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Traces of Sobolev spaces to irregular subsets of metric measure spaces

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Abstract. Given $p \in (1, \infty)$, let $(\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space with uniformly locally doubling measure μ supporting a weak local $(1, p)$-Poincaré inequality. For each $\theta \in [0, p)$ we characterize the trace space of the Sobolev $W_p^1(\mathrm{X})$-space to lower θ-codimensional content regular closed sets $S \subset \mathrm{X}$. In particular, if the space $(\mathrm{X}, \mathrm{d}, \mu)$ is Ahlfors Q-regular for some $Q \geqslant 1$ and $p \in (Q, \infty)$, then we obtain an intrinsic description of the trace-space of the Sobolev space $W_p^1(\mathrm{X})$ to arbitrary closed nonempty sets $S \subset \mathrm{X}$.

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§ 1. Introduction

The theory of Sobolev spaces on metric measure spaces $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ is an important rapidly growing area of contemporary geometric analysis. Since no additional regularity structure on X is assumed *a priori*, it is not surprising that most studies available so far are related to the *first-order Sobolev spaces* $W_p^1(\mathrm{X})$, $p \in (1, \infty)$. We refer to the recent beautiful monograph [1] and the lecture notes [2] containing an exhaustive treatment of the theory of $W_p^1(\mathrm{X})$-spaces, $p \in (1, \infty)$, and related questions. However, some natural questions concerning the spaces $W_p^1(\mathrm{X})$, $p \in (1, \infty)$, remain open. One of the most difficult and exciting among them is the so-called *trace problem*, that is, the problem of *a sharp intrinsic description of the trace-space* of the space $W_p^1(\mathrm{X})$, $p \in (1, \infty)$, to different closed sets $S \subset \mathrm{X}$. In all previously known studies this problem was considered under some extra regularity assumptions on S. In the present paper we introduce a *new sufficiently broad* class of closed sets and solve the corresponding trace problem for sets from that class.

In order to pose the problem precisely, we recall several concepts from analysis on metric measure spaces. First of all, by a *metric measure space* (an m.m.s. for short) we always mean a triple $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, where (X, d) is a *complete separable metric space* and μ is a *Borel regular* measure on (X, d) taking *finite positive values* on all balls $B_r(x)$ of radius $r \in (0, \infty)$ centred at $x \in \mathrm{X}$. Furthermore, we deal

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with m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ that are *q-admissible* for some $q \in (1, \infty)$ (see § 2.1 for details). This means that the following conditions hold true:

A) the measure μ has the uniformly locally doubling property;

B) the space X supports a weak local $(1, q)$-Poincaré inequality.

Given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ and a parameter $p \in (1, \infty)$, there are at least five approaches to the definition of the first-order Sobolev spaces $W_p^1(\mathrm{X})$ which are commonly used in the modern literature [3]–[9]. It is remarkable that in the case when X is p-admissible, all of them *are equivalent* in an appropriate sense (see § 2.2 for the details). In this paper we take as a basis the approach proposed by Cheeger [7] but in the equivalent modern form used in [3]. This approach appears to be more suitable for the questions considered in the present paper.

1.1. The statement of the problem. Recall the notion of p-capacity C_p (see Ch. I, § 1.4 in [10] for details). It is well known that (see details in § 2.3), given $p \in (1, \infty)$ and a p-admissible m.m.s. X, for every element $F \in W_p^1(\mathrm{X})$ there is a Borel representative $\overline{F}$ which has Lebesgue points everywhere except a set of p-capacity zero. Any such representative will be called a *p-sharp representative* of F. Given a set $S \subset \mathrm{X}$ of positive p-capacity, we define the *p-sharp trace* of an element $F \in W_p^1(\mathrm{X})$ to the set S as the (p-capacitary) equivalence class (modulo p-capacity zero) consisting of the pointwise restrictions of all p-sharp representatives of the element F to S and denote it by $F|_S$. In what follows we do not distinguish between $F|_S$ and the pointwise restriction of any p-sharp representative of F to S. We define the *p-sharp trace space* $W_p^1(\mathrm{X})|_S$ as the linear space of p-sharp traces $F|_S$ of all elements $F \in W_p^1(\mathrm{X})$. We equip this space with the corresponding quotient-space norm, that is, given $f \in W_p^1(\mathrm{X})|_S$, we put

$$\|f\|_{W_p^1(\mathrm{X})|_S} := \inf\{\|F\|_{W_p^1(\mathrm{X})} \colon f = F|_S\}. \tag{1.1}$$

We also introduce the *p-sharp trace operator* as a map $\operatorname{Tr}|_S \colon W_p^1(\mathrm{X}) \to W_p^1(\mathrm{X})|_S$ defined by the equality $\operatorname{Tr}|_S(F) := F|_S$ for $F \in W_p^1(\mathrm{X})$. It is not difficult to show that the map $\operatorname{Tr}|_S$ is a bounded linear operator from $W_p^1(\mathrm{X})$ to $W_p^1(\mathrm{X})|_S$. Finally, we say that $F \in W_p^1(\mathrm{X})$ is a *p-sharp extension* of a given Borel function $f \colon S \to \mathbb{R}$ if $f = F|_S$. The equality $f = F|_S$ should be interpreted in the following sense: the corresponding capacitary equivalence class of f coincides with $F|_S$. The first problem we consider in this paper can be formulated as follows.

Problem 1 (p-sharp trace problem). *Let $p \in (1, \infty)$, and let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a p-admissible metric measure space. Let $S \subset \mathrm{X}$ be a closed nonempty set with $C_p(S) > 0$.*

(Q1) *Given a Borel function $f \colon S \to \mathbb{R}$, find necessary and sufficient conditions for the existence of a p-sharp extension $F \in W_p^1(\mathrm{X})$ of the function f.*

(Q2) *Using only the geometry of the set S and the values of a function $f \in W_p^1(\mathrm{X})|_S$, compute the norm $\|f\|_{W_p^1(\mathrm{X})|_S}$ up to some universal constants.*

(Q3) *Does there exist a bounded linear operator $\operatorname{Ext}_S \colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})$, called a p-sharp extension operator, such that $\operatorname{Tr}|_S \circ \operatorname{Ext}_S = \mathrm{Id}$ on $W_p^1(\mathrm{X})|_S$?*

A warning about notation. Note that, formally, the operators $\operatorname{Tr}|_S$ and Ext_S depend implicitly on p. However, we do not complicate notation since in all main results of our paper the parameter p is fixed.

In many particular cases the concepts of the p-sharp trace space and p-sharp extension should be relaxed in an appropriate sense. For example, if a set $S \subset \mathrm{X}$ has a 'constant Hausdorff dimension', then it is natural to use the corresponding Hausdorff measure instead of C_p-capacity to describe 'negligible sets'. For example, in [11]–[18] the corresponding notions of traces of Sobolev functions were introduced with the help of the corresponding Hausdorff-type measures rather than capacities. However, the situation becomes more intricate if we deal with a set S consisting of infinitely many 'pieces of different dimensions'. Clearly, in this case the use of a single Hausdorff-type measure in the definition of traces of Sobolev spaces can lead to an ill-posed trace problem. At the same time, the use of C_p-capacities seems to be unnatural.

The above observations motivate us to introduce a more flexible concept of the trace of a Sobolev function. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space, and let $S \subset \mathrm{X}$ be a closed nonempty set. Given a Borel regular locally finite measure $\mathfrak{m}$ on X, we denote by $L_0(\mathfrak{m})$ the linear space of $\mathfrak{m}$-equivalence classes of all Borel functions $f\colon \operatorname{supp}\mathfrak{m} \to \mathbb{R}$. Assume that $\operatorname{supp}\mathfrak{m} = S$ and the measure $\mathfrak{m}$ is absolutely continuous with respect to C_p, that is, for each Borel set $E \subset S$ the equality $C_p(E) = 0$ implies the equality $\mathfrak{m}(E) = 0$. We define the $\mathfrak{m}$-trace $F|^{\mathfrak{m}}_S$ of an element $F \in W^1_p(\mathrm{X})$ to S as the $\mathfrak{m}$-equivalence class of the p-sharp trace $F|_S$. We let $W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$ denote the linear space of $\mathfrak{m}$-traces of all $F \in W^1_p(\mathrm{X})$ equipped with the corresponding quotient-space norm, that is, given $f \in W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$, we put

$$\|f\|_{W^1_p(\mathrm{X})|^{\mathfrak{m}}_S} := \inf\{\|F\|_{W^1_p(\mathrm{X})} \colon f = F|^{\mathfrak{m}}_S\}. \tag{1.2}$$

We also introduce the $\mathfrak{m}$-*trace operator* as a map $\operatorname{Tr}|^{\mathfrak{m}}_S \colon W^1_p(\mathrm{X}) \to W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$ defined by the equality $\operatorname{Tr}|^{\mathfrak{m}}_S(F) := F|^{\mathfrak{m}}_S$ for $F \in W^1_p(\mathrm{X})$. It is not difficult to show that the map $\operatorname{Tr}|^{\mathfrak{m}}_S$ is a bounded linear operator from $W^1_p(\mathrm{X})$ to $W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$. We say that $F \in W^1_p(\mathrm{X})$ is an $\mathfrak{m}$-*extension* of an element $f \in L_0(\mathfrak{m})$ if $f = F|^{\mathfrak{m}}_S$. The second problem considered in the present paper can be formulated as follows.

Problem 2 ($\mathfrak{m}$-trace problem). *Let $p \in (1, \infty)$, and let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a p-admissible metric measure space. Let $\mathfrak{m}$ be a positive locally finite Borel regular measure on X that is absolutely continuous with respect to C_p, and let $S = \operatorname{supp}\mathfrak{m}$.*

(MQ1) *Given $f \in L_0(\mathfrak{m})$, find necessary and sufficient conditions for the existence of an $\mathfrak{m}$-extension $F \in W^1_p(\mathrm{X})$ of the element f.*

(MQ2) *Using only the geometry of the set S, the properties of $\mathfrak{m}$ and the values of $f \in W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$ compute the norm $\|f\|_{W^1_p(\mathrm{X})|^{\mathfrak{m}}_S}$ up to some universal constants.*

(MQ3) *Does there exist a bounded linear operator $\operatorname{Ext}_{S,\mathfrak{m}} \colon W^1_p(\mathrm{X})|^{\mathfrak{m}}_S \to W^1_p(\mathrm{X})$, called an $\mathfrak{m}$-extension operator, such that $\operatorname{Tr}|^{\mathfrak{m}}_S \circ \operatorname{Ext}_{S,\mathfrak{m}} = \mathrm{Id}$ on $W^1_p(\mathrm{X})|^{\mathfrak{m}}_S$?*

1.2. Previously known results. As far as we know, Problem 1 has been considered only in the case $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$. Furthermore, this problem remains open in full generality, that is, in the full range $p \in (1, \infty)$. Below we briefly recall the most powerful particular results available in the literature.

(R.1.1) The results in [19] and [20] cover completely the case when $p > n$, that is, Problem 1 is solved without any additional regularity assumptions on S.

(R.1.2) In the case when $p \in (1, n]$, for each $d \in (n - p, n]$ Problem 1 has been solved for any closed lower content d-regular (or, equivalently, d-thick) set $S \subset \mathbb{R}^n$ (see [21]).

(R.1.3) Very recently, a weakened version of Problem 1 was solved by the author without any additional regularity assumption on S (see [22] and [23]).

Now we describe briefly the available results concerning Problem 2. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space. Throughout the paper we use the symbol $B_r(x)$ to denote the closed ball of radius $r \geqslant 0$ centred at $x \in \mathrm{X}$, that is,

$$B_r(x) := \{y \in \mathrm{X}\colon\ \mathrm{d}(x,y) \leqslant r\}.$$

Since in general metric spaces the behaviour of $\mu(B_r(x))$ is not so transparent, it was observed in [14], [15], [12] and [11] that codimensional analogues of the Hausdorff contents $\mathcal{H}_{\theta,\delta}$ (see § 2.1 for the corresponding precise definition) are more suitable in this case. Following [14] and [15], given an m.m.s. X and a parameter $\theta \geqslant 0$, we say that a closed set $S \subset \mathrm{X}$ is *Ahlfors-David θ-coregular* if there exist constants $c_{S,1}(\theta), c_{S,2}(\theta) > 0$ such that

$$c_{S,1}(\theta)\frac{\mu(B_r(x))}{r^{\theta}} \leqslant \mathcal{H}_{\theta}(B_r(x) \cap S) \leqslant c_{S,2}(\theta)\frac{\mu(B_r(x))}{r^{\theta}} \quad \text{for all } (x,r) \in S \times (0,1]. \tag{1.3}$$

We denote the class of all Ahlfors-David θ-coregular sets by $\mathcal{ADR}_{\theta}(\mathrm{X})$.

(R.2.1) In [18] traces of Calderón-Sobolev spaces and Hajlasz-Sobolev spaces to sets $S \in \mathcal{ADR}_0(\mathrm{X})$ were considered. It was assumed in [18] that the measure μ is globally doubling and, in addition, satisfies the reverse doubling property.

(R.2.2) In [17] traces of Besov, Lizorkin-Triebel and Hajlasz-Sobolev spaces to porous Ahlfors-David regular closed subsets of X were considered. In fact, the methods used in [17] allow one to achieve some relaxation of the Ahlfors-David θ-regularity condition by replacing it by Ahlfors-David θ-coregularity.

(R.2.3) In [14], given $\theta > 0$ and a uniform domain $\Omega \subset \mathrm{X}$ whose boundary $\partial\Omega$ satisfies the corresponding Ahlfors-David θ-coregularity condition, given $p \in (\max\{1,\theta\},\infty)$, an exact description of traces of the Newtonian-Sobolev $N^1_p(\Omega)$-spaces to $\partial\Omega$ was obtained. Furthermore, very recently a similar problem was considered for homogeneous Sobolev-type spaces or, as they are sometimes called, Dirichlet spaces $D^1_p(\Omega)$, $p \in (1,\infty)$ (see [12]).

(R.2.4) Very recently an analogue of Problem 2 for Banach-valued Sobolev mappings were studied in [24] in the case $S \in \mathcal{ADR}_0(\mathrm{X})$.

1.3. The aims of the paper. Given $X = (X, d, \mu)$, an analysis of the results mentioned in (R.2.1)–(R.2.4) shows that Problem 2 has been considered for sets $S \subset \mathrm{X}$ satisfying Ahlfors-David-type regularity conditions. In particular, methods and tools available so far have been found to be inapplicable even in the case when $S = S_1 \cup S_2$, where $S_i \in \mathcal{ADR}_{\theta_i}(\mathrm{X})$, $i = 1,2$, for $\theta_1 \neq \theta_2$ and satisfying $S_1 \cap S_2 \neq \varnothing$ and $\mathcal{H}_{\max\{\theta_1,\theta_2\}}(S_1 \cap S_2) = 0$. This elementary obstacle shows that the classes $\mathcal{ADR}_{\theta}(\mathrm{X})$, $\theta \geqslant 0$, are too narrow to build a fruitful trace theory. Hence it is natural to introduce a relaxation of the Ahlfors-David regularity condition (1.3) by replacing the Hausdorff measure by the corresponding Hausdorff content. We say that a set $S \subset \mathrm{X}$ is *lower θ-codimensional content regular* if there exists a constant $\lambda_S(\theta) \in (0,1]$ such that

$$\lambda_S(\theta)\frac{\mu(B_r(x))}{r^{\theta}} \leqslant \mathcal{H}_{\theta,r}(B_r(x) \cap S) \quad \text{for all } (x,r) \in S \times (0,1]. \tag{1.4}$$

By $\mathcal{LCR}_\theta(\mathrm{X})$ we denote the class of all lower θ-codimensional content regular subsets of X. This class is a natural generalisation of the class of all d-thick subsets of $\mathbb{R}^n$ introduced by Rychkov [16] to the case of general metric measure spaces. Indeed, in the case when $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$ and $d \in [0, n]$, a set $S \subset \mathbb{R}^n$ is d-thick in the sense of Rychkov if and only if $S \in \mathcal{LCR}_{n-d}(\mathrm{X})$. Very recently some interesting geometric properties of d-thick subsets of $\mathbb{R}^n$ were actively studied in [25]–[27]. One can show that if the measure μ has the uniformly locally doubling property, then $\mathcal{ADR}_\theta(\mathrm{X}) \subset \mathcal{LCR}_\theta(\mathrm{X})$ for each $\theta \geqslant 0$, but this inclusion is strict in general (see § 4 for the details).

The class $\mathcal{LCR}_\theta(\mathrm{X})$ is very broad. For example, in the case when $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$ any path-connected set $\Gamma \subset \mathbb{R}^n$ containing at least two distinct points belongs to $\mathcal{LCR}_{n-1}(\mathrm{X})$. Furthermore, if an m.m.s. X is Ahlfors Q-regular for some $Q > 0$, then for each $\theta \geqslant Q$ every nonempty set $S \subset \mathrm{X}$ belongs to $\mathcal{LCR}_\theta(\mathrm{X})$.

The aim of this paper is to solve Problems 1 and 2 for all closed sets $S \in \mathcal{LCR}_\theta(\mathrm{X})$ for each $\theta \in [0, p)$ (in the case when $\theta \geqslant p$ the above problems appear to be ill posed in general). We will show that our results cover all previously known results (see [18], [19], [17] and [21]). Furthermore, we provide an illustrative example (Example 8) in which Problem 2 is solved for a set formed by two Ahlfors-David regular sets of different codimensions with nonempty intersection. Note that even this elementary example was beyond the scope of the previously known techniques. Finally, as a particular case of our main results, given parameters $Q \geqslant 1$ and $p \in (Q, \infty)$, and an Ahlfors Q-regular p-admissible m.m.s. X, in Example 9 we present a solution to Problem 1 for an arbitrary closed nonempty set $S \subset \mathrm{X}$. This example gives a natural generalization of one of the main results from [19].

1.4. Statements of the main results. In order to formulate the main results of the present paper, we introduce some keystone tools.

Given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ and a parameter $\theta \geqslant 0$, we say that a sequence of locally finite Borel regular measures $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty}$ is *θ-regular* if there exists $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0, 1)$ such that the following conditions are satisfied:

(M1) there exists a closed nonempty set $S \subset \mathrm{X}$ such that

$$\operatorname{supp} \mathfrak{m}_k = S \quad \text{for all } k \in \mathbb{N}_0; \tag{1.5}$$

(M2) there exists a constant $C_1 > 0$ such that for each $k \in \mathbb{N}_0$

$$\mathfrak{m}_k(B_r(x)) \leqslant C_1 \frac{\mu(B_r(x))}{r^\theta} \quad \text{for all } x \in \mathrm{X} \text{ and all } r \in (0, \epsilon^k]; \tag{1.6}$$

(M3) there exists a constant $C_2 > 0$ such that for each $k \in \mathbb{N}_0$

$$\mathfrak{m}_k(B_r(x)) \geqslant C_2 \frac{\mu(B_r(x))}{r^\theta} \quad \text{for all } x \in S \text{ and all } r \in [\epsilon^k, 1]; \tag{1.7}$$

(M4) for each $k \in \mathbb{N}_0$ there exists $w_k \in L_\infty(\mathfrak{m}_0)$ such that $\mathfrak{m}_k = w_k \mathfrak{m}_0$ and, furthermore, there exists a constant $C_3 > 0$ such that for any $k, j \in \mathbb{N}_0$

$$\frac{\epsilon^{\theta j}}{C_3} \leqslant \frac{w_k(x)}{w_{k+j}(x)} \leqslant C_3 \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{1.8}$$

Given a nonempty closed set $S \subset X$, the class of all θ-regular sequences of measures $\{\mathfrak{m}_k\}$ satisfying (1.5) is denoted by $\mathfrak{M}_\theta(S)$. Furthermore, we say that a sequence $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ is *strongly θ-regular* if

(M5) for each Borel set $E \subset S$,

$$\varlimsup_{k\to\infty} \frac{\mathfrak{m}_k(B_{\epsilon^k}(\underline{x}) \cap E)}{\mathfrak{m}_k(B_{\epsilon^k}(\underline{x}))} > 0 \quad \text{for } \mathfrak{m}_0\text{-a.e. } \underline{x} \in E. \tag{1.9}$$

Given a nonempty closed set $S \subset X$, the class of all strongly θ-regular sequences of measures $\{\mathfrak{m}_k\}$ satisfying (1.5) is denoted by $\mathfrak{M}^{\rm str}_\theta(S)$. Condition (M5) can be considered as a multiweight generalization of the famous A_∞-condition of Muckenhoupt (cf. Ch. 5, § 5.7 in [28]). It is clear that, given $\theta \geqslant 0$, we have the inclusion $\mathfrak{M}^{\rm str}_\theta(S) \subset \mathfrak{M}_\theta(S)$. The question of the coincidence of $\mathfrak{M}^{\rm str}_\theta(S)$ and $\mathfrak{M}_\theta(S)$ is rather subtle and will be discussed in § 5.2 of our paper.

The first main result in this paper looks like an auxiliary statement. Nevertheless, this result is new, and we believe that it can be of independent interest. It can be considered as a natural and far-reaching generalization of a simple characterization of Ahlfors-David θ-coregular sets in $\mathbb{R}^n$ (see Definition 1.1 and Theorem 1 in Ch. 1 of [13]).

Theorem 1. *Given $p \in (1,\infty)$, let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a p-admissible metric measure space. Let $\theta \geqslant 0$ and let $S \subset \mathrm{X}$ be a closed nonempty set. If $S \in \mathcal{LCR}_\theta(\mathrm{X})$, then $\mathfrak{M}^{\rm str}_\theta(S) \neq \varnothing$. If $\mathfrak{M}_\theta(S) \neq \varnothing$, then $S \in \mathcal{LCR}_\theta(\mathrm{X})$.*

For an exposition of the subsequent results it will be convenient to fix *the following data*:

(D1) a parameter $p \in (1,\infty)$ and a p-admissible metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$;

(D2) a parameter $\theta \in [0,p)$ and a closed set $S \in \mathcal{LCR}_\theta(\mathrm{X})$;

(D3) a sequence of measures $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ and a parameter $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0, 1/10]$.

Given $r > 0$, we introduce *important notation* by setting

$$k(r) := \max\{k \in \mathbb{Z}\colon r \leqslant \epsilon^k\}.$$

Now we introduce several keystone functionals, which will be the main tools in obtaining different characterizations of traces of $W^1_p(\mathrm{X})$-spaces. Given $q \in [0,+\infty)$, we use the following notation. We set

$$L_q(\{\mathfrak{m}_k\}) := \bigcap_{k=0}^{\infty} L_q(\mathfrak{m}_k) \quad \text{and} \quad L^{\rm loc}_q(\{\mathfrak{m}_k\}) := \bigcap_{k=0}^{\infty} L^{\rm loc}_q(\mathfrak{m}_k).$$

Given a nonzero Borel regular locally finite measure $\mathfrak{m}$ on X and an element $f \in L^{\rm loc}_1(\mathfrak{m})$, for every bonded Borel set G with $\mathfrak{m}(G) > 0$ we put

$$\mathcal{E}_{\mathfrak{m}}(f, G) := \inf_{c\in\mathbb{R}} \frac{1}{\mathfrak{m}(G)} \int_G |f(x) - c|\, d\mathfrak{m}(x).$$

For each $r > 0$ we put

$$\widetilde{\mathcal{E}}_{\mathfrak{m}}(f, B_r(x)) := \begin{cases} \mathcal{E}_{\mathfrak{m}}(f, B_{2r}(x)) & \text{if } B_r(x) \cap \operatorname{supp} \mathfrak{m} \neq \varnothing, \\ 0 & \text{if } B_r(x) \cap \operatorname{supp} \mathfrak{m} = \varnothing. \end{cases} \tag{1.10}$$

Now, given $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, we define the $\{\mathfrak{m}_k\}$-*Calderón maximal function* as a mapping $f^{\sharp}_{\{\mathfrak{m}_k\}}\colon \mathrm{X} \to [0,+\infty]$ defined by the formula

$$f^{\sharp}_{\{\mathfrak{m}_k\}}(x) := \sup_{r\in(0,1]} \frac{1}{r}\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(r)}}(f, B_r(x)), \qquad x \in \mathrm{X}.$$

Furthermore, we consider the *Calderón functional* on the space $L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ (with values in $[0,+\infty]$) by letting, for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f) := \|f^{\sharp}_{\{\mathfrak{m}_k\}}\|_{L_p(\mu)}. \tag{1.11}$$

Note that if $\mathrm{X} = S = \mathbb{R}^n$ and $\mathfrak{m}_k := \mathcal{L}^n$ for all $k \in \mathbb{N}_0$, then the $\{\mathfrak{m}_k\}$-Calderón maximal function coincides with the classical maximal function $f^{\sharp}$ introduced by Calderón in [29]. Furthermore, in his paper [29] Calderón proved that, for $q\in(1,\infty]$, an element $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ lies in $W_q^1(\mathbb{R}^n)$ if and only if both f and $f^{\sharp}$ belong to $L_q(\mathbb{R}^n)$. This fact justifies our name for the functional $\mathcal{CN}_{p,\{\mathfrak{m}_k\}}$.

Given $c > 1$, we also introduce the *Brudnyi-Shvartsman functional* on $L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ (with values in $[0,+\infty]$) by letting, for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}(f) := \sup\left(\sum_{i=1}^{N} \frac{\mu(B_{r_i}(x_i))}{r_i^p}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(r_i)}}(f, B_{cr_i}(x_i))\big)^p\right)^{1/p}, \tag{1.12}$$

where the supremum is taken over all finite families of closed balls $\{B_{r_i}(x_i)\}_{i=1}^N$ such that:

(F1) $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \varnothing$ provided that $i \neq j$;

(F2) $0 < \min\{r_i\colon i = 1,\dots,N\} \leqslant \max\{r_i\colon i = 1,\dots,N\} \leqslant 1$;

(F3) $B_{cr_i}(x_i) \cap S \neq \varnothing$ for all $i \in \{1,\dots,N\}$.

Note that if $\mathrm{X} = S = \mathbb{R}^n$ and $\mathfrak{m}_k := \mathcal{L}^n$ for all $k \in \mathbb{N}_0$, then the functional $\mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}$ is very close in spirit to that used by Brudnyi [30] to characterize Sobolev-type spaces on $\mathbb{R}^n$. In the case when $\mathrm{X} = \mathbb{R}^n$, $p > n$, and $S \subset \mathbb{R}^n$ is an arbitrary closed nonempty set, our functional is also very close in spirit to the corresponding functionals used by Shvartsman in [19] and [20]. These observations justify our name for the functional $\mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}$.

Given a parameter $\sigma \in (0,1]$, we say that a ball $B_r(x)$ is (S,σ)-*porous* if there is a ball $B_{r'}(x') \subset B_r(x) \setminus S$ such that $r' \geqslant \sigma r$. Furthermore, given $r \in (0,1]$, we put

$$S_r(\sigma) := \{x \in S\colon B_r(x) \text{ is } (S,\sigma)\text{-porous}\}. \tag{1.13}$$

We say that S is σ-porous if $S = S_r(\sigma)$ for all $r \in (0,1]$. Porous sets arise naturally in many areas of modern geometric analysis (see, for example, the survey [31]). In the classical Euclidean settings, the porosity properties of lower content regular sets were studied in [27].

We define a natural analogue of the Besov seminorm. More precisely, given $\sigma \in (0,1]$, we introduce the *Besov functional* on $L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ by letting, for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) := \|f^{\sharp}_{\{\mathfrak{m}_k\}}\|_{L_p(S,\mu)} + \left(\sum_{k=1}^{\infty} \epsilon^{k(\theta-p)} \int_{S_{\epsilon^k}(\sigma)} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_{\epsilon^k}(x))\big)^p \, d\mathfrak{m}_k(x)\right)^{1/p}. \tag{1.14}$$

If the space X is Ahlfors Q-regular for some $Q > 0$, $S \subset \mathrm{X}$ is a closed Ahlfors-David θ-coregular set for some $\theta \in (0, Q)$, and $\mathfrak{m}_k = \mathcal{H}_\theta\lfloor_S$, $k \in \mathbb{N}_0$, then the functional $\mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}$ coincides with the corresponding Besov seminorm [17]. This justifies our name for the functional.

The second main result of this paper gives answers to questions (MQ1) and (MQ2) in Problem 2. Namely, we present several equivalent characterizations of the trace space. It is important that condition (1.8) implies that $W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0} = W_p^1(\mathrm{X})|_S^{\mathfrak{m}_k}$ for all $k \in \mathbb{N}_0$.

Theorem 2. *If* $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$, $c \geqslant 3/\epsilon$ *and* $\sigma \in (0, \epsilon^2/(4c))$, *then, given* $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, *the following conditions are equivalent*:

(i) $f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$;

(ii) $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) := \|f\|_{L_p(\mathfrak{m}_0)} + \mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f) < +\infty$;

(iii) $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) := \|f\|_{L_p(\mathfrak{m}_0)} + \mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$;

(iv) $\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) := \|f\|_{L_p(\mathfrak{m}_0)} + \mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) < +\infty$.

Furthermore, for each $c \geqslant 3/\epsilon$ *and* $\sigma \in (0, \epsilon^2/(4c))$, *for every* $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\|f\|_{W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}} \approx \mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) \approx \mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \approx \mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f), \tag{1.15}$$

where the corresponding equivalence constants do not depend on f.

In § 11 we show that the equivalence of (i) and (iv) in Theorem 2 implies Theorem 1.5 in [17] as a particular case. Furthermore, in the Euclidean settings the equivalence of (i) and (iv) strengthens the author's joint result in [21].

The third main result gives answers to questions (Q1) and (Q2) posed in Problem 1.

Theorem 3. *A p-capacitary equivalence class of a Borel function* $f\colon S \to \mathbb{R}$ *belongs to the space* $W_p^1(\mathrm{X})|_S$ *if and only if the following conditions hold*:

(A) *the* $\mathfrak{m}_0$*-equivalence class* $[f]_{\mathfrak{m}_0}$ *of* f *belongs to* $W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$;

(B) *there exists a set* $\underline{S}_f \subset S$ *with* $C_p(S \setminus \underline{S}_f) = 0$ *such that*

$$\lim_{k\to\infty} \fint_{B_{\epsilon^k}(x)} |f(\underline{x}) - f(y)|\, d\mathfrak{m}_k(y) = 0 \quad \textit{for all } \underline{x} \in \underline{S}_f. \tag{1.16}$$

Furthermore, for each $c \geqslant 3/\epsilon$ *and* $\sigma \in (0, \epsilon^2/(4c))$, *for every* $f \in W_p^1(\mathrm{X})|_S$,

$$\|f\|_{W_p^1(\mathrm{X})|_S} \approx \mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) \approx \mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \approx \mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f), \tag{1.17}$$

where the corresponding equivalence constants do not depend on f.

Note that, in contrast to Theorem 2, condition (B) in Theorem 3 is delicate and important. Roughly speaking, given $f \in L_p(\mathfrak{m}_0)$, the finiteness of the functionals (1.11), (1.12) and (1.14) is not sufficient for the existence of a p-sharp extension of f. On the other hand, the additional condition (B) allows one to relax restrictions on the sequence of measures $\{\mathfrak{m}_k\}$. Indeed, we do not require that $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$ in Theorem 3. We show in Example 9 that if X is geodesic and Ahlfors Q-regular for some $Q \geqslant 1$, then Theorem 3 gives an exact intrinsic description of the p-sharp trace space $W_p^1(\mathrm{X})|_S$ to an arbitrary closed nonempty set $S \subset \mathrm{X}$.

The fourth main result of the present paper gives answers to questions (Q3) and (MQ3) in Problems 1 and 2, respectively. Furthermore, it clarifies a deep connection between Problems 1 and 2. This connection is given by the existence of a canonical isomorphism between *a priori* different trace spaces $W_p^1(\mathrm{X})|_S$ and $W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$. This fact sheds light on the main reason why the concept of the p-sharp trace space was not used in the previous investigations. As usual, given normed linear spaces $E_1 = (E_1, \|\cdot\|_1)$ and $E_2 = (E_2, \|\cdot\|_2)$, we denote the linear space of all bounded linear mappings from E_1 to E_2 by $\mathcal{L}(E_1, E_2)$.

Theorem 4. *Let $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$. Then the following assertions holds.*

(1) *There exists an $\mathfrak{m}_0$-extension operator $\mathrm{Ext} := \mathrm{Ext}_{S,\{\mathfrak{m}_k\}} \in \mathcal{L}(W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}, W_p^1(\mathrm{X}))$.*

(2) *There exists a p-sharp extension operator $\overline{\mathrm{Ext}} := \overline{\mathrm{Ext}}_{S,\{\mathfrak{m}_k\},p} \in \mathcal{L}(W_p^1(\mathrm{X})|_S, W_p^1(\mathrm{X}))$.*

(3) *The canonical imbedding $\mathrm{I}_{\mathfrak{m}_0} : W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ that takes $f \in W_p^1(\mathrm{X})|_S$ and returns the $\mathfrak{m}_0$-equivalence class $[f]_{\mathfrak{m}_0}$ of f is an isometric isomorphism.*

(4) *We have the following commutative diagram:*

$$
\begin{array}{ccc}
W_p^1(\mathrm{X}) & \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} & W_p^1(\mathrm{X}) \\
\overline{\mathrm{Ext}}\ \uparrow\downarrow\ \mathrm{Tr}\,|_S & & \mathrm{Ext}\ \uparrow\downarrow\ \mathrm{Tr}\,|_S^{\mathfrak{m}_0} \\
W_p^1(\mathrm{X})|_S & \underset{\mathrm{I}_{\mathfrak{m}_0}^{-1}}{\overset{\mathrm{I}_{\mathfrak{m}_0}}{\rightleftarrows}} & W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}
\end{array}
$$

1.5. The keystone innovations. Note that even in the particular case of $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$ our results are new. Indeed, characterizations via Brudnyi-Shvartsman-type functionals were never considered in the literature for $p \in (1, n]$ (the case $p > n$ was considered in [19]). The keystone innovations in the present paper can be summarized as follows.

- In contrast to the classical Whitney method used in the previous investigations, we build a new extension operator by constructing, for a fixed element $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, a special approximating sequence $\{f^j\} \subset L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, and we obtain the resulting extension as the weak limit of this sequence.
- We introduce the Brudnyi-Shvartsman-type functional in metric measure settings.
- In contrast to the previously known studies related to Problem 2, we use the so-called 'vertical approach' to Sobolev spaces on metric measure spaces, introduced originally by Cheeger [7]. This gives a natural symbiosis with our new extension operator and leads to the Brudnyi-Shvartsman-type characterization of the trace space.
- We introduce the new concept of the $\mathfrak{m}$-trace of the Sobolev $W_p^1(\mathrm{X})$-space and investigate its relationships with the notion of the p-sharp trace of the space $W_p^1(\mathrm{X})$.
- We introduce the new class of sets $\mathcal{LCR}_\theta(\mathrm{X})$ which is a natural generalization of the class of d-thick sets introduced by Rychkov in [16] from the case of finite-dimensional Euclidean space $\mathbb{R}^n$ to the case of admissible metric measure spaces.

- We introduce the new class of sequences of measures $\mathfrak{M}^{\mathrm{str}}_{\theta}(\mathrm{X})$. This allows one to obtain a characterization of the $\mathfrak{m}_0$-trace space of the Sobolev $W^1_p(\mathrm{X})$-space using only the finiteness of the corresponding functionals.

1.6. The organization of the paper. The paper is organized as follows.

In § 2 we collect some classical results about metric measure spaces and Sobolev functions defined on such spaces. These results form the fundament for the subsequent exposition.

In § 3 we introduce weakly noncollapsed measures and show that they possess some sort of asymptotically doubling properties, which will be very important in proving the existence of strongly θ-regular sequences of measures in § 5.

Section 4 is devoted to some elementary properties of sets $S \in \mathcal{LCR}_{\theta}(\mathrm{X})$, $\theta \geqslant 0$. We also present some simple examples.

Section 5 is a 'technical basis' of the paper. We prove Theorem 1 and study in detail various properties of θ-regular sequences of measures. Furthermore, we present elementary examples of sets S for which one can easily construct explicit examples of strongly θ-regular sequences of measures.

Section 6 is devoted to investigations of some delicate pointwise properties of functions. This section plays a crucial role in proving that the new extension operator is a right inverse of the corresponding trace operator.

In § 7 we construct our new extension operator.

Sections 8 and 9 contain a technical foundation for the proofs of the so-called direct and reverse trace theorems, respectively.

In § 10 we prove the main results of the paper, that is, Theorems 2, 3 and 4.

We conclude our paper by § 11, where we show that the most part of the available results are mere particular cases of our main results. On the other hand, we present simple examples which do not fall into the scope of the previous investigations.

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§ 2. Preliminaries

The goal of this preliminary section is to recall some basic material related to modern analysis and to set the terminology that we adopt in this paper.

2.1. Geometric analysis background. Given a metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a set $E \subset \mathrm{X}$, we denote by $\operatorname{int} E$, $\operatorname{cl} E$ and ∂E the *interior* of E, the *closure* of E and the *boundary* of E in the metric topology of X, respectively. Unless otherwise stated, all the balls in X are assumed to be closed. More precisely, we put

$$B_r(x) := \{y \in \mathrm{X} \colon \mathrm{d}(x, y) \leqslant r\} \quad \text{for } (x, r) \in \mathrm{X} \times [0, +\infty).$$

Clearly, if one regards a given ball B just as a subset of X, then it can occur that its centre and radius are not uniquely determined. Hence, in what follows we always consider a ball B together with some fixed centre x_B and fixed radius r_B. Given a ball $B = B_r(x)$ and a constant $\lambda \geqslant 0$, we set $\lambda B := B_{\lambda r}(x)$.

Given a metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a Borel set $E \subset \mathrm{X}$, we denote by $\mathfrak{B}(E)$ the set of all Borel functions $f\colon E \to [-\infty, +\infty]$. We denote by $C(\mathrm{X})$ and $C_c(\mathrm{X})$ the linear space of all continuous real-valued and all compactly supported continuous real-valued functions, respectively. We equip these spaces with the usual sup-norm. Finally, the symbol $\mathrm{LIP}^{\mathrm{loc}}(\mathrm{X})$ ($\mathrm{LIP}(\mathrm{X})$) denotes the set of all real-valued uniformly locally Lipschitz functions (Lipschitz functions, respectively) on X, that is, $f \in \mathrm{LIP}^{\mathrm{loc}}(\mathrm{X})$ ($f \in \mathrm{LIP}(\mathrm{X})$) if and only if for each $R \in (0, +\infty)$ (for each $R \in (0, +\infty]$, respectively)

$$L_f(R) := \sup_{0<\mathrm{d}(x,y)<R} \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)} < +\infty.$$

For each function $f\colon \mathrm{X} \to \mathbb{R}$ we define its *local Lipschitz constant* $\operatorname{lip} f\colon \mathrm{X} \to [0, +\infty]$ (which is also called the *slope of* f and denoted by $|\nabla f|$) by the equality

$$\operatorname{lip} f(x) := |\nabla f|(x) := \begin{cases} \overline{\lim\limits_{y\to x}} \dfrac{|f(y) - f(x)|}{\mathrm{d}(x, y)}, & x \text{ is an accumulation point,} \\ 0, & x \text{ is an isolated point.} \end{cases} \tag{2.1}$$

It is clear that for each function $f \in \mathrm{LIP}^{\mathrm{loc}}(\mathrm{X})$ the local Lipschitz constant $\operatorname{lip} f$ is finite everywhere on X and belongs to $\mathfrak{B}(\mathrm{X})$. Below we summarize the elementary properties of the local Lipschitz constants of Lipschitz functions.

Proposition 1. *Given a metric space* $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, *the following properties hold*:

(1) *if* $f \equiv c$ *on* X *for some number* $c \in \mathbb{R}$, *then* $\operatorname{lip} f \equiv 0$ *on* X;

(2) *if* $f_1, f_2 \in \mathrm{LIP}^{\mathrm{loc}}(\mathrm{X})$, *then*

$$\operatorname{lip}(f_1 + f_2)(x) \leqslant \operatorname{lip} f_1(x) + \operatorname{lip} f_2(x) \quad \text{for } x \in \mathrm{X};$$

(3) $\operatorname{lip}(f + c) \equiv \operatorname{lip} f$ *for each function* $f \in \mathrm{LIP}^{\mathrm{loc}}(\mathrm{X})$ *and any number* $c \in \mathbb{R}$.

The following fact is well known (see Corollary 1.6 in [2], for example).

Proposition 2. *If* $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ *is a compact metric space*, *then the space* $C(\mathrm{X})$ *is separable*.

Given a metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a number $\epsilon \in (0, 1)$, for each $k \in \mathbb{Z}$ we denote by $Z_k(\mathrm{X}, \epsilon)$ an arbitrary *maximal* ϵ^k*-separated subset* of X. Furthermore, the symbol $\mathcal{A}_k(\mathrm{X}, \epsilon)$ denotes the corresponding index set, that is,

$$Z_k(\mathrm{X}, \epsilon) = \{z_{k,\alpha}\colon \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)\}. \tag{2.2}$$

It is clear that if X is separable, then $\mathcal{A}_k(\mathrm{X}, \epsilon)$ is an at most countable set. For each $k \in \mathbb{Z}$ we introduce the special family of balls $\mathcal{B}_k(\mathrm{X}, \epsilon)$ by setting

$$\mathcal{B}_k(\mathrm{X}, \epsilon) := \{B_{\epsilon^k}(z_{k,\alpha})\colon \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)\}. \tag{2.3}$$

Finally, we put

$$Z(\mathrm{X}, \epsilon) := \bigcup_{k\in\mathbb{Z}} Z_k(\mathrm{X}, \epsilon) \quad \text{and} \quad Z^{\underline{k}}(\mathrm{X}, \epsilon) := \bigcup_{k\geqslant\underline{k}} Z_k(\mathrm{X}, \epsilon), \quad \underline{k} \in \mathbb{Z}. \tag{2.4}$$

Given a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, we say that $\mathfrak{m}$ is a *measure on* X if $\mathfrak{m}$ is a Borel regular outer measure on X. A measure $\mathfrak{m}$ on X is said to be *locally finite* if $\mathfrak{m}(B_r(x)) < +\infty$ for all pairs $(x,r) \in \mathrm{X} \times [0,+\infty)$. Given a Borel set $E \subset \mathrm{X}$ and a measure $\mathfrak{m}$ on X, we define the restriction $\mathfrak{m}\lfloor_E$ of $\mathfrak{m}$ to E, as usual, by the formula

$$\mathfrak{m}\lfloor_E(F) := \mathfrak{m}(F \cap E) \quad \text{for any Borel sets } F \subset \mathrm{X}. \tag{2.5}$$

Sometimes it will be convenient to work with so-called *weighted measures*. More precisely, if $\mathfrak{m}$ is a measure on X, then we say that $\gamma \in \mathfrak{B}(\mathrm{X})$ is an $\mathfrak{m}$-weight if $\gamma(x) \geqslant 0$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. In this case $\gamma\mathfrak{m}$ should be interpreted as the measure on X defined by

$$\gamma\mathfrak{m}(E) := \int_E \gamma(x)\, d\mathfrak{m}(x) \quad \text{for every Borel set } E \subset \mathrm{X}. \tag{2.6}$$

Given a locally compact separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, following [32] we say that a sequence of measures $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty}$ on X *converges locally $*$-weakly* to a measure $\mathfrak{m}$ on X and write $\mathfrak{m}_k \rightharpoonup \mathfrak{m}$ as $k \to \infty$, if

$$\lim_{k\to\infty} \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}_k(x) = \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}(x) \quad \text{for every } \varphi \in C_c(\mathrm{X}).$$

The following fact is well known. For a detailed proof, see, for example, Corollary 1.60 in [32].

Lemma 1. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ *be a locally compact separable metric space, and let* $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty}$ *be a sequence of measures on* X *such that*

$$\sup_{k\in\mathbb{N}_0} \mathfrak{m}_k(B) < +\infty \quad \text{for every ball } B \subset \mathrm{X}. \tag{2.7}$$

Then there is a locally $*$-weakly convergent subsequence $\{\mathfrak{m}_{k_l}\}$ *of the sequence* $\{\mathfrak{m}_k\}$.

We also recall some standard properties of locally $*$-weakly convergent sequences of measures.

Proposition 3. *Let* (X, d) *be a locally compact separable metric space. If a sequence of measures* $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ *on* X *converges locally $*$-weakly to a measure* $\mathfrak{m}$ *on* X, *then for every open set* $G \subset \mathrm{X}$ *and every compact set* $F \subset \mathrm{X}$,

$$\varliminf_{k\to\infty} \mathfrak{m}_k(G) \geqslant \mathfrak{m}(G) \quad \text{and} \quad \varlimsup_{k\to\infty} \mathfrak{m}_k(F) \leqslant \mathfrak{m}(F). \tag{2.8}$$

Throughout the paper, by a *metric measure space* (an m.m.s., for short) we always mean a triple $(\mathrm{X}, \mathrm{d}, \mu)$, where (X, d) is a *complete separable metric space* and μ is a nonzero locally finite measure on X such that $\operatorname{supp}\mu = \mathrm{X}$.

Remark 1. In what follows, given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, by a measure on X we always mean a measure on the metric space (X, d).

Given a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a measure $\mathfrak{m}$ on X, we assume that the collection of $\mathfrak{m}$-measurable sets is the completion of the Borel σ-algebra with respect to $\mathfrak{m}$. Furthermore, given a Borel nonempty set $E \subset \operatorname{supp} \mathfrak{m}$ and a function $f \in \mathfrak{B}(E)$, we put

$$[f]_{\mathfrak{m}} := \{\widetilde{f}\colon E \to [-\infty, +\infty] \colon \widetilde{f}(x) = f(x) \text{ for } \mathfrak{m}\text{-a.e. } x \in E\}. \tag{2.9}$$

We put $L_0(E, \mathfrak{m}) := \{[f]_{\mathfrak{m}} \colon f \in \mathfrak{B}(E)\}$ and $L_0(\mathfrak{m}) := L_0(\operatorname{supp} \mathfrak{m}, \mathfrak{m})$. Given $p \in (0, \infty)$ and a Borel nonempty set $E \subset \operatorname{supp} \mathfrak{m}$, we let $L_p(E, \mathfrak{m})$ ($L_p^{\mathrm{loc}}(E, \mathfrak{m})$) denote the linear space of $\mathfrak{m}$-equivalence classes $[f]_{\mathfrak{m}}$ of all functions $f \in \mathfrak{B}(E)$ which are (locally) p-integrable on E with respect to the measure $\mathfrak{m}$. We let $L_\infty(E, \mathfrak{m})$ ($L_\infty^{\mathrm{loc}}(E, \mathfrak{m})$) denote the linear space of $\mathfrak{m}$-equivalence classes of all (locally) bounded on E Borel functions. For each $p \in [0, \infty]$ we set $L_p(\mathfrak{m}) := L_p(\operatorname{supp} \mathfrak{m}, \mathfrak{m})$ and $L_p^{\mathrm{loc}}(\mathfrak{m}) := L_p^{\mathrm{loc}}(\operatorname{supp} \mathfrak{m}, \mathfrak{m})$. Given $p \in [0, \infty]$, for a sequence of measures $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ on X we put

$$L_p(\{\mathfrak{m}_k\}) := \bigcap_{k=0}^{\infty} L_p(\mathfrak{m}_k) \quad \text{and} \quad L_p^{\mathrm{loc}}(\{\mathfrak{m}_k\}) := \bigcap_{k=0}^{\infty} L_p^{\mathrm{loc}}(\mathfrak{m}_k).$$

Given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, a parameter $p \in [0, \infty]$ and a Borel set $S \subset \mathrm{X}$, we use the notation $L_p(S) := L_p(\mu\lfloor_S)$.

We introduce the following important definition.

Definition 1. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ be a complete separable metric space. Given a measure $\mathfrak{m}$ on X and a Borel set $E \subset \operatorname{supp} \mathfrak{m}$, we define a mapping $\mathrm{I}_{\mathfrak{m}} \colon \mathfrak{B}(E) \to L_0(E, \mathfrak{m})$ by setting $\mathrm{I}_{\mathfrak{m}}(f) := [f]_{\mathfrak{m}}$ for each function $f \in \mathfrak{B}(E)$.

Remark 2. Typically, given a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, a measure $\mathfrak{m}$ on X, and a parameter $p \in [0, \infty]$, it will be convenient to identify the functions $\widetilde{f} \in [f]_{\mathfrak{m}}$ for each $[f]_{\mathfrak{m}} \in L_p(\mathfrak{m})$. We follow this path whenever our statements depend only on the equivalence classes without further mention, provided that this is clear from the context. In this case we use the symbol f instead of $[f]_{\mathfrak{m}}$ and the phrase 'a function f belongs to $L_p(\mathfrak{m})$' should be interpreted as $[f]_{\mathfrak{m}} \in L_p(\mathfrak{m})$. But we do not consider functions agreeing $\mathfrak{m}$-almost everywhere to be identical if we are concerned with fine pointwise properties of the single function.

Given a metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a family of sets $\mathcal{G} \subset 2^{\mathrm{X}}$, we denote by $\mathcal{M}(\mathcal{G})$ its *covering multiplicity*, that is, the minimum integer $M' \in \mathbb{N}_0 \cup \{+\infty\}$ such that every point $x \in \mathrm{X}$ belongs to at most M' sets from $\mathcal{G}$. We say that a family $\mathcal{G}$ is *disjoint* if $\mathcal{M}(\mathcal{G}) \leqslant 1$. The following proposition is elementary; we omit the proof.

Proposition 4. *Let $\mathfrak{m}$ be a measure on a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$. Let $\mathcal{G} \subset 2^{\mathrm{X}}$ be an at most countable family of sets with $\mathcal{M}(\mathcal{G}) < +\infty$. Then*

$$\sum_{G \in \mathcal{G}} \int_G |f(x)|\, d\mathfrak{m}(x) \leqslant \mathcal{M}(\mathcal{G}) \int_{\mathrm{G}} |f(x)|\, d\mathfrak{m}(x) \quad \textit{for every } f \in L_1(\mathfrak{m}\lfloor_{\mathrm{G}}), \tag{2.10}$$

where $\mathrm{G} = \bigcup \{G \colon G \in \mathcal{G}\}$.

Given a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ and a measure $\mathfrak{m}$ on X, for each $f \in L_1^{\mathrm{loc}}(\mathfrak{m})$, and every bounded Borel set $G \subset \mathrm{X}$ such that $\mathfrak{m}(G) < +\infty$ we put

$$f_{G,\mathfrak{m}} := \fint_G f(x)\, d\mathfrak{m}(x) := \begin{cases} \dfrac{1}{\mathfrak{m}(G)} \displaystyle\int_G f(x)\, d\mathfrak{m}(x), & \mathfrak{m}(G) > 0, \\ 0, & \mathfrak{m}(G) = 0. \end{cases} \tag{2.11}$$

Furthermore, we put

$$\mathcal{E}_{\mathfrak{m}}(f, G) := \inf_{c \in \mathbb{R}} \fint_G |f(x) - c|\, d\mathfrak{m}(x). \tag{2.12}$$

In order to built a fruitful theory we work with measures satisfying some restrictions.

Definition 2. Given a complete separable metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, we say that a measure $\mathfrak{m}$ on X has a *uniformly locally doubling property* if, for each $R > 0$,

$$C_{\mathfrak{m}}(R) := \sup_{r \in (0,R]} \sup_{x \in \mathrm{X}} \frac{\mathfrak{m}(B_{2r}(x))}{\mathfrak{m}(B_r(x))} < +\infty. \tag{2.13}$$

Remark 3. Clearly, we have the following chain of inequalities:

$$\mathcal{E}_{\mathfrak{m}}(f, G) \leqslant \fint_G |f(x) - f_{G,\mathfrak{m}}|\, d\mathfrak{m}(x) \leqslant \fint_G \fint_G |f(x) - f(y)|\, d\mathfrak{m}(x)\, d\mathfrak{m}(y) \leqslant 2\mathcal{E}_{\mathfrak{m}}(f, G).$$

Furthermore, if $\mathfrak{m}$ has a uniformly locally doubling property, then it follows easily from the above chain of inequalities that for each $R > 0$ and $c \geqslant 1$ there is a constant $C > 0$ such that for every pair $(x, r) \in \mathrm{X} \times (0, R]$

$$|f_{B_r(x'),\mathfrak{m}} - f_{B_{cr}(x),\mathfrak{m}}| \leqslant C\mathcal{E}_{\mathfrak{m}}(f, B_{cr}(x)) \quad \text{for } B_r(x') \subset B_{cr}(x). \tag{2.14}$$

We will sometimes use the following rough upper estimate of $\mathcal{E}_{\mathfrak{m}}(f, G)$, which is an easy consequence of Remark 3, Hölder's inequality for sums and Hölder's inequality for integrals.

Proposition 5. *If* $p \in [1, \infty)$, *then*

$$(\mathcal{E}_{\mathfrak{m}}(f, G))^p \leqslant 2^p \fint_G |f(x)|^p\, d\mathfrak{m}(x).$$

Given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, it is well known that the global doubling property of the measure μ implies the globally metric doubling property of the space (X, d) (see, for example, p. 102 in [1]). Similarly, we have the following result (we put $[c] := \max\{k \in \mathbb{Z} \colon k \leqslant c\}$).

Proposition 6. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space. If* μ *has a uniformly locally doubling property, then for all* $R > 0$ *and* $c \geqslant 1$, *any closed ball* $B = B_{cR}(x)$ *contains at most* $N_\mu(R, c) := [(C_\mu((c+1)R))^{\log_2(2c)+1}] + 1$ *disjoint closed balls of radius* R.

Proof. If $B' = B_R(x') \subset B$ then $B \subset B_{2cR}(x')$. Applying (2.13) $[\log_2(2c)]+1$ times we have

$$\mu(B) \leqslant (C_\mu((c+1)R))^{[\log_2(2c)]+1}\mu(B') \leqslant (C_\mu((c+1)R))^{\log_2(2c)+1}\mu(B').$$

If $\mathcal{B}$ is a disjoint family of closed balls of radius R in B, then $\sum\{\mu(B')\colon B' \in \mathcal{B}\} \leqslant \mu(B)$. Hence

$$\#\mathcal{B}\frac{\mu(B)}{(C_\mu((c+1)R))^{\log_2(2c)+1}} \leqslant \sum\{\mu(B')\colon B' \in \mathcal{B}\} \leqslant \mu(B).$$

As a result, we have $\#\mathcal{B} \leqslant N_\mu(R,c)$.

Given a number $\epsilon \in (0,1)$ and a family of closed balls $\mathcal{B} \subset 2^{\mathrm{X}}$, for each $k \in \mathbb{Z}$ we put

$$\mathcal{B}(k,\epsilon) := \{B \in \mathcal{B}\colon r_B \in (\epsilon^{k+1}, \epsilon^k]\}. \tag{2.15}$$

Proposition 7. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space. If* μ *has a uniformly locally doubling property, then for each* $c \geqslant 1$, $\epsilon \in (0,1)$, *and any disjoint family of closed balls* $\mathcal{B}$,

$$\mathcal{M}(\{cB\colon B \in \mathcal{B}(k,\epsilon)\}) \leqslant N_\mu\left(\epsilon^{k+1}, \frac{2c}{\epsilon}\right) \quad \textit{for every } k \in \mathbb{Z},$$

where the number $N_\mu(\epsilon^{k+1}, 2c/\epsilon)$ *is the same as in Proposition 6.*

Proof. Fix $c \geqslant 1$, $\epsilon \in (0,1)$, a disjoint family of closed balls $\mathcal{B}$ and a number $k \in \mathbb{Z}$. Consider the family $\widetilde{\mathcal{B}}(k,\epsilon)$ consisting of the closed balls whose centres are exactly the same as in the family $\mathcal{B}(k,\epsilon)$ but of radius ϵ^{k+1}. Given a point $x \in \mathrm{X}$, if $x \in cB$ for some $B \in \mathcal{B}(k,\epsilon)$, then $B \subset B_{2c\epsilon^k}(x)$. Since the family $\widetilde{\mathcal{B}}(k,\epsilon)$ is disjoint, by Proposition 6,

$$\begin{aligned}\mathcal{M}(\{cB\colon B \in \mathcal{B}(k,\epsilon)\}) &\leqslant \sup_{x\in\mathrm{X}} \sum_{B\in\mathcal{B}(k,\epsilon)} \chi_{cB}(x)\\ &\leqslant \sup_{x\in\mathrm{X}} \#\{B \in \widetilde{\mathcal{B}}(k,\epsilon)\colon B \subset B_{2c\epsilon^k}(x)\} \leqslant N_\mu\left(\epsilon^{k+1}, \frac{2c}{\epsilon}\right).\end{aligned} \tag{2.16}$$

The proof is complete.

The following proposition, which is an easy consequence of Proposition 6, is also well known (we recall that all balls are assumed to be closed).

Proposition 8. *Let* $(\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space. Let the measure* μ *have the uniformly locally doubling property. Then each ball* $B = B_r(x)$ *is a compact subset of* X.

We recall the notation (2.2) and (2.4).

Definition 3. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ be a complete separable metric space and $\epsilon \in (0,1)$. We say that a partial order $\preceq$ on $Z(\mathrm{X},\epsilon)$ is admissible if the following properties hold:

(PO1) if $z_{k,\alpha} \preceq z_{l,\beta}$ for some $k,l \in \mathbb{Z}$, then $k \geqslant l$;

(PO2) for any $l \leqslant k$ and $z_{k,\alpha} \in Z_k(\mathrm{X},\epsilon)$ there is a unique $z_{l,\beta} \in Z_l(\mathrm{X},\epsilon)$ such that $z_{k,\alpha} \preceq z_{l,\beta}$;

(PO3) if $k \in \mathbb{Z}$ and $z_{k,\alpha} \preceq z_{k-1,\beta}$, then $\mathrm{d}(z_{k,\alpha}, z_{k-1,\beta}) < \epsilon^{k-1}$;
(PO4) if $k \in \mathbb{Z}$ and $\mathrm{d}(z_{k,\alpha}, z_{k-1,\beta}) < \frac{\epsilon^{k-1}}{2}$, then $z_{k,\alpha} \preceq z_{k-1,\beta}$.

The following proposition was proved in [33].

Proposition 9. *For any complete separable metric space* $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ *and any parameter* $\epsilon \in (0,1)$ *there exists at least one admissible partial order on the set* $Z(\mathrm{X}, \epsilon)$.

According to one beautiful result of Christ [33], given a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, if the measure μ is globally doubling, then there exists a natural analogue of Euclidean dyadic cubes in $\mathbb{R}^n$. However, an analysis of the arguments in [33] shows that, in fact, the uniformly locally doubling property of μ is sufficient to establish the following result.

Proposition 10. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ *be a complete separable metric space. Let* $\epsilon \in (0, 1/10]$, *and let* $\preceq$ *be an admissible partial order on the set* $Z(\mathrm{X}, \epsilon)$. *Given* $a \in (0, 1/8]$, *for each* $k \in \mathbb{Z}$, *and every* $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$ *let the generalized dyadic cube* $Q_{k,\alpha}$ *in the space* X *be defined by the equality*

$$Q_{k,\alpha} := \bigcup_{z_{j,\beta} \preceq z_{k,\alpha}} \operatorname{int} B_{a\epsilon^k}(z_{j,\beta}). \tag{2.17}$$

Then the family $\{Q_{k,\alpha}\} := \{Q_{k,\alpha} \colon k \in \mathbb{Z}, \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)\}$ *is at most countable and has the following properties*:
(DQ1) *for each* $k \in \mathbb{Z}$ *the equality* $\mathrm{X} = \bigcup_{\alpha \in \mathcal{A}_k(\mathrm{X},\epsilon)} \operatorname{cl} Q_{k,\alpha}$ *holds*;
(DQ2) *if* $j \geqslant k$, *then either* $Q_{j,\beta} \subset Q_{k,\alpha}$ *or* $Q_{j,\beta} \cap Q_{k,\alpha} = \varnothing$;
(DQ3) *if* $l < k$ *and* $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$, *then there is a unique* $\beta \in \mathcal{A}_l(\mathrm{X}, \epsilon)$ *such that* $Q_{k,\alpha} \subset Q_{l,\beta}$;
(DQ4) $B_{\epsilon^k/8}(z_{k,\alpha}) \subset Q_{k,\alpha} \subset B_{2\epsilon^k}(z_{k,\alpha})$ *for each* $k \in \mathbb{Z}$ *and any* $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$.
If, in addition, a measure μ *on* X *has a uniformly locally doubling property, then*
(DQ5) $\mu(\partial Q_{k,\alpha}) = 0$ *for each* $k \in \mathbb{Z}$ *and every* $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$.

Given $\epsilon \in (0,1)$ and $r > 0$, we use the following important notation:

$$k(r) := k_\epsilon(r) := \max\{k \in \mathbb{Z} \colon r \leqslant \epsilon^k\}. \tag{2.18}$$

Proposition 11. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space, and let the measure* μ *have the uniformly locally doubling property. Let* $\epsilon \in (0,1)$, $\underline{k} \in \mathbb{Z}$, *and let* $\{Q_{k,\alpha}\}$ *be a family of generalized dyadic cubes. For each* $c \geqslant 1$ *there exists a constant* $C_D(c, \underline{k}) > 0$ *depending only on* $C_\mu((c + 1 + \frac{4}{\epsilon})\epsilon^{\underline{k}})$, ϵ, c *and* $\underline{k}$ *such that, for each* $x \in \mathrm{X}$ *and any* $r \in (0, \epsilon^{\underline{k}}]$,

$$\#\{\alpha \in \mathcal{A}_{k(r)}(\mathrm{X}, \epsilon) \colon \operatorname{cl} Q_{k(r),\alpha} \cap B_{cr}(x) \neq \varnothing\} \leqslant C_D(c, \underline{k}). \tag{2.19}$$

Proof. Note that if $\operatorname{cl} Q_{k(r),\alpha} \cap B_{cr}(x)$, then by property (DQ4) in Proposition 10 we have the following inclusions

$$\operatorname{cl} Q_{k(r),\alpha} \subset \left(c + \frac{4\epsilon^{k(r)}}{r}\right) B_r(x) \subset \left(c + \frac{4}{\epsilon}\right) B_r(x).$$

On the other hand, the closed balls $\frac{1}{2} B_{\epsilon^{k(r)}}(z_{k(r),\alpha})$, $\alpha \in \mathcal{A}_{k(r)}(\mathrm{X}, \epsilon)$, are disjoint. As a result, an application of Proposition 6 proves the claim.

Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space. In the case when μ has a uniformly locally doubling property, given $q \in (1, \infty)$, $\alpha \geqslant 0$ and $R > 0$, we define the *local fractional maximal function* $M^R_{q,\alpha}(f)$ of $f \in L^{\mathrm{loc}}_1(\mathrm{X})$ by

$$M^R_{q,\alpha}(f)(x) := \sup_{r \in (0,R]} r^{\alpha} \left(\fint_{B_r(x)} |f(y)|^q \, d\mu(y) \right)^{1/q}, \qquad x \in \mathrm{X}. \tag{2.20}$$

It is well known that the doubling condition coupled with Vitali's $5B$-covering lemma (see § 3.3 in [1]) allows one to prove the following proposition.

Proposition 12. *Let* $(\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space, and let the measure* μ *have a uniformly locally doubling property. Let* $p \in (1, \infty)$ *and* $q \in (1, p)$. *Then for every* $R > 0$ *there is a constant* $C > 0$ *depending only on* p, q *and* $C_{\mu}(R)$ *such that*

$$\|M^R_{q,0}(f)\|_{L_p(\mathrm{X})} \leqslant C \|f\|_{L_p(\mathrm{X})} \quad \textit{for all } f \in L_p(\mathrm{X}). \tag{2.21}$$

Given $q \in [1, \infty)$, a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ is said to *support a weak local* $(1, q)$*-Poincaré inequality* if for each $R > 0$ there are constants $C = C(R) > 0$ and $\lambda = \lambda(R) \geqslant 1$ such that for any function $f \in \mathrm{LIP}(\mathrm{X})$ (we use the notation (2.12))

$$\mathcal{E}_{\mu}(f, B_r(x)) \leqslant Cr \left(\fint_{B_{\lambda r}(x)} (\operatorname{lip} f(y))^q \, d\mu(y) \right)^{1/q} \quad \text{for all } (x, r) \in \mathrm{X} \times (0, R]. \tag{2.22}$$

Remark 4. Recall [3] that a function $g \in \mathfrak{B}(\mathrm{X})$ is said to be an *upper gradient of a function* $f \in \mathfrak{B}(\mathrm{X})$ if for every absolutely continuous curve $\gamma \colon [0, 1] \to \mathrm{X}$,

$$|f(\gamma(1)) - f(\gamma(0))| \leqslant \int_0^1 g(\gamma(s)) |\dot{\gamma}_s| \, ds,$$

where $|\dot{\gamma}_s|$ is the so-called metric speed of γ at $s \in [0, 1]$, that is,

$$|\dot{\gamma}_s| := \varlimsup_{t \to s} \frac{\mathrm{d}(\gamma(s), \gamma(t))}{|t - s|}.$$

Notice that in the literature inequality (2.22) is typically required to hold for Borel representatives of $f \in L^{\mathrm{loc}}_1(\mathrm{X})$ and their upper gradients. However, using results of [3] it is not difficult to establish the equivalence of these two approaches. In other words, inequality (2.22) holds true for all functions $f \in \mathrm{LIP}(\mathrm{X})$ if and only if, for every Borel function $f \in L^{\mathrm{loc}}_1(\mathrm{X})$ (recall Remark 2) and each upper gradient g of f,

$$\mathcal{E}_{\mu}(f, B_r(x)) \leqslant Cr \left(\fint_{B_{\lambda r}(x)} (g(y))^q \, d\mu(y) \right)^{1/q} \quad \text{for all } (x, r) \in \mathrm{X} \times (0, R]. \tag{2.23}$$

Here the positive constant C is the same as in (2.22).

If μ is doubling, then a similar result was established in Theorem 8.4.2 in [1].

In this paper we always work with a *special class* of metric measure spaces which is commonly used in modern geometric analysis.

Definition 4. Given $q \in [1,\infty)$, we say that a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ is q-admissible and write $\mathrm{X} \in \mathfrak{A}_q$ if the measure μ has a uniformly locally doubling property and X supports a weak local $(1,q)$-Poincaré inequality.

The following powerful result due to Keith and Zhong will be useful for us (see Ch. 12 in [1] for a detailed proof and historical remarks).

Proposition 13. *Let $p \in (1,\infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Then there is a parameter $q \in [1,p)$ such that $\mathrm{X} \in \mathfrak{A}_q$.*

Our assumptions about the space X under consideration are quite typical in modern geometric analysis and imply some nice properties of X. In the beautiful monograph [1] the reader can find a detailed exposition of the theory of metric measure spaces satisfying the assumptions adopted in our paper. We have the following result.

Proposition 14. *Let $\mathrm{X} \in \mathfrak{A}_p$ for some $p \in [1,\infty)$. Then the space X has the following properties*:

(1) *the metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$ is locally convex, that is, for each $R > 0$ there exists a constant $L(R) \geqslant 1$ such that any two points $x, y \in \mathrm{X}$ for which $\mathrm{d}(x,y) \leqslant R$ can be joined by a curve $\gamma_{x,y}$ of length $l(\gamma_{x,y}) \leqslant L(R)\,\mathrm{d}(x,y)$;*

(2) *for each $R > 0$ there is a number $Q = Q(R) > 0$ such that the measure μ has a relative volume decay property of order Q up to the scale R, that is, there exists a constant $\overline{C}(R,Q) > 0$ such that, for any balls $\underline{B} \subset \overline{B}$ of radii $0 < r_{\underline{B}} \leqslant r_{\overline{B}} \leqslant R$,*

$$\left(\frac{r(\underline{B})}{r(\overline{B})}\right)^{Q} \leqslant \overline{C}(R,Q)\frac{\mu(\underline{B})}{\mu(\overline{B})}; \tag{2.24}$$

(3) *for each $R > 0$ there is a number $q = q(R) > 0$ such that the measure μ has a reverse relative volume decay property of order q up to the scale R, that is, there exists a constant $\underline{C}(R,Q) > 0$ such that, for any balls $\underline{B} \subset \overline{B}$ of radii $0 < r_{\underline{B}} \leqslant r_{\overline{B}} \leqslant R$,*

$$\frac{\mu(\underline{B})}{\mu(\overline{B})} \leqslant \underline{C}(R,q)\left(\frac{r(\underline{B})}{r(\overline{B})}\right)^{q}. \tag{2.25}$$

Proof. To prove (1) it is sufficient to repeat, with appropriate technical modifications, the arguments in the proof of Theorem 8.3.2 in [1] and take Remark 4 into account.

To establish (2) one needs to modify the arguments in the proof of Lemma 8.1.13 in [1].

To prove (3) it is sufficient to use the arguments in Remark 8.1.15 in [1].

Having Proposition 14 at our disposal we formulate the following definition.

Definition 5. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space, and let μ have a uniformly locally doubling property. Given $R > 0$, we let $\mathrm{Q}_{\mu}(R)$ denote the set of all $Q > 0$ for each of which (2.24) holds. Furthermore, we set $\underline{Q}_{\mu}(R) := \inf\{Q \colon Q \in \mathrm{Q}_{\mu}(R)\}$. Similarly, we denote by $\mathrm{q}_{\mu}(R)$ the set of all $q > 0$ for which (2.25) holds. We set $\overline{q}_{\mu}(R) := \sup\{q \colon q \in \mathrm{q}_{\mu}(R)\}$.

Remark 5. It is clear that $\overline{q}_{\mu}(R) \leqslant \underline{Q}_{\mu}(R)$ for any $R > 0$. Unfortunately, given $R > 0$, in many cases there is a 'gap' between these parameters, that is, $\overline{q}_{\mu}(R)$ can be much smaller than $\underline{Q}_{\mu}(R)$. The reader can find interesting examples illustrating this phenomenon in [34].

It is well known that in Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$, the d-Hausdorff measures provide a useful tool for measuring some $\mathcal{L}^n$-negligible sets. Clearly, from Remark 5 it follows that the dependence of $\mu(B_r(x))$ on r is not powerlike in general. By this reason it is natural to construct *codimensional substitutions* for the usual Hausdorff contents and measures. More precisely, following [12], [11], [14], [15] and [17], given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ with locally uniformly doubling measure μ and a parameter $\theta \geqslant 0$, for $\delta \in (0, \infty]$, for each set $E \subset \mathrm{X}$ we put

$$\mathcal{H}_{\theta,\delta}(E) := \inf\biggl\{\sum \frac{\mu(B_{r_i}(x_i))}{(r_i)^{\theta}} : E \subset \bigcup B_{r_i}(x_i) \text{ and } r_i < \delta\biggr\}, \tag{2.26}$$

where the infimum is taken over all at most countable coverings of E by closed balls $\{B_{r_i}(x_i)\}$ of radii $r_i \in (0, \delta)$. Given $\delta > 0$, the mapping $\mathcal{H}_{\theta,\delta} \colon 2^{\mathrm{X}} \to [0, +\infty]$ is called the *θ-codimensional Hausdorff content* at the scale of δ. We define the *θ-codimensional Hausdorff measure* by the equality

$$\mathcal{H}_{\theta}(E) := \lim_{\delta \to 0} \mathcal{H}_{\theta,\delta}(E). \tag{2.27}$$

Remark 6. It is clear that, given $\theta \in [0, \underline{Q}_{\mu}(R))$, $R > 0$, the equality $\mathcal{H}_{\theta}(\varnothing) = 0$ follows from the existence of a sequence of (closed) balls $\{B_i\} := \{B_{r_i}(x_i)\}_{i=1}^{\infty}$ of radii $r_i \to 0$ as $i \to \infty$, such that $\mu(B_i)/(r_i)^{\theta} \to 0$, as $i \to \infty$. As a result, by Theorem 4.2 in [35], in this case $\mathcal{H}_{\theta} \colon 2^{\mathrm{X}} \to [0, +\infty]$ is a Borel regular outer measure on X. Clearly, the inequality $0 \leqslant \theta < \overline{q}_{\mu}$ is sufficient for this. Unfortunately, this condition is far from necessary.

The problem of finding an appropriate range of parameters for which $\mathcal{H}_{\theta}$ is a nontrivial outer measure (that is, there exist nonempty subsets of X with finite positive measure) appears to be quite subtle and depends on the particular structure of a given metric measure space. The situation is completely transparent for so-called Ahlfors Q-regular spaces, that is, when $\mu(B_r(x)) \approx r^Q$, $r > 0$, $x \in \mathrm{X}$, for some $Q \geqslant 0$ (independent on x and r). In this case $\mathcal{H}_{\theta}$ is a nontrivial outer measure in the full range of $\theta \in [0, Q)$. In the case when $\theta = Q$ the measure $\mathcal{H}_Q$ is a counting measure and $\mathcal{H}_Q(E) = +\infty$ for any infinite set E.

In what follows we use the following result from [11] (see Lemma 3.10 and the discussion after the lemma).

Proposition 15. *Let $f \in L_1^{\mathrm{loc}}(\mathrm{X})$, suppose $t > 0$, and set*

$$\Lambda_t := \biggl\{x \in \mathrm{X} \colon \varlimsup_{r \to 0} r^t \fint_{B_r(x)} |f(y)|\, d\mu(y) > 0\biggr\}.$$

Then $\mathcal{H}_t(\Lambda_t) = 0$.

There is a special class of m.m.s. for which the behaviour of $\mu(B_r(x))$ is, roughly speaking, expressed by the function r^Q for some $Q > 0$. The detailed discussion of such spaces is beyond the scope of this paper. We mention only [7], [17], [36] and [37], where the reader can find interesting results related to such spaces.

Definition 6. Given $Q > 0$, we say that a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ is *Ahlfors Q-regular* if there exist constants $c_{\mu,1}$, $c_{\mu,2} > 0$ such that

$$c_{\mu,1} r^Q \leqslant \mu(B_r(x)) \leqslant c_{\mu,2} r^Q \quad \text{for all } (x,r) \in \mathrm{X} \times [0, \operatorname{diam} \mathrm{X}).$$

2.2. Sobolev calculus on metric measure spaces. As mentioned in § 1, given an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ and a parameter $p \in (1, \infty)$, there are at least five different approaches to the definition of Sobolev-type spaces on X. In the literature the corresponding spaces are as follows: the Korevaar-Schoen-Sobolev space $KS^1_p(\mathrm{X})$ [5], [9], the Hajlasz-Sobolev space $M^1_p(\mathrm{X})$ [6], the Cheeger-Sobolev space $Ch^1_p(\mathrm{X})$ [7], the Newtonian-Sobolev space $N^1_p(\mathrm{X})$ [8] and the Sobolev space $W^1_p(\mathrm{X})$ [4], [3], [38]. The reader can also find some useful information relating to these spaces in Ch. 10 of [1], the lecture notes [2] and [39].

Remark 7. Given an arbitrary m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ and a parameter $p \in (1, \infty)$, there are canonical isometric isomorphisms between $Ch^1_p(\mathrm{X})$, $N^1_p(\mathrm{X})$ and $W^1_p(\mathrm{X})$ [3]. Furthermore, if $\mathrm{X} \in \mathfrak{A}_p$ for some $p \in (1, \infty)$, then it follows from the results of [3] and [9] that $KS^1_p(\mathrm{X}) = M^1_p(\mathrm{X}) = Ch^1_p(\mathrm{X}) = N^1_p(\mathrm{X}) = W^1_p(\mathrm{X})$, where equalities should be interpreted in the sense of the existence of canonical (not necessarily isometric in general!) isomorphisms, the corresponding norms being equivalent. In all main results of our paper we always assume that $\mathrm{X} \in \mathfrak{A}_p$ for some $p \in (1, \infty)$. Hence, when dealing with Sobolev spaces, without loss of generality one can identify (in an appropriate sense) different Sobolev spaces and use the symbol $W^1_p(\mathrm{X})$ to denote each of them. However, it will be convenient for us to use Cheeger's approach elaborated originally in [7] and modified in [3].

Keeping in mind Remark 7 we recall the approach of Cheeger to Sobolev spaces in the Lipschitz interpretation of [3].

Definition 7. Given $p \in (1, \infty)$, the *Sobolev space* $W^1_p(\mathrm{X})$ is a linear space consisting of all $F \in L_p(\mathrm{X})$ satisfying $\mathrm{Ch}_p(F) < +\infty$, where $\mathrm{Ch}_p(F)$ is the *Cheeger p-energy of F* defined by

$$\mathrm{Ch}_p(F) := \inf\left\{ \varliminf_{n\to\infty} \int_{\mathrm{X}} (\operatorname{lip} F_n)^p \, d\mu \colon \{F_n\} \subset \operatorname{LIP}(\mathrm{X}),\ F_n \to F \text{ in } L_p(\mathrm{X}) \right\}.$$

The norm in the space $W^1_p(\mathrm{X})$ is defined by

$$\|F\|_{W^1_p(\mathrm{X})} := \|F\|_{L_p(\mathrm{X})} + (\mathrm{Ch}_p(F))^{1/p}.$$

Remark 8. It is well known that for each $p \in (1, \infty)$ and any $F \in W^1_p(\mathrm{X})$ there is a well-defined nonnegative function $|\nabla F|_{*,p} \in L_p(\mathrm{X})$ (in fact, a μ-equivalence class of functions), called the *minimal p-relaxed slope* of F, which, if X is a smooth Riemannian manifold, coincides μ-almost everywhere with the modulus of the distributional differential of F. Furthermore, $\mathrm{Ch}_p(F) = \||\nabla F|_{*,p}\|_{L_p(\mathrm{X})}$ (see [3]). However, it follows from the results of [3] and [40] that, by contrast with the classical settings, the minimal p-relaxed slope can depend on p.

The following assertion will be important in the proofs of some key estimates in § 10.

Proposition 16. *Let $R > 0$, $q \in (1,\infty)$, $p \geqslant q$ and $\mathrm{X} \in \mathfrak{A}_q$. Then, for each $F \in W_p^1(\mathrm{X})$,*

$$\mathcal{E}_\mu(F, B_r(x)) \leqslant Cr\left(\fint_{B_{\lambda r}(x)} (|\nabla F|_{*,p})^q \, d\mu(y)\right)^{1/q} \quad \text{for all } (x,r) \in \mathrm{X}\times(0,R], \tag{2.28}$$

where $C = C(R)$ and $\lambda = \lambda(R)$ are the same constants as in (2.22).

Proof. According to the main results of [3], given $F \in W_p^1(\mathrm{X})$, there is a sequence $\{F_n\} \subset \mathrm{LIP}(\mathrm{X})$ such that $F_n \to F$ as $n \to \infty$ in the $L_p(\mathrm{X})$-sense and $\operatorname{lip} F_n \to |\nabla F|_{*,p}$ as $n \to \infty$ in the $L_p(\mathrm{X})$-sense. Hence, taking into account that $L_p(\mathrm{X}) \subset L_t^{\mathrm{loc}}(\mathrm{X})$ for all $t \in [1,p]$, we use (2.22) and pass to the limit as $n \to \infty$. This gives (2.28) and completes the proof.

It was shown in [38] that under mild assumptions on an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, the Sobolev space $W_p^1(\mathrm{X})$ is reflexive for every $p \in (1,\infty)$. In particular, we have the following result.

Proposition 17. *Let $p \in (1,\infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Then the Sobolev space $W_p^1(\mathrm{X})$ is reflexive.*

Remark 9. In fact, it was assumed in [38] that the metric space (X, d) is globally metrically doubling. However, a careful analysis of the proof shows that the uniformly locally doubling property of the measure μ is sufficient.

2.3. Traces of Sobolev spaces. We assume that the reader is familiar with the notion and basic properties of so-called Sobolev p-capacities C_p, $p \in (1,\infty)$ (see §§ 7.2 and 9.2 in [1] and § 1.4 in [10] for details). In fact, the main properties of p-capacities sufficient for our purposes are contained in the following proposition.

Proposition 18. *Let $p \in (1,\infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Then the following properties hold:*

(1) *for each $F \in W_p^1(\mathrm{X})$ there is a set E_F with $C_p(E_F) = 0$ such that*

$$\overline{F}(x) := \varlimsup_{r\to 0} \fint_{B_r(x)} F(y) \, d\mu(y) \in \mathbb{R} \quad \text{for all } x \in \mathrm{X}\setminus E_F, \tag{2.29}$$

and, furthermore, each $x \in \mathrm{X}\setminus E_F$ is a μ-Lebesgue point of $\overline{F}$;

(2) *if $\theta \in [0,p)$, then $C_p(E) = 0$ implies that $\mathcal{H}_\theta(E) = 0$ for any Borel set $E \subset \mathrm{X}$.*

Proof. To prove (1) one should repeat almost verbatim the arguments from the proof of Theorem 9.2.8 in [1] and note that the additional requirement $Q \geqslant 1$ was used only at the end of the proof to establish higher-order integrability.

Property (2) was proved in the recent paper [11] (see Proposition 3.11 therein).

Given a metric measure space $(\mathrm{X}, \mathrm{d}, \mu)$ and a parameter $p \in (1,\infty)$, a measure $\mathfrak{m}$ on X is said to be *absolutely continuous with respect to Sobolev p-capacity C_p* if, for any Borel set $E \subset \mathrm{X}$, the equality $C_p(E) = 0$ implies the equality $\mathfrak{m}(E) = 0$.

Definition 8. Let $p \in (1, \infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Let $S \subset \mathrm{X}$ be a Borel set such that $C_p(S) > 0$. Given an element $F \in W_p^1(\mathrm{X})$, we define the *p-sharp trace* $F|_S$ of F to the set S by the equality

$$F|_S := \{f \in \mathfrak{B}(S) \colon C_p(\{f(x) \neq \overline{F}(x)\}) = 0\},$$

where $\overline{F}$ is the representative of F defined in (2.29). Furthermore, we define the *p-sharp trace space* of the space $W_p^1(\mathrm{X})$ by the formula

$$W_p^1(\mathrm{X})|_S := \{F|_S \colon F \in W_p^1(\mathrm{X})\} \tag{2.30}$$

and equip this space with the usual quotient space norm, that is, given $f \in W_p^1(\mathrm{X})|_S$, we set

$$\|f\|_{W_p^1(\mathrm{X})|_S} := \inf\{\|F\|_{W_p^1(\mathrm{X})} \colon f = F|_S\}.$$

As already mentioned in § 1, sometimes it is useful to work with a relaxed version of the p-sharp trace space of the Sobolev $W_p^1(\mathrm{X})$-space. This motivates us to introduce the following concept.

Definition 9. Let $p \in (1, \infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Let $S \subset \mathrm{X}$ be a Borel set such that $C_p(S) > 0$. Let $\mathfrak{m}$ be a nonzero measure on X which is absolutely continuous with respect to C_p and such that $S \subset \operatorname{supp} \mathfrak{m}$. Given an element $F \in W_p^1(\mathrm{X})$, we define the $\mathfrak{m}$-*trace* $F|_S^{\mathfrak{m}}$ of F to the set S as the $\mathfrak{m}$-equivalence class of its p-sharp trace. More precisely,

$$F|_S^{\mathfrak{m}} := \{f \colon S \to \mathbb{R} \colon \mathfrak{m}(\{f(x) \neq \overline{F}(x)\}) = 0\},$$

where $\overline{F}$ is the representative of F defined in (2.29). Furthermore, we define the $\mathfrak{m}$-*trace space* of the space $W_p^1(\mathrm{X})$ by the formula

$$W_p^1(\mathrm{X})|_S^{\mathfrak{m}} := \{F|_S^{\mathfrak{m}} \colon F \in W_p^1(\mathrm{X})\}$$

and equip this space with the quotient space norm, that is, given $f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}}$, we set

$$\|f\|_{W_p^1(X)} := \inf\{\|F\|_{W_p^1(X)} \colon f = F|_S^{\mathfrak{m}}\}. \tag{2.31}$$

Having different notions of trace spaces at our disposal, it is natural to define the corresponding trace and extension operators.

Definition 10. Let $p \in (1, \infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Let $S \subset \mathrm{X}$ be a Borel set such that $C_p(S) > 0$. Let $\mathfrak{m}$ be a nonzero measure on X which is absolutely continuous with respect to C_p and such that $S \subset \operatorname{supp} \mathfrak{m}$. We define the *p-sharp trace operator* by the formula

$$\operatorname{Tr}|_S(F) = F|_S, \qquad F \in W_p^1(\mathrm{X}). \tag{2.32}$$

Furthermore, we define the $\mathfrak{m}$-*trace operator* by the equality

$$\operatorname{Tr}|_S^{\mathfrak{m}}(F) = F|_S^{\mathfrak{m}}, \qquad F \in W_p^1(\mathrm{X}). \tag{2.33}$$

Remark 10. Let us compare (2.32) and (2.33). If we identify functions that differ on a set of p-capacity zero, then one can write $\operatorname{Tr}|_S^{\mathfrak{m}} = \mathrm{I}_{\mathfrak{m}} \circ \operatorname{Tr}|_S$. This formula is correct because $\mathfrak{m}$ is absolutely continuous with respect to C_p according to our assumptions and, thus, if $f_1, f_2 \in \mathfrak{B}(S) \cap F|_S$ for some $F \in W_p^1(\mathrm{X})$, then $\mathrm{I}_{\mathfrak{m}}(f_1) = \mathrm{I}_{\mathfrak{m}}(f_2) = F|_S^{\mathfrak{m}}$.

Definition 11. Let $p \in (1,\infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Let $S \subset \mathrm{X}$ be a Borel set such that $C_p(S) > 0$. We say that a map $\operatorname{Ext}_S := \operatorname{Ext}_{S,p} \colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})$ is a *p-sharp extension operator* if it is a right inverse of $\operatorname{Tr}|_S$, and we say that a map $\operatorname{Ext}_{S,\mathfrak{m}} \colon W_p^1(\mathrm{X})|_S^{\mathfrak{m}} \to W_p^1(\mathrm{X})$ is an $\mathfrak{m}$-*extension operator* if it is a right inverse of $\operatorname{Tr}|_S^{\mathfrak{m}}$.

Remark 11. Typically, the index p will be fixed in the main theorems of the paper. For this reason we omit it from the notation for the p-sharp trace operator and the p-sharp extension operator. In contrast to this we keep the symbol $\mathfrak{m}$ in the notation of the $\mathfrak{m}$-trace operator and the $\mathfrak{m}$-extension operator. Indeed, generally speaking, given a closed set $S \subset \mathrm{X}$, there exists an infinite family of different measures whose supports coincide with the set S. As a result, by varying measures we obtain a different accuracy in the description of the trace to different 'pieces of the set S'.

Remark 12. In view of Definitions 8–10 it is clear that the p-sharp trace operator and the $\mathfrak{m}$-trace operator are linear and bounded.

§ 3. Relaxing the doubling property

Given a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, in the next sections we work frequently with measures $\mathfrak{m}$ on X that fail to have the uniformly locally doubling property (2.13). However, given a measure $\mathfrak{m}$ on X, in some cases it is sufficient to have some sort of the uniformly doubling property only for a fixed family of balls. This motivates us to introduce the following concept.

Definition 12. Given a locally finite measure $\mathfrak{m}$ on a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$, we say that $\mathfrak{m}$ has the *uniformly weak asymptotically doubling property* if, for each $c > 0$,

$$\underline{\mathrm{C}}_{\mathfrak{m}}(c) := \lim_{R\to+0} \sup_{x\in\operatorname{supp}\mathfrak{m}} \inf_{r\in(0,R]} \frac{\mathfrak{m}(B_{cr}(x))}{\mathfrak{m}(B_r(x))} < +\infty. \tag{3.1}$$

Remark 13. The word 'weak' was used in Definition 12 because in [1] one can find the notion of uniformly asymptotically doubling property, which means that, for each $c > 0$,

$$\overline{\mathrm{C}}_{\mathfrak{m}}(c) := \lim_{R\to+0} \sup_{x\in\operatorname{supp}\mathfrak{m}} \sup_{r\in(0,R]} \frac{\mathfrak{m}(B_{cr}(x))}{\mathfrak{m}(B_r(x))} < +\infty.$$

Typically, in the present paper we deal with measures that cannot degenerate too rapidly.

Definition 13. We say that a locally finite measure $\mathfrak{m}$ on a metric measure space, $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ is *weakly noncollapsed* if

$$C_{\mu}^{\mathfrak{m}} := \inf_{x\in\operatorname{supp}\mathfrak{m}} \varliminf_{r\to 0} \frac{\mathfrak{m}(B_r(x))}{\mu(B_r(x))} > 0. \tag{3.2}$$

It is well known that, given a measure $\mathfrak{m}$ on the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, there are a lot of 'doubling balls'. This fact was mentioned in [41] without a proof. We are grateful to D. M. Stolyarov who kindly shared with us the key idea of that proof. Using a similar idea we establish the following simple result, which is quite important in what follows. We recall property (2) in Proposition 14 and Definition 5.

Lemma 2. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space and let the measure* μ *have the uniformly locally doubling property. If a measure* $\mathfrak{m}$ *on* X *is weakly noncollapsed, then it satisfies the uniformly weak asymptotically doubling property. Furthermore, for all* $c > 1$ *and* $Q \in \mathrm{Q}_{\mu}(1)$,

$$\underline{\mathrm{C}}_{\mathfrak{m}}(c) \leqslant 2^{([\log_2 c]+1)Q}. \tag{3.3}$$

Proof. We fix $c > 1$ and assume for a contradiction that $\underline{\mathrm{C}}_{\mathfrak{m}}(c) > 2^{([\log_2 c]+1)Q}$. We fix $\underline{k} = [\log_2 c] + 1$ and $M \in (2^{\underline{k}Q}, \underline{\mathrm{C}}_{\mathfrak{m}}(c))$. It is clear that there exist a point $\underline{x} \in \operatorname{supp} \mathfrak{m}$ and a number $\overline{r} = \overline{r}(M, c) \in (0, 1)$ such that

$$\frac{\mathfrak{m}(B_{cr}(\underline{x}))}{\mathfrak{m}(B_r(\underline{x}))} > M \quad \text{for all } r \in (0, \overline{r}]. \tag{3.4}$$

By Definition 13 we clearly have

$$\varlimsup_{r \to 0} \frac{\mu(B_r(\underline{x}))}{\mathfrak{m}(B_r(\underline{x}))} \leqslant \frac{1}{C^{\mathfrak{m}}_{\mu}}. \tag{3.5}$$

Hence, combining (2.24) with (3.4) and (3.5), for all sufficiently large $i \in \mathbb{N}$ we obtain

$$\begin{aligned} 2^{-Q\underline{k}i} &\leqslant \overline{C}(1, Q) \frac{\mu(B_{\overline{r}/2^{i\underline{k}}}(\underline{x}))}{\mu(B_{\overline{r}}(\underline{x}))} = \overline{C}(1, Q) \frac{\mu(B_{\overline{r}/2^{i\underline{k}}}(\underline{x}))}{\mathfrak{m}(B_{\overline{r}/2^{i\underline{k}}}(\underline{x}))} \frac{\mathfrak{m}(B_{\overline{r}}(\underline{x}))}{\mu(B_{\overline{r}}(\underline{x}))} \frac{\mathfrak{m}(B_{\overline{r}/2^{i\underline{k}}}(\underline{x}))}{\mathfrak{m}(B_{\overline{r}}(\underline{x}))} \\ &\leqslant \frac{2\overline{C}(1, Q)}{C^{\mathfrak{m}}_{\mu}} \frac{\mathfrak{m}(B_{\overline{r}}(\underline{x}))}{\mu(B_{\overline{r}}(\underline{x}))} \left(\frac{1}{M}\right)^i. \end{aligned}$$

However, for sufficiently large $i \in \mathbb{N}$ the above chain of inequalities leads to a contradiction with the choice of M.

Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space. We recall the notation (2.18). Given a sequence of locally finite measures $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty}$ on X, a parameter $\epsilon \in (0, 1)$, and a Borel set $E \subset \bigcap_{k=0}^{\infty} \operatorname{supp} \mathfrak{m}_k$, for each $x \in E$ we introduce the *lower* and *upper* $(\{\mathfrak{m}_k\}, \epsilon)$*-densities* of E at x by setting

$$\underline{D}^{\{\mathfrak{m}_k\}}_E(x, \epsilon) := \varliminf_{r \to 0} \frac{\mathfrak{m}_{k_\epsilon(r)}(B_r(x) \cap E)}{\mathfrak{m}_{k_\epsilon(r)}(B_r(x))} \quad \text{and} \quad \overline{D}^{\{\mathfrak{m}_k\}}_E(x, \epsilon) := \varlimsup_{r \to 0} \frac{\mathfrak{m}_{k_\epsilon(r)}(B_r(x) \cap E)}{\mathfrak{m}_{k_\epsilon(r)}(B_r(x))}. \tag{3.6}$$

We say that $x \in E$ is an $(\{\mathfrak{m}_k\}, \epsilon)$*-density point of* E if

$$\underline{D}^{\{\mathfrak{m}_k\}}_E(x, \epsilon) = \overline{D}^{\{\mathfrak{m}_k\}}_E(x, \epsilon) = 1.$$

It is clear that if there is a measure $\mathfrak{m}$ on X such that $\mathfrak{m}_k = \mathfrak{m}$ for all $k \in \mathbb{N}_0$, then we obtain the standard lower and upper $\mathfrak{m}$-densities of E at x, which are denoted by $\underline{D}^{\mathfrak{m}}_E(x)$ and $\overline{D}^{\mathfrak{m}}_E(x)$, respectively (in this case the parameter ϵ is irrelevant and we omit it from our notation).

It is well known that if $\mathfrak{m}$ is a locally uniformly doubling measure on X, then $\mathfrak{m}$-almost every point $x \in E$ is an $\mathfrak{m}$-density point of E. Unfortunately, this is not the case if $\mathfrak{m}$ fails to have the locally uniformly doubling property. However, the following result holds.

Lemma 3. *Let* $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ *be a metric measure space with uniformly locally doubling measure* μ. *Let* $\mathfrak{m}$ *be a weakly noncollapsed measure on* X. *Then for each Borel set* $E \subset \mathrm{X}$, *any parameter* $c \geqslant 1$ *and* $\mathfrak{m}$-*almost every point* $x \in E$ *there is a sequence* $\{r_l(x)\}$ *decreasing to zero such that*

$$\varlimsup_{l\to\infty} \frac{\mathfrak{m}(B_{\max\{c,5\}r_l(x)}(x))}{\mathfrak{m}(B_{r_l(x)}(x))} \leqslant N \quad \textit{and} \quad \varliminf_{l\to\infty} \frac{\mathfrak{m}(B_{r_l(x)}(x) \cap E)}{\mathfrak{m}(B_{r_l(x)}(x))} \geqslant \frac{1}{2N}, \tag{3.7}$$

where $N = \underline{\mathrm{C}}_{\mathfrak{m}}(5\max\{c,5\})$. *In particular,* $\overline{D}^{\mathfrak{m}}_E(x) > 0$ *for* $\mathfrak{m}$-*almost every* $x \in E$.

Proof. By Lemma 2, for every point $x \in E$ there exists a sequence $r_l(x) \downarrow 0$ satisfying

$$\frac{\mathfrak{m}(B_{\max\{c,5\}r_l(x)}(x))}{\mathfrak{m}(B_{r_l(x)/5}(x))} \leqslant \underline{\mathrm{C}}_{\mathfrak{m}}(5\max\{c,5\}) = N \quad \text{for all } l \in \mathbb{N}. \tag{3.8}$$

Given $n \in \mathbb{N}$, we consider the set

$$G_n := \left\{x \in E\colon \varliminf_{l\to\infty} \frac{\mathfrak{m}(B_{r_l(x)}(x) \cap E)}{\mathfrak{m}(B_{r_l(x)}(x))} < \frac{1}{n}\right\}. \tag{3.9}$$

We show that $\mathfrak{m}(G_n) = 0$ for all $n \in \mathbb{N} \cap (2N, +\infty)$. Without loss of generality we assume that all sets G_n, $n \in \mathbb{N}$ are bounded. Since the measure $\mathfrak{m}$ is locally finite, in the rest of the proof we may assume that $\mathfrak{m}(G_n) < +\infty$ for all $n \in \mathbb{N}$. Applying the $5B$-covering lemma (see p. 60 in [1] for details) and taking the Borel regularity of the measure $\mathfrak{m}$ into account, for each $n \in \mathbb{N}$ we obtain a family of closed balls $\mathcal{B}_n = \{B_{r_{l_i}(x_i)}(x_i)\}$ such that:

(1) the family $\widetilde{\mathcal{B}}_n := \{\frac{1}{5}B\colon B \in \mathcal{B}_n\}$ is disjoint;
(2) $G_n \subset \bigcup\{B\colon B \in \mathcal{B}_n\} \subset U_{\varepsilon_n}(G_n)$ for some $\varepsilon_n > 0$;
(3) $|\mathfrak{m}(U_{\varepsilon_n}(G_n)) - \mathfrak{m}(G_n)| < \frac{1}{2n}$;
(4) $\mathfrak{m}(B) \leqslant \frac{3}{2}N\mathfrak{m}(\frac{1}{5}B)$ for all $B \in \mathcal{B}_n$;
(5) $\mathfrak{m}(B \cap E) < \frac{1}{n}\mathfrak{m}(B)$ for all $B \in \mathcal{B}_n$.

We fix an arbitrary $n > 2N$ and assume that $\mathfrak{m}(G_n) > 0$ (note that if G_n is not $\mathfrak{m}$-measurable, then we consider $\mathfrak{m}$ as an outer measure). Hence, taking $\varepsilon > 0$ small enough, from the above properties (1)–(5) we deduce

$$\begin{aligned}\mathfrak{m}(G_n) &\leqslant \sum\{\mathfrak{m}(B \cap G_n)\colon B \in \mathcal{B}_n\} \leqslant \sum\{\mathfrak{m}(B \cap E)\colon B \in \mathcal{B}_n\} \\ &\leqslant \frac{3N}{2n}\sum\left\{\mathfrak{m}\left(\frac{1}{5}B\right)\colon B \in \mathcal{B}_n\right\} \leqslant \frac{3N}{2n}\mathfrak{m}(U_{\varepsilon_n}(G_n)) \leqslant \frac{2N}{n}\mathfrak{m}(G_n).\end{aligned}$$

This contradicts the assumption $\mathfrak{m}(G_n) > 0$.

As a result, we obtain $\mathfrak{m}(G_n) = 0$ for every $n > 2N$ and complete the proof.

Now we introduce a new concept, which can be looked upon as a natural generalization of the notion of a Lebesgue point of a locally integrable function. This concept will be extremely useful in the analysis of the local behaviour of the traces of Sobolev functions. We recall (2.18).

Definition 14. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a metric measure space. Let $\{\mathfrak{m}_k\} = \{\mathfrak{m}_k\}_{k=0}^{\infty}$ be a sequence of measures on X, and let $\epsilon \in (0,1)$. Given $f \in \mathfrak{B}(\mathrm{X})$ such that $[f]_{\mathfrak{m}_k} \in L_1^{\mathrm{loc}}(\mathfrak{m}_k)$, $k \in \mathbb{N}_0$, we say that $\underline{x} \in \cap_{k=0}^{\infty} \operatorname{supp} \mathfrak{m}_k$ is an $(\{\mathfrak{m}_k\}, \epsilon)$-*Lebesgue point of* f if

$$\lim_{r\to 0} \fint_{B_r(\underline{x})} |f(\underline{x}) - f(y)|\, d\mathfrak{m}_{k_\epsilon(r)}(y) = 0. \tag{3.10}$$

We denote the set of all $(\{\mathfrak{m}_k\}, \epsilon)$-Lebesgue points of f by $\mathfrak{R}_{\{\mathfrak{m}_k\},\epsilon}(f)$.

If there is a measure $\mathfrak{m}$ on X such that $\mathfrak{m}_k = \mathfrak{m}$ for all $k \in \mathbb{N}_0$, then an $(\{\mathfrak{m}_k\}, \epsilon)$-Lebesgue point of f is called an $\mathfrak{m}$-*Lebesgue point of* f (in this case the parameter ϵ is irrelevant, and we omit it from the notation).

§ 4. Lower θ-codimensional content regular sets

Throughout this section, we fix a metric measure space $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ with uniformly locally doubling measure μ. We also recall that all balls are assumed to be closed.

The following concept was actively used in [12], [14] and [15], where problems similar to Problem 2 were considered. We recall (2.27).

Definition 15. Given $\theta \geqslant 0$, a closed set $S \subset \mathrm{X}$ is said to be *codimension θ Ahlfors-David regular* if there exist constants $c_{\theta,1}(S), c_{\theta,2}(S) > 0$ such that, for every pair $(x,r) \in S \times (0,1]$,

$$c_{\theta,1}(S)\frac{\mu(B_r(x))}{r^\theta} \leqslant \mathcal{H}_\theta(B_r(x) \cap S) \leqslant c_{\theta,2}(S)\frac{\mu(B_r(x))}{r^\theta}. \tag{4.1}$$

The class of all closed codimension θ Ahlfors-David regular sets is denoted by $\mathcal{ADR}_\theta(\mathrm{X})$.

The following proposition shows that the scale of 1 in (4.1) is not crucial. The proof is quite simple and follows easily from Proposition 6. The details are left to the reader.

Proposition 19. *Let $\theta \geqslant 0$ and $S \in \mathcal{ADR}_\theta(\mathrm{X})$. Then for each $R \geqslant 1$ there exist constants $c_{\theta,1}(S,R) > 0$ and $c_{\theta,2}(S,R) > 0$ such that, for every pair $(x,r) \in S \times (0,R]$,*

$$c_{\theta,1}(S,R)\frac{\mu(B_r(x))}{r^\theta} \leqslant \mathcal{H}_\theta(B_r(x) \cap S) \leqslant c_{\theta,2}(S,R)\frac{\mu(B_r(x))}{r^\theta}. \tag{4.2}$$

Remark 14. In the case when $\theta = 0$ sets $S \in \mathcal{ADR}_0(\mathrm{X})$ were called regular sets in [18].

Now we introduce a natural generalization of the class $\mathcal{ADR}_\theta(\mathrm{X})$. We recall (2.26).

Definition 16. Given $\theta \geqslant 0$, we say that a set $S \subset \mathrm{X}$ is *lower θ-codimensional content regular* if there exists a constant $\lambda_\theta(S) \in (0,1]$ such that, for every pair $(x,r) \in S \times (0,1]$,

$$\lambda_\theta(S)\frac{\mu(B_r(x))}{r^\theta} \leqslant \mathcal{H}_{\theta,r}(B_r(x) \cap S). \tag{4.3}$$

The class of lower θ-codimensional content regular sets is denoted by $\mathcal{LCR}_\theta(\mathrm{X})$.

Remark 15. Let $n \in \mathbb{N}$ and $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|, \mathcal{L}^n)$. It is easy to see that, given $\theta \in [0, n]$, a set S lies in $\mathcal{LCR}_\theta(\mathrm{X})$ if and only if

$$\mathcal{H}_{\theta,\infty}(B_r(x) \cap S) \geqslant \lambda_\theta(S) \frac{\mathcal{L}^n(B_r(x))}{r^\theta} \quad \text{for all } r \in (0, 1]. \tag{4.4}$$

In other words, in the classical Euclidean settings one can replace $\mathcal{H}_{\theta,r}$ by $\mathcal{H}_{\theta,\infty}$. Consequently, a set S lies in $\mathcal{LCR}_\theta(\mathbb{R}^n)$ if and only if it is an $(n-\theta)$-thick set in the sense of Rychkov [16]. Furthermore, d-thick sets, $d \in [0, n]$, were actively studied in [25] and [26], where they were called d-lower content regular. In general metric measure spaces the replacement of $\mathcal{H}_{\theta,r}$ by $\mathcal{H}_{\theta,\infty}$ in (4.3) can lead to a more narrow class of sets. The reason for this phenomenon is a 'possible gap' between parameters $\overline{q}_\mu(R)$ and $\underline{Q}_\mu(R)$ which we mentioned in Remark 5.

The following lemma was proved in [21] in the particular case of $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$. The proof in the general case is similar. We present the details for the completeness of our exposition.

Lemma 4. *Given $\theta \geqslant 0$, $\mathcal{ADR}_\theta(\mathrm{X}) \subset \mathcal{LCR}_\theta(\mathrm{X})$.*

Proof. Fix $\theta \geqslant 0$ and $S \in \mathcal{ADR}_\theta(\mathrm{X})$. Assume that $S \neq \varnothing$. Consider an arbitrary closed ball $B_r(x)$ with $x \in S$ and $r \in (0, 1]$. Let $\mathcal{B}$ be an at most countable family of closed balls such that $B_r(x) \cap S \subset \bigcup\{B \colon B \in \mathcal{B}\}$, $r_B \in (0, r)$ for all $B \in \mathcal{B}$, and

$$\sum \left\{ \frac{\mu(B)}{(r_B)^\theta} \colon B \in \mathcal{B} \right\} \leqslant 2\mathcal{H}_{\theta,r}(B_r(x) \cap S). \tag{4.5}$$

Without loss of generality we may assume that for each ball $B \in \mathcal{B}$ we have $B \cap S \neq \varnothing$. For each $B \in \mathcal{B}$ we choose an arbitrary point $\widetilde{x}_B \in B \cap S$ and consider the ball $\widetilde{B}$ of radius $2r_B$ centred at $\widetilde{x}_B$. Clearly, $B \cap S \subset \widetilde{B} \cap S$ and $\widetilde{B} \subset 4B$ for all $B \in \mathcal{B}$. Hence, using (4.5), the subadditivity property of $\mathcal{H}_\theta$ and Proposition 19 we obtain the required estimate

$$\begin{aligned} 2\mathcal{H}_{\theta,r}(B_r(x) \cap S) &\geqslant \frac{1}{(C_\mu(2))^2} \sum \left\{ \frac{\mu(\widetilde{B})}{(r_{\widetilde{B}})^\theta} \colon B \in \mathcal{B} \right\} \\ &\geqslant \frac{1}{c_{\theta,2}(S,2)(C_\mu(2))^2} \sum \{\mathcal{H}_\theta(\widetilde{B} \cap S) \colon B \in \mathcal{B}\} \\ &\geqslant \frac{1}{c_{\theta,2}(S,2)(C_\mu(2))^2} \mathcal{H}_\theta(B_r(x) \cap S) \geqslant \frac{c_{\theta,1}(S,1)}{c_{\theta,2}(S,2)(C_\mu(2))^2} \frac{\mu(B_r(x))}{r^\theta}. \end{aligned}$$

The proof is complete.

The following example demonstrates that if X is sufficiently regular, then the classes $\mathcal{LCR}_\theta(\mathrm{X})$, $\theta \geqslant 0$, are quite broad. We recall Definition 6.

Example 1. Assume that the space X is Ahlfors Q-regular for some $Q > 0$. We fix $\theta \in [\max\{0, Q-1\}, Q)$ and show that any path-connected set $S \subset \mathrm{X}$ consisting of more than one point belongs to the class $\mathcal{LCR}_\theta(\mathrm{X})$. Indeed, fix $x \in S$ and $r \in (0, 1]$ and consider two cases. In the first case $S \subset B_r(x)$. Then (clearly, $\mathcal{H}_{\theta,1}(S) > 0$)

$$\mathcal{H}_{\theta,r}(B_r(x) \cap S) = \mathcal{H}_{\theta,r}(S) \geqslant \mathcal{H}_{\theta,1}(S) r^{Q-\theta} \geqslant \frac{\mathcal{H}_{\theta,1}(S)}{c_{\mu,2}} \frac{\mu(B_r(x))}{r^\theta}. \tag{4.6}$$

In the second case there is a point $y \in S \setminus B_r(x)$. Hence there exists a curve $\gamma_{x,y}$ joining x and y. Let $\mathcal{B}$ be an at most countable family of closed balls such that $B_r(x) \cap S \subset \bigcup\{B\colon B \in \mathcal{B}\}$ and $r_B < r$ for all $B \in \mathcal{B}$. Consider the family $\underline{\mathcal{B}} := \{\operatorname{int}(2B)\colon B \in \mathcal{B}\}$. By Proposition 8 the set $\gamma_{x,y} \cap B_r(x)$ is compact. Furthermore, it is not difficult to see that there is a path-connected component $\Gamma \subset \gamma_{x,y} \cap B_r(x)$ such that $\operatorname{diam}\Gamma \geqslant r$. Hence there is a finite family $\{B_i\}_{i=1}^{N} \subset \underline{\mathcal{B}}$, $N \in \mathbb{N}$, such that $\Gamma \subset \bigcup_{i=1}^{N} \operatorname{int} 2B_i$ and for every $x, y \in \Gamma$ there is a subfamily $\{B_{i_j}\}_{j=1}^{\widetilde{N}} \subset \{B_i\}_{i=1}^{N}$ with $\widetilde{N} \leqslant N$, $i_j \in \{1, \dots, N\}$ such that $x \in \Gamma \cap \operatorname{int} 2B_{i_1}$, $y \in \Gamma \cap \operatorname{int} 2B_{i_{\widetilde{N}}}$ and $\Gamma \cap \operatorname{int} 2B_{i_j} \cap \operatorname{int} 2B_{i_{j+1}} \neq \varnothing$ for all $j \in \{1, \dots, \widetilde{N} - 1\}$. Then from the triangle inequality we deduce the crucial estimate

$$r \leqslant \operatorname{diam}(\Gamma) \leqslant \sum_{i=1}^{N} \operatorname{diam}(\Gamma \cap \operatorname{int} 2B_i) \leqslant \sum_{i=1}^{N} \operatorname{diam}(\operatorname{int} 2B_i) \leqslant \sum_{i=1}^{N} 4r_{B_i}.$$

As a result, since $Q - \theta \in (0, 1]$, we obtain

$$\sum_{B \in \mathcal{B}} \frac{\mu(B)}{(r_B)^{\theta}} \geqslant c_{\mu,1} \sum_{B \in \underline{\mathcal{B}}} (r_B)^{Q-\theta} \geqslant c_{\mu,1} \left(\sum_{i=1}^{N} r_{B_i} \right)^{Q-\theta} \geqslant c_{\mu,1} \left(\frac{r}{4} \right)^{Q-\theta}. \tag{4.7}$$

Taking the infimum in (4.7) over all families $\mathcal{B}$ we obtain

$$\mathcal{H}_{\theta,r}(B_r(x) \cap S) \geqslant c_{\mu,1} \left(\frac{r}{4} \right)^{Q-\theta} \geqslant \frac{c_{\mu,1}}{4^{Q-\theta} c_{\mu,2}} \frac{\mu(B_r(x))}{r^{\theta}}. \tag{4.8}$$

Finally, we conclude the discussion by combining (4.6) and (4.8).

Remark 16. Even in the case of $\mathrm{X} = (\mathbb{R}^2, \|\cdot\|, \mathcal{L}^2)$ it is clear that generic path-connected sets $S \subset \mathrm{X}$ can fail to satisfy the codimension 1 Ahlfors-David regularity condition. Furthermore, it was shown in [42] that relevant examples can be obtained as the graphs of locally Lipschitz functions. Coupled with Lemma 4, this shows that, given $\theta > 0$, the family $\mathcal{ADR}_{\theta}(\mathrm{X})$ can be a very poor subfamily of $\mathcal{LCR}_{\theta}(\mathrm{X})$ in general.

Example 2. Assume that X is Ahlfors Q-regular for some $Q > 0$. Then each nonempty set $S \subset \mathrm{X}$ belongs to $\mathcal{LCR}_{\theta}(\mathrm{X})$ for every $\theta \geqslant Q$. Indeed, fix $x \in S$, $r \in (0, 1]$ and an at most countable family of balls $\{B_i\}$ of radii $r_i < r$ that cover $B_r(x) \cap S$. Then

$$\sum \frac{\mu(B_i)}{(r_i)^{\theta}} \geqslant c_{\mu,1} \sum (r_i)^{Q-\theta} \geqslant c_{\mu,1} r^{Q-\theta} \geqslant \frac{c_{\mu,1}}{c_{\mu,2}} \frac{\mu(B_r(x))}{r^{\theta}}.$$

Taking the infimum over all such coverings we obtain the required result.

§ 5. θ-regular sequences of measures

Throughout this section we fix $p \in (1, \infty)$ and an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$. We recall the definitions of the classes $\mathfrak{M}_{\theta}(S)$ and $\mathfrak{M}_{\theta}^{\mathrm{str}}(S)$ given in § 1. We also recall the notation $B_r(x)$ in § 2. It is clear that there are smallest constants for which conditions (M2) and (M4) hold. We denote them by $C_{\{\mathfrak{m}_k\},1}$ and $C_{\{\mathfrak{m}_k\},3}$, respectively. In a similar way, there is a greatest constant for which (M3) holds. We denote it by $C_{\{\mathfrak{m}_k\},2}$. We use the symbol $\mathcal{C}_{\{\mathfrak{m}_k\}}$ to denote the set of these constants, that is, $\mathcal{C}_{\{\mathfrak{m}_k\}} := \{C_{\{\mathfrak{m}_k\},1}, C_{\{\mathfrak{m}_k\},2}, C_{\{\mathfrak{m}_k\},3}\}$.

5.1. Elementary properties. The following fact is an immediate consequence of (1.6) and the definition of the set functions $\mathcal{H}_\theta$, $\theta \geqslant 0$. We omit an elementary proof.

Proposition 20. *Let $\theta \geqslant 0$ and $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ for some closed set $S \subset \mathrm{X}$. Then for each $k \in \mathbb{N}_0$ the measure $\mathfrak{m}_k$ is absolutely continuous with respect to $\mathcal{H}_\theta$. Furthermore, for any Borel set $E \subset S$, for each $k \in \mathbb{N}_0$, we have $\mathfrak{m}_k(E) \leqslant C_{\{\mathfrak{m}_k\},1}\mathcal{H}_\theta(E)$.*

Given a sequence $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$, it is natural to ask whether the measures $\mathfrak{m}_k$, $k \in \mathbb{N}_0$, have the doubling property. Unfortunately, this is not the case in general. Nevertheless, we have at our disposal the following important result (we put $B_k(x) := B_{\epsilon^k}(x)$ for all $x \in \mathrm{X}$ and $k \in \mathbb{Z}$).

Theorem 5. *Let $\theta \geqslant 0$, let the closed set S belong to $\mathcal{LCR}_\theta(\mathrm{X})$, and let $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$. Then for each $c \geqslant 1$ there is a constant $C > 0$ depending on c, $\mathcal{C}_{\{\mathfrak{m}_k\}}$ and $C_\mu(2c)$ such that, for each $k \in \mathbb{N}_0$ and any $y \in S$,*

$$\frac{1}{C}\mathfrak{m}_k(B_k(y)) \leqslant \mathfrak{m}_k\left(\frac{1}{c}B_k(y)\right) \leqslant \mathfrak{m}_k(cB_k(y)) \leqslant C\mathfrak{m}_k(B_k(y)). \tag{5.1}$$

Proof. We set $\underline{k} := \min\{k \in \mathbb{Z} \colon \epsilon^k < \frac{1}{c}\}$ and consider the upper and lower bounds in (5.1) separately.

To prove the upper bound we consider two cases. In the case when $k \in \{0, \dots, \underline{k}\}$, a combination of (1.6), (1.8) and the uniformly locally doubling property of μ gives the existence of a constant $C > 0$ such that (we recall (2.3) and take Proposition 7 into account)

$$\begin{aligned}\mathfrak{m}_k(cB_k(y)) &\leqslant C\mathfrak{m}_0(cB_k(y)) \leqslant C \sum_{B \in \mathcal{B}_k(\mathrm{X},\epsilon)} \mathfrak{m}_0(B \cap cB_k(y)) \\ &\leqslant C \sum_{\substack{B \in \mathcal{B}_k(\mathrm{X},\epsilon) \\ B \cap cB_k(y) \neq \varnothing}} \mu(B) \leqslant C\mu((c+2)B_k(y)) \leqslant C\mu(B_k(y)).\end{aligned} \tag{5.2}$$

In the case when $k > \underline{k}$, using (1.6)–(1.8), and the uniformly locally doubling property of μ we obtain

$$\begin{aligned}\mathfrak{m}_k(cB_k(y)) &\leqslant C\mathfrak{m}_{k-\underline{k}}(cB_k(y)) \leqslant C\mathfrak{m}_{k-\underline{k}}(B_{k-\underline{k}}(y)) \\ &\leqslant C\frac{\mu(B_{k-\underline{k}}(y))}{\epsilon^{k-\underline{k}}} \leqslant C\frac{\mu(B_k(y))}{\epsilon^k} \leqslant C\mathfrak{m}_k(B_k(y)).\end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3) we deduce the required upper bound in (5.1).

Now we fix $k \in \mathbb{N}_0$. To prove the lower bound in (5.1), an appeal to (1.6)–(1.8) and the uniformly locally doubling property of μ gives us the required estimate

$$\begin{aligned}\mathfrak{m}_k\left(\frac{1}{c}B_k(y)\right) &\geqslant C\mathfrak{m}_{k+\underline{k}}\left(\frac{1}{c}B_k(y)\right) \geqslant C\mathfrak{m}_{k+\underline{k}}(B_{k+\underline{k}}(y)) \\ &\geqslant C\frac{\mu(B_{k+\underline{k}}(y))}{\epsilon^{k+\underline{k}}} \geqslant C\frac{\mu(B_k(y))}{\epsilon^k} \geqslant C\mathfrak{m}_k(B_k(y)).\end{aligned} \tag{5.4}$$

The proof is complete.

Theorem 5 leads to the following useful corollary (we put $B_k(x) := B_{\epsilon^k}(x)$).

Proposition 21. *Let $\theta \geqslant 0$, let the closed set S belong to $\mathcal{LCR}_\theta(\mathrm{X})$, and let $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$. Then for each $c \geqslant 1$ there exists a constant $C > 0$ such that for each $k \in \mathbb{N}_0$,*

$$\int_{cB_k(z)} \frac{1}{\mathfrak{m}_k(cB_k(y))}\, d\mathfrak{m}_k(y) \leqslant C \quad \textit{for all } z \in S. \tag{5.5}$$

Proof. Fix $k \in \mathbb{N}_0$ and $z \in S$. Note that $2cB_k(y) \supset cB_k(z)$ for all $y \in cB_k(z)$. Hence by Theorem 5,

$$\sup_{y \in cB_k(z)} \frac{1}{\mathfrak{m}_k(cB_k(y))} \leqslant \sup_{y \in cB_k(z)} \frac{C}{\mathfrak{m}_k(2cB_k(y))} \leqslant \frac{C}{\mathfrak{m}_k(cB_k(z))}.$$

Consequently,

$$\int_{cB_k(z)} \frac{1}{\mathfrak{m}_k(cB_k(y))}\, d\mathfrak{m}_k(y) \leqslant C \int_{cB_k(z)} \frac{1}{\mathfrak{m}_k(cB_k(z))}\, d\mathfrak{m}_k(y) = C. \tag{5.6}$$

The proof is complete.

5.2. Comparison of different classes of measures. Now given a closed set $S \subset \mathrm{X}$, we formulate a simple condition that is sufficient for the equality of the classes $\mathfrak{M}^{\mathrm{str}}_\theta(S)$ and $\mathfrak{M}_\theta(S)$. We recall Definition 5 and Remark 6. We also recall the notation (2.18) and (3.6) and put $k(r) := k_\epsilon(r)$.

Theorem 6. *Let $\theta \in [0, \underline{Q}_\mu(1))$, and let $S \subset \mathrm{X}$ be a closed set such that $\mathcal{H}_\theta(S) \in (0, +\infty)$. Then*

$$\mathfrak{M}^{\mathrm{str}}_\theta(S) = \mathfrak{M}_\theta(S).$$

Proof. Clearly, it is sufficient to show that $\mathfrak{M}_\theta(S) \subset \mathfrak{M}^{\mathrm{str}}_\theta(S)$. Assume that $\mathfrak{M}_\theta(S) \neq \varnothing$ and fix an arbitrary sequence of measures $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$. We also fix a Borel set $E \subset S$ and verify (1.9). We put $N := \{x \in E\colon \overline{D}^{\{\mathfrak{m}_k\}}_E(x, \epsilon) = 0\}$. Since $\theta \in [0, \underline{Q}_\mu(1))$, by Remark 6 the set function $\mathcal{H}_\theta\lfloor_S$ is a finite measure on X. Furthermore, $\mathfrak{m}_0(S) < +\infty$ by Proposition 20. If $\mathfrak{m}_0(N) > 0$, then using Egorov's theorem, given $\varepsilon > 0$, we find a compact set $K_\varepsilon \subset N$ and a number $\delta(\varepsilon) > 0$ such that $\mathfrak{m}_0(N \setminus K_\varepsilon) < \varepsilon$ and

$$\sup_{x \in K_\varepsilon} \sup_{r < \delta(\varepsilon)} \frac{\mathfrak{m}_{k(r)}(E \cap B_r(x))}{\mathfrak{m}_{k(r)}(B_r(x))} < \varepsilon. \tag{5.7}$$

By the assumptions of the lemma we have $\mathcal{H}_\theta(S) < +\infty$. Hence we find an arbitrary at most countable covering of K_ε by balls $\{B_j\}_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, of radii $r_j := r(B_j) < \delta(\varepsilon)/2$ such that

$$\sum_j \frac{\mu(B_j)}{(r_j)^\theta} \leqslant 2\mathcal{H}_{\theta,\delta(\varepsilon)/2}(K_\varepsilon) \leqslant 2\mathcal{H}_\theta(S). \tag{5.8}$$

Without loss of generality we may assume that, for each B_j, $K_\varepsilon \cap B_j \neq \varnothing$. For each j we fix a point $x_j \in B_j \cap K_\varepsilon$. We obviously have $B_j \subset B_{2r_j}(x_j) \subset 3B_j$. Hence, combining (1.6), (1.8) with (5.7), (5.8) and taking into account the uniformly locally doubling property of μ, we have

$$\begin{aligned}\mathfrak{m}_0(K_\varepsilon) &\leqslant \sum_j \mathfrak{m}_0(E\cap B_j) \leqslant C\sum_j \mathfrak{m}_{k(r_j)}(E\cap B_{2r_j}(x_j)) < \varepsilon C\sum_j \mathfrak{m}_{k(r_j)}(B_{2r_j}(x_j))\\ &\leqslant \varepsilon C\sum_j \frac{\mu(B_{2r_j}(x_j))}{(r_j)^\theta} \leqslant \varepsilon C\sum_j \frac{\mu(3B_j)}{(r_j)^\theta} \leqslant \varepsilon C\sum_j \frac{\mu(B_j)}{(r_j)^\theta} \leqslant \varepsilon C\mathcal{H}_\theta(S).\end{aligned} \tag{5.9}$$

Hence for all sufficiently small $\varepsilon > 0$ we have $\mathfrak{m}_0(K_\varepsilon) = 0$. Since $\mathfrak{m}_0(N \setminus K_\varepsilon) < \varepsilon$ and $\varepsilon > 0$ can be chosen arbitrarily, we obtain the equality $\mathfrak{m}_0(N) = 0$, completing the proof.

In the proof of the next theorem we build a simple example that exhibits a delicate difference between the classes $\mathfrak{M}^{\mathrm{str}}_\theta(S)$ and $\mathfrak{M}_\theta(S)$. Despite its simplicity, the corresponding constructions are typical and reflect the essence of the matter. One can build similar examples in higher dimensions and even in some nice classes of metric measure spaces. However, the corresponding machinery will be much less transparent.

Theorem 7. *Let* $\mathrm{X} = (\mathbb{R}^2, \|\cdot\|_2, \mathcal{L}^2)$ *and* $S := \{(x_1,x_2)\in\mathbb{R}^2 : x_1 \in [0,1],\, x_2 = 0\}$. *Then for each* $\theta \in (1,2)$ *there exists a sequence of measures* $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)\setminus\mathfrak{M}^{\mathrm{str}}_\theta(S)$.

Proof. We fix an arbitrary $\theta\in(1,2)$ and put

$$c_1(\theta) := 2\sum_{k=1}^\infty \frac{1}{k^\theta} \quad \text{and} \quad c_2(\theta) := \min_{j\in\mathbb{N}_0} \frac{2^j}{(1+j)^{\theta-1}}.$$

It is convenient to split the proof into several steps.

Step 1. Let E denote the closed Cantor-type set built recursively as follows. At the first step we put

$$E_1 := [0,1] \setminus \left(\frac{1}{2} - (2c_1(\theta))^{-1},\, \frac{1}{2} + (2c_1(\theta))^{-1}\right)$$

and

$$U_1 := \left(\frac{1}{2} - (2c_1(\theta))^{-1},\, \frac{1}{2} + (2c_1(\theta))^{-1}\right).$$

Suppose that for some $k\in\mathbb{N}$ we have already built closed sets $E_1 \supset \cdots \supset E_k$ and open sets $U_1,\ldots,U_k$ such that

$$E_i \cup \left(\bigcup_{j=1}^i U_j\right) = [0,1] \quad \text{and} \quad \mathcal{L}^1(U_i) = \frac{1}{c_1(\theta) i^\theta} \quad \text{for all } i\in\{1,\ldots,k\}.$$

Furthermore, for each $i \in \{1,\ldots,k\}$ the set E_i is a disjoint union of 2^i closed intervals $I_{i,l}$ and each U_i is a disjoint union of 2^{i-1} open intervals $J_{i,l}$. From the middle of each closed interval $I_{k,l}$ we remove an open interval of length $1/(c_1(\theta)2^k k^\theta)$ and consider the union of the remaining closed sets. Then we obtain a set E_{k+1}

and put $U_{k+1} := E_k \setminus E_{k+1}$. As a result, we obtain the sequence $\{E_k\}_{k=1}^{\infty}$ of closed sets and the sequence $\{U_k\}_{k=1}^{\infty}$ of open sets. Furthermore, for each $k \in \mathbb{N}$ we let $\mathcal{I}_k$ and $\mathcal{J}_k$ denote the corresponding families of closed and open intervals, respectively. More precisely, $E_k = \bigcup\{I\colon I \in \mathcal{I}_k\}$ and $U_k = \bigcup\{J\colon J \in \mathcal{J}_k\}$ for all $k \in \mathbb{N}$. Now we put $E := \bigcap_{n=0}^{\infty} E_n$ and define weight functions $\omega_k \in L_1([0,1])$, $k \in \mathbb{N}_0$, by (we put $U_0 := \varnothing$)

$$\omega_k(x) := \chi_E(x) + \sum_{i=0}^{k} 2^{(\theta-1)i} i^{\theta-1} \chi_{U_i}(x) + 2^{(\theta-1)k}(k+1)^{\theta-1} \sum_{i=k+1}^{\infty} \chi_{U_i}(x), \qquad x \in [0,1]. \tag{5.10}$$

Finally, we recall (2.6) and put $\mathfrak{m}_k := \omega_k \mathcal{H}^1\lfloor_S$, $k \in \mathbb{N}_0$ (here $\mathcal{H}^1$ is the usual 1-dimensional Hausdorff measure). We put $\epsilon = 1/2$ and claim that $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k\in\mathbb{N}_0} \in \mathfrak{M}_\theta(S) \setminus \mathfrak{M}_\theta^{\mathrm{str}}(S)$. This will be shown at the next steps.

Step 2. Note that $\operatorname{supp} \mathfrak{m}_k = S$ for all $k \in \mathbb{N}_0$. This verifies (M1).

Step 3. By (5.10) it is easy to see that for each $k \in \mathbb{N}_0$ and every $j \in \mathbb{N}_0$,

$$\frac{c_2(\theta)}{2^{\theta j}} \leqslant \frac{1}{2^{(\theta-1)j}(1+j)^{\theta-1}} \leqslant \frac{(k+1)^{\theta-1}}{2^{(\theta-1)j}(k+1+j)^{\theta-1}} \leqslant \frac{w_k(x)}{w_{k+j}(x)} \leqslant 1, \qquad x \in [0,1]. \tag{5.11}$$

This proves that condition (M4) is satisfied with $C_3 = \max\{1, (c_2(\theta))^{-1}\}$.

Step 4. To verify (M2) we proceed as follows. We fix arbitrary $k \in \mathbb{N}_0$, $j \geqslant k$ and $Q \in \mathcal{D}_j$ (by $\mathcal{D}_j$ here and throughout the rest of the proof we denote the family of closed dyadic intervals of length 2^{-j}). Given $i \in \mathbb{N}$, there are two cases to consider.

In the first case $(c_1(\theta))^{-1} 2^{-i} i^{-\theta} < 2^{-j}$. Since $\theta > 1$, we obviously have

$$\frac{2^{(\theta-1)i} i^{\theta-1}}{2^i i^\theta} = \frac{2^{(\theta-1)i} i^{\theta-1}}{2^{(\theta-1+2-\theta)i} i^{\theta(\theta-1+2-\theta)}} \leqslant \frac{1}{2^{(2-\theta)i}} \frac{1}{i^{\theta(2-\theta)}} \leqslant \frac{(c_1(\theta))^{2-\theta}}{2^{(2-\theta)j}}. \tag{5.12}$$

Consequently, given $J \in \mathcal{J}_i$, from (5.10) and (5.12) we obtain (since $(c_1(\theta))^{1-\theta} \leqslant 1$)

$$\frac{1}{2}\mathfrak{m}_k(Q \cap J) \leqslant \frac{1}{2}\mathfrak{m}_k(J) \leqslant \begin{cases} \dfrac{2^{(\theta-1)k}(k+1)^{\theta-1}}{c_1(\theta) 2^i i^\theta} \leqslant \dfrac{2^{(\theta-1)i} i^{\theta-1}}{c_1(\theta) 2^i i^\theta} \leqslant \dfrac{1}{2^{(2-\theta)j}}, & i > k, \\ \dfrac{2^{(\theta-1)i} i^{\theta-1}}{c_1(\theta) 2^i i^\theta} \leqslant \dfrac{1}{2^{(2-\theta)j}}, & i \leqslant k. \end{cases} \tag{5.13}$$

In the second case $(c_1(\theta))^{-1} 2^{-i} i^{-\theta} \geqslant 2^{-j}$. Since $\theta > 1$, we clearly have

$$\frac{2^{(\theta-1)i} i^{\theta-1}}{2^j} \leqslant \frac{2^{(\theta-1)i} i^{\theta-1}}{2^{(2-\theta)j} 2^{(\theta-1)j}} \leqslant \frac{2^{(\theta-1)i} i^{\theta-1}}{(c_1(\theta))^{\theta-1} 2^{(2-\theta)j} 2^{(\theta-1)i} i^{\theta(\theta-1)}} \leqslant \frac{(c_1(\theta))^{1-\theta}}{2^{(2-\theta)j}}. \tag{5.14}$$

Consequently, given $J \in \mathcal{J}_i$, from (5.10) and (5.14) we obtain (since $(c_1(\theta))^{1-\theta} \leqslant 1$)

$$\frac{1}{2}\mathfrak{m}_k(Q \cap J) \leqslant \begin{cases} \dfrac{2^{(\theta-1)k}(k+1)^{\theta-1}}{2^j} \leqslant \dfrac{2^{(\theta-1)i} i^{\theta-1}}{2^j} \leqslant \dfrac{1}{2^{(2-\theta)j}}, & i > k, \\ \dfrac{2^{(\theta-1)i} i^{\theta-1}}{2^j} \leqslant \dfrac{1}{2^{(2-\theta)j}}, & i \leqslant k, \end{cases} \tag{5.15}$$

As a result, combining (5.13) and (5.15), we have

$$\mathfrak{m}_k(Q \cap J) \leqslant \frac{1}{2^{(2-\theta)j}} \quad \text{for each } i \in \mathbb{N} \text{ for every } Q \in \mathcal{D}_j \text{ and any } J \in \mathcal{J}_i. \tag{5.16}$$

We fix a closed interval $I \in \mathcal{I}_j$ and note that $I \cap U_i = \varnothing$ for all $i \in \{1, \dots, j\}$. Hence, taking into account that for each $i \geqslant j+1$ the set $U_i \cap I$ is formed by at most 2^{i-j} open intervals of length $(c_1(\theta))^{-1}2^{-i}i^{-\theta}$, we have

$$c_1(\theta) \sum_{i=j+1}^{\infty} \mathcal{L}^1(U_i \cap I) \leqslant 2^{-j} \sum_{i=j+1}^{\infty} i^{-\theta} \leqslant \frac{2}{\theta - 1} 2^{-j} j^{1-\theta}.$$

Using this observation and keeping in mind that $\theta \in (1,2)$ and $c_1(\theta) \geqslant 2$, by (5.10) we have

$$\begin{aligned}
\mathfrak{m}_k(Q \cap I) &\leqslant \mathfrak{m}_k(I) \leqslant 2^{-j} + 2^{(\theta-1)k}(k+1)^{\theta-1} \sum_{i=j+1}^{\infty} \mathcal{L}^1(U_i \cap I) \\
&\leqslant \frac{1}{2^{(2-\theta)j}} + 2\frac{(c_1(\theta))^{-1}}{\theta - 1} \frac{2^{(\theta-1)k}(k+1)^{\theta-1}}{2^j j^{\theta-1}} \leqslant \frac{1}{2^{(2-\theta)j}} + \frac{1}{\theta-1} \frac{(j+1)^{\theta-1}}{j^{\theta-1}} \frac{1}{2^{(2-\theta)j}} \\
&\leqslant \frac{2^\theta}{\theta-1} \frac{1}{2^{(2-\theta)j}} \quad \text{for each } Q \in \mathcal{D}_j.
\end{aligned} \tag{5.17}$$

By the construction of U_i we have $\sum_{i=1}^{l} \mathcal{L}^1(U_i) < 1/2$ for all $l \in \mathbb{N}$. Hence it is easy to see that each closed interval $I \in \mathcal{I}_j$ has length greater than 2^{-j-1}. Consequently, it is easy to see that Q can intersect at most three different closed intervals in $\mathcal{I}_j$ and at most two different open intervals in $\bigcup_{i=1}^{j} \mathcal{J}_i$. As a result, given $x \in \mathbb{R}^2$ and $r \in (2^{-j-1}, 2^{-j}]$, combining the above observations with (5.16) and (5.17) we obtain

$$\mathfrak{m}_k(B_r(x)) \leqslant \sum_{Q \in \mathcal{D}_j} \mathfrak{m}_k(Q \cap B_r(x)) \leqslant \frac{2^\theta}{\theta - 1} \frac{15}{2^{(2-\theta)j}}. \tag{5.18}$$

Consequently, we conclude that $\{\mathfrak{m}_k\}$ satisfies condition (M2).

Step 5. We fix $\underline{x} \in E$ and $k \in \mathbb{N}_0$. By the construction of E there exists an interval $I_k(\underline{x}) \in \mathcal{I}_k$ such that $\underline{x} \in I_k(\underline{x})$. Hence, taking into account that for each $i \geqslant k+1$ the set $U_i \cap I_k(\underline{x})$ consists of 2^{i-k} disjoint intervals of length $(c_1(\theta))^{-1}2^{-i}i^{-\theta}$, we obtain

$$\begin{aligned}
\mathfrak{m}_k(B_{2^{-k}}(\underline{x})) &\geqslant \mathfrak{m}_k(I_k(\underline{x})) \geqslant 2^{(\theta-1)k}(k+1)^{\theta-1} \sum_{i=k+1}^{\infty} \mathcal{L}^1(U_i \cap I_k(\underline{x})) \\
&\geqslant \frac{2^{(\theta-1)k}(k+1)^{\theta-1}}{c_1(\theta)2^k} \sum_{i=k+1}^{\infty} \frac{1}{i^\theta} \geqslant \frac{1}{c_1(\theta)(\theta-1)} \frac{1}{2^{k(2-\theta)}}.
\end{aligned} \tag{5.19}$$

This observation, in combination with (5.11), easily implies condition (M3).

Step 6. By (5.10) it is easy to see that $\mathfrak{m}_k(E \cap B_{2^{-k}}(x)) \leqslant 2^{-k+1}$ for all $x \in E$ and all $k \in \mathbb{N}_0$. Since $\theta > 1$, the above observation in combination with (5.19), shows for every $x \in E$ that

$$\frac{\mathfrak{m}_k(B_{2^{-k}}(x) \cap E)}{\mathfrak{m}_k(B_{2^{-k}}(x))} \leqslant \frac{2c_1(\theta)(\theta-1)}{2^{k(\theta-1)}} \to 0, \qquad k \to \infty. \tag{5.20}$$

This proves that $\{\mathfrak{m}_k\} \notin \mathfrak{M}^{\mathrm{str}}_\theta(S)$.

The proof is complete.

5.3. Proof of Theorem 1. We begin with a necessary condition for the existence of a θ-regular sequence of measures.

Theorem 8. *Let $S \subset \mathrm{X}$ be a closed nonempty set. If $\theta \geqslant 0$ is such that $\mathfrak{M}_\theta(S) \neq \varnothing$, then $S \in \mathcal{LCR}_\theta(\mathrm{X})$.*

Proof. Let $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ and $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0,1)$. Given $r \in (0,1]$ and $x \in S$, let $\mathcal{B} = \{B_j\}_{j\in\mathbb{N}} = \{B_{r_j}(x_j)\}_{j\in\mathbb{N}}$ be a sequence of closed balls such that $r_j \in (0,r)$ for all $j \in \mathbb{N}$, $B_j \cap S \neq \varnothing$ for all $j \in \mathbb{N}$, $B_r(x) \cap S \subset \bigcup_{j\in\mathbb{N}} B_j$ and

$$\sum_j \frac{\mu(B_j)}{(r_j)^\theta} \leqslant 2\mathcal{H}_{\theta,r}(B_r(x) \cap S). \tag{5.21}$$

We recall the notation (2.18), put $k_j := k(r_j)$, $j \in \mathbb{N}$, and, for every $j \in \mathbb{N}$, fix a ball $\widetilde{B}_j$ of radius $r_{\widetilde{B}_j} = 2r_j$ centred at some point $x_j \in S \cap B_j$. It is clear that $B_j \subset \widetilde{B}_j \subset 4B_j$ for all $j \in \mathbb{N}$. Hence, using the uniformly locally doubling property of the measure μ, applying (1.6), and then taking Theorem 5 into account we obtain

$$\begin{aligned}\sum_j \frac{\mu(B_j)}{(r_j)^\theta} &\geqslant C\sum_j \frac{\mu(4B_j)}{(4r_j)^\theta} \geqslant C\sum_j \frac{\mu(\widetilde{B}_j/2)}{(r_j/2)^\theta}\\ &\geqslant C\sum_j \mathfrak{m}_{k_j}\biggl(\frac{1}{2}\widetilde{B}_j\biggr) \geqslant C\sum_j \mathfrak{m}_{k_j}(\widetilde{B}_j).\end{aligned} \tag{5.22}$$

Now we combine (5.21) with (5.22), take into account that $k(r) \leqslant k_j$ for all $j \in \mathbb{N}$, and use (1.8). This gives

$$\mathcal{H}_{\theta,r}(B_r(x) \cap S) \geqslant C\sum_j \mathfrak{m}_{k(r)}(\widetilde{B}_j) \geqslant C\mathfrak{m}_{k(r)}(B_r(x) \cap S).$$

Since $r \in (\epsilon^{k(r)+1}, \epsilon^{k(r)}]$, using Theorem 5 and (1.7) we can continue the previous estimate and obtain

$$\mathcal{H}_{\theta,r}(B_r(x) \cap S) \geqslant C\mathfrak{m}_{k(r)}(B_{\epsilon^{k(r)}}(x) \cap S) \geqslant C\frac{\mu(B_{\epsilon^{k(r)}}(x))}{\epsilon^{k(r)\theta}} \geqslant C\frac{\mu(B_r(x))}{r^\theta}. \tag{5.23}$$

Since $x \in S$ and $r \in (0,1]$ were chosen arbitrarily, the theorem follows from Definition 16.

Theorem 8 is proved.

The following result presents conditions on a set $S \subset \mathrm{X}$ that are sufficient for the existence of a strongly θ-regular sequence of measures whose supports coincide with S.

Theorem 9. *Let $\theta \geqslant 0$. If $S \in \mathcal{LCR}_{\theta}(\mathrm{X})$ is closed and nonempty, then $\mathfrak{M}^{\mathrm{str}}_{\theta}(S) \neq \varnothing$.*

Proof. We fix an arbitrary $\epsilon \in (0,1/10]$ and recall the notation (2.2)–(2.4). Since S is a closed subset of the complete separable metric space (X,d), the space $\mathrm{S} := (S, \mathrm{d}|_S)$ is a complete separable metric space (here $\mathrm{d}|_S$ is the restriction of the metric d to the set S).

Step 1. We recall Definition 3 and Proposition 9 and fix an admissible partial order on $Z(\mathrm{S},\epsilon)$. Given $k \in \mathbb{N}_0$ and $z_{k,\alpha} \in Z_k(\mathrm{S},\epsilon)$, we put

$$\widetilde{Q}_{k,\alpha} := \bigcup \{\operatorname{int} B_{\epsilon^j/8}(z_{j,\beta})\colon z_{j,\beta} \preceq z_{k,\alpha}\}. \tag{5.24}$$

Note that the $\widetilde{Q}_{k,\alpha}$, $k \in \mathbb{N}_0$, $\alpha \in \mathcal{A}_k(\mathrm{S},\epsilon)$, are open subsets in X. However, they are neither generalized dyadic cubes in X, nor generalized dyadic cubes in S. At the same time, by (2.17) $\widetilde{Q}_{k,\alpha} \cap S$ is a generalized dyadic cube in the space S for each $k \in \mathbb{N}_0$ and any $\alpha \in \mathcal{A}_k(\mathrm{S},\epsilon)$. The only reason for the introduction of such special sets $\widetilde{Q}_{k,\alpha}$ is that the 'centres' of these 'quasicubes' belong to the set S. This fact is crucial at Step 8 below.

Since $\epsilon \in (0,1/10]$, it is easy to see from (PO3) in Definition 3 and (5.24) that

$$\widetilde{Q}_{k,\alpha} \subset B_{2\epsilon^k}(z_{k,\alpha}) \quad \text{for each } k \in \mathbb{N}_0 \text{ and any } \alpha \in \mathcal{A}_k(\mathrm{S},\epsilon). \tag{5.25}$$

Repeating almost verbatim the arguments in the proof of Lemma 15 in [33] we obtain that

$$\text{if } l \geqslant k, \quad \text{then either} \quad \widetilde{Q}_{l,\beta} \subset \widetilde{Q}_{k,\alpha}, \quad \text{or} \quad \widetilde{Q}_{l,\beta} \cap \widetilde{Q}_{k,\alpha} = \varnothing. \tag{5.26}$$

Furthermore, by (5.24) and (5.26) we clearly have

$$\sum_{z_{k+1,\beta} \preceq z_{k,\alpha}} \mu(\widetilde{Q}_{k+1,\beta}) \leqslant \mu(\widetilde{Q}_{k,\alpha}) \quad \text{for each } k \in \mathbb{N}_0 \text{ and any } \alpha \in \mathcal{A}_k(\mathrm{S},\epsilon). \tag{5.27}$$

Finally, let $B = B_r(x)$ be an arbitrary closed ball of radius $r \in (0,1]$ centred at $x \in \mathrm{X}$. Let $c \geqslant 1$ be such that $B_{cr}(x) \cap S \neq \varnothing$. The same arguments as in the proof of Proposition 11 give

$$\#\{\alpha \in \mathcal{A}_{k(r)}(\mathrm{S},\epsilon)\colon \operatorname{cl}(\widetilde{Q}_{k(r),\alpha} \cap S) \cap cB \neq \varnothing\} \leqslant C_D(c,0). \tag{5.28}$$

Step 2. For each $j \in \mathbb{N}_0$ and any $\beta \in \mathcal{A}_j(\mathrm{S},\epsilon)$ we put $h_{j,\beta} := \mu(\widetilde{Q}_{j,\beta})/\epsilon^{j\theta}$. Now, for each $j \in \mathbb{N}_0$ we define a measure $\mathfrak{m}^{j,j}$ on S by the formula (by δ_x we denote the Dirac measure concentrated at $x \in S$)

$$\mathfrak{m}^{j,j} := \sum_{z_{j,\beta} \in Z_j(\mathrm{S},\epsilon)} h_{j,\beta}\delta_{z_{j,\beta}}. \tag{5.29}$$

Given $j \in \mathbb{N}$, we modify the measure $\mathfrak{m}^{j,j}$ in the following way. If $\alpha \in \mathcal{A}_{j-1}(\mathrm{S},\epsilon)$ is such that

$$\mathfrak{m}^{j,j}(\widetilde{Q}_{j-1,\alpha} \cap S) = \mathfrak{m}^{j,j}(\{z_{j,\beta}\colon z_{j,\beta} \preceq z_{j-1,\alpha}\}) > h_{j-1,\alpha},$$

then we reduce the mass of $\mathfrak{m}^{j,j}$ uniformly on $\{z_{j,\beta}\colon z_{j,\beta} \preceq z_{j-1,\alpha}\}$ until it becomes equal to $h_{j-1,\alpha}$. On the other hand, if $\alpha \in \mathcal{A}_{j-1}(\mathrm{S},\epsilon)$ is such that $\mathfrak{m}^{j,j}(\widetilde{Q}_{j-1,\alpha} \cap S) \leqslant h_{j-1,\alpha}$, then we leave $\mathfrak{m}^{j,j}$ unchanged. In this way we clearly obtain a new measure $\mathfrak{m}^{j,j-1}$. We repeat this procedure for $\mathfrak{m}^{j,j-1}$ obtaining $\mathfrak{m}^{j,j-2}$, and after j steps we obtain $\mathfrak{m}^{j,0}$. Given $j \in \mathbb{N}_0$ and $k \leqslant j$, it follows from this construction that

$$\mathfrak{m}^{j,k}(\widetilde{Q}_{i,\beta} \cap S) \leqslant h_{i,\beta} \quad \text{for each } i \in \{k,\dots,j\} \text{ for all } \beta \in \mathcal{A}_i(\mathrm{S},\epsilon). \tag{5.30}$$

By (5.27) it is clear that

$$\underline{M} := \inf_{k\in\mathbb{N}_0} \inf_{z_{k,\alpha}\in Z_k(\mathrm{S},\epsilon)} \Bigg(\sum_{z_{k+1,\beta}\preceq z_{k,\alpha}} h_{k+1,\beta}\Bigg)^{-1} h_{k,\alpha} \geqslant \epsilon^{\theta}. \tag{5.31}$$

Note that by the above construction, for each $k \in \mathbb{N}_0$ and every $j \geqslant k$ there is a family of positive constants $\{c_{j,k}(\widetilde{Q}_{j,\beta})\colon \beta \in \mathcal{A}_j(\mathrm{S},\epsilon)\}$ such that

$$\mathfrak{m}^{j,k}(\widetilde{Q}_{j,\beta} \cap S) = c_{j,k}(\widetilde{Q}_{j,\beta})\mathfrak{m}^{j,j}(\widetilde{Q}_{j,\beta} \cap S) \quad \text{for all } \beta \in \mathcal{A}_j(\mathrm{S},\epsilon). \tag{5.32}$$

Furthermore, by (5.31), for each $k \in \mathbb{N}_0$, $i \in \{0,\dots,k\}$, and every $j \geqslant k$, we have

$$\epsilon^{i\theta} c_{j,k}(\widetilde{Q}_{j,\beta}) \leqslant c_{j,k-i}(\widetilde{Q}_{j,\beta}) \leqslant c_{j,k}(\widetilde{Q}_{j,\beta}) \quad \text{for all } \beta \in \mathcal{A}_j(\mathrm{S},\epsilon). \tag{5.33}$$

Step 3. Using estimates (5.30) and (5.28) we obtain $\sup_{j\geqslant k} \mathfrak{m}^{j,k}(B) < \infty$ for every closed ball $B \subset \mathrm{X}$. Hence, by Proposition 8 and Lemma 1, for each $k \in \mathbb{N}_0$ there is a subsequence $\{\mathfrak{m}^{j_s,k}\}$ and a (Borel regular) measure $\mathfrak{m}_k$ on X such that $\mathfrak{m}^{j_s,k} \rightharpoonup \mathfrak{m}_k$ as $s \to \infty$. In fact, from the standard diagonal arguments we conclude that there is a strictly increasing sequence $\{j_l\}_{l=1}^{\infty} \subset \mathbb{N}$ such that $\mathfrak{m}^{j_l,k} \rightharpoonup \mathfrak{m}_k$ as $l \to \infty$ for every $k \in \mathbb{N}_0$ (in the case when $j_l < k$ we put formally $\mathfrak{m}^{j_l,k} := \mathfrak{m}^{k,k}$).

At the next steps we show that the sequence $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty}$ satisfies (M1)–(M5).

Step 4. From properties (M3) and (M4) verified at Steps 5 and 7 below it follows that $\mathfrak{m}_k(B_j(x)) > 0$ for every $x \in S$ and all $k, j \in \mathbb{N}_0$. This implies that condition (M1) is satisfied.

Step 5. We fix arbitrary $k, i \in \mathbb{N}_0$. By (5.32) and (5.33), for each $\varphi \in C_{\mathrm{c}}(\mathrm{X})$ and all sufficiently large $l \in \mathbb{N}$ we obtain

$$\epsilon^{i\theta} \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}^{j_l,k+i}(x) \leqslant \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}^{j_l,k}(x) \leqslant \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}^{j_l,k+i}(x).$$

Hence, passing to the limit as $l \to \infty$, we have

$$\epsilon^{i\theta} \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}_{k+i}(x) \leqslant \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}_k(x) \leqslant \int_{\mathrm{X}} \varphi(x)\, d\mathfrak{m}_{k+i}(x) \quad \text{for all } \varphi \in C_c(\mathrm{X}).$$

As a result, using the Borel regularity of the measures $\mathfrak{m}_k$, $k \in \mathbb{N}_0$, and the Radon-Nikodým theorem we see that condition (M4) is satisfied with $C_3 = 1$.

Step 6. Given $k \in \mathbb{N}_0$, we fix an arbitrary closed ball $B_r(x)$, where $x \in \mathrm{X}$ and $r \in (0, \epsilon^k]$. If $\widetilde{Q}_{k(r),\alpha} \cap B_{2r}(x) \neq \varnothing$ for some $\alpha \in \mathcal{A}_{k(r)}(\mathrm{S}, \epsilon)$, then by (5.25) we have $\widetilde{Q}_{k(r),\alpha} \subset B_{cr}(x)$ for $c = 4\epsilon^{k(r)}/r + 2$. Furthermore, since $k(r) \geqslant k$, by (5.30) we have $\mathfrak{m}^{j_l,k}(\widetilde{Q}_{k(r),\alpha}) \leqslant h_{k(r),\alpha}$ for each $\alpha \in \mathcal{A}_{k(r)}(\mathrm{S}, \epsilon)$ and all sufficiently large $l \in \mathbb{N}$. Finally, by construction we have $\mathfrak{m}^{j_l,k}(\partial\widetilde{Q}_{k(r),\alpha}) = 0$ for each $\alpha \in \mathcal{A}_{k(r)}(\mathrm{S}, \epsilon)$ and all sufficiently large $l \in \mathbb{N}$. As a result, we apply Proposition 3 to $G = \operatorname{int} B_{2r}(x)$, then take the above observations into account and, finally, use the uniformly locally doubling property of the measure μ. This gives

$$\begin{aligned}\mathfrak{m}_k(B_r(x)) &\leqslant \mathfrak{m}_k(\operatorname{int} B_{2r}(x)) \leqslant \varliminf_{l\to\infty} \mathfrak{m}^{j_l,k}(\operatorname{int} B_{2r}(x)) \\ &\leqslant \varliminf_{l\to\infty} \sum\{\mathfrak{m}^{j_l,k}(\widetilde{Q}_{k(r),\alpha}) \colon \widetilde{Q}_{k(r),\alpha} \cap B_{2r}(x) \neq \varnothing\} \leqslant C\frac{\mu(B_{cr}(x))}{r^\theta} \leqslant C\frac{\mu(B_r(x))}{r^\theta}.\end{aligned} \tag{5.34}$$

Hence condition (M2) is satisfied.

Step 7. To verify condition (M3) it is sufficient to show that there is a constant $C > 0$ such that (we put $B_k(x) := B_{\epsilon^k}(x)$ for brevity)

$$\mathfrak{m}_k(B_k(x)) \geqslant C\frac{\mu(B_k(x))}{\epsilon^{k\theta}} \quad \text{for all } k \in \mathbb{N}_0 \text{ and all } x \in S. \tag{5.35}$$

Indeed, assume that we have already proved (5.35). Then, given $k \in \mathbb{N}_0$ and $r \in [\epsilon^k, 1]$, we note that $k(r) \leqslant k$ in accordance with our notation (2.18). Hence, using (M4) and, taking into account the uniformly locally doubling property of the measure μ, for each $x \in S$ we obtain the required estimate

$$\begin{aligned}\mathfrak{m}_k(B_r(x)) &\geqslant \mathfrak{m}_{k(r)}(B_{k(r)+1}(x)) \geqslant \epsilon\, \mathfrak{m}_{k(r)+1}(B_{k(r)+1}(x)) \\ &\geqslant C\frac{\mu(B_{k(r)+1}(x))}{\epsilon^{(k(r)+1)\theta}} \geqslant C\frac{\mu(B_r(x))}{r^\theta}.\end{aligned}$$

To prove (5.35) we fix an arbitrary $k \in \mathbb{N}_0$ and $x \in S$. Using the subadditivity property of the set function $\mathcal{H}_{\theta,\epsilon^k}$ and (5.28) we find a cube $\widetilde{Q}_{k,\alpha}$, $\alpha \in \mathcal{A}_k(\mathrm{S}, \epsilon)$, such that $\operatorname{cl}(\widetilde{Q}_{k,\alpha} \cap S) \cap B \neq \varnothing$ and

$$\mathcal{H}_{\theta,\epsilon^k}(\operatorname{cl}(\widetilde{Q}_{k,\alpha} \cap S)) \geqslant \frac{1}{C_D(1,0)}\mathcal{H}_{\theta,\epsilon^k}(B_{\epsilon^k}(x) \cap S). \tag{5.36}$$

Note that for each $j \geqslant k$ and any $z_{j,\beta} \preceq z_{k,\alpha}$ such that $\beta \in \mathcal{A}_j(\mathrm{S}, \epsilon)$, there is a minimum number among all integers $s \in \{k, \dots, j\}$ for which there exists $\gamma \in \mathcal{A}_s(\mathrm{S}, \epsilon)$ such that $z_{j,\beta} \preceq z_{s,\gamma} \preceq z_{k,\alpha}$ and $\mathfrak{m}^{j,k}(\widetilde{Q}_{s,\gamma} \cap S) = h_{s,\gamma}$. Thus, there exists a disjoint finite family $\{\widetilde{Q}_{s_i,\gamma_i}\}_{i=1}^N$, where $i \in \{1, \dots, N\}$, such that

$$\mathfrak{m}^{j,k}(\widetilde{Q}_{k,\alpha} \cap S) \geqslant \sum_{i=1}^N \mathfrak{m}^{j,k}(\widetilde{Q}_{s_i,\gamma_i} \cap S) = \sum_{i=1}^N h_{s_i,\gamma_i}.$$

At the same time, by (5.24) and (5.25) we have

$$\widetilde{Q}_{k,\alpha} \cap S \subset \bigcup_{i=1}^N \operatorname{cl} \widetilde{Q}_{s_i,\gamma_i} \subset \bigcup_{i=1}^N B_{2\epsilon^{s_i}}(z_{s_i,\gamma_i}) \quad \text{and} \quad \frac{1}{8}B_{\epsilon^{s_i}}(z_{s_i,\gamma_i}) \subset \widetilde{Q}_{s_i,\gamma_i}$$

for all $j \in \{1,\ldots,N\}$. Consequently, using the uniformly locally doubling property of the measure μ in combination with Proposition 7 we obtain

$$\begin{aligned}\mathfrak{m}^{j,k}(\widetilde{Q}_{k,\alpha}\cap S) &\geqslant \sum_{i=1}^{N}\frac{\mu(\frac{1}{8}B_{\epsilon^{s_i}}(z_{s_i,\gamma_i}))}{\epsilon^{s_i\theta}} \geqslant C\sum_{i=1}^{N}\frac{\mu(4B_{\epsilon^{s_i}}(z_{s_i,\gamma_i}))}{\epsilon^{s_i\theta}}\\ &\geqslant C\sum_{i=1}^{N}\epsilon^{-s_i\theta}\sum_{B\in\mathcal{B}_{s_i}(\mathrm{X},\epsilon)}\{\mu(B)\colon B\cap B_{2\epsilon^{s_i}}(z_{s_i,\gamma_i})\neq\varnothing\} \geqslant C\mathcal{H}_{\theta,\epsilon^k}(\mathrm{cl}(\widetilde{Q}_{k,\alpha}\cap S)).\end{aligned} \tag{5.37}$$

Since the closed ball $B_{5\epsilon^k}(x)$ is a compact set and $\widetilde{Q}_{k,\alpha}\subset B_{5\epsilon^k}(x)$, we use Proposition 3 for $F = B_{5\epsilon^k}(x)$, then combine (5.36) and (5.37) and, finally, take Definition 16 into account. This gives us the crucial estimate

$$\mathfrak{m}_k(B_{5\epsilon^k}(x)) \geqslant \varlimsup_{l\to\infty}\mathfrak{m}^{j_l,k}(B_{5\epsilon^k}(x)) \geqslant \varlimsup_{l\to\infty}\mathfrak{m}^{j_l,k}(\widetilde{Q}_{k,\alpha}) \geqslant C\frac{\mu(B_{\epsilon^k}(x))}{\epsilon^{k\theta}}. \tag{5.38}$$

As a result, using (5.38) and the upper bound in Theorem 5 (we can use it because the proof of this upper bound is based on condition (M2) which has already been verified above) we arrive at (5.35) completing the proof of (M3).

Step 8. By (M4) and (M3), which were verified at Steps 5 and 7, respectively, we have $\mathfrak{m}_0(B_k(x)) \geqslant \epsilon^{k\theta}\mathfrak{m}_k(B_k(x)) \geqslant C\mu(B_k(x))$ for all $x\in S$. By Definition 13 this implies that the measure $\mathfrak{m}_0$ is weakly noncollapsed. We fix an arbitrary Borel set $E\subset S$ and recall the notation (3.6). Throughout this step we set $c = 4/\epsilon+2$ and use Lemma 3. This gives us the existence of a set $E'\subset E$ satisfying $\mathfrak{m}_0(E\setminus E') = 0$ and such that for each point $x\in E'$ one can find a sequence $\{r_l(x)\}$ strictly decreasing to zero such that (recall (3.1))

$$\overline{D}^{\mathfrak{m}_0}_E(x) \geqslant \overline{\mathrm{D}}(x) := \varliminf_{l\to\infty}\frac{\mathfrak{m}_0(B_{r_l(x)}(x)\cap E)}{\mathfrak{m}_0(B_{r_l(x)}(x))} \geqslant \frac{1}{2\underline{\mathrm{C}}_{\mathfrak{m}_0}(5c)},$$
$$\varlimsup_{l\to\infty}\frac{\mathfrak{m}_0(B_{cr_l(x)}(x))}{\mathfrak{m}_0(B_{r_l(x)}(x))} \leqslant \underline{\mathrm{C}}_{\mathfrak{m}_0}(5c).$$

Furthermore, fix $\underline{x}\in E'$ and $\varepsilon\in(0,1)$. We recall the notation (2.18) and put $r_l := r_l(\underline{x})$ and $k_l := k_\epsilon(r_l)$ for all $l\in\mathbb{N}_0$. Clearly, there is $L = L(\underline{x},\varepsilon)\in\mathbb{N}$ such that, for all $l\geqslant L$,

$$\frac{\mathfrak{m}_0(B_{r_l}(\underline{x})\cap E)}{\mathfrak{m}_0(B_{r_l}(\underline{x}))} > \left(1-\frac{\varepsilon}{8}\right)\overline{\mathrm{D}}(\underline{x}) \quad\text{and}\quad \frac{\mathfrak{m}_0(B_{cr_l}(\underline{x}))}{\mathfrak{m}_0(B_{r_l}(\underline{x}))} \leqslant 2\underline{\mathrm{C}}_{\mathfrak{m}_0}(5c). \tag{5.39}$$

Using the Borel regularity of the measure $\mathfrak{m}_0$, given $l\in\mathbb{N}$, we find an open set $\Omega_l\subset B_{2r_l}(\underline{x})$ containing $B_{r_l}(\underline{x})\cap E$ and a compact set $K_l\subset B_{r_l}(\underline{x})\cap E$ such that

$$|\mathfrak{m}_0(\Omega_l)-\mathfrak{m}_0(K_l)| \leqslant \frac{\varepsilon}{8}\overline{\mathrm{D}}(\underline{x})\mathfrak{m}_0(B_{r_l}(\underline{x})). \tag{5.40}$$

Since $\sigma_l := \mathrm{dist}(K_l,\mathrm{X}\setminus\Omega_l) > 0$ for the $\sigma_l/2$-neighbourhood $U_{\sigma_l/2}(K_l)$ of K_l, we obtain

$$K_l\subset U_{\sigma_l/2}(K_l)\subset \mathrm{cl}\,U_{\sigma_l/2}(K_l)\subset\Omega_l\subset B_{2r_l}(\underline{x}). \tag{5.41}$$

Since the index l is occupied for the sequence $\{r_l\}$ and for the sake of simplicity, we assume at this step that $\mathfrak{m}^{j,k} \rightharpoonup \mathfrak{m}_k$ as $j \to \infty$. To verify condition (M5) it is sufficient to establish the existence of a constant $C(\underline{x}) > 0$ independent on ε and l such that for each $l \geqslant L$ there is $N = N(\underline{x}, l, \varepsilon) \geqslant k_l$ such that for any $j \geqslant N$

$$\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)} \cap U_{\sigma_l/2}(K_l)) \geqslant C(\underline{x})\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)}) \quad \text{for some } \beta(j) \in \mathcal{A}_{k_l}(\mathrm{S}, \epsilon). \tag{5.42}$$

Indeed, suppose that we have already proved (5.42). Then, given $l \geqslant L$, we use (5.24) and (5.41) and apply Proposition 3 to $F = \operatorname{cl}(U_{\sigma/2}(K_l))$ and $G = \operatorname{int} B_{r_l/8}(z_{k_l,\beta})$. This gives (we use the notation $B_{k_l}(z) := B_{\epsilon^{k_l}}(z)$, $z \in \mathrm{X}$)

$$\begin{aligned}
\mathfrak{m}_{k_l}(\Omega_l) &\geqslant \mathfrak{m}_{k_l}(\operatorname{cl} U_{\sigma_l/2}(K_l)) \geqslant \varlimsup_{j\to\infty} \mathfrak{m}^{j,k_l}(\operatorname{cl} U_{\sigma_l/2}(K_l)) \geqslant \varliminf_{j\to\infty} \mathfrak{m}^{j,k_l}(U_{\sigma_l/2}(K_l)) \\
&\geqslant \varliminf_{j\to\infty} \mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)} \cap U_{\sigma_l/2}(K_l)) \geqslant C(\underline{x}) \varliminf_{j\to\infty} \mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)}) \\
&\geqslant C(\underline{x}) \varliminf_{j\to\infty} \mathfrak{m}^{j,k_l}\left(\operatorname{int} \frac{1}{8} B_{k_l}(z_{k_l,\beta(j)})\right) \geqslant C(\underline{x})\mathfrak{m}_{k_l}\left(\operatorname{int} \frac{1}{8} B_{k_l}(z_{k_l,\beta(j)})\right).
\end{aligned} \tag{5.43}$$

Since $\widetilde{Q}_{k_l,\beta(j)} \cap B_{2r_l}(\underline{x}) \neq \varnothing$, from (5.25) we obtain $B_{r_l}(\underline{x}) \subset 6B_{k_l}(z_{k_l,\beta(j)})$. The crucial fact is that $z_{k_l,\beta(j)} \in S$ and we can use Theorem 5 for $c = 6, 8$ and $y = z_{k_l,\beta(j)}$. As a result,

$$\mathfrak{m}_{k_l}(\Omega_l) \geqslant C(\underline{x})\mathfrak{m}_{k_l}\left(\operatorname{int} \frac{1}{8} B_{k_l}(z_{k_l}, \beta(j))\right) \geqslant C\mathfrak{m}_{k_l}(6B_{k_l}(z_{k_l,\beta(j)})) \geqslant C\mathfrak{m}_{k_l}(B_{r_l}(\underline{x})). \tag{5.44}$$

The constant $C > 0$ on the right-hand side of (5.44) does not depend on l and ε. Finally, since $\Omega_l \supset E \cap B_{r_l}(\underline{x})$ was chosen arbitrarily and since the measure $\mathfrak{m}_{k_l}$ is Borel regular, we obtain the required estimate $\mathfrak{m}_{k_l}(E \cap B_{r_l}(\underline{x})) \geqslant C\mathfrak{m}_{k_l}(B_{r_l}(\underline{x}))$ for a positive constant C independent of $l \in \mathbb{N}$. Since $l \geqslant L$ was chosen arbitrarily, this verifies (M5).

To prove (5.42) we proceed as follows. We fix $l \geqslant L$, apply Proposition 3 to $G = U_{\sigma_l/2}(K_l)$ and $F = B_{2r_l}(\underline{x})$ and use (5.39)–(5.41). This gives (recall that $\varepsilon \in (0, 1)$)

$$\begin{aligned}
\varliminf_{j\to\infty} \mathfrak{m}^{j,0}(U_{\sigma_l/2}(K_l)) \geqslant \mathfrak{m}_0(U_{\sigma_l/2}(K_l)) &\geqslant \left(1 - \frac{\varepsilon}{2}\right)\overline{\mathrm{D}}(\underline{x})\mathfrak{m}_0(B_{r_l}(\underline{x})) \\
&\geqslant C\mathfrak{m}_0(B_{cr_l}(\underline{x})) \geqslant C \varlimsup_{j\to\infty} \mathfrak{m}^{j,0}(B_{cr_l}(\underline{x})),
\end{aligned}$$

where $C = \overline{\mathrm{D}}(\underline{x})/(4\underline{\mathrm{C}}_{\mathfrak{m}_0}(5c))$. Hence there exists $N = N(\underline{x}, l, \varepsilon) \geqslant k_l$ such that for all $j \geqslant N$,

$$\mathfrak{m}^{j,0}(U_{\sigma_l/2}(K_l)) \geqslant \frac{\overline{\mathrm{D}}(\underline{x})}{5\underline{\mathrm{C}}_{\mathfrak{m}_0}(5c)}\mathfrak{m}^{j,0}(B_{cr_l}(\underline{x})). \tag{5.45}$$

From (5.28) and (5.41) we see that there are at most $C_D(2, 0)$ generalized dyadic cubes $\widetilde{Q}_{k_l,\beta} \cap S$ in S whose closures have nonempty intersections with $U_{\sigma_l/2}(K_l)$. Furthermore, any such cube is contained in $B_{cr_l}(\underline{x})$ together with its closure. Hence, using (5.45) we conclude that for each $j \geqslant N$ there exists $\beta(j) \in \mathcal{A}_{k_l}(\mathrm{S}, \epsilon)$ such that the inequality

$$\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)} \cap U_{\sigma_l/2}(K_l)) \geqslant C(\underline{x})\mathfrak{m}^{j,0}(B_{cr_l}(\underline{x})) \geqslant C(\underline{x})\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)}) \tag{5.46}$$

holds for $C(\underline{x}) := (5C_D(2,0)\underline{C}_{\mathfrak{m}_0}(5c))^{-1}\overline{\mathrm{D}}(\underline{x})$. As a result, taking (5.32) into account we deduce from (5.46) the required estimate

$$\begin{aligned}\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)} \cap U_{\sigma_l/2}(K_l)) &= \frac{\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)})}{\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)})}\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)} \cap U_{\sigma_l/2}(K_l)) \\ &\geqslant C(\underline{x})\frac{\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)})}{\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)})}\mathfrak{m}^{j,0}(\widetilde{Q}_{k_l,\beta(j)}) = C(\underline{x})\mathfrak{m}^{j,k_l}(\widetilde{Q}_{k_l,\beta(j)}).\end{aligned}$$

Theorem 9 is proved.

Theorem 1 follows from Theorems 8 and 9.

5.4. Some examples. In this subsection we show that for some sets $S \in \mathcal{LCR}_\theta(\mathrm{X})$, $\theta > 0$, one can easily build concrete examples of sequences $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$. For generic sets $S \in \mathcal{LCR}_\theta(\mathrm{X})$, where $\theta > 0$, finding explicit examples of sequences $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ is quite a sophisticated problem. In [42] an explicit example of $\{\mathfrak{m}_k\} \in \mathfrak{M}_1(\Gamma)$ was constructed in the case when $\Gamma \subset \mathbb{R}^2$ is a simple rectifiable plane curve of positive length. In [21] an explicit example of $\{\mathfrak{m}_k\} \in \mathfrak{M}_{n-1}(K)$ was given for the case of a single cusp K in $\mathbb{R}^n$. In fact, one can show that these sequences of measures belong to the more narrow classes $\mathfrak{M}^{\mathrm{str}}_1(\Gamma)$ (one should use Theorem 6) and $\mathfrak{M}^{\mathrm{str}}_{n-1}(K)$, respectively.

Example 3. Recall Remark 6. Let $\underline{\theta} \in [0, \underline{Q}_\mu(R))$ for some $R > 0$, and let $S \in \mathcal{ADR}_{\underline{\theta}}(\mathrm{X})$. In this case, given $\theta \geqslant \underline{\theta}$, we put $\epsilon = 1/2$ and define

$$\mathfrak{m}_k := 2^{k(\theta-\underline{\theta})}\mathcal{H}_{\underline{\theta}}\lfloor_S, \qquad k \in \mathbb{N}_0.$$

It is easy to verify that $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_\theta(S)$. Indeed, conditions (M1)–(M4) follow immediately from the construction. To verify (M5) one should repeat with minor technical modifications (keeping in mind (1.3)) the corresponding arguments from the proof of Theorem 6.2 in [35].

Example 4. Let $N \in \mathbb{N}$ and $\{\underline{\theta}_1, \dots, \underline{\theta}_N\} \subset [0, \underline{Q}_\mu(R))$. Given $i \in \{1, \dots, N\}$, let $S_i \in \mathcal{ADR}_{\underline{\theta}_i}(\mathrm{X})$. Set $\overline{\theta} := \max\{\underline{\theta}_1, \dots, \underline{\theta}_N\}$. Now we put $\epsilon = 1/2$ and, given $\theta \geqslant \overline{\theta}$, define

$$\mathfrak{m}_k := \sum_{i=1}^N 2^{k(\theta-\underline{\theta}_i)}\mathcal{H}_{\underline{\theta}_i}\lfloor_{S_i}, \qquad k \in \mathbb{N}_0. \tag{5.47}$$

Based on Example 3, we obtain $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_\theta(S)$. Indeed, properties (M1)–(M4) can be verified easily. The most delicate condition (M5) can be verified as follows. We consider the case $N = 2$. The general case is a little bit more technical but ideologically similar. For each Borel set $E \subset (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$, condition (1.9) is satisfied by (5.47) because S_1 and S_2 are closed. Now we assume that $E \subset S_1 \cap S_2$ and $\underline{\theta}_1 \geqslant \underline{\theta}_2$. Given $\underline{x} \in E$, by condition (M3) already verified we have (we use the notation adopted at the very beginning of § 5)

$$\frac{\mathfrak{m}_k(B_{2^{-k}}(\underline{x})\cap E)}{\mathfrak{m}_k(B_{2^{-k}}(\underline{x}))} \geqslant \frac{2^{k(\theta-\underline{\theta}_1)}\mathcal{H}_{\underline{\theta}_1}\lfloor_{S_1}(B_{2^{-k}}(\underline{x})\cap E)}{\mathfrak{m}_k(B_{2^{-k}}(\underline{x}))} \geqslant \frac{2^{-k\underline{\theta}_1}}{C_1\mu(B_{2^{-k}})}\mathcal{H}_{\underline{\theta}_1}\lfloor_{S_1}(B_{2^{-k}}(\underline{x})\cap E).$$

Consequently, taking into account the Ahlfors-David regularity of S_1 and using the corresponding arguments from the proof of Theorem 6.2 in [35] (with minor technical modifications) we complete the verification of (M5).

§ 6. Lebesgue points of functions

Throughout this section we fix the following data:
(D.6.1) a parameter $p \in (1,\infty)$ and an m.m.s. $\mathrm{X}=(\mathrm{X},\mathrm{d},\mu) \in \mathfrak{A}_p$;
(D.6.2) a parameter $\theta \in [0,p)$ and a closed set $S \in \mathcal{LCR}_\theta(\mathrm{X})$;
(D.6.3) a sequence $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_\theta(S)$ with $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0,1/10]$.
In this section, for any $x \in \mathrm{X}$ and $k \in \mathbb{Z}$ we use the notation $B_k(x) := B_{\epsilon^k}(x)$.

Definition 17. Given $c>0$ and $\delta>0$, we say that a family of closed balls $\mathcal{B} := \{B_{r_i}(x_i)\}_{i=1}^N$, where $N \in \mathbb{N}$, is (S,c,δ)-*nice*, if the following conditions hold:
(B1) $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \varnothing$ if $i,j \in \{1,\dots,N\}$ and $i \neq j$;
(B2) $0 < \min\{r_i : i=1,\dots,N\} \leqslant \max\{r_i : i=1,\dots,N\} \leqslant \delta$;
(B3) $B_{cr_i}(x_i) \cap S \neq \varnothing$ for all $i \in \{1,\dots,N\}$.
Furthermore, we say that $\mathcal{B}$ is an (S,c,δ)-*Whitney family* if it satisfies (B1)–(B3) and
(B4) $B \subset \mathrm{X}\setminus S$ for all $B \in \mathcal{B}$.
We will call $(S,c,1)$-nice families and $(S,c,1)$-Whitney families just (S,c)-*nice families* and (S,c)-*Whitney families*, respectively.

Remark 17. Given $\delta \in (0,1]$ and $c \geqslant 1$, every (S,c,δ)-Whitney family is an (S,c',δ')-Whitney family and every (S,c,δ)-nice family is an (S,c',δ')-nice family for any $\delta' \in [\delta,1]$ and $c' \geqslant c$.

We recall the notation (2.18) and, given a ball $B=B_r(x)$, we put $k(B) := k(r_B)$. Furthermore, we recall the notation introduced at the beginning of § 5.

Proposition 22. *Let $c \geqslant 1$ and $c' \geqslant c+1$. If a closed ball $B=B_r(x)$ in X is such that $r \in (0,1]$ and $cB \cap S \neq \varnothing$, then*

$$\frac{\mu(B)}{\mathfrak{m}_{k(B)}(c'B)} \leqslant \frac{(C_\mu(c'))^{\log_2 2c'+1}}{\epsilon^\theta}\,\frac{C_{\{\mathfrak{m}_k\},3}}{C_{\{\mathfrak{m}_k\},2}}(r_B)^\theta. \tag{6.1}$$

Proof. Note that there exists a ball $\underline{B} \subset c'B$ such that $r_{\underline{B}} = r_B$ and the centre $\underline{x}$ of $\underline{B}$ belongs to S. In this case we have $B \subset 2c'\underline{B}$. Furthermore, in accordance with our notation, $\epsilon^{k(B)+1} < r_B \leqslant \epsilon^{k(B)}$. Hence from (1.7), (1.8) and the uniformly locally doubling property of μ (we set $[c] := \max\{k \in \mathbb{Z}\colon k \leqslant c\}$) we obtain

$$\frac{\mu(B)}{\mathfrak{m}_{k(B)}(c'B)} \leqslant \frac{\mu(2c'\underline{B})}{\mathfrak{m}_{k(B)}(\underline{B})} \leqslant \frac{(C_\mu(c'))^{[\log_2 2c']+1}}{\epsilon^\theta}\,\frac{C_{\{\mathfrak{m}_k\},3}}{C_{\{\mathfrak{m}_k\},2}}(r_B)^\theta. \tag{6.2}$$

This completes the proof.

In this and the subsequent sections we need a Brudnyi-Shvartsman functional 'on small scales'. More precisely we recall (1.10) and formulate the following concept.

Definition 18. Given $\delta \in (0,1]$ and $c > 1$, we introduce the *δ-scale Brudnyi-Shvartsman functional* on $L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ (it takes values in $[0,+\infty]$) by letting

$$\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) := \|f\|_{L_p(\mathfrak{m}_0)} + \sup\biggl(\sum_{B\in\mathcal{B}^{\delta}} \frac{\mu(B)}{(r_B)^p}\bigl(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\bigr)^p\biggr)^{1/p}, \qquad (6.3)$$

where the supremum is taken over all families (S,c,δ)-nice families $\mathcal{B}^{\delta}$.

Remark 18. Keeping in mind (1.12) and the notation used in Theorems 2 and 3, for $\delta = 1$ we write $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f)$ instead of $\mathrm{BSN}^1_{p,\{\mathfrak{m}_k\},c}(f)$.

Lemma 5. *Let $\delta \in (0,1]$ and $c \geqslant 1$. Then there is a constant $C > 0$ depending only on $\delta, \mathcal{C}_{\{\mathfrak{m}_k\}}, c, \epsilon, \theta$ and $C_\mu(2c)$ such that if $\mathcal{B}_\delta$ is an arbitrary (S,c)-nice family of balls such that $r_B \geqslant \delta$ for all $B \in \mathcal{B}_\delta$, then for each $f \in L_p(\mathfrak{m}_0)$,*

$$\sum_{B\in\mathcal{B}_\delta} \frac{\mu(B)}{(r_B)^p}\bigl(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\bigr)^p \leqslant C\int_S |f(x)|^p\,d\mathfrak{m}_0(x). \qquad (6.4)$$

Proof. We fix $f \in L_p(\mathfrak{m}_0)$ and an (S,c)-nice family $\mathcal{B}_\delta$ such that $r_B \geqslant \delta$ for all $B \in \mathcal{B}_\delta$. Let $\underline{k}$ be the largest integer k satisfying the inequality $\epsilon^k \geqslant \delta$. Below we write explicitly all intermediate constants to indicate their dependence on $\underline{k}$ (and hence on δ). By Proposition 7 we have (we take into account that $N_\mu(\epsilon^k, C) \leqslant N_\mu(1,C)$ for all $k \in \{0,\dots,\underline{k}\}$ and $C > 0$)

$$\mathcal{M}(\{2cB\colon B\in\mathcal{B}_\delta\}) \leqslant \sum_{k=0}^{\underline{k}} \mathcal{M}(\{2cB\colon B\in\mathcal{B}_\delta(k,\epsilon)\}) \leqslant (1+\underline{k})N_\mu\Bigl(1,\frac{4c}{\epsilon}\Bigr). \qquad (6.5)$$

Given $k \in \{0,\dots,\underline{k}\}$, an application of Proposition 5 to $\mathfrak{m} = \mathfrak{m}_k$ yields

$$\bigl(\mathcal{E}_{\mathfrak{m}_k}(f,2cB)\bigr)^p \leqslant 2^p \fint_{2cB} |f(z)|^p\,d\mathfrak{m}_k(z) \quad \text{for any } B\in\mathcal{B}_\delta(k,\epsilon).$$

Hence, from Proposition 22 for $c' = 2c$ and (1.8), for any $k \in \{0,\dots,\underline{k}\}$ and $B \in \mathcal{B}_\delta(k,\epsilon)$ we obtain

$$\begin{aligned}\mu(B)\bigl(\mathcal{E}_{\mathfrak{m}_k}(f,2cB)\bigr)^p &\leqslant 2^p\frac{C_{\{\mathfrak{m}_k\},3}}{\epsilon^{\underline{k}\theta}}\,\frac{\mu(B)}{\mathfrak{m}_k(2cB)}\int_{2cB}|f(z)|^p\,d\mathfrak{m}_0(z)\\ &\leqslant 2^p\frac{(C_\mu(2c))^{\log_2 4c+1}}{\epsilon^{(\underline{k}+1)\theta}}\,\frac{(C_{\{\mathfrak{m}_k\},3})^2}{C_{\{\mathfrak{m}_k\},2}}(r_B)^\theta\int_{2cB}|f(z)|^p\,d\mathfrak{m}_0(z). \qquad (6.6)\end{aligned}$$

Note that the right-hand side of (6.6) depends on $\underline{k}$ but does not depend on $k \in \{0,\dots,\underline{k}\}$. Consequently, using Proposition 4, (6.5) and (6.6) we obtain (recall that $\theta \in [0,p)$)

$$\sum_{B\in\mathcal{B}_\delta} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p$$
$$\leqslant \sum_{B\in\mathcal{B}_\delta} \frac{2^p}{\delta^{p-\theta}} \frac{(C_\mu(2c))^{\log_2 4c+1}}{\epsilon^{(\underline{k}+1)\theta}} \frac{(C_{\{\mathfrak{m}_k\},3})^2}{C_{\{\mathfrak{m}_k\},2}} \int_{2cB} |f(z)|^p\,d\mathfrak{m}_0(z)$$
$$\leqslant (1+\underline{k})N_\mu\Big(1,\frac{4c}{\epsilon}\Big)\frac{2^p}{\delta^p} \frac{(C_\mu(2c))^{\log_2 4c+1}}{\epsilon^{\theta}} \frac{(C_{\{\mathfrak{m}_k\},3})^2}{C_{\{\mathfrak{m}_k\},2}} \int_{S} |f(x)|^p\,d\mathfrak{m}_0(x).$$

The proof is complete.

It is natural to ask whether the finiteness of $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f)$ for small $\delta > 0$ implies that of $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f)$. Fortunately, we have an affirmative answer.

Lemma 6. $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ *if and only* $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ *for some* $\delta \in (0,1]$.

Proof. Necessity follows from Remark 17. To prove sufficiency, given an (S,c)-nice family $\mathcal{B}$, we divide it into two subfamilies. More precisely, we put $\mathcal{B}^\delta := \{B \in \mathcal{B}\colon r_B \leqslant \delta\}$ and $\mathcal{B}_\delta := \mathcal{B}\setminus\mathcal{B}^\delta$. Now the claim follows from Lemma 5.

We start with the following lemma (we use the notation $B_k(x) := B_{\epsilon^k}(x)$, $k\in\mathbb{N}_0$, $x \in \mathrm{X}$).

Lemma 7. *Let* $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ *be such that* $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ *for some* $c > 1$ *and* $\delta \in (0,1]$. *Then there exists a Borel function* $\overline{f}\colon S \to \mathbb{R}$ *and a Borel set* $\underline{S} \subset S$ *satisfying* $\mathcal{H}_\theta(S\setminus\underline{S}) = 0$ *such that*

$$\varlimsup_{k\to\infty} \fint_{B_k(x)} |\overline{f}(x) - f(y)|\,d\mathfrak{m}_k(y) = 0 \quad \textit{for all } x \in \underline{S}. \tag{6.7}$$

Proof. We fix $\varepsilon \in (0,(p-\theta)/(2p))$ and split the proof into several steps.

Step 1. Consider the function

$$R(x) := \varlimsup_{r\to 0} \sum_{\epsilon^k<r} \mathcal{E}_{\mathfrak{m}_k}(f,B_k(x)), \qquad x\in\mathrm{X}. \tag{6.8}$$

It is clear that, given $\delta' \in (0,\delta]$, we have

$$R^p(x) \leqslant \varlimsup_{r\to 0}\Big(\sum_{\epsilon^k<r} \frac{\epsilon^{k\varepsilon}}{\epsilon^{k\varepsilon}}\mathcal{E}_{\mathfrak{m}_k}(f,B_k(x))\Big)^p \leqslant \underline{C}\,\delta'^{\varepsilon p} \sup_{\epsilon^k<\delta'} \frac{1}{\epsilon^{k\varepsilon p}}\big(\mathcal{E}_{\mathfrak{m}_k}(f,B_k(x))\big)^p. \tag{6.9}$$

Given $t > 0$, we introduce the t-superlevel set of R^p by letting $E_t := \{x \in S\colon R^p(x) \geqslant t\}$. Our aim is to show that

$$\mathcal{H}_\theta(E_t) = 0 \quad \text{for all } t > 0. \tag{6.10}$$

Now we fix arbitrary $t > 0$ and $\delta' \in (0,\delta]$. For each $x \in E_t$ we find $k_x \in \mathbb{N}_0$ such that $\epsilon^{k_x} \in (0,\delta')$ and

$$t < \underline{C}\frac{\delta'^{\varepsilon p}}{\epsilon^{k_x\varepsilon p}}\big(\mathcal{E}_{\mathfrak{m}_{k_x}}(f,B_{k_x}(x))\big)^p.$$

Clearly, the family of balls $\mathcal{B}\colon \{B_{k_x}(x)\colon x \in E_t\}$ is a covering of E_t. Using

Vitali's $5B$-covering lemma we find a disjoint subfamily $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that $E_t \subset \bigcup\{5B\colon B \in \widetilde{\mathcal{B}}\}$. Hence we have

$$\sum\left\{\frac{\mu(B)}{(r_B)^\theta}\colon B \in \widetilde{\mathcal{B}}\right\} \geqslant C\sum\left\{\frac{\mu(5B)}{(r_{5B})^\theta}\colon B \in \widetilde{\mathcal{B}}\right\} \geqslant C\mathcal{H}_{\theta,5\delta'}(E_t). \tag{6.11}$$

Note that any (S,c,δ')-nice family is also an (S,c,δ)-nice family. Furthermore, by Theorem 5 and Remark 3 it is easy to see that $\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,B) \leqslant C\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,cB)$ for all $B \in \widetilde{\mathcal{B}}$. As a result, we obtain

$$\begin{aligned} t\mathcal{H}_{\theta,5\delta'}(E_t) &\leqslant C\sum_{B\in\widetilde{\mathcal{B}}} \frac{\mu(B)}{(r_B)^\theta}\frac{(\delta')^{\varepsilon p}}{(r_B)^{\varepsilon p}}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,B)\big)^p \\ &\leqslant C(\delta')^{\varepsilon p}\sum_{B\in\widetilde{\mathcal{B}}}\frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,cB)\big)^p \leqslant C(\delta')^{\varepsilon p}\big(\mathrm{BSN}^\delta_{p,\{\mathfrak{m}_k\},c}(f)\big)^p. \end{aligned} \tag{6.12}$$

Passing to the limit as $\delta' \to 0$ and taking into account that $t > 0$ is arbitrary we obtain (6.10).

Step 2. If $l,k \in \mathbb{N}_0$ are such that $l > k$, then from Remark 3 and Theorem 5 it is easy to see that

$$\begin{aligned} &⨍_{B_l(x)} ⨍_{B_k(x)} |f(y)-f(z)|\,d\mathfrak{m}_l(y)\,d\mathfrak{m}_k(z) \\ &\quad\leqslant \sum_{i=k}^{l-1} ⨍_{B_i(x)} ⨍_{B_{i+1}(x)} |f(y)-f(z)|\,d\mathfrak{m}_i(z)\,d\mathfrak{m}_{i+1}(y) \leqslant C\sum_{i=k}^{l}\mathcal{E}_{\mathfrak{m}_i}(f,B_i(x)). \end{aligned} \tag{6.13}$$

Consider the set $\underline{S} := S \setminus \bigcup_{t>0} E_t$. Since $R(x) = 0$ for all $x \in \underline{S}$, it follows from (6.8), (6.13) that if $x \in \underline{S}$, then

$$\left\{⨍_{B_k(x)} f(y)\,d\mathfrak{m}_k(y)\right\}_{k=1}^\infty$$

is a Cauchy sequence. Hence for every $x \in \underline{S}$ there exists a finite limit

$$\overline{f}(x) := \lim_{l\to\infty} ⨍_{B_l(x)} f(z)\,d\mathfrak{m}_l(z).$$

An application of Fatou's lemma together with (6.13) leads to the required estimate

$$\begin{aligned} &\varlimsup_{k\to\infty} ⨍_{B_k(x)} |\overline{f}(x)-f(y)|\,d\mathfrak{m}_k(y) \\ &\qquad\leqslant \varlimsup_{k\to\infty}\varliminf_{l\to\infty} ⨍_{B_k(x)}\left|⨍_{B_l(x)} f(z)\,d\mathfrak{m}_l(z)-f(y)\right|d\mathfrak{m}_k(y) \\ &\qquad\leqslant \varlimsup_{k\to\infty}\varliminf_{l\to\infty} ⨍_{B_l(x)} ⨍_{B_k(x)} |f(y)-f(z)|\,d\mathfrak{m}_k(y)\,d\mathfrak{m}_k(z) \\ &\qquad\leqslant C\varlimsup_{k\to\infty}\sum_{i=k}^\infty \mathcal{E}_{\mathfrak{m}_i}(f,B_i(x)) \leqslant CR(x) = 0 \quad \text{for all } x \in \underline{S}. \end{aligned} \tag{6.14}$$

From (6.10) we obviously have $\mathcal{H}_\theta(S \setminus \underline{S}) = 0$ completing the proof.

We recall the following classical result (see Corollary 3.3.51 in [1]).

Proposition 23. *Let $\mathfrak{m}$ be a locally finite measure on* X. *Given $p \in [1,\infty)$, for each $f \in \mathfrak{B}(\mathrm{X})$ such that $[f]_{\mathfrak{m}} \in L_p(\mathfrak{m})$ and every $\varepsilon > 0$ there exists an open set $O \subset \mathrm{X}$ such that $\mathfrak{m}(O) < \varepsilon$ and $f|_{\mathrm{X} \setminus O}$ is continuous on $\mathrm{X} \setminus O$.*

Now we are ready to state the main result of this section. We recall Definition 14.

Theorem 10. *Let $f \in \mathfrak{B}(\mathrm{X})$ be a function satisfying $[f]_{\mathfrak{m}_0} \in L_p^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ and such that $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}([f]_{\mathfrak{m}_0}) < +\infty$ for some $c > 1$ and $\delta \in (0,1]$. Then*

$$\mathfrak{m}_0(S \setminus (\mathfrak{R}_{\{\mathfrak{m}_k\},\epsilon}(f))) = 0. \tag{6.15}$$

Proof. Let $\overline{f}$ and $\underline{S}$ be the same as in Lemma 7. By Proposition 20 we have $\mathfrak{m}_0(S \setminus \underline{S}) = \mathcal{H}_\theta(S \setminus \underline{S}) = 0$. Hence in order to establish (6.15) it is sufficient to show that

$$f(x) = \overline{f}(x) \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in \underline{S}. \tag{6.16}$$

We apply Proposition 23 for $\mathfrak{m} = \mathfrak{m}_0$. This gives, for each $\varepsilon > 0$, the existence of an open set $O_\varepsilon \subset \mathrm{X}$ such that $\mathfrak{m}_0(O_\varepsilon) < \varepsilon$ and $f \in C(\mathrm{X} \setminus O_\varepsilon)$. We recall (3.6) and put $S_\varepsilon := \{x \in S \setminus O_\varepsilon \colon \overline{D}^{\{\mathfrak{m}_k\}}_{S \setminus O_\varepsilon}(x,\epsilon) > 0\}$. Taking (D.6.3) and (1.9) into account, for each sequence satisfying $\varepsilon_n \downarrow 0$ as $n \to \infty$ we obtain

$$\mathfrak{m}_0\Bigl(S \setminus \bigcup_{n \in \mathbb{N}} \{S_{\varepsilon_n} \cap \underline{S}\}\Bigr) = 0. \tag{6.17}$$

Fix a sufficiently small $\varepsilon > 0$ and a point $\underline{x} \in S_\varepsilon \cap \underline{S}$. By Chebyshev's inequality, for any fixed $\sigma > 0$ we have

$$\begin{aligned} &(\mathfrak{m}_k(B_k(\underline{x})))^{-1}\mathfrak{m}_k(\{y \in B_k(x) \colon |f(y) - \overline{f}(\underline{x})| \geqslant \sigma\}) \\ &\quad \leqslant \frac{1}{\sigma} \fint_{B_k(x)} |f(y) - \overline{f}(\underline{x})| \, d\mathfrak{m}_k(y) \to 0, \qquad k \to \infty. \end{aligned} \tag{6.18}$$

We set $c(\underline{x}) := \overline{D}^{\{\mathfrak{m}_k\}}_{S \setminus O_\varepsilon}(x,\epsilon)$ for brevity. Hence, given $\sigma > 0$, there exists a sufficiently large number $k = k(\underline{x},\sigma) \in \mathbb{N}$ such that

$$\mathfrak{m}_k\biggl(\biggl\{y \in B_k(x) \colon |f(y) - \overline{f}(\underline{x})| < \frac{\sigma}{2},\ |f(y) - f(\underline{x})| < \frac{\sigma}{2}\biggr\}\biggr) \geqslant \frac{c(\underline{x})}{2}\mathfrak{m}_k(B_k(\underline{x})).$$

As a result, $|\overline{f}(\underline{x}) - f(\underline{x})| < \sigma$ by the triangle inequality. Since, given $\underline{x} \in S_\varepsilon$, one can chose $\sigma > 0$ arbitrarily small, we obtain

$$\overline{f}(\underline{x}) = f(\underline{x}) \quad \text{for all } \underline{x} \in S_\varepsilon \cap \underline{S}. \tag{6.19}$$

Finally, taking into account that $\varepsilon > 0$ can be chosen arbitrarily small and combining (6.17) with (6.19) we deduce (6.16) and complete the proof.

§ 7. Extension operator

Throughout this section we fix the following data:

(D.7.1) a parameter $p \in (1, \infty)$ and an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$;

(D.7.2) a parameter $\theta \in [0, p)$ and a closed set $S \in \mathcal{LCR}_\theta(\mathrm{X})$;

(D.7.3) a sequence of measures $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_\theta(S)$ with parameter $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0, 1/10]$.

In this section we put $B_k(x) := B_{\epsilon^k}(x)$ for each $k \in \mathbb{Z}$ and all $x \in \mathrm{X}$. Furthermore, we recall the notation (2.2) and fix a sequence $\{Z_k(\mathrm{X}, \epsilon)\} := \{Z_k(\mathrm{X}, \epsilon)\}_{k \in \mathbb{Z}}$. We recall (2.3) and put

$$\widetilde{B}_{k,\alpha} := B_{2\epsilon^k}(z_{k,\alpha}) \quad \text{for each } k \in \mathbb{Z} \text{ for every } \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon). \tag{7.1}$$

Given $k \in \mathbb{Z}$, the *kth neighbourhood of* S and the *kth layer of* S, respectively, are defined by

$$U_k(S) := \{x \in \mathrm{X} \colon \operatorname{dist}(x, S) < 5\epsilon^k\} \quad \text{and} \quad V_k(S) := U_{k-1}(S) \setminus U_k(S). \tag{7.2}$$

The advantages of such layers are clear from the following elementary proposition.

Proposition 24. *Let $k, k' \in \mathbb{Z}$ be such that $|k - k'| \geqslant 2$. Then, for any ball $B = \widetilde{B}_{k,\alpha}$ such that $B \cap V_k(S) \neq \varnothing$ and any ball $B' = \widetilde{B}_{k',\alpha'}$ such that $B' \cap V_{k'}(S) \neq \varnothing$ we have $B \cap B' = \varnothing$.*

Proof. We fix arbitrary balls B and B' satisfying the assumptions of the lemma. If $B \cap B' \neq \varnothing$, then by the triangle inequality we obtain $\operatorname{dist}(V_k(S), V_{k'}(S)) \leqslant 4(\epsilon^k + \epsilon^{k'})$. On the other hand, since $\epsilon \leqslant 1/10$, we have

$$\operatorname{dist}(V_k(S), V_{k'}(S)) \geqslant 4\epsilon^{\min\{k,k'\}} + 5\epsilon^{\max\{k,k'\}}.$$

This contradiction proves the claim.

A useful property of our space X is the following simple and known result about partitions of unity (see Lemma 2.4 in [9] for details).

Lemma 8. *There is a constant $C > 0$ depending only on $C_\mu(10)$ such that, for each $k \in \mathbb{N}_0$,*

$$0 \leqslant \varphi_{k,\alpha} \leqslant \chi_{\widetilde{B}_{k,\alpha}}, \quad \operatorname{lip} \varphi_{k,\alpha} \leqslant \frac{C}{\epsilon^k} \chi_{\widetilde{B}_{k,\alpha}} \quad \textit{for all } \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon) \tag{7.3}$$

and, furthermore,

$$\sum_{\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)} \varphi_{k,\alpha} = 1. \tag{7.4}$$

Now we establish a simple combinatorial result which is a folklore. Nevertheless, we present the details for the completeness of our exposition.

Lemma 9. *There exists a constant $\underline{N} \in \mathbb{N}$ depending only on $C_\mu(10)$ such that for each $k \in \mathbb{N}_0$ the family $\widetilde{\mathcal{B}}_k := \{\widetilde{B}_{k,\alpha} \colon \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)\}$ can be decomposed into at most $N \leqslant \underline{N}$ disjoint subfamilies $\{\widetilde{\mathcal{B}}^i_k\}_{i=1}^N$.*

Proof. We fix $k \in \mathbb{N}_0$ and put $E(0) = Z_k(\mathrm{X}, \epsilon)$. We denote a maximal $5\epsilon^k$-separated subset of $Z_k(\mathrm{X}, \epsilon)$ by $Z(1)$ and put $E(1) := Z_k(\mathrm{X}, \epsilon) \setminus Z(1)$. Arguing by induction, suppose that for some $i \in \mathbb{N}$ we have already built sets $Z(1), \ldots, Z(i)$ and $E(1), \ldots, E(i)$ in such a way that

$$E(i') = Z_k(\mathrm{X}, \epsilon) \setminus \bigcup_{l=1}^{i'} Z(l), \qquad i' \in \{1, \ldots, i\}.$$

Let $Z(i+1)$ be a maximal $5\epsilon^k$-separated subset of $E(i)$, and let $E(i+1) := E(i) \setminus Z(i+1)$. We put $\underline{N} := \lceil N_\mu(1, 24) \rceil$ (where $N_\mu(R, c)$ is the same as in Proposition 6). We show that $E(i) = \varnothing$ for each $i > \underline{N}$. Indeed, assume that there is a number $i > \underline{N}$ such that $E(i) \neq \varnothing$ and fix $\underline{x}(i) \in E(i)$. Given $i' \in \{1, \ldots, i\}$, from the maximality of $Z(i')$ and the obvious inclusion $E_i \subset E_{i'-1}$ it follows that there is a point $\underline{x}(i') \in Z(i')$ such that $\mathrm{d}(\underline{x}(i'), \underline{x}(i)) < 5\epsilon^k$. Hence

$$B_{\epsilon^k/4}(\underline{x}(i')) \subset B_{6\epsilon^k}(\underline{x}(i)).$$

As a result, since i' can be chosen arbitrarily, there exists a family

$$\mathcal{F} := \{B_{\epsilon^k/4}(\underline{x}(i')) \colon i' \in \{1, \ldots, i\}\}$$

of pairwise disjoint balls contained in the ball $B_{6\epsilon^k}(\underline{x}(i))$ such that $\#\mathcal{F} = i$. Combining this observation with Proposition 6, we obtain a contradiction. Consequently,

$$\#\{i \in \mathbb{N} \colon E(i) \neq \varnothing\} \leqslant \underline{N}.$$

It remains to note that for each $i \in \{1, \ldots, \underline{N}\}$ and any $z, z' \in Z(i)$ we have $B_{2\epsilon^k}(z) \cap B_{2\epsilon^k}(z') = \varnothing$.

The lemma is proved.

Given $k \in \mathbb{Z}$, we set

$$\mathcal{A}_k(S) := \{\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon) \colon \widetilde{B}_{k,\alpha} \cap U_{k-1}(S) \neq \varnothing\}. \tag{7.5}$$

Remark 19. Since $\epsilon \in (0, 1/10]$, it is easy to see from (7.2) and (7.5) that for each $k \in \mathbb{Z}$ and any $\alpha \in \mathcal{A}_k(S)$ there exists a point $\underline{x} \in S$ such that

$$B_{\epsilon^k}(\underline{x}) \subset \frac{3}{\epsilon} \widetilde{B}_{k,\alpha} = B_{6/\epsilon}(z_{k,\alpha}).$$

The following result is an immediate consequence of (7.2)–(7.5).

Proposition 25. *For each* $k \in \mathbb{N}_0$,

$$\chi_{U_{k-1}(S)}(x) \leqslant \sum_{\alpha \in \mathcal{A}_k(S)} \varphi_{k,\alpha}(x) \leqslant \chi_{U_{k-2}(S)}(x), \qquad x \in \mathrm{X}.$$

Now, keeping in mind Remark 19, given an element $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, for each $k \in \mathbb{N}_0$ we define a special family of numbers. More precisely, we put

$$f_{k,\alpha} := \begin{cases} \displaystyle⨍_{(3/\epsilon)\widetilde{B}_{k,\alpha}} f(x)\, d\mathfrak{m}_k(x) & \text{if } \alpha \in \mathcal{A}_k(S), \\ 0 & \text{if } \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon) \setminus \mathcal{A}_k(S). \end{cases} \tag{7.6}$$

The following simple proposition will be useful in what follows. We recall the notation (1.10).

Proposition 26. *There exists a constant $C > 0$ such that for each $f \in L_1^{\rm loc}(\{\mathfrak{m}_k\})$ and every $k \in \mathbb{N}_0$ the inequality*

$$|f_{k,\alpha} - f_{k',\beta}| \leqslant C\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\left(f, \frac{3}{\epsilon}\widetilde{B}_{k,\alpha}\right) \tag{7.7}$$

holds for each $k' \in \{k, k+1\}$ and any $\alpha \in \mathcal{A}_k(S)$ and $\beta \in \mathcal{A}_{k'}(S)$ for which $\widetilde{B}_{k,\alpha} \cap \widetilde{B}_{k',\beta} \neq \varnothing$.

Proof. We fix $f \in L_1^{\rm loc}(\{\mathfrak{m}_k\})$, numbers $k \in \mathbb{N}_0$ and $k' \in \{k, k+1\}$, and indices $\alpha \in \mathcal{A}_k(S)$ and $\beta \in \mathcal{A}_{k'}(S)$ such that $\widetilde{B}_{k,\alpha} \cap \widetilde{B}_{k',\beta} \neq \varnothing$. By (7.1) we have

$$\frac{3}{\epsilon}\widetilde{B}_{k',\beta} \subset \left(\frac{3}{\epsilon} + 4\right)\widetilde{B}_{k,\alpha} \subset \frac{6}{\epsilon}\widetilde{B}_{k,\alpha}.$$

Hence, using (7.6), (1.8), Theorem 5 and Remark 3 we obtain the required estimate

$$\begin{aligned} |f_{k,\alpha} - f_{k',\beta}| &\leqslant \fint_{(3/\epsilon)\widetilde{B}_{k,\alpha}} \fint_{(3/\epsilon)\widetilde{B}_{k',\beta}} |f(y) - f(y')|\, d\mathfrak{m}_k(y)\, d\mathfrak{m}_{k'}(y') \\ &\leqslant C \fint_{(6/\epsilon)\widetilde{B}_{k,\alpha}} \fint_{(6/\epsilon)\widetilde{B}_{k,\alpha}} |f(y) - f(y')|\, d\mathfrak{m}_k(y) d\mathfrak{m}_k(y') \leqslant C\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\left(f, \frac{3}{\epsilon}\widetilde{B}_{k,\alpha}\right). \end{aligned} \tag{7.8}$$

The proof is complete.

Given an element $f \in L_1^{\rm loc}(\{\mathfrak{m}_k\})$, for $k \in \mathbb{N}_0$ we put

$$f_k(x) := \sum_{\alpha \in \mathcal{A}_k(\mathrm{X},\epsilon)} \varphi_{k,\alpha}(x) f_{k,\alpha} = \sum_{\alpha \in \mathcal{A}_k(S)} \varphi_{k,\alpha}(x) f_{k,\alpha}, \qquad x \in \mathrm{X}. \tag{7.9}$$

Having Propositions 25 and 26 at our disposal we obtain nice pointwise estimates for the local Lipschitz constants of the functions f_k, $k \in \mathbb{N}_0$. We recall (2.1).

Proposition 27. *There exists a constant $C > 0$ such that, for each $f \in L_1^{\rm loc}(\{\mathfrak{m}_k\})$, for every $k \in \mathbb{N}_0$ and every $\underline{x} \in U_{k-1}(S)$ the inequality*

$$\operatorname{lip} f_k(\underline{x}) \leqslant \frac{C}{\epsilon^k}\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\left(f, \frac{3}{\epsilon}\widetilde{B}_{k,\underline{\alpha}}\right) \tag{7.10}$$

holds for any index $\underline{\alpha} \in \mathcal{A}_k(S)$ satisfying the condition $\widetilde{B}_{k,\underline{\alpha}} \ni \underline{x}$.

Proof. We fix $k \in \mathbb{N}_0$ and $\underline{x} \in U_{k-1}(S)$. We also fix an arbitrary index $\underline{\alpha} \in \mathcal{A}_k(S)$ such that $\underline{x} \in \widetilde{B}_{k,\underline{\alpha}}$. By property (3) in Proposition 1 and (7.4) we have

$$\operatorname{lip} f_k(\underline{x}) = \operatorname{lip}\left(f_k - \sum_{\alpha \in \mathcal{A}_k(\mathrm{X},\epsilon)} \varphi_{k,\alpha} f_{k,\underline{\alpha}}\right)(\underline{x}).$$

From (7.5) it follows that if $\alpha \in \mathcal{A}_k(\mathrm{X},\epsilon)$ and $\varphi_{k,\alpha}(\underline{x}) \neq 0$, then $\alpha \in \mathcal{A}_k(S)$. Hence we use (7.9) in combination with (7.3) and, finally, take Propositions 7 and 26 into account. This gives

$$\begin{aligned}\operatorname{lip} f_k(\underline{x}) &\leqslant \sum_{\alpha\in\mathcal{A}_k(\mathrm{X},\epsilon)} \operatorname{lip}\varphi_{k,\alpha}(\underline{x})|f_{k,\underline{\alpha}} - f_{k,\alpha}| = \sum_{\alpha\in\mathcal{A}_k(S)} \operatorname{lip}\varphi_{k,\alpha}(\underline{x})|f_{k,\underline{\alpha}} - f_{k,\alpha}| \\ &\leqslant C \sum_{\alpha\in\mathcal{A}_k(S)} \chi_{\widetilde{B}_{k,\alpha}}(\underline{x})\frac{1}{\epsilon^k}|f_{k,\underline{\alpha}} - f_{k,\alpha}| \leqslant \frac{C}{\epsilon^k}\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\Big(f,\frac{3}{\epsilon}\widetilde{B}_{k,\underline{\alpha}}\Big). \qquad (7.11)\end{aligned}$$

The proof is complete.

Proposition 27 leads to a nice estimate for the local Lipschitz constant in the L_p-norm. Recall that, given a Borel set $E \subset \mathrm{X}$ and an element $f \in L_p^{\mathrm{loc}}(\mathrm{X})$, we put $\|f\|_{L_p(E)} := \|f\|_{L_p(E,\mu)}$.

Lemma 10. *There exists a constant $C > 0$ such that for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, every $k \in \mathbb{N}_0$ and any Borel set $E \subset U_{k-1}(S)$,*

$$\|\operatorname{lip} f_k\|^p_{L_p(E)} \leqslant C \sum_{E\cap\widetilde{B}_{k,\alpha}\neq\varnothing} \frac{\mu(\widetilde{B}_{k,\alpha})}{\epsilon^{kp}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\Big(f,\frac{3}{\epsilon}\widetilde{B}_{k,\alpha}\Big)\Big)^p. \qquad (7.12)$$

Proof. By Proposition 27, for any ball $\widetilde{B}_{k,\alpha}$ such that $E\cap\widetilde{B}_{k,\alpha} \neq \varnothing$ we clearly have

$$\int_{E\cap\widetilde{B}_{k,\alpha}} (\operatorname{lip} f_k(x))^p\,d\mu(x) \leqslant C\frac{\mu(E\cap\widetilde{B}_{k,\alpha})}{\epsilon^{kp}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\Big(f,\frac{3}{\epsilon}\widetilde{B}_{k,\alpha}\Big)\Big)^p.$$

This observation in combination with (7.5) gives

$$\begin{aligned}\int_E (\operatorname{lip} f_k(x))^p\,d\mu(x) &\leqslant \sum_{\alpha\in\mathcal{A}_k(S)} \int_{E\cap\widetilde{B}_{k,\alpha}} (\operatorname{lip} f_k(x))^p\,d\mu(x) \\ &\leqslant C \sum_{E\cap\widetilde{B}_{k,\alpha}\neq\varnothing} \frac{\mu(\widetilde{B}_{k,\alpha})}{\epsilon^{kp}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}\Big(f,\frac{3}{\epsilon}\widetilde{B}_{k,\alpha}\Big)\Big)^p. \qquad (7.13)\end{aligned}$$

This completes the proof.

To construct our extension operator, given an arbitrary $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, we build a certain special sequence $\{f^j\}_{j\in\mathbb{N}}$. Informally speaking, the graph of each f^j looks like a stairway formed of elementary steps $\mathrm{St}_i[f]$, $i = 1,\dots,j$. More precisely, given $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, we set $f_0 := 0$. Furthermore, arguing by induction, for each $i \in \mathbb{N}$, we define the *elementary ith step of f* by

$$\mathrm{St}_i[f](x) := \sum_{\alpha\in\mathcal{A}_i(S)} \varphi_{i,\alpha}(x)(f_{i,\alpha} - f_{i-1}(x)), \qquad x \in \mathrm{X}. \qquad (7.14)$$

Remark 20. In view of Proposition 25 it is clear that $\operatorname{supp}\mathrm{St}_i[f] \subset U_{i-2}(S)$ for all $i \in \mathbb{N}$.

Finally, we define the *special approximating sequence* by letting, for each $j \in \mathbb{N}$,

$$f^j(x) := \sum_{i=1}^{j} \mathrm{St}_i[f](x), \qquad x \in \mathrm{X}. \tag{7.15}$$

Proposition 28. *For each point $x \in \mathrm{X} \setminus S$ there exists $j(x) \in \mathbb{N}$ such that $f^j(x) = f^{j(x)}(x)$ for all $j \geqslant j(x)$.*

Proof. For each $x \in \mathrm{X} \setminus S$, from Remark 20 and (7.15) we obtain $f^j(x) = f^{j+1}(x)$, provided that $x \in \mathrm{X} \setminus U_{j-1}(S)$. Since $U_{j+1}(S) \subset U_j(S)$ for all $j \in \mathbb{N}$, the claim follows.

Now we are ready to present our extension operator.

Definition 19. Given $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$, we put

$$\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f) := \chi_S f + \chi_{\mathrm{X} \setminus S} \lim_{j \to \infty} f^j, \tag{7.16}$$

where by $\chi_S f$ we mean an $\mathfrak{m}_0$-equivalence class and $\chi_{\mathrm{X} \setminus S} \lim_{j \to \infty} f^j$ denotes the pointwise limit of the sequence $\{f^j\}$ on the set $\mathrm{X} \setminus S$.

Remark 21. Let $N_{\mathfrak{m}_0}$ be a linear space of all functions $f \colon \mathrm{X} \to \mathbb{R}$ such that $f(x) = 0$ for $\mathfrak{m}_0$-almost all $x \in S$ and $f(x) = 0$ for all $x \in \mathrm{X} \setminus S$. By Proposition 28 formula (7.16) gives us a mapping $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}} \colon L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}) \to \mathfrak{B}(\mathrm{X})/N_{\mathfrak{m}_0}$ which is well defined and linear.

The main reason for introducing the sequence $\{f^j\}$ in this way is some remarkable pointwise properties of the steps $\mathrm{St}_i[f]$, $i \in \mathbb{N}$. More precisely, the following result holds.

Proposition 29. *Let $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$. Let $i \in \mathbb{N}$, $\underline{x} \in U_{i-2}(S)$ and $\underline{\alpha} \in \mathcal{A}_{i-1}(S)$ be such that $\underline{x} \in \widetilde{B}_{i-1,\underline{\alpha}}$. Then*

$$\widetilde{\mathrm{St}}_i[f](\underline{x}) := \sum_{\alpha \in \mathcal{A}_i(S)} \chi_{\widetilde{B}_{i,\alpha}}(\underline{x}) |f_{i,\alpha} - f_{i-1}(\underline{x})| \leqslant C \widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon} \widetilde{B}_{i-1,\underline{\alpha}}\Big), \tag{7.17}$$

where the constant $C > 0$ depends neither on f nor on i, $\underline{x}$ or $\underline{\alpha}$.

Proof. By (7.4) and (7.9) we have

$$\widetilde{\mathrm{St}}_i[f](\underline{x}) \leqslant \sum_{\alpha \in \mathcal{A}_i(S)} \sum_{\alpha' \in \mathcal{A}_{i-1}(\mathrm{X},\epsilon)} \chi_{\widetilde{B}_{i,\alpha}}(\underline{x}) \varphi_{i-1,\alpha'}(\underline{x}) |f_{i,\alpha} - f_{i-1,\alpha'}|.$$

Using the triangle inequality we have $|f_{i,\alpha} - f_{i-1,\alpha'}| \leqslant |f_{i,\alpha} - f_{i-1,\underline{\alpha}}| + |f_{i-1,\underline{\alpha}} - f_{i-1,\alpha'}|$. Hence, using (7.3) and Proposition 26 and taking Proposition 7 into account we obtain

$$\widetilde{\mathrm{St}}_i[f](\underline{x}) \leqslant \sum_{i'=i-1}^{i} \sum_{\alpha' \in \mathcal{A}_{i'}(\mathrm{X},\epsilon)} \chi_{\widetilde{B}_{i',\alpha'}}(\underline{x}) |f_{i',\alpha'} - f_{i-1,\underline{\alpha}}| \leqslant C \widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon} \widetilde{B}_{i-1,\underline{\alpha}}\Big).$$

The proof is complete.

Proposition 29 leads to useful estimates for the L_p-norms of steps and their local Lipschitz constants.

Lemma 11. *There exists $C>0$ such that, for each $i \in \mathbb{N}$, the following properties hold*:

1) *for each Borel set $E \subset U_{i-2}(S)$ and any measure ν on* X,

$$\|\mathrm{St}_i[f]\|^p_{L_p(E,\nu)} \leqslant C \sum_{\widetilde{B}_{i-1,\alpha}\cap E \neq \varnothing} \nu(\widetilde{B}_{i-1,\alpha})\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i-1,\alpha}\Big)\Big)^p;$$

2) *for each Borel set $E \subset U_{i-2}(S)$*,

$$\|\mathrm{lip}(\mathrm{St}_i[f])\|^p_{L_p(E)} \leqslant C \sum_{\widetilde{B}_{i-1,\alpha}\cap E \neq \varnothing} \frac{\mu(\widetilde{B}_{i-1,\alpha})}{\epsilon^{(i-1)p}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i-1,\alpha}\Big)\Big)^p.$$

Proof. Fix $i \in \mathbb{N}$ and a Borel set $E \subset U_{i-2}(S)$.

To prove claim 1) we note that by (7.3) we have $|\mathrm{St}_i[f](x)| \leqslant \widetilde{\mathrm{St}}_i[f](x)$ for all $x \in \mathrm{X}$. Hence an application of Proposition 29 gives

$$\begin{aligned}\|\mathrm{St}_i[f]\|^p_{L_p(E,\nu)} \leqslant \|\widetilde{\mathrm{St}}_i[f]\|^p_{L_p(E,\nu)} &\leqslant \sum_{\alpha\in\mathcal{A}_{i-1}(S)} \int_{\widetilde{B}_{i-1,\alpha}\cap E} \big(\widetilde{\mathrm{St}}_i[f](x)\big)^p \, d\nu(x)\\ &\leqslant C \sum_{\widetilde{B}_{i-1,\alpha}\cap E \neq \varnothing} \nu(\widetilde{B}_{i-1,\alpha})\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i-1,\alpha}\Big)\Big)^p. \qquad (7.18)\end{aligned}$$

To prove claim 2) we note that by (7.14), Proposition 1, (7.3) and (7.4) we have

$$\begin{aligned}\mathrm{lip}(\mathrm{St}_i[f])(x) &\leqslant \sum_{\alpha\in\mathcal{A}_i(S)} \mathrm{lip}\,\varphi_{i,\alpha}(x)|f_{i,\alpha} - f_{i-1}(x)| + \sum_{\alpha\in\mathcal{A}_i(S)} \varphi_{i,\alpha}(x)\,\mathrm{lip}\, f_{i-1}(x)\\ &\leqslant \frac{C}{\epsilon^i}\widetilde{\mathrm{St}}_i[f](x) + \mathrm{lip}\, f_{i-1}(x) \quad \text{for all } x \in \mathrm{X}.\end{aligned}$$

Hence, using Lemma 10 for $k=i-1$ and (7.18) for $\nu=\mu$ we obtain the required estimate

$$\|\mathrm{lip}(\mathrm{St}_i[f])\|^p_{L_p(E)} \leqslant C \sum_{\widetilde{B}_{i-1,\alpha}\cap E \neq \varnothing} \frac{\mu(\widetilde{B}_{i-1,\alpha})}{\epsilon^{(i-1)p}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i-1,\alpha}\Big)\Big)^p. \qquad (7.19)$$

The proof is complete.

The crucial observation is made in the following lemma.

Lemma 12. *There is a constant $C>0$ such that, for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ and every $j \in \mathbb{N}$*,

$$\|\mathrm{lip}\, f^j\|^p_{L_p(U_0(S))} \leqslant C \sum_{i=1}^{j} \sum_{\widetilde{B}_{i,\alpha}\cap \widehat{V}_i(S) \neq \varnothing} \frac{\mu(\widetilde{B}_{i,\alpha})}{\epsilon^{ip}}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_i}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i,\alpha}\Big)\Big)^p, \qquad (7.20)$$

where $\widehat{V}_i(S) := V_i(S)$ if $j \geqslant 2$, $i \in \{1,\dots,j-1\}$ and $\widehat{V}_j(S) := U_{j-1}(S)$.

Proof. For a moment fix $j \in \mathbb{N}$, $j \geqslant 2$ and $i \in \{1, \ldots, j-1\}$. Given $\underline{x} \in \widehat{V}_i(S)$, from (7.2) and Remark 20 it is clear that $\operatorname{St}_{i'}[f](\underline{x}) = 0$ for all $i' \geqslant i+2$. Hence, using (7.15) and Proposition 1 we obtain,

$$\operatorname{lip} f^j(\underline{x}) \leqslant \operatorname{lip} f_i(\underline{x}) + \operatorname{lip} \operatorname{St}_{i+1}[f](\underline{x}).$$

Thus, applying Lemmas 10 and 11 to $E = \widehat{V}_i(S)$ we deduce

$$\begin{aligned} \|\operatorname{lip} f^j\|^p_{L_p(\widehat{V}_i(S))} &\leqslant \|\operatorname{lip} f_i\|^p_{L_p(\widehat{V}_i(S))} + \|\operatorname{lip} \operatorname{St}_{i+1}[f]\|^p_{L_p(\widehat{V}_i(S))} \\ &\leqslant C \sum_{\widetilde{B}_{i,\alpha} \cap \widehat{V}_i(S) \neq \varnothing} \frac{\mu(\widetilde{B}_{i,\alpha})}{\epsilon^{ip}} \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_i}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{i,\alpha}\Big)\Big)^p. \end{aligned} \tag{7.21}$$

On the other hand, given $j \in \mathbb{N}$, an application of Lemma 10 to $E = U_{j-1}(S)$ gives

$$\begin{aligned} \|\operatorname{lip} f^j\|^p_{L_p(U_{j-1}(S))} &= \|\operatorname{lip} f_j\|^p_{L_p(U_{j-1}(S))} \\ &\leqslant C \sum_{\widetilde{B}_{j,\alpha} \cap U_{j-1}(S) \neq \varnothing} \frac{\mu(\widetilde{B}_{j,\alpha})}{\epsilon^{jp}} \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_j}\Big(f, \frac{3}{\epsilon}\widetilde{B}_{j,\alpha}\Big)\Big)^p. \end{aligned} \tag{7.22}$$

Summing inequalities (7.21) over all $i \in \{1, \ldots, j-1\}$ and then taking (7.22) into account we arrive at (7.20) completing the proof.

Lemma 13. *There exists a constant* $C > 0$ *such that, for each* $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\|f_1\|^p_{L_p(\mathrm{X})} + \|\operatorname{lip} f_1\|^p_{L_p(\mathrm{X})} \leqslant C\|f\|^p_{L_p(\mathfrak{m}_0)}. \tag{7.23}$$

Proof. Combining the first inequality in (7.3) with (7.6) and (7.9) and using Hölder's inequality we obtain

$$|f_1(x)|^p \leqslant C \sum_{\alpha \in \mathcal{A}_1(S)} \chi_{\widetilde{B}_{1,\alpha}}(x) \Big(\fint_{\frac{3}{\epsilon}\widetilde{B}_{1,\alpha}} |f(y)|\, d\mathfrak{m}_1(y)\Big)^p, \qquad x \in \mathrm{X}. \tag{7.24}$$

Similarly, taking the second inequality in (7.3) into account we have

$$(\operatorname{lip} f_1(x))^p \leqslant C \sum_{\alpha \in \mathcal{A}_1(S)} \chi_{\widetilde{B}_{1,\alpha}}(x) \Big(\fint_{\frac{3}{\epsilon}\widetilde{B}_{1,\alpha}} |f(y)|\, d\mathfrak{m}_1(y)\Big)^p, \qquad x \in \mathrm{X}. \tag{7.25}$$

Combining (7.24) with (7.25) and using Hölder's inequality we obtain

$$\|f_1\|^p_{L_p(\mu)} + \|\operatorname{lip} f_1\|^p_{L_p(\mu)} \leqslant C \sum_{\alpha \in \mathcal{A}_1(S)} \mu(\widetilde{B}_{1,\alpha}) \fint_{\frac{3}{\epsilon}\widetilde{B}_{1,\alpha}} |f(y)|^p\, d\mathfrak{m}_1(y). \tag{7.26}$$

By (7.5) we have $(6/\epsilon - 1)B_{1,\alpha} \cap S \neq \varnothing$ for all $\alpha \in \mathcal{A}_1(S)$. Hence using Proposition 22 for $c = 6/\epsilon - 1$ and $c' = 6/\epsilon$ and taking into account the uniformly locally doubling property of μ we obtain

$$\frac{\mu(\widetilde{B}_{1,\alpha})}{\mathfrak{m}_1(\frac{3}{\epsilon}\widetilde{B}_{1,\alpha})} \leqslant C \quad \text{for all } \alpha \in \mathcal{A}_1(S).$$

As a result, using this observation, Propositions 4 and 7 and taking (1.8) into account we obtain

$$\sum_{\alpha\in\mathcal{A}_1(S)} \mu(\widetilde{B}_{1,\alpha}) ⨍_{\frac{3}{\epsilon}\widetilde{B}_{1,\alpha}} |f(y)|^p\, d\mathfrak{m}_1(y) \leqslant C \sum_{\alpha\in\mathcal{A}_1(S)} \int_{\frac{3}{\epsilon}\widetilde{B}_{1,\alpha}} |f(y)|^p\, d\mathfrak{m}_1(y) \leqslant C\|f\|^p_{L_p(\mathfrak{m}_1)} \leqslant C\|f\|^p_{L_p(\mathfrak{m}_0)}. \tag{7.27}$$

Combining (7.26) and (7.27) we obtain the required estimate.

The lemma is proved.

We recall Definitions 17 and 18. We also recall (1.10) and write for brevity $k(B) := k(r_B)$. Now we introduce a new useful functional.

Definition 20. Given $f \in L_1^{\rm loc}(\{\mathfrak{m}_k\})$, we put

$$\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) := \varliminf_{\delta\to 0} \mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) + \sup\Big(\sum_{B\in\mathcal{B}} \frac{\mu(B)}{(r_B)^p}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\big)^p\Big)^{1/p}, \tag{7.28}$$

where the supremum in the second term is taken over all (S,c)-Whitney families $\mathcal{B}$.

Now we present a keystone estimate for the local Lipschitz constants of the functions f^j, $j\in\mathbb{N}$.

Theorem 11. *For each $c \geqslant 3/\epsilon$ there exists a constant $C>0$ such that*

$$\varliminf_{j\to\infty} \|\operatorname{lip} f^j\|^p_{L_p(\mathrm{X})} \leqslant C\,\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) \quad \textit{for all } f \in L_1^{\rm loc}(\{\mathfrak{m}_k\}). \tag{7.29}$$

Proof. Without loss of generality we may assume that $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$, since otherwise the inequality is trivial. We split the proof into several steps.

Step 1. We claim that for each $j\in\mathbb{N}$, $j\geqslant 2$, there is an (S,c)-Whitney family of balls $\mathcal{B}^j_1(S)$ such that (we put $k(B) := k(r_B)$, as usual, and recall (1.10))

$$\sum_{B\in\mathcal{B}^j_1(S)} \frac{\mu(B)}{(r_B)^p}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\big)^p \geqslant \frac{1}{2\underline{N}} \sum_{i=1}^{j-1} \sum_{\widetilde{B}_{i,\alpha}\cap V_i(S)\neq\varnothing} \frac{\mu(\widetilde{B}_{i,\alpha})}{\epsilon^{ip}}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_i}(f,c\widetilde{B}_{i,\alpha})\big)^p, \tag{7.30}$$

where the constant $\underline{N}$ is the same as in Lemma 9. Indeed, we split the sum on the right-hand side of (7.30) into sums over the odd and even $i\in\{1,\dots,j-1\}$, respectively. Without loss of generality we may assume that the sum over the odd indices is not smaller than the one over the even indices. Next, for each odd $i\in\{1,\dots,j-1\}$ we use Lemma 9 and divide the family $\{\widetilde{B}_{i,\alpha}\colon \widetilde{B}_{i,\alpha}\cap V_i\}$ into at most $\underline{N}$ disjoint subfamilies. For each odd $i\in\{1,\dots,j-1\}$ we choose a subfamily which maximizes the corresponding sum and denote it by $\mathcal{G}_i$. By Proposition 24 we have $\mathcal{G}_i\cap\mathcal{G}_{i'}=\varnothing$ if $i\neq i'$. Finally, we set $\mathcal{B}^j_1(S) := \bigcup\{\mathcal{G}_i\}$, where the union is taken over all odd $i\in\{1,\dots,j-1\}$. This clearly gives (7.30). On the other hand it is clear that

$$\sum_{B\in\mathcal{B}^j_1(S)} \frac{\mu(B)}{(r_B)^p}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\big)^p \leqslant \sup \sum_{B\in\mathcal{B}} \frac{\mu(B)}{(r_B)^p}\big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\big)^p, \tag{7.31}$$

where the supremum in (7.31) is taken over all (S,c)-Whitney families $\mathcal{B}$.

Step 2. Given $j \in \mathbb{N}$, by Lemma 9 there is a disjoint (S,c)-nice family $\mathcal{B}^j_2(S)$ such that

$$\sum_{B \in \mathcal{B}^j_2(S)} \frac{\mu(B)}{(r_B)^p} \big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, cB)\big)^p \geqslant \frac{1}{\underline{N}} \sum_{\alpha \in \mathcal{A}_j(S)} \frac{\mu(B_{j,\alpha})}{\epsilon^{jp}} \big(\widetilde{\mathcal{E}}_{\mathfrak{m}_j}(f, c\widetilde{B}_{j,\alpha})\big)^p. \tag{7.32}$$

By Definitions 17 and 18 we have

$$\sum_{B \in \mathcal{B}^j_2(S)} \frac{\mu(B)}{(r_B)^p} \big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, cB)\big)^p \leqslant \mathrm{BSN}^{2\epsilon^j}_{p,\{\mathfrak{m}_k\},c}(f). \tag{7.33}$$

Step 3. Using Lemma 12 and (7.30), (7.32), for any sufficiently large $j \in \mathbb{N}$ we obtain (here we use the estimate $\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_1 B) \leqslant C \widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_2 B)$ for $1 \leqslant c_1 \leqslant c_2$, which follows from Remark 3 and Theorem 5)

$$\int_{U_0(S)} (\operatorname{lip} f^j(x))^p \, d\mu(x) \leqslant C \sum_{B \in \mathcal{B}^j_1(S) \cup \mathcal{B}^j_2(S)} \frac{\mu(B)}{(r_B)^p} \big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, cB)\big)^p.$$

Combining this inequality with (7.31) and (7.33) and (7.28) we deduce

$$\varliminf_{j \to \infty} \|\operatorname{lip} f^j\|^p_{L_p(U_0(S))} \leqslant C \, \mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f). \tag{7.34}$$

Step 4. From Remark 20 and (7.15) it follows that $f_1(x) = f^j(x)$ for each $j \in \mathbb{N}$ and all $x \in \mathrm{X} \setminus U_0(S)$. Hence, using Lemma 13 we obtain

$$\varliminf_{j \to \infty} \|\operatorname{lip} f^j\|^p_{L_p(\mathrm{X} \setminus U_0(S))} = \|\operatorname{lip} f_1\|^p_{L_p(\mathrm{X} \setminus U_0(S))} \leqslant C \|f\|^p_{L_p(\mathfrak{m}_0)}. \tag{7.35}$$

Step 5. Combining (7.34) and (7.35) and taking (6.3) and (7.28) into account we obtain the required inequality (7.29).

The theorem is proved.

The finiteness of $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f)$ allows one to establish some interesting convergence properties of the sequence $\{f^j\}$. More precisely, the following assertion holds.

Theorem 12. *If* $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ *for some* $c \geqslant 3/\epsilon$ *and* $\delta \in (0,1]$, *then:*

(i) $\{f^j\}$ *converges to* f $\mathfrak{m}_0$*-almost everywhere on* S *and converges to* $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)$ *everywhere on* $\mathrm{X} \setminus S$;

(ii) $\|f^j - \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)\|_{L_p(\mathrm{X})} \to 0$ *as* $j \to \infty$;

(iii) *for each* $k \in \mathbb{N}$, $\|f^j - f\|_{L_p(\mathfrak{m}_k)} \to 0$ *as* $j \to \infty$.

Proof. Recall Remark 2. By Proposition 28 and Definition 19 the sequence $\{f^j\}$ converges to the function $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)$ everywhere on $\mathrm{X} \setminus S$. Now we recall Definition 14 and fix an arbitrary point $\underline{x} \in \mathfrak{R}_{\{\mathfrak{m}_k\},\epsilon}(f)$. Since $\mathfrak{R}_{\{\mathfrak{m}_k\},\epsilon}(f) \subset S$, from (7.9) and (7.15) it follows that $f^j(\underline{x}) = f_j(\underline{x})$ for all $j \in \mathbb{N}$. Hence using (7.3) and (7.6) and Theorem 5, for every $j \geqslant 2$ we have

$$\begin{aligned} |f(\underline{x}) - f^j(\underline{x})| &\leqslant \sum_{\alpha \in \mathcal{A}_j(S)} \varphi_{j,\alpha}(\underline{x}) |f(\underline{x}) - f_{j,\alpha}| \\ &\leqslant \sum_{\alpha \in \mathcal{A}_j(S)} \chi_{\widetilde{B}_{j,\alpha}}(\underline{x}) \fint_{\frac{3}{\epsilon}\widetilde{B}_{j,\alpha}} |f(\underline{x}) - f(y)| \, d\mathfrak{m}_j(y) \leqslant C \fint_{B_{j-2}(\underline{x})} |f(\underline{x}) - f(y)| \, d\mathfrak{m}_j(y). \end{aligned} \tag{7.36}$$

Now, using (1.8) we obtain

$$|f(\underline{x}) - f^j(\underline{x})| \leqslant C \fint_{B_{j-2}(\underline{x})} |f(\underline{x}) - f(y)| \, d\mathfrak{m}_{j-2}(y) \to 0, \qquad j \to \infty.$$

Combining this observation with Theorem 10 we arrive at assertion (i).

Now let us prove (ii). Given $i \geqslant 2$, by Lemma 9 there exists an $(S, \frac{3}{\epsilon}, 2\epsilon^{i-1})$-nice family $\mathcal{B}$ such that

$$\sum_{\widetilde{B}_{i-1,\alpha} \cap U_{i-2}(S) \neq \varnothing} \mu(\widetilde{B}_{i-1,\alpha}) \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}} \Big(f, \frac{3}{\epsilon} \widetilde{B}_{i-1,\alpha} \Big) \Big)^p \leqslant C \sum_{B \in \mathcal{B}} \mu(B) \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}} \Big(f, \frac{3}{\epsilon} B \Big) \Big)^p. \tag{7.37}$$

Given $i \geqslant 2$, by Remark 20 we have $\| \operatorname{St}_i[f] \|_{L_p(\mathrm{X})} = \| \operatorname{St}_i[f] \|_{L_p(U_{i-2}(S))}$. Hence we apply Lemma 11 to $\nu = \mu$ and $E = U_{i-2}(S)$, use (7.37) and take Definition 18 and Remark 17 into account (here we use the estimate $\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_1 B) \leqslant C \widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_2 B)$ for $1 \leqslant c_1 \leqslant c_2$, which follows from Remark 3 and Theorem 5). As a result, given $i \geqslant 2$, we obtain

$$\| \operatorname{St}_i[f] \|_{L_p(\mathrm{X})} \leqslant C \epsilon^i \operatorname{BSN}^{2\epsilon^{i-1}}_{p, \{\mathfrak{m}_k\}, c}(f).$$

Thus, by the triangle inequality, for each $l \in \mathbb{N}$ such that $\epsilon^{l-1} \leqslant \delta$ and any $m > l$ we deduce

$$\| f^l - f^m \|_{L_p(\mathrm{X})} \leqslant \sum_{i=l+1}^{m} \| \operatorname{St}_i[f] \|_{L_p(\mathrm{X})} \leqslant C \epsilon^l \operatorname{BSN}^{\delta}_{p, \{\mathfrak{m}_k\}, c}(f).$$

Consequently, $\| f^l - f^m \|_{L_p(\mathrm{X})} \to 0$ as $l, m \to \infty$. Since $L_p(\mathrm{X})$ is complete, there exists $F \in L_p(\mathrm{X})$ such that $\| F - f^j \|_{L_p(\mathrm{X})} \to 0$ as $j \to \infty$. The classical arguments ensure the existence of a subsequence $\{f^{j_s}\}$ converging to F μ-almost everywhere. On the other hand the measure μ is absolutely continuous with respect to the measure $\mathfrak{m}_0$ and, consequently, using assertion (i) already proved we obtain $F = \operatorname{Ext}_{S, \{\mathfrak{m}_k\}}(f)$ in the sense of equality almost everywhere. This proves the claim.

To establish (iii) we fix $k \in \mathbb{N}_0$. Given $i \geqslant 2$, by Lemma 9 there exists an $(S, 3/\epsilon, 2\epsilon^{i-1})$-nice family $\mathcal{B}$ such that

$$\sum_{\widetilde{B}_{i-1,\alpha} \cap U_{i-2}(S) \neq \varnothing} \mathfrak{m}_k(\widetilde{B}_{i-1,\alpha}) \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}} \Big(f, \frac{3}{\epsilon} \widetilde{B}_{i-1,\alpha} \Big) \Big)^p \leqslant C \sum_{B \in \mathcal{B}} \mathfrak{m}_k(B) \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}} \Big(f, \frac{3}{\epsilon} B \Big) \Big)^p. \tag{7.38}$$

For each $i \in \mathbb{N}$, $i \geqslant 2$, we apply Lemma 11 to $\nu = \mathfrak{m}_k$ and $E = U_{i-2}(S)$, use (7.38) and take Remark 20 into account. This gives

$$\| \operatorname{St}_i[f] \|^p_{L_p(\mathfrak{m}_k)} \leqslant C \sum_{B \in \mathcal{B}} \mathfrak{m}_k(B) \Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_{i-1}} \Big(f, \frac{3}{\epsilon} B \Big) \Big)^p.$$

Thus, by (1.6), Definition 18 and Remark 17, for each $i \geqslant 2$, $i > k$, we obtain (here we use the estimate $\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_1 B) \leqslant C\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f, c_2 B)$ for $1 \leqslant c_1 \leqslant c_2$, which follows from Remark 3 and Theorem 5)

$$\|\mathrm{St}_i[f]\|^p_{L_p(\mathfrak{m}_k)} \leqslant C\epsilon^{(p-\theta)i}\big(\mathrm{BSN}^{2\epsilon^{i-1}}_{p,\{\mathfrak{m}_k\},c}(f)\big)^p.$$

Since $\theta \in [0, p)$, given $\delta \in (0, 1]$, an application of the triangle inequality gives, for all sufficiently large $l \in \mathbb{N}$ and all $m > l$,

$$\|f^l - f^m\|_{L_p(\mathfrak{m}_k)} \leqslant \sum_{i=l+1}^{m} \|\mathrm{St}_i(f)\|_{L_p(\mathfrak{m}_k)} \leqslant C\epsilon^{((p-\theta)l)/p}\,\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f).$$

Since $L_p(\mathfrak{m}_k)$ is complete, there exists $h \in L_p(\mathfrak{m}_k)$ such that $\|h - f^j\|_{L_p(\mathfrak{m}_k)} \to 0$ as $j \to \infty$. Thus, there is a subsequence $\{f^{j_s}\}$ converging to h $\mathfrak{m}_k$-almost everywhere. Combining this fact with the above assertion (i) and (1.8) we obtain $h = f$ in the sense of equality $\mathfrak{m}_k$-almost everywhere and complete the proof of (iii).

The proof is complete.

While the finiteness of $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f)$ for small $\delta > 0$ is sufficient to control the convergence of the sequence $\{f^j\}$ in the $L_p(\mathrm{X})$-sense, it is still not sufficiently powerful to obtain an appropriate estimate for the limit in the $L_p(\mathrm{X})$-norm.

Theorem 13. *Given $c \geqslant 3/\epsilon$, there exists a constant $C > 0$ such that*

$$\|\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)\|_{L_p(\mathrm{X})} \leqslant C\,\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \quad \textit{for all } f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}). \tag{7.39}$$

Proof. The arguments used in the proof of assertion (ii) of Theorem 12 give, for any $m \geqslant 2$,

$$\|f^m - f^1\|_{L_p(\mathrm{X})} \leqslant C\,\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f).$$

On the other hand, from Lemma 13 we obtain

$$\|f^1\|_{L_p(\mathrm{X})} \leqslant C\|f\|_{L_p(\mathfrak{m}_0)}.$$

As a result, by the triangle inequality and assertion (ii) of Theorem 12,

$$\|\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)\|_{L_p(\mathrm{X})} = \lim_{l\to\infty} \|f^l\|_{L_p(\mathrm{X})} \leqslant C\,\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f).$$

The proof is complete.

Unfortunately, it is difficult to estimate $\|\,\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)\|_{L_p(\mathrm{X})}$ from above in terms of $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f)$ with a constant $C > 0$ independent on f. On the other hand, we have a weaker result which, however, is sufficient for our purposes.

Corollary 1. *Given $c \geqslant 3/\epsilon$, for each $f \in L_p(\mathfrak{m}_0)$ there exists a constant $C_f > 0$ such that*

$$\|\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)\|_{L_p(\mathrm{X})} \leqslant C_f\,\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f). \tag{7.40}$$

Proof. If $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) = +\infty$, then one can put $C_f = 1$. If $f \in L_p(\mathfrak{m}_0)$ and $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$, then by (7.28) there is $\delta = \delta(f) \in (0, 1)$ such that $\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$. Hence $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ by Lemma 6. This fact, in combination with Theorem 13, proves the claim.

Now we are ready to prove the key result of this section. We recall Definition 7.

Theorem 14. *If* $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ *for some* $c \geqslant 3/\epsilon$, *then* $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f) \in W_p^1(\mathrm{X})$. *Furthermore, there exists a constant* $C > 0$ *such that*

$$\mathrm{Ch}_p(\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)) \leqslant C\,\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) \quad \textit{for all } f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}). \tag{7.41}$$

Proof. If $\mathrm{N}_{p,\{\mathfrak{m}_k,c\}}(f) = +\infty$, then inequality (7.41) is obvious. If $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$, then by Theorem 12 and Corollary 1 we have $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f) \in L_p(\mathrm{X})$ and $f^j \to \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)$ as $j \to \infty$ in $L_p(\mathrm{X})$-sense. Furthermore, from Theorem 11 we obtain

$$\mathrm{Ch}_p(\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)) \leqslant \varliminf_{j\to\infty} \|\operatorname{lip} f^j\|_{L_p(\mathrm{X})} \leqslant C\,\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f).$$

By Definition 7 this implies that $F \in W_p^1(\mathrm{X})$ and (7.41) holds.

The proof is complete.

§ 8. Comparison of different trace functionals

The aim of this section is to compare the functionals $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}$, $\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}$, $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}$ and $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}$. Recall that these functionals are originally defined on the space $L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ and take their values in $[0,+\infty]$.

Throughout this section the following data are assumed to be fixed:

(D.8.1) a parameter $p \in (1,\infty)$ and an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$;

(D.8.2) a parameter $\theta \in [0,p)$ and a closed set $S \in \mathcal{LCR}_\theta(\mathrm{X})$;

(D.8.3) a sequence of measures $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ with parameter $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0,1/10]$.

We recall Definitions 18 and 20. The first keystone result of this section is straightforward.

Theorem 15. *For each* $c \geqslant 1$, $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant 2\,\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f)$ *for all* $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$.

Proof. In the case when $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) = +\infty$ the inequality is trivial. We fix $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ such that $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$. Since each (S,c)-Whitney family is an (S,c)-nice family, we have

$$\sup\biggl(\sum_{B\in\mathcal{B}} \frac{\mu(B)}{(r_B)^p} \bigl(\widetilde{\mathcal{E}}_{\mathfrak{m}_{k(B)}}(f,cB)\bigr)^p\biggr)^{1/p} \leqslant \mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f),$$

where the supremum on the left-hand side is taken over all (S,c)-Whitney families $\mathcal{B}$. On the other hand, by Remark 17 we have

$$\varliminf_{\delta\to 0} \mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant \mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f).$$

Combining these observations we obtain the required estimate.

The theorem is proved.

To go further we recall the notation adopted at the beginning of § 5. Furthermore, we set $B_k(x) := B_{\epsilon^k}(x)$ for all $k \in \mathbb{Z}$ and all $x \in \mathrm{X}$.

Proposition 30. *Let $c \geqslant 1$, and let $\underline{k} \in \mathbb{N}_0$ be the smallest $k' \in \mathbb{N}_0$ satisfying $\epsilon^{k'} \leqslant 1/(2c)$. Then there exists a constant $C > 0$ depending on p, θ, $\underline{k}$, c and $\mathcal{C}_{\{\mathfrak{m}_k\}}$ such that if $k \geqslant \underline{k}$, $r \in (\epsilon^{k+1}, \epsilon^k]$ and $B_r(y)$ is a closed ball such that $S \cap B_{cr}(y) \neq \varnothing$ and $B_r(x) \subset B_{cr}(y)$ for some $x \in S \cap B_{cr}(y)$, then*

$$\mathcal{E}_{\mathfrak{m}_k}(f, B_{cr}(y)) \leqslant C \inf_{z \in B_{cr}(y)} \mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f, B_{k-\underline{k}}(z)) \quad \text{for all } f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}). \tag{8.1}$$

Proof. We fix a ball $B_r(x) \subset B_{cr}(y)$ centred at $x \in S \cap cB_r(y)$. Since $r/\epsilon \geqslant \epsilon^k$, we have

$$B_{k-\underline{k}}(z) \subset \left(2c + \frac{1}{\epsilon^{\underline{k}+1}}\right) B_r(x) \quad \text{and} \quad cB_r(y) \subset B_{k-\underline{k}}(z) \quad \text{for all } z \in cB_r(y). \tag{8.2}$$

Since $x \in S$, it follows from (8.2) and Theorem 5 that

$$\sup_{z \in B_{cr}(y)} \mathfrak{m}_k(B_{k-\underline{k}}(z)) \leqslant \mathfrak{m}_k\left(\left(2c + \frac{1}{\epsilon^{\underline{k}+1}}\right) B_r(x)\right) \leqslant C\mathfrak{m}_k(B_r(x)) \leqslant C\mathfrak{m}_k(cB_r(y)).$$

Hence, by the second inclusion in (8.2), Remark 3 and (1.8), for each $z \in B_{cr}(y)$ we have

$$\begin{aligned}\mathcal{E}_{\mathfrak{m}_k}(f, cB_r(y)) &\leqslant \left(\frac{1}{\mathfrak{m}_k(cB_r(y))}\right)^2 \int_{B_{k-\underline{k}}(z)} \int_{B_{k-\underline{k}}(z)} |f(v) - f(w)| \, d\mathfrak{m}_k(v) \, d\mathfrak{m}_k(w) \\ &\leqslant C \fint_{B_{k-\underline{k}}(z)} \fint_{B_{k-\underline{k}}(z)} |f(v) - f(w)| \, d\mathfrak{m}_{k-\underline{k}}(v) \, d\mathfrak{m}_{k-\underline{k}}(w) \leqslant C\mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f, B_{k-\underline{k}}(z)).\end{aligned} \tag{8.3}$$

Since $z \in B_{cr}(y)$ was chosen arbitrarily, the claim follows.

We recall that in Theorem 2 we defined $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) := \mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f) + \|f\|_{L_p(\mathfrak{m}_0)}$ for $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$. The following assertion is the second keystone result of this section.

Theorem 16. *For each $c \geqslant 1$ there exists a constant $C > 0$ such that*

$$\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant C \, \mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) \quad \text{for all } f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}). \tag{8.4}$$

Furthermore, for each $\delta \in (0, 1/(4c)]$ there is a constant $C > 0$ (depending on δ) such that

$$\left|\mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) - \|f\|_{L_p(\mathfrak{m}_0)}\right| \leqslant C\|f^{\sharp}_{\{\mathfrak{m}_k\}}\|_{L_p(U_{(c+1)\delta}(S))}. \tag{8.5}$$

Proof. We put $\overline{\delta} := 1/(4c)$. Let $\underline{k}$ be the smallest $k \in \mathbb{N}_0$ satisfying $\epsilon^k \leqslant \overline{\delta}$.

We start with the second claim. We fix $\delta \in (0, \overline{\delta}]$. Without loss of generality we may assume that $f^{\sharp}_{\{\mathfrak{m}_k\}} \in L_p(U_{(c+1)\delta}(S))$ because otherwise inequality (8.5) is trivial. Let $\mathcal{B}^\delta$ be an arbitrary (S, c, δ)-nice family of closed balls. Since $cB \cap S \neq \varnothing$ for all $B \in \mathcal{B}^\delta$, we obtain

$$B \subset U_{(c+1)\delta}(S) \quad \text{for all } B \in \mathcal{B}^\delta. \tag{8.6}$$

We recall the notation (2.15). Given $k \geqslant \underline{k}$ and $B \in \mathcal{B}^{\delta}(k,\epsilon)$, we use Proposition 30 for c replaced by $2c$. This gives

$$\mathcal{E}_{\mathfrak{m}_k}(f,2cB) \leqslant C \inf_{z\in B} \mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f,B_{k-\underline{k}}(z)).$$

Consequently, we have

$$\frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p \leqslant C\int_B (f^{\sharp}_{\{\mathfrak{m}_k\}}(y))^p\,d\mu(y). \tag{8.7}$$

Combining (8.6) and (8.7) and taking into account that $\mathcal{B}^{\delta}$ is a disjoint family of balls we obtain

$$\begin{aligned}\sum_{B\in\mathcal{B}^{\delta}} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p &\leqslant C\sum_{k\geqslant\underline{k}}\ \sum_{B\in\mathcal{B}^{\delta}(k,\epsilon)}\int_B (f^{\sharp}_{\{\mathfrak{m}_k\}}(y))^p\,d\mu(y)\\ &\leqslant C\int_{U_{(c+1)\delta}(S)} (f^{\sharp}_{\{\mathfrak{m}_k\}}(x))^p\,d\mu(x).\end{aligned}$$

Since $\mathcal{B}^{\delta}$ was chosen arbitrarily, the claim follows from (6.3).

To prove (8.4), given an (S,c)-nice family of closed balls $\mathcal{B}$, we split it into two subfamilies. The first, $\mathcal{B}^1$, consists of the balls of radius greater than or equal to $\overline{\delta}$ and the second is $\mathcal{B}^2 := \mathcal{B}\setminus\mathcal{B}^1$. Applying Lemma 5 we obtain

$$\sum_{B\in\mathcal{B}^1} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p \leqslant C\int_S |f(y)|^p\,d\mathfrak{m}_0(y). \tag{8.8}$$

On the other hand, using (8.5) just proved and taking (1.11) into account we obtain

$$\sum_{B\in\mathcal{B}^2} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p \leqslant C\big(\mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f)\big)^p. \tag{8.9}$$

It remains to combine (8.8) and (8.9) and take into account that $\mathcal{B}$ was chosen arbitrarily. This proves the first claim.

The proof is complete.

The following lemma is an important ingredient for comparing the functionals $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}$ and $\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}$. Recall that in Theorem 2 we set

$$\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) := \|f\|_{L_p(\mathfrak{m}_0)} + \mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) \quad \text{for } f\in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}).$$

Lemma 14. *For each $c \geqslant 1$ and every $\sigma\in(0,\epsilon^2/(4c))$ there exists $C>0$ (depending on σ) such that for each (S,c)-Whitney family of closed balls $\mathcal{B}$*

$$\begin{aligned}\Sigma := \sum_{B\in\mathcal{B}} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f,2cB)\big)^p\\ \leqslant C\big(\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f)\big)^p \quad \textit{for all } f\in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\}).\end{aligned} \tag{8.10}$$

Proof. We put $\overline{\sigma} := \epsilon^2/(4c)$ and let $\underline{k}$ be the smallest $k \in \mathbb{N}_0$ satisfying $\epsilon^k \leqslant 1/(4c)$. Since $\mathcal{B}$ is an (S,c)-Whitney family, we have $cB \cap S \neq \varnothing$ for all $B \in \mathcal{B}$. We recall the notation (2.15). Given $k \geqslant \underline{k}$ and $B \in \mathcal{B}(k,\epsilon)$, it is easy to see that $B \subset B_{(2c+1)r_B}(x) \subset B_{\epsilon^{k-\underline{k}}}(x)$ for all $x \in 2cB \cap S$. If $B \in \mathcal{B}(k,\epsilon)$ for some $k \geqslant \underline{k}$, then

$$r_B \geqslant \frac{\epsilon^2}{4c}\epsilon^{k-\underline{k}}.$$

Hence for each $k \geqslant \underline{k}$ and every $\sigma \in (0, \overline{\sigma}]$ we have

$$2cB \cap S \subset S_{k-\underline{k}}(\sigma) \quad \text{for all } B \in \mathcal{B}(k,\epsilon). \tag{8.11}$$

Given $k \geqslant \underline{k}$ and $B \in \mathcal{B}(k,\epsilon)$, by Proposition 30 (for $2c$ instead of c) we have

$$\mathcal{E}_{\mathfrak{m}_k}(f, 2cB) \leqslant C \inf_{y \in 2cB} \mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f, B_{k-\underline{k}}(y)). \tag{8.12}$$

It follows from (1.7) and (1.8) that

$$\frac{\mu(B)}{(r_B)^p} \leqslant C\mathfrak{m}_{k-\underline{k}}(B) \leqslant C\mathfrak{m}_{k-\underline{k}}(2cB \cap S).$$

This estimate in combination with (8.12) leads to the inequality

$$\frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_k}(f,2cB)\big)^p \leqslant C \int_{2cB\cap S} \big(\mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f, B_{k-\underline{k}}(y))\big)^p \, d\mathfrak{m}_{k-\underline{k}}(y).$$

As a result, using (8.11) and taking Propositions 4 and 7 into account we derive

$$\begin{aligned} &\sum_{k=\underline{k}+1}^{\infty} \sum_{B \in \mathcal{B}(k,\epsilon)} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_k}(f,2cB)\big)^p \\ &\qquad \leqslant C \sum_{k=\underline{k}+1}^{\infty} \int_{S_{k-\underline{k}}(\sigma)} \big(\mathcal{E}_{\mathfrak{m}_{k-\underline{k}}}(f, B_{k-\underline{k}}(y))\big)^p \, d\mathfrak{m}_{k-\underline{k}}(y) \leqslant C\big(\mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f)\big)^p. \end{aligned} \tag{8.13}$$

On the other hand, by Lemma 5 we clearly have

$$\sum_{k=0}^{\underline{k}} \sum_{B \in \mathcal{B}(k,\epsilon)} \frac{\mu(B)}{(r_B)^p}\big(\mathcal{E}_{\mathfrak{m}_k}(f,2cB))\big)^p \leqslant C \int_S |f(z)|^p \, d\mathfrak{m}_0(z). \tag{8.14}$$

Combining (8.13) and (8.14) we obtain (8.10).

The lemma is proved.

Now we are ready to formulate and prove the third keystone result of this section.

Theorem 17. *Given $c \geqslant 1$, there exists a constant $C > 0$ such that for each $\sigma \in (0, \epsilon^2/(4c))$ the inequality*

$$\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant C\, \mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) \tag{8.15}$$

holds for any $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ satisfying $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) < +\infty$.

Proof. We fix an arbitrary $\sigma \in (0, \epsilon^2/(4c))$. If $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) < +\infty$, then by Theorem 16 we have

$$\varlimsup_{\delta \to 0} \mathrm{BSN}^{\delta}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant C\Big(\varlimsup_{\delta \to 0} \|f^{\sharp}_{\{\mathfrak{m}_k\}}\|_{L_p(U_{(c+1)\delta}(S))} + \|f\|_{L_p(\mathfrak{m}_0)}\Big)$$
$$= C\big(\|f^{\sharp}_{\{\mathfrak{m}_k\}}\|_{L_p(S)} + \|f\|_{L_p(\mathfrak{m}_0)}\big). \tag{8.16}$$

On the other hand, by Lemma 14 we obtain

$$\sup \sum_{B \in \mathcal{B}} \frac{\mu(B)}{(r(B))^p} \big(\mathcal{E}_{\mathfrak{m}_{k(B)}}(f, 2cB)\big)^p \leqslant C\big(\mathrm{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f)\big)^p, \tag{8.17}$$

where the supremum is taken over all (S, c)-Whitney families $\mathcal{B}$. Collecting estimates (8.16) and (8.17) we obtain (8.15).

The theorem is proved.

Finally, the fourth keystone result of this section reads as follows. We recall (1.11) and (1.14).

Theorem 18. *For each* $\sigma \in (0, 1)$ *there exists a constant* $C > 0$ *such that*

$$\mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) \leqslant C\mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f) \quad \textit{for all } f \in L^{\mathrm{loc}}_1(\{\mathfrak{m}_k\}). \tag{8.18}$$

Proof. We recall the notation (1.13) and (2.3). We put $S_k(\sigma) := S_{\epsilon^k}(\sigma)$ and $\mathcal{B}_k := \mathcal{B}_k(\mathrm{X}, \epsilon)$, $k \in \mathbb{N}_0$, for brevity. We set $B_k(x) := B_{\epsilon^k}(x)$ for $k \in \mathbb{N}_0$ and $x \in \mathrm{X}$, as usual. For each $k \in \mathbb{N}_0$ and for any ball $B \in \mathcal{B}_k$ that has a nonempty intersection with $S_k(\sigma)$ we fix a point $x_B \in B \cap S_k(\sigma)$ and a ball $B' = B'(B) \subset B_k(x_B) \setminus S$ of radius $r_{B'} \geqslant \sigma\epsilon^k$.

Since $\epsilon^{-1} \geqslant 10$, given $k \in \mathbb{N}$ and $B \in \mathcal{B}_k$, for any point $z \in B'(B)$ we have $B_{k-1}(z) \supset 2B_k(y)$ and $2B_k(y) \supset B$ for all $y \in B \cap S_k(\sigma)$. Hence, by Proposition 30 (for $x = y$, $c = 2$ and $\underline{k} = 1$)

$$\mathcal{E}_{\mathfrak{m}_k}(f, 2B_k(y)) \leqslant C\mathcal{E}_{\mathfrak{m}_{k-1}}(f, B_{k-1}(z)) \quad \text{for all } y \in B \cap S_k(\sigma) \text{ and all } z \in B'.$$

On the other hand, by Remark 3 and Theorem 5 we have $\mathcal{E}_{\mathfrak{m}_k}(f, B_k(y)) \leqslant C\mathcal{E}_{\mathfrak{m}_k}(f, 2B_k(y))$ for $C > 0$ independent of f, k and y. These observations, in combination with the uniformly locally doubling property of μ and (1.6), give

$$\epsilon^{k(\theta-p)} \int_{B \cap S_k(\sigma)} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(y))\big)^p \, d\mathfrak{m}_k(y)$$
$$\leqslant \epsilon^{k(\theta-p)} \mathfrak{m}_k(B) \inf_{z \in \frac{1}{2}B'} \big(\mathcal{E}_{\mathfrak{m}_{k-1}}(f, B_{k-1}(z))\big)^p$$
$$\leqslant C\mu(B') \inf_{z \in \frac{1}{2}B'} (f^{\sharp}_{\{\mathfrak{m}_k\}}(z))^p \leqslant C \int_{\frac{1}{2}B'} (f^{\sharp}_{\{\mathfrak{m}_k\}}(z))^p \, d\mu(z). \tag{8.19}$$

Given $k \in \mathbb{N}$, the key property of the balls $B'(B)$, $B \in \mathcal{B}_k$, is that for some $\underline{l} = \underline{l}(\sigma) \in \mathbb{N}$ (because $\epsilon \in (0, 1/10]$)

$$\frac{1}{2}B'(B) \subset U_{2\epsilon^k}(S) \setminus U_{\epsilon^{k+\underline{l}}}(S) \quad \text{for each } B \in \mathcal{B}_k \text{ such that } B \cap S_k(\sigma) \neq \varnothing. \tag{8.20}$$

Furthermore, since $\frac{1}{2}B'(B) \subset 3B$ for all $B \in \mathcal{B}_k$ and the family $\{\frac{1}{2}B\colon B \in \mathcal{B}_k\}$ is disjoint by Proposition 7, we have

$$\mathcal{M}\biggl(\biggl\{\frac{1}{2}B'(B)\colon B \in \mathcal{B}_k \text{ and } B \cap S_k(\sigma) \neq \varnothing\biggr\}\biggr) \leqslant C. \tag{8.21}$$

Consequently, using (8.19)–(8.21) and Proposition 4, for each $k \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\epsilon^{k(\theta-p)} \int_{S_k(\sigma)} \bigl(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\bigr)^p \, d\mathfrak{m}_k(x) \\
&\quad\leqslant \epsilon^{k(\theta-p)} \sum_{B \in \mathcal{B}_k} \int_{B \cap S_k(\sigma)} \bigl(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\bigr)^p \, d\mathfrak{m}_k(x) \\
&\quad\leqslant C \sum_{\substack{B \in \mathcal{B}_k \\ B \cap S_k(\sigma) \neq \varnothing}} \int_{\frac{1}{2}B'(B)} (f^{\sharp}_{\{\mathfrak{m}_k\}}(y))^p \, d\mu(y) \leqslant C \int_{U_{k-1}(S) \setminus U_{k+1}(S)} (f^{\sharp}_{\{\mathfrak{m}_k\}}(y))^p \, d\mu(y).
\end{aligned}$$

As a result, we have

$$\sum_{k=1}^{\infty} \epsilon^{k(\theta-p)} \int_{S_k(\sigma)} \bigl(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\bigr)^p \, d\mathfrak{m}_k(x) \leqslant C \int_{U_0(S)} (f^{\sharp}_{\{\mathfrak{m}_k\}}(y))^p \, d\mu(y). \tag{8.22}$$

Combining (8.22) with (1.11) and (1.14) we obtain (8.18).

The theorem is proved.

§ 9. Trace inequalities for the Riesz potentials

The aim of this section is to establish a Hedberg-Wolff-type inequality. Of course, the results of this section are not surprising for experts. Nevertheless, the author has not succeeded in finding a precise reference in the literature. We present the details for completeness.

Throughout the whole section we fix the following data:

(D.9.1) a parameter $q \in [1, \infty)$ and an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_q$;

(D.9.2) a locally finite measure $\mathfrak{m}$ on X, a parameter $\epsilon \in (0, 1/10]$ and a number $\underline{k} \in \mathbb{Z}$.

Having Proposition 10 at our disposal we fix a family $\{Q_{k,\alpha}\}$ of generalized dyadic cubes in X and introduce the *essential part* of X by letting $\underline{\mathrm{X}} := \bigcap_{k=\underline{k}}^{\infty} \bigcup_{\alpha \in \mathcal{A}_k(\mathrm{X},\epsilon)} Q_{k,\alpha}$. Furthermore, given a generalized dyadic cube $Q_{k,\alpha}$ in X, we put

$$\widehat{Q}_{k,\alpha} := \bigcup \{\operatorname{cl} Q_{k,\alpha'}\colon \operatorname{cl} Q_{k,\alpha'} \cap 5B_{\epsilon^k}(z_{k,\alpha}) \neq \varnothing\}. \tag{9.1}$$

From Propositions 7 and 10 we easily obtain the following assertion.

Proposition 31. *There exists a constant $C > 0$ such that*

$$\mathcal{M}(\{\widehat{Q}_{k,\alpha}\colon \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)\}) \leqslant C \quad \textit{for all } k \geqslant \underline{k}.$$

From (9.1) and properties (DQ2) and (DQ3) in Proposition 10 we immediately obtain the following result.

Proposition 32. *For each $k \geqslant \underline{k}$ and $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$*

$$\bigcup\{\widehat{Q}_{j,\beta} \colon Q_{j,\beta} \subset Q_{k,\alpha}\} \subset \widehat{Q}_{k,\alpha} \quad \text{for all } j \geqslant k.$$

Given a point $x \in \underline{\mathrm{X}}$ and a number $k \geqslant \underline{k}$, there is a unique $\alpha(x) \in \mathcal{A}_k(\mathrm{X}, \epsilon)$ for which $x \in Q_{k,\alpha(x)}$. In what follows we set

$$Q_{k,\alpha}(x) := Q_{k,\alpha(x)} \quad \text{and} \quad \widehat{Q}_{k,\alpha}(x) := \widehat{Q}_{k,\alpha(x)}. \tag{9.2}$$

Given a Borel set $E \subset \mathrm{X}$ such that $\mu(E) > 0$, we put

$$a_{\mathfrak{m}}(E) := \frac{\mathfrak{m}(E)}{\mu(E)} \operatorname{diam} E. \tag{9.3}$$

Given $R \in (0, \epsilon^{\underline{k}}]$, the *restricted Riesz potential* of $\mathfrak{m}$ is the mapping $I^R[\mathfrak{m}] \colon \mathrm{X} \to [0, +\infty]$ defined by

$$I^R[\mathfrak{m}](x) := \sum_{\epsilon^k \leqslant R} a_{\mathfrak{m}}(B_{\epsilon^k}(x)), \qquad x \in \mathrm{X}\,. \tag{9.4}$$

We also introduce the *restricted dyadic Riesz potential* of the measure $\mathfrak{m}$ by the formula

$$\widehat{I}^R[\mathfrak{m}](x) := \begin{cases} \sum\limits_{\epsilon^k \leqslant R} a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha}(x)), & x \in \underline{\mathrm{X}}, \\ 0, & x \in \mathrm{X} \setminus \underline{\mathrm{X}}. \end{cases} \tag{9.5}$$

Given $p \in (1, \infty)$, we set $p' := p/(p-1)$. Given a Borel set $E \subset \mathrm{X}$ and a parameter $R \in (0, \epsilon^{\underline{k}}]$, the *restricted energy* and the *restricted dyadic energy* of the measure $\mathfrak{m}$ are defined by

$$\mathfrak{E}^R_p[\mathfrak{m}](E) := \int_E \left(I^R[\mathfrak{m}](x)\right)^{p'} d\mu(x) \quad \text{and} \quad \widehat{\mathfrak{E}}^R_p[\mathfrak{m}](E) := \int_E \left(\widehat{I}^R[\mathfrak{m}](x)\right)^{p'} d\mu(x). \tag{9.6}$$

By (DQ4) in Proposition 10 it is clear that

$$\widehat{Q}_{k,\alpha} \subset 9B_{\epsilon^k}(z_{k,\alpha}). \tag{9.7}$$

Hence, by the uniformly locally doubling property of μ there is a constant $C > 0$ such that for each $R \in (0, \epsilon^{\underline{k}}]$,

$$I^R[\mathfrak{m}](x) \leqslant C \widehat{I}^R[\mathfrak{m}](x) \quad \text{for all } x \in \underline{\mathrm{X}}. \tag{9.8}$$

The following elementary observation will be important in the proof of Theorem 19 below.

Proposition 33. *Let $j \geqslant \underline{k}$ and $\beta \in \mathcal{A}_j(\mathrm{X}, \epsilon)$. Then*

$$\sum_{k'=k}^{j} \sum_{Q_{k',\alpha} \supset Q_{j,\beta}} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha}) \leqslant \inf_{x \in Q_{j,\beta} \cap \underline{\mathrm{X}}} \widehat{I}^{\epsilon^k}[\mathfrak{m}](x), \qquad k \in \{\underline{k}, \ldots, j\}.$$

Proof. Indeed, by (9.2)

$$\sum_{k'=k}^{j}\sum_{Q_{k',\alpha}\supset Q_{j,\beta}} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha}) = \sum_{k'=k}^{j} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha}(x)) \quad \text{for all } x \in Q_{j,\beta}. \tag{9.9}$$

Combining this observation with (9.5) we obtain the required estimate.

The following three assertions will be crucial in what follows. In fact, the corresponding proofs are based on ideas very similar to the ones used in the proof of Proposition 2.2 in [43]. However, in order to realise these ideas in our case we must make some modifications because of the lack of Euclidean structure. We present the details.

Lemma 15. *Let* $p \in (1,\infty)$. *Then there exists a constant* $C>0$ *such that for each* $R \in (0, \epsilon^{\underline{k}}]$ *the inequality*

$$\widehat{\mathfrak{E}}^{R}_{p}[\mathfrak{m}](E) \leqslant p' \sum_{\epsilon^k \leqslant R}\sum_{\alpha\in\mathcal{A}_k(\mathrm{X},\epsilon)} a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha}) \int_{Q_{k,\alpha}\cap E} (\widehat{I}^{\epsilon^k}[\mathfrak{m}](x))^{p'-1}\,d\mu(x) \tag{9.10}$$

holds for any Borel set $E \subset \mathrm{X}$.

Proof. First of all, we claim that

$$(\widehat{I}^{R}[\mathfrak{m}](x))^{p'} \leqslant p' \sum_{\epsilon^k\leqslant R} a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha}(x))(\widehat{I}^{\epsilon^k}[\mathfrak{m}](x))^{p'-1}, \qquad x \in \underline{\mathrm{X}}. \tag{9.11}$$

Indeed, in the case when $\widehat{I}^{R}[\mathfrak{m}](x) = +\infty$ the inequality is obvious. Assume that $\widehat{I}^{R}[\mathfrak{m}](x) < +\infty$. Recall that for any $s \geqslant 1$ the elementary inequality $\beta^s - \alpha^s \leqslant s(\beta-\alpha)\beta^{s-1}$ holds for all real numbers $0 \leqslant \alpha \leqslant \beta$. Hence, given $x \in \underline{\mathrm{X}}$, if $k \in \mathbb{Z}$ is such that $\epsilon^k \leqslant R$, then

$$\begin{aligned}\Big(\sum_{j\geqslant k} a_{\mathfrak{m}}(\widehat{Q}_{j,\alpha}(x))\Big)^{p'} &- \Big(\sum_{j\geqslant k+1} a_{\mathfrak{m}}(\widehat{Q}_{j,\alpha}(x))\Big)^{p'}\\ &\leqslant p'(a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha}(x)))\Big(\sum_{j\geqslant k} a_{\mathfrak{m}}(\widehat{Q}_{j,\alpha}(x))\Big)^{p'-1}.\end{aligned}$$

Clearly, if $\widehat{I}^{R}[\mathfrak{m}](x) < +\infty$, then we have $\widehat{I}^{\epsilon^k}[\mathfrak{m}](x) \to 0$ as $k \to \infty$. Thus, the standard telescopic-type arguments, in combination with the above inequality, lead to (9.11). As a result, combining (9.5), (9.6) and (9.11) and taking into account (DQ5) in Proposition 10 we arrive at the required estimate.

The lemma is proved.

Now we can estimate the restricted dyadic energy from above in the case when $p \geqslant 2$.

Lemma 16. *Let* $p \in [2,\infty)$. *Then there exists a constant* $C>0$ *such that, for each* $R \in (0,\epsilon^{\underline{k}}]$ *and any Borel set* $E \subset \mathrm{X}$,

$$\widehat{\mathfrak{E}}^{R}_{p}[\mathfrak{m}](E) \leqslant C \sum_{\epsilon^k\leqslant R}\sum_{Q_{k,\alpha}\cap E\neq\varnothing} \epsilon^k \mathfrak{m}(\widehat{Q}_{k,\alpha})\big(a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha})\big)^{p'-1}. \tag{9.12}$$

Proof. We fix $R \in (0, \epsilon^{\underline{k}}]$ and a Borel set $E \subset \mathrm{X}$. We have $p' \in (1, 2]$, and so $p' - 1 \leqslant 1$. We change the sum and the integral, use (DQ1), (DQ2) and (DQ5) in Proposition 10, and take (9.3), (9.5) and (9.7) into account. As a result, given $k \geqslant \underline{k}$ and $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$, we obtain

$$\frac{1}{\mu(Q_{k,\alpha})} \int_{Q_{k,\alpha} \cap E} (\widehat{I}^{\epsilon^k}[\mathfrak{m}](x))^{p'-1} \, d\mu(x) \leqslant \left(\fint_{Q_{k,\alpha}} \sum_{j \geqslant k} a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x)) \, d\mu(x) \right)^{p'-1}$$
$$\leqslant 9^{p'-1} \left(\frac{1}{\mu(Q_{k,\alpha})} \sum_{j \geqslant k} \epsilon^j \sum_{Q_{j,\beta} \subset Q_{k,\alpha}} \mathfrak{m}(\widehat{Q}_{j,\beta}) \right)^{p'-1}.$$

Given $j \geqslant k$, using Propositions 31 and 32 we obtain $\sum_{Q_{j,\beta} \subset Q_{k,\alpha}} \mathfrak{m}(\widehat{Q}_{j,\beta}) \leqslant C\mathfrak{m}(\widehat{Q}_{k,\alpha})$. Hence, using the above inequality, (DQ4) in Proposition 10 and the uniformly locally doubling property of the measure μ, we derive

$$\frac{1}{\mu(Q_{k,\alpha})} \int_{Q_{k,\alpha} \cap E} (\widehat{I}^{\epsilon^k}[\mathfrak{m}](x))^{p'-1} \, d\mu(x) \leqslant C \left(\epsilon^k \frac{\mathfrak{m}(\widehat{Q}_{k,\alpha})}{\mu(Q_{k,\alpha})} \right)^{p'-1} \leqslant C (a_{\mathfrak{m}}(\widehat{Q}_{k,\alpha}))^{p'-1}. \tag{9.13}$$

Finally, using Lemma 15 and (9.13), we obtain (9.12) and complete the proof.

Now we are ready to establish the keystone estimate.

Theorem 19. *Let $p \in (1, \infty)$. Then there exists a constant $C > 0$ such that, for each $k \geqslant \underline{k}$,*

$$\widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}) \leqslant C \sum_{j=k}^{\infty} \sum_{Q_{j,\beta} \subset Q_{k,\alpha}} \epsilon^j \mathfrak{m}(\widehat{Q}_{j,\beta}) (a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}))^{p'-1}$$
$$\textit{for all } \alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon). \tag{9.14}$$

Proof. In the case when $p \in [2, \infty)$ the assertion of the theorem follows from Lemma 16.

Consider the case when $p \in (1, 2)$. We fix $k \geqslant \underline{k}$, $\alpha \in \mathcal{A}_k(\mathrm{X}, \epsilon)$ and argue by induction. More precisely, the base of induction is that (9.14) holds for $p' \in (1, l]$, where $l = 2$. We are going to show that (9.14) holds for each $p' > 1$. We assume that (9.14) is proved for $p' \in (1, l]$, for some $l \in \mathbb{N} \cap [2, \infty)$, and show that (9.14) holds for all $p' \in (1, l + 1]$. We use Lemma 15, and then take into account that $p' - 1 \in (1, l]$. As a result, we obtain

$$\widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}) \leqslant p' \sum_{k'=k}^{\infty} \sum_{Q_{k',\alpha'} \subset Q_{k,\alpha}} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha'}) \int_{Q_{k',\alpha'}} (\widehat{I}^{\epsilon^{k'}}[\mathfrak{m}](x))^{p'-1} \, d\mu(x)$$
$$\leqslant C \sum_{k'=k}^{\infty} \sum_{Q_{k',\alpha'} \subset Q_{k,\alpha}} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha'}) \sum_{j=k'}^{\infty} \sum_{Q_{j,\beta} \subset Q_{k',\alpha'}} \epsilon^j \mathfrak{m}(\widehat{Q}_{j,\beta}) (a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}))^{p'-2}.$$

Changing the order of summation we obtain

$$\widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}) \leqslant C \sum_{j=k}^{\infty} \sum_{Q_{j,\beta} \subset Q_{k,\alpha}} \epsilon^j \mathfrak{m}(\widehat{Q}_{j,\beta}) (a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}))^{p'-2} \sum_{k'=k}^{j} \sum_{Q_{k',\alpha'} \supset Q_{j,\beta}} a_{\mathfrak{m}}(\widehat{Q}_{k',\alpha'}). \tag{9.15}$$

Hence, by Proposition 33, (9.3) and (9.7), and the uniformly locally doubling property of μ,

$$\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha}) \leqslant C\sum_{j=k}^{\infty}\sum_{Q_{j,\beta}\subset Q_{k,\alpha}}\int_{Q_{j,\beta}}\frac{\epsilon^j\mathfrak{m}(\widehat{Q}_{j,\beta})}{\mu(Q_{j,\beta})}(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}))^{p'-2}\widehat{I}^{\epsilon^k}[\mathfrak{m}](x)\,d\mu(x)$$
$$\leqslant C\int_{Q_{k,\alpha}}\sum_{j=k}^{\infty}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{p'-1}\big(\widehat{I}^{\epsilon^k}[\mathfrak{m}](x)\big)\,d\mu(x). \tag{9.16}$$

An application of Hölder's inequality for sums with exponents $q=(p'-1)/(p'-2)$ and $q'=p'-1$ gives

$$\sum_{j=k}^{\infty}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{p'-1}=\sum_{j=k}^{\infty}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{1/(p'-1)}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{p'-2+(p'-2)/(p'-1)}$$
$$\leqslant C\big(\widehat{I}^{\epsilon^k}[\mathfrak{m}](x)\big)^{1/(p'-1)}\bigg(\sum_{j=k}^{\infty}(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x)))^{p'}\bigg)^{(p'-2)/(p'-1)}. \tag{9.17}$$

Now we plug (9.17) into (9.16) and apply Hölder's inequality for integrals with exponents $p'-1$ and $(p'-1)/(p'-2)$. This gives

$$\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})$$
$$\leqslant C\big(\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})\big)^{1/(p'-1)}\bigg(\int_{Q_{k,\alpha}}\sum_{j=k}^{\infty}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{p'}\,d\mu(x)\bigg)^{(p'-2)/(p'-1)}.$$

As a result, if $\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})<+\infty$, then we have the required inequality

$$\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})\leqslant C\int_{Q_{k,\alpha}}\sum_{j=k}^{\infty}\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}(x))\big)^{p'}\,d\mu(x)$$
$$\leqslant C\sum_{j=k}^{\infty}\sum_{Q_{j,\beta}\subset Q_{k,\alpha}}\epsilon^j\mathfrak{m}(\widehat{Q}_{j,\beta})(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta}))^{p'-1}. \tag{9.18}$$

To remove the assumption $\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})<+\infty$ we proceed as follows. Given $l\in\mathbb{N}$, we consider the lth truncation of the restricted Riesz potential $\widehat{I}^{\epsilon^l,\epsilon^k}[\mathfrak{m}]$ obtained by summing only over $l\leqslant k'\leqslant k$ in (9.5) (note that here we use the index k' instead of k in (9.5)). Clearly, the corresponding truncations of the restricted dyadic energies $\widehat{\mathfrak{E}}^{\epsilon^l,\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})$ are finite for all $l\in\mathbb{N}$. Repeating the above arguments with minor changes, for any fixed $l\in\mathbb{N}$ we obtain an analogue of (9.18) for $\widehat{\mathfrak{E}}^{\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})$ replaced by $\widehat{\mathfrak{E}}^{\epsilon^l,\epsilon^k}_p[\mathfrak{m}](Q_{k,\alpha})$. Then we pass to the limit as l goes to infinity.

The theorem is proved.

Given $R>0$, the *restricted Wolff potential* is a mapping $\mathcal{W}^R_p[\mathfrak{m}]\colon \mathrm{X}\to[0,+\infty]$ defined by

$$\mathcal{W}^R_p[\mathfrak{m}](x):=\sum_{\epsilon^k\leqslant R}\Big(\epsilon^{kp}\frac{\mathfrak{m}(B_{\epsilon^k}(x))}{\mu(B_{\epsilon^k}(x))}\Big)^{p'-1},\qquad x\in\mathrm{X}. \tag{9.19}$$

Now we are ready to establish a Hedberg-Wolff-type inequality.

Corollary 2. *There are constants* $c_1, c_2 > 1$ *depending on* ϵ *only such that the following holds. Given* $p \in (1, \infty)$, *there is a constant* $C > 0$ *such that for each* $R \in (0, \epsilon^{\underline{k}}]$ *and every Borel set* $E \subset \mathrm{X}$,

$$\mathfrak{E}_p^R[\mathfrak{m}](E) \leqslant C \int_{U_{c_2 R}(E)} \mathcal{W}_p^{c_1 R}[\mathfrak{m}](y)\, d\mathfrak{m}(y), \tag{9.20}$$

where $U_{c_2 R}(E) := \{y \in \mathrm{X} \colon \inf_{x \in E} \mathrm{d}(y,x) < c_2 R\}$.

Proof. We fix $R \in (0, \epsilon^{\underline{k}}]$, recall the notation (2.18), and put $k := k_\epsilon(R)$. By (9.6), (9.8) and (DQ5) in Proposition 10,

$$\mathfrak{E}_p^R[\mathfrak{m}](E) \leqslant C \sum_{Q_{k,\alpha} \cap E \neq \varnothing} \widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}). \tag{9.21}$$

From (DQ1) in Proposition 10 and (9.1), (9.7), given $j \geqslant k$, it is clear that, for any generalized dyadic cube $Q_{j,\beta}$ in X we have

$$B_{5\epsilon^j}(z_{j,\beta}) \subset \widehat{Q}_{j,\beta} \subset B_{18\epsilon^j}(y) \subset B_{27\epsilon^j}(z_{j,\beta}) \quad \text{for all } y \in \widehat{Q}_{j,\beta}.$$

Hence, using the uniformly locally doubling property of the measure μ we obtain

$$\epsilon^j \mathfrak{m}(\widehat{Q}_{j,\beta})\big(a_{\mathfrak{m}}(\widehat{Q}_{j,\beta})\big)^{p'-1} \leqslant C \int_{\widehat{Q}_{j,\beta}} \left(\epsilon^{jp} \frac{\mathfrak{m}(B_{18\epsilon^j}(y))}{\mu(B_{18\epsilon^j}(y))}\right)^{p'-1} d\mathfrak{m}(y).$$

Combining this estimate with Theorem 19 we deduce

$$\widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}) \leqslant C \int_{\mathrm{X}} \sum_{j=k}^{\infty} \sum_{Q_{j,\beta} \subset Q_{k,\alpha}} \chi_{\widehat{Q}_{j,\beta}}(y) \left(\epsilon^{jp} \frac{\mathfrak{m}(B_{18\epsilon^j}(y))}{\mu(B_{18\epsilon^j}(y))}\right)^{p'-1} d\mathfrak{m}(y). \tag{9.22}$$

For each $j \geqslant k$ we put $k(j) := k_\epsilon(18\epsilon^j)$ (we use the notation (2.18)). By the uniformly locally doubling property of μ it is easy to see that

$$\epsilon^j \frac{\mathfrak{m}(B_{18\epsilon^j}(y))}{\mu(B_{18\epsilon^j}(y))} \leqslant C \epsilon^{k(j)} \frac{\mathfrak{m}(B_{\epsilon^{k(j)}}(y))}{\mu(B_{\epsilon^{k(j)}}(y))}.$$

Hence, letting $c_1 = 18\epsilon^k/R$ and taking into account (9.19) and Proposition 31, we can continue (9.22). This gives

$$\widehat{\mathfrak{E}}_p^{\epsilon^k}[\mathfrak{m}](Q_{k,\alpha}) \leqslant C \int_{\widehat{Q}_{k,\alpha}} \mathcal{W}_p^{c_1 R}[\mathfrak{m}](y)\, d\mathfrak{m}(y). \tag{9.23}$$

We put $c_2 = 11\epsilon^k/R$. By (9.7) we have $\widehat{Q}_{k,\alpha} \subset U_{c_2 R}(E)$, provided that $Q_{k,\alpha} \cap E \neq \varnothing$. Hence combining (9.21) and (9.23), and taking Proposition 4 into account we obtain (9.20).

The corollary is proved.

§ 10. Proofs of the main results

In this section we prove Theorems 2–4. We recall Proposition 13.

Throughout this section we fix the following data:

(D.10.1) a parameter $p \in (1, \infty)$, an m.m.s. $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$ and a parameter $q \in (1, p)$ such that $\mathrm{X} \in \mathfrak{A}_q$;

(D.10.2) a parameter $\theta \in [0, q)$ and a set $S \in \mathcal{LCR}_\theta(\mathrm{X})$;

(D.10.3) a sequence of measures $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ with parameter $\epsilon = \epsilon(\{\mathfrak{m}_k\}) \in (0, 1/10]$.

Again, we recall the notation (2.18) and (2.20).

Lemma 17. *Let $\alpha > \theta/p$. Then for each $R > 0$ there is a constant $C > 0$ such that*

$$\|M^R_{q,\alpha}(g)\|_{L_p(\mathfrak{m}_0)} \leqslant C\|g\|_{L_p(\mathrm{X})} \quad \textit{for all } g \in L_p(\mathrm{X}). \tag{10.1}$$

Proof. Given $g \in L_p(\mathrm{X})$, let

$$\mathrm{X}_0(g) := \Bigl\{x \in \mathrm{X}\colon \lim_{\widetilde{R} \to 0} M^{\widetilde{R}}_{p,\alpha}(g) = 0\Bigr\}.$$

By Proposition 15 and Hölder's inequality, $\mathcal{H}_{\alpha p}(\mathrm{X} \setminus \mathrm{X}_0(g)) = 0$. Since $\alpha p > \theta$, by (2.27) we have $\mathcal{H}_\theta(\mathrm{X} \setminus \mathrm{X}_0(g)) = 0$. By Proposition 20, we have $\mathfrak{m}_0(\mathrm{X} \setminus \mathrm{X}_0(g)) = 0$. Using this observation and Hölder's inequality, we see that $M^R_{q,\alpha}(g)(x) < +\infty$ for all $x \in \mathrm{X}_0(g)$. Given $x \in \mathrm{X}_0(g)$, we fix $r_x \in (0, R]$ such that

$$(r_x)^\alpha \biggl(\fint_{B_{r_x}(x)} |g(y)|^q \, d\mu(y)\biggr)^{1/q} > \frac{1}{2} M^R_{q,\alpha}(g)(x).$$

We put $\mathcal{G} := \{B_{r_x}(x)\colon x \in \mathrm{X}_0(g)\}$. Given $k \in \mathbb{N}_0$, we put

$$E_k := \biggl\{x \in \mathrm{X}_0(g)\colon r_x \in \biggl(\frac{R}{2^{k+1}}, \frac{R}{2^k}\biggr]\biggr\} \quad \text{and} \quad \mathcal{G}_k := \{B_{r_x}(x)\colon x \in E_k\}.$$

Obviously, $E_k \cap E_j = \varnothing$ for $k \neq j$ and $\bigcup_{k \in \mathbb{N}_0} E_k = \mathrm{X}_0(g)$. Hence

$$\int_{\mathrm{X}} (M^R_{q,\alpha}(g)(x))^p \, d\mathfrak{m}_0(x) = \sum_{k=0}^{\infty} \int_{E_k} (M^R_{q,\alpha}(g)(x))^p \, d\mathfrak{m}_0(x). \tag{10.2}$$

Given $k \in \mathbb{N}_0$, using Vitali's $5B$-covering lemma we find a disjoint family of balls $\widetilde{\mathcal{G}}_k \subset \mathcal{G}_k$ such that $E_k \subset \bigcup\{5B\colon B \in \widetilde{\mathcal{G}}_k\}$. Using the uniformly locally doubling property of μ we obtain

$$\begin{aligned}
&\int_{E_k} (M^R_{q,\alpha}(g)(x))^p \, d\mathfrak{m}_0(x) \\
&\qquad \leqslant 2^p \sum_{B \in \widetilde{\mathcal{G}}_k} \int_{5B \cap E_k} (r_x)^{\alpha p} \biggl(\fint_{B_{r_x}(x)} |g(y)|^q \, d\mu(y)\biggr)^{p/q} d\mathfrak{m}_0(x) \\
&\qquad \leqslant C \sum_{B \in \widetilde{\mathcal{G}}_k} \mathfrak{m}_0(5B) \biggl(\frac{R}{2^k}\biggr)^{\alpha p} \biggl(\fint_{7B} |g(y)|^q \, d\mu(y)\biggr)^{p/q}.
\end{aligned}$$

By Theorem 5 and (1.6) we have $\mathfrak{m}_0(5B) \leqslant C2^{k\theta}\mu(B)$ for all $B \in \widetilde{\mathcal{G}}_k$. Consequently, using Hölder's inequality and Propositions 4 and 7 we deduce

$$\int_{E_k} (M^R_{q,\alpha}(g)(x))^p \, d\mathfrak{m}_0(x) \leqslant C2^{k(\theta-\alpha p)} \sum_{B\in\widetilde{\mathcal{G}}_k} \mu(5B) \fint_{7B} |g(y)|^p \, d\mu(y)$$
$$\leqslant C2^{k(\theta-\alpha p)} \sum_{B\in\widetilde{\mathcal{G}}_k} \int_{7B} |g(y)|^p \, d\mu(y) \leqslant C2^{k(\theta-\alpha p)} \int_{\mathrm{X}} |g(y)|^p \, d\mu(y). \qquad (10.3)$$

Since $\alpha p > \theta$, a combination of (10.2) and (10.3) gives (10.1).

Lemma 17 is proved.

The following result is crucial for our analysis. We recall (2.11), and throughout this section we put $F_G := F_{G,\mu}$. Furthermore, we set $B_k(x) := B_{\epsilon^k}(x)$, as usual.

Theorem 20. *For each $c \geqslant 1$ and $\widetilde{p} \in (q,p]$ there exist $C > 0$ and $\widetilde{c} \geqslant c$ such that, if $B_k(x') \subset cB_k(x)$ for some $k \in \mathbb{N}_0$, $x' \in S$ and $x \in \mathrm{X}$, then*

$$\fint_{cB_k(x)} |F|^{\mathfrak{m}_0}_S(y) - F_{cB_k(x)}| \, d\mathfrak{m}_k(y) \leqslant C\epsilon^k \left(\fint_{\widetilde{c}B_k(x)} (|\nabla F|_{*,p}(y))^{\widetilde{p}} \, d\mu(y) \right)^{1/\widetilde{p}} \qquad (10.4)$$

for all $F \in W^1_p(\mathrm{X})$.

Proof. Fix $c \geqslant 1$. We also fix $k \in \mathbb{N}_0$, $x' \in S$ and $x \in \mathrm{X}$ such that $B_k(x') \subset cB_k(x)$. Throughout, we put $B := B_k(x)$.

Step 1. By Definition 9, for $\mathfrak{m}_0$-almost all $y \in cB \cap S$ we have

$$|F|^{\mathfrak{m}_0}_S(y) - F_{cB}| = \lim_{i\to\infty} |F_{B_i(y)} - F_{cB}| \leqslant |F_{B_k(y)} - F_{cB}| + \sum_{i=k}^{\infty} |F_{B_i(y)} - F_{B_{i+1}(y)}|. \qquad (10.5)$$

Combining Remark 3 with Proposition 16, for $\mathfrak{m}_0$-almost all $y \in cB \cap S$ we obtain

$$\sum_{i=k}^{\infty} |F_{B_i(y)} - F_{B_{i+1}(y)}| \leqslant C \sum_{i=k}^{\infty} \epsilon^i \left(\fint_{\lambda B_i(y)} (|\nabla F|_{*,p}(v))^q \, d\mu(v) \right)^{1/q}. \qquad (10.6)$$

We recall (2.20). Using the uniformly locally doubling property of μ it is easy to see that for each $i \geqslant k$,

$$\left(\fint_{\lambda B_i(y)} (|\nabla F|_{*,p}(v))^q \, d\mu(v) \right)^{1/q} \leqslant C \inf_{z\in B_i(y)} \left(\fint_{(\lambda+1)B_i(z)} (|\nabla F|_{*,p}(v))^q \, d\mu(v) \right)^{1/q}$$
$$\leqslant C \fint_{B_i(y)} M^{(\lambda+1)\epsilon^i}_{q,0}(|\nabla F|_{*,p})(z) \, d\mu(z) \quad \text{for all } y \in cB. \qquad (10.7)$$

As a result, using estimates (10.6) and (10.7) and taking (2.6) and (9.4) into account we obtain

$$\fint_{cB} \sum_{i=k}^{\infty} |F_{B_i(y)} - F_{B_{i+1}(y)}| \, d\mathfrak{m}_k(y) \leqslant C \left(\fint_{cB} I^{\epsilon^k}[M^{(\lambda+1)\epsilon^k}_{q,0}(|\nabla F|_{*,p})\mu](y) \, d\mathfrak{m}_k(y) \right). \qquad (10.8)$$

On the other hand $B_k(y) \subset (c+1)B$ for all $y \in cB$. Hence, using Remark 3, Proposition 16 and Hölder's inequality we obtain

$$|F_{B_k(y)} - F_{cB}| \leqslant C\epsilon^k \left(\fint_{\lambda(c+1)B} (|\nabla F|_{*,p}(z))^{\widetilde{p}}\, d\mu(z) \right)^{1/\widetilde{p}} \quad \text{for all } y \in cB. \tag{10.9}$$

Step 2. From the standard duality arguments it is clear that, given a constant $\underline{C} > 0$, a parameter $R > 0$ and locally finite measures ν and σ on X,

$$\|I^R[g\nu]\|_{L_1(\sigma)} \leqslant \underline{C}\|g\|_{L_{\widetilde{p}}(\nu)} \quad \text{for all nonnegative } g \in L_{\widetilde{p}}(\nu)$$

if and only if (we set $\widetilde{p}' := \widetilde{p}/(\widetilde{p}-1)$ as usual)

$$\|I^R[h\sigma]\|_{L_{\widetilde{p}'}(\nu)} \leqslant \underline{C}\|h\|_{L_\infty(\sigma)} \quad \text{for all nonnegative } h \in L_\infty(\sigma).$$

Note that in (10.8) we work only with the part of the measure μ concentrated on $(c+2+\lambda)B$. Now we set $\widetilde{c} := \max\{c+2+\lambda, \lambda(c+1)\}$ and apply the duality arguments given above to the measures $\sigma = \mathfrak{m}_k\lfloor_{cB}$ and $\nu = \mu\lfloor_{\widetilde{c}B}$ and to $g = M^{(\lambda+1)\epsilon^k}_{q,0}(|\nabla F|_{*,p}\chi_{\widetilde{c}B})$. This gives

$$\begin{aligned} &\int_{cB} I^{\epsilon^k}[M^{(\lambda+1)\epsilon^k}_{q,0}(|\nabla F|_{*,p}\chi_{\widetilde{c}B})\mu](y)\, d\mathfrak{m}_k(y) \\ &\qquad \leqslant \underline{C}\left(\int_{\widetilde{c}B} (M^{(\lambda+1)\epsilon^k}_{q,0}(|\nabla F|_{*,p}\chi_{\widetilde{c}B}))^{\widetilde{p}}\, d\mu(y) \right)^{1/\widetilde{p}} \end{aligned} \tag{10.10}$$

for the constant $\underline{C} := \left(\mathfrak{E}^{\epsilon^k}_{\widetilde{p}}[\mathfrak{m}_k](\widetilde{c}B)\right)^{1/\widetilde{p}'}$.

Step 3. By Corollary 2 we have

$$(\underline{C})^{\widetilde{p}'} \leqslant C \int_{(c_2+\widetilde{c})B} \mathcal{W}^{c_1\epsilon^k}_{\widetilde{p}}[\mathfrak{m}_k](y)\, d\mathfrak{m}_k(y).$$

Hence, using (9.19) and (1.6) and Theorem 5 we obtain

$$(\underline{C})^{\widetilde{p}'} \leqslant C \int_{(c_2+\widetilde{c})B} \sum_{\epsilon^i \leqslant c_1\epsilon^k} (\epsilon^{i(\widetilde{p}-\theta)})^{\widetilde{p}'-1}\, d\mathfrak{m}_k(y) \leqslant C\mu(B)\epsilon^{k(\widetilde{p}-\theta)\widetilde{p}'/\widetilde{p}-k\theta}. \tag{10.11}$$

Step 4. Since $B_k(x') \subset cB$, we have $B \subset (c+1)B_k(x')$. Hence, using (1.7) and the uniformly locally doubling property of μ we obtain

$$\frac{1}{\mathfrak{m}_k(cB)} \leqslant \frac{1}{\mathfrak{m}_k(B_k(x'))} \leqslant C\frac{\epsilon^{k\theta}}{\mu(B_k(x'))} \leqslant C\frac{\epsilon^{k\theta}}{\mu((c+1)B_k(x'))} \leqslant C\frac{\epsilon^{k\theta}}{\mu(B)}. \tag{10.12}$$

Step 5. Combining (10.5) with (10.8)–(10.12) and taking Proposition 12 into account we obtain (10.4).

The theorem is proved.

Now we recall (1.11) and establish the following powerful estimate.

Corollary 3. *There exists a constant* $C > 0$ *such that*

$$\mathcal{CN}_{p,\{\mathfrak{m}_k\}}(F|_S^{\mathfrak{m}_0}) \leqslant C\||\nabla F|_{*,p}\|_{L_p(\mathrm{X})} \quad \textit{for all } F \in W_p^1(\mathrm{X}) \tag{10.13}$$

and, furthermore,

$$\|F|_S^{\mathfrak{m}_0}\|_{L_p(\mathfrak{m}_0)} \leqslant C\|F\|_{W_p^1(\mathrm{X})} \quad \textit{for all } F \in W_p^1(\mathrm{X}). \tag{10.14}$$

Proof. We fix $\widetilde{p} \in (q,p)$ and put $f := F|_S^{\mathfrak{m}_0}$ for brevity. Using (2.12) and Theorem 20 for $c = 2$, for each ball $B_k(x)$, $k \in \mathbb{N}_0$, such that $B_k(x) \cap S \neq \varnothing$ (we set $B_k(x) := B_{\epsilon^k}(x)$) we obtain

$$\begin{aligned}\mathcal{E}_{\mathfrak{m}_k}(f, 2B_k(x)) &\leqslant \fint_{2B_k(x)} |f(y) - F_{2B_k(x)}|\, d\mathfrak{m}_k(y) \\ &\leqslant C\epsilon^k \left(\fint_{\widetilde{c}B_k(x)} (|\nabla F|_{*,p})^{\widetilde{p}}\, d\mu(y) \right)^{1/\widetilde{p}}.\end{aligned} \tag{10.15}$$

From Theorem 5 it is easy to see that $f^{\sharp}_{\{\mathfrak{m}_k\}}(x) \leqslant C \sup_{k\in\mathbb{N}_0} \epsilon^{-k}\mathcal{E}_{\mathfrak{m}_k}(f, 2B_k(x))$ for all $x \in \mathrm{X}$. Hence, using (10.15) we obtain

$$f^{\sharp}_{\{\mathfrak{m}_k\}}(x) \leqslant C M^{\widetilde{c}}_{\widetilde{p},0}(|\nabla F|_{*,p})(x) \quad \text{for all } x \in \mathrm{X}\,.$$

As a result, an application of Proposition 12 gives (10.13).

We recall the notation (2.3) and fix a family $\mathcal{B} := \mathcal{B}_0(\mathrm{X}, \epsilon)$. By Hölder's inequality,

$$\begin{aligned}\int_S |f(x)|^p\, d\mathfrak{m}_0(x) &\leqslant C \sum_{\substack{B\in\mathcal{B}\\ B\cap S\neq\varnothing}} \int_B |f(x)|^p\, d\mathfrak{m}_0(x) \\ &\leqslant C \sum_{\substack{B\in\mathcal{B}\\ B\cap S\neq\varnothing}} \left(\int_B |f(x) - F_{2B}|^p\, d\mathfrak{m}_0(x) + \frac{\mathfrak{m}_0(B)}{\mu(2B)} \int_{2B} |F(x)|^p\, d\mu(x) \right).\end{aligned} \tag{10.16}$$

Given $B \in \mathcal{B}$ such that $B \cap S \neq \varnothing$, by the triangle inequality we have

$$\begin{aligned}&\int_B |f(x) - F_{2B}|^p\, d\mathfrak{m}_0(x) \\ &\qquad \leqslant \int_B |F_{B_0(x)} - F_{2B}|^p\, d\mathfrak{m}_0(x) + \int_B |f(x) - F_{B_0(x)}|^p\, d\mathfrak{m}_0(x).\end{aligned} \tag{10.17}$$

By Remark 3, $|F_{B_0(x)} - F_{2B}| \leqslant C\mathcal{E}_{\mu}(F, 2B)$ for all $x \in B \cap S$. Hence, by Proposition 16,

$$\int_B |F_{B_0(x)} - F_{2B}|^p\, d\mathfrak{m}_0(x) \leqslant C\mathfrak{m}_0(B) \fint_{2\lambda B} (|\nabla F|_{*,p})^p\, d\mu(y). \tag{10.18}$$

By Definition 9, given $\delta \in (0, 1/2)$, we have

$$|f(x) - F_{B_0(x)}| \leqslant \sum_{i=0}^{\infty} \frac{\epsilon^{i\delta}}{\epsilon^{i\delta}} |F_{B_i(x)} - F_{B_{i+1}(x)}| \quad \text{for} \quad \mathfrak{m}_0\text{-a.e.} \quad x \in S. \tag{10.19}$$

Using Hölder's inequality for sums, Remark 3, Proposition 16 and taking (2.20) into account, for $\mathfrak{m}_0$-almost all $x \in S$ we obtain

$$\begin{aligned} |f(x) - F_{B_0(x)}|^p &\leqslant C \sum_{i=0}^{\infty} \epsilon^{-ip\delta} |F_{B_i(x)} - F_{B_{i+1}(x)}|^p \\ &\leqslant C \sum_{i=0}^{\infty} \epsilon^{i(p-p\delta)} \biggl(\fint_{\lambda B_i(x)} (|\nabla F|_{*,p}(y))^q \, d\mu(y) \biggr)^{p/q} \leqslant C \bigl(M^{\lambda}_{q,1-2\delta}(|\nabla F|_{*,p})(x) \bigr)^p. \end{aligned} \tag{10.20}$$

By (1.6), $\mathfrak{m}_0(B) \leqslant C\mu(B)$ for all $B \in \mathcal{B}$. We combine estimates (10.16)–(10.20), and take Propositions 4 and 7 into account. Finally, choosing $\delta > 0$ sufficiently small so that $p - 2p\delta > \theta$ we use Lemma 17 for $\alpha = 1 - 2\delta$. This gives (10.14).

The corollary is proved.

We should warn the reader that the following result is not a consequence of Theorem 20. Indeed, at this moment it has not yet been proved that $\operatorname{Ext}_{S,\{\mathfrak{m}_k\}}$ is a right inverse of $\operatorname{Tr}|^{\mathfrak{m}_0}_S$.

Theorem 21. *Assume that $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_{\theta}(S)$. Then for each $c \geqslant 3/\epsilon$ there exists a constant $C > 0$ such that for every $f \in L^{\mathrm{loc}}_1(\{\mathfrak{m}_k\})$ satisfying $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$, each $k \in \mathbb{N}_0$ and any ball $B = B_{\epsilon^k}(x)$ such that $x \in S$,*

$$\fint_B |f(y) - F_B| \, d\mathfrak{m}_k(y) \leqslant C\epsilon^k \biggl(\fint_B (|\nabla F|_{*,p}(y))^p \, d\mu(y) \biggr)^{1/p}, \tag{10.21}$$

where $F := \operatorname{Ext}_{S,\{\mathfrak{m}_k\}}(f)$.

Proof. We fix $f \in L^{\mathrm{loc}}_1(\{\mathfrak{m}_k\})$ such that $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$. By Theorem 14 we have $F = \operatorname{Ext}_{S,\{\mathfrak{m}_k\}}(f) \in W^1_p(\mathrm{X})$. Consequently, the right-hand side of inequality (10.21) makes sense. We recall the concept of a special approximating sequence introduced in (7.15). By Theorem 11 there is a constant $C > 0$ independent on f and a subsequence $\{f^{j_s}\}$ of the sequence $\{f^j\}$ such that for all sufficiently large $s \in \mathbb{N}$ we have

$$\|\operatorname{lip} f^{j_s}\|_{L_p(\mathrm{X})} \leqslant C\, \mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f).$$

At the same time, by Theorem 12 and Corollary 1 we have

$$\sup_{j \in \mathbb{N}} \|f^j\|_{L_p(\mathrm{X})} \leqslant C_f\, \mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty.$$

By Definition 7 this implies that $\mathrm{Ch}_p(f^{j_s}) < +\infty$ (while we do not claim that the functions f^j, $j \in \mathbb{N}$, are Lipschitz, yet it is not difficult to show that these functions are uniformly locally Lipschitz, and so, multiplying them by the corresponding Lipschitz cut-off functions, one can easily obtain the finiteness of the corresponding Cheeger p-energies) and, moreover, by Remark 8 $\||\nabla f^{j_s}|_{*,p}\|_{L_p(\mathrm{X})} \leqslant C\, \mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f)$ for all sufficiently large $s \in \mathbb{N}$, where the constant $C > 0$ is independent of f and s. Hence the sequence $\{f^{j_s}\}$ is bounded in $W^1_p(\mathrm{X})$. In view of Proposition 17 there exists a weakly convergent subsequence of the sequence $\{f^{j_s}\}$.

By Mazur's lemma there is an increasing sequence $\{N_l\}_{l\in\mathbb{N}}$ and a sequence of convex combinations $\widetilde{f}^{N_l} := \sum_{i=0}^{M_l} \lambda^i_{N_l} f^{N_l+i}$, where $\lambda^i_{N_l} \geqslant 0$ and $\sum_{i=0}^{M_l} \lambda^i_{N_l} = 1$, such that

$$\|F - \widetilde{f}^{N_l}\|_{W^1_p(\mathrm{X})} \to 0 \quad \text{as } l \to \infty.$$

Since, given $l \in \mathbb{N}$, the function $\widetilde{f}^{N_l}$ is continuous, the $\mathfrak{m}_0$-trace of $\widetilde{f}^{N_l}$ to S is an $\mathfrak{m}_0$-equivalence class of the pointwise restriction of $\widetilde{f}^{N_l}$ to S. Hence from Theorem 20 we obtain

$$\fint_B |\widetilde{f}^{N_l}|^{\mathfrak{m}_0}_S(y) - \widetilde{f}^{N_l}_B|\, d\mathfrak{m}_k(y) \leqslant \epsilon^k \left(\fint_B (|\nabla \widetilde{f}^{N_l}|_{*,p}(y))^p \, d\mu(y) \right)^{1/p}. \tag{10.22}$$

By Theorem 12 the sequence $\{f^j\}$ converges both in $L_p(\mathrm{X})$ (hence in $L^{\mathrm{loc}}_1(\mathfrak{m}_k)$) and $L_p(\mathfrak{m}_k)$ to F and f, respectively. Clearly, the same holds true for the sequence $\{\widetilde{f}^{N_l}\}$. As a result, passing to the limit in (10.22) we obtain the required estimate.

The theorem is proved.

Now we are ready to show that our extension operator $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}$ is the right inverse of the $\mathfrak{m}_0$-trace operator $\mathrm{Tr}\,|^{\mathfrak{m}_0}_S$. We recall Definition 14.

Corollary 4. *Assume that* $\{\mathfrak{m}_k\} \in \mathfrak{M}^{\mathrm{str}}_\theta(S)$. *If* $f \in \mathfrak{B}(\mathrm{X})$ *is a function such that* $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}([f]_{\mathfrak{m}_0}) < +\infty$ *for some* $c \geqslant 3/\epsilon$, *then*

$$\lim_{k\to\infty} \fint_{B_k(x)} |f(x) - \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)(y)|\, d\mu(y) = 0 \quad \textit{for } \mathcal{H}_p\textit{-a.e.} \quad x \in \mathcal{R}_{\{\mathfrak{m}_k\},\epsilon}(f). \tag{10.23}$$

In particular, $\mathrm{Tr}\,|^{\mathfrak{m}_0}_S \circ \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}([f]_{\mathfrak{m}_0}) = [f]_{\mathfrak{m}_0}$.

Proof. We put $F := \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}([f]_{\mathfrak{m}_0})$. It is clear that, given $k \in \mathbb{N}_0$ and $x \in S$, we have

$$\begin{aligned}\fint_{B_k(x)} |f(x) - F(y)|\, d\mu(y) &\leqslant \left| f(x) - \fint_{B_k(x)} f(y)\, d\mathfrak{m}_k(y) \right| \\ &+ \fint_{B_k(x)} \fint_{B_k(x)} |f(y) - F(y)|\, d\mathfrak{m}_k(y)\, d\mu(y) =: \sum_{i=1}^{2} R^i_k(x).\end{aligned} \tag{10.24}$$

By Definition 14 we have $\lim_{k\to\infty} R^1_k(x) = 0$ for all $x \in \mathcal{R}_{\{\mathfrak{m}_k\},\epsilon}(f)$. On the other hand, combining Remark 3, Propositions 15, and Theorem 21, we obtain

$$(R^2_k(x))^p \leqslant C\epsilon^{kp} \fint_{B_k(x)} (|\nabla F|_{*,p}(y))^p \, d\mu(y) \to 0, \quad k \to \infty, \quad \text{for } \mathcal{H}_p\text{-a.e. } x \in S. \tag{10.25}$$

Combining (10.24) and (10.25) we derive (10.23). Finally, to prove the second claim it is sufficient to use Theorem 10, Proposition 20 and take into account that the inequality $p > \theta$ implies that $\mathcal{H}_\theta\lfloor_S$ is absolutely continuous with respect to $\mathcal{H}_p\lfloor_S$.

The corollary is proved.

We recall the concept of a p-sharp representative introduced in § 2.3.

Proposition 34. *Given $F \in W_p^1(\mathrm{X})$, let $\overline{F}$ be an arbitrary p-sharp representative of F. Then $C_p(S \setminus \mathcal{R}_{\{\mathfrak{m}_k\},\epsilon}(\overline{F})) = 0$.*

Proof. One should repeat almost verbatim the proof of Lemma 4.3 in [21] using (at appropriate places) Propositions 18, 15 and Theorem 20 of this paper instead of Propositions 2.4, 3.1 and Theorem 3.1 from [21].

Proof of Theorems 2–4. We fix $c \geqslant 3/\epsilon$ and $\sigma \in (0, \epsilon^2/(4c))$ and split the proof into several steps.

Step 1. We recall Definition 20 and fix a function $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$ such that $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$. By Theorem 14 we have $F := \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f) \in W_p^1(\mathrm{X})$. From Corollary 4 we conclude that $f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ and $f = \mathrm{Tr}\,|_S^{\mathfrak{m}_0}(F)$. Hence from Corollary 3 we obtain $\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f) < +\infty$. By Theorem 16 this implies that $\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) < +\infty$ for any $c \geqslant 1$. Combining these observations with Theorem 15 we prove the equivalence of (i), (ii) and (iii) in Theorem 2. An application of Theorems 17 and 18 verifies the equivalence of (i)–(iv) in Theorem 2.

Step 2. By Theorems 13–16 we have (we put $F = \mathrm{Ext}_{S,\{\mathfrak{m}_k\}}(f)$)

$$\|f\|_{W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}} \leqslant \|F\|_{W_p^1(\mathrm{X})} \leqslant C\,\mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \leqslant C\,\mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f). \tag{10.26}$$

From Remark 8, Theorem 14 and Corollary 3 we obtain

$$C^{-1}\mathcal{CN}_{p,\{\mathfrak{m}_k\}}(f) \leqslant \||\nabla F|_{*,p}\|_{L_p(\mathrm{X})} \leqslant C\,\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f). \tag{10.27}$$

By (10.26) and (10.27) and Theorem 15 we have $\mathrm{N}_{p,\{\mathfrak{m}_k\},c}(f) \approx \mathrm{BSN}_{p,\{\mathfrak{m}_k\},c}(f) \approx \mathrm{CN}_{p,\{\mathfrak{m}_k\}}(f)$. Finally, combining this fact with (10.26) and Theorems 17 and 18 we obtain (1.15). As a result, we complete the proof of Theorem 2, and, furthermore, we prove assertion (1) in Theorem 4.

Step 3. Now we prove the first claim in Theorem 3. If $f \in W_p^1(\mathrm{X})|_S$, then by Definitions 8 and 9 it is clear that $\mathrm{I}_{\mathfrak{m}_0}(f) \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$. Consequently, condition (A) in Theorem 3 holds. Furthermore, condition (B) in Theorem 3 holds by Proposition 34. Conversely, assume that a function $f \in \mathfrak{B}(S)$ is such that conditions (A) and (B) in Theorem 3 hold true. Note that the corresponding assertions in §§ 7, 8 and 10, Theorem 2 and assertion (1) in Theorem 4 remain valid if we replace the requirement $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$ by the requirement $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(S)$ in combination with condition (B) in Theorem 3. By Theorem 2, Definition 14 and Corollary 4 we obtain $f \in W_p^1(\mathrm{X})|_S$.

Step 4. By Definitions 8 and 9 it is clear that $\mathrm{I}_{\mathfrak{m}_0}\colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ is a continuous embedding. From Definitions 8 and 9 and Remark 10 it follows that for each $[f] \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ there is a representative $f \in W_p^1(\mathrm{X})|_S$. Hence $\mathrm{I}_{\mathfrak{m}_0}$ is a surjection. Now we show that the mapping $\mathrm{I}_{\mathfrak{m}_0}$ is injective on $W_p^1(\mathrm{X})|_S$. Assume that, given $f \in W_p^1(\mathrm{X})|_S$, we have $\mathrm{I}_{\mathfrak{m}_0}(f)(x) = 0$ for $\mathfrak{m}_0$-almost every $x \in \mathrm{X}$. Consequently, by (B) in Theorem 3 this implies that $f(x) = 0$ everywhere on S except on a set of p-capacity zero, that is, $f = 0$ in the sense of $W_p^1(\mathrm{X})|_S$. As a result, from Definitions 8 and 9 it follows easily that $\mathrm{I}_{\mathfrak{m}_0}$ is an isometric isomorphism. This verifies (3) in Theorem 4.

Step 5. We put $\overline{\mathrm{Ext}}_{S,\{\mathfrak{m}_k\},p} := \mathrm{Ext}_{S,\{\mathfrak{m}_k\}} \circ \mathrm{I}_{\mathfrak{m}_0}$. By Remark 10 this gives a well-defined linear operator $\overline{\mathrm{Ext}}_{S,\{\mathfrak{m}_k\},p}\colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})$. Furthermore, since

$\mathrm{I}_{\mathfrak{m}_0}\colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ is an isomorphism, from Definition 10 we see that $\mathrm{Tr}\,|_S = (\mathrm{I}_{\mathfrak{m}_0})^{-1} \circ \mathrm{Tr}\,|_S^{\mathfrak{m}_0}$ and conclude that the diagram in Theorem 4 is commutative. Since $\mathrm{I}_{\mathfrak{m}_0}\colon W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ is an isometric isomorphism, it follows that $\|\overline{\mathrm{Ext}}_{S,\{\mathfrak{m}_k\},p}\| = \|\,\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}\,\|$. Combining this fact with (1.15) we obtain (1.17) and complete the proof of Theorems 3 and 4.

§ 11. Examples

In this section we show that many results related to Problems 1 and 2 and available in the literature are particular cases of Theorems 2–4 in this paper. In addition, we present a model example (Example 8) which does not fall into the scope of the previously known investigations. In fact, in some examples we present only sketches of the corresponding proofs, leaving the routine verifications to the reader.

Example 5. First of all, we note that in the particular case of $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_2, \mathcal{L}^n)$, by Theorem 3 and assertion (2) of Theorem 4 we obtain a clarification of the results in [21]. Indeed, in contrast to Theorem 3, the criterion presented in Theorem 2.1 of [21] was based on a more subtle Besov-type norm. Furthermore, characterizations via Brudnyi-Shvartsman-type functionals were not considered in [21].

For the next examples we recall the notation (1.13) and Definition 6.

Example 6. Let $p \in (1,\infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Assume, in addition, that the metric measure space X is Ahlfors Q-regular for some $Q > 0$. Let $\theta \in (0, \min\{p, Q\})$ and $S \in \mathcal{ADR}_\theta(\mathrm{X})$. Since $\theta > 0$, we have $\mu(S) = 0$. By Theorem 5.3 in [37] there exists $\sigma > 0$ such that for each $\epsilon \in (0,1]$ we have $S_{\epsilon^k}(\sigma) = S$ for all $k \in \mathbb{N}_0$, that is, the set S is porous. We put $\mathfrak{m}_k = \mathcal{H}_\theta\lfloor_S$ for all $k \in \mathbb{N}_0$. In accordance with Example 3, we have $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$. As a result, using the equivalence between (i) and (iv) in Theorem 2 and assertion (1) of Theorem 4 we obtain the following criterion due to Saksman and Soto [17] (see Theorems 1.5 and 1.7 therein).

A function $f \in L_p(\mathcal{H}_\theta\lfloor_S)$ belongs to the space $W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta}$ if and only if the Besov seminorm of f is finite, that is,

$$\|f\|^p_{B^{1-\theta/p}_{p,p}(S)} := \sum_{k=0}^{\infty} 2^{k(p-\theta)} \int_S (\mathcal{E}_{\mathcal{H}_\theta}(f, B_{2^{-k}}(x)))^p \, d\mathcal{H}_\theta(x) < +\infty.$$

Furthermore,

$$\|F\|_{W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta}} \approx \|f\|_{L_p(\mathcal{H}_\theta\lfloor_S)} + \|f\|_{B^{1-\theta/p}_{p,p}(S)},$$

where the equivalence constants are independent of f. Moreover, there exists an $\mathcal{H}_\theta\lfloor_S$-extension operator $\mathrm{Ext}_S \in \mathcal{L}(W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta}, W_p^1(\mathrm{X}))$.

Example 7. Let $p \in (1,\infty)$, $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$ and $S \in \mathcal{ADR}_0(\mathrm{X})$. We recall Example 3 and note that by letting $\mathfrak{m}_k = \mu\lfloor_S$ for $k \in \mathbb{N}_0$ we obtain $\{\mathfrak{m}_k\} \in \mathfrak{M}_0^{\mathrm{str}}(S)$. We recall a combinatorial result, which is a slight modification of Theorem 2.6 in [18].

Proposition 35. *Let $\mathcal{B}$ be an (S,c)-Whitney family of balls. Then there exist constants $c_1, c_2 > 0$ and $\tau \in (0,1)$ and a family $\mathcal{U} := \{U(B)\colon B \in \mathcal{B}\}$ of Borel subsets of S such that $U(B) \subset c_1 B$, $\mu(U(B)) \geqslant \tau\mu(B)$ for all $B \in \mathcal{B}$ and $\mathcal{M}(\{U(B)\colon B \in \mathcal{B}\}) \leqslant c_2$.*

Based on Proposition 35, one can repeat the arguments used in Example 6.1 of [21] with minor modifications and deduce, for each $\sigma \in (0,1)$, the existence of a constant $C > 0$ such that, for every $f \in L_1^{\mathrm{loc}}(\mu\lfloor_S)$,

$$\sum_{k=1}^{\infty} 2^{k(p-\theta)} \int_{S_k(\sigma)} \big(\mathcal{E}_{\mu\lfloor_S}(f, B_{2^{-k}}(x))\big)^p \, d\mu\lfloor_S(x) \leqslant C(\|f\|_{L_p(\mu\lfloor_S)} + \|f^{\sharp}_{\mu\lfloor_S}\|_{L_p(\mu\lfloor_S)}). \tag{11.1}$$

Hence, using the equivalence between (i) and (iv) in Theorem 2 and assertion (1) of Theorem 4 we arrive at Shvartsman's criterion [18].

A function $f \in L_p(\mu\lfloor_S)$ belongs to the space $W_p^1(\mathrm{X})|_S^{\mu}$ if and only if $f^{\sharp}_{S,\mu} \in L_p(\mu\lfloor_S)$. Furthermore,

$$\|f\|_{W_p^1(\mathrm{X})|_S^{\mu}} \approx \|f\|_{L_p(\mu\lfloor_S)} + \|f^{\sharp}_{\mu\lfloor_S}\|_{L_p(\mu\lfloor_S)},$$

where the equivalence constants are independent of f. Moreover, there exists a $\mu\lfloor_S$-extension operator $\mathrm{Ext}_S \in \mathcal{L}(W_p^1(\mathrm{X})|_S^{\mu}, W_p^1(\mathrm{X}))$.

Now we present a model example, which exhibits interesting effects arising in the case when we describe the trace spaces to closed sets consisting of pieces of different dimensions.

Example 8. Let $p \in (1,\infty)$ and $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu) \in \mathfrak{A}_p$. Assume that X is Ahlfors Q-regular for some $Q > 1$. Let $\underline{B} = B_R(x)$ be a closed ball of radius $R > 0$ centred at $x \in \mathrm{X}$ and $\gamma\colon [0,1] \to \mathrm{X}$ be a rectifiable curve such that $\Gamma := \gamma([0,1]) \in \mathcal{ADR}_{Q-1}(\mathrm{X})$, $\Gamma \cap \underline{B} = \gamma(0) = \{\underline{x}\}$ for some point $\underline{x} \in \partial \underline{B}$ and, furthermore, for some $\kappa > 0$, $\operatorname{dist}(\gamma(t), \underline{B}) \geqslant \kappa l(\gamma([0,t]))$ for all $t \in [0,1]$. We put $S := \underline{B} \cup \Gamma$ and recall Example 4. For each $k \in \mathbb{N}_0$ we set $\mathfrak{m}_k := 2^{k(Q-1)}\mu\lfloor_{\underline{B}} + \mathcal{H}_{Q-1}\lfloor_{\Gamma}$. Hence we obtain $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k=0}^{\infty} \in \mathfrak{M}^{\mathrm{str}}_{Q-1}(S)$.

Given $k \in \mathbb{N}_0$, we introduce the kth gluing functional by letting, for each $f \in L_1^{\mathrm{loc}}(\{\mathfrak{m}_k\})$,

$$\mathrm{gl}_k(f, \underline{x}) := \fint_{\underline{B} \cap B_k(\underline{x})} \fint_{\Gamma \cap B_k(\underline{x})} |f(y) - f(z)| \, d\mu(y) \, d\mathcal{H}_{Q-1}(z). \tag{11.2}$$

We put $\underline{k} := \min\{k \in \mathbb{N}\colon 2^k > 1/\kappa\}$ and $\alpha := 1 - (Q-1)/p$. Using Remark 3 and letting $B_k(x) := B_{2^{-k}}(x)$ and $S_k(\sigma) := S_{2^{-k}}(\sigma)$ it is easy to see that, given $\sigma \in (0,1)$,

$$\left(\sum_{k=1}^{\underline{k}+1} 2^{k\alpha p} \int_{S_k(\sigma)} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\big)^p \, d\mathfrak{m}_k(x)\right)^{1/p} \leqslant C\big(\|f\|_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|_{L_p(\mathcal{H}_{Q-1}\lfloor_{\Gamma})}\big). \tag{11.3}$$

Using the arguments presented in Examples 6 and 7 and taking into account that for each $k > \underline{k}$ we have $S \cap B_k(x) = \underline{B} \cap B_k(x)$ for all $x \in \underline{B} \setminus B_{k-\underline{k}}(\underline{x})$ and $S \cap B_k(x) = \Gamma \cap B_k(x)$ for all $x \in \Gamma \setminus B_{k-\underline{k}}(\underline{x})$, one can show that, given a sufficiently small $\sigma \in (0,1)$ (depending on Γ),

$$\sum_{k=\underline{k}+2}^{\infty} 2^{k\alpha p} \int_{S_k(\sigma) \setminus B_{k-\underline{k}}(\underline{x})} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\big)^p d\mathfrak{m}_k(x) \leqslant C\big(\|f^{\sharp}_{\mu\lfloor_{\underline{B}}}\|^p_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|^p_{B^{\alpha}_{p,p}(\Gamma)}\big). \tag{11.4}$$

Since $1/\epsilon \geqslant 10$, for $x \in S_k(\sigma) \cap B_{k-\underline{k}}(\underline{x})$ and $k > \underline{k}$ we have $B_k(x) \subset B_{k-\underline{k}-1}(\underline{x})$ and $B_{k-\underline{k}-1}(\underline{x}) \subset B_{k-\underline{k}-2}(x)$. By Remark 3 and Theorem 5, $\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x)) \leqslant C\,\mathrm{gl}_{k-\underline{k}-1}(f, \underline{x})$. Furthermore, it is easy to see that $\mathfrak{m}_k(B_{k-\underline{k}}(\underline{x})) \leqslant C2^{-k}$. As a result,

$$\int_{S_k(\sigma)\cap B_{k-\underline{k}}(\underline{x})} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\big)^p \, d\mathfrak{m}_k(x) \leqslant C2^{-k} \big(\mathrm{gl}_{k-\underline{k}-1}(f, \underline{x})\big)^p. \tag{11.5}$$

On the other hand, given $x \in S_k(\sigma) \cap B_{k-\underline{k}}(\underline{x})$ and $k > \underline{k}+1$, we have $B_{k-\underline{k}}(\underline{x}) \subset B_{k-\underline{k}-1}(x)$ and $B_{k-\underline{k}-1}(x) \subset B_{k-\underline{k}-2}(\underline{x})$. Hence, from Theorem 5 and Remark 3 it is easy to obtain $\mathrm{gl}_{k-\underline{k}-1}(f, \underline{x}) \leqslant C\mathcal{E}_{\mathfrak{m}_k}(f, B_{k-\underline{k}-2}(x))$. If $\sigma \in (0,1)$ is sufficiently small, then one can show that $\mathfrak{m}_k(S_k(\sigma) \cap B_k(\underline{x})) \approx 2^{-k}$. As a result,

$$\big(\mathrm{gl}_{k-\underline{k}}(f, \underline{x})\big)^p \leqslant C2^k \int_{S_k(\sigma)\cap B_k(\underline{x})} \big(\mathcal{E}_{\mathfrak{m}_k}(f, B_{k-\underline{k}-1}(x))\big)^p \, d\mathfrak{m}_k(x). \tag{11.6}$$

At the same time it follows from Examples 6 and 7 that there is $C > 0$ such that for all $F \in W_p^1(\mathbb{R}^n)$, for $f = F|_S^{\mathfrak{m}_0}$ we have

$$\|f\|_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|_{L_p(\mathcal{H}_{Q-1}\lfloor_\Gamma)} + \|f^\sharp_{\mu\lfloor_{\underline{B}}}\|_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|_{B^\alpha_{p,p}(\Gamma)} \leqslant C\|F\|_{W_p^1(\mathbb{R}^n)}. \tag{11.7}$$

Finally, combining (11.3)–(11.7) it is easy to deduce the following result from Theorem 2 and assertion (1) of Theorem 4.

A function $f \in L_p(\mu\lfloor_{\underline{B}}) \cap L_p(\mathcal{H}_\theta\lfloor_\Gamma)$ *belongs to* $W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$ *if and only if* $f^\sharp_{\mu\lfloor_{\underline{B}}} \in L_p(\mu\lfloor_{\underline{B}})$, $f \in B^{1-(Q-1)/p}_{p,p}(\Gamma)$ *and*

$$(\mathcal{GL}(f, \underline{x}))^p := \sum_{k=1}^{\infty} 2^{k(p-Q)} (\mathrm{gl}_k(f, \underline{x}))^p < +\infty.$$

Furthermore,

$$\begin{aligned} \|f\|_{W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}} \approx{}& \|f\|_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|_{L_p(\mathcal{H}_\theta\lfloor_\Gamma)} \\ &+ \|f^\sharp_{\mu\lfloor_{\underline{B}}}\|_{L_p(\mu\lfloor_{\underline{B}})} + \|f\|_{B^{1-(Q-1)/p}_{p,p}(\Gamma)} + \mathcal{GL}(f, \underline{x}), \end{aligned} \tag{11.8}$$

where the equivalence constants are independent of f. *Moreover, there exists an* $\mathfrak{m}_0$*-extension operator* $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}} \in \mathcal{L}(W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}, W_p^1(\mathrm{X}))$.

We conclude by presenting a natural generalization of Theorem 1.2 from [19].

Example 9. Let $\mathrm{X} = (\mathrm{X}, \mathrm{d}, \mu)$ be a geodesic metric measure space. Assume that X is Ahlfors Q-regular for some $Q \geqslant 1$. Furthermore, assume that $p \in (Q, \infty)$ and $\mathrm{X} \in \mathfrak{A}_p$. Also fix a parameter $\theta \in [Q, p)$ and a nonempty closed set $S \subset \mathrm{X}$. For simplicity we assume that $S \subset B_1(\underline{x})$ for some $\underline{x} \in \mathrm{X}$. According to Example 2, we have $S \in \mathcal{LCR}_\theta(\mathrm{X})$.

Combining results from [3] with Theorem 9.1.15 from [1] we obtain that each $F \in W_p^1(\mathrm{X})$ has a *continuous representative* $\overline{F}$ such that for each closed ball B,

$$\sup_{x\in B} |\overline{F}(x) - F_B| \leqslant C r_B \left(⨍_B (|\nabla F|_{*,p}(y))^p \, d\mu(y) \right)^{1/p}. \tag{11.9}$$

Furthermore, it is clear that, given $F \in W_p^1(\mathrm{X})$, the p-sharp trace $F|_S$ is well defined and coincides with the pointwise restriction of $\overline{F}$ to the set S.

From Remark 3 it is easy to see that if a ball $B_r(x)$ of radius $r \in (0,1]$ centred at $x \in \mathrm{X}$ is such that $B_{cr}(x) \cap S \neq \varnothing$ for some $c \geqslant 1$, then for each sequence $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta^{\mathrm{str}}(S)$,

$$\mathcal{E}_{\mathfrak{m}_{k(r)}}(f, B_{2cr}(x)) \leqslant \sup_{y,z \in B_{2cr}(x) \cap S} |f(y) - f(z)| \quad \text{for all } f \in C(S). \tag{11.10}$$

We define the modified *Brudnyi-Shvartsman-type functional* on $C(S)$ (with values in $[0,+\infty]$) as follows. Given a function $f \in C(\mathrm{X})$, we put

$$\widetilde{\mathcal{BSN}}_p(f) := \sup \biggl(\sum_{i=1}^N \frac{\mu(B_{r_i}(x_i))}{r_i^p} \sup_{y,z \in B_{60r_i}(x_i) \cap S} |f(y) - f(z)|^p \biggr)^{1/p},$$

where the supremum is taken over all finite disjoint families of closed balls $\{B_i\}_{i=1}^N = \{B_{r_i}(x_i)\}_{i=1}^N$ in X of radii $r_i \in (0,1]$, $i = 1, \dots, N$. In the particular case of $\mathrm{X} = (\mathbb{R}^n, \|\cdot\|_\infty, \mathcal{L}^n)$ our functional is very similar to that used in [19]. The only difference is that in [19] the corresponding coefficient of dilation of balls was 11 rather than 60. Using (11.9) and Proposition 7 it is easy to see that

$$\widetilde{\mathcal{BSN}}_p(F|_S) + \inf_{z \in S} |F|_S(z)| \leqslant C\|F\|_{W_p^1(\mathrm{X})} \quad \text{for all } F \in W_p^1(\mathrm{X}). \tag{11.11}$$

On the other hand, if $x_m \in S$ is a minimum point of f and $x_M \in S$ is a maximum point of f, then it is easy to see that $|f(x_m) - f(x_M)| \leqslant (\mu(B_1(\underline{x})))^{-1/p} \widetilde{\mathcal{BSN}}_p(f)$. As a result, by (11.10)

$$\begin{aligned} \mathrm{BSN}_{p,\{\mathfrak{m}_k\},30}(f) &\leqslant \widetilde{\mathcal{BSN}}_p(f) + \sup_{x \in S} |f(x)| \\ &\leqslant C\Bigl(\widetilde{\mathcal{BSN}}_p(f) + \inf_{z \in S} |f(z)|\Bigr), \qquad f \in C(S). \end{aligned} \tag{11.12}$$

Finally, we apply Theorem 3 for $\epsilon = 1/10$ and $c = 30$ and use (11.11) and (11.12). This leads to the following criterion.

A function $f \in C(S)$ belongs to the space $W_p^1(\mathrm{X})|_S$ if and only if $\widetilde{\mathcal{BSN}}_p(f) < +\infty$. Furthermore,

$$\|f\|_{W_p^1(\mathrm{X})|_S} \approx \widetilde{\mathcal{BSN}}_p(f) + \inf_{z \in S} |f(z)|,$$

where the corresponding equivalence constants do not depend on f. Moreover, there exists a p-sharp extension operator $\mathrm{Ext}_{S,p} \in \mathcal{L}(W_p^1(\mathrm{X})|_S, W_p^1(\mathrm{X}))$.

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