OPTIMAL HOLES FOR THE $W^{1,\infty}$ POINCARÉ INEQUALITY

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Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^N$, we study the sharp constant $\lambda(A)$ of the classical Poincaré inequality for functions in $W^{1,\infty}(\Omega)$ that vanish in a hole $A \subset \Omega$. Then, we prove existence of an optimal hole $A^\star$ that minimizes the Poincaré constant $\lambda(A)$, under some penalization on the volume of the hole A. Moreover, we give a geometrical characterization of this optimal hole $A^\star$. In addition, we will consider the same shape optimization problem but in the case where the penalization is given by the Hausdorff measure of a rectifiable curve A.

On the other hand, we will also study the best constant $\lambda(A)$ of the Poincaré-Wirtinger inequality for functions $u \in W^{1,\infty}(\Omega)$ such that $\int_A u = 0$. And, we will also consider a more general version of this Poincaré inequality where characteristic function of A in the constraint $\int_A u = 0$ is replaced by a probability measure ν (i.e., $\int_\Omega u \,\mathrm{d}\nu = 0$). Moreover, we will show existence of an optimal hole $A^\star$ that optimizes $\lambda(A)$, among all subsets $A \subset \Omega$ of prescribed volume, and we will also characterize it. If the penalization on A is part of the functional, then we discuss the cases where an optimal hole $A^\star$ exists and others where it does not.

Finally, we will prove existence, uniqueness and regularity of the optimal density $\nu^\star$ that minimizes the Poincaré constant $\overline{\lambda}(\nu)$ plus some penalization $F(\nu)$, among all probability measures ν over Ω.

1. Introduction

Poincaré inequalities have been studied by many authors and there is actually an extensive literature on this subject which we do not attempt to summarise here. For an overview, see the articles [9, 12, 13, 14, 16, 17, 1, 7] and the references therein where the main purpose was to find the smallest constant Λ (or at least to show some bounds on Λ) in the Poincaré inequality.

Let Ω be a compact domain in $\mathbb{R}^N$ and A be a subset of Ω. Then, we will consider the following Poincaré inequality:

$$||u||_\infty \leq \Lambda ||\nabla u||_\infty, \tag{1.1}$$

where $u \in W^{1,\infty}(\Omega)$ with $u = 0$ on A. It is clear that the sharp constant Λ in the inequality (1.1) is given by the following formula:

$$\Lambda = \sup\left\{\frac{||u||_\infty}{||\nabla u||_\infty} \,:\, u \neq 0 \in W^{1,\infty}(\Omega),\ u = 0 \text{ on } A\right\}. \tag{1.2}$$

We note that some similar problems to (1.2) have been already studied in the literature due to many applications in several branches of applied mathematics (see [11, 6]). First, our aim will be to know the dependence of the Poincaré constant $\Lambda := \Lambda(A)$ on the geometry of A. In [5], the authors consider the case when $A = \partial\Omega$ and, they prove by studying the limit of the first eigenvalue of the $p-$Laplacian as $p \to \infty$, that $\Lambda(\partial\Omega) = \max\{d(x, \partial\Omega) : x \in \Omega\}$. Moreover, they find at the same time that Problem (1.2) is nothing else than the dual of an optimal transport problem to the boundary (see also [8]). In addition, the authors of [3] showed (using a different method than the one in [5]) a formula for the best Poincaré constant Λ that extends the one in [5] to the case of a general bounded set $A \subset \Omega$. More precisely, one can prove that

$$\Lambda(A) = \max\{d(x, A) : x \in \Omega\}$$

while $u(x) := d(x, A)$ maximizes (1.2). In Section 2, we will introduce an alternative proof for this result that we consider much simpler than those in [3, 5] and which will be based on a simple duality trick.

On the other hand, an interesting problem that we consider in the present paper will be to find an optimal hole $A^\star$ that gives rise to an optimal Poincaré constant $\Lambda^\star := \Lambda(A^\star)$ among a class of admissible sets $A \subset \Omega$. More precisely, we will consider the following shape optimization problem:

$$\min\left\{\lambda(A) - \alpha|A| \,:\, A \subset \Omega\right\} \tag{1.3}$$

where $\lambda(A) := [\Lambda(A)]^{-1}$ and $\alpha > 0$ is fixed. In [3], the authors have already studied the easiest version of Problem (1.3), where the Lebesgue measure of the hole A was assumed to be bounded from below by a constant $m > 0$:

$$\min\left\{\lambda(A) \,:\, A \subset \Omega,\ |A| \geq m\right\}. \tag{1.4}$$

The main goal of [3] was to show existence of an optimal hole $A^\star$ for Problem (1.4) and, to characterize it. To be more precise, they prove that $A^\star$ is the complement of a ball centered at some point on $\partial\Omega$. However, the existence of a solution for Problem (1.4) does not imply that Problem (1.3) has a solution as well. Yet, we will also show in Section 3 that an optimal hole $A^\star$ for Problem (1.3) always exists. We note that our proof of existence here is simpler than the one in [3]. Moreover, we will also consider the case when the penalization on the hole A is given by its $\mathcal{H}^1$−Hausdorff measure:

$$\min\left\{\Lambda(A) + \alpha\,\mathcal{H}^1(A) \,:\, A \subset \Omega\right\}. \tag{1.5}$$

For a subset $A \subset \Omega$, another possibility will be to consider instead of (1.1) the following Poincaré inequality:

$$||u||_\infty \leq \Lambda||\nabla u||_\infty, \tag{1.6}$$

among all functions $u \in W^{1,\infty}(\Omega)$ such that $\int_A u = 0$. In this case, the sharp Poincaré constant $\Lambda := \Lambda(A)$ is given by the following:

$$\Lambda = \sup\left\{\frac{||u||_\infty}{||\nabla u||_\infty} \,:\, u \neq 0 \in W^{1,\infty}(\Omega),\ \int_A u = 0\right\}. \tag{1.7}$$

In [2], the authors consider the Poincaré inequality (1.6) but in the case when $A = \Omega$. In particular, they prove that $\Lambda(\Omega) = \max\{\fint_\Omega d(x, z) \,:\, x \in \Omega\}$. Moreover, they show in [2, Proposition 3.1], the following upper and lower bounds on $\Lambda(\Omega)$:

$$\frac{1}{2}\operatorname{diam}(\Omega) \leq \Lambda(\Omega) \leq \frac{N}{N+1}\operatorname{diam}(\Omega). \tag{1.8}$$

In Section 2, we will extend the result of [2, Theorem 1.1] to an arbitrary set $A \subset \Omega$. More precisely, using a duality argument, we will show that

$$\Lambda(A) = \max\left\{ \fint_A d(x,z) : x \in \Omega \right\}.$$

We note that our proof also seems simpler than that of [2]. In addition, we will prove the following upper/lower bounds:

$$R^\star \leq \Lambda(A) \leq \operatorname{diam}(\Omega), \text{ for all } A \subset \Omega,$$

where $R^\star$ is the radius of the smallest ball containing Ω. In fact, one can also show that these two bounds are sharp.

Moreover, we will also consider the shape optimization problem (1.3) (or (1.4)) but when $\Lambda(A)$ is given by (1.7) instead of (1.2). The existence of an optimal hole $A^\star$ in this case is much delicate due to the lack of semicontinuity of the functional with respect to the weak$\star-$convergence in $L^\infty(\Omega)$. To be more precise, we will show in Section 4 the existence of a minimizer for Problem (1.4), for all $m > 0$.

However, we will see that there is a critical value $\alpha_\star$ which depends on the geometry of the domain Ω such that the solution of Problem (1.3) exists as soon as $\alpha > \alpha_\star$. In particular, an optimal hole $A^\star$ for Problem (1.3) does not exist if $\alpha < \alpha_\star$. In addition, we will characterize the optimal hole $A^\star$ by showing that it will be again the complement of a ball centered at some point $z^\star \in \partial\Omega$. In order to prove existence of such an optimal hole $A^\star$, the idea will be to replace the characteristic functions χ_A having $|A| \geq m$ with probability measures having bounded densities $0 \leq \nu \leq m^{-1}$, i.e. we consider the following relaxation of (1.3):

$$\min\left\{ \overline{\lambda}(\nu) \,:\, 0 \leq \nu \leq m^{-1},\ \int_\Omega \nu = 1 \right\} \tag{1.9}$$

where $\overline{\lambda}(\nu) := [\overline{\Lambda}(\nu)]^{-1}$ and $\overline{\Lambda}(\nu)$ is the following sharp Poincaré constant for functions u in $W^{1,\infty}(\Omega)$ with $\int_\Omega u \,\mathrm{d}\nu = 0$:

$$\overline{\Lambda}(\nu) = \sup\left\{ \frac{||u||_\infty}{||\nabla u||_\infty} \,:\, u \neq 0 \in W^{1,\infty}(\Omega),\ \int_\Omega u \,\mathrm{d}\nu = 0 \right\}.$$

First, we will show that Problem (1.9) has a minimizer $\nu^\star$. Then, we will prove that $\nu^\star$ is a characteristic function of some set $A^\star \subset \Omega$, which turns out to be an optimal hole for Problem (1.4).

On the other hand, we will also consider the following shape optimization problem (where $\Lambda(A)$ is always given by (1.7)):

$$\min\left\{ \Lambda(A) \,:\, A \subset \Omega,\ |A| \geq m \right\}. \tag{1.10}$$

The existence of an optimal hole $A^\star$ for this problem (1.10) seems to be much complicated without adding any extra constraint on A. Yet, assuming convexity of the admissible sets A in (1.10) will help us in proving existence of such an optimal hole $A^\star$. However, it will be difficult to characterize optimal holes in this case. But, we will be able to show some symmetry property on $A^\star$ provided that the domain Ω is symmetric.

In Section 5, we will consider the problem of minimizing the Poincaré constant $\overline{\lambda}(\nu)$ among all densities ν with $\int \nu = 1$, but in the case where we have an additional penality term on ν:

$$\min\left\{\overline{\lambda}(\nu)+\int_\Omega f(\nu)\;:\;\int_\Omega \nu=1\right\} \tag{1.11}$$

where $f:\mathbb{R}^+\mapsto\mathbb{R}^+$ is an appropriate given function. In fact, the existence of a minimizer $\nu^\star$ for this problem is not difficult thanks to some results which were already proved in [4] on the lower semicontinuity of the penality term that we are using here. Moreover, we will show under some assumptions on f that this minimizer $\nu^\star$ is unique and smooth.

2. Sharp Poincaré inequality via Optimal Transport

Given a compact set $\Omega\subset\mathbb{R}^N$ with Lipschitz boundary, we denote by $d(x,y)$ the geodesic distance in Ω between the points x and y, that is defined as the infimum of the lengths of rectifiable curves in Ω that join x and y:

$$d(x,y):=\inf\left\{\int_0^1|\alpha'(t)|\,\mathrm{d}t\;:\;\alpha\in\mathrm{Lip}([0,1],\Omega),\;\alpha(0)=x\;\text{ and }\;\alpha(1)=y\right\}.$$

2.1. **The classical Poincaré inequality.** Let A be a subset of Ω. The Poincaré inequality for functions in $W^{1,\infty}(\Omega)$ and vanishing on A reads as follows: there is a constant $\Lambda<\infty$ such that for every $u\in W^{1,\infty}(\Omega)$ with $u=0$ on A, we have

$$||u||_\infty\le\Lambda||\nabla u||_\infty. \tag{2.1}$$

Now, our aim will be to characterize the sharp constant $\Lambda(A)$ in this Poincaré inequality (2.1). Set

$$\lambda(A):=\min\left\{\frac{||\nabla u||_\infty}{||u||_\infty}\;:\;u\neq 0\in W^{1,\infty}(\Omega),\;u=0\;\text{ on }\;A\right\}. \tag{2.2}$$

It is clear that $\Lambda(A)=[\lambda(A)]^{-1}$. Moreover, we have the following:

Theorem 2.1. *The function $u_A(z):=d(z,A)$ minimizes (2.2) and, the following equality holds:*

$$\lambda(A)=[\max\{d(z,A)\;:\;z\in\Omega\}]^{-1}.$$

Proof. For every $u\in W^{1,\infty}(\Omega)$ with $u=0$ on A, we define its normalized gradient function $\tilde{u}:=\frac{u}{||\nabla u||_\infty}$. So, we have

$$||\nabla\tilde{u}||_\infty=1,\;\;||\tilde{u}||_\infty=\frac{||u||_\infty}{||\nabla u||_\infty}\quad\text{and}\quad\tilde{u}=0\;\text{ on }\;A.$$

Hence,

$$\begin{aligned}\lambda(A)&:=\min\left\{{||\tilde{u}||_\infty}^{-1}\;:\;\tilde{u}\in W^{1,\infty}(\Omega),\;||\nabla\tilde{u}||_\infty=1,\;\tilde{u}=0\;\text{ on }\;A\right\}\\&=\left[\sup\left\{||u||_\infty\;:\;u\in W^{1,\infty}(\Omega),\;||\nabla u||_\infty=1,\;u=0\;\text{ on }\;A\right\}\right]^{-1}\end{aligned} \tag{2.3}$$

$$= \left[\sup \left\{ \sup \left\{ \int_{\Omega\setminus A} u \,\mathrm{d}\delta_z \ : \ u \in W^{1,\infty}(\Omega\setminus A),\ ||\nabla u||_\infty = 1,\ u = 0 \text{ on } \partial A \right\} \ : \ z \in \Omega\setminus A \right\} \right]^{-1}.$$

Let us compute

$$\sup \left\{ \int_{\Omega\setminus A} u \,\mathrm{d}\delta_z \ : \ u \in W^{1,\infty}(\Omega\setminus A),\ ||\nabla u||_\infty = 1,\ u = 0 \text{ on } \partial A \right\}. \tag{2.4}$$

Thanks to [8, Section 2.2], one can see that this problem (2.4) is the dual of the optimal transport problem of the mass δ_z to the Dirichlet region ∂A:

$$\min \left\{ \int_{\Omega\setminus A\times\partial A} d(x,y)\,\mathrm{d}\gamma \ : \ (\Pi_x)_\#\gamma = \delta_z \text{ and } \mathrm{spt}[(\Pi_y)_\#\gamma] \subset \partial A \right\}. \tag{2.5}$$

To see that, let u be an admissible function in (2.4) and γ be a transport plan in (2.5). Then, we have

$$\int_{\Omega\setminus A} u \,\mathrm{d}\delta_z = \int_{\Omega\setminus A\times\partial A} [u(x) - u(y)]\,\mathrm{d}\gamma(x,y) \le \int_{\Omega\setminus A\times\partial A} d(x,y)\,\mathrm{d}\gamma.$$

Hence,

$$\sup(2.4) \le \min(2.5).$$

Now, set $u_A(x) := d(x, A)$ (we note that $u_A = 0$ on A and u_A is $1-$Lipschitz with respect to the geodesic distance and so, one has $||\nabla u_A||_\infty = 1$) and $\gamma_z := \delta_z \otimes \delta_{p(z)}$, where $p(z)$ is any projection point of z onto ∂A, i.e.

$$p(z) \in \mathrm{argmin}\{d(z,y) \ : \ y \in \partial A\}.$$

We have

$$\int_{\Omega\setminus A} u_A \,\mathrm{d}\delta_z = \int_{\Omega\setminus A\times\partial A} d(x,y)\,\mathrm{d}\gamma_z = d(z, A).$$

Hence, u_A is a maximizer in (2.4) while γ_z is an optimal transport plan in (2.5). In particular, we get that

$$\sup(2.4) = \min(2.5) = d(z, A).$$

Recalling (2.3), this implies that

$$\lambda(A) = [\max\{d(z,A) \ : \ z \in \Omega\}]^{-1}.$$

Finally, we note also that $\lambda(A) = \frac{||\nabla u_A||_\infty}{||u_A||_\infty}$. Hence, this means that u_A minimizes Problem (2.2). □

Remark 2.1. *For any set $A \subset \Omega$, it is easy to see that we have $\lambda(\overline{A}) = \lambda(A)$, where $\overline{A}$ denotes the closure of A. Therefore, in Section 3, we will always assume that A is a closed set.*

2.2. **The Poincaré-Wirtinger inequality.** Let $A \subset \Omega$ be such that $|A| > 0$ (where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$). The Poincaré inequality for functions in $W^{1,\infty}(\Omega)$ having zero mean value on A can be stated as follows: there is a constant $\Lambda < \infty$ such that for every $u \in W^{1,\infty}(\Omega)$ with $\int_A u = 0$, we have

$$||u||_\infty \le \Lambda||\nabla u||_\infty. \tag{2.6}$$

Again, we will try to find the best constant $\Lambda(A)$ in this inequality (2.6). For this aim, we define

$$\lambda(A) = [\Lambda(A)]^{-1} = \min\left\{ \frac{||\nabla u||_\infty}{||u||_\infty} \ : \ u \neq 0 \in W^{1,\infty}(\Omega), \ \int_A u = 0 \right\}. \tag{2.7}$$

Moreover, we may consider a more general version of (2.7) by considering a probability measure ν instead of the uniform probability mesure over A. More precisely, take $\nu \in \mathcal{P}(\Omega)$. Then, we know that there is a constant $\overline{\Lambda} < \infty$ such that for every $u \in W^{1,\infty}(\Omega)$ with $\int_\Omega u \,\mathrm{d}\nu = 0$, we have

$$||u||_\infty \leq \overline{\Lambda}\, ||\nabla u||_\infty. \tag{2.8}$$

In particular, we see that the optimal constant $\overline{\Lambda}(\nu)$ in the Poincaré inequality (2.8) is given by the following:

$$\overline{\lambda}(\nu) = [\overline{\Lambda}(\nu)]^{-1} = \min\left\{ \frac{||\nabla u||_\infty}{||u||_\infty} \ : \ u \neq 0 \in W^{1,\infty}(\Omega), \ \int_\Omega u \,\mathrm{d}\nu = 0 \right\}. \tag{2.9}$$

It is clear that

$$\lambda(A) = \overline{\lambda}(\nu_A)$$

where

$$\nu_A := \frac{\chi_A}{|A|}.$$

Theorem 2.2. *Let ν be a probability measure over Ω. Then, there exists a point $x_\nu \in \Omega$ such that $u_\nu(z) := d(z, x_\nu) - \int_\Omega d(y, x_\nu)\, \nu(y)$ minimizes (2.9). Moreover, we have the following equality:*

$$\overline{\lambda}(\nu) = \left[\max\left\{ \int_\Omega d(x,z)\, \nu(z) \ : \ x \in \Omega \right\} \right]^{-1} = \left[\int_\Omega d(x_\nu, z)\, \nu(z) \right]^{-1}.$$

Proof. First, we see that

$$\overline{\lambda}(\nu) = \min\left\{ \min\left\{ \frac{||\nabla u||_\infty}{|u(x)|} \ : \ u \in W^{1,\infty}(\Omega), \ \int_\Omega u \,\mathrm{d}\nu = 0 \right\} \ : \ x \in \Omega \right\}.$$

Using the normalization $\tilde{u} := \frac{u}{||\nabla u||_\infty}$, we get that

$$\begin{aligned}
\overline{\lambda}(\nu) &= \min\left\{ \left[\sup\left\{ |\tilde{u}(x)| \ : \ \tilde{u} \in W^{1,\infty}(\Omega), \ ||\nabla \tilde{u}||_\infty = 1, \ \int_\Omega \tilde{u}\, \nu = 0 \right\} \right]^{-1} \ : \ x \in \Omega \right\} \\
&= \min\left\{ \left[\sup\left\{ \left| u(x) - \int_\Omega u \,\mathrm{d}\nu \right| \ : \ u \in W^{1,\infty}(\Omega), \ ||\nabla u||_\infty = 1 \right\} \right]^{-1} \ : \ x \in \Omega \right\} \\
&= \min\left\{ \left[\sup\left\{ \int_\Omega u \,\mathrm{d}\nu - \int_\Omega u \,\mathrm{d}\delta_x \ : \ u \in W^{1,\infty}(\Omega), \ ||\nabla u||_\infty = 1 \right\} \right]^{-1} \ : \ x \in \Omega \right\}.
\end{aligned}$$

Now, we compute

$$\sup\left\{ \int_\Omega u \,\mathrm{d}\nu - \int_\Omega u \,\mathrm{d}\delta_x \ : \ u \in W^{1,\infty}(\Omega), \ ||\nabla u||_\infty = 1 \right\}. \tag{2.10}$$

We note that (2.10) is nothing else than the dual of the classical Monge-Kantorovich problem between the source measure ν and the target measure δ_x:

$$\min\left\{\int_{\Omega\times\Omega} d(z,y)\,\mathrm{d}\gamma \,:\, (\Pi_z)_{\#}\gamma=\nu \;\text{ and }\; (\Pi_y)_{\#}\gamma=\delta_x\right\}. \tag{2.11}$$

If u is an admissible function in (2.10) and γ is a transport plan in (2.11) then we have the following:

$$\int_\Omega u\,\mathrm{d}\nu-\int_\Omega u\,\mathrm{d}\delta_x=\int_{\Omega\times\Omega}[u(z)-u(y)]\,\mathrm{d}\gamma(z,y)\le\int_{\Omega\times\Omega} d(z,y)\,\mathrm{d}\gamma.$$

Then, one has

$$\sup\,(2.10)\le\min\,(2.11).$$

Set $u_\nu(z):=d(z,x)-\int_\Omega d(y,x)\,\nu(y)$ (it is clear that $\int_\Omega u_\nu\,\mathrm{d}\nu=0$ and $||\nabla u_\nu||_\infty=1$) and $\gamma_x:=\nu\otimes\delta_x$. We have

$$\int_\Omega u_\nu\,\mathrm{d}\nu-\int_\Omega u_\nu\,\mathrm{d}\delta_x=\int_\Omega d(z,x)\,\nu(z)=\int_{\Omega\times\Omega} d(z,y)\,\mathrm{d}\gamma_x.$$

This implies that u_ν maximizes (2.10) and γ_x minimizes (2.11). As a consequence of that, we have

$$\sup\,(2.10)=\min\,(2.11)=\int_\Omega d(z,x)\,\nu(z).$$

Hence,

$$\overline{\lambda}(\nu)=\left[\max\left\{\int_\Omega d(x,z)\,\nu(z)\;:\;x\in\Omega\right\}\right]^{-1}. \quad\square$$

Then, we get the following:

Corollary 2.3. *Let $A\subset\Omega$ be such that $|A|>0$. Then, there is a point $x^\star\in\Omega$ such that $u_A(z):=d(z,x^\star)-\frac{1}{|A|}\int_A d(y,x^\star)$ minimizes (2.7). In addition, the following equality holds:*

$$\lambda(A)=\left[\max\left\{\frac{1}{|A|}\int_A d(z,x)\;:\;x\in\Omega\right\}\right]^{-1}=\left[\frac{1}{|A|}\int_A d(z,x^\star)\right]^{-1}.$$

Proof. Set $\nu_A:=\frac{\chi_A}{|A|}$. Then, we recall that $\lambda(A)=\overline{\lambda}(\nu_A)$. Thanks to Theorem 2.2, we infer that

$$\lambda(A)=\left[\max\left\{\int_\Omega d(x,z)\,\nu_A(z)\;:\;x\in\Omega\right\}\right]^{-1}. \quad\square$$

We close this section by the following:

Remark 2.2. *Notice that $\lambda(A)$ is not sensitive by adding any negligible set (with respect to the Lebesgue measure) to A. On the other hand, one can see that in general $\lambda(\overline{A})\neq\lambda(A)$ and so, we can not assume that A is always a closed set, unless ∂A is negligible.*

Remark 2.3. *Since the map $x\mapsto\int_\Omega d(x,z)\,\nu(z)$ is strictly convex, then its maximum is attained on $\partial\Omega$ and so, we have*

$$\overline{\lambda}(\nu)=\left[\max\left\{\int_\Omega d(x,z)\,\nu(z)\;:\;x\in\partial\Omega\right\}\right]^{-1}.$$

3. The classical Poincaré inequality: existence and characterization of optimal holes

In this section, our aim will be to find a region $A \subset \Omega$ that optimizes the sharp Poincaré constant $\lambda(A)$ (see (2.2)). Fix $\alpha > 0$, then we consider the following shape optimization problem:

$$\min\left\{\lambda(A) - \alpha|A| \,:\, A \subset \Omega\right\} \tag{3.1}$$

where we recall that $\lambda(A) = [\max\{d(x, A) : x \in \Omega\}]^{-1}$. Notice that Problem (3.1) has a trivial solution $A^\star$ if $\alpha \leq 0$. Let $x^\star \in \partial\Omega$ be such that there is a point $z^\star \in \partial\Omega$ with $d(z^\star, x^\star) = \mathrm{diam}(\Omega)$ (here, $\mathrm{diam}(\Omega)$ is with respect to the geodesic distance). So, it is sufficient to take $A^\star = \{x^\star\}$. Indeed, one has $\lambda(A^\star) = [\mathrm{diam}(\Omega)]^{-1}$ while it is easy to see that $\lambda(A) \geq [\mathrm{diam}(\Omega)]^{-1}$, for any set $A \subset \Omega$. Coming back to the case when $\alpha > 0$, there is no trivial solution since when the set A increases then both terms $\lambda(A)$ and $|A|$ increase.

On the other hand, we recall that the following problem (which is sometime much easier to study than (3.1) as we will see in Section 4)

$$\min\left\{\lambda(A) \,:\, A \subset \Omega,\ |A| = m\right\} \tag{3.2}$$

has been already studied in [10] where the authors proved existence of a solution $A^\star$. More precisely, they show that every optimal hole in (3.2) is a set of the form $A^\star = \Omega \backslash B(z^\star, R^\star)$, for some point $z^\star \in \partial\Omega$. Here, we will introduce an alternative proof of this result that we consider simpler than the one in [10].

Proposition 3.1. *There exists an optimal hole $A^\star$ for Problem (3.1). In addition, there is a point $z^\star \in \partial\Omega$ such that, up to a negligible set, $A^\star = \Omega \backslash B(z^\star, R^\star)$ for some $R^\star > 0$.*

Proof. Let $\{A_n\}_n$ be a minimizing sequence. Recalling Remark 2.1, one can assume that A_n is closed, for all $n \in \mathbb{N}$. On the other hand, by Theorem 2.1, we know that $u_n(x) := d(x, A_n)$ is a minimizer for Problem (2.2), for all $n \in \mathbb{N}$. Up to a subsequence, we see that $A_n \to A^\star$ in the Hausdorff sense and so, $u_n \to u^\star := d(x, A^\star)$ uniformly in Ω. So, we get that

$$\lambda(A^\star) = ||u^\star||_\infty{}^{-1} = \lim_n ||u_n||_\infty^{-1} = \lim_n \lambda(A_n).$$

On the other hand, the volume (Lebesgue measure) is upper semicontinuous with respect to the Hausdorff convergence. Hence, we have

$$\limsup_n |A_n| \leq |A^\star|.$$

Consequently, we infer that

$$\lambda(A^\star) - \alpha|A^\star| \leq \liminf_n\, [\lambda(A_n) - \alpha|A_n|].$$

Hence, $A^\star$ minimizes Problem (3.1). Now, let $z^\star \in \Omega \backslash A^\star$ be such that $\lambda(A^\star) = [d(z^\star, \partial A^\star)]^{-1}$. Set $R^\star = d(z^\star, \partial A^\star)$ and $A := \Omega \backslash B(z^\star, R^\star)$. Then, it is clear that $A^\star \subset A$ while by definition of A, we have $\lambda(A) = [R^\star]^{-1} = \lambda(A^\star)$. Assume that $|A \backslash A^\star| > 0$. Then, we get that

$$\lambda(A^\star) - \alpha|A^\star| \leq \lambda(A) - \alpha|A| = \lambda(A^\star) - \alpha|A^\star| - \alpha|A \backslash A^\star|,$$

which is a contradiction. Hence, $|A\backslash A^\star| = 0$ and so, $A^\star = \Omega\backslash B(z^\star, R^\star)$. In addition, thanks to the fact that $\lambda(A) - \alpha|A|$ is invariant by translation, then it is not difficult to see that the point $z^\star$ must be located on $\partial\Omega$. This concludes the proof. $\square$

Remark 3.1. *Fix a constant $0 < m < |\Omega|$. Then, we consider the following problem:*

$$\min\left\{\lambda(A) \,:\, A \subset \Omega,\ |A| \geq m\right\}. \tag{3.3}$$

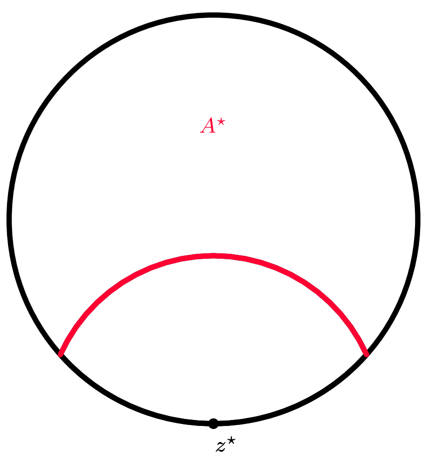
Similarly to Proposition 3.1, one can show that Problem (3.3) has an optimal hole $A^\star$. In addition, we have $|A^\star| = m$ since if this is not the case (i.e. $|A^\star| > m$) then one can reduce the set $A^\star$ by removing an $\varepsilon-$neighborhood of $\partial A^\star$ so that we get a subset $A \subset\subset A^\star$ such that $|A| \geq m$ (this is possible if $\varepsilon > 0$ is small enough). But, it is clear that $\lambda(A) < \lambda(A^\star)$, which is in contradiction with the optimality of $A^\star$. In particular, we see that this optimal hole $A^\star$ minimizes (3.2). In addition, there will be some $\alpha = \alpha(m) > 0$ such that this set $A^\star$ also minimizes Problem (3.1). Thanks to Proposition 3.1, this yields that $A^\star := \Omega\backslash B(z^\star, R^\star)$, for some point $z^\star \in \partial\Omega$ such that $\lambda(A^\star) = [d(z^\star, \partial A^\star)]^{-1} := {R^\star}^{-1}$.

Let us introduce two simple examples:

Example 3.1.1. *Assume that Ω is the unit ball. Fix $0 < m < |\Omega| = \pi$. Let $z^\star$ be any point on $\partial\Omega$. Then, it is clear that there exists a unique $R^\star > 0$ such that $|\Omega \setminus B(z^\star, R^\star)| = m$. More precisely, this $R^\star$ is the unique solution of the following equation:*

$$\pi - R^\star\sqrt{4 - R^{\star 2}} + \int_{-R^\star\sqrt{1-\frac{R^{\star 2}}{4}}}^{R^\star\sqrt{1-\frac{R^{\star 2}}{4}}} \left(\sqrt{R^{\star 2} - s^2} + \sqrt{1 - s^2}\,\right) = m.$$

The set $A^\star := \Omega\backslash B(z^\star, R^\star)$ will be optimal for Problem (3.2). In particular, we see that we have infinitely many optimal holes.



Example 3.1.2. *Let $\Omega := [0,1]^2$ be the unit cube. Fix $0 < m < |\Omega| = 1$. Let $z^\star$ and $R^\star$ be defined as in Proposition 3.1. In fact, it is not difficult to see that $z^\star$ must be a corner point. By symmetry, we may assume that $z^\star = 0$. If $m > 1 - \frac{\pi}{4}$, set $R^\star = \sqrt{\frac{4(1-m)}{\pi}}$. Then, we have $|\Omega\backslash B(0, R^\star)| = m$. Moreover, one has*

$$\lambda(\Omega\backslash B(0, R^\star)) = \frac{1}{R^\star} = \sqrt{\frac{\pi}{4(1-m)}}.$$

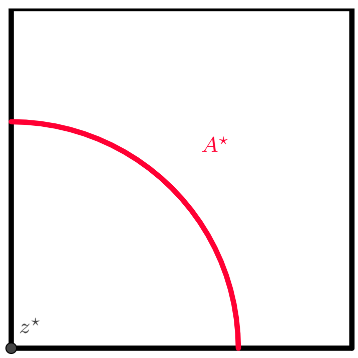
Now, assume that $m \leq 1 - \frac{\pi}{4}$*. Let* $R^\star > 0$ *be such that* $|\Omega \backslash B(0, R^\star)| = m$*. It is easy to see that* $R^\star$ *is the solution of the following equation:*

$$\frac{\pi R^{\star 2}}{4} - 2\int_1^{R^\star} \sqrt{R^{\star 2} - s^2}\,\mathrm{d}s = 1 - m.$$

In this case, we also have that

$$\lambda(\Omega \backslash B(0, R^\star)) = \frac{1}{R^\star}.$$

In particular, we see that $R^\star \to \sqrt{2}$ *when* $m \to 0$*. Moreover, we have exactly four optimal holes.*



Set

$$\Lambda(A) = [\lambda(A)]^{-1} = \max\{d(x, A) : x \in \Omega\}.$$

Then, our aim now is to minimize the Poincaré constant $\Lambda(A)$ (or equivalently, to maximize $\lambda(A)$), but again in the presence of some suitable penalization term on the hole A. First, we note that the case when the penalization is given by the Lebesgue measure of A, i.e. when we consider the following problem:

$$\min\left\{\Lambda(A) - \alpha|A| \,:\, A \subset \Omega\right\},$$

is trivial. If $\alpha \geq 0$, it is clear that this problem has a trivial solution which is $A^\star = \Omega$. On the other hand, if $\alpha < 0$ then any negligible set $A^\star$ which is dense in Ω turns out to be an optimal hole. Since $|A|$ is not sensitive by adding a negligible set while this is not the case for the constant $\Lambda(A)$, then it is much natural here to add a penalization on the $\mathcal{H}^1-$Hausdorff measure of A. More precisely, we consider now the following problem (where $\alpha > 0$ is fixed):

$$\min\left\{\Lambda(A) + \alpha\,\mathcal{H}^1(A) \,:\, A \subset \Omega \text{ is closed and connected}\right\}. \tag{3.4}$$

Proposition 3.2. *Problem (3.4) has an optimal hole* $A^\star$*. Moreover, if* $S^\star \subset \Omega$ *is the set of points* $x \in \Omega$ *such that* $\Lambda(A^\star) = d(x, A^\star) := R^\star$*, then any simple component of* $A^\star \backslash \bigcup_{x \in S^\star} \overline{B(x, R^\star)}$ *is a line segment.*

Proof. Let $\{A_n\}_n$ be a minimizing sequence. Set $u_n(x) := d(x, A_n)$, for every $n \in \mathbb{N}$. We recall that $\Lambda(A_n) = ||u_n||_\infty$. Up to a subsequence, we see that there is a set $A^\star \subset \Omega$ such that $A_n \to A^\star$ in the Hausdorff sense and so, $u_n \to u^\star = d(x, A^\star)$ uniformly in Ω. In particular, this implies that $\Lambda(A_n) \to \Lambda(A^\star)$. On the other hand, thanks to the fact that the measure $\mathcal{H}^1$ is lower semicontinuous with respect to the Hausdorff convergence over the class of all compact connected subsets of Ω, then we also have

$$\mathcal{H}^1(A^\star) \le \liminf_n \mathcal{H}^1(A_n).$$

Consequently, we get that

$$\Lambda(A^\star) + \alpha\,\mathcal{H}^1(A^\star) \le \liminf_n \Big[\Lambda(A_n) + \alpha\,\mathcal{H}^1(A_n)\Big].$$

Hence, $A^\star$ minimizes Problem (3.4). Now, we prove the second statement. Fix a point x_0 on some simple arc of $A^\star$ in the interior of Ω and assume that $x_0 \notin \bigcup_{x\in S^\star} \overline{B(x,R^\star)}$. After a rotation and translation of axes, one can assume that x_0 is the origin. Let $w^\star(s)$, $s \in (-\delta,\delta)$, be a parametrization of $A^\star$ around x_0. Then, it is obvious that $w^\star$ minimizes the following problem:

$$\min\Big\{\mathcal{J}(w) + \alpha\,\mathcal{H}^1(A_w) : w(\pm\delta) = w^\star(\pm\delta) \;\text{ and }\; A_w \subset \Omega\Big\}$$

where

$$\mathcal{J}(w) = \max\{d(z,A_w) \,:\, z \in \Omega\},\;\; A_w := (A^\star \backslash \{w^\star(s) : s \in (-\delta,\delta)\}) \cup \{w(s) : s \in (-\delta,\delta)\}.$$

Let η be a smooth function on $(-\delta,\delta)$ with $\eta(\pm\delta) = 0$. Then, by the minimality of $w^\star$, we have that

$$\mathcal{J}(w^\star) + \alpha\,\mathcal{H}^1(A_{w^\star}) \le \mathcal{J}(w^\star + \varepsilon\eta) + \alpha\,\mathcal{H}^1(A_{w^\star+\varepsilon\eta}), \tag{3.5}$$

for all ε small enough. If $\delta > 0$ is small enough, then it is easy to see that for any $s \in [-\delta,\delta]$, we also have $w^\star(s) \notin \bigcup_{x\in S^\star} \overline{B(x,R^\star)}$. On the other hand, assume that for every ε, there is a point $z_\varepsilon \in \Omega$ and $s_\varepsilon \in [-\delta,\delta]$ such that

$$\mathcal{J}(w^\star + \varepsilon\eta) = d(z_\varepsilon, A_{w^\star+\varepsilon\eta}) = d(z_\varepsilon, w^\star(s_\varepsilon) + \varepsilon\,\eta(s_\varepsilon)). \tag{3.6}$$

Yet, it is clear that $w^\star(s_\varepsilon) + \varepsilon\,\eta(s_\varepsilon) \to w^\star(s)$ and $z_\varepsilon \to z^\star$, where $s \in [-\delta,\delta]$ and $z^\star \in \Omega$. Since $A_{w^\star+\varepsilon\eta} \to A^\star$ in the Hausdorff sence, then $u_\varepsilon(z) := d(z, A_{w^\star+\varepsilon\eta})$ converges uniformly to $u^\star(z) := d(z,A^\star)$. In particular, we have

$$\lim_{\varepsilon\to 0} \mathcal{J}(w^\star + \varepsilon\eta) = \mathcal{J}(w^\star). \tag{3.7}$$

Hence, passing to the limit in (3.6) when $\varepsilon \to 0$ and using the equality (3.7), we get that

$$\mathcal{J}(w^\star) = d(z^\star, w^\star(s)).$$

So, $z^\star \in S^\star$ and $w^\star(s) \in \partial B(z^\star, R^\star)$. Yet, this is a contradiction. Consequently, there exists a point $z^\star \in \Omega$ such that $\mathcal{J}(w^\star + \varepsilon\,\eta) = \mathcal{J}(w^\star) = d(z^\star, A^\star)$, for all ε. By (3.5), this yields that

$$\mathcal{H}^1(A_{w^\star}) \le \mathcal{H}^1(A_{w^\star+\varepsilon\eta}).$$

Since η is arbitrary, this implies that $A^\star$ is a line segment between $w^\star(-\delta)$ and $w^\star(\delta)$. This concludes the proof. □

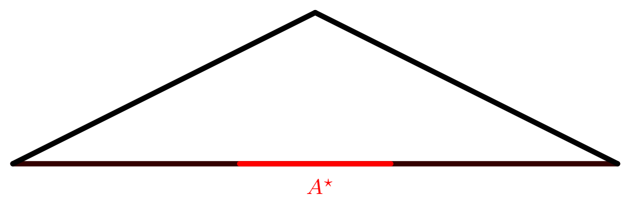
Remark 3.2. *Fix a constant $L \ge 0$. Then, we consider the shape optimization problem:*

$$\min\Big\{\Lambda(A) \,:\, A \subset \Omega \;\textit{ is closed and connected, }\; \mathcal{H}^1(A) \le L\Big\}. \tag{3.8}$$

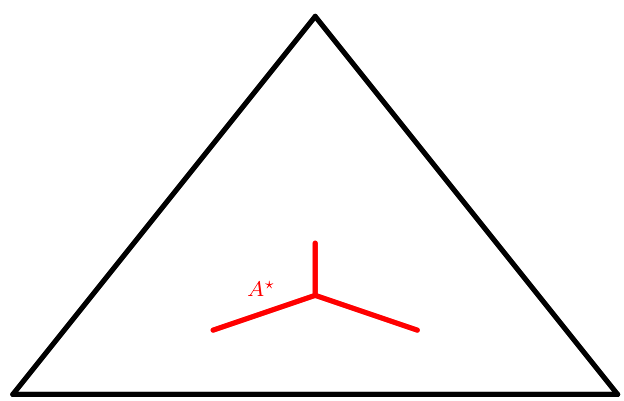
Similarly to the proposition 3.2, one can see that Problem (3.8) has an optimal hole $A^\star$. Moreover, $A^\star$ has a full length (i.e., $\mathcal{H}^1(A^\star) = L$) since if $\mathcal{H}^1(A^\star) < L$ then one can add *an arc of length $\varepsilon > 0$ (small enough) to $A^\star$ so that we get a set A with $\mathcal{H}^1(A) \le L$ and $\Lambda(A) < \Lambda(A^\star)$, which is a contradiction. This yields that $A^\star$ is also an optimal hole for the following problem:*

$$\min\left\{\Lambda(A)\,:\, A \subset \Omega \text{ is closed and connected, } \mathcal{H}^1(A) = L\right\}.$$

Example 3.2.1. *Let Ω be the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,\delta)$, where $0<\delta<1$. Let $A^\star$ be an optimal hole in Problem (3.8). If $L=0$ then it is easy to see that $A^\star = \{(0,0)\}$. Now, assume that $0 < L < 2(1-\delta)$, then it is not difficult to check that $A^\star = [-\frac{L}{2}, \frac{L}{2}] \times \{0\}$.*



We note also that if $\delta \ge 1$ then the situation becomes much complicated since one can see that the optimal hole $A^\star$ will be made of three line segments (two of them having the same length while the third has a length greater or smaller than the others depending on the values of δ and L) connected through a triple point which will be the center of the smallest enclosing ball of Ω. The same difficulty arises when L is large, where we expect $A^\star$ to be a network.



4. The Poincaré-Wirtinger inequality: existence and characterization of optimal holes

Fix a constant $0 < m < |\Omega|$. Then, we consider the following shape optimization problem:

$$\min\left\{\lambda(A)\,:\, A \subset \Omega,\ |A| \ge m\right\} \tag{4.1}$$

where

$$\lambda(A) = \left[\max\left\{\frac{1}{|A|}\int_A d(x,z)\ :\ x \in \Omega\right\}\right]^{-1}.$$

We note that it is not easy to prove existence of an optimal hole for this problem due to the fact that if we take a minimizing sequence $\{A_n\}_n$, then all what can we say about the convergence of this sequence is that $\chi_{A_n} \rightharpoonup^\star \nu^\star$ in $L^\infty(\Omega)$, for some function $0 \le \nu^\star \le 1$. On the other hand, if we assume that the sets A_n are closed then we will have Hausdorff convergence of $\{A_n\}_n$ to some set $A^\star$. However, the Hausdorff convergence may not imply convergence in the sense of Lebesgue measure and so, the functional $\lambda(A)$ is not a priori lower semicontinuous with respect to the Hausdorff convergence. To avoid this difficulty, let us consider instead the following problem which is a relaxation of Problem (4.1):

$$\min\left\{\overline{\lambda}(\nu)\ :\ 0 \le \nu \le m^{-1},\ \int_\Omega \nu = 1\right\} \tag{4.2}$$

where

$$\overline{\Lambda}(\nu) = (\overline{\lambda}(\nu))^{-1} = \max\left\{\int_\Omega d(x,z)\,\nu(z)\ :\ x \in \partial\Omega\right\}.$$

First, we start by introducing the following:

Lemma 4.1. *Let ν be a probability measure over Ω. Then, we have the following bounds:*

$$diam(\Omega)/2 \le \overline{\Lambda}(\nu) \le diam(\Omega).$$

Proof. The second inequality is trivial. For the first inequality, take two points $x,\, x' \in \partial\Omega$ such that $d(x,x') = \mathrm{diam}(\Omega)$. Then, we have the following

$$\overline{\Lambda}(\nu) \ge \int_\Omega d(x,z)\,\nu(z) \ge \int_\Omega [d(x,x') - d(x',z)]\nu(z) \ge d(x,x') - \overline{\Lambda}(\nu). \qquad \square$$

Lemma 4.2. *The map $\nu \mapsto \overline{\lambda}(\nu)$ is continuous with respect to the weak* convergence of measures.*

Proof. Fix $\mu,\, \nu \in \mathcal{P}(\Omega)$. Let $x^\star \in \Omega$ be such that $\overline{\Lambda}(\mu) = \int_\Omega d(x^\star,z)\,\mu(z)$. Then, we have

$$\overline{\Lambda}(\mu) - \overline{\Lambda}(\nu) \le \int_\Omega d(x^\star,z)\,[\mu-\nu](z) \le W_1(\mu,\nu),$$

where the second inequality comes from the fact that $\phi(z) := d(x^\star,z)$ is 1-Lipschitz and that we have

$$\sup\left\{\int_\Omega u\,\mathrm{d}[\mu-\nu]\ :\ u \in W^{1,\infty}(\Omega),\ ||\nabla u||_\infty = 1\right\}$$
$$= \min\left\{\int_{\Omega\times\Omega} d(x,y)\,\mathrm{d}\gamma\ :\ (\Pi_x)_\#\gamma = \mu \quad\text{and}\quad (\Pi_y)_\#\gamma = \nu\right\} := W_1(\mu,\nu).$$

Hence, we get that

$$|\overline{\Lambda}(\mu) - \overline{\Lambda}(\nu)| \le W_1(\mu,\nu).$$

Yet, it is well known (see, for instance, [15]) that the Wasserstein distance W_1 is continuous with respect to the weak* convergence of measures. $\square$

From Lemma 4.1, we know that $\overline{\lambda}(\nu) \ge 1/\mathrm{diam}(\Omega)$. Yet, we can also show the following:

Lemma 4.3. *There is a point $z^\star \in \partial\Omega$ such that the dirac measure $\nu^\star := \delta_{z^\star}$ minimizes*

$$\min\left\{\overline{\lambda}(\nu)\,:\,\nu\in\mathcal{P}(\Omega)\right\}. \tag{4.3}$$

In addition, we have the equality:

$$\inf\left\{\lambda(A)\,:\,A\subset\Omega,\ |A|>0\right\} = \inf\,(4.3) = [diam(\Omega)]^{-1}. \tag{4.4}$$

However, there is a sequence of sets A_n such that $|A_n|\to 0$ and $\lambda(A_n)\to[diam(\Omega)]^{-1}$. In particular, an optimal hole $A^\star$ in (4.4) does not exist.

Proof. Let $x^\star\in\partial\Omega$ be such that there is a point $z^\star\in\partial\Omega$ with $d(x^\star,z^\star)=\mathrm{diam}(\Omega)$. Set $\nu^\star:=\delta_{z^\star}$. So, it is clear that

$$\overline{\lambda}(\nu^\star)=\left[\max\left\{\int_\Omega d(x,z)\,\nu^\star(z)\,:\,x\in\Omega\right\}\right]^{-1}=\left[\max\left\{d(x,z^\star)\,:\,x\in\Omega\right\}\right]^{-1}=[d(x^\star,z^\star)]^{-1}.$$

On the other hand, it is clear that one can always find a sequence of sets $A_n\subset\Omega$ with $|A_n|>0$ such that $A_n\to\{z^\star\}$ in the Hausdorff sense and $\nu_{A_n}:=\frac{\chi_{A_n}}{|A_n|}\rightharpoonup^\star\nu^\star$ in the sense of measures. In particular, by Lemma 4.2, we get that

$$\lambda(A_n)=\overline{\lambda}(\nu_{A_n})\to\overline{\lambda}(\nu^\star).\quad\square$$

However, we will be able to prove existence of an optimal hole $A^\star$ that minimizes the Poincaré constant $\lambda(A)$ as soon as we assume a lower bound on the Lebesgue measure of the set A. For this aim, we start by the following:

Proposition 4.4. *The relaxed problem (4.2) has a minimizer $\nu^\star$. Moreover, there exists a subset $A^\star\subset\Omega$ such that $\nu^\star=m^{-1}\cdot\chi_{A^\star}$.*

Proof. Let $\{\nu_n\}_n$ be a minimizing sequence. Up to a subsequence, we have $\nu_n\rightharpoonup^\star\nu^\star$ with $0\le\nu^\star\le m^{-1}$ and $\int_\Omega\nu^\star=1$. Thanks to Lemma 4.2, one has

$$\overline{\lambda}(\nu_n)\to\overline{\lambda}(\nu^\star).$$

Hence, $\nu^\star$ is a minimizer. Now, we will show that the set $\{0<\nu^\star<m^{-1}\}$ is Lebesgue negligible. We note that there are at most countably many connected components C_n, $n\in\mathbb{N}$, of the support of $\nu^\star$ which are of positive Lebesgue measure. So, it is sufficient to show that for every n, the set $E_\delta\cap C_n$, where $E_\delta:=\{\delta<\nu^\star<m^{-1}-\delta\}$, is Lebesgue negligible for all $\delta>0$. Assume that this is not the case, i.e. there exists $\delta>0$ such that $|E_\delta\cap C_n|>0$.

Let h be a bounded function with $\mathrm{spt}(h)\subset E_\delta\cap C_n$ and $\int_\Omega h=0$. For $\varepsilon>0$ small enough, it is clear that $0\le\nu^\star+\varepsilon h\le m^{-1}$ and $\int_\Omega(\nu^\star+\varepsilon h)=1$. By the minimality of $\nu^\star$, we get that

$$\overline{\lambda}(\nu^\star)\le\overline{\lambda}(\nu^\star+\varepsilon h). \tag{4.5}$$

Fix a point $x^\star\in\partial\Omega$ such that $\overline{\lambda}(\nu^\star)=[\int_\Omega d(x^\star,z)\,\nu^\star(z)]^{-1}$. Recalling Theorem 2.2, we have by (4.5) that

$$\left[\int_\Omega d(x^\star,z)\,\nu^\star(z)\right]^{-1}\le\left[\int_\Omega d(x^\star,z)\,[\nu^\star(z)+\varepsilon h(z)]\right]^{-1}.$$

Hence,

$$\int_\Omega d(x^\star,z)\,h(z)\le 0.$$

Since h is arbitrary and $x^\star$ does not depend on h, so we infer that there is a constant c such that the following holds:

$$d(x^\star, z) = c, \quad \text{for a.e. } z \in E_\delta \cap C_n.$$

In other words, this means that the set $E_\delta \cap C_n$ is contained in the sphere of center $x^\star$ and radius c. Hence, $|E_\delta \cap C_n| = 0$. Yet, this is a contradiction.

As a consequence of that, we have $|\{0 < \nu^\star < m^{-1}\}| = 0$ and so, there is a subset $A^\star \subset \Omega$ such that $\nu^\star = m^{-1} \cdot \chi_{A^\star}$. $\square$

Proposition 4.5. *We have* $\min(4.1) = \min(4.2)$*. Moreover, there exists an optimal hole* $A^\star$ *for Problem* (4.1) *with* $|A^\star| = m$*. In particular,* $A^\star$ *solves*

$$\min\left\{\lambda(A) \,:\, A \subset \Omega,\ |A| = m\right\}. \tag{4.6}$$

In addition, there is a point $x^\star \in \partial\Omega$ *and a constant* $R^\star > 0$ *such that we have* $A^\star = \Omega \setminus B(x^\star, R^\star)$*.*

Proof. First, it is clear that for any subset $A \subset \Omega$ with $|A| \geq m$, the measure $\nu_A := \frac{\chi_A}{|A|}$ is admissible in (4.2), i.e. $0 \leq \nu_A \leq m^{-1}$ and $\int_\Omega \nu_A = 1$. Moreover, we have $\overline{\lambda}(\nu_A) = \lambda(A)$. Consequently, we get that

$$\min(4.2) \leq \min(4.1). \tag{4.7}$$

Yet, by Proposition 4.4, we know that there is a subset $A^\star \subset \Omega$ such that $\nu^\star = m^{-1} \cdot \chi_{A^\star}$ minimizes Problem (4.2). Since $\int_\Omega \nu^\star = 1$, then $|A^\star| = m$. Moreover, we have

$$\overline{\lambda}(\nu^\star) = \overline{\lambda}\left(\frac{\chi_{A^\star}}{|A^\star|}\right) = \lambda(A^\star).$$

Recalling (4.7), we infer that $A^\star$ is an optimal hole for Problem (4.1). In addition, we have the equality:

$$\min(4.1) = \min(4.2).$$

Now, we prove the last statement. Fix a point $x_0 \in \partial A^\star \cap \mathring{\Omega}$. One can assume, after a rotation and translation of axes, that $x_0 = 0$ with $\delta\, e_N \in A^\star$, where $e_N :=< 0, ..., 0, 1 >$ and $\delta > 0$ is small enough. Let $w^\star(\bar{z})$, $\bar{z} \in B_\delta := B^{N-1}(0, \delta)$, be a parametrization of $\partial A^\star$ around x_0. Then, we see that $w^\star$ minimizes

$$\min\left\{\lambda(A_w) : w = w^\star \text{ on } \partial B_\delta,\ A_w \subset \Omega \text{ and } \int_{B_\delta} w = \int_{B_\delta} w^\star\right\}$$

where

$$A_w := (A^\star \backslash G(w^\star)) \cup G(w), \quad G(w) := \{(\bar{z}, w(\bar{z})) \,:\, \bar{z} \in B_\delta\}.$$

Let η be a smooth function on B_δ with $\eta = 0$ on ∂B_δ and $\int_{B_\delta} \eta = 0$. Thanks to the minimality of $w^\star$, we have

$$\lambda(A_{w^\star}) \leq \lambda(A_{w^\star + \varepsilon\eta}), \quad \text{for all } \varepsilon \text{ small enough.} \tag{4.8}$$

Pick any point $x^\star \in \partial\Omega$ such that

$$\lambda(A_{w^\star}) = \left[\frac{1}{|A^\star|}\int_{A^\star} d(x^\star, z)\right]^{-1}.$$

From (4.8), we have

$$\left[\frac{1}{|A^\star|}\int_{A^\star} d(x^\star, z)\right]^{-1} \le \left[\frac{1}{|A_{w^\star+\varepsilon\eta}|}\int_{A_{w^\star+\varepsilon\eta}} d(x^\star, z)\right]^{-1}.$$

Since $|A_{w^\star+\varepsilon\eta}| = |A_{w^\star}|$, this means that

$$\Phi(\varepsilon) := \int_{B_\delta}\int_0^{w^\star(\bar z)+\varepsilon\eta(\bar z)} d(x^\star, (\bar z, s))\,\mathrm{d}s\,\mathrm{d}\bar z$$

reaches a minimum at $\varepsilon = 0$ and so, we must have $\Phi'(0) = 0$. Computing the derivative of Φ, this implies that

$$\int_{B_\delta} d(x^\star, (\bar z, w^\star(\bar z)))\,\eta(\bar z)\,\mathrm{d}\bar z = 0.$$

Since η is an arbitrary function with zero mean value, then we infer that there is a constant $R > 0$ such that

$$d(x^\star, (\bar z, w^\star(\bar z))) = R, \text{ for all } \bar z \in B_\delta.$$

In other words, any connected part of $\partial A^\star$ in the interior of Ω is a part of a sphere centered at $x^\star$ and with some radius $R > 0$. However, we will show that there is exactly one connected part of $\partial A^\star$ inside Ω. More precisely, we claim that $A^\star = \Omega\backslash B(x^\star, R^\star)$, for some $R^\star > 0$. Since $\varepsilon = 0$ is a minimizer of Φ, then we also have $\Phi''(0) \ge 0$. This yields that

$$\Phi''(0) = \int_{B_\delta}[\nabla d(x^\star, (\bar z, w^\star(\bar z)))\cdot e_N]\,\eta(\bar z)^2\,\mathrm{d}\bar z \ge 0. \tag{4.9}$$

Yet, η is arbitrary. Then, by (4.9), we infer that

$$\nabla d(x^\star, (\bar z, w^\star(\bar z)))\cdot e_N \ge 0, \text{ for all } \bar z \in B_\delta.$$

Recalling that $\delta e_N \in A^\star$, this implies that $\nabla d(x^\star, x_0)$ is the interior normal vector to $\partial A^\star$ at the point x_0. Consequently, the claim is proved. □

On the other hand, we may also consider the problem of minimizing the Poincaré constant $\lambda(A)$ but in the case where the penalization on A is posed this time in the functional (not in the constraint):

$$\min\left\{\lambda(A) - \alpha\,|A| \,:\, A\subset\Omega\right\}. \tag{4.10}$$

Now, the question will be to check whether this problem has an optimal hole or not. But, in the following example, we will give a negative answer by showing that a solution to Problem (4.10) does not exist as soon as $\alpha \in]0, \alpha_\star[$, for some critical value $\alpha_\star$. However, we may expect to have existence of solutions if $\alpha > \alpha_\star$. Moreover, let us define

$$\alpha^\star := \sup\left\{\frac{\lambda(\Omega) - \lambda(A)}{|\Omega| - |A|} \,:\, A\subset\Omega\right\}.$$

If $\alpha^\star < \infty$, then it is easy to see that for all $\alpha \ge \alpha^\star$, the whole domain Ω is an optimal hole in Problem (4.10).

But, it is not clear how to show existence of an optimal hole in the case when $\alpha \in]\alpha_\star, \alpha^\star[$, where the optimal hole becomes non-trivial (i.e., $A^\star \neq \Omega$). It seems that this critical value $\alpha_\star$ depends on the geometry of Ω in some abstract way. Before introducing more details, let us consider the following example:

Example 4.5.1. *Assume that $\Omega := \{(R\cos(\beta), R\sin(\beta)) : 0 \le R \le 1,\ 0 \le \beta \le \beta_0\}$ is a thin sector (so, $\beta_0 > 0$ is small enough). First, let us compute $\lambda(\Omega)$. By Theorem 2.2, we know that*

$$\lambda(\Omega) = \left[\max\left\{\frac{1}{|\Omega|}\int_\Omega d(x,z)\ :\ x \in \partial\Omega\right\}\right]^{-1}.$$

Fix $x \in \partial\Omega$. After a rotation and translation of axes, one can assume that x is the origin. Then, using polar coordinates, we have the following:

$$\frac{1}{|\Omega|}\int_\Omega d(x,z) = \frac{\int_{\theta_{\min}}^{\theta_{\max}}\int_0^{\tau(\theta)} r^2\,\mathrm{d}r\,\mathrm{d}\theta}{\int_{\theta_{\min}}^{\theta_{\max}}\int_0^{\tau(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta} \le \frac{2}{3},$$

for some $0 \le \theta_{\min} \le \theta_{\max} \le \pi$ while $\tau(\theta)$ is the maximal distance from x to a point $z \in \partial\Omega$ in the direction $e_\theta :=< \cos(\theta), \sin(\theta) >$. On the other hand, we also have

$$\frac{1}{|\Omega|}\int_\Omega d(0,z) = \frac{1}{|\Omega|}\int_0^{\beta_0}\int_0^1 R^2\,\mathrm{d}R\,\mathrm{d}\beta = \frac{2}{3}.$$

Then,

$$\lambda(\Omega) = \frac{3}{2}.$$

Fix $0 < m \le |\Omega|$. Then, thanks to Proposition 4.5, it is not difficult to see that $A_m := \Omega \backslash B(0, R_m)$, where $R_m = \sqrt{1 - \frac{2m}{\beta_0}}$, is the optimal hole in Problem (4.6). Let us compute the value of $\lambda(A_m)$. Similarly to the estimates above, we see that

$$\lambda(A_m) = \left[\max\left\{\frac{1}{|A_m|}\int_{A_m} d(x,z)\ :\ x \in \Omega\right\}\right]^{-1} = \left(\frac{1}{m}\int_{A_m} d(0,z)\right)^{-1}.$$

Yet, one has

$$\left(\frac{1}{m}\int_{A_m} d(0,z)\right)^{-1} = \left(\frac{1}{m}\int_0^{\beta_0}\int_{R_m}^1 r^2\,\mathrm{d}r\,\mathrm{d}\beta\right)^{-1} = \frac{3}{2}\,\frac{R_m + 1}{R_m^2 + R_m + 1}.$$

Hence,

$$\lambda(A_m) = \frac{3}{2}\,\frac{1 + \sqrt{1 - \frac{2m}{\beta_0}}}{\sqrt{1 - \frac{2m}{\beta_0}} + 2 - \frac{2m}{\beta_0}}.$$

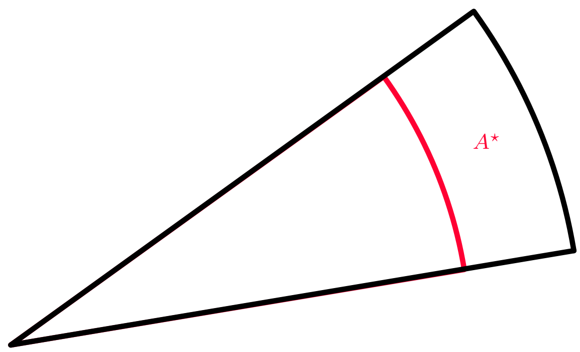
Set $s := \sqrt{1 - \frac{2m}{\beta_0}}$, then we have

$$\inf\left\{\frac{3}{2}\,\frac{1 + \sqrt{1 - \frac{2m}{\beta_0}}}{\sqrt{1 - \frac{2m}{\beta_0}} + 2 - \frac{2m}{\beta_0}} - \alpha\, m : 0 < m \le \frac{\beta_0}{2}\right\}$$

$$= \frac{3 - \alpha\,\beta_0}{2} \ +\ \inf\left\{\frac{\alpha\,\beta_0}{2}\, s^2 - \frac{3}{2}\,\frac{s^2}{s^2 + s + 1} : 0 \le s < 1\right\}.$$

After some tedious computations, one can check that if $\alpha \leq \frac{1}{2\beta_0}$ *then the infimum is attained at* $s = 1$ *(so,* $m = 0$*). This implies that a solution for Problem* (4.10) *does not exist. On the other hand, if* $\alpha \geq \frac{3}{\beta_0}$ *then the minimum is attained at* $s = 0$ *(so,* $m = \frac{\beta_0}{2}$*) and so,* $A^\star = \Omega$ *is a solution for Problem* (4.10)*. Finally, if* $\alpha \in]\frac{1}{2\beta_0}, \frac{3}{\beta_0}[$ *then there is a unique* $s_0 \in]0,1[$ *(resp.* $0 < m_0 < \frac{\beta_0}{2}$*) such that the minimum is attained at* s_0 *(resp.* m_0*) and so,* $A^\star = A_{m_0}$ *is an optimal hole for Problem* (4.10)*. Hence, in this example, the critical values are:*

$$\alpha_\star = \frac{1}{2\beta_0} \qquad \textit{and} \qquad \alpha^\star = \frac{3}{\beta_0}.$$



In order to prove existence of a solution to Problem (4.10), we will consider instead the following relaxation:

$$\min \left\{ \overline{\lambda}(\overline{\nu}) - \alpha \int_\Omega \nu \,:\, 0 \leq \nu \leq 1, \ \int_\Omega \nu > 0 \right\} \tag{4.11}$$

where

$$\overline{\nu} = \frac{\nu}{\int_\Omega \nu}.$$

Let us define the critical value $\alpha_\star$ as follows:

$$\alpha_\star = \inf \left\{ \frac{\overline{\lambda}(\overline{\nu}) - [\mathrm{diam}(\Omega)]^{-1}}{\int_\Omega \nu} \,:\, 0 \leq \nu \leq 1, \ \int_\Omega \nu > 0 \right\}.$$

Proposition 4.6. *We have* $\min(4.10) = \min(4.11)$*. Moreover, there exists a minimizer* $\nu^\star$ *for the relaxation* (4.11) *as soon as* $\alpha > \alpha_\star$*. In addition, there is a subset* $A^\star \subset \Omega$ *such that* $\nu^\star = \chi_{A^\star}$ *and, this set* $A^\star$ *turns out to be an optimal hole for Problem* (4.10)*. If* $\alpha < \alpha_\star$*, then Problem* (4.11) *does not admit any minimizer. And, a solution for Problem* (4.10) *does not exist in this case.*

Proof. Let $\{\nu_n\}_n$ be a minimizing sequence in Problem (4.11). Up to a subsequence, we have $\nu_n \rightharpoonup^\star \nu^\star$ in $L^\infty(\Omega)$ with $0 \leq \nu^\star \leq 1$. In particular, $\int_\Omega \nu_n \to \int_\Omega \nu^\star$. Assume that $\int_\Omega \nu^\star > 0$. Hence, we get

$$\overline{\nu_n} \rightharpoonup^\star \overline{\nu^\star}.$$

By Lemma 4.2, we infer that

$$\overline{\lambda}(\overline{\nu_n}) \to \overline{\lambda}(\overline{\nu^\star}).$$

Consequently, $\nu^\star$ minimizes Problem (4.11). But, it remains to check if $\int_\Omega \nu^\star > 0$ or not. Assume that $\int_\Omega \nu^\star = 0$. Since $\overline{\nu_n} \rightharpoonup^\star \mu$, so we get that

$$\overline{\lambda}(\overline{\nu_n}) - \alpha \int_\Omega \nu_n \to \overline{\lambda}(\mu).$$

In particular, this implies that for all functions $0 \le \nu \le 1$ with $\int_\Omega \nu > 0$, we have the following:

$$\overline{\lambda}(\overline{\nu}) - \alpha \int_\Omega \nu \ge \mathrm{diam}(\Omega)^{-1}.$$

Then, $\alpha \le \alpha_\star$. Hence, Problem (4.11) reaches a minimum if $\alpha > \alpha_\star$. In particular, we see that $\nu^\star$ minimizes

$$\min\left\{\overline{\lambda}(\overline{\nu}) \,:\, 0 \le \nu \le 1,\ \int_\Omega \nu = \int_\Omega \nu^\star\right\}.$$

Yet, this implies that $\nu^\star / \int_\Omega \nu^\star$ also minimizes

$$\min\left\{\overline{\lambda}(\nu) \,:\, 0 \le \nu \le \left(\int_\Omega \nu^\star\right)^{-1},\ \int_\Omega \nu = 1\right\}. \tag{4.12}$$

Recalling Proposition 4.4, we get that there is a set $A^\star \subset \Omega$ such that $\frac{\nu^\star}{\int_\Omega \nu^\star} = (\int_\Omega \nu^\star)^{-1} \cdot \chi_{A^\star}$ and so, $\nu^\star = \chi_{A^\star}$. For every set $A \subset \Omega$, set $\nu_A := \chi_A$. Then, we have

$$\lambda(A^\star) - \alpha|A^\star| = \overline{\lambda}(\overline{\nu^\star}) - \alpha \int_\Omega \nu^\star \le \overline{\lambda}(\overline{\nu_A}) - \alpha \int_\Omega \nu_A = \lambda(A) - \alpha|A|.$$

Consequently, $A^\star$ minimizes Problem (4.10) and, we have $\min\,(4.10) = \min\,(4.11)$. Finally, assume that $\alpha < \alpha_\star$. Then, for every $0 \le \nu \le 1$ with $\int_\Omega \nu > 0$, we have

$$\overline{\lambda}(\overline{\nu}) - \alpha \int_\Omega \nu > \mathrm{diam}(\Omega)^{-1}.$$

But, it is easy to check that in this case one has

$$\inf\left\{\overline{\lambda}(\overline{\nu}) - \alpha \int_\Omega \nu \,:\, 0 \le \nu \le 1,\ \int_\Omega \nu > 0\right\} = \mathrm{diam}(\Omega)^{-1}.$$

Hence, a solution for Problem (4.11) does not exist. In the same way, we see also that there is no optimal hole in (4.10). ☐

Remark 4.1. *Notice that from the definition of the critical value $\alpha_\star$, we always have the following estimate:*

$$\alpha_\star \le \frac{\lambda(\Omega) - [diam(\Omega)]^{-1}}{|\Omega|} \le \frac{[diam(\Omega)]^{-1}}{|\Omega|}.$$

Hence, for every $\alpha > \frac{[diam(\Omega)]^{-1}}{|\Omega|}$, Problem (4.10) has an optimal hole $A^\star$.

Now, our aim is to maximize $\lambda(A)$ (or equivalently, to minimize the Poincaré constant $\Lambda(A) = \lambda(A)^{-1}$), under some penalization on A. To be more precise, we consider the following optimization problem ($m > 0$ is fixed):

$$\min\left\{\Lambda(A) \,:\, A \subset \Omega,\ |A| \ge m\right\} \tag{4.13}$$

where we recall that

$$\Lambda(A) = \max\left\{\frac{1}{|A|}\int_A d(x,z) \,:\, x \in \Omega\right\}.$$

In fact, we do not have a priori compactness in (4.13) without additional assumptions on A. However, if we assume that the admissible sets A are closed then we get compactness but the remaining issue will still be that the functional $\Lambda(A)$ is not lower semicontinuous with respect to the Hausdorff convergence. On the other hand, we note that if we consider a relaxation of (4.13) using bounded functions $0 \leq \nu \leq m^{-1}$ instead of sets A with $|A| \geq m$ (see (4.2)) then it will be also difficult to show that a minimizer $\nu^\star$ is a characteristic function (and possibly, this is not even the case). All these facts make the existence of a solution for this problem a delicate question!

From Lemma 4.1, we recall that

$$\mathrm{diam}(\Omega)/2 \leq \Lambda(A) \leq \mathrm{diam}(\Omega), \text{ for all } A \subset \Omega.$$

Thanks to Lemma 4.3, we also know that

$$\sup\left\{\Lambda(A) \,:\, A \subset \Omega,\ |A| > 0\right\} = \mathrm{diam}(\Omega).$$

Conversely, we also find the minimal value of $\Lambda(A)$. In fact, we may think that the minimal value of $\Lambda(A)$ should be $\mathrm{diam}(\Omega)/2$ but we will see in the next result that this is not really the case.

Lemma 4.7. *Let $R^\star$ be the radius of the minimal enclosing ball of Ω. Then, we have the following equality:*

$$\inf\{\Lambda(A) : A \subset \Omega\} = \min\{\overline{\Lambda}(\nu) : \nu \in \mathcal{P}(\Omega)\} = R^\star. \tag{4.14}$$

Moreover, the measure $\nu^\star := \delta_{z^\star}$, where $z^\star$ is the center of the minimal enclosing ball, is a minimizer in (4.14). But, an optimal hole $A^\star$ does not exist in (4.14).

Proof. First of all, by an approximation argument and thanks to Lemma 4.2, it is easy to see that we have the following inequality:

$$\inf\{\Lambda(A) : A \subset \Omega\} \leq \min\{\overline{\Lambda}(\delta_z) : z \in \Omega\}.$$

Fix a point $z \in \Omega$. Then, we have

$$\overline{\Lambda}(\delta_z) = \max\{d(x,z) : x \in \partial\Omega\}.$$

Consequently, we get that

$$\min\{\overline{\Lambda}(\delta_z) : z \in \Omega\} = \min\{\max\{d(z,x) : x \in \partial\Omega\} : z \in \Omega\} = R^\star.$$

Hence,

$$\min\{\overline{\Lambda}(\nu) : \nu \in \mathcal{P}(\Omega)\} \leq \inf\{\Lambda(A) : A \subset \Omega\} \leq R^\star. \tag{4.15}$$

On the other hand, we claim that

$$\min\{\overline{\Lambda}(\nu) : \nu \in \mathcal{P}(\Omega)\} \geq R^\star. \tag{4.16}$$

Translating Ω, we may assume without loss of generality that $z^\star = 0$. Let $\mathcal{F} := \{x_1, ..., x_n\}$ be a family of points on $\overline{B(z^\star, R^\star)} \cap \partial\Omega$ such that $z^\star$ is in the interior of the convex hull of $\mathcal{F}$. Hence, there will be a family of constants $\{\lambda_k : 1 \leq k \leq n\}$ such that

$$\sum_{k=1}^n \lambda_k\, x_k = 0, \quad \lambda_k \geq 0 \ \text{ (for every } 1 \leq k \leq n), \quad \sum_{k=1}^n \lambda_k = 1.$$

Now, we see that

$$\min\{\overline{\Lambda}(\nu) : \nu \in \mathcal{P}(\Omega)\} = \min\left\{ \max\left\{ \int_{\Omega} d(x,z)\, \nu(z) \ : \ x \in \partial\Omega \right\} : \nu \in \mathcal{P}(\Omega) \right\}$$
$$\geq \ \min\left\{ \sum_{k=1}^{n} \lambda_k \int_{\Omega} d(x_k,z)\, \nu(z) : \nu \in \mathcal{P}(\Omega) \right\}.$$

Yet, we have

$$\sum_{k=1}^{n} \lambda_k \int_{\Omega} d(x_k,z^\star)\, \nu(z) \ \leq \ \sum_{k=1}^{n} \lambda_k \int_{\Omega} d(x_k,z)\, \nu(z).$$

This follows immediately from the fact that the function $\Phi(z) := \sum_{k=1}^{n} \lambda_k\, d(x_k,z)$ is clearly convex while $\nabla\Phi(z^\star) = 0$ and so, we have

$$\Phi(z^\star) = \sum_{k=1}^{n} \lambda_k\, d(x_k,z^\star) \leq \Phi(z) = \sum_{k=1}^{n} \lambda_k\, d(x_k,z).$$

Then, this implies that

$$\min\left\{ \sum_{k=1}^{n} \lambda_k \int_{\Omega} d(x_k,z)\, \nu(z) : \nu \in \mathcal{P}(\Omega) \right\} = \sum_{k=1}^{n} \lambda_k\, d(z^\star,x_k) = R^\star.$$

Hence, the claim (4.16) is proved. Finally, combining (4.16) with (4.15), this concludes the proof. □

Coming back to Problem (4.13), we will add another constraint (namely, convexity) on the admissible sets A which makes the problem easier to solve. To be more precise, we consider now the following version of (4.13):

$$\min\left\{ \Lambda(A) \ : \ A \subset \Omega \ \text{ is convex}, \ \ |A| \geq m \right\}. \tag{4.17}$$

Then, we have the following:

Proposition 4.8. *Problem (4.17) has an optimal hole $A^\star$.*

Proof. Let $\{A_n\}_n$ be a minimizing sequence in Problem (4.17). Recalling Remark 2.2, one can assume that A_n is closed, for all $n \in \mathbb{N}$. Hence, up to a subsequence, $A_n \to A^\star$ in the Hausdorff sense and $A^\star$ is clearly convex. In addition, thanks to the convexity of the sets A_n, we also have $|A_n| \to |A^\star|$. In particular, one has $|A^\star| \geq m$. On the other hand, it is easy to check that $\frac{\chi_{A_n}}{|A_n|} \to \frac{\chi_{A^\star}}{|A^\star|}$ in $L^1(\Omega)$ and so,

$$\Lambda(A_n) \to \Lambda(A^\star).$$

Consequently, this implies that $A^\star$ is an optimal hole for Problem (4.17). Notice that the convexity of A here is a sufficient condition to get lower semicontinuity of the functional $\Lambda(A)$. □

Remark 4.2. *Let us consider instead of (4.17) the following problem (which is somehow more complicated to study):*

$$\min\left\{ \Lambda(A) - \alpha|A| \ : \ A \subset \Omega \ \textit{ is convex} \right\}. \tag{4.18}$$

Let us check if this problem admits a solution or not. Let $\{A_n\}_n$ be a minimizing sequence converging to some convex set $A^\star$. If $|A^\star| > 0$ then similarly to Proposition 4.8, we infer that $A^\star$ is an optimal hole for Problem (4.18). Now, assume that $|A^\star| = 0$. Since $\nu_{A_n} = \frac{\chi_{A_n}}{|A_n|} \rightharpoonup^\star \nu^\star$, for some $\nu^\star \in \mathcal{P}(\Omega)$, then $\overline{\Lambda}(\nu_{A_n}) \to \overline{\Lambda}(\nu^\star)$. In particular, one has

$$\Lambda(A_n) - \alpha|A_n| = \overline{\Lambda}(\nu_{A_n}) - \alpha|A_n| \to \overline{\Lambda}(\nu^\star).$$

Yet, thanks to Lemma 4.7, we know that $\overline{\Lambda}(\nu^\star) \geq R^\star$. Hence, this implies that for all convex sets $A \subset \Omega$, we have

$$\Lambda(A) - \alpha|A| \geq R^\star.$$

Now, let us define the critical value $\alpha_\star$ as follows:

$$\alpha_\star = \inf\left\{\frac{\Lambda(A) - R^\star}{|A|} \,:\, A \subset \Omega \;\; \textit{is convex}\right\}.$$

Then, for any $\alpha > \alpha_\star$, Problem (4.18) reaches a minimum. If $\alpha < \alpha^\star$, then it is not difficult to see that the infimum of (4.18) equals $R^\star$ and so, a solution for Problem (4.18) does not exist. We note also that

$$\alpha_\star \leq \frac{\Lambda(\Omega) - R^\star}{|\Omega|} \leq \frac{\textit{diam}(\Omega) - R^\star}{|\Omega|}.$$

On the other hand, it is difficult to characterize an optimal hole $A^\star$ for Problem (4.18). However, we will be able to show a symmetry property on this optimal hole assuming that the domain Ω is symmetric.

For this aim, let us recall the definition of the Steiner symmetrization of a compact convex set $A \subset \mathbb{R}^N$ relative to an hyperplane L which will be represented by $z_1 = 0$. For any point $\bar{z} := (0, z_2, ..., z_N) \in L$, we will denote by $L_{\bar{z}}^\perp$ the line segment passing by $\bar{z}$ and orthogonal to L, i.e.

$$L_{\bar{z}}^\perp := \{\bar{z} + z_1\, e_1 \,:\, z_1 \in \mathbb{R}\},$$

where $e_1 :=< 1, 0, ..., 0 >$. Let $m_{\bar{z}} = |A \cap L_{\bar{z}}^\perp|$ be the measure (i.e., total length) of the slice $L_{\bar{z}}^\perp \cap A$. Replacing each slice by the interval centered on L with the same length yields the symmetrized domain $S_L(A)$:

$$S_L(A) := \left\{\bar{z} + z_1\, e_1 \,:\, \bar{z} + w_1\, e_1 \in A \text{ for some } w_1 \in \mathbb{R} \;\text{ and }\; -\frac{m_{\bar{z}}}{2} \leq z_1 \leq \frac{m_{\bar{z}}}{2}\right\}.$$

It is clear that $S_L(A)$ is symmetric with respect to L, $S_L(A)$ is convex and, $|S_L(A)| = |A|$. We define the interval $I_{\bar{z}} = L_{\bar{z}}^\perp \cap A$, $I'_{\bar{z}}$ the reflection of $I_{\bar{z}}$ with respect to L and, $S_L(I_{\bar{z}}) = [-\frac{m_{\bar{z}}}{2}, \frac{m_{\bar{z}}}{2}]$.

Then, we have the following:

Proposition 4.9. *Assume that $S_L(\Omega) = \Omega$, for some hyperplane L. Let $A^\star$ be an optimal hole in Problem (4.17). Then, we also have $S_L(A^\star) = A^\star$.*

Proof. First of all, we may assume after a rotation and translation of axes that L is the hyperplane $z_1 = 0$. For every $x \in \partial\Omega$, we have

$$\int_{A^\star} d(x, z) = \int_{\bar{z} \in L} \int_{I_{\bar{z}}} d(x, (z_1, \bar{z}))\, \mathrm{d}z_1\, \mathrm{d}\bar{z}.$$

Let A' (resp. x') be the symmetry of $A^\star$ (resp. x) with respect to L. Then, it is easy to see that

$$\int_{A'} d(x',z) = \int_{\bar{z}\in L}\int_{I'_{\bar{z}}} d(x',(z_1,\bar{z}))\,\mathrm{d}z_1\,\mathrm{d}\bar{z} = \int_{A^\star} d(x,z).$$

In particular, we have

$$\Lambda(A') = \Lambda(A^\star).$$

On the other hand, one has

$$\int_{S_L(A^\star)} d(x,z) = \int_{\bar{z}\in L}\int_{S_L(I_{\bar{z}})} d(x,(z_1,\bar{z}))\,\mathrm{d}z_1\,\mathrm{d}\bar{z}.$$

Assume that $I_{\bar{z}} = \{(1-t)p+tq : 0\le t\le 1\}$, where $(p,\bar{z})$, $(q,\bar{z})\in\partial A^\star$. Then, for every $z_1\in S_L(I_{\bar{z}})$, we see that there is a unique $t\in[0,1]$ such that $z_1 = \frac{(1-2t)p+(2t-1)q}{2}$. In particular, we have

$$d\Big(x,\Big(\frac{(1-2t)p+(2t-1)q}{2},\bar{z}\Big)\Big) \le \frac{d(x,((1-t)p+tq,\bar{z}))}{2} + \frac{d(x,(-tp+(t-1)q,\bar{z}))}{2}.$$

Hence,

(4.19)
$$\int_{\bar{z}\in L}\int_{S_L(I_{\bar{z}})} d(x,(z_1,\bar{z}))\,\mathrm{d}z_1\,\mathrm{d}\bar{z} \le \frac{1}{2}\int_{\bar{z}\in L}\int_{I_{\bar{z}}} d(x,(z_1,\bar{z}))\,\mathrm{d}z_1\,\mathrm{d}\bar{z} + \frac{1}{2}\int_{\bar{z}\in L}\int_{I'_{\bar{z}}} d(x,(z_1,\bar{z}))\,\mathrm{d}z_1\,\mathrm{d}\bar{z}$$
$$= \frac{1}{2}\int_{A^\star} d(x,z) + \frac{1}{2}\int_{A'} d(x,z) \le \frac{|A^\star|}{2}\,\Lambda(A^\star) + \frac{|A'|}{2}\,\Lambda(A') = |A^\star|\,\Lambda(A^\star).$$

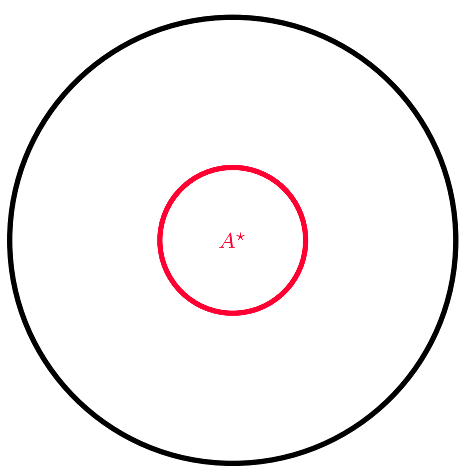
Since the point $x\in\partial\Omega$ is arbitrary and $|S_L(A^\star)| = |A^\star|$, then this yields that we have the following:

$$\Lambda(S_L(A^\star)) \le \Lambda(A^\star).$$

Consequently, $S_L(A^\star)$ is also an optimal hole in (4.17). In addition, the inequality in (4.19) becomes equality and so, we must have $S_L(A^\star) = A^\star$. □

As a simple consequence of that, we get the following:

Corollary 4.10. *Assume that $\Omega := B(0,R)$. Then, the optimal hole $A^\star$ is a ball centered at the origin and with radius $R^\star \le R$.*



Proof. This follows immediately from Proposition 4.9 since $A^\star$ has to be symmetric with respect to any line passing through the origin. $\square$

Let us come back to the original problem (4.13) and assume that it admits an optimal hole $A^\star$, then it seems that one can show that $A^\star$ is convex. In the sequel, we will show this convexity result by characterizing $A^\star$ in the case where Ω is a polygonal domain.

Proposition 4.11. *Assume that Ω is a simplex with $n-$hyperplanes of symmetry. If $A^\star$ is an optimal hole for Problem (4.13), then there will be a constant $c > 0$ such that*

$$A^\star = \left\{x \in \Omega \,:\, \sum_{i=1}^{2n} d(x, x_i) \leq c\right\}$$

where $\mathcal{F} := \{x_i \,:\, 1 \leq i \leq 2n\}$ is the set of vertices on $\partial\Omega$ such that, for every $1 \leq i \leq 2n$, we have

$$\Lambda(A^\star) = \frac{1}{|A^\star|}\int_{A^\star} d(x_i, z).$$

Proof. Let $\{L_j \,:\, 1 \leq j \leq n\}$ be the hyperplanes of symmetry for Ω. These hyperplanes divide $A^\star$ into $2n-$essentially disjoint sets $A^\star_k$, $1 \leq k \leq 2n$. Moreover, it is clear that each of these sets $A^\star_k$ can be obtained from $A^\star_1$ after a finite number of symmetries with respect to some of these hyperplanes L_j; we will denote by $S_k : \Omega \mapsto \Omega$ these maps that transform $A^\star_1$ into $A^\star_k$. Let us assume that $A^\star_1$ is bounded by $\partial A^\star$ and the hyperplanes L_1, L_2. Fix a point $a_1 \in (\partial A^\star_1 \cap \mathring{\Omega})\backslash\{L_1 \cup L_2\}$. After a rotation and translation of axes, we may assume that $a_1 = 0$ with $\delta e_N \in A^\star$, where $e_N :=< 0, ..., 0, 1 >$ and $\delta > 0$ is small enough. Let $w^\star(\bar{z})$, $\bar{z} \in B_\delta := B^{N-1}(0, \delta)$, be a parametrization of $\partial A^\star$ around a_1. Set

$$G_1(w^\star) := \{(\bar{z}, w^\star(\bar{z})) \,:\, \bar{z} \in B_\delta\}.$$

On the other hand, we define $a_k := S_k(a_1)$ and $G_k(w^\star) := S_k(G_1)$, for every $1 \leq k \leq 2n$. Then, it is clear that $w^\star$ minimizes the following problem:

$$\min\left\{\Lambda(A_w) \,:\, w = w^\star \text{ on } B_\delta,\ A_w \subset \Omega \text{ and } \int_{B_\delta} w = \int_{B_\delta} w^\star\right\}$$

where

$$A_w := (A^\star\backslash G(w^\star)) \cup G(w), \quad G(w) := \bigcup_{k=1}^{2n} G_k(w).$$

Let η be a smooth function on B_δ such that $\eta = 0$ on ∂B_δ and $\int_{B_\delta} \eta = 0$. Thanks to the minimality of $w^\star$, we have

$$\Lambda(A_{w^\star}) \leq \Lambda(A_{w^\star+\varepsilon\eta}), \text{ for all } \varepsilon \text{ small enough.}$$

Let $x^\star \in \partial\Omega$ be any vertex such that

$$\Lambda(A^\star) = \frac{1}{|A^\star|}\int_{A^\star} d(x^\star, z).$$

Since Ω is a simplex and $A_{w^\star+\varepsilon\eta}$ is symmetric with respect to all the hyperplanes L_j, then it is easy to see that for all ε small enough, we have

$$\Lambda(A_{w^\star+\varepsilon\eta}) = \frac{1}{|A_{w^\star+\varepsilon\eta}|}\int_{A_{w^\star+\varepsilon\eta}} d(x^\star, z).$$

Consequently, we have

$$\int_{A_{w^\star+\varepsilon\eta}} d(x^\star, z) - \int_{A_{w^\star}} d(x^\star, z) \geq 0.$$

But,

$$\int_{A_{w^\star+\varepsilon\eta}} d(x^\star, z) - \int_{A_{w^\star}} d(x^\star, z)$$
$$= -\sum_{k=1}^{2n} \Big[\int_{B_\delta} \int_0^{w^\star(\bar z)+\varepsilon\eta(\bar z)} d(x^\star, S_k(\bar z, s)) - \int_{B_\delta} \int_0^{w^\star(\bar z)} d(x^\star, S_k(\bar z, s)) \Big].$$

Moreover, one has

$$\lim_{\varepsilon\to 0} \frac{\int_{A_{w^\star+\varepsilon\eta}} d(x^\star, z) - \int_{A_{w^\star}} d(x^\star, z)}{\varepsilon} = 0.$$

Then, we get

$$\int_{B_\delta} \Big(\sum_{k=1}^{2n} d(x^\star, S_k(\bar z, w^\star(\bar z))) \Big) \eta(\bar z) = 0.$$

Yet, it is clear that

$$d(x^\star, S_k(\bar z, w^\star(\bar z))) = d(S_k^{-1}(x^\star), (\bar z, w^\star(\bar z))), \text{ for all } \bar z \in B_\delta.$$

Hence,

$$\int_{B_\delta} \Big(\sum_{k=1}^{2n} d(S_k^{-1}(x^\star), (\bar z, w^\star(\bar z))) \Big) \eta(\bar z) = 0. \tag{4.20}$$

Since η is an arbitrary function with $\int_{B_\delta} \eta = 0$, then by (4.20) we infer that there is a constant $c > 0$ such that

$$\sum_{k=1}^{2n} d(S_k^{-1}(x^\star), z) = c.$$

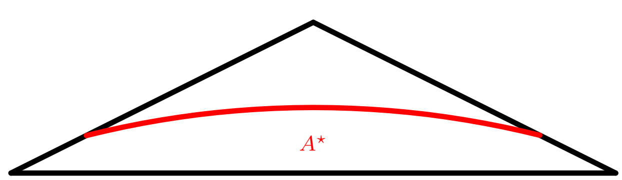
But, thanks to Proposition 4.9, it is not difficult to see that $\mathcal{F} = \{S_k^{-1}(x^\star) : 1 \leq k \leq 2n\}$. This concludes the proof. □

Example 4.11.1. *Assume that Ω is the triangle determined by $(-L, 0)$, $(L, 0)$ and $(0, \varepsilon)$, where $0 < \varepsilon << L$. For ε small enough, it is not difficult to see that $\mathcal{F} = \{(-L, 0), (L, 0)\}$ (see the definition of $\mathcal{F}$ in Proposition 4.11). Hence, by Proposition 4.11, we know that $\partial A^\star$ is given by the following equation:*

$$|x - (L, 0)| + |x + (L, 0)| = c.$$

We note that $c > 2L$. Hence, the boundary of $A^\star$ in the interior of Ω is an arc of the ellipse with equation:

$$\Big(1 - \frac{4L^2}{c^2}\Big) x_1^2 + x_2^2 = \frac{c^2}{4} - L^2.$$



5. Regularity of the optimal measure in the Poincaré-Wirtinger inequality

In this last section, we minimize the Poincaré constant $\overline{\lambda}(\nu)$ among all densities $\nu \in L^1(\Omega)$ but in the case where there is an additional penality term which is given by a convex functional $F(\nu)$. More precisely, we consider the following problem:

$$\min \left\{ \overline{\lambda}(\nu) + F(\nu) \,:\, \nu \in \mathcal{P}(\Omega) \right\} \tag{5.1}$$

where

$$F(\nu) := \begin{cases} \int_\Omega f(\nu(x)) & \text{if } \nu << \mathcal{L}^d, \\ +\infty & \text{otherwise} \end{cases}$$

and $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is assumed to be lower semicontinuous and convex, with $f(0) = 0$ and superlinear at infinity, that is $\lim\limits_{t \to +\infty} \frac{f(t)}{t} = +\infty$.

Then, we have the following:

Proposition 5.1. *There exists a density $\nu^\star$ that minimizes Problem 5.1.*

Proof. Let $\{\nu_n\}_n$ be a minimizing sequence in Problem (5.1). Up to a subsequence, we know that $\nu_n \rightharpoonup^\star \nu^\star$. From Lemma 4.2, we get that

$$\overline{\lambda}(\nu_n) \to \overline{\lambda}(\nu^\star).$$

Thanks to [4] and to our assumptions on f, we also have that the functional F is lower semicontinuous with respect to the weak* convergence of measures. Hence, we infer that

$$\overline{\lambda}(\nu^\star) + F(\nu^\star) \le \liminf_n \,[\,\overline{\lambda}(\nu_n) + F(\nu_n)].$$

Consequently, $\nu^\star$ is a minimizer in Problem (5.1). □

Under certain assumptions on the function f, we will be able also to show uniqueness of this minimizer $\nu^\star$ by characterizing it and, to study at the same time its regularity. Then, we close this paper by the following result:

Proposition 5.2. *Assume that f is C^1 and strictly convex. Then, $\nu^\star$ is the unique minimizer in Problem 5.1. Moreover, there exist a point $x^\star \in \partial\Omega$ and a constant $c \in \mathbb{R}$ such that we have the following:*

$$\nu^\star(x) = f'^{-1}\left(\max\left\{ \frac{d(x^\star, x)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2} + c, f'(0) \right\} \right). \tag{5.2}$$

In particular, $\nu^\star$ is a continuous density over Ω. In addition, $\nu^\star$ is Lipschitz as soon as $f'' \ge \delta > 0$.

Proof. Let $\mu \in \mathcal{P}(\Omega)$ be such that $\mu << \mathcal{L}^d$. For every $\varepsilon > 0$, set $\nu_\varepsilon := (1-\varepsilon)\nu^\star + \varepsilon\mu$. Then, it is clear that $\nu_\varepsilon \in \mathcal{P}(\Omega)$. Thanks to the minimality of $\nu^\star$, we have

$$\overline{\lambda}(\nu^\star) + F(\nu^\star) \le \overline{\lambda}(\nu_\varepsilon) + F(\nu_\varepsilon).$$

Now, let $x^\star \in \partial\Omega$ be a point such that $\overline{\lambda}(\nu^\star) = [\int_\Omega d(x^\star, z)\, \nu^\star(z)]^{-1}$. Recalling (2.2), we get that

$$\left[\int_\Omega d(x^\star, z)\, \nu^\star(z)\right]^{-1} + F(\nu^\star) \leq \left[\int_\Omega d(x^\star, z)\, \nu_\varepsilon(z)\right]^{-1} + F(\nu_\varepsilon).$$

Hence,

$$\frac{\int_\Omega d(x^\star, z)\, (\nu^\star - \mu)(z)}{[\int_\Omega d(x^\star, z)\, \nu_\varepsilon(z)][\int_\Omega d(x^\star, z)\, \nu^\star(z)]} + \frac{\int_\Omega (f(\nu_\varepsilon(z)) - f(\nu^\star(z)))}{\varepsilon} \geq 0.$$

Since f is a C^1 convex function, then we have

$$\frac{\int_\Omega d(x^\star, z)\, (\nu^\star - \mu)(z)}{[\int_\Omega d(x^\star, z)\, \nu_\varepsilon(z)][\int_\Omega d(x^\star, z)\, \nu^\star(z)]} + \int_\Omega f'(\nu^\star(z))[\mu(z) - \nu^\star(z)] \geq 0.$$

Passing to the limit when $\varepsilon \to 0^+$, we get

$$\frac{\int_\Omega d(x^\star, z)\, (\nu^\star - \mu)(z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2} + \int_\Omega f'(\nu^\star(z))[\mu(z) - \nu^\star(z)] \geq 0.$$

Consequently,

$$\int_\Omega \left[f'(\nu^\star(z)) - \frac{d(x^\star, z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2}\right] \mu(z) \geq \int_\Omega \left[f'(\nu^\star(z)) - \frac{d(x^\star, z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2}\right] \nu^\star(z).$$

Since μ is an arbitrary probability measure with $\mu << \mathcal{L}^d$, then there will be a constant c such that

$$f'(\nu^\star(z)) - \frac{d(x^\star, z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2} \geq c$$

with

$$f'(\nu^\star(z)) - \frac{d(x^\star, z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2} = c \quad \text{ on } \quad \{\nu^\star > 0\}.$$

Hence,

$$f'(\nu^\star(z)) = \max\left\{c + \frac{d(x^\star, z)}{[\int_\Omega d(x^\star, z)\, \nu^\star(z)]^2}, f'(0)\right\}.$$

Thanks to the fact that f is assumed to be strictly convex (so, f' is invertible), we get (5.2). Finally, the last statement follows immediately from (5.2). □

References

[1] G. Acosta and R. Dur'an, An optimal Poincaré inequality in L^1 for convex domains, *Proc. AMS* 132(1), 195–202, 2003.

[2] J. J. Bevan, J. Deane and S. Zelik, A sharp Poincaré inequality for functions in $W^{1,\infty}(\Omega;\mathbb{R})$, *Proceedings of the American Mathematical Society*, 2022.

[3] J. F. Bonder, J. D. Rossi, C-B Schönlieb, The best constant and extremals of the Sobolev embeddings in domains with holes: The L^∞ case, *Illinois J. Math.*, 52 (4) 1111–1121, 2008.

[4] G. Bouchitté and G. Buttazzo, New lower semicontinuity results for nonconvex functionals defined on measures, *Nonlinear Analysis* 15, no. 7, 679–692, 1990.

[5] T. Champion, L. De Pascale and Chloe Jimenez, The $\infty-$Eigenvalue problem and a problem of Optimal Transportation, *Communications in Applied Analysis*, 13, no. 4, 547–566, 2009.

[6] A. Cherkaev and E. Cherkaeva, Optimal design for uncertain loading condition Homogenization, 193–213, *Ser. Adv. Math. Appl. Sci.*, 50, World Sci. Publishing, River Edge, NJ, 1999.

[7] A. Cianchi, A sharp form of Poincaré inequalities on balls and spheres, *Z. Angew. Math. Phys.* 40, no. 4, 558–569, 1989.

[8] S. Dweik and F. Santambrogio, Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques, *Control, Optimisation and Calculus of Variations, 2016.*

[9] L. C. Evans, *Partial Differential Equations, Graduate Studies in Mathematics*, vol. 19, American Mathematical Society, Providence, Rhode Island, 1998.

[10] J. Fernandez Bonder, J. D. Rossi and C. B. Schönlieb, The best constant and extremals of the Sobolev embeddings in domains with holes: the L^∞ case, *Illinois Journal of Mathematics*, Vol. 52(4), 1111-1121, 2008.

[11] A. Henrot, Minimization problems for eigenvalues of the Laplacian, *J. Evol. Equ.*, 3 (3), 443–461, 2003.

[12] A. Kuznetsov and A. Nazarov, Sharp constants in the Poincaré, Steklov and related inequalities (a survey), *Mathematika* 61 (2015), no. 2, 328–344.

[13] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, Grundlehren der Mathematischen Wissenschaften, 342, Springer, Heidelberg, 2011.

[14] L. E. Payne and H. F. Weinberger, An optimal Poincaré inequality for convex domains, *Arch. Rat. Mech. Anal.* 5, 286–292, 1960.

[15] F. Santambrogio, *Optimal Transport for Applied Mathematicians,* in *Progress in Nonlinear Differential Equations and Their Applications,* 87, Birkhäuser Basel, 2015.

[16] G. Szeg'o, Inequalities for certain membranes of a given area, *J. Rational Mech. Anal.* 3, 343–356, 1954.

[17] H. F. Weinberger, An isoperimetric inequality for the N-dimensional free membrane problem, *J. Rational Mech. Anal.* 5, 633–636 1956.

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