SOME RESULTS ON PARTIAL DIFFERENTIAL EQUATIONS AND ASIAN OPTIONS

E. BARUCCI
Dipartimento di Statistica e Matematica Applicata all'Economia
Università di Pisa
ebarucci@ec.unipi.it

S. POLIDORO
Dipartimento di Matematica
Università degli Studi di Bologna
polidoro@dm.unibo.it

V. VESPRI
Dipartimento di Matematica "Ulisse Dini"
Università degli Studi di Firenze
vespri@poincare.dma.unifi.it

We analyze partial differential equations arising in the evaluation of Asian Options. The equations are strongly degenerate partial differential equations in three dimensions. We show that the solution of the no arbitrage partial differential equation is sufficiently regular and standard numerical methods can be employed to approximate it.

Keywords: Ultraparabolic partial differential equations, Hypoellitic equations, Finite difference methods, Contingent claims valuation, Asian Options.

1. Introduction

The valuation of path dependent (Asian) European contingent claims, i.e., claims whose final payoff depends on the history of the underlying asset price, is a difficult task in mathematical finance. Only in some simple cases the no-arbitrage price of a path dependent contingent claim computed as the expected final payoff under the risk neutral probability measure is obtained in closed form, for some results see Ref.29. For example, considering a lognormal stochastic process for the underlying asset price, we have that its geometric average is still characterized by a lognormal probability density, but things change dramatically considering the arithmetic average, only recently its moments have been determined in Ref.15. In some cases, introducing path dependent variables the state space can be augmented in such a way that the no-arbitrage price of the claim is obtained as the solution of a Cauchy problem.

When a closed form solution for the price of the contingent claim is not available, we have to rely upon numerical methods. Different methodologies have been applied in the literature to address the problem. The approaches can be split in two groups: methods aiming at estimating the conditional expectation of the final payoff and methods aiming at solving numerically the Cauchy problem associated with the no-arbitrage Partial Differential Equation (PDE).

Looking at the methods focusing on the estimation of the expectation of the final payoff, we have the following: Monte Carlo simulations (Ref.19 and 30), fast Fourier transform to calculate the density of the sum of random variables as the convolution of the density functions (Ref.8). In Ref.15 a closed form solution of the no-arbitrage price of the Arithmetic average fixed strike price is obtained through the inversion of a Laplace transform. Considering a lognormal diffusion process for the underlying asset, the pricing density can be approximated by substituting the arithmetic average of the price with the geometric average which preserves the lognormality for the average. The approximation is obtained by equating the moments of the true distribution and of the approximating distribution, see Ref.21 for an approximation up to the second moment and Ref.31 for an approximation up to fourth moment, or by adjusting the strike price to correct for the misspricing, see Ref.32. In Ref.7 an upper-bound to the approximation error occurred using this procedure is proposed.

The standard approximation methods based on partial differential equations require some regularity conditions of the solution of the no arbitrage PDE. Considering path dependent contingent claims in a general setting, the PDE is a strongly degenerate parabolic equation in three dimensions (time, the underlying asset price and the path dependent variable). In this setting, the needed regularity seemed out of reach. To avoid this difficulty, many authors considered a two dimensional second order partial differential equation which is obtained from the original one through a change of variable (*similarity reduction method*) when the contingent claim final payoff has a particular form (see Refs.18,1,12,28,33). This method covers a large set of contingent claim contracts, including arithmetic Asian options, but not a contingent claim characterized by a general final payoff.

A general payoff and therefore a PDE in three dimensions is considered in Refs.3,4,5. In Ref.5 a numerical method is proposed based on some analysis arguments and on some probabilistic remarks. Convergence of the numerical scheme is proved by means of the central limit theorem. In Ref.3 and 4 a method based on the notion of viscosity solution is developed. Convergence of the numerical scheme is proved through Dini's theorem. The advantage of these two methods, with respect to the classical ones, is that they don't require the regularity of the coefficients of the PDE, on the other hand they don't give any explicit convergence estimate.

This paper aims at showing that the solution of the no arbitrage partial differential equation is in fact sufficiently regular and that the standard numerical methods can be employed to approximate it. We first present theoretical results concerning existence, uniqueness and regularity of the solution, then the theoretical results are coupled with results about the capability of numerical methods to approximate the no-arbitrage price of the contingent claim. Unlike the above mentioned methods, a convergence estimate is explicitly given. The results obtained for the geometric average build on the results obtained in Ref.27. The numerical method proposed in this paper is a classical finite-difference method. The main problem encountered when applying this method to the arithmetic average case is the lack of suitable a-priori estimates near infinity for the partial derivatives of the solution.

We concentrate our attention on arithmetic and geometric average options, some insights are given for more general path dependent options and assuming a stochastic process for the underlying asset price characterized by constant elasticity variance.

The paper is organized as follows. In Section 2 we present the path dependent contingent claims framework, then we consider in detail the arithmetic and the geometric average options and the other examples discussed above. In Section 3 we state the analytic and the numerical results for the Cauchy problem related to the geometric average options, while in Section 4 the arithmetic average options are considered. In both sections we first present the theoretical results and then in two subsections we describe the numerical methods. Finally, Section 5 contains some remarks about the general case.

2. Pricing Path Dependent Contingent Claims

We consider a standard complete markets economy. We take as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which is right continuous and such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

There are two assets traded in two markets: a risk-free asset and a risky asset. The risky asset price follows the Itô stochastic differential equation:

$$dS_t = \mu_s(t,S)dt + \sigma_s(t,S)dW_t, \quad S_0 > 0, \tag{2.1}$$

where W_t is a Brownian motion in $\mathbb{R}$, $\mathcal{F}_t$ is the natural filtration of W_t. Let the classical conditions ensuring existence and uniqueness of (2.1) be satisfied. The risk-free asset is a bond whose price evolves as follows:

$$dB_t = r(t)B_t dt, \quad B_0 > 0. \tag{2.2}$$

$r(t)$ is the risk-free rate at time t. Note that if we directly refer to the risk-neutral probability measure associated with the Brownian motion W then $\mu_s = r$. The analysis can be extended to a Brownian motion in $\mathbb{R}^n$ still with complete markets, for simplicity we concentrate our attention to the scalar case.

Following Ref.33 and 5 we consider the class of European path dependent contingent claims defined by the path dependent variable $A_t = \Psi([S_\tau]_{\tau \le t})$ which is assumed to follow the stochastic differential equation

$$dA_t = \mu_a(t, S_t, A_t)dt + \sigma_a(t, S_t, A_t)dW_t \tag{2.3}$$

and with final payoff function $\Omega(S_T, A_T)$.

The no-arbitrage price of the contingent claim is given by the expected value of the final payoff under the risk neutral probability measure. The price can also be obtained as the solution $V(t, S_t, A_t)$ of the following Cauchy problem:

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2} + \mu_a\frac{\partial V}{\partial A} + \frac{1}{2}\sigma_a^2\frac{\partial^2 V}{\partial A^2} + \sigma_s\sigma_a\frac{\partial^2 V}{\partial S \partial A} = 0 \tag{2.4}$$

with

$$V(T, S_T, A_T) = \Omega(S_T, A_T). \tag{2.5}$$

In what follows we restrict our attention to the class of path dependent options characterized by a payoff which is a function of the entire history of the underlying asset price $(S_t,\ 0 \le t \le T)$. We do not consider *forward start* options.

Many path dependent contingent claims are traded in financial markets, for a survey see Ref.29. Among them we recall the following:

- *Arithmetic average floating strike call option*:

$$\Omega(S_T, A_T) = \max(S_T - \frac{A_T}{T}, 0),\ A_t = \int_0^t S_\tau d\tau,\ \mu_a = S,\ \sigma_a = 0;$$

- *Arithmetic average fixed strike call option*:

$$\Omega(S_T, A_T) = \max(\frac{A_T}{T} - E, 0),\ A_t = \int_0^t S_\tau d\tau,\ \mu_a = S,\ \sigma_a = 0;$$

- *Geometric average floating strike call option*:

$$\Omega(S_T, A_T) = \max(S_T - e^{\frac{A_T}{T}}, 0),\ A_t = \int_0^t \log(S_\tau)d\tau,\ \mu_a = \log(S),\ \sigma_a = 0;$$

- *Geometric average fixed strike call option*:

$$\Omega(S_T, A_T) = \max(e^{\frac{A_T}{T}} - E, 0),\ A_t = \int_0^t \log(S_\tau)d\tau,\ \mu_a = \log(S),\ \sigma_a = 0.$$

In the first two contracts the path dependent variable is given by the arithmetic average of the underlying asset price, in the last two contracts the path dependent variable is given by the geometric average of the underlying asset price. In practice only contracts written on the arithmetic average are exchanged in financial markets. However, it is useful to evaluate contracts written on the geometric average for two main reasons: the problem is easier and it provides, under some conditions, an approximation to the no arbitrage price of a claim written on the arithmetic average.

3. Geometric Average Options

First of all we consider a lognormal stochastic process for the underlying asset price, i.e., $\mu_s = \mu S$ and $\sigma_s = \sigma S$, where μ and σ are two non zero constants. Then the Cauchy problems for the price of the Geometric average floating strike call option and of the Geometric average fixed strike call option become

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \log(S)\frac{\partial V}{\partial A} = 0 \tag{3.1}$$

with

$$V(T, S, A) = \max(S - e^{\frac{A}{T}}, 0), \tag{3.2}$$

or

$$V(T, S, A) = \max(e^{\frac{A}{T}} - E, 0). \tag{3.3}$$

Letting $x = \frac{\sqrt{2}}{\sigma}\log(S), y = \frac{\sqrt{2}}{\sigma}A$ and

$$u(x, y, t) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x + \left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2 t} V\left(T - t, e^{\frac{\sigma x}{\sqrt{2}}}, \frac{\sigma y}{\sqrt{2}}\right),$$

we see that the differential equation (3.1) is equivalent to

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \tag{3.4}$$

and that the *final* conditions (3.2) and (3.3) become the *initial* conditions

$$u(x, y, 0) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x} \max\left\{e^{\frac{\sigma x}{\sqrt{2}}} - e^{\frac{\sigma y}{\sqrt{2}T}}, 0\right\} \tag{3.5}$$

and

$$u(x, y, 0) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x} \max\left\{e^{\frac{\sigma y}{\sqrt{2}T}} - E, 0\right\}. \tag{3.6}$$

Equation (3.4) has been extensively studied in the literature. Let's recall some known results. Equation (3.4) was first considered by Kolmogorov (see Ref.20), then in the introduction of Ref.17 concerning hypoelliptic operators. See Ref.22 and its bibliography for an exhaustive survey of results about equations like (3.4).

In Ref.20 it is shown that the function Γ defined by

$$\Gamma(x, y, t; \xi, \eta, \tau) = \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)} - \frac{3}{(t-\tau)^3}\left(y - \eta - \frac{t-\tau}{2}(x+\xi)\right)^2\right) \tag{3.7}$$

if $t > \tau, \Gamma(x,y,t;\xi,\eta,\tau) = 0$ if $t \leq \tau$, is a fundamental solution of (3.4). We stress that Γ has some important properties that are distinctive of the heat kernel: for every fixed $(\xi,\eta,\tau) \in \mathbb{R}^3$, Γ is a C^∞ function in the variables $(x,y,t) \in \left(\mathbb{R}^3 \backslash \{(\xi,\eta,\tau)\}\right)$ and, for every $t > \tau$, it is an "anisotropic" Gauss kernel in the variables (x,y), in the sense that the quadratic form in the argument of the exponential function is strictly negative.

In order to simplify the notations, in the following we shall denote by L the differential operator

$$L \equiv \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial t} \tag{3.8}$$

and by $\mathcal{S}_T$ the set $\mathbb{R} \times \mathbb{R} \times]0,T[$. For a given function φ, that is continuous on $\mathbb{R}^2$, we shall say that u is a solution to the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } (x,y,t) \in \mathcal{S}_T, \\ u(x,y,0) = \varphi(x,y) & \text{in } (x,y) \in \mathbb{R}^2, \end{cases} \tag{3.9}$$

if the equation $Lu = 0$ is satisfied in every point of $\mathcal{S}_T$ and $u(x,y,t) \to \varphi(x_0,y_0)$ as $(x,y,t) \to (x_0,y_0,0)$, for every $(x_0,y_0) \in \mathbb{R}^2$. The following result holds (see Ref.26).

Theorem 3.1 *Let $\varphi \in C(\mathbb{R}^2)$, such that*

$$\int_{\mathbb{R}^2} e^{-c(x^2+y^2)}\varphi(x,y)\ dx\ dy < \infty \tag{3.10}$$

for some positive constant c. Then the function u defined as

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma(x,y,t;\xi,\eta,0)\varphi(\xi,\eta)\ d\xi\ d\eta \tag{3.11}$$

is a solution to the Cauchy problem (3.9). Moreover, if u and v are two solutions of (3.9) and satisfy the following condition

$$\int_{\mathcal{S}_T} e^{-c(x^2+y^2)}|u(x,y,t) - v(x,y,t)|\ dx\ dy\ dt < \infty,$$

for some constant c, then $u \equiv v$.

Remark 3.2 *Note that the condition* (3.10) *is satisfied whenever the payoff is a continuous function such that $0 \leq V(T,S,A) \leq S$. Theorem 3.1 ensures that, among all the solutions of* (3.9), *the function defined by* (3.11) *is the unique solution corresponding to a price V such that $0 \leq V(t,S,A) \leq S$. This condition is satisfied by most of the contingent claim contracts.*

Remark 3.3 *The results stated above can be generalized to the equation*

$$a(x,y,t)\frac{\partial^2 u}{\partial x^2} + b(x,y,t)\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \tag{3.12}$$

where a and b are bounded and Hölder continuous functions and $a(x,y,t) \geq a_0 > 0$ for every (x,y,t) (see Ref.25; of course in (3.11) *the function Γ has to be replaced* *by the fundamental solution of* (3.12)*). Thus we may consider equation* (3.1) *with coefficients* r *and* σ *depending on* t *and* S.

3.1. *Numerical approximation*

In order to provide a numerical approximation of the solution of (3.9), we extend some results obtained in Ref.27, where a boundary value problem for a bounded subset of $\mathbb{R}^3$ was considered. Let's define the "discrete" operator

$$L_G u\,(x,y,t) = -\frac{u(x,y,t) - u(x,y+\delta x,t-\delta)}{\delta} + \frac{u(x-h,y+\delta x,t-\delta) - 2u(x,y+\delta x,t-\delta) + u(x+h,y+\delta x,t-\delta)}{h^2},$$

that is an approximation of the operator L in the sense that

$$L_G u = Lu + (h^2+\delta)O(h,\delta)$$

(here $O(h,\delta)$ denotes a bounded function). The operator L_G is well defined in the grid

$$G = \Big\{(j\Delta_x, k\Delta_y, n\Delta_t) \in \mathbb{R}^3 \, : j,k,n \in \mathbb{Z}\Big\},$$

with $\Delta_x = h$, $\Delta_t = \delta$ and $\Delta_y = t\delta$ and the following approximation result holds.

Theorem 3.4 *Let* u *be a solution of (3.9),* u_G *be a solution of* $L_G u_G = 0$ *in* $G \cap \mathcal{S}_T, u_G = \varphi$ *in* $G \cap \{t=0\}$*. Then, for every* $\varepsilon_1, \varepsilon_2 > 0$ *and for every* H *compact subset of* $\mathcal{S}_T$ *there exists a grid* G*, verifying the following stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \le \frac{1}{2}, \tag{3.13}$$

such that

$$\max_{(x,y,t)\in G\cap H} |u(x,y,t) - u_G(x,y,t)| \le \varepsilon_1,$$

$$\max_{(x,y,t)\in G\cap H} \left| \frac{\partial u}{\partial x}(x,y,t) - \frac{u_G(x+\Delta_x,y,t) - u_G(x,y,t)}{\Delta_x} \right| \le \varepsilon_2.$$

Moreover $\varepsilon_1 = O\big((\Delta_x)^2\big)$ *and* $\varepsilon_2 = O\big(\Delta_x\big)$.

Remark 3.5 *The techniques employed in the proof can be easily adapted to the study of equation* (3.12)*. In that case the stability condition becomes*

$$\frac{\Delta_t}{(\Delta_x)^2} \le \frac{1}{2a_0}, \quad \big(\textit{where } a_0 = \sup a(x,y,t)\big).$$

It is convenient to assume that the coefficients a *and* b *are differentiable functions and that their derivatives are locally Hölder continuous.*

Proof of Theorem 3.4. The proof relies on the following fact: if w is a solution of the discrete Cauchy problem

$$\begin{cases} L_G w = g & \text{in } G \cap \mathcal{S}_T, \\ u = \psi & \text{on } G \cap \{t = 0\}, \end{cases} \tag{3.14}$$

G satisfies the stability condition (3.13) and g and ψ are bounded functions, then

$$\sup_{G \cap \mathcal{S}_T} |w| \leq \sup_{G \cap \{t=0\}} |\psi| + T \sup_{G \cap \mathcal{S}_T} |g|. \tag{3.15}$$

In order to prove the above inequality, we denote

$$w_{j,k}^n = w\big(j\Delta_x, k\Delta_y, n\Delta_t\big) \quad j, k, n \in \mathbb{Z},$$

and we observe that $L_G w = g$ in $G \cap \mathcal{S}_T$ is equivalent to

$$w_{j,k+j}^{n+1} = \frac{\Delta_t}{(\Delta_x)^2} w_{j-1,k}^n + \left(1 - 2\frac{\Delta_t}{(\Delta_x)^2}\right) w_{j,k}^n + \frac{\Delta_t}{(\Delta_x)^2} w_{j+1,k}^n + \Delta_t g_{j,k}^n,$$

for every $j, k, n \in \mathbb{Z}$ such that $0 \leq n\Delta_t \leq T$. Thus, since condition (3.13) is equivalent to $\left(1 - 2\frac{\Delta_t}{(\Delta_x)^2}\right) \geq 0$, we have

$$|w_{j,k+j}^{n+1}| \leq \frac{\Delta_t}{(\Delta_x)^2} |w_{j-1,k}^n| + \left(1 - 2\frac{\Delta_t}{(\Delta_x)^2}\right) |w_{j,k}^n| + \frac{\Delta_t}{(\Delta_x)^2} |w_{j+1,k}^n| + \Delta_t |g_{j,k}^n|,$$

then

$$\sup_{(j,k) \in \mathbb{Z}^2} |w_{j,k}^{n+1}| \leq \sup_{(j,k) \in \mathbb{Z}^2} |w_{j,k}^n| + \Delta_t \sup_{(j,k) \in \mathbb{Z}^2} |g_{j,k}^n|,$$

for every $n \in \mathbb{Z}$ such that $0 \leq n\Delta_t \leq T$ and the proof of (3.15) follows from an induction argument.

We next complete the proof of Theorem 3.4 by using inequality (3.15). We first show that it is sufficient to approximate the solution of a Cauchy problem corresponding to a continuous initial data $\widetilde{\varphi}$ with compact support, then we prove the convergence result assuming some further regularity on the solution, finally we remove these additional conditions.

Step 1 For every $R > 0$ we consider a function $\widetilde{\varphi} \in C(\mathbb{R}^2)$ such that $\widetilde{\varphi}(x, y) = \varphi(x, y)$ when $|(x, y)| \leq R$, $\widetilde{\varphi}(x, y) = 0$ when $|(x, y)| \geq 2R$. Let H be a compact subset of $\mathbb{R}^2 \times]0, \infty[$. For every $\varepsilon > 0$, there exists $R > 0$ such that, if $\widetilde{u}$ denotes the solution to the problem

$$\begin{cases} Lu = 0 & \text{in } \{t > 0\} \\ u = \widetilde{\varphi} & \text{on } \{t = 0\}, \end{cases} \tag{3.16}$$

then

$$\max_{z \in H} |u(z) - \widetilde{u}(z)| \leq \varepsilon/4. \tag{3.17}$$

The proof of this assertion is a consequence of formula (3.11) and of the fact that $|\varphi(\xi,\eta)| \le e^{c|(\xi,\eta)|}$, while

$$\Gamma(z;\xi,\eta,0) \le c_0 e^{-c_1|(\xi,\eta)|^2}$$

for every $z \in H$ and for $(\xi,\eta) \in \mathbb{R}^2, |(\xi,\eta)| \ge R$, (where c_0, c_1 and R are suitable positive constants. For a detailed proof of the last inequality see Lemma 3.1 in Ref.26).

Step 2 Suppose that

$$\sup_{\{t>0\}} \left|\frac{\partial^4 \widetilde{u}}{\partial x^4}\right| \le M, \quad \sup_{\{t>0\}} \left|\left(x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\right)^2 \widetilde{u}\right| \le M \tag{3.18}$$

for some $M \in \mathbb{R}$ and denote by $\widetilde{u}_G$ the solution to the discrete problem

$$\begin{cases} L_G v = 0 & \text{in } G \cap \{t>0\} \\ v = \widetilde{\varphi} & \text{in } G \cap \{t=0\}. \end{cases} \tag{3.19}$$

Then, since

$$L_G\widetilde{u} = L\widetilde{u} + \frac{\Delta_t}{2}\left(x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\right)^2 \widetilde{u}(x_1,y_1,t_1) + \frac{\Delta_x^2}{24}\frac{\partial^4 \widetilde{u}}{\partial x^4}(x_2,y_2,t_2) + \frac{\Delta_x^2}{24}\frac{\partial^4 \widetilde{u}}{\partial x^4}(x_3,y_3,t_3),$$

for every $(x,y,t) \in G \cap \{t>0\}$ and for suitable $(x_1,y_1,t_1),(x_2,y_2,t_2),(x_3,y_3,t_3) \in \{t>0\}$, we have

$$|L_G\widetilde{u}| \le \left(\frac{\Delta_t}{2} + \frac{\Delta_x^2}{12}\right) M.$$

Using this inequality in (3.15) we find that, if Δ_x and Δ_t satisfy the stability condition (3.13), then

$$|\widetilde{u}_G(x,y,t) - \widetilde{u}(x,y,t)| \le \frac{TM}{3}\Delta_x^2, \tag{3.20}$$

for any $(x,y,t) \in G \cap H$. As a consequence, if we choose Δ_x and Δ_t small enough, we obtain

$$\sup_{G\cap H} |\widetilde{u}_G - \widetilde{u}| \le \varepsilon/4.$$

Step 3 We next remove the additional hypothesis (3.18). First of all, we note that, as a consequence of the representation formula (3.11), the function $\widetilde{u}$ is uniformly continuous; then there exists $t_0 > 0$ such that

$$\sup_{\{t\ge 0\}} |\widetilde{u}(x,y,t) - \widetilde{u}(x,y,t+t_0)| \le \varepsilon/4. \tag{3.21}$$

Moreover, again by (3.11), we get the following estimates

$$\sup_{\{t\ge t_0\}} \left|\frac{\partial^4 \widetilde{u}}{\partial x^4}\right| \le c(t_0)\max|\widetilde{\varphi}|, \quad \sup_{\{t\ge t_0\}} \left|\left(x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\right)^2 \widetilde{u}\right| \le c(t_0)\max|\widetilde{\varphi}|. \tag{3.22}$$

Consider now the function $v(x,y,t) = \widetilde{u}(x,y,t+t_0)$: it is a solution to the problem

$$\begin{cases} Lv = 0 & \text{in } \{t > 0\} \\ v(x,y,0) = \widetilde{u}(x,y,t_0) & \text{for } (x,y) \in \mathbb{R}^2, \end{cases} \tag{3.23}$$

its derivatives are bounded, then, by Step 2,

$$\sup_{G\cap\{t\geq 0\}} |v_G - v| \leq \varepsilon/4, \tag{3.24}$$

where (obviously) v_G denotes the solution of the problem

$$\begin{cases} L_G v = 0 & \text{in } G \cap \{t > 0\}, \\ v(x,y,0) = \widetilde{u}(x,y,t_0) & \text{for } (x,y,0) \in G. \end{cases} \tag{3.25}$$

Again by (3.21) we have

$$\sup_{(x,y,0)\in G} |v_G(x,y,0) - \widetilde{u}_G(x,y,0)| \leq \frac{\varepsilon}{4},$$

hence, using (3.15), we obtain

$$\sup_{G\cap\{t\geq 0\}} |v_G - \widetilde{u}_G| \leq \frac{\varepsilon}{4}.$$

¿From this inequality, from (3.17), (3.24) and from (3.21) we obtain the first claim.

The second claim can be proved in the same manner. Using (3.17) and the representation formula (3.11) we readily obtain

$$\sup_{H\cap\{t\geq t_0\}} \left| \frac{\partial u}{\partial x} - \frac{\partial \widetilde{u}}{\partial x} \right| \leq \frac{\varepsilon}{4},$$

while from (3.20) it follows that

$$\left| \frac{\widetilde{u}(x+\Delta_x,y,t) - \widetilde{u}(x,y,t)}{\Delta_x} - \frac{\widetilde{u}_G(x+\Delta_x,y,t) - \widetilde{u}_G(x,y,t)}{\Delta_x} \right| \leq$$
$$\left| \frac{\widetilde{u}(x,y,t) - \widetilde{u}_G(x,y,t)}{\Delta_x} \right| + \left| \frac{\widetilde{u}(x+\Delta_x,y,t) - \widetilde{u}_G(x+\Delta_x,y,t)}{\Delta_x} \right| \leq \frac{2Mt}{3}\Delta_x.$$

Arguing as above, we get from this inequality

$$\sup_{(x,y,t)\in H\cap\{t\geq t_0\}} \left| \frac{\partial u}{\partial x}(x,y,t) - \frac{u_G(x+\Delta_x,y,t) - u_G(x,y,t)}{\Delta_x} \right| \leq \frac{\varepsilon}{2} + \widetilde{M}\Delta_x,$$

with $\widetilde{M}$ depending on M and on H and this completes the proof of Theorem 3.4 $\square$.

4. Arithmetic Average Options

Assuming that the underlying asset price follows a lognormal stochastic process, i.e., $\mu_s = \mu S$ and $\sigma_s = \sigma S$, then the Cauchy problems for the price of the Arithmetic average floating strike call option and of the Arithmetic average fixed strike call option become

$$-rV+\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+S\frac{\partial V}{\partial A}=0 \tag{4.1}$$

with

$$V(T,S,A)=\max\left(S-\frac{A}{T},0\right), \tag{4.2}$$

or

$$V(T,S,A)=\max\left(\frac{A}{T}-E,0\right). \tag{4.3}$$

We first simplify equation (4.1) by a plain change of function: letting

$$u(x,y,t)=x^m e^{qt}V\left(T-\frac{2t}{\sigma^2},x,\frac{2y}{\sigma^2}\right),\quad m=\frac{r}{\sigma^2},\ q=m^2+m, \tag{4.4}$$

the differential equation (4.1) is transformed into

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t}. \tag{4.5}$$

Because of the coefficient x^2, the above differential equation is characterized by a difficulty that does not appear in the Kolmogorov equation (3.4). Indeed, as stated in Remark 3.3, we are able to deal with continuous coefficient that are bounded and bounded away from zero functions, but x^2 does not satisfy such assumption. Actually, this kind of degeneracy also appears in equation (3.1), but the change of variable $x=\log(S)$ allows us to avoid this problem. If we apply the same idea to (4.1), we transform it into a PDE like

$$\frac{\partial^2 u}{\partial x^2}+e^x\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t}, \tag{4.6}$$

and the difficulty is shifted on the coefficient of the derivative $\frac{\partial u}{\partial y}$.

It seems us that equation (4.5) is more tractable than (4.6), then we study our Cauchy problem transformed by (4.4), namely

$$\begin{cases} x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t} & \text{for } (x,y,t)\in\Omega_T \\ u(x,y,0)=\varphi(x,y) & \text{for } (x,y)\in\mathbb{R}^+\times\mathbb{R}, \end{cases} \tag{4.7}$$

where $\Omega_T=\mathbb{R}^+\times\mathbb{R}\times]0,\frac{\sigma^2}{2}T[$ and

$$\varphi(x,y)=x^m\max\left(x-\frac{2|y|}{\sigma^2 T},0\right) \tag{4.8}$$

in the case (4.2) (we are interested in positive values of y),

$$\varphi(x,y)=x^m\max\left(\frac{2y}{\sigma^2 T}-E,0\right) \tag{4.9}$$

in the case (4.3).

We prove an existence result for the problem (4.7) by using an approach proposed by Gleit in Ref.16, for the non-uniformly parabolic equation in two variables

$$x^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{4.10}$$

that consists in solving a sequence of Cauchy problems in suitable bounded sets, that fill the half-plane, and in passing to the limit.

In order to state our existence result, we introduce the notion of super- and sub-solution. We shall say that u is *supersolution* to the problem (4.7) if

$$\begin{cases} x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} \leq \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_T \\ u(x,y,0) \geq \varphi(x,y) & \text{for } (x,y) \in \mathbb{R}^+ \times \mathbb{R} \end{cases} \tag{4.11}$$

and that u is *subsolution* to (4.7) if the condition (4.11) holds with the reverse inequalities.

Theorem 4.6 *Let $\varphi \in C(\mathbb{R}^+ \times \mathbb{R}), \overline{u}, \underline{u}$ be supersolution and subsolution, respectively, to problem (4.7) and suppose that $\underline{u} \leq \overline{u}$ in Ω_T. Then there exists a solution u to the problem (4.7) such that*

$$\underline{u} \leq u \leq \overline{u} \quad in\ \Omega_T. \tag{4.12}$$

About the condition $\underline{u} \leq \overline{u}$ assumed in the previous Theorem, we note that it is not trivially satisfied, unless we make some condition on the growth of $\underline{u}$ and $\overline{u}$. We will come to this question and related problems in Subsection 4.1 (see Theorem 4.13 and Proposition 4.15).

Theorem 4.6 transforms the Cauchy problem (4.7) into the problem of finding a subsolution and a supersolution, then it can be seen as an *equivalence* result. When we consider a Cauchy problem (4.7) corresponding to an Asian option, it is not very difficult to find a super- and a sub-solution, since we expect that the value of the option V is positive and lesser than the price of the stock S. Following this hint, we next provide a couple of super- and sub-solutions for the initial data φ defined in (4.8) and in (4.9).

Remark 4.7 *Theorem 4.6 ensures that there exists a solution to the Cauchy problem (4.7), corresponding to the initial condition (4.8), provided that we choose $\underline{u} \equiv 0$ and $\overline{u}(x,y,t) = x^{m+1}e^{qt}$. It is sufficient to check that $\underline{u}(x,y,0) \leq \varphi(x,y) \leq \overline{u}(x,y,0)$ for every $(x,t) \in \mathbb{R}^+ \times \mathbb{R}$.*

Analogously, when we consider the initial condition (4.9), we may choose $\underline{u} \equiv 0$ and $\overline{u}(x,y,t) = \frac{\sigma^2}{2T}e^{kt}x^m\sqrt{x^2+y^2}$, $k = m^2 + 2m + 2$.

In the sequel we will state some results which ensure that the solutions above considered are unique (see Remark 4.14).

The proof of Theorem 4.6 is postponed in Subsection 4.1. Subsection 4.1 also contains a result by Aronson and Besala concerning the uniqueness of the solution to (4.7) (see Theorem 4.13 below). We also show in Proposition 4.15 that the hypotheses of Theorem 4.13 can be appreciably weakened, when we consider the differential equation (4.5). In Subsection 4.2 we adapt the finite difference method proposed in Section 3 to the solution of (4.7).

We conclude this Section with a remark due to Ingersoll in Ref.18, that in some cases allows us to reduce the Cauchy problem (4.7) in another form, for which the PDE is a nondegenerate parabolic equation in only one space variable.

Suppose that the final condition is a homogeneous function of degree 1 in the variables S, A:

$$V(T, S, A) = S \cdot V\left(T, 1, \frac{A}{S}\right)$$

(this condition is satisfied, for example, when we consider the initial data (4.2), but not for (4.3)). Denote by W the solution to the Cauchy problem

$$\begin{cases} x^2 \frac{\partial^2 W}{\partial x^2} + (1-x)\frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} = 0 & \text{for } (x,t) \in \mathbb{R}^+ \times]0, T[, \\ W(T, x) = V(T, 1, x), & \text{for } x > 0, \end{cases} \tag{4.13}$$

then the function $V(t, S, A) = S \cdot W\left(t, \frac{A}{S}\right)$ is a solution to the problem corresponding to the PDE (4.1).

4.1. *The Cauchy problem*

As previously declared, we prove Theorem 4.6 by solving a sequence of boundary value problems in suitable bounded sets.

Note that the differential operator in (4.5) can be seen as a second order operator with non-negative characteristic form of the type

$$L \equiv \sum_{i,j=1}^{n} \partial_{x_i}\left(a_{i,j}(x,t)\partial_{x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\partial_{x_i}, \quad \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq 0 \quad \forall \xi \in \mathbb{R}^n. \tag{4.14}$$

We recall that a general theory for the boundary value problem for operators of this kind was proposed by Fichera in Ref.14, where the problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega, \end{cases} \tag{4.15}$$

was considered for a domain $\Omega \subset \mathbb{R}^n$ such that its boundary (that Fichera denotes by Σ) is a finite union of C^1 surfaces. Fichera pointed out that the problem (4.15) is well posed provided that the boundary condition is prescribed in the set $\Sigma_2 \cup \Sigma_3$, where Σ_3 is the *noncharacteristic* part of Σ (*i.e.* $\sum_{i,j=1}^{n} a_{i,j}\nu_i\nu_j > 0$, where ν is the inward normal to Σ) and Σ_2 is the subset of $\Sigma \backslash \Sigma_3$ where the first order part of the operator *points inside* Ω (more precisely: $\sum_{i=1}^{n} b_i\nu_i < 0$). We refer to Ref.23 for a general theory concerning differential operators (4.14).

The boundary value problem (4.15) for Hörmander type operators was also studied by Bony in Ref.6, in the framework of the potential theory. A framework that allows to consider arbitrary open set, without any regularity assumption on the boundary Σ (see Ref.9 for a complete treatment of the subject). The potential theory gives a unique solution to (4.15) that, in the sequel, will be said "generalized solution". Such a solution is a $C^\infty(\Omega)$ function, but it is not *a priori* clear in what part of $\partial\Omega$ it takes the prescribed value. According to the usual terminology, we say that a point $z_0 \in \partial\Omega$ is a *regular* point if, for every given $f \in C(\partial\Omega)$, the generalized solution u to the problem (4.15) satisfies $u(z) \to f(z_0)$ as $z \to z_0$ $(z \in \Omega)$.

In Ref.24 Manfredini gave some sufficient and some necessary geometric conditions for the regularity of $\partial\Omega$ when the operator L is the one defined in (4.5). We next recall some results in Ref.24 which will be useful in the sequel.

Theorem 4.8 (Theorem 1.5 in Ref.24) *Let Ω be a bounded open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \{x \neq 0\}$ let $f \in C(\partial\Omega)$ and let L be the operator defined in* (4.5). *Then the problem* (4.15), *has a generalized solution.*

Note that the hypothesis $\overline{\Omega} \subset \{x \neq 0\}$ does not explicitly appear in Ref.24, but it is assumed the condition (that actually is equivalent) that the coefficient of the derivative $\frac{\partial^2}{\partial x^2}$ is bigger than some positive constant.

Proposition 4.9 (Theorem 1.4 in Ref.24) *Let Ω be a bounded open set such that $\overline{\Omega} \subset \{x \neq 0\}$ and let u be a solution of $Lu = 0$ in Ω, where L is the operator defined in* (4.5). *Then, for every open set Ω' such that $\overline{\Omega}' \subset \Omega$ there exist two positive constants c and α, that do not depend on u, such that*

$$|u(z) - u(w)| \leq c|z - w|^\alpha \sup_\Omega |u|,$$

for every $z, w \in \Omega'$.

We next recall a sufficient condition for the regularity of the boundary for the problem (4.15) (see Theorems 6.1 and 6.3 in Ref.24). In that order we say that the vector $\nu \in \mathbb{R}^3$ is an *outer normal vector to the open set Ω at the point $z_0 \in \partial\Omega$* if $B_{|\nu|}(z_0 + \nu) \subset \mathbb{R}^3 \backslash \Omega$ (here $B_r(z)$ denotes the Euclidean ball with center z and radius r).

Proposition 4.10 *Let Ω be an open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \{x \neq 0\}$ and let $(x_0, y_0, t_0) \in \partial\Omega$. Consider the boundary value problem* (4.15) *where L is the operator defined in* (4.5). *If there exists an outer normal vector $\nu = (\nu_x, \nu_y, \nu_t)$ such that one of the following conditions is satisfied:*

i) $\nu_x \neq 0$,

ii) $\nu_x = 0$, *but* $x_0\nu_y - \nu_t > 0$ *and there exists a positive constant δ such that* $x_0^2\delta^2 \leq x_0\nu_y - \nu_t$ *and that*

$$\Big\{(x,y,t) \in \mathbb{R}^3 : \delta^2(x-x_0)^2 + (y-y_0-\delta^2\nu_y)^2 + (t-t_0-\delta^2\nu_t)^2 \leq \delta^4\Big\} \subset \mathbb{R}^3\backslash\Omega,$$

then (x_0, y_0, t_0) is a regular point.

Remark 4.11 *When $\partial\Omega$ is a C^1 surface, in a neighborhood of (x_0, y_0, t_0), then condition (i) is equivalent to $(x_0, y_0, t_0) \in \Sigma_3$, according to the definition by Fichera. Condition (ii) implies $(x_0, y_0, t_0) \in \Sigma_2$.*

We are now ready to prove Theorem 4.6.

Proof of Theorem 4.6. We first define a sequence of bounded open sets (Ω_k) that fill Ω_T and consider a suitable boundary value problem on every Ω_k. We choose the sets Ω_k to easily check the regularity of their boundary points. For this purpose it is not important that $\partial\Omega_k$ is a smooth surface, then we consider a sequence of boxes.

For any $k \in \mathbb{N}$ we set $\Omega_k =]1/(k+1), k+1[\times] - (k+1), k+1[\times]0, T[$ and we consider a continuous function $\chi_k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, such that

$$\begin{aligned} \chi_k(x,y) &= 0 \quad \forall\, (x,y) \notin]1/(k+1), k+1[\times] - (k+1), k+1[, \\ \chi_k(x,y) &= 1 \quad \forall\, (x,y) \in]1/k, k[\times] - k, k[, \end{aligned}$$

and $0 \le \chi_k(x,y) \le 1$ otherwise. Finally, we define

$$\varphi_k(x,y,t) = \chi_k(x,y)\varphi(x,y) + \big(1 - \chi_k(x,y)\big)\underline{u}(x,y,t) \tag{4.16}$$

in $\partial\Omega$. By Theorem 4.8 there exists a generalized solution u_k to the boundary value problem

$$\begin{cases} Lu = 0 & \text{for } (x,y,t) \in \Omega_k \\ u = \varphi_k & \text{for } (x,y,t) \in \partial\Omega_k. \end{cases} \tag{4.17}$$

Moreover Proposition 4.10 ensures that every point in the set

$$\begin{aligned} &\big\{(x,y,t) \in \partial\Omega_k : t = 0\big\} \cup \big\{(x,y,t) \in \partial\Omega_k : x = 1/(k+1)\big\} \cup \\ &\qquad \big\{(x,y,t) \in \partial\Omega_k : x = k+1\big\} \cup \big\{(x,y,t) \in \partial\Omega_k : y = k+1\big\} \end{aligned}$$

is regular. Besides, from the maximum principle (see for example Ref.6, Proposition 3.1) it follows that

$$\underline{u} \le u_k \le \overline{u} \quad \text{in} \quad \Omega_k. \tag{4.18}$$

We can now build a solution to (4.7) by using the sequence $\big(u_k\big)_{k\in\mathbb{N}}$. For every $k \in \mathbb{N}$ we let

$$A_k = \big\{(x,y,t) \in \Omega_k : T/(3k) < t < T\big(1 - 1/(3k)\big)\big\}$$

and we note that

$$\Omega_T = \bigcup_{k\in\mathbb{N}} A_k. \tag{4.19}$$

¿From (4.18) it follows that the sequence $\big(u_k\big)_{k\ge 2}$ is bounded in $\overline{A}_1$ and, from Proposition 4.9, we see that it is also equicontinuous, thus there exists a subsequence $\big(u_{k_{1,j}}\big)_{j\in\mathbb{N}}$ that uniformly converges to some function $v_1 \in C(\overline{A}_1)$. We note that, as a consequence, v_1 is a *weak* solution to $Lu=0$ in A_1 (then, by Hörmander's results Ref.17, it is also a $C^\infty(A_1)$ function and classical solution to $Lu=0$) moreover

$$\underline{u} \le v_1 \le \overline{u} \quad \text{in} \quad A_1.$$

We next apply the same argument to the sequence $\left(u_{k_{1,j}}\right)_{j\in\mathbb{N}}$ on the set $\overline{A}_2$ and we obtain a subsequence $\left(u_{k_{2,j}}\right)_{j\in\mathbb{N}}$ that converges in $C(\overline{A}_2)$ to some function v_2, that is a classical solution to $Lu=0$ and such that $\underline{u} \le v_2 \le \overline{u}$ in A_2. Note that, since v_2 is the limit of a subsequence of $\left(u_{k_{1,j}}\right)_{j\in\mathbb{N}}$, it must coincide with v_1 in A_1.

We next proceed by induction: for every $m \in \mathbb{N}$ we consider the sequence $\left(u_{k_{m-1,j}}\right)_{j\in\mathbb{N}}$ on the set $\overline{A}_m$ and we extract from it a subsequence $\left(u_{k_{m,j}}\right)_{j\in\mathbb{N}}$ converging in $C(\overline{A}_m)$ to some function v_m, classical solution of $Lu=0$, such that $\underline{u} \le v_m \le \overline{u}$ in A_m and that it equals v_{m-1} on the set A_{m-1}.

We then define a function u in the following way: for every $(x,y,t) \in \Omega_T$ we choose $m \in \mathbb{N}$ such that $(x,y,t) \in A_m$ and we set $u(x,y,t) = v_m(x,y,t)$. Note that the definition is well posed, since, if $(x,y,t) \in A_n$, then $v_n(x,y,t) = v_m(x,y,t)$. Moreover u is a classical solution to $Lu=0$ in Ω_T and satisfies $\underline{u} \le u \le \overline{u}$ in Ω_T.

To conclude the proof of Theorem 4.6 we have to verify that, for any $(x_0,y_0) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$\lim_{(x,y,t)\to(x_0,y_0,0)} u(x,y,t) = \varphi(x_0,y_0). \tag{4.20}$$

We already observed that, if $k \in \mathbb{N}$ is such that $(x_0,y_0,0) \in \partial\Omega_k$, then Proposition 4.10 ensures that every solution u_k assumes the boundary data φ in $(x_0,y_0,0)$. This fact is not sufficient to guarantee that also the function u has the same property, nevertheless the proof of Proposition 4.10 is based on the use of "barriers", that give an estimate of the rate of convergence as (x,y,t) goes to $(x_0,y_0,0)$, that is *uniform* with respect to k. From such uniform estimate we can get (4.20) □.

Remark 4.12 *As a consequence of* (4.18) *we find that* $u_{k+1} \ge \varphi_k$ *at every regular point of* $\partial\Omega_k$, *thus* $u_{k+1} \ge u_k$ *in* Ω_k. *In other words, the sequence* $\left(u_k\right)$ *is increasing.*

If we define

$$\psi_k(x,y,t) = \chi_k(x,y)\varphi(x,y) + \big(1-\chi_k(x,y)\big)\overline{u}(x,y,t), \tag{4.21}$$

and we consider the boundary value problem

$$\begin{cases} Lv = 0 & \text{for } (x,y,t) \in \Omega_k \\ v = \psi_k & \text{for } (x,y,t) \in \partial\Omega_k. \end{cases} \tag{4.22}$$

we find a decreasing sequence $\left(v_k\right)$ *that converges, uniformly on compact sets, to a function* v *that is a solution to* (4.7), *satisfying* $\underline{u} \le v \le \overline{u}$ *in* Ω.

As we will see lather, under some conditions there is only a solution to (4.7) *such that* $\underline{u} \le u \le \overline{u}$. *In this case both sequences* $\left(u_k\right)$ *and* $\left(v_k\right)$ *converge to* u.

Aiming to deal with the uniqueness of the solution to the Cauchy problem for operators of the form (4.14), we first recall a result by Aronson and Besala contained in Theorem B of Ref. 2.

Theorem 4.13 (Theorem B in Ref.2) *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary unbounded open domain and let L be the differential operator*

$$L \equiv \sum_{i,j=1}^{n} a_{i,j}(x,t)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b_i(x,t)\partial_{x_i} + c(x,t) - \partial_t,$$

in the strip $S = \mathbb{R}^n \times [0,T]$. Suppose that the coefficients of L satisfy

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq 0, \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0,T],\ \xi \in \mathbb{R}^n,$$

and

$$|a_{i,j}| \leq A(|x|^2+1), \quad |b_i| \leq B(|x|^2+1)^{1/2}, \quad |c| \leq C$$

in $\overline{\Omega} \times [0,T]$ for some positive constant A, B and C. If u is a regular solution of $Lu \leq 0$ in $\Omega \times (0,T]$ such that

$$u(x,t) \geq 0, \quad \textit{for} \quad (x,t) \in \left\{\partial\Omega \times [0,T]\right\} \cup \left\{\Omega \times \{t=0\}\right\}$$

and

$$u(x,t) \geq -M \exp\left\{k \log(|x|^2+1)+1\right\}^2$$

in $\Omega \times (0,T]$ for some positive constant M and k, then $u(x,t) \geq 0$ in $\overline{\Omega} \times [0,T]$.

Remark 4.14 *The above result ensures that each solution u to the Cauchy problems considered in Remark 4.7 is the* unique *solution that satisfies (4.12). Indeed, in both cases, we have $\underline{u}(0,y,t) = 0$ and $\overline{u}(0,y,t) = 0$, then we implicitly assumed that $u(0,y,t) = 0$ for every $(y,t) \in \mathbb{R} \times [0,T]$.*

In the particular case of the operator (4.5), the assumption $u(0,y,t) = 0$ considered above is not necessary. We next state a comparison result for the problem (4.7) that improves Theorem 4.13 in that sense. We first define the following function in $\mathbb{R}^+ \times \mathbb{R}$

$$\psi(x,y) = \log\left(x^2+y^2+1\right) - \log(x) \tag{4.23}$$

and we note that

$$\psi(x,y) \xrightarrow[\|(x,y)\|\to\infty]{} \infty, \quad \psi(x,y) \xrightarrow[(x,y)\to(0,y_0)]{} \infty, \tag{4.24}$$

for every $y_0 \in \mathbb{R}$.

Proposition 4.15 *Let $u \in C(\overline{\Omega}_T)$, satisfy*

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} \leq \frac{\partial u}{\partial t}$$

in Ω_T*, and* $u(x,y,0) \geq 0$ *for every* $(x,y) \in \mathbb{R}^+ \times \mathbb{R}$ *and assume that there exist two positive constants* M *and* k *such that*

$$u(x,y,t) \geq -Me^{k\psi^2(x,y)},$$

for every $(x,y,t) \in \Omega_T$ *(*ψ *denotes the function defined in (4.23)). Then* $u \geq 0$ *in* Ω_T*.*

We do not give the proof of the above statement since it follows the same lines of the proof of Theorem 4.13, making use of the function defined in (4.23) instead of the one proposed in Ref.2.

4.2. *Numerical approximation*

In this Section we adapt the approximation method employed in the study of geometric average options to the Cauchy problem (4.7). The main technical difficulties are due to the lack of suitable a priori estimates near $x = 0$ and as $x \to \infty$ for the derivatives of the solution. In order to avoid such difficulties, it is convenient to approximate one of the functions u_k defined in (4.17). Since $u_k \to u$, uniformly on every compact set, by approximating u_k we also approximate u.

Note that, since the convergence of (u_k) relies on a compactness argument, it does not give an explicit dependence of the approximation error ε in terms of k. To bypass this problem, we exploit the monotonicity of the sequence (u_k) and (v_k) defined in (4.22).

Theorem 4.16 *Assume that* $\varphi, \underline{u}, \overline{u}$ *satisfy the hypotheses of Theorem 4.6 and suppose that the solution* u *of* (4.7) *given by Theorem 4.6 is the unique solution such that* $\underline{u} \leq u \leq \overline{u}$*. For a given* $k \in \mathbb{N}$ *let* u_k *and* v_k *be the solution of* (4.17) *and* (4.22)*, respectively. Then for every* $\varepsilon > 0$ *there exists a grid* G*, satisfying the stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2(k+1)^2},$$

such that

$$(u_k)_G - \varepsilon \leq u_k \leq u \leq v_k \leq (v_k)_G + \varepsilon \quad \text{in } G \cap \Omega_k.$$

Proof. We only need to approximate u_k and v_k, then the result is a trivial consequence of Remark 4.12 □.

As noticed above, we are not able to choose *a priori* u_k and v_k in order to have a satisfactory approximation of u in terms of $(u_k)_G$ and $(v_k)_G$, but the monotonicity of (u_k) and (v_k) allows us to establish *a posteriori* the accuracy of the approximation.

5. The General Case

Let us consider now the Cauchy problem associated with the partial differential equation (2.4) in its general form. In this context there is no hope to prove the regularity properties of the solution that allow us to demonstrate the approximation results obtained above. In fact, since the matrix of the coefficients of the second derivatives in (2.4)

$$\frac{1}{2}\begin{pmatrix} \sigma_a^2 & \sigma_a\sigma_s \\ \sigma_a\sigma_s & \sigma_s^2 \end{pmatrix} \tag{5.1}$$

is positive semidefinite with an eigenvalue equal to zero, it may be possible to convert the PDE into another nondegenerate PDE in a submanifold. In that case the Cauchy problem should be studied with different techniques. As observed in Ref.5, it is possible to characterize such equation through Lie Algebra. To this end we rewrite equation (2.4) as follows

$$Lu = \frac{1}{2}X^2u + Yu, \tag{5.2}$$

where X and Y are the following directional derivatives:

$$X = \sigma_a\frac{\partial}{\partial x} + \sigma_s\frac{\partial}{\partial y}, \quad Y = \widetilde{\mu}_a\frac{\partial}{\partial x} + \widetilde{\mu}_s\frac{\partial}{\partial y} - \frac{\partial}{\partial t}, \tag{5.3}$$

($\widetilde{\mu}_a$ e $\widetilde{\mu}_s$ are coefficients that depend on $\sigma_a, \sigma_s, \mu_a$ e μ_s).

We are now able to recognize equations which admit regular solutions. In fact, when the commutator $[X,Y](= XY - YX)$ is zero at every point of some open subset of $\mathbb{R}^3$, then it follows from the Frobenius theorem that, after a change of coordinates, the operator only acts on two variables, while the last one may be considered as a constraint. In that case the solutions of (5.2) are not necessarily smooth; on the other hand, when the change of variables is performed, the differential equation becomes nondegenerate and the classical numerical methods can be applied.

If otherwise the commutator $[X,Y]$ never vanishes, by the results in Ref.17, every solution of (5.2) is a C^∞ function. In that case it seems possible to employ the numerical techniques used above. Among operators of this kind there are the ones corresponding to *Path Dependent Options in a Constant Elasticity Variance Environment* which will be considered in next subsection.

5.1. *Path Dependent Options in a Constant Elasticity Variance Environment*

In the above sections we have considered a geometrical brownian motion for the underlying asset price. This type of diffusion process is heavily employed in mathematical finance because it allows for a great tractability. The main drawback is that observed data are not well described by this process. Starting from this observation in Ref. 10 and 11 the following diffusion process for the underlying asset price was proposed:

$$dS = \mu S dt + \sigma S^{\varrho} dz,$$

with $0 \leq \varrho \leq 1$. In this setting the Cauchy problem (2.4)-(2.5) becomes

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^\alpha \frac{\partial^2 V}{\partial S^2} + \mu_a\frac{\partial V}{\partial A} = 0, \tag{5.4}$$

with

$$V(T, S_T, A_T) = \Omega(S_T, A_T), \tag{5.5}$$

where $\alpha = 2\varrho$ and $\mu_a = S$ (arithmetic average) or $\mu_a = \log(S)$ (geometrc average).

The technique used in the study of the problem corresponding to the arithmetic average options can be employed in the study of the Cauchy problem (5.4)-(5.5).

Theorem 5.17 *Let $\varphi \in C(\mathbb{R}^+ \times \mathbb{R}), a, b$ and c be locally Hölder continuous functions in $\Omega_T(\equiv \mathbb{R}^+ \times \mathbb{R}\times]0,T[)$, with $a > 0$. Moreover, let $\overline{u}, \underline{u}$ be, respectively, supersolution and subsolution to the problem*

$$\begin{cases} a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + cu = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_T, \\ u(x,y,0) = \varphi(x,y) & \text{for } (x,y) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{5.6}$$

such that $\underline{u} \leq \overline{u}$ in Ω_T. Then there exists a solution u to (5.6) such that

$$\underline{u} \leq u \leq \overline{u} \quad \text{in } \Omega_T. \tag{5.7}$$

In order to state our numerical result we consider, as in the Proof of Theorem 4.6, $\Omega_k(=]1/(k+1), k+1[\times]-(k+1), k+1[\times]0,T[)$ and the functions φ_k, ψ_k defined in (4.16) and (4.21), for $k \in \mathbb{N}$. Let u_k be the solution to the boundary value problem

$$\begin{cases} a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + cu = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_k, \\ u = \varphi_k & \text{for } (x,y,t) \in \partial\Omega_k, \end{cases} \tag{5.8}$$

and v_k be the solution of

$$\begin{cases} a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + cu = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_k, \\ u = \psi_k & \text{for } (x,y,t) \in \partial\Omega_k. \end{cases} \tag{5.9}$$

Theorem 5.18 *Let u be a solution to (5.6), u_k and v_k be the solutions to* (5.8) *and* (5.9)*, respectively. Then, for every $\varepsilon > 0$ there exists a grid G, satisfying the stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2a_0}, \quad \big(\text{where } a_0 = \sup a(x,y,t)\big), \tag{5.10}$$

such that

$$\big(u_k\big)_G - \varepsilon \leq u_k \leq u \leq v_k \leq \big(v_k\big)_G + \varepsilon \quad \text{in } G \cap \Omega_k.$$

Remark 5.19 *When $\mu_a(S) = S$, we can write the Cauchy problem corresponding to Path Dependent Options in a Constant Elasticity Variance Environment in the form* (5.6) *by choosing the coefficients $a = \frac{1}{2}\sigma^2 x^\alpha, b = xr, c = -r$. In the case $\mu_a(S) = \log(S)$, in order to rewrite equation* (5.4) *in the form (5.6), it is convenient to perform the chenge of variables $x = \log(S)$.*

We do not give the proof of Theorems 5.17 and 5.18, since it is essentially the same as the proof of Theorems 4.6 and 4.16, respectively.

6. Conclusions

In this paper we have analyzed the Cauchy problem associated with the no arbitrage price of an Asian option. We have shown that the solution of the Cauchy problem and therefore the no arbitrage price of an Asian contingent claim has enough regularity so that classical finite difference methods can be applied to approximate the solution. An estimate of the approximation error is obtained.

These results are useful in an applied perspective because a closed form solution for the no arbitrage price of Asian options is not available in general. Numerical methods aiming at approximating the Cauchy problem solution have been employed in the mathematical finance literature without an adequate analysis of their convergence and of their approximation error. This is the void filled in our paper.

Acknowledgments

The first author was supported by MURST and University of Pisa research funds. The second author was supported by the University of Bologna: fund for selected research topics. We acknowledge a referee for comments and suggestions useful to improve the paper. The usual discliamers apply.

References

1. B. Alziary, J. P. Décamps, P.F. Koehl, *A P.D.E. approach to Asian options: analytical and numerical evidence*, Journal of Banking anf Finance **21** (1997), 613-640.
2. D. G. Aronson, P. Besala, *Uniqueness of positive solution of parabolic equations with unbounded coefficients,* Colloquium Mathematicum **18** (1967), 125–135.
3. G. Barles, *Convergence of Numerical schemes for degenerate parabolic equations arising in finance theory,* **Numerical Methods in Finance**, L. Rogers and Talay, D. eds., (Cambridge University Press, 1996).
4. G. Barles, Ch. Daher, M. Romano, *Convergence of numerical schemes for problems arising in finance theory,* Math. Models and Meth. in Appl. Sciences, **5** (1995), 125–143.
5. J. Barraquand, T. Pudet, *Pricing of American Path-Dependent Contingent Claims*, Mathematical Finance (1996), 17-51.
6. J. -M. Bony, *Principe de maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés,* Ann. Inst. Fourier, Grenoble **19,1** (1969), 277–304.
7. L. Bouaziz, E. Briys and M. M. Crouhy, *The pricing of forward-starting asian options*, Journal of Banking anf Finance **18** (1994), 823-839.
8. A. Caverhill and L. Clewlow, *valuing average rate Asian options*, RISK **3**, 25-29.
9. C. Constantinescu, A. Cornea, **Potential theory on harmonic spaces,** (Springer-Verlag, Berlin, 1972).
10. J. Cox, S. Ross, *The valuation of options for alternative stochastic processes,* Journal of Financial Economics, 3 (1976), 145-166.

11. J. Cox, M. Rubinstein, *Options Markets,* (Englewood Cliffs 1985).
12. J. Dewynne, P. Wilmott, *Asian options as linear complementarity problems*, Advances in Futures and Options Research, **8** (1995), 145-173.
13. A. Eydeland and H. Geman, *Asian options revisited: inverting the Laplace transform*, RISK, 1995.
14. G. Fichera, *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine,* Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1956), 1–30.
15. H. Geman and M. Yor, *Bessel Processes, Asian Options, and Perpetuities*, Mathematical Finance **3** (1993), 349-375.
16. A. Gleit, *Valuation of General Contingent Claims*, Journal of Financial Economics **6** (1978), 71-87.
17. L. Hormander, *Hypoelliptic second order differential equations,* Acta Math. **119** (1967), 147–171.
18. J. Ingersoll, **Theory of Financial Decision Making** (Rowman & Littlefield 1987).
19. A. Kenna and A. Vorst, *A pricing method for options based on average asset values*, Journal of Banking and Finance **14** (1990) 113-129.
20. A. N. Kolmogorov, *Zur theorie der steigen zufälligen prozesse,* Math. Ann. **108** (1933) 149-160.
21. E. Levy, *Pricing Europan average rate currency options*, Journal of International Money and Finance, **11** (1990), 474-491.
22. E. Lanconelli, S. Polidoro, *On a class of hypoelliptic evolution operators,* Rend. Sem. Mat. Univ. Pol. Torino **52,1** (1994), 29–63.
23. O. A. Oleĭnik, E. V. Radkevič, **Second order equations with nonnegative characteristic form,** (American Mathematical Society, Providence, Rhode Island Plenum Press, New York-London).
24. M. Manfredini, *The Dirichlet problem for a class of ultraparabolic equations,* Adv. Differential Equations, **2,5** (1997), 831–866.
25. S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type,* Le Matematiche, **49** (1994), 53–105.
26. S. Polidoro, *Uniqueness and representation theorems for solutions of Kolmogorov-Fokker-Planck equations,* Rendiconti di Matematica, VII, **15** (1995), 535–560.
27. S. Polidoro, C. Mogavero, *A finite difference method for a boundary value problem related to the Kolmogorov equation,* Calcolo **32,3-4** (1995), 193–205.
28. C. Rogers and Z. Shi, *The Value of an Asian Option*, Journal of Applied Probability **32** (1995), 1077-1088.
29. M. Rubinstein, **Exotic Options**, (University of California at Berkeley 1991).
30. A. Ruttiens, *Average-rate Options, Classical Replica*, RISK, **3** (1990), 33-36.
31. S. Turnbull and L. Wakeman, *A Quick Algorithm for Pricing European Average Options* Journal of Financial and Quantitative Analysis **26** (1991), 377-389.
32. T. Vorst, *Analytical boundaries and approximations of the prices and hedge ratios of average exchange rate options*, (working paper, 1993), Erasmus University, Rotterdam.
33. P. Wilmott, J. Dewynne and S. Howison, **Option Pricing**, (Oxford Financial Press),1993.