2.3. *Let $f \in \mathrm{L}^2(X)$. The following hold:*

(i) if $\mathfrak{m}(X) < \infty$ and $\int_X f \,\mathrm{d}\mathfrak{m} = 0$, then $\|\mathsf{H}_t f\|_{\mathrm{L}^2} \le e^{-\lambda_1 t}\|f\|_{\mathrm{L}^2}$ for all $t \ge 0$;

(ii) if $\mathfrak{m}(X) = \infty$, then $\|\mathsf{H}_t f\|_{\mathrm{L}^2} \le e^{-\lambda_0 t}\|f\|_{\mathrm{L}^2}$ for all $t \ge 0$.

Proof. To prove (i), we can assume $\mathfrak{m}(X) = 1$. By definition of λ_1 in (2.18), we have

$$2\lambda_1 \int_X |\mathsf{H}_t f|^2 \,\mathrm{d}\mathfrak{m} \le 2\int_X |\mathsf{D}(\mathsf{H}_t f)|_w^2 \,\mathrm{d}\mathfrak{m} = -2\int_X \mathsf{H}_t f\,\Delta(\mathsf{H}_t f)\,\mathrm{d}\mathfrak{m} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_X |\mathsf{H}_t f|^2 \,\mathrm{d}\mathfrak{m}$$

for all $t > 0$, thanks to (2.9), (2.6) and (2.2). Hence (i) follows by Grönwall's Lemma. The proof of (ii) similarly follows by exploiting the definition in (2.17) and is thus omitted. □

2.10. BV **functions and Cheeger constants.** We say that $f \in \mathrm{BV}(X) = \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ if $f \in \mathrm{L}^1(X)$ and there exists $(f_k)_{k\in\mathbb{N}} \subset \mathrm{Lip}_{bs}(X)$ such that $f_k \to f$ in $\mathrm{L}^1(X)$ and

$$\sup_{k\in\mathbb{N}} \int_X |\mathsf{D} f_k| \,\mathrm{d}\mathfrak{m} < \infty.$$

We thus let

$$\mathsf{Var}(f) = \inf\left\{\liminf_{k\to\infty} \int_X |\mathsf{D} f_k| \,\mathrm{d}\mathfrak{m} : f_k \in \mathrm{Lip}_{bs}(X),\ f_k \to f \text{ in } \mathrm{L}^1(X)\right\} \tag{2.19}$$

be the *total variation of f*. We write $\mathsf{Per}(A) = \mathsf{Var}(\chi_A)$ whenever $\chi_A \in \mathrm{BV}(X)$.

In analogy with (2.17) and (2.18), we consider

$$h_0(X) = \inf\left\{\frac{\mathsf{Per}(A)}{\mathfrak{m}(A)} : A \subset X \text{ Borel subset with } 0 < \mathfrak{m}(A) < \infty\right\} \tag{2.20}$$

and

$$h_1(X) = \inf\left\{\frac{\mathsf{Per}(A)}{\mathfrak{m}(A)} : A \subset X \text{ Borel subset with } 0 < \mathfrak{m}(A) \le \frac{\mathfrak{m}(X)}{2}\right\}. \tag{2.21}$$

The definition in (2.21) corresponds to the one introduced in [22]. We observe that, if $\mathfrak{m}(X) < \infty$, then $h_0(X) = 0$.

For future convenience, we recall the following simple estimate, proved in [28, Lem. 3.2].

Lemma 2.4. *If* $\mathfrak{m}(X) < \infty$*, then*

$$h_1(X) \le \inf\left\{2\,\mathsf{Per}(\{f>0\})\,\frac{\|f\|_{\mathrm{L}^\infty}}{\|f\|_{\mathrm{L}^1}} : f \in \mathrm{L}^\infty(X) \text{ such that } \int_X f\,\mathrm{d}\mathfrak{m} = 0\right\}. \tag{2.22}$$

2.11. **Main assumptions.** We conclude this section by summarizing the main assumptions we are going to use throughout the rest of the paper. We let $(X, \mathsf{d}, \mathfrak{m})$ be a metric-measure space satisfying the following properties:

(P1) (X, d) is a complete and separable metric space;
(P2) $\mathfrak{m}$ is a non-negative Borel-regular measure on X satisfying $\operatorname{supp}\mathfrak{m} = X$ and (2.5);
(P3) $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, i.e., (2.7) holds.

3. Quantitative Lipschitz smoothing property and implicit inequalities

3.1. **Quantitative Lipschitz smoothing property.** We let $\mathsf{c}\colon (0,\infty) \to (0,\infty)$ be a Borel function.

Definition 3.1 (L^∞-to-Lip)**.** We say that the heat semigroup $(\mathsf{H}_t)_{t\ge0}$ satisfies the L^∞*-to-*Lip *contraction property with Lipschitz constant* c (or is c-Lip, for short) if

$$f \in \mathrm{L}^\infty(X) \implies \mathsf{H}_t f \in \mathrm{Lip}_b(X) \text{ with } \mathrm{Lip}(\mathsf{H}_t f) \le \mathsf{c}(t)\,\|f\|_{\mathrm{L}^\infty} \text{ for all } t>0. \tag{c-Lip}$$

In this section, we assume that the linear heat semigroup $(\mathsf{H}_t)_{t\ge0}$ is c-Lip.

3.2. **Dual semigroup.** For any $f \in \mathrm{L}^1(X)$ such that $f \ge 0$, we define

$$\mathsf{H}_t^\star(f\mathfrak{m}) = (\mathsf{H}_t f)\,\mathfrak{m} \in \mathscr{M}(X) \quad \text{for } t \ge 0. \tag{3.1}$$

In the following result, we show that the semigroup $\mathsf{H}_t^\star$ in (3.1) can be extended to finite Borel measures on X. Here and in the following, we let $\mathrm{rba}(X)$ be the space of *bounded, Borel regular, finitely-additive measures* on X.

Theorem 3.2. *If* $\mu \in \mathscr{M}(X)$ *and* $t>0$*, then there exists a unique* $\mathsf{H}_t^\star\mu \in \mathrm{rba}(X)$ *with* $|\mathsf{H}_t^\star\mu|(X) \le |\mu|(X)$ *such that*

$$\int_X \mathsf{H}_t f\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mathsf{H}_t^\star\mu \quad \textit{for all } f \in \mathrm{C}_b(X). \tag{3.2}$$

If (X,d) *is locally compact, then* $\mathsf{H}_t^\star\mu \in \mathscr{M}(X)$*, with* $|\mathsf{H}_t^\star\mu| \ll \mathfrak{m}$ *and* $\mathsf{H}_t^\star\mu \ge 0$ *if* $\mu \ge 0$*.*

Proof. Let $t>0$ be fixed. Thanks to (c-Lip), the map $\mathcal{F}\colon \mathrm{C}_b(X) \to \mathbb{R}$ given by

$$\mathcal{F}(f) = \int_X \mathsf{H}_t f\,\mathrm{d}\mu, \quad \text{for } f \in \mathrm{C}_b(X),$$

defines a a linear and continuous functional on $\mathrm{C}_b(X)$ such that, by (2.4),

$$|\mathcal{F}(f)| \le \|\mathsf{H}_t f\|_{\mathrm{L}^\infty}\,|\mu|(X) \le \|f\|_{\mathrm{L}^\infty}\,|\mu|(X) \quad \text{for } f \in \mathrm{C}_b(X).$$

By [1, Th. 14.10], we hence get that $\mathsf{H}_t^\star\mu \in \mathrm{rba}(X)$. If (X,d) is locally compact, then the restriction of $\mathcal{F}$ to $\mathrm{C}_c(X)$, the space of continuous functions with compact support, is a linear continuous operator on $\mathrm{C}_c(X)$. By [1, Ths. 14.12 and 14.14], we hence get that $\mathsf{H}_t\mu \in \mathscr{M}(X)$ with $\mathsf{H}_t\mu \ge 0$ if $\mu \ge 0$. To conclude, we just need to prove that $|\mathsf{H}_t\mu| \ll \mathfrak{m}$. Let $K \subset X$ be a compact set such that $\mathfrak{m}(K)=0$. We can find a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathrm{C}_c(X)$, $f_k(x) = [1 - k\,\mathsf{d}(x,K)]^+$, $x \in X$, such that $\chi_K \le f_k \le \chi_H$ for $k \in \mathbb{N}$, where $H = \{x \in X : \mathsf{d}(x,K) \le 1\}$, and $f_k(x) \to \chi_K(x)$ for all $x \in X$ as $k \to \infty$. Since also $f_k \to \chi_K$ in $\mathrm{L}^2(X)$ as $k \to \infty$, we can apply the Dominated Convergence Theorem twice to infer that

$$\mathsf{H}_t^\star\mu(K) = \int_K \mathrm{d}\mathsf{H}_t^\star\mu = \lim_{k\to\infty}\int_X f_k\,\mathrm{d}\mathsf{H}_t^\star\mu = \lim_{k\to\infty}\int_X \mathsf{H}_t f_k\,\mathrm{d}\mu = \int_X \mathsf{H}_t\chi_K\,\mathrm{d}\mu = 0.$$

By inner regularity, we thus get $\mathsf{H}_t^\star\mu(A) = 0$ on any Borel set $A \subset X$ with $\mathfrak{m}(A) = 0$. □

By Theorem 3.2, if (X, d) is locally compact, then for each $x \in X$ there exists a non-negative density $\mathsf{p}_t[x] \in \mathrm{L}^1(X)$ such that

$$\mathsf{H}_t^\star\delta_x = \mathsf{h}_t[x]\,\mathfrak{m}, \quad \text{for all } t > 0. \tag{3.3}$$

Therefore, according to (3.2), if $f \in \mathrm{C}_b(X)$, then

$$\mathsf{H}_t f(x) = \int_X f\,\mathsf{h}_t[x]\,\mathrm{d}\mathfrak{m} \quad \text{for all } t > 0.$$

The following result collects the basic properties of the density $\mathsf{h}_t[\,\cdot\,]$. Its proof is very similar to that of [53, Lem. 3.24] and is thus omitted.

Corollary 3.3. *Let (X, d) be locally compact and let $t > 0$. The following hold:*
(i) $\mathsf{H}_s(\mathsf{h}_t[x]) = \mathsf{h}_{s+t}[x]$ $\mathfrak{m}$-a.e. in X, for each $x \in X$ and $s \geq 0$;
(ii) $\mathsf{h}_t[x](y) = \mathsf{h}_t[y](x)$ for $\mathfrak{m}$-a.e. $x, y \in X$.

Remark 3.4. Theorem 3.2 and Corollary 3.3 have been obtained under stronger Bakry–Émery-type properties [6, 53] in possibly not locally compact metric spaces.

3.3. W_1-L^1 **regularization.** In the following result, we provide a comparison between L^1 and W_1 distances of non-negative functions.

Theorem 3.5. *If $f_0, f_1 \in \mathrm{L}^1(X)$ with $f_0, f_1 \geq 0$, then*

$$\|\mathsf{H}_t(f_0 - f_1)\|_{\mathrm{L}^1} \leq \mathsf{c}(t)\,\mathsf{W}_1(f_0\mathfrak{m}, f_1\mathfrak{m}) \quad \textit{for all } t > 0. \tag{3.4}$$

In addition, provided that (X, d) is locally compact, if $\mu_0, \mu_1 \in \mathscr{P}_1(X)$, then

$$|\mathsf{H}_t^\star(\mu_0 - \mu_1)|(X) \leq \mathsf{c}(t)\,\mathsf{W}_1(\mu_0, \mu_1) \quad \textit{for all } t > 0, \tag{3.5}$$

and so, as a consequence,

$$\|\mathsf{h}_t[x] - \mathsf{h}_t[y]\|_{\mathrm{L}^1} \leq \mathsf{c}(t)\,\mathsf{d}(x, y) \quad \textit{for all } x, y \in X,\ t > 0. \tag{3.6}$$

Proof. Let $t > 0$ be fixed. Given $g \in \mathrm{L}^\infty(X)$, by (2.8), (2.1) and (c-Lip), we can estimate

$$\begin{aligned}\int_X g\,\mathsf{H}_t(f_0 - f_1)\,\mathrm{d}\mathfrak{m} &= \int_X \mathsf{H}_t g\,\mathrm{d}(f_0\mathfrak{m} - f_1\mathfrak{m}) \leq \mathrm{Lip}(\mathsf{H}_t g)\,\mathsf{W}_1(f_0\mathfrak{m}, f_1\mathfrak{m})\\ &\leq \mathsf{c}(t)\,\|g\|_{\mathrm{L}^\infty}\,\mathsf{W}_1(f_0\mathfrak{m}, f_1\mathfrak{m}),\end{aligned}$$

readily yielding (3.4). To prove (3.5), we argue as follows. By [57, Th. 6.18], we can find $\mu_0^k = f_0^k\mathfrak{m}$ and $\mu_1^k = f_1^k\mathfrak{m}$ in $\mathscr{P}_1(X)$, with $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty}\mathsf{W}_1(\mu_0^k, \mu_0) = \lim_{k\to\infty}\mathsf{W}_1(\mu_1^k, \mu_1) = 0. \tag{3.7}$$

By (3.4), we know that

$$\|\mathsf{H}_t(f_0^k - f_1^k)\|_{\mathrm{L}^1} \leq \mathsf{c}(t)\,\mathsf{W}_1(\mu_0^k, \mu_1^k) \quad \text{for all } k \in \mathbb{N}.$$

Given $g \in \mathrm{C}_b(X)$ with $\|g\|_{\mathrm{L}^\infty} \le 1$, by (2.8) and (3.2) we can estimate

$$\|\mathsf{H}_t(f_0^k - f_1^k)\|_{\mathrm{L}^1} \ge \int_X g\,\mathsf{H}_t(f_0^k - f_1^k)\,\mathrm{d}\mathfrak{m} = \int_X \mathsf{H}_t g\,\mathrm{d}(\mu_0^k - \mu_1^k),$$

so that

$$\int_X \mathsf{H}_t g\,\mathrm{d}(\mu_0^k - \mu_1^k) \le \mathsf{c}(t)\,\mathsf{W}_1(\mu_0^k, \mu_1^k) \quad \text{for all } k \in \mathbb{N}$$

whenever $g \in \mathrm{C}_b(X)$ with $\|g\|_{\mathrm{L}^\infty} \le 1$. Thanks to (c-Lip), $\mathsf{H}_t g \in \mathrm{C}_b(X)$. Thus, recalling [57, Th. 6.9], we can exploit (3.7) to pass to the limit as $k \to \infty$ and get that

$$\int_X \mathsf{H}_t g\,\mathrm{d}(\mu_0 - \mu_1) \le \mathsf{c}(t)\,\mathsf{W}_1(\mu_0, \mu_1)$$

whenever $g \in \mathrm{C}_b(X)$ with $\|g\|_{\mathrm{L}^\infty} \le 1$. Recalling the definition in (3.2), we get that

$$\int_X g\,\mathrm{d}\mathsf{H}_t^\star(\mu_0 - \mu_1) \le \mathsf{c}(t)\,\mathsf{W}_1(\mu_0, \mu_1)$$

whenever $g \in \mathrm{C}_b(X)$ with $\|g\|_{\mathrm{L}^\infty} \le 1$, readily yielding (3.5). The validity of (3.6) hence easily follows by recalling the definition of $\mathsf{h}_t[\,\cdot\,]$ in (3.3) and applying (3.5) to $\mu_0 = \delta_x$ and $\mu_1 = \delta_y$, $x, y \in X$, completing the proof. □

Remark 3.6. In $\mathsf{RCD}(K,\infty)$ spaces, Theorem 3.5 has been proved in [5, Cor. 6.6]. Inequality (3.5) can be equivalently rephrased as follows. If $\mu_0, \mu_1 \in \mathscr{P}_1(X)$, then $\mathsf{He}_1(\mathsf{H}_t^\star\mu_0, \mathsf{H}_t^\star\mu_1) \le \mathsf{c}(t)\,\mathsf{W}_1(\mu_0, \mu_1)$ for all $t > 0$, where He_1 denotes the 1-*Matusita–Hellinger distance*, see [44, Th. 5.2.] and [28] and the references therein.

3.4. **Quantitative L^2 contraction estimate.** From now on, we assume that

$$\mathsf{c} \in \mathrm{L}^1_{\mathrm{loc}}([0, +\infty)) \tag{3.8}$$

and we define $\mathsf{C}\colon [0,\infty) \to [0,\infty)$ by letting

$$\mathsf{C}(t) = \int_0^t \mathsf{c}(s)\,\mathrm{d}s \quad \text{for all } t \ge 0. \tag{3.9}$$

We warn the reader that (3.8) is not restrictive and holds in the settings considered in Sections 4 and 5.

The following result, generalizing [10, Th. 4.1], provides a quantification of the L^2 contraction property (2.3) of the heat semigroup on sufficiently smooth functions.

Theorem 3.7. *If $f \in \mathrm{W}^{1,2}(X) \cap \mathrm{L}^\infty(X)$ is such that $|\mathsf{D}f|_w \in \mathrm{L}^1(X)$, then*

$$\|f\|_{\mathrm{L}^2}^2 - \|\mathsf{H}_{t/2} f\|_{\mathrm{L}^2}^2 \le \mathsf{C}(t)\,\|f\|_{\mathrm{L}^\infty} \||\mathsf{D}f|_w\|_{\mathrm{L}^1} \quad \textit{for all } t \ge 0.$$

Proof. Taking $g = f$ in Lemma 2.1 and using (c-Lip), we can estimate

$$\begin{aligned}\int_X f(f - \mathsf{H}_t f)\,\mathrm{d}\mathfrak{m} &= \int_0^t \int_X \mathsf{D}f \cdot \mathsf{D}\mathsf{H}_s f\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \le \int_0^t |\mathsf{D}f|_w \,\mathrm{Lip}(\mathsf{H}_s f)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\ &\le \int_0^t |\mathsf{D}f|_w\,\mathsf{c}(s)\,\|f\|_{\mathrm{L}^\infty}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \mathsf{C}(t)\,\|f\|_{\mathrm{L}^\infty} \int_X |\mathsf{D}f|_w\,\mathrm{d}\mathfrak{m}\end{aligned}$$

and the conclusion follows by the symmetry and the semigroup property of $(\mathsf{H}_t)_{t\ge0}$. □

As [10, Th. 4.1], Theorem 3.7 can be refined provided that the heat semigroup $(\mathsf{H}_t)_{t\ge0}$ is ϑ-*ultracontractive*, i.e., for some Borel function $\vartheta\colon(0,\infty)\to(0,\infty)$, it holds that

$$\|\mathsf{H}_t f\|_{\mathrm{L}^\infty}\le\vartheta(t)\,\|f\|_{\mathrm{L}^1}\quad\text{for all } t>0. \tag{3.10}$$

Precisely, we get the following interpolation inequality for bounded BV functions.

Corollary 3.8. *Under* (3.10)*, if* $f\in\mathrm{BV}(X)\cap\mathrm{L}^\infty(X)$*, then*

$$\|f\|^2_{\mathrm{L}^2}\le\inf_{t>0}\left(\vartheta(\tfrac{t}{2})\,\|f\|^2_{\mathrm{L}^1}+\mathsf{C}(t)\,\|f\|_{\mathrm{L}^\infty}\mathsf{Var}(f)\right).$$

Proof. We begin by observing that, by (2.3) and (3.10),

$$\|\mathsf{H}_{t/2}f\|^2_{\mathrm{L}^2}\le\|\mathsf{H}_{t/2}f\|_{\mathrm{L}^1}\,\|\mathsf{H}_{t/2}f\|_{\mathrm{L}^\infty}\le\vartheta(\tfrac{t}{2})\,\|f\|^2_{\mathrm{L}^1}.$$

Owing to Theorem 3.7, we hence plainly get that

$$\|f\|^2_{\mathrm{L}^2}\le\vartheta(\tfrac{t}{2})\,\|f\|^2_{\mathrm{L}^1}+\mathsf{C}(t)\,\|f\|_{\mathrm{L}^\infty}\||\mathsf{D}f|\|_{\mathrm{L}^1}\quad\text{for all } t>0, \tag{3.11}$$

whenever $f\in\mathrm{Lip}_{bs}(X)$. In view of (2.19), we can find $(f_k)_{k\in\mathbb{N}}\subset\mathrm{Lip}_{bs}(X)$ such that $f_k\to f$ in $\mathrm{L}^1(X)$ as $k\to\infty$ and

$$\mathsf{Var}(f)=\lim_{k\to\infty}\int_X|\mathsf{D}f_k|\,\mathrm{d}\mathfrak{m}.$$

Up to a truncation, we can also assume that $\|f_k\|_{\mathrm{L}^\infty}\le\|f\|_{\mathrm{L}^\infty}$ for $k\in\mathbb{N}$. The conclusion hence follows by applying (3.11) to each f_k and then passing to the limit as $k\to\infty$. ☐

Remark 3.9. The ultracontractivity property (3.10) is available in a wide range of settings, such as Markov spaces supporting a Sobolev inequality [9, Sect. 6.3], hence $\mathsf{RCD}(K,N)$ spaces with $N<\infty$ [37, Rem. 5.17] and sub-Riemannian manifolds [35]. For $\mathsf{RCD}(K,\infty)$ spaces with a uniform lower bound on the measure of balls, see [30, Prop. 2.4].

3.5. **Caloric-type Poincaré inequality and compactness.** The following result gives a *caloric-type Poincaré inequality* for BV functions.

Theorem 3.10. *If* $f\in\mathrm{BV}(X)$*, then*

$$\|f-\mathsf{H}_t f\|_{\mathrm{L}^1}\le\mathsf{C}(t)\,\mathsf{Var}(f)\quad\textit{for all } t\ge0. \tag{3.12}$$

Proof. We can find $f_k\in\mathrm{Lip}_{bs}(X)$ such that $f_k\to f$ in $\mathrm{L}^1(X)$ as $k\to\infty$ and

$$\mathsf{Var}(f)(X)=\lim_{k\to+\infty}\int_X|\mathsf{D}f_k|\,\mathrm{d}\mathfrak{m}. \tag{3.13}$$

In particular, $f_k\in\mathrm{W}^{1,2}(X)$ with $|\mathsf{D}f_k|_w\le|\mathsf{D}f|$ $\mathfrak{m}$-a.e. in X. Now, given $g\in\mathrm{W}^{1,2}\cap\mathrm{L}^\infty(X)$, by Lemma 2.1 we can estimate

$$\int_X g\,(f_k-\mathsf{H}_t f_k)\,\mathrm{d}\mathfrak{m}=\int_0^t\int_X\mathsf{D}f_k\cdot\mathsf{D}\mathsf{H}_s g\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\le\||\mathsf{D}f_k|\|_{\mathrm{L}^1}\int_0^t|\mathsf{D}\mathsf{H}_s g|_w\,\mathrm{d}s.$$

Since $\mathsf{H}_s g\in\mathrm{Lip}_b(X)$ with $|\mathsf{D}\mathsf{H}_s g|_w\le|\mathsf{D}\mathsf{H}_s g|\le\mathsf{c}(s)\,\|g\|_{\mathrm{L}^\infty}$ for all $s\in(0,t)$ thanks to (c-Lip), we can write

$$\int_X g\,(f_k-\mathsf{H}_t f_k)\,\mathrm{d}\mathfrak{m}\le\||\mathsf{D}f_k|\|_{\mathrm{L}^1}\,\|g\|_{\mathrm{L}^\infty}\int_0^t\mathsf{c}(s)\,\mathrm{d}s=\mathsf{C}(t)\,\|g\|_{\mathrm{L}^\infty}\,\||\mathsf{D}f_k|\|_{\mathrm{L}^1}$$

whenever $g \in \mathrm{W}^{1,2} \cap \mathrm{L}^\infty(X)$. Now, given $g \in \mathrm{L}^\infty(X)$, by a plain approximation argument exploiting [53, Lem. 3.2], we can find $g_j \in \mathrm{W}^{1,2} \cap \mathrm{L}^\infty(X)$ such that $g_n \overset{*}{\rightharpoonup} g$ in $\mathrm{L}^\infty(X)$. Consequently, we get that

$$\int_X g\,(f_k - \mathsf{H}_t f_k)\,\mathrm{d}\mathfrak{m} \le \mathsf{C}(t)\,\|g\|_{\mathrm{L}^\infty}\,\||\mathsf{D} f_k|\|_{\mathrm{L}^1}$$

whenever $g \in \mathrm{L}^\infty(X)$. The conclusion hence readily follows by (3.13). □

As a consequence of Theorem 3.10, we can prove the following compactness result for uniformly bounded BV funcitons.

Corollary 3.11 (Compactness)**.** *Let (X, d) be a proper metric space. If $(f_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(X)$ is such that*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{\mathrm{L}^\infty} + \mathsf{Var}(f_k) < \infty$$

then there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ such that $f_{k_j} \to f$ in $\mathrm{L}^1_{\mathrm{loc}}(X)$.

Proof. Define $f_{k,n} = \mathsf{H}_{\frac{1}{n}} f_k$ for $k, n \in \mathbb{N}$ and note that, in virtue of (c-Lip), $f_{k,n} \in \mathrm{Lip}_b(X)$ with $\|f_{k,n}\|_{\mathrm{L}^\infty} \le M$ and $\mathrm{Lip}(f_{k,n}) \le \mathsf{c}\left(\frac{1}{n}\right) M$, where $M = \sup_{k\in\mathbb{N}} \|f_k\|_{\mathrm{L}^\infty} < \infty$. In particular, for each $n \in \mathbb{N}$ fixed, the sequence $(f_{k,n})_{k\in\mathbb{N}} \subset \mathrm{Lip}_b(X)$ is equi-bounded and equi-Lipschitz. By Arzelà–Ascoli's Theorem, we can thus find a sequence $(k_j)_{j\in\mathbb{N}}$ such that $(f_{k_j,n})_{j\in\mathbb{N}}$ is uniformly convergent on any bounded $U \subset X$. Consequently, we can exploit Theorem 3.10 to estimate

$$\begin{aligned}\limsup_{i,j\to\infty} \int_U |f_{k_i} - f_{k_j}|\,\mathrm{d}\mathfrak{m} &\le \limsup_{i,j\to\infty} \int_U |f_{k_i,n} - f_{k_j,n}|\,\mathrm{d}\mathfrak{m} \\ &\quad + \limsup_{i,j\to\infty} \int_U |f_{k_i} - f_{k_i,n}| + |f_{k_j} - f_{k_j,n}|\,\mathrm{d}\mathfrak{m} \\ &\le 2\,\mathsf{C}\left(\tfrac{1}{n}\right) \sup_{k\in\mathbb{N}} \mathsf{Var}(f_k)\end{aligned}$$

for any bounded $U \subset X$. Since $n \in \mathbb{N}$ is arbitrary and $L^1(U)$ is a Banach space, this proves that $(f_{k_i})_{j\in\mathbb{N}}$ converges in $L^1(U)$ for any bounded $U \subset X$. Up to extracting a further subsequence (which we do not relabel for simplicity), we can find $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ such that $f_{k_i} \to f$ in $L^1_{\mathrm{loc}}(X)$, yielding the conclusion. □

By combining Theorems 3.5 and 3.10, we get the following interpolation estimate for the L^1 norm of a BV function.

Corollary 3.12. *If $f \in \mathrm{BV}(X)$, then*

$$\|f\|_{\mathrm{L}^1} \le \mathsf{c}(t)\,\mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) + \mathsf{C}(t)\,\mathsf{Var}(f) \quad \text{for all } t > 0. \tag{3.14}$$

Proof. By Theorems 3.5 and 3.10, we can estimate

$$\begin{aligned}\mathsf{c}(t)\,\mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) &\ge \|\mathsf{H}_t(f^+ - f^-)\|_{\mathrm{L}^1} \ge \|f\|_{\mathrm{L}^1} - \|f - \mathsf{H}_t f\|_{\mathrm{L}^1} \\ &\ge \|f\|_{\mathrm{L}^1} - \mathsf{C}(t)\,\mathsf{Var}(f)\end{aligned}$$

readily yielding the conclusion. □

3.6. **Implicit indeterminacy estimate.** The next result provides an implicit indeterminacy estimate, which, in few words, quantifies the relation between the Wasserstein distance of positive and negative parts of an $\mathrm{L}^1 \cap \mathrm{L}^\infty$ function and the size of its zero set.

Theorem 3.13. *If $\mathfrak{m}(X) < \infty$ and $f \in \mathrm{L}^\infty(X, \mathfrak{m})$, then*

$$\|f\|_{\mathrm{L}^1} \le \mathsf{c}(t)\, \mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) + 2\sqrt{\mathsf{C}(t)\, \|f\|_{\mathrm{L}^\infty}\, \|f\|_{\mathrm{L}^1}\, \mathsf{Per}(\{f > 0\})} \quad \text{for all } t \ge 0. \tag{3.15}$$

To prove Theorem 3.13, we need the following preliminary result.

Lemma 3.14. *If $A \subset X$ is an $\mathfrak{m}$-measurable set with $\mathfrak{m}(A) < \infty$, then*

$$\int_{A^c} \mathsf{H}_t(\chi_A)\, \mathrm{d}\mathfrak{m} \le \frac{1}{2}\, \mathsf{C}(t)\, \mathsf{Per}(A) \quad \text{for all } t \ge 0. \tag{3.16}$$

Moreover, if $\mathfrak{m}(X) < \infty$ and $f \in \mathrm{L}^\infty(X)$, then

$$\int_X \sqrt{\mathsf{H}_t(f^+)\, \mathsf{H}_t(f^-)}\, \mathrm{d}\mathfrak{m} \le \sqrt{\mathsf{C}(t)\, \|f\|_{\mathrm{L}^\infty}\, \|f\|_{\mathrm{L}^1}\, \mathsf{Per}(\{f > 0\})} \quad \text{for all } t \ge 0. \tag{3.17}$$

Proof. Since $\chi_A \in \mathrm{BV}(X)$, from Theorem 3.10 we immediately get

$$\begin{aligned} C(t)\, \mathsf{Per}(A) \ge \|\chi_A - \mathsf{H}_t(\chi_A)\|_{\mathrm{L}^1} &= \int_A (1 - \mathsf{H}_t(\chi_A))\, \mathrm{d}\mathfrak{m} + \int_{A^c} \mathsf{H}_t(\chi_A)\, \mathrm{d}\mathfrak{m} \\ &= \int_X (1 - \mathsf{H}_t(\chi_A))\, \mathrm{d}\mathfrak{m} - \int_{A^c} 1\, \mathrm{d}\mathfrak{m} + 2\int_{A^c} \mathsf{H}_t(\chi_A)\, \mathrm{d}\mathfrak{m} = 2\int_{A^c} \mathsf{H}_t(\chi_A)\, \mathrm{d}\mathfrak{m}, \end{aligned}$$

yielding (3.16). Concerning (3.17), since $f^- \le \|f^-\|_{\mathrm{L}^\infty} \chi_{\{f \le 0\}}$, by (2.4), the Cauchy–Schwarz inequality, the mass-preservation property (2.6) and the previous (3.16), we get

$$\begin{aligned} \left(\int_{\{f>0\}} \sqrt{\mathsf{H}_t(f^+)\, \mathsf{H}_t(f^-)}\, \mathrm{d}\mathfrak{m} \right)^2 &\le \|f^-\|_{\mathrm{L}^\infty} \left(\int_{\{f>0\}} \sqrt{\mathsf{H}_t(f^+)\, \mathsf{H}_t(\chi_{\{f \le 0\}})}\, \mathrm{d}\mathfrak{m} \right)^2 \\ &\le \|f^-\|_{\mathrm{L}^\infty}\, \|H_t(f^+)\|_{\mathrm{L}^1} \int_{\{f>0\}} \mathsf{H}_t(\chi_{\{f \le 0\}})\, \mathrm{d}\mathfrak{m} \\ &\le \frac{1}{2}\, \mathsf{C}(t)\, \|f^-\|_{\mathrm{L}^\infty}\, \|f^+\|_{\mathrm{L}^1}\, \mathsf{Per}(\{f > 0\}). \end{aligned}$$

Similarly, we can also estimate

$$\left(\int_{\{f \le 0\}} \sqrt{\mathsf{H}_t(f^+)\, \mathsf{H}_t(f^-)}\, \mathrm{d}\mathfrak{m} \right)^2 \le \frac{1}{2}\, \mathsf{C}(t)\, \|f^+\|_{\mathrm{L}^\infty}\, \|f^-\|_{\mathrm{L}^1}\, \mathsf{Per}(\{f \le 0\}),$$

and the conclusion readily follows by observing that $\mathsf{Per}(\{f > 0\}) = \mathsf{Per}(\{f \le 0\})$, $\|f^\pm\|_{\mathrm{L}^\infty} \le \|f\|_{\mathrm{L}^\infty}$ and $\|f^+\|_{\mathrm{L}^1} + \|f^-\|_{\mathrm{L}^1} = \|f\|_{\mathrm{L}^1}$. □

We can now give the proof of Theorem 3.13.

Proof of Theorem 3.13. By (3.4), we have

$$\|\mathsf{H}_t(f^+ - f^-)\|_{\mathrm{L}^1} \le \mathsf{c}(t)\, \mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}). \tag{3.18}$$

Since $|a - b| \ge a + b - 2\sqrt{ab}$ whenever $a, b \ge 0$, from (3.17) we get

$$\begin{aligned} \|\mathsf{H}_t(f^+ - f^-)\|_{\mathrm{L}^1} &\ge \int_X \mathsf{H}_t(f^+) + \mathsf{H}_t(f^-) - 2\sqrt{\mathsf{H}_t(f^+)\mathsf{H}_t(f^-)}\, \mathrm{d}\mathfrak{m} \\ &\ge \|f\|_{\mathrm{L}^1} - 2\sqrt{\mathsf{C}(t)\, \|f\|_{\mathrm{L}^\infty}\, \|f\|_{\mathrm{L}^1}\, \mathsf{Per}(\{f > 0\})} \end{aligned} \tag{3.19}$$

and the conclusion follows by combining (3.18) and (3.19). □

3.7. **Implicit estimates for eigenfunctions.** Theorems 3.5 and 3.13 can be exploited to obtain implicit lower bounds on the (perimeter of the) nodal set and an indeterminacy-type inequality for eigenfunctions.

Theorem 3.15. *If f_λ is a λ-eigenfunction, then*

$$\mathsf{Per}(\{f_\lambda > 0\})\,\|f_\lambda\|_{\mathrm{L}^\infty} \ge \frac{(1-e^{-\lambda t})^2}{4\,\mathsf{C}(t)}\,\|f_\lambda\|_{\mathrm{L}^1} \quad \text{for all } t>0 \tag{3.20}$$

and

$$\mathsf{W}_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \ge \frac{e^{-\lambda t}}{\mathsf{c}(t)}\,\|f_\lambda\|_{\mathrm{L}^1} \quad \text{for all } t>0. \tag{3.21}$$

Proof. The proof of (3.20) is the same of (3.15), since one just need to replace (3.18) with

$$\|\mathsf{H}_t(f_\lambda^+ - f_\lambda^-)\|_{\mathrm{L}^1} = \|\mathsf{H}_t f_\lambda\|_{\mathrm{L}^1} = e^{-\lambda t}\,\|f_\lambda\|_{\mathrm{L}^1} \tag{3.22}$$

by Lemma 2.2. Inequality (3.21) is again a consequence of Lemma 2.2, together with Theorem 3.5. □

One can get rid of the L^∞ norm in the lower bound (3.20) as soon as the heat semigroup $(\mathsf{H}_t)_{t\ge 0}$ is ϑ-ultracontractive as in (3.10). Precisely, we have the following result.

Corollary 3.16. *Under (3.10), if f_λ is a λ-eigenfunction, then*

$$\mathsf{Per}(\{f_\lambda > 0\}) \ge \sup_{t>0} \frac{e^{-\lambda t}(1-e^{-\lambda t})^2}{4\,\vartheta(t)\,\mathsf{C}(t)}. \tag{3.23}$$

Proof. Thanks to Lemma 2.2 and (3.10), we can estimate

$$\|f_\lambda\|_{\mathrm{L}^\infty} = e^{\lambda t}\|\mathsf{H}_t f_\lambda\|_{\mathrm{L}^\infty} \le e^{\lambda t}\,\vartheta(t)\,\|f_\lambda\|_{\mathrm{L}^1} \quad \text{for all } t>0,$$

which, combined with (3.20), easily yields (3.23). □

3.8. **Implicit Buser inequality.** We conclude this section with the following result, yielding an implicit Buser inequality for the Cheeger constants $h_0(X)$ and $h_1(X)$.

Theorem 3.17. *The following hold:*

(i) if $\mathfrak{m}(X) < \infty$, then $h_1(X) \ge \sup_{t>0} \left\{ \frac{1-e^{-\lambda_1 t}}{\mathsf{C}(t)} \right\}$;

(ii) if $\mathfrak{m}(X) = \infty$, then $h_0(X) \ge 2 \sup_{t>0} \left\{ \frac{1-e^{-\lambda_0 t}}{\mathsf{C}(t)} \right\}$.

Proof. We start by observing that, by Theorem 3.10, we have

$$\begin{aligned}\mathsf{C}(t)\,\mathsf{Per}(A) \ge \|\chi_A - \mathsf{H}_t(\chi_A)\|_{\mathrm{L}^1} &= \int_A (1-\mathsf{H}_t(\chi_A))\,\mathrm{d}\mathfrak{m} + \int_{A^c} \mathsf{H}_t(\chi_A)\,\mathrm{d}\mathfrak{m} \\ &= 2\,\mathfrak{m}(A) - 2\int_A \mathsf{H}_t(\chi_A)\,\mathrm{d}\mathfrak{m} = 2\,\mathfrak{m}(A) - 2\left\|\mathsf{H}_{t/2}(\chi_A)\right\|_{\mathrm{L}^2}^2\end{aligned} \tag{3.24}$$

for any $\mathfrak{m}$-measurable set $A \subset X$, thanks to (2.4), (2.6), (2.11) and the semigroup property. We prove the two statements separately.

Proof of (i). Assume $\mathfrak{m}(X) = 1$ without loss of generality. Since $\mathsf{H}_t(1) = \mathfrak{m}(X) = 1$ because of (2.6), we immediately get that

$$\int_X \mathsf{H}_{t/2}(\chi_A - \mathfrak{m}(A))\,\mathrm{d}\mathfrak{m} = 0.$$

We can hence apply Lemma 2.3(i) to get

$$\left\|\mathsf{H}_{t/2}(\chi_A)\right\|_2^2 = \mathfrak{m}(A)^2 + \left\|\mathsf{H}_{t/2}(\chi_A - \mathfrak{m}(A))\right\|_2^2 \le \mathfrak{m}(A)^2 + e^{-\lambda_1 t}\left\|\chi_A - \mathfrak{m}(A)\right\|_2^2. \tag{3.25}$$

By direct computation, we can write

$$\|\chi_A - \mathfrak{m}(A)\|_2^2 = \mathfrak{m}(A)\,(1 - \mathfrak{m}(A)),$$

so that, by combining (3.24) with (3.25), we get that

$$\mathsf{C}(t)\,\mathsf{Per}(A) \ge 2\,\mathfrak{m}(A)\,(1 - \mathfrak{m}(A))\left(1 - e^{-\lambda_1 t}\right) \quad \text{for every } t > 0. \tag{3.26}$$

The conclusion hence follows by recalling the definition in (2.21).

Proof of (ii). We can bound the last term in the chain (3.24) using Lemma 2.3(ii). The conclusion hence immediately follows by the definition in (2.20). □

4. Quantitative Lipschitz smoothing with controls and explicit bounds

4.1. **Quantitative Lipschitz smoothing with controls.** In the following, we give a power-logarithmic upper bound on the Lipschitz constant $\mathsf{c}(t)$ in Definition 3.1.

Definition 4.1 (L^∞-to-Lip with controls)**.** We say that c is *controlled by the triplet* $(M, a, b) \in [0,\infty)^2 \times (0,1)$ if

$$\mathsf{c}(t) \le M\,\frac{(1 + |\log(t)|)^a}{t^b} \quad \text{for all } t \in (0,1]. \tag{4.1}$$

Consequently, we say that $(\mathsf{H}_t)_{t\ge0}$ is c-Lip *with controls* (M, a, b) if $(\mathsf{H}_t)_{t\ge0}$ is c-Lip as in Definition 3.1 and c is controlled by the triplet (M, a, b) as in (4.1).

Some comments are now in order. In most of the settings of interest, the bound in (4.1) holds with $a = 0$ and $b = \frac{1}{2}$, see the discussion in Section 5. This is, for instance, the case of $\mathsf{RCD}(K,\infty)$ spaces, in which the constant $M > 0$ may depend on $K \in \mathbb{R}$. Moreover, the bound (4.1) should be understood in an *operative sense*, meaning that it allows us to obtain the inequalities in a manageable explicit form. In most of the cases, this analysis is enough, but in some specific situations—such as the Buser inequality in $\mathsf{RCD}(K,\infty)$ spaces with $K > 0$ [29]—the *exact* form of the function $\mathsf{c}(t)$ allows to recover *sharp* results (i.e., inequalities which are equalities in some non-trivial cases).

The following result collects some elementary estimates following from Definition 4.1.

Lemma 4.2. *Let c be controlled by the triplet (M, a, b). For every $\varepsilon > 0$ there exists $T = T(\varepsilon, a) \in (0,1) > 0$, depending on ε and a only, such that*

$$\mathsf{c}(t) \le \frac{M}{t^{b+\varepsilon}} \quad \textit{for all } t \in (0,T]. \tag{4.2}$$

Consequently, the primitive function C is well defined and, setting $\widetilde{M} = \frac{M}{1-b-\varepsilon}$, it satisfies

$$\mathsf{C}(t) \le \frac{\widetilde{M}}{t^{b+\varepsilon-1}} \quad \textit{for all } t \in (0,T] \textit{ and } \varepsilon < 1 - b. \tag{4.3}$$

Proof. It follows by letting $T \in (0,1)$ be the smallest solution of $(1+|\log(T)|)^a = T^{-\varepsilon}$. □

In the rest of this section we assume that $\{\mathsf{H}_t\}_{t\geq 0}$ is c-Lip with controls (M,a,b). This kind of control is motivated by the examples, as we further comment in Section 5.

4.2. **Explicit indeterminacy estimate.** We begin with the following explicit version of the indeterminacy estimate in Theorem 3.13.

Theorem 4.3. *If $\mathfrak{m}(X) < \infty$ and $h_1(X) > 0$, then, for every $\varepsilon \in (0, 1-b)$, there exists a constant $C = C(M,a,b,\varepsilon,h_1(X)) > 0$ such that*

$$\mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq C\left(\frac{\|f\|_{\mathrm{L}^1}}{\|f\|_{\mathrm{L}^\infty}\mathsf{Per}(\{f>0\})}\right)^{\frac{b+\varepsilon}{1-b-\varepsilon}}\|f\|_{\mathrm{L}^1} \tag{4.4}$$

for every $f \in \mathrm{L}^\infty(X)$ satisfying $\int_X f\,\mathrm{d}\mathfrak{m} = 0$. In addition, if $a = 0$, then there exists $C = C(M,b,h_1(X)) > 0$ such that (4.4) holds with $\varepsilon = 0$.

Proof. We exploit (3.15) in combination with the bounds in Lemma 4.2 and the choice

$$t = \vartheta\, T\left(\frac{2\,\|f\|_{\mathrm{L}^\infty}\mathsf{Per}(\{f>0\})}{h_1(X)\,\|f\|_{\mathrm{L}^1}}\right)^{\frac{1}{b+\varepsilon-1}} \tag{4.5}$$

where $\vartheta \in (0,1]$ has to be chosen later. Note that $t \in (0,T)$ follows from (2.22). Hence

$$\mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq \frac{t^{b+\varepsilon}\|f\|_{\mathrm{L}^1}}{M}\left(1-(\vartheta T)^{\frac{1-b-\varepsilon}{2}}\sqrt{2\widetilde{M}h_1(X)}\right)$$

and the conclusion follows from the definition in (4.5) by choosing ϑ sufficiently small. The proof in the case $a = 0$ is simpler and is thus omitted. □

Remark 4.4. Non-optimal indeterminacy estimates were considered in [20, 55]. In the class of closed Riemannian manifolds, the exponent $\frac{b+\varepsilon}{1-b-\varepsilon}$ in (4.4) can be replaced by 1, and no smaller exponent is possible. In this form, the inequality was proved in the more general setting of essentially non-branching $\mathsf{CD}(K,N)$ spaces with $N < \infty$ in [21] and in $\mathsf{RCD}(K,\infty)$ spaces in [28]. Indeterminacy estimates with optimal exponent 1 and best possible multiplicative constant $C > 0$ were recently achieved in [31] for spaces with simple geometry.

4.3. **Explicit estimates for eigenfunctions.** We now provide an explicit version of the bounds given in Theorem 3.15.

We begin with the following explicit version of the first part of Theorem 3.15.

Theorem 4.5. *For every $\varepsilon \in (0,1-b)$, there exist $\lambda_0 = \lambda_0(a,\epsilon) > 0$ and a constant $C = C(M,a,b,\varepsilon) > 0$ such that*

$$\mathsf{Per}(\{f_\lambda > 0\}) \geq C\lambda^{1-b-\varepsilon}\frac{\|f_\lambda\|_{\mathrm{L}^1(X)}}{\|f_\lambda\|_{\mathrm{L}^\infty(X)}} \tag{4.6}$$

for every f_λ λ-eigenfunction with $\lambda \geq \lambda_0$. In addition, if $a = 0$, then there exist $\lambda_0 > 0$ and $C = C(M,b,\lambda_0) > 0$ such that (4.6) holds with $\varepsilon = 0$.

Proof. We choose $T \in (0,1)$ as in (4.3) and exploit (3.20) for the admissible choice $t = \frac{T\lambda_0}{\lambda}$. In combination with (4.3), we thus obtain

$$e^{-T\lambda_0} - 1 + 2\frac{\widetilde{M}^{\frac{1}{2}}\lambda^{\frac{b+\varepsilon-1}{2}}}{T^{\frac{b+\varepsilon-1}{2}}\lambda_0^{\frac{b+\varepsilon-1}{2}}}\,\mathsf{Per}(\{f_\lambda > 0\})^{\frac{1}{2}}\left(\frac{\|f_\lambda\|_{\mathrm{L}^\infty}}{\|f_\lambda\|_{\mathrm{L}^1}}\right)^{\frac{1}{2}} \geq 0 \tag{4.7}$$

and (4.6) follows by rearranging. The proof for $a = 0$ is simpler and thus omitted. □

Remark 4.6. On an N-dimensional closed Riemannian manifold, inequality (4.6) can be coupled with the sharp bound $\|f_\lambda\|_{\mathrm{L}^\infty} \leq C\lambda^{\frac{N-1}{4}}\|f_\lambda\|_{\mathrm{L}^1}$, e.g. see [52], (here and below, $C > 0$ is a constant independent of λ which may vary from line to line) to recover the lower bound

$$\mathsf{Per}(\{f_\lambda > 0\}) \geq C\,\lambda^{\frac{3-N}{4}}$$

obtained in [23, 52, 54]. Noteworthy, our approach to establish the lower bound on the nodal set is different from the ones employed in [23, 52, 54] and, up to our knowledge, is new. The sharp lower bound

$$\mathsf{Per}(\{f_\lambda > 0\}) \geq C\sqrt{\lambda}$$

conjectured by Yau has been proved in [43]. On compact $\mathsf{RCD}(K,N)$ spaces, one can instead exploit in (4.6) the bound $\|f_\lambda\|_{\mathrm{L}^\infty} \leq C\lambda^{\frac{N}{4}}\|f_\lambda\|_{\mathrm{L}^1}$ obtained in [7, Prop. 7.1], to achieve

$$\mathsf{Per}(\{f_\lambda > 0\}) \geq C\,\lambda^{\frac{1-N}{4}}, \tag{4.8}$$

which improves the previously best-known estimate given in [21, Th. 1.5]. Theorem 4.5, as well as Corollary 3.16, provides a lower bound on the size of nodal sets in several sub-Riemannian structures, also see the recent work [33] for a related discussion.

We can now pass to the following explicit version of the second part of Theorem 3.15.

Theorem 4.7. *For every $\varepsilon \in (0, 1-b)$, there exist $\lambda_0 = \lambda_0(a,\varepsilon) > 0$ and a constant $C = C(M,a,b,\varepsilon) > 0$ such that*

$$\mathsf{W}_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{C}{\lambda^{b+\varepsilon}}\|f_\lambda\|_{\mathrm{L}^1(X)} \tag{4.9}$$

for every f_λ λ-eigenfunction with $\lambda \geq \lambda_0$. In addition, if $a = 0$, then there exist $\lambda_0 > 0$ and $C = C(M,b,\lambda_0) > 0$ such that (4.9) holds with $\varepsilon = 0$.

Proof. We choose $T \in (0,1)$ as in (4.3) and exploit (3.21) for the admissible choice $t = \frac{T\lambda_0}{\lambda}$. In combination with (4.2), we thus obtain

$$\mathsf{W}_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{e^{-T\widetilde{\lambda}}\,T^{b+\varepsilon}\,\widetilde{\lambda}^{b+\varepsilon}}{M\lambda^{b+\varepsilon}}\|f_\lambda\|_{\mathrm{L}^1(X)} = \frac{\widetilde{C}}{\lambda^{b+\varepsilon}}\|f_\lambda\|_{\mathrm{L}^1(X)}$$

yielding (4.9). The proof for $a = 0$ is simpler and thus omitted. □

Remark 4.8. In [55] it was conjectured that, on any closed Riemannian manifolds, there exist some constants $C_2 \geq C_1 > 0$ such that

$$\frac{C_2}{\sqrt{\lambda}}\|f_\lambda\|_{\mathrm{L}^1(X)} \geq \mathsf{W}_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{C_1}{\sqrt{\lambda}}\|f_\lambda\|_{\mathrm{L}^1(X)}. \tag{4.10}$$

The left-hand side of (4.10) was confirmed in [20], while the right-hand side was established in [28] in the more general context of $\mathsf{RCD}(K,\infty)$ spaces (also see [48] for an alternative proof of the right-hand side of (4.10) for closed Riemannian manifolds). Theorem 4.7 yields the right-hand side of (4.10) any time (4.1) holds with $a = 0$ and $b = \frac{1}{2}$.

4.4. **Explicit Buser inequality.** We now pass to the explicit version of the Buser inequalities provided in Theorem 3.17.

Theorem 4.9. *For every $\varepsilon \in (0, 1-b)$, there exist constants $C_{1,i} = C_{1,i}(M, a, b, \varepsilon) > 0$ and $C_{2,i} = C_{2,i}(M, b, \varepsilon) > 0$, $i = 0, 1$, such that the following hold:*

(i) if $\mathfrak{m}(X) < \infty$, then $\lambda_1 \le \max\left\{C_{1,1} h_1(X), C_{2,1} h_1(X)^{\frac{1}{1-b-\varepsilon}}\right\}$;

(ii) if $\mathfrak{m}(X) = \infty$, then $\lambda_0 \le \max\left\{C_{1,0} h_0(X), C_{2,0} h_0(X)^{\frac{1}{1-b-\varepsilon}}\right\}$.

In addition, if $a = 0$, then (i) and (ii) hold for $\varepsilon = 0$ for some $C_{1,i} = C_{1,i}(M, b) > 0$.

Proof. We just prove (i), the other case (ii) being analogous. Since $\mathfrak{m}(X) < \infty$, we apply Theorem 3.17(i). If $T\lambda_1 \ge 1$, then we choose $t = 1/\lambda_1$ so that, recalling (4.3), we find

$$h_1(X) \ge \frac{1-e^{-1}}{\widetilde{M}} \lambda_1^{1-b-\varepsilon}. \tag{4.11}$$

If $T\lambda_1 < 1$ instead, then we simply choose $t = T$ and get

$$h_1(X) \ge \frac{1-e^{-\lambda_1 T}}{\widetilde{M}} T^{\varepsilon+b-1} = \frac{T^{\varepsilon+b}}{\widetilde{M}} \frac{1-e^{-\lambda_1 T}}{\lambda_1 T} \lambda_1 \ge \frac{T^{\varepsilon+b}}{\widetilde{M}} (1-e^{-1}) \lambda_1 \tag{4.12}$$

since $r \mapsto \frac{1-e^{-r}}{r}$ is decreasing for $r \in (0,1]$. The conclusion thus follows by rearranging and combining (4.11) with (4.12). If $a = 0$, then one directly choose $t = 1/\lambda_1$ if $\lambda_1 \ge 1$ and $t = 1$ otherwise, with no need to appeal to (4.3). The simple details are omitted. □

Remark 4.10. Upper bounds on the first eigenvalue in terms of the Cheeger constant of the space were firstly proved in [19] in the setting of closed Riemannian manifolds. An alternative proof based on heat semigroup techniques was given in [41], and subsequently improved in [42] to a dimension-free estimate. The strategy of [41, 42] was later refined in [29], yielding sharp estimates in $\mathsf{RCD}(K, \infty)$ spaces, with equality cases discussed in [30]. It is worth noticing that the lower bound $4\lambda_1 \ge h_1^2(X)$ for $\mathfrak{m}(X) < \infty$ (respectively, $4\lambda_0 \ge h_0^2(X)$ for $\mathfrak{m}(X) = \infty$) on the first eigenvalue in terms of the Cheeger constant was noticed independently by Maz'ja and Cheeger [22, 45], and it is known to hold on any metric measure space [29, Th. 4.2] and even in more general settings [34, Sect. 6.1]. We also refer to [15, Sect. 3.4] for a lower bound in the sub-Riemannian context.

4.5. **Explicit interpolation estimate.** We conclude this section with the following explicit version of Corollary 3.12. We need to reinforce Definition 4.1 with a stronger control on c, that is, we require a power-type upper bound for *all* times.

Theorem 4.11. *If $(\mathsf{H}_t)_{t \ge 0}$ is c-Lip with c such that*

$$c(t) \le \frac{M}{t^b} \quad \text{for all } t > 0 \tag{4.13}$$

for some $(M, b) \in (0, \infty) \times (0, 1)$, then there exists $C = C(M, b) > 0$ such that

$$\|f\|_{\mathrm{L}^1} \le C\, \mathrm{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m})^{1-b} \,\mathsf{Var}(f)^b \tag{4.14}$$

for every $f \in \mathrm{BV}(X)$.

Proof. From (4.13), we immediately get that

$$\mathsf{C}(t) \le \frac{\widetilde{M}}{t^{b-1}} \quad \text{for all } t>0, \tag{4.15}$$

for some $\widetilde{M} = \widetilde{M}(M,b) > 0$. Combining (4.13) and (4.15) with (3.14), we get that

$$\|f\|_{\mathrm{L}^1} \le \frac{M}{t^b}\,\mathsf{W}_1(f^+\mathfrak{m}, f^-\mathfrak{m}) + \frac{\widetilde{M}}{t^{b-1}}\,\mathsf{Var}(f) \quad \text{for all } t>0.$$

The inequality (4.14) hence follows by choosing $t = \frac{\mathsf{W}_1(f^+\mathfrak{m},f^-\mathfrak{m})}{\mathsf{Var}(f)}$. □

Remark 4.12. In the context of smooth weighted Riemannian manifolds with non-negative weighted Ricci curvature, inequality (4.14) is essentially contained in [17, 18].

5. Examples

In this last section, we provide a brief overview of the settings where our results apply.

5.1. **Weak Bakry–Émery condition.** Let $(X,\mathsf{d},\mathfrak{m})$ be a metric-measure space satisfying the properties (P1), (P2) and (P3) listed in Section 2.11. Following [53, Def. 3.4], $(X,\mathsf{d},\mathfrak{m})$ satisfies the *weak Bakry–Émery condition* with respect to some Borel function $\kappa\colon[0,\infty)\to(0,\infty)$ such that $\kappa,\kappa^{-1}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,\infty))$, $\mathsf{BE}_w(\kappa,\infty)$ for short, if

$$|\mathsf{DH}_t f|_w^2 \le \kappa^2(t)\,\mathsf{H}_t(|\mathsf{D}f|_w^2) \quad \mathfrak{m}\text{-a.e. in } X \tag{5.1}$$

for all $f\in \mathrm{W}^{1,2}(X)$ and $t>0$ (where $\kappa(0)=1$ for simplicity). Adding the *Sobolev-to-Lipschitz property*, i.e.,

(P4) if $f\in\mathrm{W}^{1,2}(X)$ is such that $|\mathsf{D}f|_w\le 1$, then f admits a continuous representative $\widetilde{f}$ such that $\widetilde{f}\in\mathrm{Lip}(X)$ with $\mathrm{Lip}(\widetilde{f})\le 1$,

to the assumptions in Section 2.11, and by combining [53, Cor. 3.21] with a plain approximation argument, we easily infer that $(\mathsf{H}_t)_{t\ge0}$ satisfies (c-Lip) with

$$\mathsf{c}(t) \le \left(2\int_0^t \kappa^{-2}(s)\,\mathrm{d}s\right)^{-2} \quad \text{for all } t\ge 0. \tag{5.2}$$

According to [53, Cor. 3.7], if (5.1) is met by some Borel function κ such that

$$\limsup_{t\to0^+}\kappa(t)<\infty, \tag{5.3}$$

then the optimal function $\kappa_\star$ satisfying (5.1) is such that

$$\kappa_\star(t)\le Me^{-Kt} \quad \text{for all } t\ge 0 \tag{5.4}$$

for some $M\ge1$ and $K\in\mathbb{R}$. Therefore, assuming (5.3), we can plug (5.4) in (5.2) and get (c-Lip) with $\mathsf{c}(t)\le M\sqrt{j_K(t)}$ for all $t>0$, where

$$j_K(t) = \begin{cases} \dfrac{K}{e^{2Kt}-1} & \text{for } K\ne 0\\[2ex] \dfrac{1}{2t} & \text{for } K=0. \end{cases}$$

In particular, the bound (4.1), as well as the stronger (4.13), are both satisfied with $b=\frac12$.

5.2. **Synthetic constant lower curvature bounds.** The class of $\mathsf{RCD}(K,\infty)$ spaces, $K \in \mathbb{R}$, meet (5.1) with $\kappa(t) = e^{-Kt}$ for $t \geq 0$ and thus are a particular instance of spaces satisfying a weak Bakry–Émery condition [5]. The validity of (c-Lip) in $\mathsf{RCD}(K,\infty)$ spaces with $\mathsf{c}(t) = \sqrt{j_K(t)}$ for $t > 0$ has been established in [5, Th. 6.5], and subsequently improved to $\mathsf{c}(t) = \sqrt{\frac{2}{\pi}\, j_K(t)}$ for $t > 0$ in [29, Prop. 3.1], which is sharp for $t \to 0^+$ as a consequence of the results in [30]. Our results hence encode the ones in [28, 29].

5.3. **Variable lower curvature bounds.** In [16], the authors studies the consequences of the variable lower bound $\mathrm{Ric}_g(x)(v,v) \geq k(x)\,|v|^2$, for every $x \in M$ and $v \in T_xM$, on a smooth, geodesically complete, non-compact and connected Riemannian manifold (M,g) without boundary, where $k\colon M \to [0,\infty)$ is a continuous function. Under a suitable integrability assumption on the negative part k^- of the function k (precisely, see [16, Eq. (1.1)], as well as the definition of the *Kato class* in [16, Def. 1.2]), in [16, Th. 1.1(iii)] they establish (c-Lip) with $\mathsf{c}(t) = \sqrt{8}\, t^{-1/2}\, \alpha_{k^-}(t)$ for $t > 0$, where $\alpha_{k^-} \geq 1$ is a function depending on the integrability condition imposed on k^-.

5.4. **Sub-Riemannian manifolds.** Sub-Riemannian manifolds (endowed with a smooth volume form) are infinitesimally Hilbertian spaces that do not satisfy the $\mathsf{CD}(K,\infty)$ property for any $K \in \mathbb{R}$ [50]. Nevertheless, numerous sub-Riemannian manifolds do enjoy (c-Lip): non-abelian *Carnot groups* [40, 46] and the Grushin plane [59], both with $\mathsf{c}(t) = C\sqrt{j_0(t)}$ for $t > 0$ with $C > 1$, and the $\mathbb{SU}(2)$ *group* [11] with $\mathsf{c}(t) = C\sqrt{j_K(t)}$ for $t > 0$ with $C > 1$ and $K > 0$. Noteworthy, (c-Lip) has also been proved in [12, Cor. 3.3] and [27, Cor. 4.11] under suitable generalized CD-type conditions [13, 47].

5.5. **Other settings.** We believe that our approach may be naturally adapted to several other frameworks. Here we only mention that L^∞-to-Lip contraction inequalities analogous to (c-Lip) have been established relatively to *metric graphs* [14], *diamond fractals* [2], the *rearranged stochastic heat equation* [25], and the *Dyson Brownian motion* [56]. Noteworthy, in some of this frameworks [2, 14, 25], the function c has a (possibly, non-sharp) power-logarithmic-type behavior as in Definition 4.1.

Remark 5.1. The extension of our analysis to *extended* metric-measure spaces requires some caution. We only mention that, in this more general framework, there exist bounded Lipschitz functions (with respect to the *extended* distance) which are not even measurable, see [26, Exam. 3.4]. We thank Lorenzo Dello Schiavo for pointing this issue to us.

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(N. De Ponti) Università di Milano-Bicocca, Dipartimento di Matematica e Applicazioni, via Roberto Cozzi 55, 20125 Milano (MI), Italy

Email address: nicolo.deponti@unimib.it

(G. Stefani) Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, 34136 Trieste (TS), Italy

Email address: gstefani@sissa.it or giorgio.stefani.math@gmail.com