CYLINDRICAL ESTIMATES FOR THE CHEEGER CONSTANT AND APPLICATIONS

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Abstract. We prove a lower bound for the Cheeger constant of a cylinder $\Omega \times (0, L)$, where Ω is an open and bounded set. As a consequence, we obtain existence of minimizers for the shape functional defined as the ratio between the first Dirichlet eigenvalue of the p-Laplacian and the p-th power of the Cheeger constant, within the class of bounded convex sets in any $\mathbb{R}^N$. This positively solves open conjectures raised by Parini (*J. Convex Anal.* (2017)) and by Briani–Buttazzo–Prinari (*Ann. Mat. Pura Appl.* (2023)).

1. Introduction

The Cheeger constant of an open, bounded set $\Omega \subseteq \mathbb{R}^N$ is defined as

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \,:\, E \subseteq \Omega,\, |E| > 0 \right\} .$$

The constant owes its name to Cheeger, who in [2] used its Riemannian counterpart to provide a lower bound to the first eigenvalue of the Dirichlet Laplace–Beltrami operator. In the Euclidean setting, denoting by $\lambda_p(\Omega)$ the first eigenvalue of the Dirichlet p-Laplacian, one has

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p , \tag{1.1}$$

and, assuming Ω to be regular enough (Lipschitz suffices), one has that $\lambda_1(\Omega) = h(\Omega)$, as first proved in [7]. Inequality (1.1) is very robust, as noticed even earlier by Maz'ya [13, 14], and can be extended to many different settings, see, e.g., [3] and the references therein. For an overview of the Cheeger problem, we refer the interested reader to the surveys [11, 15]. Rearranging (1.1), one obtains

$$F_p[\Omega] := \frac{\lambda_p^{1/p}(\Omega)}{h(\Omega)} \geq \frac{1}{p} .$$

Thence, one has a lower bound to the spectral operator $F_p[\,\cdot\,]$. It is therefore natural to wonder whether this bound is attained (it is actually not), and in general if the minimization of $F_p[\,\cdot\,]$ has solutions in some suitable class of subsets of $\mathbb{R}^N$. The first step in this direction was made in [16], where Parini proved existence of minimizers among the class of convex subsets of the plane $\mathbb{R}^2$, in the case $p = 2$. Later on, Ftouhi [4] provided a different proof of existence, along with a sufficient criterion to determine whether minimizers among convex subsets of $\mathbb{R}^N$ exist. Recently, Buttazzo, Briani and Prinari extended this criterion to general p, and proved existence for any p among convex subsets of the plane $\mathbb{R}^2$, see [1].

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In this paper, by exploiting the above-mentioned criterion, we are able to show existence of minimizers of $F_p[\,\cdot\,]$ among convex subsets of $\mathbb{R}^N$, for any $N \geq 2$ and $p > 1$, see Theorem 3.3. The proof relies on some cylindrical estimates on the Cheeger constant. The key one, see Theorem 2.1, is the following. Given an open and bounded subset Ω of $\mathbb{R}^N$, not necessarily convex, and given $L \geq 1$, we show that

$$h(\Omega) + \frac{c}{L} \leq h(\Omega_L) < h(\Omega) + \frac{2}{L}\,, \tag{1.2}$$

where $\Omega_L := \Omega \times (0, L)$ and $c > 0$ is a constant depending only on Ω. In other words, we can estimate both from above and from below the Cheeger constant of the $(N+1)$-dimensional cylinder Ω_L with the Cheeger constant of its cross-section plus a non-zero term that goes like the inverse of the height of the cylinder. The proof of Theorem 3.3 essentially comes by combining (1.2) with the results of [1, 4].

The structure of the paper is quite simple. In Section 2 we prove the cylindrical estimates, refer to Theorem 2.1, while in Section 3 we use them to obtain Theorem 3.3.

2. Estimates

This section is devoted to show the key estimate (1.2) for cylinders. First of all, we fix a quick notation that will be used through the rest of the paper. Given any set $\Omega \subseteq \mathbb{R}^N$, for any $L > 0$ the cylinder $\Omega \times (0, L) \subseteq \mathbb{R}^{N+1}$ will be denoted by Ω_L. Moreover, for any subset $C \subseteq \Omega_L$ and $t \in [0, L]$ we will denote by C_t the horizontal section of C at height t, that is,

$$C_t := \begin{cases} C \cap (\Omega \times \{t\}), & t \in (0, L)\,, \\ \partial C \cap (\Omega \times \{t\}), & t \in \{0, L\}\,. \end{cases}$$

Moreover, with some abuse of notation we shall write $C_t \subseteq \Omega$, in place of $\Pi_N(C_t) \subseteq \Omega$, being Π_N the restriction to the first N coordinates. Notice also that, by the well known Vol'pert Theorem (see for instance [5, Theorem 6.2]), if C is a set of finite perimeter in Ω_L then for a.e. $t \in [0, L]$ the section C_t is a well-defined set with finite perimeter in $\mathbb{R}^N$.

The main result of the section reads as follows.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. There exists a constant $c = c(\Omega) > 0$ such that for any $L \geq 1$ one has*

$$h(\Omega) + \frac{c}{L} \leq h(\Omega_L) \leq h(\Omega) + \frac{2}{L}\,. \tag{1.2}$$

Before proving this result, we need to show the following preliminary lemma. As recalled, by Vol'pert Theorem, if $D \subset \Omega_L$ is a set of finite perimeter, then almost every section D_t is well-defined. The lemma says that for those values $t \in [0, L]$ such that the section is not well-defined, one can however "identify" it either as the unique limit of sections above it (i.e., for $s \searrow t$) or as the unique limit (possibly different from the previous one) of sections below it (i.e., for $s \nearrow t$).

Lemma 2.2. *Let D be any set of finite perimeter contained in the cylinder Ω_L. Then, for every $0 \leq t < L$, there exists a set $D_t^+ \subseteq \Omega$ such that*

$$\lim_{s \searrow t} \left| D_s \Delta D_t^+ \right| = 0\,. \tag{2.1}$$

Analogously, for every $0 < t \leq L$ *there exists* $D_t^- \subseteq \Omega$ *such that*

$$\lim_{s \nearrow t} \left| D_s \Delta D_t^- \right| = 0 \,.$$

Proof. Let us fix $0 \leq t < L$, and let $s_j \searrow t$ be any monotone decreasing sequence converging to t and such that the section D_{s_j} is well-defined for every j. By projection, we know that

$$P\Big(D; \Omega \times \big(s_{j+1}, s_j\big)\Big) \geq \left| D_{s_{j+1}} \Delta D_{s_j} \right| .$$

As a consequence, taking the sum on j, and recalling that D has finite perimeter, we have that

$$+\infty > P(D) \geq \sum_{j \in \mathbb{N}} P\Big(D; \Omega \times \big(s_{j+1}, s_j\big)\Big) \geq \sum_{j \in \mathbb{N}} \left| D_{s_{j+1}} \Delta D_{s_j} \right| . \tag{2.2}$$

We now extract a subsequence $\{\sigma_n\}$ of $\{s_j\}$ as follows. By the finiteness of the sum in (2.2), for any integer $n \geq 1$ there exists an index j_n so that the tail of the series is bounded from above as

$$\sum_{j \geq j_n} \left| D_{s_{j+1}} \Delta D_{s_j} \right| < \frac{1}{2^n} \,. \tag{2.3}$$

Moreover, we can also assume that $\{j_n\}$ is strictly monotone, and that

$$j_n \text{ and } n \text{ have the same parity for each } n. \tag{2.4}$$

We let the sequence $\{\sigma_n\} := \{s_{j_n}\}$, and we notice that, owing to (2.3) and the set-wise triangular inequality $A \Delta B \subseteq A \Delta C \cup C \Delta B$, the sections of D individuated by the sequence of heights $\{\sigma_n\}$ satisfy

$$\left| D_{\sigma_{n+1}} \Delta D_{\sigma_n} \right| \leq \sum_{j=j_n}^{j_{n+1}-1} \left| D_{s_{j+1}} \Delta D_{s_j} \right| < \frac{1}{2^n} \,, \qquad \forall\, n \in \mathbb{N} \,. \tag{2.5}$$

Calling now, for every $k \in \mathbb{N}$,

$$F_k := \bigcap_{n \geq k} D_{\sigma_n} \,, \qquad\qquad G_k := \bigcup_{n \geq k} D_{\sigma_n} \,,$$

we trivially have that $F_k \subseteq D_{\sigma_k} \subseteq G_k$. Moreover, one can easily prove the set equality

$$G_k \setminus F_k = \bigcup_{n \geq k} \left(D_{\sigma_{n+1}} \Delta D_{\sigma_n} \right) \,,$$

which combined with (2.5) yields the estimate $\left| G_k \setminus F_k \right| \leq 2^{1-k}$. Defining now the set D_t^+ as

$$D_t^+ := \bigcup_{k \in N} F_k = \bigcap_{k \in \mathbb{N}} G_k \,,$$

we have that

$$\begin{aligned} \lim_{k \to \infty} \left| D_{\sigma_k} \Delta D_t^+ \right| &= \lim_{k \to \infty} \left| D_{\sigma_k} \setminus D_t^+ \right| + \left| D_t^+ \setminus D_{\sigma_k} \right| \\ &\leq \lim_{k \to \infty} \left| D_{\sigma_k} \setminus F_k \right| + \lim_{k \to \infty} \left| G_k \setminus D_{\sigma_k} \right| = \lim_{k \to \infty} \left| G_k \setminus F_k \right| = 0 \,. \end{aligned} \tag{2.6}$$

This property is in principle weaker than (2.1), because the set D_t^+ might depend on the particular choice of the sequence $\{s_j\}$ and of the subsequence $\{\sigma_n\}$. Therefore, it suffices to show that any choice of the sequence and of the relative subsequence yields the same limit set D_t^+. Consider two monotone sequences $s'_j \searrow t$ and $s''_j \searrow t$ satisfying

$$\lim_{j\to\infty} \big|D_{s'_j} \Delta D'\big| = \lim_{j\to\infty} \big|D_{s''_j} \Delta D''\big| = 0$$

for two different sets D', D'', and define a monotone sequence $s_j \searrow t$ in such a way $\{s_{2j}\}$ is a subsequence of $\{s'_j\}$ and $\{s_{2j+1}\}$ is a subsequence of $\{s''_j\}$, which is clearly possible. Reasoning as before, one can find a subsequence $\{\sigma_n\}$ of $\{s_j\}$ and a set D_t^+ such that (2.6) holds, but by construction and thanks to (2.4) we deduce that D_t^+ must coincide both with D' and D'', which is impossible since D' and D'' are different. The contradiction shows that actually D_t^+ does not depend on the sequence, so that we have obtained (2.1).

The existence of the set D_t^- can be of course obtained exactly in the same way, so the proof is complete. □

Remark 2.3. Notice that the above lemma works for any set of finite perimeter. In other words, the lone finiteness of the perimeter implies the existence of all the "upper" and all the "lower" sections. Moreover, every $t \in (0, L)$ for which they do not coincide contributes for a quantity $|D_t^+ \Delta D_t^-|$ in perimeter for D. Hence, there are at most countable many such t.

We are now ready to present the proof of the main estimate of this section.

Proof of Theorem 2.1. The proof will be divided in some steps.
Step I. A weaker inequality.
In this first step, we prove an estimate which is weaker than (1.2), namely,

$$h(\Omega) \le h(\Omega_L) \le h(\Omega) + \frac{2}{L}\,. \tag{2.7}$$

We start by proving the upper bound. We let F be a Cheeger set for Ω, that is, a set realizing the infimum in the definition of the Cheeger constant, and whose existence is ensured by the boundedness of Ω, refer to [11, Proposition 3.5(iii)]. It is then enough to notice that

$$h(\Omega_L) \le \frac{P\big(F\times[0,L]\big)}{\big|F\times[0,L]\big|} = \frac{LP(F)+2|F|}{L|F|} = \frac{P(F)}{|F|} + \frac{2}{L} = h(\Omega) + \frac{2}{L}\,,$$

so that the right inequality in (2.7) is proved, and so it also is the right one in (1.2).

We now turn our attention to the lower bound. This estimate is very easy to prove and it is not needed to prove the stronger one (1.2), yet it contains the basic ideas we shall exploit to prove our main result. Let $D \subseteq \Omega_L$ be any set of finite perimeter. Then, as noticed above, by the Vol'pert Theorem the section D_t is a well-defined set with finite perimeter in $\mathbb{R}^N$ for a.e. $t \in [0, L]$. In particular, $D_t \subseteq \Omega$ and then $P(D_t) \ge h(\Omega)|D_t|$, where by $P(D_t)$ and $|D_t|$ we denote the perimeter and the measure of D_t in $\mathbb{R}^N$, that is, $\mathscr{H}^{N-1}(\partial^* D_t)$ and $\mathscr{H}^N(D_t)$ respectively. A simple integration gives then

$$P(D) \ge \int_0^L P(D_t)\,\mathrm{d}t \ge \int_0^L h(\Omega)|D_t|\,\mathrm{d}t = h(\Omega)\int_0^L |D_t|\,\mathrm{d}t = h(\Omega)|D|\,, \tag{2.8}$$

thus $P(D)/|D| \ge h(\Omega)$ for every $D \subseteq \Omega_L$ and then the left inequality in (2.7) is proved.

We remark that the first inequality in (2.8) is very sloppy. Indeed, it remains true adding on the right the measure of the section D_0^+ given by Lemma 2.2. This simple observation will be the starting point of the proof of the stronger inequality.

Step II. The "minimal volume" v, the gap ε, the height τ and the volume up to τ, V.
This step is devoted to define two positive quantities v and ε, which depend only on Ω, and two other quantities τ and V that depend on the choice of one Cheeger set of the cylinder Ω_L.

As recalled in the previous step, in view of the boundedness of Ω, there exist Cheeger sets for Ω. In general there might be several different Cheeger sets, but their measure cannot be too small. In particular, given any Cheeger set C of Ω, one has

$$|C| \geq \omega_N \left(\frac{N}{h(\Omega)}\right)^N , \tag{2.9}$$

refer to [11, Proposition 3.5(v)]. We now let

$$v := \inf\Big\{|C| \,:\, C \text{ is a Cheeger set for } \Omega\Big\},$$

and we have that $v = v(\Omega) > 0$ in view of (2.9). We then let

$$\varepsilon := \inf\left\{\frac{P(E)}{|E|},\, E \subseteq \Omega,\, |E| \leq v/2\right\} - h(\Omega)\,.$$

It is simple to notice that $\varepsilon > 0$. Indeed, let E be any subset of Ω with $|E| \leq v/2$. By definition of v, E is not a Cheeger set for Ω, and then

$$\frac{P(E)}{|E|} > h(\Omega)\,.$$

This only shows that $\varepsilon \geq 0$. Argue now by contradiction and assume the existence of a sequence of sets $E_j \subseteq \Omega$ such that $|E_j| \leq v/2$ for every j, and

$$\frac{P(E_j)}{|E_j|} \to h(\Omega)\,. \tag{2.10}$$

The sequence of the characteristic functions χ_{E_j} is then bounded in $BV(\Omega)$, so that, up to a subsequence, we can assume that χ_{E_j} is weakly-star converging in $BV(\Omega)$ to some function φ. However, since Ω is bounded the convergence is strong in L^1, thus φ is the characteristic function of some set E_∞ with $|E_\infty| > 0$, as otherwise (2.10) would be easily contradicted by using the isoperimetric inequality. Then, the lower semicontinuity of the perimeter implies that E_∞ is a Cheeger set for Ω with measure less than $v/2$, which is impossible by definition of v. Hence, we conclude that $\varepsilon > 0$.

Let us now fix a Cheeger set C^* of Ω_L. Notice that for almost every $t \in [0, L]$, $(C^*)^+_t$ and $(C^*)^-_t$ coincide by Remark 2.3, and by Vol'pert Theorem they also coincide with C^*_t. We define the two following quantities

$$\tau := \operatorname{ess\,inf}\big\{t \in [0, L],\, |C^*_t| \geq v/2\big\}, \qquad V := \Big|C^* \cap \big(\Omega \times [0, \tau]\big)\Big|,$$

which depend also on the choice of the Cheeger set C^*. Nevertheless, we will be able to check the validity of (1.2), independently from such a choice, exhibiting a constant c depending only on the minimal volume v, the gap ε, the measure $|\Omega|$, and the Cheeger constant $h(\Omega)$ and thus, definitively only on the set Ω.

<u>Step III.</u> *The proof of the inequality.*
We shall now refine (2.8), arguing in different ways depending on the measure of the upper section of C^* at height zero, which for the sake of convenience we denote by C_0^* in place of $(C^*)_0^+$.

(i) The case $|C_0^*| > v/4$. We first assume that the bottom section of C^* is not too small, namely, that $|C_0^*| > v/4$. Then, by Fubini and recalling that $P(C_t^*) \geq h(\Omega)|C_t^*|$ for all $t \in [0, L]$, we readily have

$$h(\Omega_L) = \frac{P(C^*)}{|C^*|} \geq \frac{\displaystyle\int_0^L P(C_t^*)\,\mathrm{d}t + |C_0^*|}{\displaystyle\int_0^L |C_t^*|\,\mathrm{d}t} \geq \frac{\displaystyle\int_0^L h(\Omega)|C_t^*|\,\mathrm{d}t + |C_0^*|}{\displaystyle\int_0^L |C_t^*|\,\mathrm{d}t} \geq h(\Omega) + \frac{v}{4|\Omega|L}\,, \tag{2.11}$$

so the required estimate is obtained in this case.

(ii) The case $|C_0^*| \leq v/4$*: when* $V > v(8h(\Omega))^{-1}$. Notice that, by construction and by Step II, $P(C_t^*) \geq (h(\Omega)+\varepsilon)|C_t^*|$ for almost every $0 \leq t \leq \tau$, while $P(C_t^*) \geq h(\Omega)|C_t^*|$ for almost every $\tau \leq t \leq L$. Therefore, again by integration we get

$$\begin{aligned} h(\Omega_L) = \frac{P(C^*)}{|C^*|} &\geq \frac{\displaystyle\int_0^\tau P(C_t^*)\,\mathrm{d}t + \int_\tau^L P(C_t^*)\,\mathrm{d}t}{|C^*|} \\ &\geq \frac{\displaystyle\int_0^L h(\Omega)|C_t^*|\,\mathrm{d}t + \int_0^\tau \varepsilon|C_t^*|\,\mathrm{d}t}{|C^*|} = h(\Omega) + \frac{\varepsilon V}{|C^*|} > h(\Omega) + \frac{\varepsilon v}{8h(\Omega)|\Omega|L}\,, \end{aligned} \tag{2.12}$$

and then the required estimate is obtained also in this case.

(iii) The case $|C_0^*| \leq v/4$*: when* $V \leq v(8h(\Omega))^{-1}$ *and* $\tau < L$. We now assume that $\tau < L$, so that by definition of τ there is a sequence of $t_n \searrow \tau$ such that $C_{t_n}^*$ is well-defined and $|C_{t_n}^*| \geq v/2$. By projection for any n, we have

$$P\Big(C^*; \big(\Omega \times (0, t_n)\big)\Big) \geq \big||C_{t_n}^*| - |C_0^*|\big| = \frac{v}{2} - |C_0^*| \geq \frac{v}{4}\,.$$

Therefore, arguing as in the previous steps, we get

$$\begin{aligned} h(\Omega_L) = \frac{P(C^*)}{|C^*|} &\geq \frac{P\Big(C^*; \big(\Omega \times (0,t_n)\big)\Big) + \displaystyle\int_{t_n}^L P(C_t^*)\,\mathrm{d}t}{|C^*|} \\ &\geq \frac{\dfrac{v}{4} + \displaystyle\int_{t_n}^L h(\Omega)|C_t^*|\,\mathrm{d}t}{|C^*|} = \frac{\dfrac{v}{4} + h(\Omega)\left(|C^*| - V - \displaystyle\int_\tau^{t_n} |C_t^*|\,\mathrm{d}t\right)}{|C^*|}\,. \end{aligned}$$

Letting now $n \to +\infty$, we have

$$h(\Omega_L) \geq h(\Omega) + \frac{v - 4h(\Omega)V}{4|C^*|} \geq h(\Omega) + \frac{v}{8|C^*|} \geq h(\Omega) + \frac{v}{8|\Omega|L}\,, \tag{2.13}$$

so the required estimate is obtained also in this case.

(iv) The case $|C_0^*| \leq v/4$*: when* $V \leq v(8h(\Omega))^{-1}$ *and* $\tau = L$. Being $\tau = L$, one has $|C_t^*| < v/2$ for almost every $t \in [0, L]$. In this case, the estimate $P(C_t^*) \geq \big(h(\Omega)+\varepsilon\big)|C_t^*|$ is true

for every $0 \le t \le L$, and then arguing as usual this time we get, using also that $L \ge 1$,

$$h(\Omega_L) \ge h(\Omega) + \varepsilon \ge h(\Omega) + \frac{\varepsilon}{L}\,. \tag{2.14}$$

We are now ready to conclude. By putting together (2.11), (2.12), (2.13) and (2.14), the claim follows by choosing $c = c(\Omega)$ as

$$c = \min\left\{\frac{v}{8|\Omega|}\,;\,\frac{\varepsilon v}{8h(\Omega)|\Omega|}\,;\,\varepsilon\right\},$$

independently from the Cheeger set C^* chosen for the proof. □

Remark 2.4. In dimension 2, there is only one kind of cylinder: rectangles. In this case, the constant is well-known and there is a formula to compute it depending only on the length of the sides of the rectangle, see the discussion after [8, Theorem 3] together with the correction done in [6, Open problem 1]. Our estimates are obviously consistent with such a formula. In the planar setting, similar estimates have been proved for "strips" (2d waveguides), that one can think of as bended rectangles, refer to [10, Theorem 3.2] and also to [12, Theorem 3.2].

Remark 2.5. In [9] the authors consider unbounded waveguides, that is, roughly speaking cylinders whose spine is the image of a generic unbounded curve γ rather than a straight line. In [9, Remark 1] they essentially prove the upper bound (1.2) for the bounded waveguides $\gamma([0,L]) \oplus B_r$ (topped with two half-balls), while they give a weaker lower bound independent of the length L, see [9, Theorem 1].

3. Application

In this last section we exploit the cylindrical estimates of Theorem 2.1 to prove some properties on the shape functional $F_p[\,\cdot\,]$ defined as

$$F_p[E] := \frac{\lambda_p^{1/p}(E)}{h(E)}\,,$$

for $p \in (1, +\infty]$ with the convention that for $p = +\infty$ we let $\lambda_p^{1/p}(E) = \rho(E)^{-1}$, where this latter denotes the *inradius* of the set E. Throughout the section we shall denote by $\mathbb{K}^N$ the class of convex subsets of $\mathbb{R}^N$. For the sake of notation, we also let

$$\widetilde{m}_N := \inf_{E \subset \mathbb{R}^N} F_p[E]\,, \qquad m_N := \inf_{E \in \mathbb{K}^N} F_p[E]\,,$$

without stressing the dependence on p as this plays no role in the following.

Theorem 3.1. *For any fixed $p \in (1, +\infty]$, the following hold: if there exist bounded minimizers of $F_p[\,\cdot\,]$ among sets*

(i) *in the Euclidean space $\mathbb{R}^N$, then $\widetilde{m}_{N+1} < \widetilde{m}_N$;*
(ii) *in the class of convex sets $\mathbb{K}^N$, then $m_{N+1} < m_N$.*

Proof. We only prove point (i), as the proof of point (ii) is completely analogous. Let us denote by Ω a minimizer of the functional in $\mathbb{R}^N$, and let us consider the cylinders Ω_L with cross-section Ω and height $L \ge 1$.

Let us start with the case $1 < p < +\infty$. The upper bounds to $\lambda_p^{1/p}(\Omega_L)$ proved in [1, Lemma 2.4] imply that for $L \gg 1$, one has

$$\lambda_p^{1/p}(\Omega_L) \leq \lambda_p^{1/p}(\Omega) + O\left(\frac{1}{L^{\min\{p,2\}}}\right).$$

Combining this inequality with the lower bound to $h(\Omega_L)$ in (1.2) for $L \gg 1$ have that

$$\widetilde{m}_{N+1} \leq \frac{\lambda_p^{1/p}(\Omega_L)}{h(\Omega_L)} \leq \frac{\lambda_p^{1/p}(\Omega)}{h(\Omega)} \cdot \frac{1 + O\left(1/L^{\min\{p,2\}}\right)}{1 + \frac{c}{Lh(\Omega)}} < \frac{\lambda_p^{1/p}(\Omega)}{h^p(\Omega)} = \widetilde{m}_N .$$

If otherwise $p = +\infty$, one has that for L large enough $\rho(\Omega) = \rho(\Omega_L)$, and thus one can conclude in the same manner still owing to our main estimate (1.2). □

The above theorem becomes particularly useful when combined with the existence criterion [1, Theorem 3.6], which was first devised in [4, Theorem 1.2] in the case $p = 2$. We stress that the criterion only works when dealing with convex sets. For the sake of convenience, we recall it below along with a sketch of the proof, also highlighting how convexity plays a major role.

Theorem 3.2 (Theorem 3.6 of [1]). *For any fixed $p \in (1, +\infty]$, if $m_{N+1} < m_N$ holds, then there exists a bounded minimizer of $F_p[\,\cdot\,]$ over $\mathbb{K}^{N+1}$.*

Sketch of the proof of Theorem 3.2. First notice that, using the same argument of the proof of Theorem 3.1 with the weaker lower bound (2.7) to $h(\Omega_L)$, one has that $m_{N+1} \leq m_N$.

Second, the key observation behind the criterion is that, refer to [1, Proposition 3.5], if a sequence $\{E_j\}_j$ of equimeasurable $(N+1)$-dimensional sets is such that the sequence of diameters $\{\operatorname{diam} E_j\}_j$ is unbounded, then $m_N \leq \liminf_j F_p[E_j]$. Taking this for granted, if the strict inequality $m_{N+1} < m_N$ holds, one can rule out that minimizing sequence $\{E_j\}_j$ have unbounded diameters. Therefore, and here convexity matters, one can invoke the Blaschke Selection Principle and extract a subsequence converging in the Hausdorff metric to a bounded, convex set, which is easily shown to be a minimizer.

For the sake of completeness, we briefly sketch also the proof of the key observation, which also relies on the convexity of the sets $\{E_j\}_j$.

Fixed any j, up to a translation and a rotation, one can assume that both the origin and the point $(0, \ldots, 0, \operatorname{diam} E_j)$ belong to ∂E_j. We consider the section $\omega_j := E_j \cap \{\, x_{N+1} = t_j \,\}$, chosen as the section attaining

$$\inf_{t \in [0, \operatorname{diam} E_j]} \lambda_p(E_j \cap \{\, x_{N+1} = t \,\}),$$

which exists in virtue of the Hausdorff continuity of the sections $E_j \cap \{\, x_{N+1} = t \,\}$ in $\mathbb{R}^N$ (being E_j convex), and the continuity of λ_p with respect to the Hausdorff metric. We mention that, up to a reflection, one can also assume that

$$t_j \geq \frac{\operatorname{diam} E_j}{2}. \tag{3.1}$$

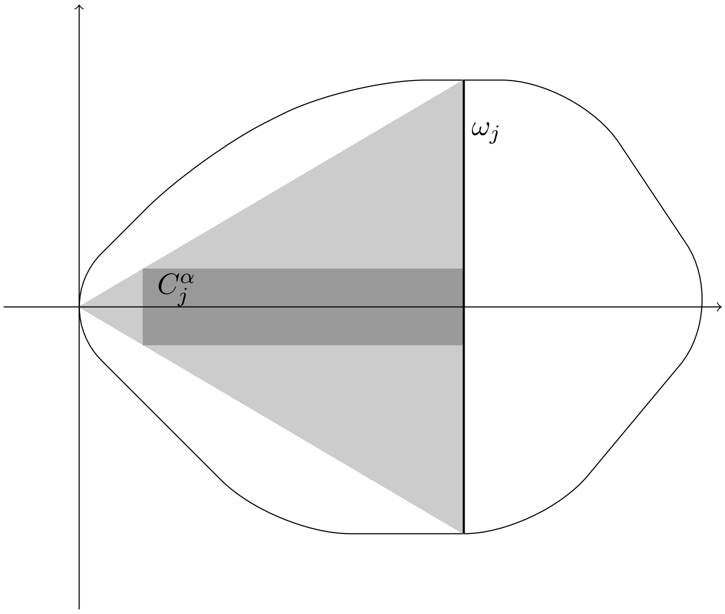

FIGURE 1. The cylinder C_j^α that is used to prove Theorem 3.2.

Owing to the fact that the sections $E_j \cap \{ x_{N+1} = 0 \}$ and $E_j \cap \{ x_{N+1} = \operatorname{diam} E_j \}$ are empty, and owing to Fubini it can be seen that

$$\lambda_p(E_j) \geq \lambda_p(\omega_j)\,, \tag{3.2}$$

refer to [1, Lemma 3.3]. Fixed a parameter $\alpha \in (0,1)$, one now considers the cylinders $C_j^\alpha \subset E_j$ of base $\alpha\omega_j$ and height $t_j(1-\alpha)$ contained in the cone given by the convex envelope of the origin with ω_j, see also Figure 1. By the monotonicity of the Cheeger constant, we then have

$$h(C_j^\alpha) \geq h(E_j). \tag{3.3}$$

Using now (3.2) and (3.3), multiplying and dividing by $h(\omega_j)$, using the scaling properties of the Cheeger constant, and owing to the upper estimate in (1.2) one has

$$F_p[E_j] = \frac{\lambda_p^{1/p}(E_j)}{h(E_j)} \geq \alpha m_N \frac{h(\omega_j)}{h(\omega_j) + \frac{2\alpha}{(1-\alpha)t_j}}\,.$$

Taking now the lim inf as $j \to +\infty$, using (3.1), and then letting $\alpha \to 1$, the claim follows. □

It should be immediately clear that combining this criterion with our Theorem 3.1(ii), it follows that existence of minimizers over $\mathbb{K}^M$ implies existence over $\mathbb{K}^N$ for any $N \geq M$. Thence, to conclude existence in all dimensions it would suffice to prove it for $N = 1$. We show how the induction works and that it can start in the next theorem.

Theorem 3.3. *For any fixed $p \in (1, +\infty]$, the sequence $\{m_N\}$ is strictly decreasing, and there exist bounded minimizers of $F_p[\,\cdot\,]$ in $\mathbb{K}^N$ for any $N \in \mathbb{N}$.*

Proof. Assume that a minimizer exists in $\mathbb{K}^N$. Then Theorem 3.1(ii) implies that the infimum over $\mathbb{K}^{N+1}$ is strictly less than that in $\mathbb{K}^N$. In turn, Theorem 3.2 implies that a minimizer exists over $\mathbb{K}^{N+1}$. Therefore, the existence of minimizers in dimension $N = 1$ would immediately imply both the strictly monotone decreasing behavior of the sequence and the existence of minimizers over $\mathbb{K}^N$ for all $N \in \mathbb{N}$. It is well-known that any interval minimizes the functional over $\mathbb{K}^1$, refer for instance to [1, Proposition 2.1], thus the claim follows. □

Remark 3.4. So far the theorem had only been proved in the 2-dimensional case, and minimizers conjectured to exist in any dimension, and we here give a positive answer. We refer to [16, Proposition 5.2] for $p = 2$, where it is also conjectured that the minimum is attained by the square, and to [1, Theorem 3.8] for general p.

Finally, it is worth noticing that the strict monotonicity proved in Theorem 3.3 implies that Cheeger's inequality can possibly be attained only asymptotically. It is easy to see that this is the case and that for any p a sequence of N-dimensional unit balls $\{B_1^N\}$ achieves the equality in the limit. In particular, for $p = 2$, this follows from the explicit knowledge of the first eigenvalue of the N-dimensional ball and its asymptotic behavior, refer to [17] and the computations carried out in [4, Theorem 1.2]. For general p it follows from the estimates of [1, Lemma 2.3] and the computations carried out in [1, Theorem 2.6], which give

$$\frac{1}{p} \le m_N \le F_p[B_1^N] \sim \frac{1}{p}.$$

Thus, the following holds.

Corollary 3.5. *Cheeger's inequality $F_p[E] \ge \frac{1}{p}$, among convex sets $E \in \mathbb{K}^N$, is saturated asymptotically.*

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