ON THE MINIMALITY OF THE WINTERBOTTOM SHAPE

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Abstract. In this short note we prove that the Winterbottom shape [5] is a volume-constraint minimizer of the corresponding anisotropic capillary functional.

1. Introduction

In this paper we study volume-constraint minimizers of the anisotropic capillary functional in the upper half-space $\Omega := \{x = (x_1, \dots, x_n) : \; x_n > 0\} \subset \mathbb{R}^n$:

$$\mathcal{C}_{\Phi,\beta}(E) := P_\Phi(E,\Omega) - \beta \int_{\partial\Omega} \chi_E d\mathcal{H}^{n-1}, \quad E \in BV(\Omega; \{0,1\}),$$

where Φ is an anisotropy – a positively-one homogeneous convex function in $\mathbb{R}^n$ satisfying

$$c_\Phi |x| \le \Phi(x) \le C_\Phi |x|, \quad x \in \mathbb{R}^n, \tag{1.1}$$

for some $C_\Phi \ge c_\Phi > 0$,

$$P_\Phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \Phi(\nu_E) d\mathcal{H}^{n-1}$$

is the Φ-perimeter of E in Ω, $\partial^* E$ is the reduced boundary and ν_E is the generalized outer unit normal of E, β is constant – a reative adhesion constant of $\partial\Omega$, and χ_E is the interior trace of E along $\partial\Omega$, i.e.,

$$\int_{\partial\Omega} \chi_E d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial\Omega \cap \partial^* E).$$

Recall that $E \in BV(\Omega; \{0,1\})$ if and only if $E \in BV(\mathbb{R}^n; \{0,1\})$, and in particular, $\chi_E \in L^1(\partial\Omega)$ (see e.g. [1]). It is well-known that if $\beta \le -\Phi(-\mathbf{e}_n)$, where $\mathbf{e}_n := (0, \dots, 0, 1)$, then up to a translation the unique volume-constrained minimizer of $\mathcal{C}_{\Phi,\beta}$ is a translation of the Wulff shape $W^\Phi := \{\Phi^o \le 1\}$ of Φ in Ω, where

$$\Phi^o(x) := \max_{\Phi(y)=1} \; \langle x, y \rangle$$

is the dual anisotropy, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. On the other hand, if $\beta \ge \Phi(\mathbf{e}_n)$, then one can readily check that

$$\inf_{E\in BV(\Omega;\{0,1\}),\,|E|=1} \; \mathcal{C}_{\Phi,\beta}(E) = \begin{cases} -\infty & \text{if } \beta > \Phi(\mathbf{e}_n), \\ 0 & \text{if } \beta = \Phi(\mathbf{e}_n), \end{cases}$$

and the minimum problem does not admit a solution. In case $\beta = 0$ from [2, Theorem 1.3] we deduce the following relative isoperimetric inequality in Ω :

$$\frac{P_\Phi(E,\Omega)}{|E|^{\frac{n-1}{n}}} \ge \frac{P_\Phi(W^\Phi,\Omega)}{|\Omega \cap W^\Phi|^{\frac{n-1}{n}}}, \quad E \in BV(\Omega, \{0,1\}), \tag{1.2}$$

for $0 < |E \cap \Omega| < +\infty$. It turns out (see [2, p. 2979]) that the equality in (1.2) holds if and only if $E = b + rW^\Phi$ for some $b \in \partial\Omega$ and $r > 0$, i.e., E is a horizontal translation of scaled Wulff shapes. In particular, the set $W_0^\Phi := \Omega \cap W^\Phi$ is a unique (up to a horizontal translation) solution to the minimum problem

$$\inf_{E\in BV(\Omega;\{0,1\}),|E|=|W_0^\Phi|} \; \mathcal{C}_{\Phi,0}(E).$$

Date: February 3, 2024.
2010 *Mathematics Subject Classification.* 53C44, 49Q20, 35A15, 35D30, 35D35.
Key words and phrases. anisotropy, capillarity functional, Wulff shape, Winterbottom shape, minimizer.

More generally, for $\beta \in (-\Phi(-\mathbf{e}_n), \Phi(\mathbf{e}_n))$ Winterbottom in [5] constructed an equilibrium shape of crystalls atop other material, which can be defined as

$$W_\beta^\Phi := \Omega \cap W^\Phi(-\beta \mathbf{e}_n), \tag{1.3}$$

where $W^\Phi(z) = z + W^\Phi$. As we have seen earlier, the "half" Wulff shape W_0^Φ (which is also the Winterbottom shape with $\beta = 0$) is not only an equilibrium, but also a global volume-constraint minimizer of $\mathcal{C}_{\Phi,0}$. The following result shows that this property is true also for other values of β.

Theorem 1.1. *For any $\beta \in (-\Phi(-\mathbf{e}_n), \Phi(\mathbf{e}_n))$*

$$\inf_{E \in BV(\Omega;\{0,1\}),\, |E| = W_\beta^\Phi} \mathcal{C}_{\Phi,\beta}(E) = \mathcal{C}_{\Phi,\beta}(W_\beta^\Phi). \tag{1.4}$$

The equality holds if and only if $E = W^\Phi(b - \beta \mathbf{e}_n)$ for some $b \in \partial\Omega$. Equivalently,

$$\frac{\mathcal{C}_{\Phi,\beta}(E)}{|E|^{\frac{n-1}{n}}} \ge \frac{\mathcal{C}_{\Phi,\beta}(W_\beta^\Phi)}{|W_\beta^\Phi|^{\frac{n-1}{n}}}, \qquad E \in BV(\Omega;\{0,1\}), \tag{1.5}$$

and the equality holds if and only if $E = \Omega \cap (b - r\beta\mathbf{e}_n + rW^\Phi)$ for some $r > 0$ and $b \in \partial\Omega$.

Thus, the volume-constraint minimizers of $\mathcal{C}_{\Phi,\beta}$ are precisely the horizontal translations of W_β^Φ. This result is well-known in the Euclidean case $\Phi = |\cdot|$ (see, for example, [4, Theorem 19.21]). To the best of my knowledge, there is no literature on the minimality of W_β^Φ except for cases where $\beta = 0$ or Φ is Euclidean.

Acknowledgement. I thank Guido De Philippis and Francesco Maggi for useful discussions, especially, for showing their (unpublished) short notes on a generalization of Theorem 1.1 to more general cones with vertex at origin. I acknowledge support from the FWF Stand-Alone project P33716.

2. Proof of Theorem 1.1

Owing (1.2) we provide an elementary proof of this result using only properties of anisotropies in $\mathbb{R}^n$. We divide the prove into smaller steps.

Step 1: Introducing new anisotropies. Fix any $\beta \in (-\Phi(-\mathbf{e}_n), \Phi(\mathbf{e}_n))$ and $\eta^\pm \in \partial\Phi(\pm\mathbf{e}_n)$, i.e.[1],

$$\langle \eta^\pm, \pm\mathbf{e}_n \rangle = \Phi(\pm\mathbf{e}_n) \quad \text{and} \quad \Phi^o(\eta^\pm) = 1, \tag{2.1}$$

where ∂f is the subdifferential of a convex function f. Consider the functions

$$\Psi_\beta(x) := \begin{cases} \Phi(x) - \beta \left\langle x, \frac{\eta^+}{\Phi(\mathbf{e}_n)} \right\rangle & \text{if } \beta \ge 0, \\ \Phi(x) + \beta \left\langle x, \frac{\eta^-}{\Phi(-\mathbf{e}_n)} \right\rangle & \text{if } \beta < 0, \end{cases} \qquad x \in \mathbb{R}^n, \tag{2.2}$$

where for shortness we drop the dependence of Ψ_β on Φ and the choice of $\eta^\pm$. Notice that such a technique of "absorbing" the relative adhesion coefficient into the anisotropy was already used in [3].

Let us show that Ψ_β is an anisotropy in $\mathbb{R}^n$. Indeed, the convexity and positive one-homogeneity of Ψ_β are obvious. Let us show that there exists $C_{\Psi_\beta} \ge c_{\Psi_\beta} > 0$ such that

$$c_{\Psi_\beta}|x| \le \Psi_\beta(x) \le C_{\Psi_\beta}|x|, \quad x \in \mathbb{R}^n. \tag{2.3}$$

Indeed, by (1.1)

$$\sup_{|x|=1} \Psi_\beta(x) \le \sup_{|x|=1} \Phi(x) + \tfrac{|\beta| \max\{|\eta^+|,|\eta^-|\}}{\Phi(\mathbf{e}_n)} \le C_\Phi + \tfrac{|\beta| \max\{|\eta^+|,|\eta^-|\}}{\Phi(\mathbf{e}_n)} =: C_{\Psi_\beta}.$$

Thus, the second inequality in (2.3) holds.

On the other hand, by the Young inequality[2] and the second equality in (2.1)

$$\left\langle x, \tfrac{\eta^\pm}{\Phi(\pm\mathbf{e}_n)} \right\rangle \le \tfrac{\Phi(x)\Phi^o(\eta^\pm)}{\Phi(\pm\mathbf{e}_n)} = \tfrac{1}{\Phi(\pm\mathbf{e}_n)}\,\Phi(x). \tag{2.4}$$

[1] Since a priori we are not assuming the regularity of Φ, there could be more than one possible choice of $\eta^\pm$. Moreover, since we are not assuming the evenness of Φ, in general we cannot claim $\eta^+ = -\eta^-$.

[2] I.e., $\langle x, y\rangle \le \Phi(x)\Phi^o(y)$ for all $x, y \in \mathbb{R}^n$.

Now if $\beta \ge 0$, then by (2.4) and (1.1)

$$\Psi_\beta(x) \ge \Big(1 - \tfrac{\beta}{\Phi(\mathbf{e}_n)}\Big)\,\Phi(x) \ge \tfrac{\Phi(\mathbf{e}_n)-\beta}{\Phi(\mathbf{e}_n)}\, c_\Phi |x|,$$

and similarly, if $\beta < 0$,

$$\Psi_\beta(x) \ge \tfrac{\Phi(-\mathbf{e}_n)+\beta}{\Phi(-\mathbf{e}_n)}\, c_\Phi |x|.$$

Thus,

$$c_{\Psi_\beta} := c_\Phi \min\Big\{\tfrac{\Phi(\mathbf{e}_n)-|\beta|}{\Phi(\mathbf{e}_n)}, \tfrac{\Phi(-\mathbf{e}_n)-|\beta|}{\Phi(-\mathbf{e}_n)}\Big\} > 0$$

and the first inequality in (2.3) holds. Therefore, Ψ_β is an anisotropy in $\mathbb{R}^n$.

Step 2: A representation of the capillary functional. Let us show

$$\mathcal{C}_{\Phi,\beta} = P_{\Psi_\beta}(\cdot,\Omega).$$

Indeed, since $\eta^\pm$ are constant, by the divergence theorem

$$0 = \int_E \operatorname{div} \eta^\pm dx = \int_{\Omega\cap\partial^* E} \big\langle \eta^\pm, \nu_E\big\rangle\, d\mathcal{H}^{n-1} - \int_{\partial\Omega\cap\partial^* E} \big\langle \eta^\pm, \mathbf{e}_n\big\rangle\, d\mathcal{H}^{n-1}.$$

Thus,

$$\begin{aligned} P_{\Psi_\beta}(E,\Omega) =& \int_{\Omega\cap\partial^* E} \Phi(\nu_E) d\mathcal{H}^{n-1} \mp \tfrac{\beta}{\Phi(\pm\mathbf{e}_n)} \int_{\Omega\cap\partial^* E} \big\langle \nu_E, \eta^\pm\big\rangle\, d\mathcal{H}^{n-1} \\ =& P_\Phi(E,\Omega) - \beta\, \tfrac{\langle \eta^\pm, \pm\mathbf{e}_n\rangle}{\Phi(\pm\mathbf{e}_n)} \int_{\partial\Omega} \chi_E d\mathcal{H}^{n-1} \\ =& P_\Phi(E,\Omega) - \beta \int_{\partial\Omega} \chi_E d\mathcal{H}^{n-1} = \mathcal{C}_{\Phi,\beta}(E), \end{aligned}$$

Step 3: Wulff shapes of Ψ_β and Φ. We claim

$$W^{\Psi_\beta} = \mp \tfrac{\beta\eta^\pm}{\Phi(\pm\mathbf{e}_n)} + W^\Phi, \tag{2.5}$$

where if $\beta \ge 0$, we take "+" sign, otherwise we take "-" sign. Indeed, assume that $\beta \ge 0$ and take any x with $\Psi_\beta^o(x) = 1$, where Ψ_β^o is the dual of Ψ_β (since Ψ_β is an anisotropy, its dual is well-defined and also is an anisotropy). We claim that

$$\Phi^o\Big(x + \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)}\Big) = 1. \tag{2.6}$$

Let $\xi \in \partial\Psi_\beta^o(x)$, i.e., $\langle x, \xi\rangle = 1$ and $\Psi_\beta(\xi) = 1$. Then one can readily check that $x \in \partial\Psi_\beta(\xi)$. Hence, using the explicit expression of Ψ_β in (2.2) we can compute its subdifferential:

$$\partial\Psi_\beta(\theta) = \partial\Phi(\theta) - \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)} \tag{2.7}$$

at each $\theta \in \mathbb{R}^n \setminus \{0\}$, and get

$$x = \zeta - \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)} \quad \text{for some } \zeta \in \partial\Phi(\xi).$$

Thus,

$$\Phi^o\Big(x + \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)}\Big) = \Phi^o(\zeta) = 1.$$

On the other hand, if (2.6) holds, then $\zeta := x + \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)} \in \partial\Phi(\xi)$ for some $\xi \ne 0$. This and (2.7) implies $x \in \partial\Psi_\beta(\xi)$, i.e., $\Psi_\beta^o(x) = 1$. Thus,

$$\Phi^o\Big(x + \tfrac{\beta\eta^+}{\Phi(\mathbf{e}_n)}\Big) = 1 \qquad \Longleftrightarrow \qquad \Psi_\beta^o(x) = 1.$$

Since both Wulff shapes are convex and their boundaries coincide, this implies (2.5).

The case $\beta < 0$ is analogous.

Step 4: Translated Wulff shapes. Let us show that the translated W^Φ in (2.5) is a horizontal translation of truncated Wulf shapes W_β^Φ in (1.3). Indeed, consider the vector

$$b := \mp \tfrac{\beta\eta^\pm}{\Phi(\pm\mathbf{e}_n)} + \beta\mathbf{e}_n.$$

By (2.1)

$$\langle b, \mathbf{e}_n\rangle = -\frac{\beta\big\langle \eta^{\pm}, \pm\mathbf{e}_n\big\rangle}{\Phi(\pm\mathbf{e}_n)} + \beta = 0,$$

and hence, $b \in \partial\Omega$. Therefore, the translated Wulff shape $\mp\frac{\beta\eta^{\pm}}{\Phi(\pm\mathbf{e}_n)} + W^\Phi$ is a horizontal translation of the translated Wulff shape $-\beta\mathbf{e}_n + W^\Phi$.

Step 5: Minimality of truncated Wulff shape W_β^Φ in (1.3). Applying (1.2) with Ψ_β we find

$$\frac{P_{\Psi_\beta}(E,\Omega)}{|E|^{\frac{n-1}{n}}} \ge \frac{P_{\Psi_\beta}(W^{\Psi_\beta},\Omega)}{|\Omega\cap W^{\Psi_\beta}|^{\frac{n-1}{n}}}, \qquad E \in BV(\Omega;\{0,1\}). \tag{2.8}$$

The equality holds iff $E = b + rW^{\Psi_\beta}$ for some $r > 0$ and $b \in \partial\Omega$. By steps 3 and 4, W_β^Φ is a horizontal translation of

$$W^{\Psi_\beta} = W^\Phi(\mp\tfrac{\beta\eta^{\pm}}{\Phi(\pm\mathbf{e}_n)}) = b_0 + W^\Phi(-\beta\mathbf{e}_n) \tag{2.9}$$

for some $b_0 \in \partial\Omega$. In particular, we can use W_β^Φ in place of $\Omega \cap W^{\Psi_\beta}$ in (2.8). Moreover, by step 2 $P_{\Psi_\beta}(\cdot,\Omega) = \mathcal{C}_{\Phi,\beta}$, and hence, we can represent (2.8) as

$$\frac{\mathcal{C}_{\Phi,\beta}(E)}{|E|^{\frac{n-1}{n}}} \ge \frac{\mathcal{C}_{\Phi,\beta}(W_\beta^\Phi)}{|W_\beta^\Phi|^{\frac{n-1}{n}}}, \qquad E \in BV(\Omega;\{0,1\}),$$

which is (1.5).

Step 6: Conclusion of the proof of Theorem 1.1. Since $\partial\Omega$ is a (horizontal) hyperplane, the set of all horizontal translations form an additive group. In particular, by step 5 and (2.9) the sets

$$E := b + rW^{\Psi_\beta} = b + rW^\Phi(\mp\tfrac{\beta\eta^{\pm}}{\Phi(\pm\mathbf{e}_n)}) = (b + b_0 r) + rW^\Phi(-\beta\mathbf{e}_n)$$

are the only ones preserving the equality in (2.8), or equivalently, the equality in (1.5) holds if and only if $E = \Omega \cap (b + W^\Phi(-\beta r\mathbf{e}_n))$ for some $b \in \partial\Omega$ and $r > 0$. Finally, the assertions related to the equality (1.4) directly follows from (1.5).

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