A lower semicontinuity result for polyconvex functionals in SBV

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Abstract

We prove a semicontinuity theorem for an integral functional made up by a polyconvex energy and a surface term. Our result extends to the BV framework a well known result by John Ball.

1 Introduction

In this paper we study the lower semicontinuity of the integral functional

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{J_u} \varphi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{1.1}$$

in $SBV(\Omega, \mathbb{R}^m)$, the space of special functions of bounded variation (see [2, 3, 4, 7]), with respect to the $L^1(\Omega, \mathbb{R}^m)$ topology. Here $\Omega \subset \mathbb{R}^n$ is an open set, f is a Carathéodory function, J_u is the jump set of u, ν_u is the measure theoretic normal to J_u, and u^+, u^- are the one-sided traces of u on both sides of J_u (see Section 2).

We assume that the integrand $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to [0, \infty)$ is polyconvex in the last variable, and that satisfies the growth condition

$$f(x, u, A) \geq \sum_{k=1}^{n\wedge m} \beta_k |\mathrm{adj}_k A|^{p_k} \qquad \text{for every } A \in \mathbb{R}^{nm}\,, \tag{1.2}$$

where $\beta_k > 0$ for all k and

$$p_1 \geq 2, \qquad p_k \geq \frac{p_1}{p_1 - 1} \quad \text{if } k = 2, \dots, n \wedge m - 1, \qquad p_{n\wedge m} > 1\,,$$

where $\mathrm{adj}_k A$ denotes the vector whose components are the minors of the matrix A of order k.

Then, if the function φ fulfils the conditions of Theorem 3.1, we prove in Theorem 3.5 that the functional F in (1.1) is $L^1(\Omega, \mathbb{R}^m)$ lower semicontinuous in $SBV(\Omega, \mathbb{R}^m)$.

Theorem 3.5 extends a result by Ambrosio in [4] where, for polyconvex functions f, the lower semicontinuity of the functional F is proved under the growth assumption

$$f(x, u, A) \geq \beta_1 |A|^{p_1}\,, \qquad \beta_1 > 0\,, \quad p_1 > n \wedge m \tag{1.3}$$

for every $A \in \mathbb{R}^{nm}$ (see [4], Corollary 4.9).

The study of this problem was motivated by the analysis of variational models for fracture mechanics. In particular, in the framework of Griffith's theory of fractures in brittle materials, the energy required to produce a crack is proportional to the crack surface. Moreover, the elastic deformation of the material outside the crack is modeled by an elastic energy independent of the crack [17]. A weak formulation for the energy has been introduced in [6] in the framework of the theory of SBV functions. The simplest energy functional in the isotropic case takes the form

$$F(u) = \int_\Omega f(\nabla u)\,dx + \eta\mathcal{H}^2(J_u)\,, \tag{1.4}$$

where $n = m = 3$, Ω now denotes the reference configuration of an elastic body, and f denotes the stored energy function of the elastic material. The fracture toughness $\eta > 0$ is a constant given by Griffith's criterion for fracture initiation (see references in [6]). If $u \in SBV(\Omega, \mathbb{R}^m)$, the gradient ∇u and the jump set J_u are defined in the sense of approximate limits (see Section 2). Hence, in the context of fracture mechanics, ∇u and J_u provide weak notions of the deformation gradient and of the fracture surface, respectively.

A relevant case is when f is the stored energy function of an hyperelastic material. If the material is isotropic, there exists [11] a symmetric function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ satisfying $f(\nabla u) = \Phi(v_1, v_2, v_3)$, where v_i denote the eigenvalues of the matrix $[(\nabla u)(\nabla u)^T]^{1/2}$, where $(\nabla u)^T$ is the transpose of ∇u.

A large class of hyperelastic materials are the Ogden materials [11, 19] which are expressed as

$$\begin{aligned}\Phi(v_1, v_2, v_3) \;&=\; \sum_{i=1}^{M} a_i(v_1^{\alpha_i} + v_2^{\alpha_i} + v_3^{\alpha_i}) \\ &+\; \sum_{j=1}^{N} b_j((v_1v_2)^{\delta_j} + (v_1v_3)^{\delta_j} + (v_2v_3)^{\delta_j}) + \psi(v_1v_2v_3)\,,\end{aligned}$$

where $a_i > 0, \alpha_i > 1$ for $i \in \{1, \dots, M\}$, $b_j > 0, \delta_j > 1$ for $j \in \{1, \dots, N\}$, and $\psi : (0, +\infty) \to \mathbb{R}$ is convex and satisfies

$$\lim_{s\to 0^+} \psi(s) = \lim_{s\to +\infty} \psi(s) = +\infty.$$

A variational model of brittle fracture based on the Ogden materials has been considered in [15]. In [11] it has been proved that the stored energy function of Ogden materials is polyconvex and satisfies the inequality

$$f(A) \geq \beta_1|A|^{p_1} + \beta_2|\mathrm{adj}_2 A|^{p_2} + \psi(\det A) \qquad \text{for every } A \in M^{3\times 3}\,,$$

where

$$\beta_1, \beta_2 > 0\,, \qquad p_1 = \max_{i=1,\dots,M} \alpha_i\,, \qquad p_2 = \max_{j=1,\dots,N} \delta_j\,. \tag{1.5}$$

In particular, if $\alpha_i \geq 2$ and $\delta_i \geq \alpha_i/(\alpha_i - 1)$, then the stored energy function of Ogden materials satisfies the growth condition (1.2) (see also Remark 3.6).

By using our lower semicontinuity result and the compactness theorem in SBV due to Ambrosio [2, 5], an existence result follows for boundary value problems in fracture mechanics. For instance, in [6] the Dirichlet boundary conditions have been formulated penalizing the possible fracture at the boundary. The weak formulation of the minimum problem for the functional (1.4) takes the form

$$(\mathcal{P}) \qquad \min\left\{\int_\Omega f(\nabla u)\,dx + \eta\mathcal{H}^2(J_u) + \eta\mathcal{H}^2(J_{u\cup u_0}) :\ u \in SBV(\Omega, \mathbb{R}^3)\right\}\,,$$

where Ω is a bounded Lipschitz set, u_0 is the boundary datum which is assumed to be the outer trace on $\partial\Omega$ of a function belonging to $SBV(\Omega', \mathbb{R}^3)$ with $\Omega \subset\subset \Omega'$, and $J_{u\cup u_0} = \{x \in \partial\Omega : u^+(x) \neq u_0(x)\}$, $u^+(x)$ being the inner trace of u at $x \in \partial\Omega$ [6].

Using the growth condition (1.3) required by the semicontinuity theorem of [4], an existence result for problem $(\mathcal{P})$ is given in [6] under the assumption $p_1 > 3$, which means $\max_i \alpha_i > 3$ in the framework of Ogden materials. However, it is desirable that the exponents α_i can be lowered in such a way that $\max_i \alpha_i = 2$ in order to cover the important subclass of Ogden materials given by the Mooney-Rivlin compressible materials, characterized by $M = N = 1$, $\alpha_1 = \delta_1 = 2$ [11, 12].

Using (1.2) and (1.5) our semicontinuity theorem permits us to use $\max_i \alpha_i = 2$, so that it improves the results obtained in [4, 6]. It should also be noted that the resulting application in fracture mechanics represents the counterpart in the SBV framework of the results of Ball [8] in nonlinear elasticity.

Finally the semicontinuity theorem has some implications for numerical computations. The numerical solution of variational problems involving the functional (1.1) is a difficult task due to the presence of the jump part of the energy. The variational approximation by means of Γ-convergence [10, 14] has been used to design numerical methods for problems in fracture mechanics when the bulk energy is modeled by linear elasticity [9]. A Γ-convergence theorem has been proved in [16] for the approximation of the functional F by means of a family of elliptic functionals which admit a discretization by means of finite elements. The proof of the $\liminf$ inequality of Γ-convergence in [16] is based on the lower semicontinuity theorem of [4] and then makes use of the same growth condition. By using our semicontinuity theorem the validity of the $\liminf$ inequality is consequently extended to the condition (1.2). Numerical experiments on brittle fracture in a class of Ogden materials has been performed in [15], by using the elliptic approximation of [16].

2 Definitions and preliminaries

In this section we recall some basic definitions and preliminary results on BV and SBV functions. We refer to [7] for all the properties of BV and SBV functions used in the sequel.

For any $x \in \mathbb{R}^n$ and any $\rho > 0$, $B_\rho(x)$ denotes the ball of radius ρ centered in x. However, we write simply B_ρ when the ball is centered at the origin. The unit sphere ∂B_1 is denoted by $\mathbb{S}^{n-1}$. In the following Ω will always denote an open subset of $\mathbb{R}^n$. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of the Borel subsets of Ω; for any $E \in \mathcal{B}(\Omega)$, $\mathcal{L}^n(E)$ stands for the Lebesgue measure of E. We set $\omega_n = \mathcal{L}^n(B_1)$.
As usual, the space $M^{n\times m}$ of $n \times m$ matrices is identified with the euclidean space $\mathbb{R}^{nm}$.

Let u be a function in $L^1_{\rm loc}(\Omega, \mathbb{R}^m)$. We say that u is *approximately continuous* at the point $x \in \Omega$ if there exists $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |u(y) - \tilde{u}(x)|\, dy = 0\ .$$

The set C_u of all points where u is approximately continuous is a Borel set. Let $S_u := \Omega \setminus C_u$. We say that a point $x \in S_u$ is an *approximate jump point* for u if there exist $u^+(x), u^-(x) \in \mathbb{R}^m$ and $\nu_u(x) \in \mathbb{S}^{n-1}$ such that $u^-(x) \neq u^+(x)$ and

$$\lim_{r\to 0} \fint_{B_r^+(x;\nu_u(x))} |u(y) - u^+(x)|\, dy = 0, \qquad \lim_{r\to 0} \fint_{B_r^-(x;\nu_u(x))} |u(y) - u^-(x)|\, dy = 0\ ,$$

where $B_r^+(x;\nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B_r^-(x;\nu_u(x))$ is defined similarly. Also the set $J_u \subset S_u$ of all the approximate jump points is a Borel set. Moreover, $\nu_u(x)$ can be oriented in such a way that the function $(u^+, u^-, \nu_u) : J_u \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1}$ is a Borel function.

Given a point $x \in C_u$, we say that u is *approximately differentiable* at x if there exists $\nabla u(x) \in \mathbb{R}^{nm}$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^{n+1}} \int_{B_\rho(x)} |u(y) - \tilde{u}(x) - \nabla u(x)(y-x)| \, dy = 0 \, .$$

The vector $\nabla u(x)$ is called the *approximate differential* of u at x. The set of points in C_u where the approximate differential of u exists is a Borel set denoted by $\mathcal{D}_u$.

A function $u \in L^1(\Omega, \mathbb{R}^m)$ is said to be of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^{nm}$-vector valued measure and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega, \mathbb{R}^m)$. If $u \in BV(\Omega, \mathbb{R}^m)$, $D^a u$ denotes the *absolutely continuous* part of Du with respect to the Lebesgue measure $\mathcal{L}^n$. The singular part $D^s u$ can be split in two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \qquad D^c u = D^s u - D^j u \, .$$

Furthermore, it can be proved that

$$D^a u = \nabla u \mathcal{L}^n \llcorner \mathcal{D}_u, \qquad D^c u = Du \llcorner (C_u \setminus \mathcal{D}_u), \qquad D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u \, ,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ (see [7, Proposition 3.92]).

In the sequel we are going to use the notion of *weak* convergence in BV*. Let $u_h, u \in BV(\Omega, \mathbb{R}^m)$. We say that $\{u_h\}$ converges weakly* in $BV(\Omega, \mathbb{R}^m)$ to u if $u_h \to u$ in $L^1(\Omega, \mathbb{R}^m)$ and the derivatives Du_h converge weakly* in the sense of measure to Du, i.e.

$$\lim_{h \to \infty} \int_\Omega \phi \, dDu_h = \int_\Omega \phi \, dDu \qquad \text{for all } \phi \in C_0(\Omega) \, .$$

Finally, we say that $u \in BV(\Omega, \mathbb{R}^m)$ is a *special function of bounded variation*, if the Cantor part $D^c u$ of its derivative is null. The space of special functions of bounded variation is denoted by $SBV(\Omega, \mathbb{R}^m)$. We recall the following compactness result in SBV due to Ambrosio (see [7, Theorem 4.8]).

Theorem 2.1 *Let Ω be bounded and $\{u_h\}$ a sequence in $SBV(\Omega, \mathbb{R}^m)$ such that*

$$\sup_{h \in \mathbb{N}} \left\{ \|u_h\|_{L^\infty(\Omega, \mathbb{R}^m)} + \int_\Omega \psi(|\nabla u_h|) \, dx + \mathcal{H}^{n-1}(J_{u_h}) \right\} < \infty \, ,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous, increasing function such that $\lim_{t \to \infty} \psi(t)/t = \infty$. Then, there exists a subsequence $\{u_{h_j}\}$ weakly converging in $BV(\Omega, \mathbb{R}^m)$ to a function $u \in SBV(\Omega, \mathbb{R}^m)$. Moreover, the approximate gradients ∇u_{h_j} converge weakly in $L^1(\Omega, \mathbb{R}^{nm})$ to ∇u.*

Let μ be a nonnegative, finite Radon measure in B_1. The *local maximal function* $M(\mu)$ is defined by

$$M(\mu)(x) = \sup \left\{ \frac{\mu(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} : 0 < \rho < 1 - |x| \right\}.$$

If μ is absolutely continuous with respect to $\mathcal{L}^n$ and ϕ is its density, we write simply $M(\phi)$ in place of $M(\phi \mathcal{L}^n)$. Moreover, if ϕ belongs to $L^p(B_1)$ for some $p > 1$, then (see [20])

$$\int_{B_1} M^p(\phi) \, dx \le c \int_{B_1} \phi^p \, dx \, , \tag{2.1}$$

for a suitable constant c depending only on n and p.

Using maximal functions, one can get a Lusin-type approximation of BV functions by means of Lipschitz functions. Next theorem, due to Ambrosio (see [4, Theorem 2.3]), is the BV counterpart of a similar result proved for $W^{1,p}$ functions in [1].

Theorem 2.2 *Let u be a function from $BV(B_1, \mathbb{R}^m) \cap L^\infty(B_1, \mathbb{R}^m)$ and $\lambda > 0$. Set*

$$E := \{x \in B_1 : M(|Du|)(x) \leq \lambda\}.$$

Then, for any $\rho \in (0,1)$, there exists a Lipschitz function $v : B_\rho \to \mathbb{R}^m$ such that $u(x) = v(x)$ for $\mathcal{L}^n$-a.e. $x \in E \cap B_\rho$ and

$$\mathrm{Lip}(v, B_\rho) \leq c(n)m\lambda + \frac{2m\|u\|_\infty}{1-\rho},$$

for some constant $c(n)$, depending only on n. Moreover, if $u \in SBV(B_1)$ and $|\nabla u| \in L^p(B_1)$ for some $p > 1$, then, for any $C \in \mathcal{B}(B_1)$,

$$\mathcal{L}^n(\{x \in C : M(|Du|)(x) > 2\lambda\}) \leq \lambda^{-p} \int_{C \cap \{M(|\nabla u|) > \lambda\}} M^p(|\nabla u|)\, dx + \frac{2c(n)\|u\|_{L^\infty(B_1)}}{\lambda} \mathcal{H}^{n-1}(J_u).$$

Next result, also known as Chacon's biting lemma, allows to recover some equi-integrability from a sequence which is only bounded in L^1 (see e.g. [7, Lemma 5.32]).

Lemma 2.3 *Let Ω be bounded and let $\{\phi_h\}$ be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$. Then, there exists a subsequence $\{\phi_{h_j}\}$ and a decreasing sequence $E_r \subset \mathcal{B}(\Omega)$, such that $\mathcal{L}^n(E_r) \to 0$ as $r \to \infty$ and the sequence $\{\phi_{h_j}\}$ is equi-integrable in $\Omega \setminus E_r$ for any $r \in \mathbb{N}$.*

Let us state also the following simple lemma (see Theorem 1.2 of [2]).

Lemma 2.4 *Let $\{\phi_h\} \subset L^1(\Omega, \mathbb{R}^m)$ be an equi-integrable sequence, $\phi \in L^1(\Omega, \mathbb{R}^m)$, and let us assume that*

$$\liminf_{h \to +\infty} \int_\Omega |\phi_h - w|\, dx \geq \int_\Omega |\phi - w|\, dx,$$

for every $w \in L^1(\Omega, \mathbb{R}^m)$. Then ϕ_h weakly converges to ϕ in $L^1(\Omega, \mathbb{R}^m)$.

We recall that a function $f : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is said to be a *Carathéodory function* if $x \longmapsto f(x,p)$ is measurable for any $p \in \mathbb{R}^k$ and $p \longmapsto f(x,p)$ is continuous for $\mathcal{L}^n$-a.e. $x \in \Omega$. In this paper we deal with Carathéodory functions $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$, $n, m > 1$, so that $k = m + nm$. Namely, we consider polyconvex functions in the third variable.

We say that a function $f : \mathbb{R}^{nm} \to \mathbb{R}$ is *polyconvex* if there exists a convex function $g : \mathbb{R}^\tau \to \mathbb{R}$ such that

$$f(A) = g(\mathcal{M}(A)) \qquad \text{for all } A \in \mathbb{R}^{nm},$$

where $\mathcal{M}(A)$ is the vector whose components are all the minors of the matrix A and $\tau = \tau(n,m)$ is the dimension of $\mathcal{M}(A)$. For $k = 1, \dots, n \wedge m$, we denote by $\mathrm{adj}_k A$ the vector whose components are the minors of the matrix A of order k. We denote the dimension of $\mathrm{adj}_k A$ by τ_k. Notice that $\tau_k = \binom{n}{k}\binom{m}{k}$.

We recall that a polyconvex function $f : \mathbb{R}^{nm} \to \mathbb{R}$ is also *quasiconvex*, i.e.

$$⨍_{B_1} f(A + \nabla\varphi)\, dx \geq f(A) \qquad \text{for any } A \in \mathbb{R}^{nm} \text{ and any } \varphi \in C_0^1(B_1, \mathbb{R}^m).$$

The following lower semicontinuity result is well known. A proof can be found, e.g. in [18] (see Theorem 4.5).

Theorem 2.5 *Let $g(x, v, z)$ be a nonnegative Carathéodory function from $\Omega \times \mathbb{R}^m \times \mathbb{R}^k$, convex in z for $\mathcal{L}^n$-a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^m$. Let $v_h, v \in L^1(\Omega, \mathbb{R}^m)$, $z_h, z \in L^1(\Omega, \mathbb{R}^k)$. If v_h converges to v strongly in $L^1(\Omega, \mathbb{R}^m)$ and z_h converges to z weakly in $L^1(\Omega, \mathbb{R}^k)$, then*

$$\int_\Omega g(x, v(x), z(x))\, dx \leq \liminf_{h \to \infty} \int_\Omega g(x, v_h(x), z_h(x))\, dx.$$

3 The main result

We study the lower semicontinuity with respect to the $L^1(\Omega,\mathbb{R}^m)$ topology of the integral functional defined in $SBV(\Omega,\mathbb{R}^m)$ by

$$F(u)=\int_\Omega f(x,u,\nabla u)\,dx+\int_{J_u}\varphi(u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}\,. \tag{3.1}$$

We assume that the integrand $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^{nm}\to[0,\infty)$ is polyconvex in the last variable, namely that

$$f(x,u,A)=g(x,u,\mathcal{M}(A))\qquad\text{for every } A\in\mathbb{R}^{nm}\,, \tag{3.2}$$

where $g(x,u,z):\Omega\times\mathbb{R}^m\times\mathbb{R}^\tau\to[0,\infty)$ is a Carathéodory function, convex in z for $\mathcal{L}^n$-a.e. $x\in\Omega$ and for every $u\in\mathbb{R}^m$. Moreover, we assume that

$$f(x,u,A)\geq\sum_{k=1}^{n\wedge m}\beta_k|\mathrm{adj}_kA|^{p_k}\qquad\text{for every } A\in\mathbb{R}^{nm}\,, \tag{3.3}$$

where $\beta_k>0$ for all k and the exponents p_k satisfy the following inequalities

$$p_1\geq 2,\qquad p_k\geq\frac{p_1}{p_1-1}\quad\text{if } k=2,\dots,n\wedge m-1,\qquad p_{n\wedge m}>1\,. \tag{3.4}$$

Concerning the surface integral in (3.1), we assume that $\varphi:\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{S}^{n-1}\to[0,\infty)$ is *jointly convex*, i.e. there exists a sequence of continuous functions $g_h:\mathbb{R}^m\to\mathbb{R}^n$ such that

$$\varphi(u,v,\nu)=\sup_{h\in\mathbb{N}}\langle g_h(u)-g_h(v),\nu\rangle\qquad\text{for all } (u,v,\nu)\in\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{S}^{n-1}\,. \tag{3.5}$$

Examples of jointly convex functions are easily constructed (see [7, Example 5.23]) by taking functions of the the form $\varphi(u,v,\nu)=\theta(|u-v|)\psi(\nu)$, where $\theta:\mathbb{R}\to[0,+\infty)$ is a lower semicontinuous, increasing and subadditive function and $\psi:\mathbb{R}^n\to[0,+\infty)$ is an even, positively 1-homogeneous convex function.

Theorem 3.1 *Let $\varphi:\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{S}^{n-1}\to[0,\infty)$ be a jointly convex function with the property that there exist $M>0$ and $\gamma>0$ such that*

$$\varphi(u,v,\nu)\geq\gamma\qquad\textit{for all } u,v\in B_M,\ u\neq v,\ \textit{and every } \nu\in\mathbb{S}^{n-1}\,. \tag{3.6}$$

Let $\{u_h\}$ be a a sequence in $SBV(\Omega,\mathbb{R}^m)$ converging in $L^1(\Omega,\mathbb{R}^m)$ to $u\in SBV(\Omega,\mathbb{R}^m)$, such that $\{|\nabla u_h|\}$ is equi-integrable and, for any $h\in\mathbb{N}$, $|u_h(x)|<M$ for $\mathcal{L}^n$-a.e. $x\in\Omega$. Then

$$\int_{J_u}\varphi(u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}\leq\liminf_{h\to\infty}\int_{J_{u_h}}\varphi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}^{n-1}\,.$$

For the proof of Theorem 3.6 see [7, Theorem 5.22]. The next result can be found in [13] (see Theorem 2.10 in Chapter 4).

Theorem 3.2 *Let Ω be bounded and u_h,u functions from $W^{1,2}(\Omega,\mathbb{R}^m)$ such that $u_h\rightharpoonup u$ weakly in $W^{1,2}(\Omega,\mathbb{R}^m)$. Assume that*

$$\sup_{h\in\mathbb{N}}\sum_{k=1}^{n\wedge m}\int_\Omega|\mathrm{adj}_k\nabla u_h|^{p_k}\,dx<\infty$$

and that the exponents p_k satisfy (3.4). Then, for all $k=1,\dots,n\wedge m$, $\mathrm{adj}_k\nabla u_h\rightharpoonup\mathrm{adj}_k\nabla u$ weakly in $L^{p_k}(\Omega,\mathbb{R}^{\tau_k})$.

Remark 3.3 Notice that there are cases when the conclusion of Theorem 3.2 holds if $p_{n\wedge m} = 1$. This is certainly true if $p_1 > n \wedge m$, since in this case the sequence $\{\mathrm{adj}_{n\wedge m} \nabla u_h\}$ is bounded in $L^{\frac{p_1}{n\wedge m}}(\Omega, \mathbb{R}^{\tau_{n\wedge m}})$ and $p_1/n \wedge m > 1$.
Assume now $p_1 < n \wedge m$. From the estimate

$$
|\mathrm{adj}_k A| \leq C_k |\mathrm{adj}_{k-1} A|^{\frac{k}{k-1}} , \tag{3.7}
$$

which holds for any $A \in \mathbb{R}^{nm}$ and any $k = 2, \dots, n \wedge m$, with a suitable constant C_k depending only on k, n, m, we get that the sequence $\{\mathrm{adj}_{n\wedge m} \nabla u_h\}$ is bounded in $L^q(\Omega, \mathbb{R}^{\tau_{n\wedge m}})$, where $q = \frac{n\wedge m-1}{n\wedge m} p_{n\wedge m-1}$ is strictly bigger than 1. Hence the conclusion of Theorem 3.2 still holds in this case. In the limit case $p_1 = n \wedge m$, using (3.7) again, we get the same conclusion by assuming that $p_{n\wedge m-1} > n \wedge m/(n \wedge m - 1)$.
It can be shown by easy counterexamples that these assumptions are essentially sharp, though they can be replaced by completely different ones, based on other algebraic properties of minors.

Our main result is stated in the next theorem which is the SBV counterpart of Theorem 3.2.

Theorem 3.4 *Let Ω be bounded and u_h, u functions from $SBV(\Omega, \mathbb{R}^m)$ such that $u_h \to u$ in $L^1(\Omega, \mathbb{R}^m)$. Assume that*

$$
\sup_{h\in\mathbb{N}} \Big\{ \|u_h\|_{L^\infty(\Omega,\mathbb{R}^m)} + \sum_{k=1}^{n\wedge m} \int_\Omega |\mathrm{adj}_k \nabla u_h|^{p_k}\, dx + \mathcal{H}^{n-1}(J_{u_h}) \Big\} < \infty\,, \tag{3.8}
$$

where the exponents p_k satisfy (3.4). Then, for all $k = 1, \dots, n \wedge m$, $\mathrm{adj}_k \nabla u_h \rightharpoonup \mathrm{adj}_k \nabla u$ weakly in $L^{p_k}(\Omega, \mathbb{R}^{\tau_k})$.

Notice that when u is a Sobolev map, all minors of ∇u can be written in divergence form and thus the weak convergence of $\mathrm{adj}_k \nabla u_h$ to $\mathrm{adj}_k \nabla u$ in $\mathcal{D}'(\Omega)$ is obtained via an integration by parts. This argument clearly fails when the functions u_h, u are assumed to be in SBV and this is indeed the main difficulty one has to face in order to prove Theorem 3.4.
Thus, the idea of the proof is to show that, for any function w in L^1, the functional

$$
u \mapsto \int_\Omega |\mathrm{adj}_k \nabla u(x) - w(x)|\, dx \tag{3.9}
$$

is L^1-lower semicontinuous in $SBV(\Omega, \mathbb{R}^m)$ and then to apply Lemma 2.4 to deduce the weak convergence of the minors $\mathrm{adj}_k \nabla u_h$ to $\mathrm{adj}_k \nabla u$. To this aim, we start with the observation that if all u_h and u were Sobolev functions then the lower semicontinuity of functional (3.9) would follow from Theorems 3.2 and 2.5 and then we deduce the general case with an induction argument based on the number of components of the functions u_h which are in SBV.

¿From the weak convergence result stated in Theorem 3.4, making use of Theorems 2.5 and 3.1 one gets at once the lower semicontinuity of polyconvex integrals of the type (1.1).

Theorem 3.5 *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to [0, +\infty)$ be a Carathéodory function, satisfying the assumptions (3.2), (3.3) and (3.4), and let φ as in Theorem 3.1. If $u_h, u \in SBV(\Omega, \mathbb{R}^m)$, $u_h \to u$ in L^1 and $\sup_h \|u_h\|_{L^\infty(\Omega,\mathbb{R}^m)} < M$, then*

$$
F(u) \leq \liminf_{h\to+\infty} F(u_h)\,.
$$

Remark 3.6 The observations made in Remark 3.3 yield that Theorem 3.5 still holds if $p_{n\wedge m} = 1$, provided that either $p_1 \neq n \wedge m$ or $p_1 = n \wedge m$ and $p_{n\wedge m-1} > n \wedge m/(n \wedge m - 1)$.

4 Proof of the semicontinuity theorem

This section is devoted to the proof of Theorem 3.4. As we already observed in the previous section, Theorem 3.5 is a straightforward consequence of Theorems 2.5, 3.1 and 3.4. We leave the details to the reader.

First, we deal with a simpler situation in which the limit function u is affine and the jump sets of the approximating functions u_h vanish as $h \to \infty$. The general case will be then recovered by means of a blow-up argument.

Lemma 4.1 *Let u_h and u satisfy the same assumptions of Theorem 3.4 and let $\Omega = B_1$. Moreover, assume that u is an affine function, i.e. $u(x) = Ax + b$, for some $A \in \mathbb{R}^{nm}$, $b \in \mathbb{R}^m$, and that $\mathcal{H}^{n-1}(J_{u_h}) \to 0$ as $h \to \infty$. Then, for every $k = 1, \ldots, n \wedge m$, any nonnegative function $a \in L^\infty(B_1)$ and any function $w \in L^1(B_1, \mathbb{R}^{\tau_k})$,*

$$\int_{B_1} a(x)|\mathrm{adj}_k \nabla u - w(x)|\, dx \le \liminf_{h\to\infty} \int_{B_1} a(x)|\mathrm{adj}_k \nabla u_h - w(x)|\, dx\,. \tag{4.1}$$

Though differently stated, conclusion (4.1) of Lemma 4.1 is equivalent to saying that for all $k = 1, \ldots, n \wedge m$

$$\mathrm{adj}_k \nabla u_h \rightharpoonup \mathrm{adj}_k \nabla u \qquad \text{weakly in } L^{p_k}(B_1, \mathbb{R}^{\tau_k})\,. \tag{4.2}$$

In fact, (4.2) immediately implies (4.1) by semicontinuity (apply for instance Theorem 2.5 with $g(x,z) = a(x)|z - w(x)|$, $z_h = \mathrm{adj}_k \nabla u_h$ and $z = \mathrm{adj}_k \nabla u$). Viceversa, since for all k the sequence $\{\mathrm{adj}_k \nabla u_h\}$ is bounded in $L^{p_k}(B_1, \mathbb{R}^{\tau_k})$, hence it is equi-integrable, the validity of (4.1) for all functions w yields, via Lemma 2.4, that $\mathrm{adj}_k \nabla u_h \rightharpoonup \mathrm{adj}_k \nabla u$ weakly in $L^1(B_1, \mathbb{R}^{\tau_k})$. Then, a simple compactness argument shows that this weak convergence holds indeed in $L^{p_k}(B_1, \mathbb{R}^{\tau_k})$. However, since the proof of Lemma 4.1 uses an induction argument, having the induction assumption stated in the form (4.1) makes the argument of the proof somewhat cleaner.

Proof of Lemma 4.1. From the assumption (3.8), using Theorem 2.1 and a standard compactness argument, we get easily that $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^{p_1}(B_1, \mathbb{R}^{nm})$. Hence, (4.1) follows, for $k = 1$, by the same semicontinuity argument used above to show that (4.2) implies (4.1).
Let us fix $k \ge 2$. To prove (4.1), we use an induction argument on the number of components of u_h belonging to $W^{1,p_1}(B_1)$. In fact, if all components of each function u_h belong to $W^{1,p_1}(B_1)$, Theorem 3.2 yields that $\mathrm{adj}_k \nabla u_h \rightharpoonup \mathrm{adj}_k \nabla u$ in $L^{p_k}(B_1, \mathbb{R}^{\tau_k})$. Hence, (4.1) follows again by semicontinuity.
Assume now that (4.1) holds true whenever the last $m - j$ components of each u_h belong to $W^{1,p_1}(B_1)$, for some $j = 0, \ldots, m-1$. We are now going to prove that (4.1) still holds if only the last $m - j - 1$ components are in $W^{1,p_1}(B_1)$, i.e. $u_h^1, \ldots, u_h^{j+1} \in SBV(B_1)$, while $u_h^{j+2}, \ldots, u_h^m \in W^{1,p_1}(B_1)$. To this aim, let us fix w and let us assume, without loss of generality, that the lim inf on the right-hand side of (4.1) is indeed a limit and that $u_h(x) \to u(x)$ for $\mathcal{L}^n$-a.e. $x \in B_1$.
From the assumption (3.8) and from the estimate (2.1) it follows that $\{M^{p_1}(|\nabla u_h^{j+1}|)\}$ is bounded in $L^1(B_1)$, hence Lemma 2.3 applies. Therefore, passing to a (not relabelled) subsequence, we may assume that for any $\varepsilon > 0$ there exist a Borel set $C_\varepsilon \subset B_1$ and a positive number $\delta < \omega_n$ such that for any Borel set $C \subset B_1 \setminus C_\varepsilon$, with $\mathcal{L}^n(C) < \delta$, we have for all h

$$\int_C M^{p_1}(|\nabla u_h^{j+1}|)\, dx < \varepsilon\,. \tag{4.3}$$

Let us fix $\varepsilon > 0$ and $\rho \in (0,1)$ such that $\mathcal{L}^n(B_\rho) > \delta$. For any h and any $\lambda > 0$, let us denote by $u^{j+1}_{h,\lambda}$ the Lipschitz approximation of u^{j+1}_h in B_ρ provided by Theorem 2.2. Hence $u^{j+1}_{h,\lambda}(x) = u^{j+1}_h(x)$ $\mathcal{L}^n$-a.e in $B_\rho \setminus E_{h,\lambda}$, where

$$E_{h,\lambda} = \{x \in B_1 : M(|Du^{j+1}_h|) > \lambda\}\,.$$

Moreover,

$$\mathrm{Lip}(u^{j+1}_{h,\lambda}, B_\rho) \le c_1\Big(\lambda + \frac{1}{1-\rho}\Big) \tag{4.4}$$

and, for every $C \in \mathcal{B}(B_1)$,

$$\mathcal{L}^n(E_{h,\lambda} \cap C) \le \Big(\frac{2}{\lambda}\Big)^{p_1} \int_{C\cap\{M(|\nabla u^{j+1}_h|)>\lambda/2\}} M^{p_1}(|\nabla u^{j+1}_h|)\,dx + \frac{c_1}{\lambda}\mathcal{H}^{n-1}(J_{u^{j+1}_h})\,, \tag{4.5}$$

where c_1 is a constant depending only on n and $\sup_h \|u_h\|_{L^\infty(B_1,\mathbb{R}^m)}$. Notice that from (4.5) and (2.1) it follows that there exists λ_ε such that $\mathcal{L}^n(E_{h,\lambda}) < \delta$, for all $\lambda > \lambda_\varepsilon$ and every h. Therefore, from (4.3) we get in particular that

$$\int_{E_{h,\lambda}\setminus C_\varepsilon} M^{p_1}(|\nabla u^{j+1}_h|)\,dx < \varepsilon \qquad \text{for all } \lambda > \lambda_\varepsilon \text{ and every } h \in \mathbb{N}\,. \tag{4.6}$$

Let us fix also $\lambda > \max\{\lambda_\varepsilon, 1\}$. From (4.4) and from the fact that $|u^{j+1}_{h,\lambda}(x)| \le M$ in $B_\rho \setminus E_{h,\lambda}$ (which is not empty, since $\mathcal{L}^n(B_\rho) > \delta > \mathcal{L}^n(E_{h,\lambda})$), it follows that $\{u^{j+1}_{h,\lambda}\}$ is bounded in $L^\infty(B_\rho)$. Therefore, passing possibly to another (and again not relabelled) subsequence, we may assume that $\{u^{j+1}_{h,\lambda}\}$ converges weakly* in $W^{1,\infty}(B_\rho)$ to a Lipschitz function u^{j+1}_λ. Moreover, since for any $h \in \mathbb{N}$

$$\mathcal{L}^n(\{x \in B_\rho : u^{j+1}_{h,\lambda} \ne u^{j+1}_h\}) \le \mathcal{L}^n(E_{h,\lambda})\,,$$

by the lower semicontinuity of the functional $v \to \mathcal{L}^n(\{x \in B_\rho : v(x) \ne 0\})$ with respect to the $\mathcal{L}^n$-a.e. convergence, the a.e. convergence of u_h to u and (4.5), we get that

$$\mathcal{L}^n(\{x \in B_\rho : u^{j+1}_\lambda \ne u^{j+1}\}) \le \frac{c_2}{\lambda}\,, \tag{4.7}$$

where c_2 is a positive constant independent of h, λ and ε.
We can now estimate, for any $h \in \mathbb{N}$,

$$\begin{aligned}\int_{B_1} a(x)|\mathrm{adj}_k\nabla u_h - w(x)|dx &\ge \int_{(B_\rho\setminus E_{h,\lambda})\setminus C_\varepsilon} a(x)|\mathrm{adj}_k(\nabla u^1_h, \dots, \nabla u^j_h, \nabla u^{j+1}_{h,\lambda}, \dots, \nabla u^m_h) - w(x)|dx \\ &= \int_{B_\rho} a(x)\chi_{B_\rho\setminus C_\varepsilon}(x)|\mathrm{adj}_k(\nabla u^1_h, \dots, \nabla u^j_h, \nabla u^{j+1}_{h,\lambda}, \dots, \nabla u^m_h) - w(x)|\,dx \\ &\quad - \int_{E_{h,\lambda}\setminus C_\epsilon} a(x)|\mathrm{adj}_k(\nabla u^1_h, \dots, \nabla u^j_h, \nabla u^{j+1}_{h,\lambda}, \dots, \nabla u^m_h) - w(x)|\,dx\,.\end{aligned}$$

Letting $h \to \infty$ on both sides of this inequality and using the fact that (4.1) (with $a(x)$ replaced by $\chi_{B_\rho\setminus C_\varepsilon}(x)$) holds true if the last $m-j$ components of each u_h are in $W^{1,p_1}(B_\rho)$, we get that

$$\begin{aligned}&\liminf_{h\to+\infty}\int_{B_1} a(x)|\mathrm{adj}_k\nabla u_h - w(x)|\,dx \\ &\qquad \ge \int_{B_\rho\setminus C_\varepsilon} a(x)|\mathrm{adj}_k(\nabla u^1, \dots, \nabla u^j, \nabla u^{j+1}_\lambda, \dots, \nabla u^m) - w(x)|\,dx - \limsup_{h\to+\infty} I^\varepsilon_{h,\lambda}\,,\end{aligned} \tag{4.8}$$

where

$$I^{\varepsilon}_{h,\lambda}=\int_{E_{h,\lambda}\setminus C_\epsilon} a(x)|\mathrm{adj}_k(\nabla u^1_h,\dots,\nabla u^j_h,\nabla u^{j+1}_{h,\lambda},\dots,\nabla u^m_h)-w(x)|\,dx\,.$$

In order to estimate $I^{\varepsilon}_{h,\lambda}$, we recall (4.4), (4.5), and (4.6), thus getting

$$\begin{aligned}
I^{\varepsilon}_{h,\lambda} &\le c_1\Big(\lambda+\frac{1}{1-\rho}\Big)\int_{E_{h,\lambda}\setminus C_\varepsilon}|\mathrm{adj}_{k-1}(\nabla u^1_h,\dots,\nabla u^j_h,\nabla u^{j+2}_h,\dots\nabla u^m_h)|\,dx+\int_{E_{h,\lambda}}|w|\,dx\\
&\le c_1\Big(\lambda+\frac{1}{1-\rho}\Big)\Big[\Big(\int_{B_1}|\mathrm{adj}_{k-1}(\nabla u^1_h,\dots,\nabla u^j_h,\nabla u^{j+2}_h,\dots\nabla u^m_h)|^{p_{k-1}}dx\Big)^{\frac{1}{p_{k-1}}}\Big(\mathcal{L}^n(E_{h,\lambda}\setminus C_\varepsilon)\Big)^{1-\frac{1}{p_{k-1}}}\Big]\\
&\quad+\omega(\lambda)\\
&\le c_3\Big(\lambda+\frac{1}{1-\rho}\Big)\Big(\frac{1}{\lambda^{p_1}}\int_{(E_{h,\lambda}\setminus C_\varepsilon)\cap\{M(|\nabla u^{j+1}_h|)>\lambda/2\}}M^{p_1}(|\nabla u^{j+1}_h|)dx+\frac{1}{\lambda}\mathcal{H}^{n-1}(J_{u^{j+1}_h})\Big)^{1-\frac{1}{p_{k-1}}}+\omega(\lambda)\\
&\le c_3\Big(\lambda+\frac{1}{1-\rho}\Big)\Big(\frac{\varepsilon}{\lambda^{p_1}}+\frac{1}{\lambda}\mathcal{H}^{n-1}(J_{u^{j+1}_h})\Big)^{1-\frac{1}{p_{k-1}}}+\omega(\lambda)\,,
\end{aligned}$$

where $\omega(\lambda)$ is a quantity, indipendent of ε and h, converging to 0 as λ goes to infinity, and c_3 is a constant depending only on n and $\sup_h\|u_h\|_{L^\infty(B_1,\mathbb{R}^m)}$. Letting h go to infinity in the previous estimate, and denoting by $\sigma(\varepsilon)$ a quantity converging to zero as ε tends to zero, we get

$$\limsup_{h\to\infty} I^{\varepsilon}_{h,\lambda}\le c_3\lambda^{1-p_1+\frac{p_1}{p_{k-1}}}\sigma(\varepsilon)+\frac{c_3}{\lambda^{p_1-\frac{p_1}{p_{k-1}}}}\frac{\sigma(\varepsilon)}{1-\rho}+\omega(\lambda)\le c_3\Big(\sigma(\varepsilon)+\frac{\sigma(\varepsilon)}{1-\rho}\Big)+\omega(\lambda),$$

where the first inequality is due to the fact that $\mathcal{H}^{n-1}(J_{u_h})\to 0$ as $h\to\infty$ and the second inequality is due to the fact that $\lambda\ge 1$ and $1-p_1+\frac{p_1}{p_{k-1}}\le 0$, according to the fact that $p_{k-1}\ge\frac{p_1}{p_1-1}$.
In conclusion, recalling (4.8), we have that if $\lambda\ge\max\{\lambda_\varepsilon,1\}$, then

$$\begin{aligned}
\liminf_{h\to\infty}\int_{B_1}a(x)|\mathrm{adj}_k\nabla u_h-w(x)|dx &\ge \int_{B_\rho\setminus C_\varepsilon}a(x)|\mathrm{adj}_k(\nabla u^1,\dots,\nabla u^j,\nabla u^{j+1}_\lambda,\dots,\nabla u^m)-w(x)|dx\\
&\quad-\frac{2c_3\sigma(\varepsilon)}{1-\rho}-\omega(\lambda)\,.
\end{aligned}$$

Therefore, we have in particular that

$$\liminf_{h\to\infty}\int_{B_1}a(x)|\mathrm{adj}_k\nabla u_h-w(x)|dx\ge\int_{(B_\rho\setminus C_\varepsilon)\cap\{u^{j+1}_\lambda=u^{j+1}\}}a(x)|\mathrm{adj}_k\nabla u-w(x)|dx-\frac{2c_3\sigma(\varepsilon)}{1-\rho}-\omega(\lambda)\,.$$

Recalling (4.7), we let first λ go to ∞, then ε to zero, and $\rho\to 1$, thus obtaining (4.1) when the last $m-j-1$ components of u_h belong to $W^{1,p_1}(B_1)$. The statement of the lemma then follows by induction. □

We are now in position to accomplish the proof of Theorem 3.4, which, in view of what we have proved in Lemma 4.1, is obtained by a blow-up argument.

Proof of Theorem 3.4. As in the proof of Lemma 4.1 we may assume, without loss of generality, that $k\ge 2$, since the assertion for the case $k=1$ follows at once from Theorem 2.1.

Thus, let us fix $k \geq 2$. From the assumption (3.8) and from Lemma 2.4 it follows that in order to prove the weak convergence in $L^{p_k}(\Omega, \mathbb{R}^{\tau_k})$ of $\mathrm{adj}_k \nabla u_h$ to $\mathrm{adj}_k \nabla u$ it is enough to show that for any $w \in L^1(\Omega, \mathbb{R}^{\tau_k})$

$$\int_\Omega |\mathrm{adj}_k \nabla u - w| \, dx \leq \liminf_{h \to \infty} \int_\Omega |\mathrm{adj}_k \nabla u_h - w| \, dx \,. \tag{4.9}$$

Let us fix w and assume, without loss of generality, that the lim inf on the right-hand side of (4.9) is a limit. Passing possibly to a (not relabelled) subsequence and observing that the sequence $|\mathrm{adj}_k \nabla u_h - w|$ is equi-integrable, we may assume that there exist a nonnegative Radon measure in Ω, say μ, and a function $g_{k,w} \in L^1(\Omega)$ such that

$$\mathcal{H}^{n-1} \llcorner J_{u_h} \rightharpoonup \mu \quad \text{weakly*}, \qquad |\mathrm{adj}_k \nabla u_h - w| \rightharpoonup g_{k,w} \quad \text{weakly in } L^1(\Omega) \,. \tag{4.10}$$

Moreover, from the assumption (3.8) we may also assume that there exists a nonnegative Radon measure ν such that

$$\sum_{k=1}^{n \wedge m} |\mathrm{adj}_k \nabla u_h|^{p_k} \llcorner \mathcal{L}^n \rightharpoonup \nu \qquad \text{weakly* in } \Omega \,. \tag{4.11}$$

Clearly, (4.9) is proved if we show that

$$g_{k,w}(x) \geq |\mathrm{adj}_k \nabla u(x) - w(x)| \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega \,. \tag{4.12}$$

To this aim, let us consider the points $x \in \Omega$ such that

$$\lim_{\rho \to 0} \frac{\mu(B_\rho(x))}{\rho^{n-1}} \;=\; 0 \,, \tag{4.13}$$

$$\lim_{\rho \to 0} \frac{\nu(B_\rho(x))}{\rho^{n}} \;<\; \infty \,, \tag{4.14}$$

and set

$$G = \{x \in \Omega : \, x \text{ is a Lebesgue point for } u, \, w \text{ and } g_{k,w}, \, \nabla u(x) \text{ exists and (4.13) holds}\} \,.$$

Notice that since the set of points in Ω where (4.13) does not hold is a Borel set of σ-finite $\mathcal{H}^{n-1}$-measure (see e.g. [7, Theorem 2.56]) and the set where (4.14) does not hold has $\mathcal{L}^n$ measure zero, we have that $\mathcal{L}^n(\Omega \setminus G) = 0$.
Let us fix $x_0 \in G$ and choose an infinitesimal sequence ρ_j such that $\mu(\partial B_{\rho_j}(x_0)) = 0$ for all j. From the strong convergence of u_h to u and from (4.10), (4.11), it follows that there exists a strictly increasing sequence of integers h_j, such that

$$\begin{cases} \dfrac{1}{\rho_j^{n+1}} \displaystyle\int_{B_{\rho_j}(x_0)} |u_{h_j}(x) - u(x)| \, dx < \dfrac{1}{j} \,, \\ \dfrac{1}{\rho_j^{n}} \left| \displaystyle\int_{B_{\rho_j}(x_0)} |\mathrm{adj}_k \nabla u_{h_j}(x) - w(x)| \, dx - \displaystyle\int_{B_{\rho_j}(x_0)} g_{k,w}(x) \, dx \right| < \dfrac{1}{j} \,, \\ \sup\limits_{j \in \mathbb{N}} \dfrac{1}{\rho_j^{n}} \displaystyle\sum_{k=1}^{n \wedge m} \int_{B_{\rho_j}(x_0)} |\mathrm{adj}_k \nabla u_{h_j}|^{p_k} \, dx < \infty \,, \\ \dfrac{1}{\rho_j^{n-1}} \left| \mathcal{H}^{n-1}(J_{u_{h_j}} \cap B_{\rho_j}(x_0)) - \mu(B_{\rho_j}(x_0)) \right| < \dfrac{1}{j} \,, \end{cases} \tag{4.15}$$

where the last inequality follows from the fact that since $\mu(\partial B_{\rho_j}(x_0)) = 0$ for all j, then we have that $\mathcal{H}^{n-1}(J_{u_h} \cap B_{\rho_j}(x_0)) \to \mu(B_{\rho_j}(x_0)$ as $h \to \infty$. Let us set

$$v_j(y) = \frac{u_{h_j}(x_0 + \rho_j y) - u(x_0)}{\rho_j} \qquad \text{for all } j \in \mathbb{N} \text{ and } y \in B_1 .$$

Notice that from $(4.15)_1$ we have

$$\begin{aligned}
\int_{B_1} |v_j(y) - \nabla u(x_0)y|dy &\le \int_{B_1} \frac{|u_{h_j}(x_0+\rho_j y) - u(x_0+\rho_j y)|}{\rho_j} dy + \int_{B_1} \Big| \frac{u(x_0+\rho_j y) - u(x_0)}{\rho_j} - \nabla u(x_0)y \Big| dy \\
&\le \frac{1}{\rho_j^{n+1}} \int_{B_{\rho_j}(x_0)} |u_{h_j}(x) - u(x)|\, dx + \frac{1}{\rho_j^{n+1}} \int_{B_{\rho_j}(x_0)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|\, dx \\
&\le \frac{1}{j} + \frac{1}{\rho_j^{n+1}} \int_{B_{\rho_j}(x_0)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|\, dx ,
\end{aligned}$$

hence $v_j \to \nabla u(x_0)y$ in $L^1(B_1, \mathbb{R}^m)$. Moreover, $(4.15)_4$ yields that

$$\mathcal{H}^{n-1}(J_{v_j} \cap B_1) = \frac{1}{\rho_j^{n-1}} \Big[\mathcal{H}^{n-1}(J_{u_{h_j}} \cap B_{\rho_j}(x_0)) - \mu(B_{\rho_j}(x_0)) \Big] + \frac{\mu(B_{\rho_j}(x_0))}{\varrho_j^{n-1}} < \frac{1}{j} + \frac{\mu(B_{\rho_j}(x_0))}{\varrho_j^{n-1}}$$

and thus we have also that $\mathcal{H}^{n-1}(J_{v_j} \cap B_1) \to 0$ as $j \to \infty$. Therefore, since from $(4.15)_3$ we have in particular

$$\sup_{j \in \mathbb{N}} \sum_{k=1}^{n \wedge m} \int_{B_1} |\mathrm{adj}_k \nabla v_j|^{p_k}\, dy < \infty ,$$

the conclusion of Lemma 4.1 holds. Thus, recalling $(4.15)_2$, we have

$$\begin{aligned}
|\mathrm{adj}_k \nabla u(x_0) - w(x_0)| &\le \liminf_{j \to \infty} \fint_{B_1} |\mathrm{adj}_k \nabla v_j(y) - w(x_0)| dy = \liminf_{j \to \infty} \fint_{B_{\rho_j}(x_0)} |\mathrm{adj}_k \nabla u_{h_j}(x) - w(x_0)| dx \\
&\le \liminf_{j \to \infty} \fint_{B_{\rho_j}(x_0)} |\mathrm{adj}_k \nabla u_{h_j}(x) - w(x)| dx + \lim_{j \to \infty} \fint_{B_{\rho_j}(x_0)} |w(x) - w(x_0)| dx \\
&= \lim_{j \to \infty} \fint_{B_{\rho_j}(x_0)} g_{k,w}(x)\, dx = g_{k,w}(x_0) .
\end{aligned}$$

Hence (4.12) is proved and the proof is complete. □

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