CRYSTALLINE HEXAGONAL CURVATURE FLOW OF NETWORKS: SHORT-TIME, LONG-TIME AND SELF-SIMILAR EVOLUTIONS

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Abstract. We study the crystalline curvature flow of planar networks with a single hexagonal anisotropy. After proving the local existence of a classical solution for a rather large class of initial conditions, we classify the homothetically shrinking solutions having one bounded component. We also provide an example of network shrinking to a segment with multiplicity two.

1. Introduction

Crystalline evolution, more generally, *geometric interface motions* in which surface tension acts as a main driving force, model many processes in material sciences such as phase transformation, grain growth, crystal growth, ion beam and chemical etching *etc.*, and therefore, became the topic of many papers (see e.g. [5, 10, 11, 12, 13, 21, 24, 26, 34, 38, 39] and references therein). In the planar case, the interface is usually represented by a family of curves bounding different regions (phases, or grains) and moving in a nonequilibrium state [15, 16, 23, 35]. In simplified models, the motion of these curves is described by a geometric equation relating, for instance, the normal velocity of the interface to its curvature. When the interface is represented by a single closed curve, i.e., the two phase case, such an evolution is usually called *anisotropic curve shortening flow* (see e.g., [2, 4, 3, 22]). In presence of more than two phases in the plane, the interface is called a *network*, and consists of a set of curves with multiple (typically triple) junctions.

The anisotropic curvature evolution of a network $\mathcal{S} \subset \mathbb{R}^2$ is the formal gradient flow of the energy functional (the anisotropic length, or weighted ϕ-length)

$$\ell_\phi(\mathcal{S}) := \int_{\mathcal{S}} \phi^o(\nu_{\mathcal{S}})\, d\mathcal{H}^1,$$

where $\nu_{\mathcal{S}}$ is a unit normal vector field to $\mathcal{S}$ and the energy density ϕ^o, sometimes called surface tension and initially defined on $\mathbb{S}^1$, is extended on $\mathbb{R}^2$ in a one-homogeneous way to a norm $\phi^o : \mathbb{R}^2 \to [0, +\infty)$. This gradient flow is well-posed when ϕ^o is smooth and elliptic (for instance Euclidean) and $\mathcal{S}$ is a finite union of sufficiently smooth curves with boundary, satisfying a suitable balance condition at triple junctions. In this case the network evolves, at least for short-times, by its anisotropic curvature in normal direction; furthermore, several qualitative properties and long time behaviour are known, see for instance [9, 27, 30, 32, 33].

A challenging case is when ϕ^o is crystalline, i.e., its unit ball B^{ϕ^o} is a (centrally symmetric) polygon, hence with facets and corners. Here the phases are expected to be mostly polygonal, and to evolve under a sort of nonlocal (i.e., crystalline) curvature. A further mathematical obstruction to the study of long-time behaviour of the flow is the possible appearence of nonpolygonal curves arising from triple junctions during the evolution [7]. Even more difficult is the case when the curve Σ_{ij} separating phase i and phase j has its own anisotropy ϕ^o_{ij}, and the corresponding total length is the sum of all corresponding weighted ϕ_{ij}-lengths $\ell_{\phi_{ij}}(\Sigma_{ij})$.

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When each ϕ^o_{ij} is crystalline, this is a model for polycrystalline materials in the plane [17, 20]; see also [9] for more.

In this paper we study short-time and long-time crystalline curvature flow of networks (Definition 3.1) with a single anisotropy whose Wulff shape B^ϕ (the dual body of the unit ball B^{ϕ^o} of ϕ^o) is a regular hexagon with two horizontal facets (see Figure 2). Such an assumption on ϕ brings a lot of simplifications, and makes it possible an almost complete analysis, which would be too complicated and probably not available for a generic regular polygonal anisotropy. It is worth to mention that most of the techniques developed here can be adapted to a rather general anisotropic setting.

Our main interests in the present paper are in short-time existence of the ϕ-curvature flow, in singularity formation at the maximal time, and in homothetically shrinking solutions and, as a matter of fact, in an analysis of the conical critical points and conical local minimizers of ℓ_ϕ. These problems are nonlocal, and the starting point is a rigorous definition of crystalline curvature of the network (i.e., the velocity of the flow), based on the notion of Lipschitz Cahn-Hoffman (CH) field (Section 2.5) satisfying a balance condition at the multiple junctions (see (2.5)); here we mainly follow [7, 9], where crystalline curvature is defined in a multi-anisotropic setting, and triple junctions might interact in the definition of crystalline curvature.

Since the anisotropy is polygonal, we mostly restrict the flow to initial simple networks $\mathcal{S}^0$–polygonal networks whose segments/half-lines are parallel to some facet of the Wulff shape, and admit only triple junctions with a 120^o-balance condition. Segments of $\mathcal{S}^0$ are expected to evolve by parallel translation in normal direction, whereas the half-lines stay still. However, the notion of ϕ-regular flow does not restrict to initial networks with the 120^o-condition. For instance, any critical network (Definition 2.7) is a stationary solution, and there are many (even minimal) conical critical networks with three half-lines not satisfying the 120^o-condition at the junctions (Theorem 2.13).

As observed in [7], not all polygonal networks preserve their topology during the flow, and new segments or curves can arise at some time (in a continuous manner) from multiple junctions. To prevent such phenomena, which make difficult the description of the subsequent flow, since one looses the description via a system of ordinary differential equations, as in [7, 9] we need some topological assumptions on the initial network. In contrast to [7, 9] where polycrystalline networks consisting of three polygonal curves made by one segment and one half-line meeting at a single triple junction were considered, our simple networks admit an arbitrary finite number of triple junctions.

Our main existence result reads as follows (see Theorem 3.5).

Theorem 1.1. *For any simple network $\mathcal{S}^0$, there exists the unique ϕ-curvature flow $\{\mathcal{S}(t)\}_{t\in[0,T^\dagger)}$ starting from $\mathcal{S}^0$ on a maximal time interval $[0,T^\dagger)$. Moreover, if $T^\dagger < +\infty$, then some segment of $\mathcal{S}(t)$ vanishes as $t \nearrow T^\dagger$.*

As mentioned earlier, critical networks are examples of initial networks for which $T^\dagger = +\infty$. In Example 5.1 we provide a noncritical $\mathcal{N}^0$ for which $T^\dagger = +\infty$.

Simple networks admit the following remarkable property. If we partition a simple network into connected graphs by removing all simple (not triple) vertices, then for each graph G containing at least one triple junction, *either* a minimal CH field is constant along each segment/half-line of G and coincides with some vertex of the Wulff shape, *or* its values never coincide on G with vertices of the Wulff shape, except at the removed simple vertices (see Lemma 3.7). In particular, those graphs whose segments have a constant minimal CH field, do not evolve by translation, i.e., stays still. Thus, to prove Theorem 1.1 we just need to study the evolution of the heights from the remaining graphs.

This observation can be generalized to networks with junctions of higher degree provided that the segments/half-lines forming those junctions have zero ϕ-curvature (see Theorem 4.2).

Such higher degree junctions could appear as a singularity after a collapse of two (or more) triple junctions, and Theorem 4.2 sometimes allows us to restart the flow after singularities in a regular manner (the topology of the network now may change, see Corollary 4.3). However, it is worth to mention that unlike the networks containing only triple junctions with the 120^o-condition, the networks admitting higher degree junctions or triple junctions not satisfying the 120^o-condition are not simple, and may not be reached generically, for instance, by weak solutions.

The next question is, of course, the asymptotic behaviour of the flow at a singular time. In the Euclidean case the blow-up behaviour of the rescaled networks can be established by means of Huisken's monotonicity formula [25, 32]; then, using parabolic rescaling, one approaches some limiting network as $t \nearrow T$, which may admit various singularities such as the loose of the $120°$ condition (collapse of triple junctions), collapse of some curve (higher multiplicity), or even collapse of a phase to a point or to a segment. To our best knowledge, at the moment there are no examples of networks reducing to a network with higher multiplicity (see the multiplicity-one conjecture in [31]). In our crystalline setting, we can provide explicit examples leading to those phenomena (see Section 5), including higher multiplicity segments (Example 5.2), and it is worth to mention that if after vanishing of some segments, the resulting network remains simple (not necessarily parallel to the initial one and possibly with multiple junctions), then the flow restarts until a subsequent singularity is reached.

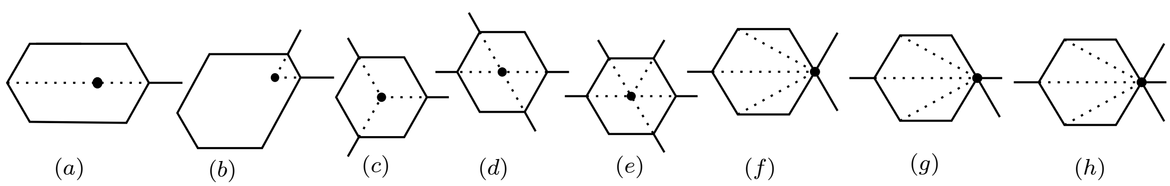

Fig. 1. All possible self-shrinkers with one bounded phase.

Our next main result is a classification of self-shrinkers with a single bounded phase (see Section 6 for more precise statements and the assumption on the topology of the initial network).

Theorem 1.2. *Up to a rotation and mirror reflection, there are only eight different simple self-shrinkers possibly with multiple junctions (see Figure 1 (a)-(h)).*

Notice that we do not need a priori any symmetry assumption (for instance, Figure 1 (b) has no symmetry lines). Recall that such a classification was done in the Euclidean case in [14] where the authors, under some symmetry assumptions, characterize six different self-shrinking networks having only one bounded phase.

As in [9], we do not treat here weak (i.e., generalized) flows: for this broad argument we refer the reader to [6, 8, 28, 40].

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2. Preliminaries

In this section we introduce the notation and the definitions used throughout the paper.

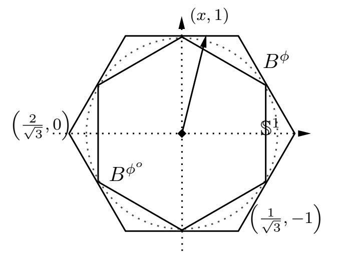

Fig. 2. The Wulff shape B^ϕ of sidelength $\frac{2}{\sqrt{3}}$, and its dual B^{ϕ^o}.

2.1. **Notation.** Unless otherwise stated, all sets we consider are subsets of $\mathbb{R}^2$. We choose the standard oriented basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ and denote by (x_1, x_2) the coordinates of $x \in \mathbb{R}^2$ with respect to this basis. $\mathrm{Int}(A)$ is the interior of $A \subset \mathbb{R}^2$. $D_r(x)$ stands for the open (Euclidean) disc in $\mathbb{R}^2$ centered at $x \in \mathbb{R}^2$ of radius $r > 0$, and for shortness, set $D_r := D_r(0)$. By $\mathcal{H}^1$ we denote the one-dimensional Hausdorff measure in $\mathbb{R}^2$.

2.2. **Anisotropies.** An (even) *anisotropy* is a nonnegative, positively one-homogeneous even convex function ϕ in $\mathbb{R}^2$ satisfying $\{\phi = 0\} = \{0\}$ (i.e., ϕ is a norm). In what follows, we fix the anisotropy ϕ in $\mathbb{R}^2$ whose closed unit ball (also called Wulff shape) $B^\phi := \{\phi \le 1\}$ is the regular hexagon circumscribed to the unit circle centered at the origin, with two horizontal facets (see Figure 2). The closed unit ball B^{ϕ^o} (also called Frank diagram) of the dual anisotropy

$$\phi^o(\xi) := \sup_{\eta \in \mathbb{R}^2,\, \phi(\eta)=1} \xi \cdot \eta, \quad \xi \in \mathbb{R}^2,$$

is also a regular hexagon inscribed to the unit circle with two vertical facets, as in Figure 2. The (six) facets of ∂B^ϕ and of ∂B^{ϕ^o} are closed.

By the definition of ϕ^o the following Young inequality holds:

$$\xi \cdot \eta \le \phi^o(\xi)\phi(\eta), \quad \xi, \eta \in \mathbb{R}^2. \tag{2.1}$$

We write

$$B_R^\phi := \{x \in \mathbb{R}^2 :\ \phi(x) \le R\} \quad \text{and} \quad \mathring{B}_R^\phi := \{x \in \mathbb{R}^2 :\ \phi(x) < R\}, \quad R > 0,$$

with $B^\phi = B_1^\phi$.

2.3. **Curves.** We call a closed set Γ in $\mathbb{R}^2$ a *curve*[1] if there exists an interval I of the form $[0,1]$, $[0,1)$ or $(0,1)$, and an absolutely continuous function $\gamma : I \to \mathbb{R}^2$ such that $\gamma(I) = \Gamma$. The function γ is called a *parametrization* of Γ. In this paper we consider only *embedded* curves, i.e., the map $\gamma : (0,1) \to \mathbb{R}^2$ is injective (and sometimes we identify the map γ with the set Γ). When $I = [0,1]$ and $\gamma(0) = \gamma(1)$, we say Γ is *closed*. When γ is C^1 (resp. Lipschitz) and $|\gamma'| > 0$ in I (resp. a.e. in I), the map γ is called a *regular parametrization* of Γ. A curve Γ is $C^{k+\alpha}$ for some $k \ge 0$ and $\alpha \in [0,1]$, $k + \alpha \ge 1$, if it admits a regular $C^{k+\alpha}$-parametrization. The tangent line to Γ at a point $p \in \Gamma$ is denoted by $T_p\Gamma$ (provided it exists). The (Euclidean) unit tangent vector to Γ at p is denoted by $\tau_\Gamma(p)$ and the unit normal vector is $\nu_\Gamma(p) = \tau_\Gamma(p)^\perp$, where $^\perp$ is the counterclockwise 90^o rotation. When there is no risk of confusion, we simply write τ and ν in place of τ_Γ and ν_Γ. If $p = \gamma(x)$ and γ is differentiable at x, then

$$\tau(p) = \frac{\gamma'(x)}{|\gamma'(x)|} \qquad \text{and} \qquad \nu(p) = \frac{\gamma'(x)^\perp}{|\gamma'(x)|}.$$

Unless otherwise stated, we choose tangent vectors in the direction of the parametrization. In particular, two oriented segments/half-lines are called *parallel* provided they lie on parallel straight lines and their unit normals coincide.

[1]We include unbounded curves without endpoints (case $I = (0,1)$) such as straight lines, parabolas, a union of two half-lines etc. meeting at one point, and unbounded curves with just one boundary point (case $I = [0,1)$) such as half-lines, half-parabolas etc., and finally, compact curves with two endpoints (possibly coinciding) such as segments, circles, arcs of circles etc.

A curve $\Gamma = \gamma(I)$ is *polygonal* if for any $[a,b] \Subset I$ the curve $\gamma([a,b])$ is a finite union of segments. Any polygonal curve is a union of segments and at most two half-lines. A curve Γ is (locally) *rectifiable* if for any $[a,b] \Subset I$ the supremum

$$\sup_{a=t_0<t_1<\ldots<t_n=b} \quad \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i+1})|$$

is finite; equivalently, if and only if for any $[a,b] \Subset I$ and $\epsilon > 0$ there exists a polygonal curve $\sigma : I \to \mathbb{R}^2$ such that

$$\sup_{x\in[a,b]} |\gamma(x) - \sigma(x)| < \epsilon.$$

By definition any polygonal curve is rectifiable. Using [18, Lemmas 3.2, 3.5] one checks that a curve Γ is rectifiable if for any $[a,b] \Subset I$ one has $\mathcal{H}^1(\gamma([a,b])) < +\infty$ and any rectifiable curve Γ admits a unit tangent vector τ (and a corresponding unit normal ν) $\mathcal{H}^1$-a.e. along Γ.

The ϕ-length of Γ in an open set $\Omega \subset \mathbb{R}^2$ is defined[2] as

$$\ell_\phi(\Gamma, \Omega) := \int_{\Omega\cap\Gamma} \phi^o(\nu)\, d\mathcal{H}^1.$$

When $\Omega = \mathbb{R}^2$, we simply write

$$\ell_\phi(\Gamma) := \ell_\phi(\Gamma, \mathbb{R}^2).$$

2.4. **Tangential divergence of a vector field.** The *tangential divergence* of a vector field $g \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ over an embedded Lipschitz curve Γ is defined as

$$\mathrm{div}_\tau g(p) = \nabla g(p)\tau(p) \cdot \tau(p) \quad \text{for } \mathcal{H}^1\text{-a.e. } p \in \Gamma.$$

The tangential divergence can also be introduced using parametrizations. More precisely, if $\gamma \in \mathrm{Lip}(I;\mathbb{R}^2)$ is a regular parametrization of Γ and $g : \Gamma \to \mathbb{R}^2$ is a Lipschitz vector field along Γ, i.e., $g \circ \gamma \in \mathrm{Lip}(I;\mathbb{R}^2)$, then

$$\mathrm{div}_\tau\, g(p) = \frac{[g\circ\gamma]'(x) \cdot \gamma'(x)}{|\gamma'(x)|^2}, \qquad p = \gamma(x)$$

at points of differentiability. One can readily check that the tangential divergence is independent of the parametrization.

2.5. **ϕ-regular curves.** Let Γ be a rectifiable curve, with $\mathcal{H}^1$-almost everywhere defined unit normal $\nu(p)$. A vector field $N : \Gamma \to \partial B^\phi$ is called a *Cahn-Hoffman field* (CH field) if

$$N \cdot \nu = \phi^o(\nu) \quad \mathcal{H}^1\text{-a.e. on } \Gamma, \tag{2.2}$$

namely $N \in \partial\phi^o(\nu)$, where ∂ stands for the subdifferential. Notice that reversing the orientation of the curve translates into a change of sign of ν and of the corresponding CH field, which is always "co-directed" as the unit normal. In what follows we shorthand $\nu^{\phi^o} := \frac{\nu}{\phi^o(\nu)}$.

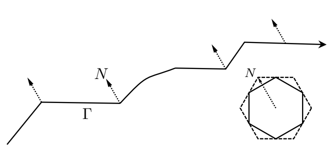

FIG. 3. A curve Γ admitting a constant CH field N.

Definition 2.1 (**Lipschitz ϕ-regular curve**)**.** We say the curve Γ is *Lipschitz ϕ-regular* (*ϕ-regular*, for short) if it admits a *Lipschitz* CH field.

A typical example of ϕ-regular curve is a polygonal curve with 120^o-angle between adjacent segments/half-lines. However, ϕ-regular curves need not be polygonal; for instance, the six arcs of unit circle in Figure 2 having the same endpoints as the facets of ∂B^{ϕ^o} are ϕ-regular.

[2]The ϕ-length coincides with the Minkowski content of Γ in Ω, defined by means of the distance function induced by ϕ.

Proposition 2.2. *A rectifiable curve Γ admits a constant CH field if and only if there is a facet $F \subset \partial B^{\phi^o}$ such that $\nu^{\phi^o}(x) \in F$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$.*

Proof. Since B^{ϕ^o} is a regular hexagon, the CH field on each facet $F \subset \partial B^{\phi^o}$ is the unique closest vertex N of B^ϕ (see Figures 2 and 3). Thus, if $\nu_\Gamma \in F$ $\mathcal{H}^1$-a.e. on Γ, then N provides a constant CH field on Γ. Conversely, suppose Γ admits a constant CH field N, i.e., $\phi(N) = 1$ and $N \cdot \nu^{\phi^o} = 1$ $\mathcal{H}^1$-a.e. on Γ. Thus,

$$N \cdot [\nu^{\phi^o}(x) - \nu^{\phi^o}(y)] = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x, y \in \Gamma. \tag{2.3}$$

We have two cases:

Case 1: $\nu^{\phi^o}(x)$ is constant. In this case Γ is a straight line, and as S we take any facet of ∂B^{ϕ^o} which contains $\nu^{\phi^o}(x)$.

Case 2: $\nu^{\phi^o}(x)$ is not constant. By (2.3) the difference $\nu^{\phi^o}(x) - \nu^{\phi^o}(y)$ lies on a straight line L orthogonal to N. If L contains some facet F of ∂B^{ϕ^o}, then by (2.3) $\nu^{\phi^o}(x) \in L \cap \partial B^\phi = F$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$. On the other hand, if L intersects two facets F_1 and F_2 of ∂B^{ϕ^o}, then up to a $\mathcal{H}^1$-negligible set, we can write $\Gamma = X_1 \cup X_2$, where $\nu(x) \in F_i$ for any $x \in X_i$. Clearly, $\nu(x) = L \cap F_i$ for any $x \in X_i$, and by nonconstancy, $\nu(x)$ cannot belong to the intersection $F_1 \cap F_2$. Let N_1 and N_2 be the vertices of B^ϕ, closest to F_1 and F_2. Then easy geometric arguments show that $N_i \cdot \nu^{\phi^o}(x) = 1$ for any $x \in X_i$ and $Q \cdot \nu^{\phi^o}(x) < 1$ for $x \in X_i$ and $Q \in \partial B^\phi \setminus \{N_i\}$. This implies there is no $N \in \partial B^\phi$ such that $\nu^{\phi^o}(x) \cdot N = 1$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$, a contradiction. $\square$

From Proposition 2.2 it follows, in particular, that there exists $k \in \mathbb{Z}$ such that a $60^o k$-rotation of Γ is the generalized[3] graph of a monotone function. We mention also that Proposition 2.2 holds for any crystalline anisotropy whose Wulff-shape is a regular polygon with an even number of facets, provided one changes appropriately the 60^o-rotation condition.

Remark 2.3. We shall frequently use the following:

(a) if sides $[AB]$ and $[BC]$ of the triangle ABC are parallel to adjacent facets of B^ϕ, then

$$\ell_\phi([AC]) = \ell_\phi([AB]) + \ell_\phi([BC]);$$

(b) if $ABCD$ is a trapezoid[4] with sides parallel to facets of B^ϕ such that $[AB]$ and $[CD]$ are parallel and $\mathcal{H}^1([AB]) < \mathcal{H}^1([CD])$, then

$$\ell_\phi([CD]) = \ell_\phi([AB]) + \ell_\phi([BC]);$$

(c) if ABC is a regular triangle with sides parallel to three (nonadjacent) facets of B^ϕ, then for any $X \in [AB]$

$$\ell_\phi([AB]) = \ell_\phi([BC]) = \ell_\phi([CA]) = \ell_\phi([CX]).$$

Lemma 2.4 (**Curves with constant CH field**). *Let Γ be a Lipschitz curve admitting a constant CH field N, and let $X, Y \in \Gamma$. Let $\Sigma \subset \Gamma$ and $S := [XY]$ be respectively the arc of Γ and the segment connecting X and Y. Then*

$$\ell_\phi(\Sigma) = \ell_\phi(S).$$

Moreover, N is a (constant) CH field also for S.

Proof. By the minimality of segments[5] $\ell_\phi(\Sigma) \geq \ell_\phi(S)$. Let us prove the converse inequality. By assumption $\Sigma = \gamma([0,1])$ for some $\gamma \in \mathrm{Lip}([0,1]; \mathbb{R}^2)$ with $|\gamma'| > 0$ a.e. on $[0,1]$. Let

$$J := \{x \in [0,1] : \ \gamma(x) \in S\}.$$

[3] I.e., possibly with vertical parts.

[4] A convex quadrangle whose two opposite sides are parallel.

[5] A consequence of Jensen's inequality, see e.g. [19].

Clearly, J is a nonempty closed set so that $[0,1] \setminus J = \cup_j (a_j b_j)$ is an open set, (a_j, b_j) its connected components. By the continuity of γ, $X_j := \gamma(a_j)$ and $Y_j := \gamma(b_j)$ belong to S and Σ does not intersect the (relative) interior of the segment $S_j := [X_j Y_j]$. Consider the bounded open set C whose boundary consisting of the two rectifiable curves $\Sigma_j := \gamma([a_j, b_j])$ and S_j. We may assume that the parametrization of S_j is oriented from X_j to Y_j so that the outer unit normal ν_C of C on Σ_j coincides with ν_{Σ_j} and on S_j with $-\nu_S$. Applying the divergence theorem with the constant vector field $\xi : \mathbb{R}^2 \to \mathbb{R}^2$, $\xi := N$, we get

$$0 = \int_C \mathrm{div}\xi \, dx = \int_{\Sigma_j} \nu_\Sigma \cdot \xi d\mathcal{H}^1 - \int_{S_j} \nu_S \cdot N d\mathcal{H}^1.$$

Hence by (2.2) and the Young inequality (2.1) we get

$$\int_{\Sigma_j} \phi^o(\nu_\Sigma) d\mathcal{H}^1 = \int_\Sigma \nu_\Sigma \cdot \xi d\mathcal{H}^1 = \int_{S_j} \nu_S \cdot N d\mathcal{H}^1 \le \int_{S_j} \phi^o(\nu_S) d\mathcal{H}^1. \tag{2.4}$$

Therefore,

$$0 \le \ell_\phi(\Sigma) - \ell_\phi(S) = \sum_j \Big(\int_{\Sigma_j} \phi^o(\nu_\Sigma) d\mathcal{H}^1 - \int_{S_j} \phi^o(\nu_S) d\mathcal{H}^1 \Big) \le 0.$$

Hence, the inequality in (2.4) is in fact an equality. Since both N and ν_S are constant vector fields satisfying $\nu_S \cdot N = \phi^o(\nu_S)$, by definition N is a CH field also for S. □

2.6. **Networks and polygonal networks.** An *oriented network* (a network, for short) is a closed set $\mathcal{S} \subset \mathbb{R}^2$ consisting of finitely many curves $\{\Gamma_i\}_{i=1}^M$ whose relative interiors are pairwise disjoint and the endpoints of each curve Γ_i is also an endpoint of at least two other curves in case Γ_i is not closed, or of another curve in case Γ_i is closed. We call such an endpoint an m-multiple junction (m-junction, for short); m is called the degree of the junction. The orientation of the network is given by the unit normals to each curve (defined via parametrization). Clearly, a network is a connected set. When all curves are polygonal with finitely many segments, $\mathcal{S}$ is called *polygonal*, and the endpoints of half-lines and of segments of $\mathcal{S}$ are called *vertices* of $\mathcal{S}$. A vertex is *simple* if it is not a multiple junction.

In the special case a polygonal network is a finite union of half-lines starting at the same point, it is called *conical*. We write a polygonal network $\mathcal{S} = \cup_i \Gamma_i$ frequently as a union $\mathcal{S} = \cup_j S_j$ of its relatively closed segments/half-lines, where S_j is a segment/half-line of a unique Γ_i with $\nu_{S_j} = \nu_{\Gamma_i}$.

The ϕ-length of a network $\mathcal{S} = \bigcup_{i=1}^M \Gamma_i$ in an open set $\Omega \subseteq \mathbb{R}^2$ is defined as

$$\ell_\phi(\mathcal{S}, \Omega) = \sum_{i=1}^M \ell_\phi(\Gamma_i, \Omega)$$

(hence, possibly, $+\infty$).

Definition 2.5 (**Admissible network**). A polygonal network $\mathcal{S} = \cup_i S_i$ is *admissible* if each segment/half-line is parallel to some facet of B^ϕ, and the angle at any simple vertex of $\mathcal{S}$ is 120^o.

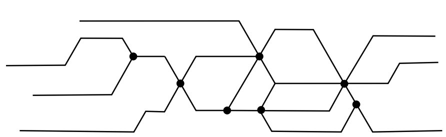

FIG. 4. An admissible network containing m-junctions for $m = 3, 4, 5, 6$.

Since B^ϕ is a regular hexagon, the degree of a junction of an admissible network is at most 6 (see Figure 4). Definition 2.5 is similar to the one in the two-phase case, where segments not parallel to facets of the Wulff shape are not considered, or to the multiphase case in [7, Definition 4.10], where (nonpolygonal) curves are excluded. For technical reasons, in [7, Definition 4.10] admissible networks may contain only triple junctions and each curve of an admissible network should contain at least one segment. However, unlike [9], in the present paper the half-lines of the network may end at a triple junction and admissible networks may contain multiple junctions. In particular, this includes Brakke-type spoons (a network consisting of the union of a closed curve and a half-line) and non-Lipschitz sets such as the union of two Wulff-shapes touching at a single point.

Definition 2.6 (**ϕ-regular networks and CH fields**). An oriented network $\mathcal{S} = \cup_i \Gamma_i$ is called *Lipschitz ϕ-regular* (*ϕ-regular*, for short) if every Γ_i is ϕ-regular, i.e., it admits a Lipschitz CH field N_i, and if X is a junction which is an endpoint of $m \geq 3$-curves $\Gamma_{i_1}, \dots, \Gamma_{i_m}$ then the *balance condition*

$$\sum_{j=1}^{m} (-1)^{\sigma_j} N_{i_j}(X) = 0, \tag{2.5}$$

holds, where $\sigma_j = 0$ if Γ_{i_j} is oriented from X, and $\sigma_j = 1$ otherwise (see Figure 5).

We call the map N, defined as $N_{|\Gamma_i} = N_i$, a *Cahn-Hoffman field* (shortly, a CH field) on $\mathcal{S}$.

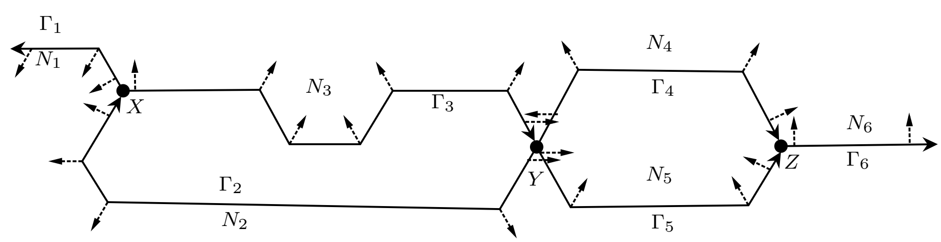

FIG. 5. A ϕ-regular admissible network (with m-junctions, $m = 3, 4$) consisting of the union of six polygonal curves Γ_i with a CH field. At the triple junction X we have $N_1 - N_2 + N_3 = 0$, since Γ_1 and Γ_3 exit from X, while Γ_2 enters to X. Thus, the balance condition (2.5) holds with $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = 1$. Similarly, $N_2 - N_3 + N_4 + N_5 = 0$ at the quadruple junction Y and $-N_4 - N_5 + N_6 = 0$ at the triple junction Z.

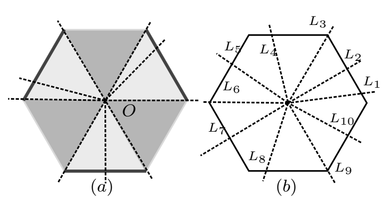

FIG. 6

The collection of all CH fields over $\mathcal{S}$ will be denoted by CH($\mathcal{S}$). In the polygonal ϕ-regular case, when $\mathcal{S} = \cup_i S_i$ and $N \in$ CH($\mathcal{S}$), we abuse the notation $N_i := N\big|_{S_i}$.

Definition 2.7 (**Critical network**). A ϕ-regular network $\mathcal{S} = \cup_i \Gamma_i$ is called a *critical point* of the ϕ-length, or shortly a *critical network*, if it admits a CH field constant over each curve Γ_i (the constant typically depends on i).

By definition, any network consisting of just one curve without boundary which admits a constant CH field, is critical. Next, consider a conical network $\mathcal{S}$ consisting of $n \geq 3$ half-lines starting from the same point, say the origin O. When $n = 3$, $\mathcal{S}$ is called a conical *triod*. Unlike networks in the Euclidean setting, the non-strict convexity of B^ϕ allows several conical networks.

Lemma 2.8 (**Conical critical networks**). *A conical triod is critical if and only if its three half-lines intersect three non-adjacent facets of ∂B^ϕ. More generally, a conical network with $n \geq 3$ half-lines is critical if and only if there exist two integers $l_1, l_2 \geq 0$ such that its half-lines can be divided into l_1 pairwise disjoint groups of triplets and l_2 pairwise disjoint groups of doublets in a way that three half-lines in each triplet intersect three non-adjacent facets of ∂B^ϕ and two half-lines in each doublet intersect two opposite facets of ∂B^ϕ.*

We omit the proof of this elementary lemma. In Figure 6 (a) three non-adjacent facets of ∂B^ϕ are highlighted. Clearly, the remaining three facets are also non-adjacent. Notice that the conical network in Figure 6 (b), consisting of 10 half-lines, is critical. Indeed, if we group the half-lines as (L_1, L_5, L_9), (L_2, L_7), (L_3, L_8) and (L_4, L_6, L_{10}), then these triplets and doublets satisfy the assertion of Lemma 2.8 with $l_1 = l_2 = 2$.

From Lemma 2.8 and the symmetry of B^ϕ, up to a rotation by an integer multiple of $\pm 60^o$ and a mirror reflection, there are eight possible admissible conical networks, seven of which are critical (see Figure 7). Notice that the network in case (a) is not ϕ-regular, because at the triple junction we cannot define any triple satisfying the balance condition (2.5).

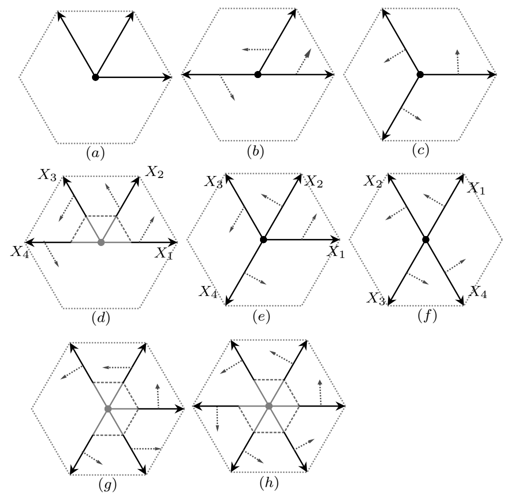

Fig. 7. Up to a rotation and a mirror reflection, there are exactly eight possible ϕ-regular conical networks, the last seven being critical. Notice that in networks (d), (g) and (e) if we replace segments starting from the multiple junction with facets of a Wulff shape of sufficiently small radius, then the length of the network inside the larger Wulff shape strictly decreases.

Example 2.9. Consider the conical ϕ-regular networks $\mathcal{S}$ with four half-lines L_1, L_2, L_3, L_4 crossing ∂B^ϕ at vertices X_1, X_2, X_3, X_4, respectively. As mentioned above, up to a rotation and a mirror reflection, $\mathcal{S}$ can be only one of the networks drawn in Figure 7 (d)-(f). Since the half-lines are oriented from the quadruple junction, we can immediately check that a balance condition at quadruple junction for any admissible CH field N over $\mathcal{S}$ implies $N_1 + N_3 = 0$ and $N_2 + N_4 = 0$. One can readily check that: in case of "W" in Figure 7 (d) N is uniquely defined by $N_1 = -N_3 = (\frac{1}{\sqrt{3}}, 1)$ and $N_2 = -N_4 = (-\frac{1}{\sqrt{3}}, 1)$; in case of "$\Psi$" in Figure 7 (e), $N_1 = -N_3 = (\frac{1}{\sqrt{3}}, 1)$ but N_2 $(= -N_4)$ can be arbitrarily chosen (still satisfying the constraints); in case of "X" in Figure 7 (f), both N_1 $(= -N_3)$ and N_2 $(= -N_4)$ can be arbitrarily chosen satisfying the constraints.

2.7. **ϕ-minimal networks.** Let $\mathcal{S}$ be a network and $\Omega \subseteq \mathbb{R}^2$ an open set; a network $\mathcal{A}$ is called a compact perturbation of $\mathcal{S}$ in Ω provided $\mathcal{S}\Delta\mathcal{A} \Subset \Omega$.

Definition 2.10 (**Local minimizers and minimal networks**)**.** A polygonal admissible network $\mathcal{S}$ is called *a local minimizer* of the ϕ-length (shortly *a local minimizer*) in an open set $U \subset \mathbb{R}^2$ if

$$\ell_\phi(\mathcal{S}, U) \le \ell_\phi(\mathcal{A}, U)$$

for any *compact perturbation* $\mathcal{A}$ of $\mathcal{S}$ in U, i.e., for any network $\mathcal{A}$ such that $\mathcal{S}\Delta\mathcal{A} \Subset U$. If $\mathcal{S}$ is a local minimizer in every bounded open subset of $\mathbb{R}^2$, we call it *ϕ-minimal (shortly, minimal).*

Notice that to check the ϕ-minimality of a network, it is enough to show its local minimality in every disc or every ϕ-ball.

Remark 2.11. Compact perturbations of a network are still networks, in particular they are connected. However, they do not need to be polygonal; unlike minimal partitions in [6, 8], they *need not preserve* the number of phases (or regions).

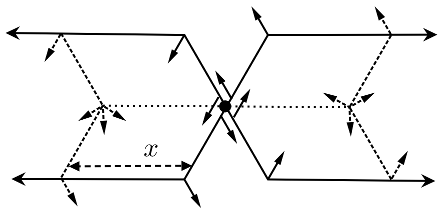

FIG. 8. A critical nonminimal network $\mathcal{S}^0$ and its length-decreasing compact perturbation. This network is a local minimizer in every disc centered at the quadruple junction and not intersecting the half-lines. However, the dotted network $\mathcal{N}^0$ (obtained from $\mathcal{S}^0$ with a "large" perturbation) has ϕ-length strictly smaller than the one of $\mathcal{S}^0$ (in addition, it satisfies the interior to the constraint condition in the sense of Definition 3.8). Unlike $\mathcal{S}^0$, the segments of $\mathcal{N}^0$ have nonzero Φ-curvatures, and during the flow they slide away from the quadruple junction. The evolution of $\mathcal{S}^0$ and $\mathcal{N}^0$ will be considered in Example 5.1.

Clearly, not every critical network is minimal (see Figure 7 (d), (g) and (h) and Figure 8), however every minimal network is critical. Indeed, if Γ_i does contain a small arc Γ' with endpoints $X, Y \in \Gamma_i$ such that any $N \in \mathrm{CH}(\mathcal{S})$ is not constant over Γ', then using a calibration argument as in Proposition 2.2, one checks that the segment $[XY]$ has strictly less ϕ-length than Γ'. Thus, replacing Γ' with $[XY]$ (since Γ' is small, such a replacement still produces a network) we get a compact perturbation of $\mathcal{S}$ which has strictly less ϕ-length in any disc containing Γ'.

Example 2.12. Let $\mathcal{S}$ consist a unique curve Γ without boundary, admitting a constant CH vector field N. As we mentioned above, $\mathcal{S}$ is critical. Let us show that $\mathcal{S}$ is also minimal. Let $R > 0$ and $\mathcal{A}$ be any compact perturbation of $\mathcal{S}$ in $D_R := D_R(O)$, and let $X, Y \in \partial D_R \cap \Gamma$ be such that the curve $\Gamma' \subset \Gamma$ connecting X and Y is the maximal, i.e., any other curve Γ'' with endpoint at ∂D_R satisfies $\Gamma'' \subseteq \Gamma'$. By Lemma 2.4 $\ell_\phi(\Gamma') = \ell_\phi([XY])$. As $X, Y \in \mathcal{A}$ and $\mathcal{A}$ is (arcwise) connected, there exists a curve $\Sigma' \subset \mathcal{A}$ of $\mathcal{A}$ connecting X to Y. By the anisotropic minimality of segments

$$\ell_\phi(\Sigma') \geq \ell_\phi([XY]) = \ell_\phi(\Gamma').$$

Since $\mathcal{A} \setminus D_R = \mathcal{S} \setminus D_R$, for any $\overline{R} > R$ such that $\Gamma' \cup (\mathcal{A}\Delta\mathcal{S}) \Subset D_{\overline{R}}$ we have

$$\ell_\phi(\mathcal{A}, D_{\overline{R}}) = \ell_\phi(\mathcal{A} \setminus \Sigma', D_{\overline{R}}) + \ell_\phi(\Sigma') \geq \ell_\phi(\mathcal{S} \setminus \Gamma', D_{\overline{R}}) + \ell_\phi(\Gamma') = \ell_\phi(\mathcal{S}, D_{\overline{R}}).$$

Thus, $\mathcal{S}$ is minimal.

Next we study the minimality of some critical conical networks.

Theorem 2.13 (**Minimal conical triods**). *Any conical critical triod is minimal.*

Proof. Let $\mathcal{S}$ be a conical critical triod – a network consisting of a union of three (different) half-lines L_1, L_2, L_3 starting at the origin O and crossing three non-adjacent facets of B^ϕ.

Let $R > 0$ and $\mathcal{A}$ be any compact perturbation of $\mathcal{S}$ in $\mathring{B}_R^\phi$ and let $X_i := L_i \cap \partial B_R^\phi$. Since $B_R^\phi \cap \mathcal{A}$ is connected there exists a point $T \in \mathcal{A}$ and three curves $\Gamma_1, \Gamma_2, \Gamma_3 \subset B_R^\phi \cap \mathcal{A}$ with disjoint relative interiors such that Γ_i connects X_i and T so that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ form a partition of $\mathring{B}_R^\phi$ with the same boundary conditions as $\mathcal{S}$. For, take any curve $\Gamma' \subset B_R^\phi \cap \mathcal{A}$ connecting X_1 to X_3, and a curve $\Gamma'' \subset B_R^\phi \cap \mathcal{A}$ connecting X_2 to Γ'. Let T be the first intersection of Γ'' with Γ' so that its subcurve Γ_2 connecting X_2 to Γ' (at T) is minimal. Notice that T divides Γ' into two subcurves Γ_1 and Γ_3 connecting T to X_1 and X_3. Let $\tilde{\mathcal{A}} := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup (\mathcal{S} \setminus \mathring{B}_R^\phi)$. Let $N_1, N_2, N_3 \in \partial B^\phi$ be constant vectors such that $N_i \cdot \nu_i = \phi^o(\nu_i)$ on L_i and $N_1 + N_2 + N_3 = 0$ at O. Define

$$\xi_1 := N_3, \quad \xi_2 := 0, \quad \xi_3 := -N_1,$$

so that $\xi_i - \xi_j = N_k$ for $ijk \in \{123, 231, 312\}$. These ξ_i play the role of a (constant) paired calibration [29, Theorem 3.2], and thus $\ell_\phi(\mathcal{S}, \mathring{B}_R^\phi) \leq \ell_\phi(\tilde{\mathcal{A}}, \mathring{B}_R^\phi) \leq \ell_\phi(\mathcal{A}, \mathring{B}_R^\phi)$. Hence, $\mathcal{S}$ is minimal. $\square$

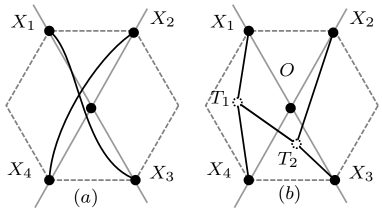

FIG. 9.

In the case of four half-lines we use a different method (see Remark 2.15).

Theorem 2.14 (**Minimality of 4-junctions**)**.** *Let $\mathcal{S}$ be a conical network consisting of four half-lines starting from the origin O, parallel to facets of B^ϕ and not lying in a half-plane. Then $\mathcal{S}$ is minimal.*

Proof. Up to a rotation we have only two possibilities, see Figure 7 (e) and (f). Since the ideas are the same, we only prove the minimality of $\mathcal{S}$ in Figure 7 (f). Fix $R>0$, take a compact perturbation $\mathcal{A}$ of $\mathcal{S}$ in $\mathring{B}_R^\phi$ and let $X_1,X_2,X_3,X_4\in\partial B_R^\phi$ be the intersection points of the half-lines of $\mathcal{S}$ with ∂B_R^ϕ. Since $B_R^\phi\cap\mathcal{A}$ is connected, there exist curves $\Gamma_1,\Gamma_2\subset B_R^\phi\cap\mathcal{A}$ connecting X_1 with X_3 and X_2 with X_4. Notice that if $\mathcal{H}^1(\Gamma_1\cap\Gamma_2)=0$ (see Figure 9 (a)), then using the additivity of the ϕ-length and the minimality of segments we get

$$\ell_\phi(\mathcal{A},\mathring{B}_R^\phi)\ge\ell_\phi(\Gamma_1\cup\Gamma_2)=\ell_\phi(\Gamma_1)+\ell_\phi(\Gamma_2)\ge\ell_\phi([X_1X_3])+\ell_\phi([X_2X_4])=\ell_\phi(\mathcal{S},\mathring{B}_R^\phi).$$

Hence, we may assume $\mathcal{H}^1(\Gamma_1\cap\Gamma_2)>0$ (see Figure 9 (b)). Let $T_1,T_2\in\Gamma_1\cap\Gamma_2$ be two points such that the subcurves $\Gamma_1'\subset\Gamma_1$ and $\Gamma_2'\subset\Gamma_2$ connecting T_1 and T_2 are maximal. Notice that T_1 and T_2 may coincide. These two points divide Γ_1 and Γ_2 into three parts: Γ_1 is divided into (X_1,T_1), (T_1,T_2) and (T_2,X_3), and Γ_2 is divided into (X_2,T_1), (T_1,T_2) and (T_2,X_4). Notice that by maximality the subcurves ending at X_i have disjoint relative interiors.

By the minimality of segments and assumption $\mathcal{H}^1(\Gamma_1\cap\Gamma_2)>0$, we may replace those subcurves with segments with the same endpoints so that T_1,T_2 are two triple junctions of segments so that $T_1\neq T_2$ (see Figures 9 (b) and 10); notice that unlike the subcurves, interiors of those segments may intersect.

Assume first T_1 is a triple junction of segments $[X_1T_1]$, $[X_4T_1]$ and $[T_1T_2]$, and T_2 is a triple junction of segments $[X_2T_2]$, $[X_3T_2]$ and $[T_1T_2]$ as in Figure 9 (b). In this case by the minimality of segments

$$\ell_\phi(\mathcal{A},\mathring{B}_R^\phi)\ge\ell_\phi([X_1T_1X_4])+\ell_\phi([X_2T_2X_3])\ge\ell_\phi([X_1X_4])+\ell_\phi([X_2X_3]).\tag{2.6}$$

Since both sides of the triangles X_1OX_4 and X_2OX_3 ending at O are parallel to adjacent facets of B^ϕ, by Remark 2.3 (a) we have

$$\ell_\phi([X_1X_4])=\ell_\phi([OX_1])+\ell_\phi([OX_4])\quad\text{and}\quad\ell_\phi([X_2X_3])=\ell_\phi([OX_2])+\ell_\phi([OX_3]).$$

Thus, placing these equalities into (2.6) we get

$$\ell_\phi(\mathcal{A},\mathring{B}_R^\phi)\ge\sum_{i=1}^4\ell_\phi([OX_i])=\ell_\phi(\mathcal{S},\mathring{B}_R^\phi).$$

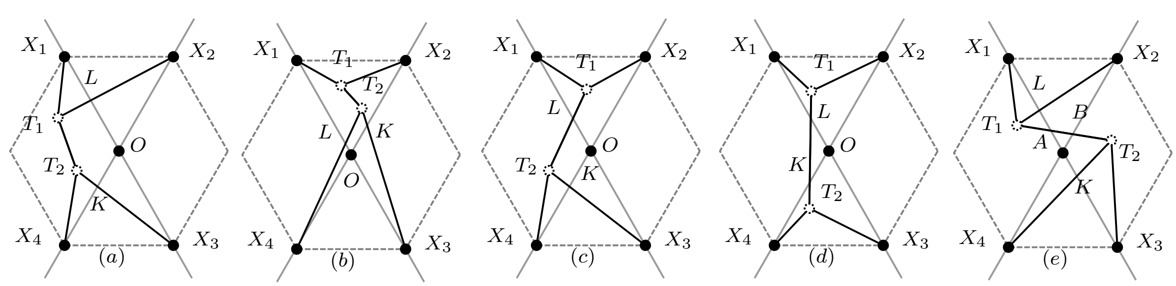

FIG. 10. Some possible locations of T_1 and T_2.

Now we turn to a more involved case, where T_1 is a triple junction of segments $[X_1T_1]$, $[X_2T_1]$ and $[T_1T_2]$ and T_2 is a triple junction of $[X_4T_2]$, $[X_3T_2]$ and $[T_1T_2]$. According to the location T_1 and T_2, as well as to the symmetry of $\mathcal{S}$, we have the following five possibilities.

Case 1: T_1 and T_2 belong to the left parallelogram with three vertices at X_1, O and X_4, see Figure 10 (a). Let $\{L\} := [OX_1] \cap [T_1X_2]$ and $\{K\} := [OX_4] \cap [T_2X_3]$. By the minimality of segments and Remark 2.3 (a)

$$\ell_\phi([X_1T_1T_2X_4]) \geq \ell_\phi([X_1X_4]) = \ell_\phi([X_1OX_4]) = \ell_\phi([X_1O]) + \ell_\phi([OX_4]),$$

where $[ABC\ldots]$ is the polygonal curve made of segments $[AB], [BC], \ldots$. On the other hand, by Remark 2.3 (c) and the monotonicity of the ϕ-length,

$$\ell_\phi([T_1X_2]) \geq \ell_\phi([LX_2]) = \ell_\phi([OX_2]), \quad \ell_\phi([T_2X_3]) \geq \ell_\phi([KX_3]) = \ell_\phi([OX_3]).$$

Therefore,

$$\ell_\phi(\mathcal{A}, \mathring{B}_R^\phi) \geq \ell_\phi([X_1T_1T_2X_4]) + \ell_\phi([T_1X_2]) + \ell_\phi([T_2X_3]) \geq \sum_{i=1}^4 \ell_\phi([OX_i]) = \ell_\phi(\mathcal{S}, \mathring{B}_R^\phi).$$

Case 2: T_1 and T_2 lie in the upper triangle X_1OX_2, see Figure 10 (b). Let $\{L\} := [T_2X_4] \cap [X_1X_3]$ and $\{K\} := [T_2X_3] \cap [X_4X_2]$. By Remark 2.3 (a) applied with the triangles T_2KO and KOX_4,

$$\ell_\phi([T_2X_4]) = \ell_\phi([T_2L]) + \ell_\phi([LOX_4]) = \ell_\phi([T_2LO]) + \ell_\phi([OX_4]).$$

Similarly,

$$\ell_\phi([T_2X_3]) = \ell_\phi([T_2K]) + \ell_\phi([KOX_3]) = \ell_\phi(T_2KO) + \ell_\phi(OX_3).$$

Next, by the minimality of segments, $\ell_\phi([T_1T_2LO]) \geq \ell_\phi([T_1O])$. Thus,

$$\begin{aligned}\ell_\phi([T_2X_4]) + \ell_\phi([T_2X_3]) + \ell_\phi([T_1T_2]) \geq & \ell_\phi(T_1T_2LO) + \ell_\phi(OX_4) + \ell_\phi(OX_3) \\ \geq & \ell_\phi(T_1O) + \ell_\phi(OX_4) + \ell_\phi(OX_3).\end{aligned}$$

Consider the network $\mathcal{B}$ consisting of segments $[X_1T_1]$, $[X_2T_1]$, $[T_1O]$ and $[OX_4]$. Clearly, it is a compact perturbation of the conical triod $\mathcal{C}$ consisting of the three half-lines starting from the origin and passing through X_1, X_2 and X_4, respectively. By Proposition 2.13 $\mathcal{C}$ is minimal, and thus, $\ell_\phi(\mathcal{B}, \mathring{B}_R^\phi) \geq \ell_\phi(\mathcal{C}, \mathring{B}_R^\phi)$. Equivalently,

$$\ell_\phi([X_1T_1]) + \ell_\phi([X_2T_1]) + \ell_\phi([T_1O]) \geq \ell_\phi([OX_1]) + \ell_\phi([OX_2]).$$

Then

$$\begin{aligned}\ell_\phi(\mathcal{A}, \mathring{B}_R^\phi) \geq & \ell_\phi([T_2X_4]) + \ell_\phi([T_2X_3]) + \ell_\phi([X_1T_1]) + \ell_\phi([X_2T_1]) + \ell_\phi([T_1T_2]) \\ \geq & \ell_\phi([X_1T_1]) + \ell_\phi([X_2T_1]) + \ell_\phi(T_1O) + \ell_\phi(OX_4) + \ell_\phi(OX_3) \\ \geq & \sum_{i=1}^4 \ell_\phi([OX_i]) = \ell_\phi(\mathcal{S}, \mathring{B}_R^\phi).\end{aligned}$$

Case 3: T_1 lies in the upper triangle X_1OX_2 and T_2 lies in the left parallelogram with three vertices X_1, O, X_4, Figure 10 (c). Let $\{L\} := [OX_1] \cap [T_1T_2]$ and $\{K\} := [OX_4] \cap [T_2X_3]$. Then

$$\ell_\phi([T_1T_2X_4]) \geq \ell_\phi([T_1L]) + \ell_\phi([LX_4]) = \ell_\phi([T_1LO]) + \ell_\phi([OX_4]) \geq \ell_\phi([T_1O]) + \ell_\phi([OX_4]),$$

and

$$\ell_\phi([T_2X_3]) \geq \ell_\phi([KX_3]) = \ell_\phi([OX_3]).$$

Moreover, as in case 2

$$\ell_\phi([X_1T_1]) + \ell_\phi([X_2T_1]) + \ell_\phi([T_1O]) \geq \ell_\phi([OX_1]) + \ell_\phi([OX_2]).$$

Summing these inequalities we get

$$\ell_\phi(\mathcal{A}, \mathring{B}_R^\phi) \geq \sum_{i=1}^{4} \ell_\phi([OX_i]) = \ell_\phi(\mathcal{S}, \mathring{B}_R^\phi).$$

Case 4: T_1 lies in the upper triangle X_1OX_2 and T_2 lies in the lower triangle X_4OX_3, Figure 10 (d). Let $L \in [X_1OX_2] \cap [T_1T_2]$ and $K \in [X_4OX_3] \cap [T_1T_2]$. By Remark 2.3 (a)

$$\ell_\phi([T_1T_2]) = \ell_\phi([T_1LOKT_2]) \geq \ell_\phi([T_1O]) + \ell_\phi([T_2O]).$$

Now applying the minimality of conical critical triods as in case 2 we get

$$\ell_\phi([X_1T_1]) + \ell_\phi([X_2T_1]) + \ell_\phi([OT_1]) \geq \ell_\phi([OX_1]) + \ell_\phi([OX_2])$$

and

$$\ell_\phi([X_4T_2]) + \ell_\phi([X_3T_2]) + \ell_\phi([OT_2]) \geq \ell_\phi([OX_3]) + \ell_\phi([OX_4]).$$

Summing these inequalities we get $\ell_\phi(\mathcal{A}, \mathring{B}_R^\phi) \geq \ell_\phi(\mathcal{S}, \mathring{B}_R^\phi)$.

Case 5: T_1 lies in the left parallelogram and T_2 lies in the right parallelogram, Figure 10 (e). Let $A \in [T_1T_2] \cap [X_1X_3]$, $B \in [T_1T_2] \cap [X_2X_4]$, $L \in [OX_1] \cap [T_1X_2]$ and $K \in [OX_3] \cap [T_2X_4]$. Then

$$\ell_\phi([T_1X_2]) \geq \ell_\phi([LX_2]) = \ell_\phi([OX_2]), \quad \ell_\phi([T_2X_4]) \geq \ell_\phi([KX_4]) = \ell_\phi([OX_4])$$

and

$$\ell_\phi([X_1T_1T_2X_3]) \geq \ell_\phi([X_1X_3]).$$

Summing these inequalities we get $\ell_\phi(\mathcal{A}, \mathring{B}_R^\phi) \geq \ell_\phi(\mathcal{S}, \mathring{B}_R^\phi)$. □

Remark 2.15. In the case of a quadruple junction we cannot produce a paired calibration made of constant fields. Indeed, let half-lines $L_1, \ldots, L_4$ start from the origin and parallel to the facets of ∂B^ϕ as in Figure 7 (e)-(f), and let $N \in \mathrm{CH}(\mathcal{S})$. As observed in Example 2.9, at quadruple junction we have $N_1 + N_3 = 0 = N_2 + N_4$ and additionally, $N_1 = -N_3 = (-\frac{2}{\sqrt{3}}, 0)$ for Figure 7 (e). Suppose we can choose vectors $\xi_i \in \mathbb{R}^2$, $i = 1, \ldots, 4$, such that $\phi(\xi_i - \xi_j) \leq 1$ for all i, j and $N_i = \xi_i - \xi_{i+1}$, with $\xi_5 = \xi_1$. In case of "X" (Figure 7 (f)), N_1 and N_2 lie in the facets of B^ϕ with common vertex $(-\frac{2}{\sqrt{3}}, 0)$, and thus, $N_1 + N_2 \in B^\phi$ if and only if $N_1 = (-\frac{1}{\sqrt{3}}, -1)$ and $N_2 = (-\frac{1}{\sqrt{3}}, 1)$. However, in this case, $N_1 + N_4 = (0, 2) \notin B^\phi$, i.e., $\phi(\xi_4 - \xi_2) = \phi(N_1 + N_4) > 1$. A similar reasoning applies in case of "Ψ" (Figure 7 (e)).

Applying Remark 2.3 (a) and (c) we immediately find that conical minimal networks $\mathcal{S}$ with a quadruple junction are not "isolated": in fact, in an arbitrarily small neighborhood[6] of the quadruple junction, we can find a compact perturbation $\mathcal{A}$ of $\mathcal{S}$ having the same ϕ-length as $\mathcal{S}$ (see Figure 11). Similar perturbations can be done for a ϕ-regular conical minimal triods

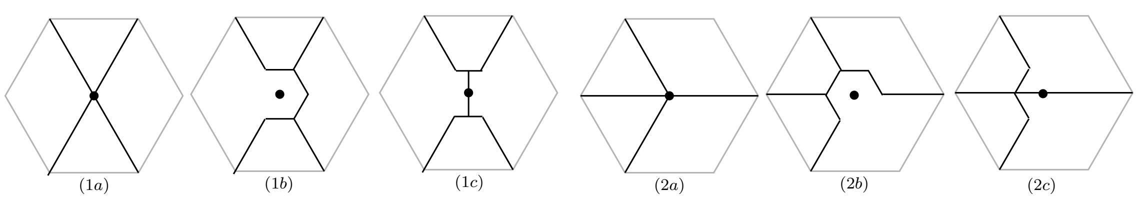

FIG. 11.

$\mathcal{S}$ not satisfying the 120^o-condition. However, for a ϕ-regular conical triod with the 120^o-condition the following condition holds: if $\mathcal{A}$ is any compact perturbation of $\mathcal{S}$ in D_R with a triple junction different from $\mathcal{S}$, then $\ell_\phi(\mathcal{S}, D_R) < \ell_\phi(\mathcal{A}, D_R)$.

[6] Differently from the network in Figure 8, where the perturbation is not "small".

2.8. **ϕ-curvature of an admissible network.** In what follows we work only with polygonal admissible networks. As we have seen after Definition 2.6, conical triods as in Figure 7 (a) do not admit a CH field (no balance condition at the triple junction), and hence, not every admissible network is ϕ-regular. The following lemma shows that without such "noncritical" triple junctions any admissible network is ϕ-regular.

Theorem 2.16 (**Existence of a minimal CH field**). *Let $\mathcal{S} = \cup_{i=1}^n S_i$ be a polygonal admissible network whose triple junctions are locally contained in conical critical triods. Then $\mathcal{S}$ is ϕ-regular. Moreover, there exists a minimizer N^0 of the problem*

$$\min_{N\in \mathrm{CH}(\mathcal{S})} \sum_{i=1}^n \int_{S_i} \Big[\mathrm{div}_\tau N_{|_{S_i}}\Big]^2 \phi^o(\nu_{S_i})\, d\mathcal{H}^1, \tag{2.7}$$

and for any S_i the number $\mathrm{div}_\tau N^0{}_{|_{S_i}}$ does not depend on N^0.

Proof. As in [9, Remark 4.7], it is enough to determine a CH field N at the endpoints of each S_i and then linearly interpolate it inside S_i. Therefore, given a segment/half-line S_i, we first define a vector field $N_i := N_{|S_i}$ with $N_i \in \partial\phi^o(\nu_{S_i})$ at the endpoints of S_i.

Take any vertex X of $\mathcal{S}$. Assume that X is a common endpoint of only two segments/half-lines, say S_1 and S_2. The definition is standard: since the angle between S_1 and S_2 is 120^o, we define $N_1(X) = N_2(X) = V$ resp. $N_1(X) = N_2(X) = -V$, where V is the vertex of B^ϕ whose adjacent facets have outer resp. inner unit normals ν_{S_1} and ν_{S_2}.

Next, suppose X is a triple junction, say a common endpoint of three segments/half-lines S_1, S_2 and S_3. Recall that, up to a rotation and a mirror reflection, we have only two critical triple junctions (see Figure 7 (b) and (c)). In either case we can choose three vectors $N_1(X)$, $N_2(X)$, $N_3(X)$ satisfying $N_i(X) \in \partial\phi^o(\nu_{S_i})$ such that

$$\sum_{i=1}^3 (-1)^{\sigma_i} N_i(X) = 0,$$

where $\sigma_i = 0$ if S_i oriented from X, and $\sigma_i = 1$ otherwise.

Similarly, if X is an m-junction of, say, $S_1, \dots, S_m$ for some $m = 4,5,6$, then the vectors $N_i(X) \in \partial\phi^o(\nu_{S_i})$ and numbers $\sigma_i \in \{0,1\}$ for $i = 1,\dots,m$ with

$$\sum_{i=1}^m (-1)^{\sigma_i} N_i(X) = 0$$

can be defined, for instance, as in Figure 7 (d)-(h). We omit the details.

Now we extend N_i to the relative interior of S_i. If S_i is a half-line starting from a vertex X of $\mathcal{S}$, we define

$$N_i(Z) := N_i(X) \quad \text{for all } Z \in S_i.$$

If $S_i = [XY]$ is a segment, then we extend N_i linearly inside S_i as

$$N_i(Z) := N_i(X) + \lambda[N_i(Y) - N_i(X)], \quad Z := X + \mathcal{H}^1(S_i)\lambda\tau_{S_i}, \quad \lambda \in [0,1],$$

where τ_{S_i} is the tangent of S_i – the clockwise 90^o-rotation of ν_{S_i}. Now defining $N_{|_{S_i}} = N_i$, we get $N \in \mathrm{CH}(\mathcal{S})$ and

$$\mathrm{div}_\tau N_{|_{S_i}} = \begin{cases} 0 & \text{if } S_i \text{ is a half-line,}\\ \frac{[N_i(Y)-N_i(X)]\cdot\tau_{S_i}}{\mathcal{H}^1(S_i)} & \text{if } S_i = [XY] \text{ is a segment.}\end{cases} \tag{2.8}$$

Thus, $\mathrm{CH}(\mathcal{S})$ is nonempty and so $\mathcal{S}$ is ϕ-regular.

Finally, we prove that the minimum problem (2.7) admits a solution. Let $(N^k) \subset \mathrm{CH}(\mathcal{S})$ be a minimizing sequence and consider the sequence $(N_i^k(X))_k$ at each endpoint X of a segment/half-line S_i in $\mathcal{S}$. Since ∂B^ϕ is compact and the number of vertices of $\mathcal{S}$ is finite, up to a (not relabelled) subsequence, $N_i^k(X) \to N_i(X)$ for some $N_i(X) \in \partial B^\phi$. Clearly, $N_i^k(X) = N_i(X)$ if X is not a multiple junction, and letting $k \to +\infty$ in the equalities

$$\sum_i (-1)^{\sigma_i} N_i^k(X) = 0 \quad \text{and} \quad N_i^k(X)\cdot \nu_{S_i} = \phi^o(\nu_{S_i})$$

we obtain

$$\sum_i (-1)^{\sigma_i} N_i(X) = 0 \quad \text{and} \quad N_i(X)\cdot \nu_{S_i} = \phi^o(\nu_{S_i})$$

at multiple junctions. As above, let us extend N_i in the (relative) interior of S_i, linearly interpolating its values at the endpoints, and denote by N^0 such an extension. Then denoting by M the left-hand-side of (2.7) we get

$$\begin{aligned}
M = & \lim_{k\to+\infty} \sum_{i=1}^n \int_{S_i} \big[\mathrm{div}_\tau N_i^k\big]^2 \phi^o(\nu_{S_i})\, d\mathcal{H}^1 \\
\ge & \lim_{k\to+\infty} \sum_{S_i\,=\,[X_iY_i]\text{ segment}} \phi^o(\nu_{S_i})\mathcal{H}^1(S_i)\Big[\tfrac{1}{\mathcal{H}^1(S_i)}\int_{S_i} \mathrm{div}_\tau N_i^k\, d\mathcal{H}^1\Big]^2 \\
= & \lim_{k\to+\infty} \sum_{S_i\,=\,[X_iY_i]\text{ segment}} \phi^o(\nu_{S_i}) \tfrac{|N_i^k(Y_i)-N_i^k(X_i)|^2}{\mathcal{H}^1(S_i)} \\
= & \sum_{S_i\,=\,[X_iY_i]\text{ segment}} \phi^o(\nu_{S_i}) \tfrac{|N_i^0(X_i)-N_i^0(Y_i)|^2}{\mathcal{H}^1(S_i)} \\
= & \sum_{i=1}^n \int_{S_i} \big[\mathrm{div}_\tau N^0\big]^2 \phi^o(\nu_{S_i})\, d\mathcal{H}^1,
\end{aligned}$$

where in the first inequality we used the Jensen inequality, in the second equality the definition of tangential divergence and the fundamental theorem of calculus and in the last equality we used (2.8). This implies $N^0 \in \mathrm{CH}(\mathcal{S})$ is a minimizer.

The independence of $\mathrm{div}_\tau N^0$ on N^0 follows from the strict convexity of the functional in (2.7) in the tangential divergence. □

We call any minimizer N of (2.7) a *minimal CH field* (of $\mathcal{S}$).

Remark 2.17. Let $\mathcal{S} = \cup_i S_i$ be as in Theorem 2.16 and N^0 be any minimal CH field. Then:

- N^0 is uniquely defined at the simple vertices of $\mathcal{S}$: if the simple vertex X is a common endpoint of S_i and S_j, then $N^0(X)$ is defined as V resp. $-V$, where V is the vertex of B^ϕ whose adjacent facets are parallel to S_i and S_j, and ν_{S_i} and ν_{S_j} are the outer resp. inner unit normals to B^ϕ;
- $N_i^0 := N^0_{|S_i}$ is constant on half-lines of $\mathcal{S}$;
- If S_i is a segment, then $N^0_{|S_i}$ is linear on S_i, and N^0 solves the minimum problem

$$\inf_{N\in \mathrm{CH}(\mathcal{S})} \sum_{S_i\,=\,[X_iY_i]\text{ segment}} \phi^o(\nu_{S_i}) \tfrac{|N_i(Y_i)-N_i(X_i)|^2}{\mathcal{H}^1(S_i)}; \tag{2.9}$$

- If, as in [9], we call the number

$$\kappa^\phi_{S_i} := \mathrm{div}_\tau N_i^0$$

the *ϕ-curvature* (or crystalline curvature) of S_i, then $\kappa^\phi_{S_i}\nu_{S_i}$ is independent of the orientation of S_i and the ϕ-curvature of any half-line of $\mathcal{S}$ is 0. Furthermore, as in the two-phase case [22], if the network forms locally a convex set around the segment $S_i := [X_iY_i]$, not ending at a multiple junction, and ν_{S_i} is directed "inward" resp. "outward" to that set, then

$$\kappa^\phi_{S_i} = -\frac{|N^0(Y_i)-N^0(X_i)|}{\mathcal{H}^1(S_i)} \quad \text{resp.} \quad \kappa^\phi_{S_i} = \frac{|N^0(Y_i)-N^0(X_i)|}{\mathcal{H}^1(S_i)}.$$

When $\mathcal{S}$ is not locally convex around S_i, then $\kappa^\phi_{S_i} = 0$.

From now on, when we speak about the crystalline curvature of a network, we always assume that the triple junctions are locally contained in conical critical triods.

Example 2.18. In general, (2.7) may have more than one minimizer. For instance, consider an admissible triod of half-lines S_1, S_2, S_3 meeting at X at 120^o-angles. Then $N_1(X)$ can be any vector in the facet of B^ϕ parallel to S_1 having the same unit normal. Clearly, $N_2(X)$ and $N_3(X)$ are uniquely chosen (satisfying the balance condition). Then the locally constant vector field $N_i := N_i(X)$ is a minimal CH field. In particular, in this case there are infinitely many minimal CH fields.

Lemma 2.19 (**ϕ-curvature-balance condition**). *Let $\mathcal{S}$ be a polygonal admissible network containing at least one segment, X be a triple junction of three segments/half-lines S_1, S_2, S_3 of $\mathcal{S}$ meeting at 120^o-angles and suppose that there exists a minimal CH field whose values at X do not coincide with vertices of B^ϕ. Then*

$$(-1)^{\sigma_1}\kappa^\phi_{S_1} + (-1)^{\sigma_2}\kappa^\phi_{S_2} + (-1)^{\sigma_3}\kappa^\phi_{S_3} = 0, \tag{2.10}$$

where $\sigma_i = 0$ if S_i is oriented from X and $\sigma_i = 1$ otherwise.

Proof. Let N^0 be a minimal CH field as in the statement. Without loss of generality we assume that S_1, S_2, S_3 are oriented from X so that

$$N^0_1(X) + N^0_2(X) + N^0_3(X) = 0.$$

Since $\mathcal{S}$ contains at least one segment and X is a triple junction, at least one S_i, say S_1, is a segment $[XY_1]$. First assume that both S_2 and S_3 are half-lines, and in this case we define a CH field N on $\mathcal{S}$ as follows: we set $N = N^0$ on all segments/half-lines S_i of $\mathcal{S}$ with $i \ge 4$ (if any), and $N_i = N_{|S_i}$ is constant on S_1, S_2, S_3 with $N_1 = N^0_1(Y_1)$ and N_2 and N_3 are unique constant vectors satisfying $N^0_1(Y_1) + N_2 + N_3 = 0$. Obviously, N is also a CH field minimizing the functional in (2.7), and by the uniqueness of the tangential divergence of the minimizers, $\kappa^\phi_{S_i} = 0$ for $i = 1, 2, 3$. Hence, (2.10) holds.

Now, assume $S_2 = [XY_2]$ is a segment and S_3 is a half-line. Clearly, $\kappa^\phi_{S_3} = 0$. Let $\overline{N} \in \mathrm{CH}(\mathcal{S})$ be any CH field such that $\overline{N} = N^0$ on all S_i with $i \ge 4$. Let V_1 be a vertex of B^ϕ closest to $N_1(X)$, and V_2, V_3 be other two vertices directed as $N_2(X)$ and $N_3(X)$, basically obtained rotating V_1 by $\pm 120^o$. Let us define

$$x := |\overline{N}_1(X) - V_1| = |\overline{N}_2(X) - V_2| \in [0, \tfrac{2}{\sqrt{3}}], \quad a_1 := |\overline{N}_1(Y_1) - V_1|, \quad a_2 := |\overline{N}_2(Y_2) - V_2|.$$

Then by the minimality of N^0 (see also the proof of Lemma 3.7 below) $x = x^0 := |N^0_1(X) - V_1|$ is a minimizer of the function

$$f(x) := \frac{(x-a_1)^2}{\mathcal{H}^1(S_1)} + \frac{(x-a_2)^2}{\mathcal{H}^1(S_2)}.$$

By assumption $N^0_1(X)$ does not coincide with vertices of B^ϕ, and thus, $x^0 \in (0, \frac{2}{\sqrt{3}})$. Therefore, it is an interior critical point of f, i.e.,

$$f'(x^0) = \frac{2(x^0-a_1)}{\mathcal{H}^1(S_1)} + \frac{2(x^0-a_2)}{\mathcal{H}^1(S_2)} = 0.$$

Now observing $\frac{x^0-a_1}{\mathcal{H}^1(S_1)} = -\frac{(N_1^0(Y_1)-N_1^0(X)]\cdot\tau_{S_1}}{\mathcal{H}^1(S_1)} = -\kappa^\phi_{S_1}$ and $\frac{x^0-a_2}{\mathcal{H}^2(S_2)} = -\frac{(N_2^0(Y_2)-N_2^0(X)]\cdot\tau_{S_2}}{\mathcal{H}^2(S_2)} = -\kappa^\phi_{S_2}$ (for the first equality use that $N_1^0(X), V_1, N_1^0(Y_1)$ lie on the same facet of B^ϕ and for the second one, see (2.8)), we deduce

$$\kappa^\phi_{S_1} + \kappa^\phi_{S_2} + \kappa^\phi_{S_3} = \kappa^\phi_{S_1} + \kappa^\phi_{S_2} = 0.$$

The case when all S_1, S_2, S_3 are segments is treated similarly. □

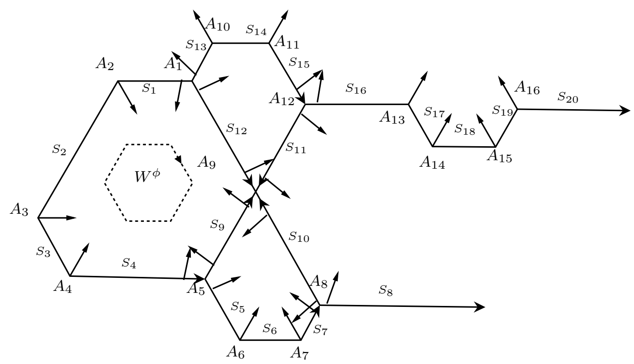

FIG. 12.

2.9. **Computation of ϕ-curvature.** As an example, let us compute the ϕ-curvature of the segments of the network $\mathcal{S}$ in Figure 12, where arrows at the endpoints of the polygonal curves show the orientation. Given $N \in \mathrm{CH}(\mathcal{S})$, write as usual $N_i := N_{|S_i}$. As mentioned in Remark 2.17, the ϕ-curvature of the half-lines S_{20} and S_8 are 0.

First, consider the segment $S_2 := [A_2A_3]$. This segment does not end at any multiple junction, $N_2(A_2) := (\frac{1}{\sqrt3}, -1)$ and $N_2(A_3) := (\frac{2}{\sqrt3}, 0)$. The curve $[A_1A_2A_3A_4]$ (and hence $\mathcal{S}$) is locally convex around S_2. As shown in Figure 12, the unit normal of S_2 is directed "inward", and hence, by Remark 2.17 the ϕ-curvature of S_2 is negative and is equal to $-\frac{|N_2(A_3)-N_2(A_2)|}{\mathcal{H}^1(S_2)} = -\frac{2}{\sqrt3\mathcal{H}^1(S_2)}$, where $\frac{2}{\sqrt3}$ is the sidelength of B^ϕ. Similarly, $\kappa^\phi_S = -\frac{2}{\sqrt3\mathcal{H}^1(S)}$ for $S \in \{S_3, S_6, S_{18}\}$. On the other hand, the curve $[A_1A_{10}A_{11}A_{12}]$ (and hence $\mathcal{S}$) is locally convex around $S_{14} = [A_{10}A_{11}]$ and the unit normal of S_{14} is directed "outward". Thus, the ϕ-curvature of this segment is positive and equals to $\frac{2}{\sqrt3\mathcal{H}^1(S_{14})}$. Notice that $\mathcal{S}$ is not locally convex around S_{17} and S_{19}, and therefore, the ϕ-curvature of these segments is zero.

Now consider the segments ending at multiple junctions, for instance, at the triple junction A_1. Let

$$x_1 := |N_1(A_2) - N_1(A_1)| \in [0, \tfrac{2}{\sqrt3}].$$

In view of [7, Lemma 2.16], given $N_1(A_1)$ we can uniquely define $N_{13}(A_1)$ and $N_{12}(A_1)$ to fulfill the balance condition

$$N_1 + N_{13}(A_1) + N_{12}(A_1) = 0,$$

the signs "+" are chosen because of the orientations of S_1, S_{13} and S_{12} with respect to A_1. Then using the symmetry of B^ϕ one can readily check that

$$\big|N_{13}(A_1) - (-1,0)\big| = \big|N_{12}(A_1) - (1,0)\big| = x_1.$$

In particular,

$$|N_{13}(A_{10}) - N_{13}(A_1)| = \tfrac{2}{\sqrt3} - x_1.$$

Next consider the triple junction A_{12}. Setting

$$x_2 := |N_{15}(A_{12}) - (1,0)| = |N_{16}(A_{12}) - (\tfrac{1}{\sqrt3}, 1)| = |N_{11}(A_{12}) - (1,0)| \in [0, \tfrac{2}{\sqrt3}],$$

we have

$$|N_{15}(A_{12}) - N_{15}(A_{11})| = |N_{16}(A_{13}) - N_{16}(A_{12})| = \frac{2}{\sqrt3} - x_2.$$

Similarly, for the triple junction A_8 we have

$$x_3 := |N_{10}(A_8) - (-1,0)| = |N_8(A_8) - (-\tfrac{1}{\sqrt{3}}, 1)| = |N_7(A_8) - (-1,0)| \in [0, \tfrac{2}{\sqrt{3}}]$$

and

$$|N_7(A_8) - N_7(A_7)| = x_3,$$

whereas for the triple junction A_5 we have

$$x_4 := |N_9(A_5) - (-1,0)| = |N_4(A_5) - (\tfrac{1}{\sqrt{3}}, 1)| = |N_5(A_5) - (1,0)| \in [0, \tfrac{2}{\sqrt{3}}]$$

and

$$|N_4(A_5) - N_4(A_4)| = |N_5(A_6) - N_5(A_5)| = x_4.$$

Finally, we turn to the quadruple junction A_9. According to Figure 12 let

$$\big|N_{12}(A_9) - (1,0)\big| = x_5 \in [0, \tfrac{2}{\sqrt{3}}], \quad \big|N_{11}(A_9) - (1,0)\big| = x_6 \in [0, \tfrac{2}{\sqrt{3}}].$$

Since all segments $S_{19}, S_{20}, S_{21}, S_{22}$ enter (are directed to) the quadruple junction, one can readily check (see also Example 2.9) that the balance condition

$$[N_{12} + N_{11} + N_9 + N_{10}]\big|_{A_9} = 0$$

holds if and only if $N_{12}(A_9) = -N_{10}(A_9)$ and $N_{11}(A_9) = -N_9(A_9)$. Thus,

$$|N_{12}(A_9) - N_{12}(A_1)| = |x_5 - x_1|, \quad |N_{11}(A_9) - N_{11}(A_{12})| = |x_6 - x_2|,$$

and

$$|N_{10}(A_9) - N_{10}(A_8)| = |x_5 - x_3|, \quad |N_9(A_9) - N_9(A_5)| = |x_6 - x_4|.$$

In view of these observations N_0 is a solution of the minimum problem (2.9) if and only if $x^0 := (x_1^0, \ldots, x_6^0)$, defined as above with $N = N^0$, minimizes the quadratic function

$$\begin{aligned} g(x) := {} & \tfrac{x_1^2}{\mathcal{H}^1(S_1)} + \tfrac{(\frac{2}{\sqrt{3}} - x_1)^2}{\mathcal{H}^1(S_{13})} + \tfrac{(x_5 - x_1)^2}{\mathcal{H}^1(S_{12})} + \tfrac{(\frac{2}{\sqrt{3}} - x_2)^2}{\mathcal{H}^1(S_{15})} + \tfrac{(\frac{2}{\sqrt{3}} - x_2)^2}{\mathcal{H}^1(S_{16})} \\ & + \tfrac{(x_6 - x_2)^2}{\mathcal{H}^1(S_{11})} + \tfrac{x_3^2}{\mathcal{H}^1(S_7)} + \tfrac{(x_5 - x_3)^2}{\mathcal{H}^1(S_{10})} + \tfrac{x_4^2}{\mathcal{H}^1(S_5)} + \tfrac{x_4^2}{\mathcal{H}^1(S_4)} + \tfrac{(x_6 - x_4)^2}{\mathcal{H}^1(S_9)} \end{aligned}$$

among all $x := (x_1, \ldots, x_6) \in [0, \frac{2}{\sqrt{3}}]^6$.

Example 2.20. Suppose furthermore that in Figure 12

$$\mathcal{H}^1(S_{12}) = \mathcal{H}^1(S_{11}) = \mathcal{H}^1(S_9) = \mathcal{H}^1(S_{10}) = 1$$

and

$$\mathcal{H}^1(S_1) = \mathcal{H}^1(S_{13}) = \mathcal{H}^1(S_{15}) = \mathcal{H}^1(S_{16}) = \mathcal{H}^1(S_4) = \mathcal{H}^1(S_5) = \mathcal{H}^1(S_7) = 1 - \epsilon$$

for some $\epsilon \in (0,1)$. Then

$$\begin{aligned} g(x) = {} & \tfrac{x_1^2 + (\frac{2}{\sqrt{3}} - x_1)^2 + 2(\frac{2}{\sqrt{3}} - x_2)^2 + x_3^2 + 2x_4^2}{1 - \epsilon} \\ & + (x_5 - x_1)^2 + (x_6 - x_2)^2 + (x_5 - x_3)^2 + (x_6 - x_4)^2. \end{aligned}$$

Since g is nonnegative and quadratic, solving the linear system $\nabla g(x) = 0$ we find that the minimizer x^0 is uniquely defined as

$$x_1^0 := \tfrac{2(3-\epsilon)}{\sqrt{3}(7-3\epsilon)}, \quad x_2^0 := \tfrac{5-\epsilon}{\sqrt{3}(3-\epsilon)}, \quad x_3^0 := \tfrac{2(1-\epsilon)}{\sqrt{3}(7-3\epsilon)},$$

$$x_4^0 := \tfrac{1-\epsilon}{\sqrt{3}(3-\epsilon)}, \quad x_5^0 := \tfrac{2(2-\epsilon)}{\sqrt{3}(7-3\epsilon)}, \quad x_6^0 := \tfrac{1}{\sqrt{3}}.$$

Thus the values of the ϕ-curvature of the segments ending at the quadruple junction are

$$\kappa_{S_{12}}^\phi = \kappa_{S_{10}}^\phi = \tfrac{2}{\sqrt{3}(7-3\epsilon)}, \quad \kappa_{S_{11}}^\phi = -\kappa_{S_9}^\phi = \tfrac{2}{\sqrt{3}(3-\epsilon)} \neq \kappa_{S_{12}}^\phi,$$

therefore, if these segments were translating in the direction of their unit normals with velocity equal to their ϕ-curvature, then the quadruple junction should break into two triple junctions, and the network instantaneously changes its topology.

We can construct other networks containing multiple junctions which exhibit such an unstable behaviour, see Figure 13.

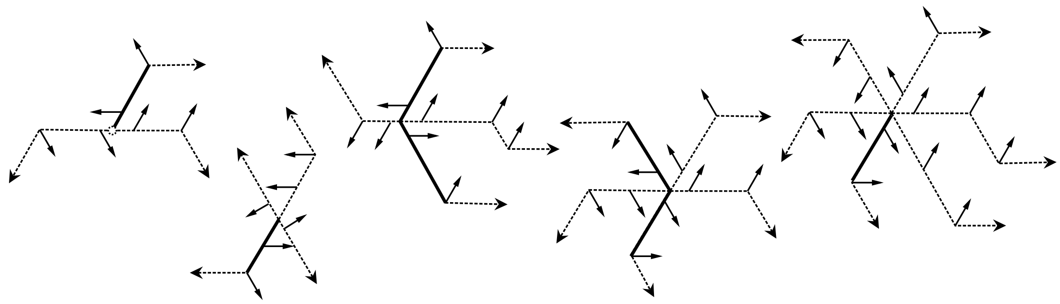

FIG. 13. Networks with multiple junctions at which at least one segment (in bold) has nonzero ϕ-curvature (for a suitable choice of the the lengths of the segments).

2.10. **Parallel networks.** Following the two-phase case we assume that segments in polygonal networks during the flow translate parallel. As in [7, 9] this encourages the following definition.

Definition 2.21 (**Parallel network**). Let $\mathcal{S} := \cup_i S_i$ be a polygonal network consisting of a union of N segments and M half-lines. A polygonal network $\overline{\mathcal{S}}$ is called *parallel* to $\mathcal{S}$ provided that:

- $\overline{\mathcal{S}} := \cup_i \overline{S}_i$ is a union of N segments and M half-lines and each S_i is parallel to $\overline{S}_i$ (so that $\nu_{S_i} = \nu_{\overline{S}_i}$);
- if S_i is a segment, then $\overline{S}_i$ is also a segment;
- if S_i is a half-line, then $\overline{S}_i$ is also a half-line and $S_i \Delta \overline{S}_i$ is bounded (hence S_i and $\overline{S}_i$ lie on the same straight line);
- if $m \geq 2$ segments/half-lines $S_{i_1}, \ldots, S_{i_m}$ have a common endpoint (for instance they form a simple vertex for $m = 2$ or an m-junction for $m \geq 3$), then so do $\overline{S}_{i_1}, \ldots, \overline{S}_{i_m}$.

If $\mathcal{S} = \cup_i S_i$ and $\overline{\mathcal{S}} = \cup_i \overline{S}_i$ are parallel and $S_i \cap S_j \neq \emptyset$ for some $i \neq j$, then $\overline{S}_i \cap \overline{S}_j \neq \emptyset$, and the angle between S_i and S_j equals the angle between $\overline{S}_i$ and $\overline{S}_j$ at their common point. In particular, any network parallel to an admissible network is itself admissible.

Definition 2.22 (**Distance vectors**).

- Let S, T be two parallel straight lines. A vector $H(S,T) \in \mathbb{R}^2$ satisfying $T = S + H(S,T)$ is called a *distance vector* from S to T. For any interval $S_1 \subseteq S$ and interval $T_1 \subseteq T$ we write $H(S_1, T_1) := H(S,T)$. The distance from S_1 to T_1 is defined as

  $$\mathrm{dist}(S_1, T_1) := |H(S_1, T_1)|.$$

  In what follows we frequently refer to the number

  $$h := H(S_1, T_1) \cdot \nu_{S_1}$$

  as the *(signed) height* from S_1 to T_1. Note that $H(S_1, T_1) = h\nu_{S_1}$.
- The *distance* between two parallel networks $\mathcal{S} := \cup_{i=1}^n S_i$ and $\widehat{\mathcal{S}} := \cup_{i=1}^n \widehat{S}_i$ is given by

  $$\mathrm{dist}(\mathcal{S}, \widehat{\mathcal{S}}) := \max_{1 \leq i \leq n} \mathrm{dist}(S_i, \widehat{S}_i).$$

3. ϕ-curvature flow of admissible ϕ-regular networks

Recalling the definition of admissible network (Definition 2.5) and of ϕ-regular network (Definition 2.6), we can now introduce the ϕ-curvature flow.

Definition 3.1 (**ϕ-curvature flow**)**.** Let $\mathcal{S}^0$ be a ϕ-regular (polygonal) admissible network, and $T \in (0,+\infty]$. A family $\{\mathcal{S}(t)\}_{t\in[0,T)}$ is called a regular *ϕ-curvature flow in* $[0,T)$ *(a ϕ-regular flow, for short) starting from* $\mathcal{S}^0$ provided that:

(a) $\mathcal{S}(t)$ is parallel to $\mathcal{S}^0$ for all $t \in [0,T)$;

(b) if $\mathcal{S}^0 = \cup_i S_i^0$ and $\mathcal{S}(t) = \cup_i S_i(t)$, then the heights

$$h_i(\cdot) := H(S_i(\cdot), S_i^0) \cdot \nu_{S_i^0}$$

belong to $C^1((0,T)) \cap C^0([0,T))$ and satisfy

$$\frac{d}{dt}\, h_i(t) = -\phi^o(\nu_{S_i(t)})\kappa^{\phi}_{S_i(t)}, \quad t \in (0,T), \tag{3.1}$$

for any $i = 1, \dots, n$.

By the admissibility of $\mathcal{S}(t)$ we have $\phi^o(\nu_{S_i(t)}) = 1$ for all $t \in [0,T)$, so (3.1) reads as

$$\frac{d}{dt}\, h_i(t) = -\kappa^{\phi}_{S_i(t)}, \quad t \in (0,T). \tag{3.2}$$

Remark 3.2.

(a) By our sign conventions, the ϕ-curvature of the segments of any convex ϕ-regular hexagon $\mathcal{S}^0$ is nonnegative and hence the ϕ-curvature flow $\mathcal{S}(\cdot)$ starting from $\mathcal{S}^0$ shrinks the hexagon.

(b) If $\mathcal{S}^0$ is a critical network (Definition 2.7), then the ϕ-curvature of its segments/half-lines is 0. Therefore, the stationary flow $\mathcal{S}(t) := \mathcal{S}^0$ is the unique ϕ-curvature flow starting from $\mathcal{S}^0$ in $[0,+\infty)$.

(c) Being a gradient flow of the ϕ-length, the ϕ-curvature flow is expected to decrease the ϕ-length. However, this is not the case for nonminimal critical networks, see for instance the network in Figure 8.

(d) Example 2.20 (see also Figure 13) shows that not every admissible network admits a regular ϕ-curvature flow: the ϕ-curvature flow instantaneously should change the topological structure and create new segments or curves (see also [7]).

Definition 3.3 (**Simple network**)**.** An admissible network is called *simple* if it only contains triple junctions meeting at 120^o-angles.

Remark 3.4. In [7, 9] a network with a single triple junction X formed by three polygonal curves, each of which is the union of a segment and a half-line, and possibly with three anisotropies, is called *stable* provided the values of the unique minimal CH field at X do not coincide with vertices of the corresponding Wulff shapes. Definition 3.3 and the next theorem generalize [7, 9] in our single anisotropic case to a wider class of networks which may contain several triple junctions, some of which may not be stable.

The main result of this section is the following

Theorem 3.5 (**Existence and uniqueness of the ϕ-curvature flow**)**.** *Let $\mathcal{S}^0$ be a simple network. Then there exist $T^\dagger \in (0,+\infty]$ and a unique family $\{\mathcal{S}(t)\}_{t\in[0,T^\dagger)}$ of simple networks such that $\mathcal{S}(\cdot)$ is the ϕ-curvature flow starting from $\mathcal{S}^0$. Moreover, if $T^\dagger < +\infty$ then some segment of $\mathcal{S}(t)$ vanishes as $t \nearrow T^\dagger$.*

We need some auxiliary results, which actually provide the steps of the proof, concluded in Section 4.

Lemma 3.6 (Quadratic minimization). *For $n \ge 1$ let $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ and $\{c_{ij}\}_{i,j=1}^n$ be finite sets of nonnegative numbers such that*

- *$c_{ij} = c_{ji}$, $c_{ii} = 0$ for all $i, j = 1, \dots, n$;*
- *if $n \ge 2$, the square matrix $C := (c_{ij})$ is irreducible, i.e., for every $i \ne j$, there exists $m_{ij} \ge 1$ such that $[C^{m_{ij}}]_{ij} > 0$. Equivalently, the unoriented graph $\mathcal{G}$ with n nodes and adjacency matrix*[7] *$(\delta_{c_{ij},0})$ is connected, where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ if $\alpha \ne \beta$* [37, Chapter 1].

Consider the quadratic function

$$\psi(x) := \sum_{i=1}^n a_i x_i^2 + \sum_{i=1}^n b_i (d - x_i)^2 + \sum_{1 \le i < j \le n} c_{ij} (x_i - x_j)^2, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then, for $d > 0$,

$$\min_{x \in [0,d]^n} \psi(x) = 0 \quad \Longleftrightarrow \quad \text{either all } a_i \text{ are zero or all } b_i \text{ are zero.}$$

Furthermore, if $\min_{x\in[0,d]^n} \psi(x) > 0$ then ψ has a unique minimizer and this minimizer lies in $(0, d)^n$.

Proof. If $\sum_{i=1}^n b_i = 0$, i.e., all b_i are 0, then $(0, \dots, 0) \in [0, d]^n$ is the minimizer of ψ and the minimum is 0. On the other hand, if $\sum_{i=1}^n a_i = 0$, i.e., all a_i are 0, then $(d, \dots, d) \in [0, d]^n$ is the minimizer of ψ and again the minimum is 0. As we shall see, these are the only cases when the minimizer belongs to the boundary of $[0, d]^n$.

Assume that $\sum_{i=1}^n b_i > 0$ and $\sum_{i=1}^n a_i > 0$. If $n = 1$ so that $a_1 b_1 > 0$, then $\min_{x_1\in[0,d]} \psi(x_1) > 0$ and its unique minimizer $x_1^0 := \frac{b_1 d}{a_1+b_1}$ belongs to $(0, d)$. Therefore, further, we may suppose that $n \ge 2$.

Consider the equation $\nabla\psi(x) = 0$ for $x \in \mathbb{R}^n$, i.e.,

$$\Big(a_i + b_i + \sum_{j=1}^n c_{ij}\Big) x_i - \sum_{j=1}^n c_{ij} x_j = b_i d, \qquad i = 1, \dots, n. \tag{3.3}$$

Let $A := (A_{ij})$ be the square matrix whose entries are

$$A_{ii} = a_i + b_i + \sum_{l=1}^n c_{il}, \quad A_{ij} = -c_{ij}, \quad i, j = 1, \dots, n, \quad i \ne j.$$

We have

$$Ax^T \cdot x^T = \sum_{i,j=1}^n A_{ij} x_i x_j = \sum_{i=1}^n (a_i + b_i) x_i^2 + \tfrac{1}{2} \sum_{i \ne j} c_{ij} (x_i - x_j)^2 \ge 0. \tag{3.4}$$

We claim that $Ax^T \cdot x^T = 0$ if and only if $x = 0$, and so, A is positive definite. Indeed, since the graph $\mathcal{G}$ associated to the matrix $C = (c_{ij})$ is connected, the last sum in (3.4) is zero if and only if $x_i = x_j$ for all $i \ne j$. Indeed, by connectedness, we can reach from the vertex $i = 1$ to each vertex $i > 1$ of the graph through the vertices $i_1 = 1, i_2, \dots, i_m = i$. Then $c_{i_1 i_2}, c_{i_2 i_3}, \dots, c_{i_{m-1} i_m} > 0$, and hence, $x_1 = x_{i_2} = \ldots = x_{i_{m-1}} = x_i$. Since $\sum_i (a_i + b_i) > 0$ by assumption, this implies $Ax^T \cdot x^T = 0$ iff $x = 0$. In particular all diagonal elements A_{ii} of A are positive.

Let us show that all entries $\hat{a}_{ij}$ of A^{-1} are also positive. Indeed, let D be the matrix formed by the diagonal elements of A, i.e., $D := A + C$. By assumption $c_{ij} \ge 0$, hence the entries of

[7] The adjacency matrix of an (oriented or unoriented) graph is the matrix (α_{ij}), where $\alpha_{ij} = 1$ if there is an edge from the node i to node j, and $\alpha_{ij} = 0$ otherwise.

the matrix $X := D^{-1/2}CD^{-1/2}$ are nonnegative. Moreover, since $D > C$ in the sense of linear operators, for any $x \in \mathbb{S}^{n-1}$

$$Xx^T \cdot x^T = C(D^{-1/2}x)^T \cdot (D^{-1/2}x)^T = Cy^T \cdot y^T < Dy^T \cdot y^T = |D^{1/2}y|^2 = |x|^2 = 1,$$

where $y := D^{-1/2}x \neq 0$. Therefore, the norm $\|X\| = \sup_{x\in\mathbb{R}^n,\,\|x\|=1} Xx^T \cdot x^T$. satisfies $\|X\| < 1$. Then the matrix $I - X$ is invertible and its inverse is given by the Neumann series,

$$A^{-1} = D^{-1/2}(I - X)^{-1}D^{-1/2} = \sum_{k\geq 0} D^{-1/2}X^kD^{-1/2}.$$

Clearly, the entries of $D^{-1/2}X^kD^{-1/2}$ are nonnegative for all $k \geq 0$. Since C is irreducible with nonzero elements, and $D^{-1/2}$ is diagonal with positive diagonal elements, X is also irreducible. In particular, all entries of X^m for some $m \geq 1$ are positive and hence, all elements of A^{-1} are also positive.

Therefore, the system (3.3) has a unique solution

$$x^T := A^{-1}b^T, \quad b = (b_1d, \dots, b_nd);$$

and recalling that $b_id \geq 0$ with $\sum_{i=1}^n b_i > 0$, we deduce $x_i^0 > 0$ for all $i = 1, \dots, n$. This unique solution provides the unique minimizer of ψ.

To prove $x_i^0 < d$, we apply the previous argument to the quadratic function

$$\psi^*(x_1, \dots, x_n) = \psi(d - x_1, \dots, d - x_n)$$

(in this case we use $\sum a_i > 0$) and conclude that the unique minimizer y^0 of ψ^* satisfies $y_i^0 > 0$. By uniqueness, this implies $d - y_i^0 = x_i^0$ and hence, $x_i^0 \in (0, d)$ for all $i = 1, \dots, n$. $\square$

Let $\mathcal{S}$ be a simple network, and divide $\mathcal{S}$ into connected graphs $G_1, \dots, G_m$ removing all simple vertices. Since $\mathcal{S}$ contains only triple junctions, each G_i is either a single segment/half-line or a union of segments/half-lines at some triple junction.

Lemma 3.7. *Let N^0 be a minimal CH field of the simple network $\mathcal{S}$. Suppose G_i contains at least one triple junction. Then for G_i the following holds:*

- *either $\mathrm{div}_\tau N^0 = 0$ on all segments/half-lines of G_i, and in this case the values of N^0 at all triple junctions of G_i can be chosen as (three distinct) vertices of B^ϕ,*
- *or the values of N^0 at all triple junctions of G_i do not coincide with any vertex of B^ϕ*

(see Figure 14).

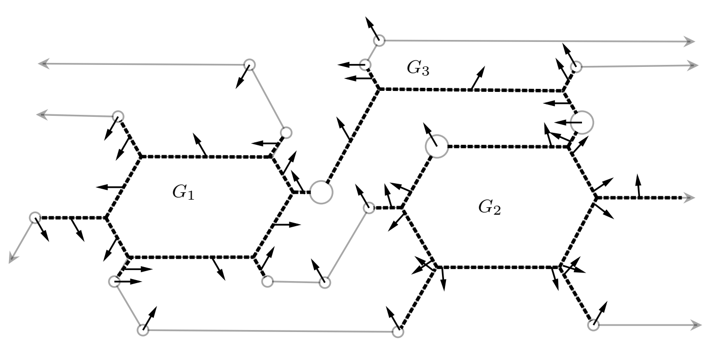

FIG. 14. A simple network parsed into connected graphs G_1, G_2, G_3, by removing all simple vertices (small circles), and a possible CH field. Here G_1, G_2, G_3, consisting of a union of bold dotted lines, are the only graphs containing triple junctions (other graphs are either isolated segments or half-lines). Notice that G_1 and G_3 admit a (locally) constant CH field.

Proof. We assume that $i = 1$ and G_1 contains at least one segment, and G_1 has exactly $r \geq 1$ triple junctions, $X_1, X_2, \ldots, X_r$. Let $N \in \mathrm{CH}(\mathcal{S})$. As we observed in Remark 2.17, the values of N at the simple vertices of $\mathcal{S}$ are uniquely defined. Consider any X_k, which is a junction of segments/half-lines, say, S_{k_1}, S_{k_2} and S_{k_3}, and assume that $S_{k_1} = [X_{k_1} Y_{k_1}]$ is a segment (oriented from X_{k_1}). Let V_k^1 be the vertex of B^ϕ such that $N_{k_1}(X_{k_1}), N_{k_1}(Y_{k_1})$ and V_k^1 lie in the same facet of B^ϕ. As in Section 2.9 define

$$x_k := |N_{k_1}(X_k) - V_k^1| \in [0, \tfrac{2}{\sqrt{3}}].$$

Repeating the same arguments in Section 2.9, the minimum problem leading to N^0 is reduced to minimizing the function

$$\begin{aligned}\psi(x_1, \ldots, x_r) = &\sum_{k=1}^r \Big(\tfrac{\alpha_k^1}{\mathcal{H}^1(S_{k_1})} + \tfrac{\alpha_k^2}{\mathcal{H}^1(S_{k_2})} + \tfrac{\alpha_k^3}{\mathcal{H}^1(S_{k_3})} \Big) x_k^2 \\ &+ \sum_{k=1}^r \Big(\tfrac{\beta_k^1}{\mathcal{H}^1(S_{k_1})} + \tfrac{\beta_k^2}{\mathcal{H}^1(S_{k_2})} + \tfrac{\beta_k^3}{\mathcal{H}^1(S_{k_3})} \Big) \big(\tfrac{2}{\sqrt{3}} - x_k \big)^2 + \sum_{1 \leq i < j \leq r} \tfrac{\gamma_{ij}\,(x_i - x_j)^2}{\mathcal{H}^1(S_{i,j})} \end{aligned} \tag{3.5}$$

in the cube $[0, \frac{2}{\sqrt{3}}]^r$, where $\alpha_k^i, \beta_k^i \in \{0, 1\}$, $\gamma_{ij} \in \{0, 1\}$ and $\gamma_{ij} = 1$ if and only if there is a segment $S_{i,j}$ of $\mathcal{S}$ connecting X_i and X_j, and for each k, these coefficients are uniquely defined depending only on how segments/half-lines $S_{k_1}, S_{k_2}, S_{k_3}$, forming a junction at X_k, behave at their other endpoints. Indeed, for shortness setting $\gamma_{ii} = 0$, let another endpoint Y of S_{k_1} be a simple vertex and $N_{k_1}(Y)$ bisects the interior resp. exterior angle of $\mathcal{S}$ at this vertex. Then $\beta_k^1 = 0$ and $\alpha_k^1 = 1$ resp. $\beta_k^1 = 1$ and $\alpha_k^1 = 0$, and also $\gamma_{kl} = \gamma_{lk} = 0$ for all l. On the other hand, if the other endpoint of S_{k_1} is another triple junction, say, X_l, then $S_{k,l} := S_{k_1}$, $\gamma_{kl} = 1$ and $\alpha_k^1 = \beta_k^1 = 0$. The same applies also to S_{k_2} and S_{k_3}. In particular, for each x_k, at most three of $\{\alpha_k^i, \beta_k^i, \gamma_{ki}\}_{i=1}^r$ can be nonzero. Moreover, by the connectedness of G_1, the matrix (γ_{ki}) is irreducible.

Now by Lemma 3.6 either $\min \psi = 0$, which is possible if and only if either all $\alpha_k^i = 0$ (so that $x_1 = \ldots = x_r = 0$ minimizes ψ) or $\beta_k^i = 0$ (so that $x_1 = \ldots = x_r = \frac{2}{\sqrt{3}}$ minimizes ψ), or $\min \psi > 0$ so that the unique minimizer $(x_1, \ldots, x_r)$ is an interior point of the cube $[0, \frac{2}{\sqrt{3}}]^r$. By definition $\min\{x_k, \frac{2}{\sqrt{3}} - x_k\}$ measures the distance between the values of N^0 at X_k and the corresponding vertices of B^ϕ, therefore, in case $\min \psi > 0$ those values do not coincide with the vertices of B^ϕ. □

Definition 3.8 (ic-triple junctions and bc-triple junctions). Let $\mathcal{S}$ be a simple network, and X be a triple junction of X. X is called *interior to the constraint* (shortly, an ic-triple junction) if any minimal CH field at X do not coincide with vertices of B^ϕ. X is called at the *boundary of the constraint* (shortly, a bc-triple junction) if there is a minimal CH field having values at X coinciding with some vertices of B^ϕ.

The following lemma shows that ic-triple junctions and bc-triple junctions are preserved in parallelness.

Lemma 3.9 (Preserving parallelness). *Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ be two parallel simple networks, and let $\{G_i\}$ and $\{\overline{G}_i\}$ be their partitions into connected graphs as above. Then, for any $i = 1, \ldots, r$, both G_i and $\overline{G}_i$ can contain either only ic-triple junctions or only bc-triple junctions.*

Proof. Let G_i contain only bc-triple junctions. By Lemma 3.7 all segments/half-lines have zero ϕ-curvature. By parallelness, we can choose the same CH field along the segments/half-lines of $\overline{G}_i$ so that all its segments/half-lines have also zero ϕ-curvature. In particular, again by Lemma 3.7 all triple junctions of $\overline{G}_i$ are bc-triple junctions. This argument shows also that if G_i contains only ic-triple junctions then $\overline{G}_i$ cannot contain bc-triple junctions. □

Lemmas 3.7 and 3.9 suggest that bc-triple junctions do not move.

Definition 3.10 (**The class** $\Xi(\mathcal{S})$)**.** Given a simple network $\mathcal{S}$, we denote by $\Xi(\mathcal{S})$ the collection of all networks $\mathcal{T}$ parallel to $\mathcal{S}$ such that if X is a bc-triple junction of $\mathcal{S}$, then X is a (bc)-triple junction also for $\mathcal{T}$.

Thus, by definition, the ϕ-curvature flow $\{\mathcal{S}(t)\}_{t\in[0,T)}$ starting from $\mathcal{S}^0$ is a subset of $\Xi(\mathcal{S}^0)$, i.e., $\mathcal{S}(t)\in\Xi(\mathcal{S})$ for all times $t\in[0,T)$.

3.1. **Parallel networks.** Now we consider the problem of reconstructing a parallel network from a given set of heights. We shall see later in Lemma 3.14 that the heights at triple junctions cannot have too much freedom.

Theorem 3.11 (**Reconstruction**)**.** *Let $\mathcal{S}:=\cup_{i=1}^n S_i$ be a simple network and define*

$$\Delta_1 := \frac{1}{3\sqrt{3}}\min_{1\le i\le n}\mathcal{H}^1(S_i), \qquad \Delta_2 := \frac{1}{6}\min_{S_i\cap S_j=\emptyset} d(S_i,S_j). \tag{3.6}$$

Let $\{h_i\}_{i=1}^n$ be any set of real numbers such that:

(a) *if S_i is either a segment at a bc-triple junction or a half-line, then $h_i=0$;*

(b) *if S_i,S_j,S_k form a ic-triple junction, then*

$$(-1)^{\sigma_i}h_i + (-1)^{\sigma_j}h_j + (-1)^{\sigma_k}h_k = 0,$$

where $\sigma_i,\sigma_j,\sigma_k\in\{0,1\}$, and $\sigma_s=0$ is S_s if oriented from the triple junction and 1 *otherwise;*

(c) $|h_j|\le\min\{\Delta_1,\Delta_2\}$.

Then there exists a unique network $\mathcal{T}:=\cup_{i=1}^n T_i$, parallel to $\mathcal{S}$ such that $\mathcal{T}\in\Xi(\mathcal{S})$ and

$$H(S_i,T_i)\cdot\nu_{S_i} = h_i, \quad i=1,\dots,n.$$

Assumption (a) in Theorem 3.11 says that any bc-triple junction of $\mathcal{S}$ is also a bc-triple junction for $\mathcal{T}$. Later in Lemma 3.14 we shall see that assumption (b) allows us to construct a parallel triple junction. Finally, assumption (c) prevents self-intersections of segments in $\mathcal{T}$, see Figure 15 (c).

We postpone the proof after some auxiliary results. The following lemma defines the distance between the vertices of two parallel cones of opening angle 120^o, knowing the heights between the corresponding parallel lines.

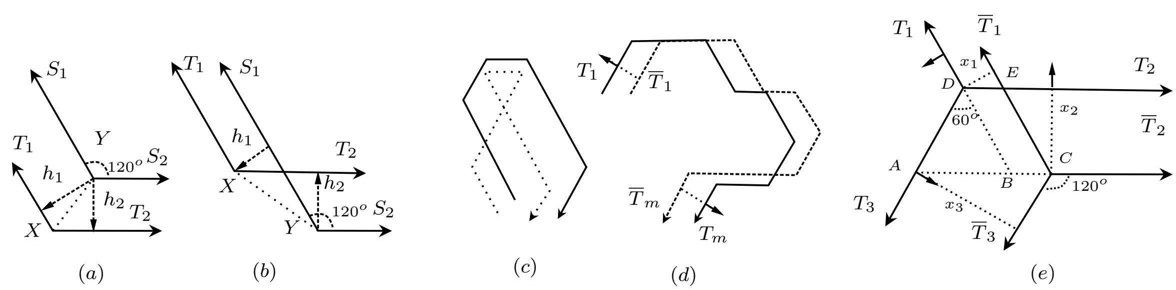

FIG. 15.

Lemma 3.12. *Let S_1,S_2 and T_1,T_2 be two pairs of segments/half-lines with common starting points X and Y respectively, such that S_i is parallel to T_i, the angle between S_1 and S_2 is 120^o, and let $h_i:=H(S_i,T_i)\cdot\nu_{S_i}$ (see Figure 15 (a)-(b)). Then*

$$|XY| = \frac{2}{\sqrt{3}}\sqrt{|h_1|^2+|h_2|^2-h_1h_2}. \tag{3.7}$$

Proof. Assume without loss of generality the case of Figure 15 (a), and let α be the angle between T_1 and $[XY]$. Then

$$|XY| = \tfrac{|h_1|}{\sin\alpha} = \tfrac{|h_2|}{\sin(120^o-\alpha)}.$$

From the last equality we get

$$|h_1|\cot\alpha = |h_1|\cot 120^o + \tfrac{|h_2|}{\sin 120^o} = -\tfrac{|h_1|}{\sqrt{3}} + \tfrac{2|h_2|}{\sqrt{3}}$$

and hence,

$$|XY|^2 = |h_1|^2 + |h_1|^2\cot^2\alpha = |h_1|^2 + \tfrac{(2|h_2|-|h_1|)^2}{3} = \tfrac{4|h_1|^2+4|h_2|^2-4|h_1||h_2|}{3}.$$

Since $|h_1||h_2| = h_1h_2$ we get (3.7). □

Note that if S is an (oriented) segment or half-line and $h\in\mathbb{R}$, then there exists a unique straight line L parallel to S such that $H(S,L)\cdot\nu_S = h$. However, given $(h_1,\dots,h_n)\in\mathbb{R}^n$ and an oriented polygonal curve Γ, consisting of a union of $n\ge 2$ segments $S_1,\dots,S_n$, not always one can define a polygonal curve $\overline{\Gamma} := \cup_{i=1}^n \overline{S}_i$, parallel to Γ, satisfying $H(S_i,\overline{S}_i)\cdot\nu_{S_i} = h_i$ for all $i=1,\dots,n$, see Figure 15 (c). Indeed, if some $|h_i|$ are large, then the relative interiors of two of the $\overline{S}_i$'s may intersect.

To retain the injectivity of $\overline{\Gamma}$, we have to ensure the smallness of all $|h_i|$. This is done in the next lemma.

Lemma 3.13 (**Injectivity**). *Let Γ be an oriented polygonal curve consisting of n-segments/half-lines $T_1,\dots,T_n$ with the 120^o-angle between T_i and T_{i+1} at their common point for $i=1,\dots,n-1$ and let $\overline{T}_1,\dots,\overline{T}_n$ be the n-segments/half-lines such that T_i and $\overline{T}_i$ are parallel, the endpoint of $\overline{T}_j$ is an endpoint of $\overline{T}_{j+1}$, and*

$$\mathrm{dist}(T_i,\overline{T}_i)\le \delta^0 := \frac{1}{3\sqrt{3}}\min_{1\le j\le n}\mathcal{H}^1(T_j)$$

for all $i=1,\dots,n$. Then $\cup_{j=2}^{n-1}\overline{T}_j$ is also a polygonal curve (without self-intersections).

Observe that we cannot state the injectivity of $\cup_{j=1}^n\overline{T}_j$, since a priori we have no information on $\overline{T}_1$ and $\overline{T}_n$.

Proof. We only need to show that $\Sigma := \cup_{j=2}^{n-1}\overline{T}_j$ has no self-intersections. Recalling that self-intersections start after some segment disappears, it is enough to show that any segment in the union Σ of segments satisfying the assumptions of the lemma has positive length. Note that by the parallelness, $\nu_{T_i} = \nu_{\overline{T}_i}$ and let

$$h_i := H(T_i,\overline{T}_i)\cdot\nu_{T_i},\quad i=1,\dots,n.$$

Direct computations show (see Figure 15 (d) and also [22, Eq. 3.3]) that

$$\mathcal{H}^1(\overline{T}_i) = \mathcal{H}^1(T_i) - \tfrac{2h_{i-1}-2h_i+2h_{i+1}}{\sqrt{3}} \tag{3.8}$$

for $i=2,\dots,n-1$. Thus, if $|h_j|\le\delta^0$ for all $1\le j\le n$, then

$$\Big|\tfrac{2h_{i-1}-2h_i+2h_{i+1}}{\sqrt{3}}\Big| \le 2\sqrt{3}\delta^0 \le \tfrac{2}{3}\mathcal{H}^1(T_i),$$

for all $2\le i\le n-1$, and hence,

$$\mathcal{H}^1(\overline{T}_i)\ge \tfrac{1}{3}\,\mathcal{H}^1(T_i) > 0,$$

and thus, Σ cannot have self-intersections. □

Lemma 3.13 has a further implication: if $\Gamma := \cup_{i=1}^n T_i$ is the polygonal curve in Lemma 3.13 and $(h_1,\dots,h_n)$ is an n-tuple of real numbers satisfying $|h_i| \le \delta^0$, then there exists a unique polygonal curve $\Sigma := \cup_{i=2}^{n-1} \overline{T}_i$ with $\overline{T}_i$ parallel to T_i and $H(T_i,\overline{T}_i)\cdot \nu_{T_i} = h_i$ for any $i = 2,\dots,n-1$. We can also define $\overline{T}_1$ and $\overline{T}_n$, parallel to T_1 and T_n, respectively, with $H(T_i,\overline{T}_i)\cdot\nu_{T_i} = h_i$ for $i\in\{1,n\}$, however, $\overline{T}_1$ and $\overline{T}_n$ are not uniquely defined (because there is no information on their length).

Next, we study how distances behave in parallel triple junctions.

Lemma 3.14 (**Heights compatibility**). *Let T_1, T_2, T_3 be segments/half-lines parallel to some segments of B^ϕ and forming a triple junction with 120^o-angles as in Figure 15 (e) and oriented out of their triple junction. Let $\overline{T}_1, \overline{T}_2, \overline{T}_3$ be another triple of segments/half-lines forming a triple junction such that T_i and $\overline{T}_i$ are parallel (and so $\nu_{T_i} = \nu_{\overline{T}_i}$). Then the corresponding heights $h_i := H(T_i,\overline{T}_i)\cdot\nu_{T_i}$, $i=1,2,3$, satisfy*

$$h_1 + h_2 + h_3 = 0. \tag{3.9}$$

Conversely, if real numbers h_1,h_2,h_3 satisfy (3.9), then there exists a unique triple $\overline{T}_1, \overline{T}_2, \overline{T}_3$ of half-lines such that $\overline{T}_i$ is parallel to T_i and $H(T_i,\overline{T}_i)\cdot\nu_{T_i} = h_i$.

Thus, the knowledge of two heights at a triple junction allows to determine uniquely the third one. Note that if any of T_i oriented towards to the triple junction, then the corresponding height h_i in (3.9) appears with the "–" sign.

Proof. By symmetry, we may assume that T_i and $\overline{T}_i$ are as in Figure 15 (e), i.e., $h_3 \ge 0$, $h_1 \le 0$, $h_2 \le 0$ and let $x_i = |h_i|$. Then

$$\mathcal{H}^1([AC]) = \tfrac{x_3}{\sin 60^o} = \tfrac{2x_3}{\sqrt{3}}. \tag{3.10}$$

Similarly,

$$\mathcal{H}^1([AC]) = \mathcal{H}^1([AB]) + \mathcal{H}^1([BC]) = \mathcal{H}^1([EC]) + \mathcal{H}^1([DE]) = \tfrac{2x_2}{\sqrt{3}} + \tfrac{2x_1}{\sqrt{3}}, \tag{3.11}$$

and hence, from (3.10) and (3.11) it follows $x_3 = x_1 + x_2$ i.e., (3.9). The proof in the other cases is similar.

To prove the last assertion let us take two half-lines $\overline{T}_1, \overline{T}_2$ starting from common point X, parallel to T_1, T_2, respectively, and satisfying $H(T_i,\overline{T}_i)\cdot\nu_{T_i} = h_i$ for $i=1,2$. Let $\overline{T}_3$ be any segment/half-line starting from X and parallel to T_3 and define $h_3' := H(T_3,\overline{T}_3)\cdot\nu_{T_3}$. By the first part of the proof $h_1 + h_2 + h_3' = 0$. On the other hand, by assumption (3.9), $h_1+h_2+h_3 = 0$, and thus, $h_3 = h_3'$. □

Now we are ready to construct parallel networks.

3.2. **Proof of Theorem 3.11.** *Step 1.* For each $i = 1,\dots,n$, let L_i be the straight line parallel to S_i and satisfying $H(S_i,L_i)\cdot\nu_{S_i} = h_i$.

We define subsets T_i of L_i as follows. First consider any half-line S_i of $\mathcal{S}$ and let S_j be any other segment/half-line of $\mathcal{S}$ having a common endpoint with S_i. Then the lines L_i and L_j intersect at a unique point separating both lines into two half-lines. Let $T_i \subset L_i$ be the one parallel to S_i. By assumption $h_i = 0$ so that by construction both T_i and S_i lie on the same line L_i. Thus, by parallelness, $T_i \Delta S_i$ is bounded.

Next, let S_j be a segment and S_i, S_j, S_k be a polygonal line (in the same order). Since the angles between the common points of S_i, S_j and S_j, S_k are 120^o, the lines L_i and L_k cut from L_j a segment, which we call T_j. Notice that a priori we do not know if T_j is parallel to S_j (because it could be oriented oppositely). We repeat this argument with each segment S_j of $\mathcal{S}$ and construct all segments $T_j \subset L_j$.

Step 2. Let us find some estimates for $\{T_i\}$. First observe that by construction if S_i and S_j, $i \neq j$, have a common vertex, then so do T_i and T_j. Let $\{X\} = S_i \cap S_j$ and $\{Y\} = T_i \cap T_j$. By Lemma 3.12 and the definition of Δ_2,

$$|XY| = \tfrac{2}{\sqrt{3}}\sqrt{|h_i|^2 + |h_j|^2 - h_ih_j} \le 2\max\{|h_i|, |h_j|\} \le 2\Delta_2 \le \tfrac13 \min_{S_k\cap S_l=\emptyset} d(S_k, S_l).$$

In particular,

$$d(S_i, T_i), d(S_j, T_j) \le |XY| \le \tfrac13 \min_{S_k\cap S_l=\emptyset} d(S_k, S_l). \tag{3.12}$$

We claim that if $S_i \cap S_j = \emptyset$, then $T_i \cap T_j = \emptyset$. Indeed, by (3.12) and the triangle inequality if $S_i \cap S_j = \emptyset$, then

$$d(S_i, S_j) \le d(S_i, T_i) + d(T_i, T_j) + d(T_j, S_j) \le d(T_i, T_j) + \tfrac23 d(S_i, S_j),$$

and hence, $d(T_i, T_j) \ge \frac13 d(S_i, S_j) > 0$.

Step 3. Let $\Gamma := S_{i_1} \cup \ldots \cup S_{i_m}$ be any polygonal curve of $\mathcal{S}$ (recall that $\mathcal{S}$ is a finite union of polygonal curves). We claim that the union $\Gamma' := S_{i_1} \cup \ldots \cup S_{i_m}$ is also a polygonal curve. Indeed, by the definition of Δ_1 and assumption (c) we can apply Lemma 3.13 to get that the union $\Gamma'' := S_{i_2} \cup \ldots \cup S_{i_{m-1}}$ is a polygonal curve without self-intersections. By step 2 we know that T_{i_1} resp. T_{i_m} have a common vertex with only T_{i_2} resp. $T_{i_{m-1}}$, and both T_{i_1} and T_{i_m} does not intersect the interior of Γ''.

Now if $S_{i_1} \cap S_{i_m} = \emptyset$, then by step 2, $T_{i_1} \cap T_{i_m} = \emptyset$, and hence, Γ' is injective. On the other hand, if S_{i_1} and S_{i_m} have a common vertex so that Γ is a closed curve, then by step 2 so do T_{i_1} and T_{i_m}. Since these segments do not intersect the interior of Γ'' and $T_{i_2} \cap T_{i_{m-1}} = \emptyset$ (in case $m \ge 4$), Γ' is also a closed injective curve.

Step 4. Let S_i, S_j, S_k form a triple junction X. If X is a bc-triple junction, then by assumption $h_i = h_j = h_k = 0$, and therefore, by construction X is a triple junction of T_i, T_j, T_k. On the other hand, if X is a ic-triple junction, then by assumption (b) and Lemma 3.14 T_i, T_j, T_k also form an ic-triple junction.

Step 5. Let Γ_1 and Γ_2 be any curves of $\mathcal{S}$ and let Γ_1' and Γ_2' be two corresponding curves in $\mathcal{T}$ (defined in step 3). Since Γ_1 and Γ_2 can intersect only at the endpoints, by step 2 the interiors of Γ_1' and Γ_2' cannot have a common point. Thus, $\mathcal{T}$ is a network parallel to $\mathcal{S}$.

Since $h_i = 0$ if S_i is a half-line or a segment at a bc-triple junction, $\mathcal{T} \in \Xi(\mathcal{S})$. Finally, the uniqueness of $\mathcal{T}$ follows from the uniqueness of lines $\{L_i\}$. □

3.2.1. *Expression of some quantities of parallel networks by a given set of heights.* Given a simple network $\mathcal{S} := \cup_{i=1}^n S_i$, let $\Delta_1, \Delta_2 > 0$ be as in (3.6). Consider arbitrary real numbers $\{h_j\}_{j=1}^n$ satisfying assumptions (a)-(c) of Theorem 3.11 so that there exists a unique $\overline{\mathcal{S}} := \cup_{i=1}^n \overline{S}_i \in \Xi(\mathcal{S})$ such that $H(S_i, \overline{S}_i) \cdot \nu_{S_i} = h_i$ for any $i = 1, \ldots, n$. By (3.8) for any segment $\overline{S}_i$ of $\overline{\mathcal{S}}$, we have

$$\mathcal{H}^1(\overline{S}_i) = \mathcal{H}^1(S_i) - \tfrac{2}{\sqrt3}\sum_{k=1}^n \beta_k h_k, \tag{3.13}$$

where $\beta_k \in \{0, \pm1\}$ and only three of them are nonzero and depend only on the orientation of the segments/half-lines of $\mathcal{S}$.

Next, let us study the ϕ-curvature of the segments of $\overline{\mathcal{S}}$. Namely, we claim that there exist $\Delta_3 > 0$ and real-analytic functions $\{u_i\}_{i=1}^n$ in $(-2\Delta_3, 2\Delta_3)^n$ depending only on $\mathcal{S}$, such that

$$|u_i(h_1, \ldots, h_n)| \le \gamma_0 \sum_{j=1}^n |h_j|, \quad |u_i(h_1', \ldots, h_n') - u_i(h_1'', \ldots, h_n'')| \le \gamma_0 \sum_{j=1}^n |h_j' - h_j''| \tag{3.14}$$

for some $\gamma_0 = \gamma_0(\mathcal{S}) > 0$ and all $\{h_j, h_j', h_j''\}$ with $|h_j|, |h_j'|, |h_j''| \le \min\{\Delta_1, \Delta_2, \Delta_3\}$, for which

$$\kappa_{\overline{S}_i}^{\phi} = \kappa_{S_i}^{\phi} + u_i(h_1, \dots, h_n). \tag{3.15}$$

Indeed, let G resp. $\overline{G}$ be a connected graph associated to $\mathcal{S}$ resp. $\overline{\mathcal{S}}$ as in Lemma 3.7 containing at least one triple junction. By that lemma we know that all triple junctions are either ic-triple junctions or bc-triple junctions, simultaneously. If G contains only ic-triple junctions, then by Lemma 3.9 so does $\overline{G}$ and hence, by Lemma 3.7 all segments/half-lines S_i and $\overline{S}_i$ in both G and $\overline{G}$ have zero ϕ-curvature, and in this case we define $u_i \equiv 0$ in (3.15). Therefore, we may assume that G contains only ic-triple junctions. Write $G = \cup_{l=1}^m S_{k_l}$ and $\overline{G} = \cup_{l=1}^m \overline{S}_{k_l}$, and let the quadratic functions ψ and $\overline{\psi}$ be defined as in (3.5) and associated to $\mathcal{S}$ and $\overline{\mathcal{S}}$, respectively. By stability, the minimizers x^0 and $\overline{x}^0$ of ψ and $\overline{\psi}$ lie in $(0, 2/\sqrt{3})^r$, where r is the number of the triple junctions in G, and thus, solves the nondegenerate linear systems $\nabla\psi(x^0) = 0$ and $\nabla\overline{\psi}(\overline{x}^0) = 0$. In particular, there exists an analytic function Ψ depending only on $\mathcal{S}$ such that

$$x^0 = \Psi\Big(\tfrac{1}{\mathcal{H}^1(S_{k_1})}, \dots, \tfrac{1}{\mathcal{H}^1(S_{k_m})}\Big), \qquad \overline{x}^0 = \Psi\Big(\tfrac{1}{\mathcal{H}^1(\overline{S}_{k_1})}, \dots, \tfrac{1}{\mathcal{H}^1(\overline{S}_{k_m})}\Big).$$

By (3.13) and the analyticity of Ψ there exists $\Delta_3 > 0$ depending only on $\{\mathcal{H}^1(S_{k_l})\}_{l=1}^m$ and the structure of $\mathcal{S}$ such that

$$\overline{x}^0 = x^0 + \widehat{\Psi}(h_1, \dots, h_n), \qquad \sup_{1\le j\le n} |h_j| \le \Delta_3,$$

where $\widehat{\Psi}$ is a real-analytic (vector-valued) function in $(-2\Delta_3, 2\Delta_3)^n$. This representation implies the existence of a family $\{u_{k_l}\}_{l=1}^m$ of real-analytic functions in $(-2\Delta_3, 2\Delta_3)^n$ which satisfies (3.15).

Recall that by the ϕ-curvature-balance condition of Lemma 2.19, if S_i, S_j, S_k form an ic-triple junction, then

$$(-1)^{\sigma_i} u_i + (-1)^{\sigma_j} u_j + (-1)^{\sigma_k} u_k \equiv 0, \tag{3.16}$$

where $\sigma_i, \sigma_j, \sigma_k$ are 0 or 1 depending on whether the corresponding segment enters to or exits from the triple junction.

4. Proof of Theorem 3.5

Step 1. Fix a simple network $\mathcal{S}^0 := \cup_{i=1}^n S_i^0$, let $\Delta_1, \Delta_2 > 0$ be as in (3.6), applied with $\mathcal{S} = \mathcal{S}^0$, and let $\Delta_3 > 0$ and real-analytic functions $\{u_i\}_{i=1}^n$ (depending only $\mathcal{S}^0$) be as above satisfying (3.14), (3.15) and (3.16). Define

$$\Delta_0 := \min\{\Delta_1, \Delta_2, \Delta_3\} > 0.$$

For any $T > 0$ let $\mathcal{B}_T$ be the collection of all n-tuples $h := (h_1, \dots, h_n)$ of continuous functions in $[0, T]$ such that

(1) $h_i(0) = 0$ for all $i = 1, \dots, n$,
(2) if S_i is a half-line or a segment at a bc-triple junction, then $h_i \equiv 0$ in $[0, T]$,
(3) if S_i^0, S_j^0, S_k^0 form an ic-triple junction, then

$$(-1)^{\sigma_i} h_i + (-1)^{\sigma_j} h_j + (-1)^{\sigma_k} h_k = 0 \quad \text{in } [0, T],$$

where $\sigma_i, \sigma_j, \sigma_k$ are 0 or 1 depending on whether the corresponding segment/half-line enters to or exits from the triple junction.

Since the assumptions (1)-(3) are linear, $\mathcal{B}_T$ is a Banach space with respect to the norm

$$\|h\|_\infty := \sum_{i=1}^n \max_{t\in[0,T]} |h_j(t)|.$$

Let

$$T_0 := \tfrac{1}{n}\,\min\Big\{\tfrac{\Delta_0}{1+\max_i|\kappa^\phi_{S_i^0}|+\gamma_0\Delta_0},\,\tfrac{1}{1+\gamma_0}\Big\} > 0,$$

where γ_0 is as in (3.14), and

$$\mathcal{K} := \{h\in\mathcal{B}_{T_0}:\ \|h\|_\infty\le\Delta_0\}.$$

Clearly, $\mathcal{K}$ is a closed convex subset of $\mathcal{B}_{T_0}$. For any $h\in\mathcal{K}$ and $i=1,\dots,n$ let us define

$$\Phi_i[h](t) = -\int_0^t \Big(\kappa^\phi_{S_i^0} + u_i(h_1(s),\dots,h_n(s))\Big)\,ds,\qquad t\in[0,T_0].$$

Note that $\Phi := (\Phi_1,\dots,\Phi_n)$ maps $\mathcal{K}$ into itself. Indeed, clearly, $\Phi_i[h]\in C^0[0,T_0]$ and $\Phi_i[h](0)=0$ for each $i=1,\dots,n$ and all $h\in\mathcal{K}$. If S_i^0 is a half-line or a segment at a bc-triple junction, then $\kappa^\phi_{S_i^0}=0$ and by definition $u_i\equiv 0$, hence, $\Phi_i[h]\equiv 0$ for all $i=1,\dots,n$. Next, if S_i^0,S_j^0,S_k^0 form a ic-triple junction, then by the ϕ-curvature-balance condition (2.10) and (3.16)

$$(-1)^{\sigma_i}\Phi_i[h] + (-1)^{\sigma_j}\Phi_j[h] + (-1)^{\sigma_k}\Phi_k[h]\equiv 0\qquad \text{in } [0,T].$$

Furthermore, by the definition of T_0,

$$\|\Phi_i[h]\|_\infty \le \Big(|\kappa^\phi_{S_i^0}| + \gamma_0\|h\|_\infty\Big)T_0 \le \tfrac{1}{n}\,\Delta_0,\qquad h\in\mathcal{K},$$

and hence, $\|\Phi[h]\|_\infty\le\Delta_0$. In particular, $\Phi[h]\in\mathcal{K}$ whenever $h\in\mathcal{K}$.

Finally, since $\kappa^\Phi_{S_i^0}$ is constant, by the second relation in (3.14)

$$\|\Phi_i[h]-\Phi_i[\overline{h}]\|_\infty \le \gamma_0\|h-\overline{h}\|_\infty T_0,\quad h,\overline{h}\in\mathcal{K},$$

and therefore, by the definition of T_0

$$\|\Phi[h]-\Phi[\overline{h}]\|_\infty \le \tfrac{\gamma_0}{1+\gamma_0}\,\|h-\overline{h}\|_\infty,\qquad h,\overline{h}\in\mathcal{K},$$

and Φ is a contraction in $\mathcal{K}$. By the Banach fixed-point theorem, there exists a unique $h\in\mathcal{K}$ such that $\Phi[h]=h$ in $[0,T_0]$. By the definition of Φ and the analyticity of u_i, for each $i=1,\dots,n$ we have $h_i\in C^\infty[0,T_0]$ and

$$h_i' = -\kappa^\phi_{S_i^0} - u_i(h_1,\dots,h_n)\quad\text{in } [0,T_0]. \tag{4.1}$$

In view of assumptions (1)-(3) on $\mathcal{B}_{T_0}$ and inequality $|h_i(t)|\le\Delta_0\le\min\{\Delta_1,\Delta_2\}$ for all $i=1,\dots,n$ and $t\in[0,T_0]$ we can apply Theorem 3.11 to find a network $\mathcal{S}(t)\in\Xi(\mathcal{S}^0)$ parallel to $\mathcal{S}^0$. Since $|h_i(t)|\le\Delta_3$, to compute the ϕ-curvatures of $S_i(t)$ we can use (3.15), which combined with (4.1) gives

$$h_i' = -\kappa^\phi_{S_i(t)}\quad\text{in } [0,T_0].$$

Since $h_i(0)=0$ for all $i=1,\dots,n$, $\mathcal{S}(0)=\mathcal{S}^0$, and therefore, $\mathcal{S}(\cdot)$ is the ϕ-curvature flow starting from $\mathcal{S}^0$. The uniqueness of $\{\mathcal{S}(\cdot)\}$ in $[0,T_0]$ follows from the uniqueness of the fixed point.

Step 2. Let $T^\dagger$ be the maximal time for which the regular flow $\mathcal{S}(t)$ starting from $\mathcal{S}^0$ exists for all times $t\in[0,T^\dagger)$. We have two possibilities:

- $T^\dagger=+\infty$, i.e., the flow $\mathcal{S}(t)$ exists for all times $t\ge 0$;
- $T^\dagger<+\infty$. In this case,

  $$\liminf_{t\nearrow T^\dagger} S_i(t) = 0$$

  for some $i=1,\dots,n$. Indeed, otherwise the limit network $\mathcal{S}(T^\dagger)$ (for instance defined as a Kuratowski limit of sets) would be simple, and the heights $h_i\in C^1[0,T^\dagger]$. In particular, we could apply step 1 with $\mathcal{S}^0 := \mathcal{S}(T^\dagger-\epsilon)$ for sufficiently small $\epsilon>0$ and find $T_0>0$

independent of ϵ such that the regular flow $\mathcal{S}(\cdot)$ exists also in $[T^\dagger - \epsilon, T^\dagger + T_0]$. But this contradicts to the maximality of $T^\dagger$.

□

Remark 4.1. The (geometric) uniqueness of a ϕ-regular curvature flow starting from a simple network implies, remarkably, that the flow *preserves the (axial, rotational and mirror) symmetries* of the initial network.

4.1. **Some extensions of Theorem 3.5 to networks with multiple junctions.** In view of the proof of Theorem 3.5, its assertions remain valid even if the initial network has junctions with degree ≥ 3 provided that the concurring segments/half-lines have zero ϕ-curvature. To this aim let us call an admissible network $\mathcal{S}$ *simple with multiple junctions* if either segments/half-lines form a triple junction with 120^o or a $m \geq 4$-junction with zero ϕ-curvature.

Theorem 4.2. *Let $\mathcal{S}^0$ be any simple network with multiple junctions. Then there exists $T^\dagger \in (0, +\infty]$ and a unique family $\{\mathcal{S}(t)\}_{t\in[0,T^\dagger)}$ of parallel networks such that $\mathcal{S}(\cdot)$ is the ϕ-curvature flow starting from $\mathcal{S}^0$. Moreover, if $T^\dagger < +\infty$ then some segment of $\mathcal{S}(t)$ vanishes as $t \nearrow T^\dagger$.*

As mentioned earlier, the proof of this theorem runs along the lines of Theorem 3.5, since the $m \geq 4$-junctions do not move, therefore, we omit it. This theorem in some cases allows to restart the flow even after some segments at the maximal time vanish.

Corollary 4.3 (**Restarting the flow**). *Let $\mathcal{S}^0$ be a simple network with multiple junctions and $\{\mathcal{S}(t)\}_{t\in[0,T^\dagger)}$ be the unique ϕ-curvature flow starting from $\mathcal{S}^0$. Assume that the Kuratowski-limit*

$$\mathcal{S}(T^\dagger) := \lim_{t\nearrow T^\dagger} \mathcal{S}(t)$$

is well-defined and $\mathcal{S}(T^\dagger)$ is simple with multiple junctions. Then there exist $T^\ddagger \in (0, +\infty]$ and a unique ϕ-curvature flow $\{\mathcal{S}(t)\}_{t\in[T^\dagger,T^\ddagger)}$ starting from $\mathcal{S}(T^\dagger)$.

Notice that, for $t \in [T^\dagger, T^\ddagger)$, the networks $\mathcal{S}(t)$ are not parallel to $\mathcal{S}^0$.

Remark 4.4. In Example 5.1 we will see that networks with quadruple junctions can produce a regular ϕ-curvature flow, in which quadruple junctions do not move. However, they are not stable under small (simple) perturbations – in physical situations those quadruple junctions can split into two triple junctions to become stable under small perturbations (see Figure 8). We expect such a phenomenon to appear, when defining the evolution using the minimizing movement method (see [1, 8, 6] in the case of two phases, and in the case of more phases).

5. Examples

Example 5.1. Consider the evolution of the simple network $\mathcal{S}^0$ (with a quadruple junction) in Figure 8, where we assume that segments at the quadruple junction have length $a > 0$. By criticality, $\mathcal{S}^0$ does not move, i.e., its unique ϕ-curvature evolution (in the sense of Theorem 4.2) is constant $\mathcal{S}(t) \equiv \mathcal{S}^0$. Now, let us parse the quadruple junction into two triple junctions at distance $2x > 0$ (dotted) and denote the obtained network by $\mathcal{N}^0 = \mathcal{N}^0(x)$. By symmetry and uniqueness of the geometric flow, the horizontal segment does not translate up or down (it has vanishing ϕ-curvature) and non-horizontal segments have constant curvature $\pm\frac{1}{\sqrt{3}a}$ (independent of x). Therefore, these segments move linearly to infinity in the direction of the corresponding half-lines. In particular, in both cases the flow $\{\mathcal{N}(t)\}_{t\in[0,T^\dagger)}$ (given by Theorem 3.5) uniquely exists with $T^\dagger = +\infty$ and in addition $\mathcal{N}(t)$ is simple for any $t \geq 0$.

In Example 5.1 the maximal existence time $T^\dagger$ of the flow if infinite, as opposite to the following example.

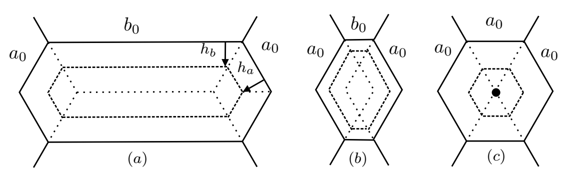

Fig. 16.

Example 5.2. Let us consider the simple networks $\mathcal{S}^0$ in Figure 16, consisting of a hexagon, symmetric with respect to the horizontal axis and clockwise oriented, and of four half-lines starting at the endpoints of the horizontal segments. Denoting by $b(t)$ and $a(t)$ the length of horizontal and lateral segments of $\mathcal{S}(t)$, we can compute the heights from $\mathcal{S}^0$ (whose sidelengths are b_0 and a_0) as

$$h_a = -\tfrac{\sqrt{3}}{2}\,\tfrac{a_0+b_0-a-b}{2}, \qquad h_b = -\tfrac{\sqrt{3}}{2}\,(a_0-a),$$

and the ϕ-curvature of all segments is equal to $\frac{2}{\sqrt{3}(b+2a)}$. Thus, the ϕ-curvature equation (3.1) is equivalent to the system

$$\begin{cases} \tfrac{\sqrt{3}}{4}\,(a'+b') = -\tfrac{2}{\sqrt{3}(b+2a)}, \\ \tfrac{\sqrt{3}}{2}\,a' = -\tfrac{2}{\sqrt{3}(b+2a)}, \end{cases}$$

which implies $b-a=b_0-a_0$.

- Consider Figure 16 (a), where $b_0>a_0$. In this case $b(t)>a(t)$ for all times $t\in[0,T^\dagger)$, and hence, $a(t)\to 0^+$ as $t\nearrow T^\dagger$. At the maximal time the network $\mathcal{S}(T^\dagger)$ is the union of a horizontal segment of length b_0-a_0 and four half-lines starting from the endpoints of this segment. Clearly, $\mathcal{S}(T^\dagger)$ is critical and admissible.
- In Figure (b), $b_0<a_0$, and hence, $b(t)<a(t)$ for all $t\in[0,T^\dagger)$. Then $b(t)\to 0^+$ as $t\nearrow T^\dagger$ and at the maximal time the network $\mathcal{S}(T^\dagger)$ is the union of a rhombus and four half-lines starting from the vertical vertices of the rhombus. Clearly, $\mathcal{S}(T^\dagger)$ is critical and admissible.
- In Figure (c), $b_0=a_0$, and hence, $b(t)=a(t)$ for all $t\in[0,T^\dagger)$. Thus, $a(t)\to 0^+$ as $t\nearrow T^\dagger$ and at the maximal time the hexagon shrinks to a point, and the limiting network $\mathcal{S}(T^\dagger)$ is a minimal conical admissible network forming an "X"-type quadruple junction. Later in Section 6 we will show that this evolution is indeed a self-shrinker.

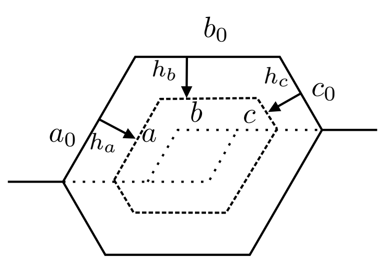

Fig. 17.

In Figure 16 (a) at the maximal time two horizontal segments of the hexagon collapse to the same segment, which has therefore *multiplicity two*.

Example 5.3. Consider the network $\mathcal{S}_0$ in Figure 17, and assume that the opposite sides of the initial hexagon are equal. Since $\mathcal{S}_0$ has a mirror symmetry, by Remark 4.1 so does its unique ϕ-curvature flow $\mathcal{S}(t)$. In the notation of Figure 17 the ϕ-curvature equation is

$$\begin{cases} h_a' = h_c' = \tfrac{2}{\sqrt{3}(a+c)}, \\ h_b' = \tfrac{2}{\sqrt{3}b}. \end{cases} \tag{5.1}$$

Moreover, it is possible to check that $a_0-c_0=a-c$, $h_a=h_c=\frac{b_0+c_0-b-c}{2}$, and therefore from (5.1) we get

$$(2a-a_0+c_0)(a'+b') = 2a'b.$$

Thus, representing $b=F(a)$ we can write this equality as

$$F'(a) = \tfrac{2}{2a-a_0+c_0}\,F(a) - 1,$$

which has the unique solution

$$b = F(a) = (2a - a_0 + c_0)\Big(\tfrac{b_0}{a_0+c_0} + \ln \tfrac{a_0+c_0}{2a-a_0+c_0}\Big). \tag{5.2}$$

(a) First consider the case $a_0 > c_0$. Then $a_0 - c_0 = a - c$, $a \geq a_0 - c_0 > 0$ and hence $a(T^\dagger) > 0$ and $b(T^\dagger) > 0$. Therefore, $c(T^\dagger) = 0$ and $\mathcal{S}(T^\dagger)$ is a union of two half-lines and a parallelogram, which is noncritical.

(b) In case $a_0 = c_0$, $\mathcal{S}^0$ has a horizontal symmetry and (5.2) is represented as

$$b = 2a\Big(\tfrac{b_0}{2a_0} + \ln a_0\Big) - 2a \ln a,$$

and hence $a(T^\dagger) = b(T^\dagger) = 0$, i.e., $\mathcal{S}(t)$ converges to the straight line (the hexagon disappears).

Next we analyse some examples of simple networks with multiple junctions.

Example 5.4. Consider the three situations in Figure 18.

- In case (a), at time $T^\dagger > 0$ the two lateral segments adjacent to $S_1(t)$ disappear first and thus, $\mathcal{S}(T^\dagger)$ looses 120^o-condition at both quadruple junctions. Similar situation happens in case (c), where only one of the lateral segments (provided that they are short enough) disappears at time $T^\dagger > 0$, destroying the 120^o-condition on the right quadruple junction. In both cases $\mathcal{S}(T^\dagger)$ is not simple anymore.

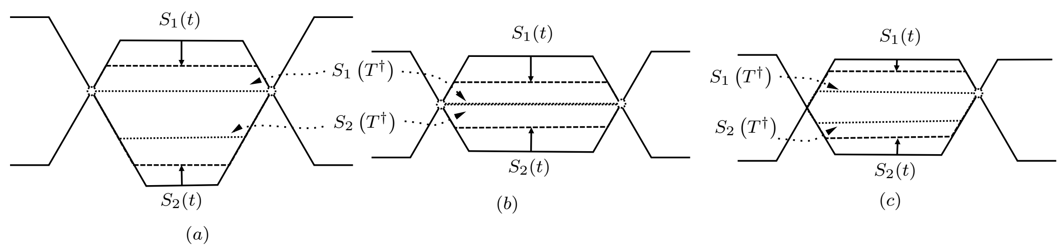

FIG. 18.

- In case (b), $\mathcal{S}^0$ is axially symmetric with respect to the horizontal and vertical axes. Since the flow preserves those symmetries, as $t \nearrow T^\dagger$ both segments $S_1(t)$ and $S_2(t)$ converge to the horizontal segment connecting the two quadruple junctions. Thus, $\mathcal{S}(T^\dagger)$ becomes a network having all axial symmetries of $\mathcal{S}^0$, but having two triple junctions connected by a segment.

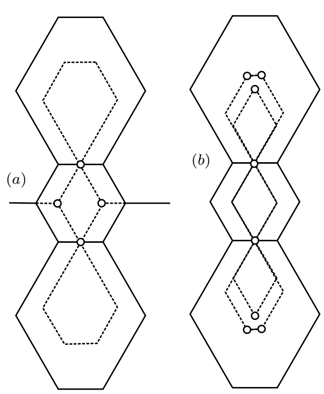

FIG. 19.

In Example 5.4 we observed the transition of quadruple junctions into triple junctions. Now we analyze the converse situation.

Example 5.5. Consider the networks in Figure 19, differing each other only by the two horizontal half-lines starting from the vertices of the mid-hexagon in the left figure. Both networks are simple and axially symmetric. Therefore $\mathcal{S}(T^\dagger)$ becomes axially symmetric, but the two triple junctions above and two triple junctions below join forming quadruple junctions (two horizontal segments of the mid hexagon disappear at $T^\dagger$). However, in (a) the quadruple junctions are linked to the triple junctions by a segment which makes nonzero the ϕ-curvature of those segments, and thus, invalidating the simpleness. Whereas in (b) $\mathcal{S}(T^\dagger)$ becomes simple with two quadruple junctions, but not parallel to the initial one. Thus, we may continue the flow after $T^\dagger$, until some other maximal time $T^\ddagger > T^\dagger$, at which two horizontal segments of pentagons above and below disappear, and hence, $\mathcal{S}(T^{\ddagger})$ ceases to be simple.

6. Homothetically shrinking solutions with one bounded component

All homotheties we consider have center at the origin of the coordinates.

Definition 6.1 (**Self-shrinker**). A family $\{\mathcal{S}(t)\}_{t\in[0,T)}$ of admissible polygonal networks evolving by ϕ-curvature is called a *homothetically shrinking solution* starting from $\mathcal{S}^0$, if there exists a strictly decreasing function $r:[0,T)\to[0,1]$ such that

$$\lim_{t\to T^-} r(t)=0 \quad \text{and} \quad \mathcal{S}(t)=r(t)\mathcal{S}^0 \text{ for all } t\in[0,T).$$

For shortness, we call $\mathcal{S}^0$ a *self-shrinker*.

In this section we classify the homothetically shrinking solutions starting from a polygonal admissible network $\mathcal{S}^0$, partitioning $\mathbb{R}^2$ into phases with only one bounded component consisting of a convex hexagon $\mathbf{A}$ (see Figure 20); when necessary, the hexagon is parametrized counterclockwise. By definition, any half-line of a self-shrinker must lie on a straight line passing through the origin and, being $\mathbf{A}$ convex, the homothety center cannot be in its exterior. Furthermore, if the origin is located in the (relative) interior of some segment $S=[XY]$ of $\mathbf{A}$, then by the self-similarity, that segment does not move (it has zero ϕ-curvature). In this case, by the convexity of $\mathbf{A}$ at least one endpoint of S, say X, should be the starting point of a half-line, lying on the same line with S. Then one can readily check that another segment of $\mathbf{A}$ sharing common point X with S should also have zero ϕ-curvature, i.e., it does not move, which falsifies the self-similarity of $\mathcal{S}^0$. Therefore, we only have to deal with two situations: the origin is either in the interior of $\mathbf{A}$ (see Figure 20) or it is one of the vertices (see Figure 23).

Recall that without half-lines a network $\mathcal{S}^0=\partial\mathbf{A}$ is a self-shrinker if and only if $\mathbf{A}=B^{\phi}_{R_0}$ for some $R_0>0$. Since the (signed) height is $\frac{\sqrt{3}}{3}(R_0-R(t))$ and the ϕ-curvature of any segment of $\mathbf{A}$ is $\frac{2}{\sqrt{3}R(t)}$, where $\frac{2}{\sqrt{3}}$ is the sidelength of the unit Wulff shape, the radius $R(t)$ of $\mathcal{S}(t)=\partial B^{\phi}_{R(t)}$ is given by

$$R(t):=\sqrt{R_0^2-\tfrac{8}{3}t}, \quad t\in[0,\tfrac{3R_0^2}{8}),$$

so that the function $r(t):=\sqrt{1-\frac{8t}{3R_0^2}}$ satisfies $\mathcal{S}(t)=r(t)\mathcal{S}^0$.

The main result of this section is the following classification theorem (see Figure 1).

Theorem 6.2 (**Classification of shrinkers**). *Let a network $\mathcal{S}^0$ be a union of an admissible convex hexagon $\mathbf{A}$ and $n\ge 1$-half-lines.*

(a) *Suppose the origin is an interior point of $\mathbf{A}$. Then $n\le 5$ and up to a rotation and a mirror reflection:*
  - *if $n=1$ (Brakke-type spoon), then the sidelengths of $\mathbf{A}$ are $a,2a,a,a,2a,a$ for some $a>0$, and the unique half-line starts from a vertex of $\mathbf{A}$ at which both adjacent segments have length a. In this case the origin divides the largest diagonal of $\mathbf{A}$ in proportion $1:2$ starting from the half-line, and*

$$\mathcal{S}(t)=r(t)\mathcal{S}^0 \qquad \text{with} \quad r(t)=\sqrt{1-\tfrac{2t}{3a_o^2}}, \quad t\in[0,\tfrac{3a_o^2}{2}); \tag{6.1}$$

  - *if $n=2$, then the sidelengths of $\mathbf{A}$ are approximately $a, 2.94771a$, $2.33925a$, $1.60847a$, $2.33924a$, $2.94772a$ for some $a>0$, and the two half-lines start from the endpoints of the segment of length a. Moreover (6.1) holds;*

– *if* $n = 3$, *then* $\mathbf{A}$ *is a regular hexagon and the three half-lines form a* 120^o*-angle. Moreover* (6.1) *holds;*
– *if* $n = 4$, *then* $\mathbf{A}$ *is a regular hexagon and the (prolongations of) four half-lines form an* X. *Moreover*

$$\mathcal{S}(t) = r(t)\mathcal{S}^0 \qquad \text{with} \quad r(t) = \sqrt{1-\tfrac{4t}{9a_o^2}}, \quad t \in [0, \tfrac{9a_o^2}{4}); \tag{6.2}$$

– *if* $n = 5$, *then* $\mathbf{A}$ *is a regular hexagon. Moreover* (6.2) *holds.*

(b) *Suppose the origin is a vertex (say* A_1*) of* $\mathbf{A}$. *Then* $\mathbf{A}$ *is homothetic to a Wulff shape, at* A_1 *there are at least one half-line parallel to adjacent segments at* A_1, *and there is another half-line at vertex* A_4 *opposite to* A_1 *(see Figures 23 (a) and (c)).*

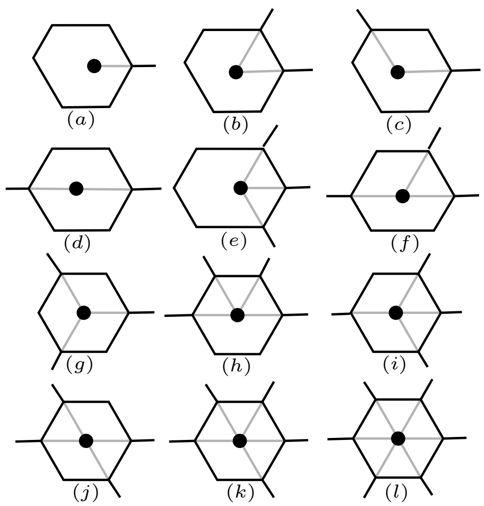

FIG. 20.

6.1. **Homothety center inside A.** In this section we assume that the homothety center – the origin – is an interior point of $\mathbf{A}$ so that all straight lines containing the half-lines pass through the origin and bisect the corresponding angles of $\mathbf{A}$. Accordingly, we write $\mathcal{S}^0 = \cup_{i=1}^n S_i$, where $S_1, \ldots, S_6$ are the sides of $\mathbf{A}$ (counterclockwise order) with lengths $a_1, \ldots, a_6$ respectively, and $S_7, \ldots, S_n$ are half-lines, $n \geq 7$ (see Figure 21). Note that $n \leq 12$.

Since the origin is inside $\mathbf{A}$, there might be at most one half-line emanating from each vertex. In particular, there are at least one and at most six such half-lines. Up to a rotation and mirror reflection, there are twelve possible variants (see Figure 20); in what follows we characterize which ones are a self-shrinker.

We use the notation of Figure 21 and, as in Figure 20, we always assume that A_1 is a triple junction of segments S_1, S_6 and the half-line S_7, and $\partial\mathbf{A}$ is oriented counterclockwise, so that the heights h_i and $\overline{h}_i$ are nonnegative. Clearly, $\theta_4 + \overline{\theta}_4 = 120^o$, and $\theta_1 = \overline{\theta}_1 = 60^o$ so that $h_1 = \overline{h}_1$. Let $a_1 := a > 0$ and $h_1 = \overline{h}_1 =: h \geq 0$.

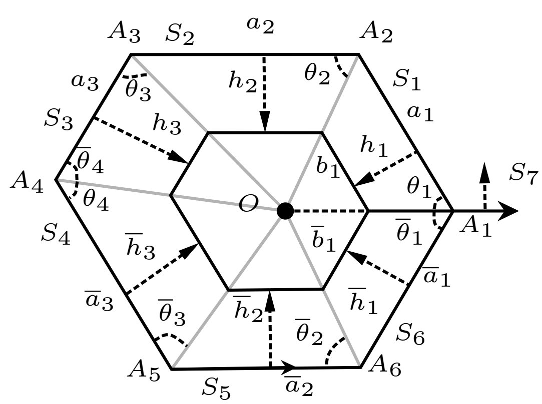

FIG. 21.

We proceed as follows. We express the sidelengths a_i and $\overline{a}_i$ of $\mathbf{A}$, for $i = 2, \ldots, 6$, by means of a and the angles $\theta_i, \overline{\theta}_i$ (see (6.5)). Similarly, we represent the remaining heights by means of h and $\theta_i, \overline{\theta}_i$ (see (6.3)). The driving reason here is that, by the homothety, θ_i and

$\overline{\theta}_i$ are independent of time, and therefore the equation $h_i' = -\kappa^\phi_{S_i} = \frac{c_i}{\mathcal{H}^1(S_i)}$ (see (3.2)) can be rewritten only using a, h and those angles; here c_i are some numbers depending only on the structure of the network (see (6.6)-(6.11) below), which somehow account for how many triple junctions "linked" to each other are present. These curvature equations for segments $S_1, \dots, S_6$ imply a system with respect to θ_i and $\overline{\theta}_i$, see (6.12). Since c_i changes as the number of half-lines in the network increases, we need to analyze each network in Figure 20 separately.

Preliminary computations. Now we enter into the details. Let us introduce the notation

$$\omega_i := \frac{\sin(60^o + \theta_i)}{\sin\theta_i}, \quad \overline{\omega}_i := \frac{\sin(60^o + \overline{\theta}_i)}{\sin\overline{\theta}_i}, \quad i = 1, 2, 3, 4.$$

Clearly $\omega_i > 0$ and $\overline{\omega}_i > 0$, since $\theta_i \in (0, 120^o)$. We express a_i, $\overline{a}_i$, h_i and $\overline{h}_i$ by means of a, h, ω_i and $\overline{\omega}_i$. We have:

$$\begin{cases} h_1 = h, \\ h_2 = \frac{\sin\theta_2}{\sin(60^o+\theta_2)} h = \frac{h}{\omega_2}, \\ h_3 = \frac{\sin\theta_3}{\sin(60^o+\theta_3)} \frac{\sin\theta_2}{\sin(60^o+\theta_2)} h = \frac{h}{\omega_2\omega_3}, \end{cases} \qquad \begin{cases} \overline{h}_1 = h, \\ \overline{h}_2 = \frac{\sin\overline{\theta}_2}{\sin(60^o+\overline{\theta}_2)} h = \frac{h}{\overline{\omega}_2}, \\ \overline{h}_3 = \frac{\sin\overline{\theta}_3}{\sin(60^o+\overline{\theta}_3)} \frac{\sin\overline{\theta}_2}{\sin(60^o+\overline{\theta}_2)} h = \frac{h}{\overline{\omega}_2\overline{\omega}_3}. \end{cases} \tag{6.3}$$

Since $\theta_4 + \overline{\theta}_4 = 120^o$ and $h_3 \sin\theta_4 = \overline{h}_3 \sin\overline{\theta}_4$, from (6.3) the angles θ_i satisfy the identities

$$\frac{\sin\theta_4}{\sin(60^o + \theta_4)} = \frac{\sin(60^o + \overline{\theta}_4)}{\sin\overline{\theta}_4}, \quad \frac{\sin(60^o + \overline{\theta}_3)}{\sin(60^o + \theta_3)} \frac{\sin(60^o + \overline{\theta}_2)}{\sin(60^o + \theta_2)} = \frac{\sin\overline{\theta}_2}{\sin\theta_2} \frac{\sin\overline{\theta}_3}{\sin\theta_3} \frac{\sin\overline{\theta}_4}{\sin\theta_4}$$

or, in terms of ω_i and $\overline{\omega}_i$,

$$\omega_4\overline{\omega}_4 = 1, \quad \overline{\omega}_3\overline{\omega}_2 = \omega_4\omega_3\omega_2, \quad \omega_3\omega_2 = \overline{\omega}_4\overline{\omega}_3\overline{\omega}_2. \tag{6.4}$$

Next, using the law of sines and the equalities

$$\frac{\sin(60^o + \theta_{i+1} - \theta_i)}{\sin\theta_{i+1}\sin\theta_i} = \frac{2}{\sqrt{3}}(\omega_{i+1}\omega_i - \omega_{i+1} + 1), \quad \frac{\sin(60^o + \overline{\theta}_{i+1} - \overline{\theta}_i)}{\sin\overline{\theta}_{i+1}\sin\overline{\theta}_i} = \frac{2}{\sqrt{3}}(\overline{\omega}_{i+1}\overline{\omega}_i - \overline{\omega}_{i+1} + 1),$$

we represent the sidelengths of $\mathbf{A}$ as follows:

$$\begin{aligned} a_1 =& a, \\ a_2 =& \frac{\sqrt{3}\sin(60^o + \theta_3 - \theta_2)}{2\sin\theta_2\sin(60^o + \theta_3)} a = \frac{\omega_3\omega_2 - \omega_3 + 1}{\omega_3} a, \\ a_3 =& \frac{\sqrt{3}\sin(60^o + \theta_4 - \theta_3)}{2\sin(60^o + \theta_3)\sin(60^o + \theta_4)} a = \frac{\omega_3\omega_4 - \omega_4 + 1}{\omega_3\omega_4} a, \\ \overline{a}_1 =& \frac{\sin\overline{\theta}_2}{\sin(60^o + \overline{\theta}_2)} \frac{\sin(60^o + \theta_2)}{\sin\theta_2} a = \frac{\omega_2}{\overline{\omega}_2} a, \\ \overline{a}_2 =& \frac{\sqrt{3}\sin(60^o + \overline{\theta}_3 - \overline{\theta}_2)\sin(60 + \theta_2)}{2\sin\theta_2\sin(60^o + \overline{\theta}_2)\sin(60^o + \overline{\theta}_3)} a = \frac{\overline{\omega}_3\overline{\omega}_2 - \overline{\omega}_3 + 1}{\omega_3\omega_4} a, \\ \overline{a}_3 =& \frac{\sqrt{3}\sin(60^o + \overline{\theta}_4 - \overline{\theta}_3)\sin\overline{\theta}_2\sin(60^o + \theta_2)}{2\sin\theta_2\sin(60^o + \overline{\theta}_2)\sin(60^o + \overline{\theta}_3)\sin(60^o + \overline{\theta}_4)} a \\ =& \frac{\overline{\omega}_3\overline{\omega}_4 - \overline{\omega}_4 + 1}{\omega_3} a = \frac{\omega_3 + \omega_4 - 1}{\omega_3\omega_4} a, \end{aligned} \tag{6.5}$$

where in the last equality we used $\omega_4\overline{\omega}_4 = 1$.

The ϕ-curvature of S_i is given by

$$\kappa^\phi_{S_i} = -\frac{c_i}{\mathcal{H}^1(S_i)},$$

where $c_i \geq 0$ is computed using similar arguments of Section 2.9. Indeed, neglecting for the moment the half-lines and assuming $S_i = S_{i-6}$ for $i > 6$, one can show that $\mathcal{S}^0$ is simple and:

- if both endpoints A_i and A_{i+1} of S_i are simple vertices of $\mathcal{S}^0$, then
$$c_i := \frac{2}{\sqrt{3}}; \tag{6.6}$$
- if S_i and S_{i+1} join at a triple junction, and both other endpoints are simple vertices, then
$$c_j = \frac{2\mathcal{H}^1(S_j)}{\sqrt{3}(\mathcal{H}^1(S_i) + \mathcal{H}^1(S_{i+1}))}, \quad j = i, i+1; \tag{6.7}$$
- if (S_i, S_{i+1}) and (S_{i+1}, S_{i+2}) join at two triple junctions[8], and the other endpoints of S_i and S_{i+2} are simple vertices, then
$$c_j = \frac{2\mathcal{H}^1(S_j)}{\sqrt{3}(\mathcal{H}^1(S_i) + \mathcal{H}^1(S_{i+1}) + \mathcal{H}^1(S_{i+2}))}, \quad j = i, i+1, i+2; \tag{6.8}$$
- if (S_i, S_{i+1}), (S_{i+1}, S_{i+2}) and (S_{i+2}, S_{i+3}) join at three triple junctions, and the other endpoints of S_i and S_{i+3} are simple vertices, then
$$c_j = \frac{2\mathcal{H}^1(S_j)}{\sqrt{3}\sum_{l=i}^{i+3}\mathcal{H}^1(S_l)}, \quad j = i, i+1, i+2, i+3; \tag{6.9}$$
- if (S_i, S_{i+1}), (S_{i+1}, S_{i+2}), (S_{i+2}, S_{i+3}) and (S_{i+3}, S_{i+4}) join at four triple junctions, and the other endpoints of S_i and S_{i+4} are simple vertices, then
$$c_j = \frac{2\mathcal{H}^1(S_j)}{\sqrt{3}\sum_{l=i}^{i+4}\mathcal{H}^1(S_l)}, \quad j = i, i+1, i+2, i+3, i+4. \tag{6.10}$$
Note that if there are only four triple junctions, then $c_{i+5} = \frac{2}{\sqrt{3}}$;
- if (S_i, S_{i+1}), (S_{i+1}, S_{i+2}), (S_{i+2}, S_{i+3}), (S_{i+3}, S_{i+4}) and (S_{i+4}, S_{i+5}) join at five triple junctions, and the other endpoints of S_i and S_{i+5} are simple vertices, then
$$c_j = \frac{2\mathcal{H}^1(S_j)}{\sqrt{3}\sum_{l=i}^{i+5}\mathcal{H}^1(S_l)}, \quad j = i, i+1, i+2, i+3, i+4, i+5. \tag{6.11}$$

These computations show that in homothetic networks these numbers do not change. In particular, with the notation of Figure 21 (d), the ϕ-curvature equation for segments S_i and S_{7-i} in the homothetically shrinking solution can be represented as
$$h_i' = -\kappa_{S_i}^\phi = \frac{c_i}{a_i} \quad \text{resp.} \quad \overline{h}_i' = -\kappa_{S_{7-i}}^\phi = \frac{c_{7-i}}{\overline{a}_i}, \quad i = 1, 2, 3.$$

By homothety, the angles θ_i and $\overline{\theta}_i$ are independent of time, and hence, using the equalities (6.3) and the representations of a_i with a in (6.5) we obtain six equalities
$$ah' = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6, \tag{6.12}$$

[8]I.e. the pairs (S_i, S_{i+1}) of segments and one half-line form a triple junction.

where

$$\gamma_i := \begin{cases} c_1 & \text{for } i=1,\\ \frac{2c_2\sin(60^o+\theta_2)\sin(60^o+\theta_3)}{\sqrt{3}\sin(60^o+\theta_3-\theta_2)} = \frac{c_2\omega_2\omega_3}{\omega_2\omega_3-\omega_3+1} & \text{for } i=2,\\ \frac{2c_3\sin(60^o+\theta_2)\sin^2(60^o+\theta_3)\sin(60^o+\theta_4)}{\sqrt{3}\sin\theta_2\sin\theta_3\sin(60^o+\theta_4-\theta_3)} = \frac{c_3\omega_2\omega_3^2\omega_4}{\omega_4\omega_3-\omega_4+1} & \text{for } i=3,\\ \frac{2c_4\sin\theta_2\sin^2(60^o+\overline{\theta}_2)\sin^2(60^o+\overline{\theta}_3)\sin(60^o+\overline{\theta}_4)}{\sqrt{3}\sin^2\overline{\theta}_2\sin\overline{\theta}_3\sin(60^o+\theta_2)\sin(60^o+\overline{\theta}_4-\overline{\theta}_3)} = \frac{c_4\omega_2\omega_3^2\omega_4}{\overline{\omega}_4\overline{\omega}_3-\overline{\omega}_4+1} & \text{for } i=4,\\ \frac{2c_5\sin\theta_2\sin^2(60^o+\overline{\theta}_2)\sin(60^o+\overline{\theta}_3)}{\sqrt{3}\sin\overline{\theta}_2\sin(60^o+\theta_2)\sin(60^o+\overline{\theta}_3-\overline{\theta}_2)} = \frac{c_5\overline{\omega}_2\omega_3\omega_4}{\overline{\omega}_3\overline{\omega}_2-\overline{\omega}_3+1} & \text{for } i=5,\\ \frac{c_6\sin\theta_2\sin(60^o+\overline{\theta}_2)}{\sin\overline{\theta}_2\sin(60^o+\theta_2)} = \frac{c_6\overline{\omega}_2}{\omega_2} & \text{for } i=6. \end{cases}$$

These equations provide a necessary and sufficient condition for $\mathcal{S}^0$ to be a self-shrinker.

Now, we examine the networks in Figure 20.

One half-line case. Let us study self-shrinking Brakke-type spoons $\mathcal{S}^0$ as in Figure 20 (a), i.e., in the notation of Figure 21, the only half-line starts from the vertex A_1 of the hexagon $\mathbf{A}$. Since $\theta_1 = \overline{\theta}_1 = 60^o$,

$$\omega_1 = \overline{\omega}_1 = 1, \quad \omega_4\overline{\omega}_4 = 1, \quad \omega_4\omega_3\omega_2 = \overline{\omega}_3\overline{\omega}_2, \quad \overline{\omega}_2\overline{\omega}_3\omega_4 = \omega_2\omega_3.$$

The sidelengths of $\mathbf{A}$ are represented by means of ω_i and $\overline{\omega}_i$ as

$$a_1 = a, \quad a_2 = \frac{\omega_3\omega_2-\omega_3+1}{\omega_3}a, \quad a_3 = \frac{\omega_4\omega_3-\omega_4+1}{\omega_3\omega_4}a, \quad \overline{a}_1 = \frac{\omega_2}{\omega_2}a$$

and

$$\overline{a}_2 = \frac{\overline{\omega}_3\overline{\omega}_2-\overline{\omega}_3+1}{\omega_3\omega_4}a = \frac{\omega_4\omega_3\omega_2-\overline{\omega}_3+1}{\omega_3\omega_4}a, \quad \overline{a}_3 = \frac{\overline{\omega}_4\overline{\omega}_3-\overline{\omega}_4+1}{\omega_3}a = \frac{\omega_4+\overline{\omega}_3-1}{\omega_3\omega_4}a.$$

Next, recalling the definitions of c_i in (6.6) and (6.7), we obtain the following representations of γ_i:

$$\gamma_1 = \gamma_6 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2}{\omega_2+\overline{\omega}_2}, \quad \gamma_2 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_3}{\omega_2\omega_3-\omega_3+1}, \quad \gamma_3 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_3^2\omega_4}{\omega_3\omega_4-\omega_4+1},$$

and

$$\gamma_4 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_3^2\omega_4}{\overline{\omega}_3\overline{\omega}_4-\overline{\omega}_4+1}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2\omega_3\omega_4}{\overline{\omega}_2\overline{\omega}_3-\overline{\omega}_3+1}.$$

First, from the equality $\gamma_1 = \gamma_2$ we deduce $\overline{\omega}_2 = \frac{\omega_2^2\omega_3}{1-\omega_3}$. Moreover, from $\gamma_3 = \gamma_4$ we get $\overline{\omega}_3 = \omega_3\omega_4^2 - \omega_4^2 + 1$ and from $\gamma_2 = \gamma_3$ we get $\omega_4 = \frac{1}{\omega_2\omega_3^2-\omega_3^2+1}$. Inserting the values found of $\overline{\omega}_2$ and $\overline{\omega}_3$ in the equality $\gamma_2 = \gamma_5$ we obtain

$$\frac{1}{\omega_2\omega_3-\omega_3+1} = \frac{\omega_3\omega_4}{(1-\omega_3)(\omega_4-\omega_4\omega_3+\omega_3\omega_2)}.$$

This equality is equivalent to $\omega_4(1-2\omega_3+\omega_3^2) = \omega_3^2\omega_2^2$. Inserting here the earlier values found of ω_4 we obtain the following fourth order equation:

$$\omega_3^4(\omega_2^3-\omega_2^2)+\omega_3^2(\omega_2^2-1)+2\omega_3-1=0. \tag{6.13}$$

On the other hand, inserting the values of $\overline{\omega}_2, \overline{\omega}_3$ and ω_4 in the equation $\omega_4\omega_3\omega_2 = \overline{\omega}_3\overline{\omega}_2$ we obtain another fourth order equation:

$$\omega_3^4(\omega_2^3-2\omega_2^2+\omega_2)+\omega_3^3(\omega_2-1)+\omega_3^2(2\omega_2^2-3\omega_2+1)+\omega_3(\omega_2+1)-1=0. \tag{6.14}$$

Subtracting (6.14) from (6.13) we get

$$\omega_3^2(\omega_2-\omega_2^2)+\omega_3^2(\omega_2-1)+\omega_3(\omega_2^2-3\omega_2+2)+(\omega_2-1)=0. \tag{6.15}$$

From this equality we deduce $\omega_2 = 1$. Inserting this in (6.13) we find $\omega_3 = \frac{1}{2}$. Let us check whether there are other solutions. Factoring out (6.15) the term $\omega_2 - 1$ we get

$$-\omega_3^2\omega_2 + \omega_3^2 + \omega_3(\omega_2 - 2) + 1 = 0. \tag{6.16}$$

Adding this to (6.13) we establish

$$\omega_3^4(\omega_3^3 - \omega_3^2) - \omega_3^3\omega_2 + \omega_3^2\omega_2^2 + \omega_2\omega_3 = 0.$$

Recalling $\omega_2, \omega_3 > 0$, the last equation is equivalent to

$$\omega_3^3(\omega_2^2 - \omega_2) - \omega_3^2 + \omega_2\omega_3 + 1 = 0. \tag{6.17}$$

Subtracting (6.16) from (6.17) gives

$$\omega_3^2\omega_2^2 - 2\omega_3 + 2 = 0. \tag{6.18}$$

Multiplying (6.16) by -2 and adding to (6.18) gives

$$2\omega_3^3\omega_2 + \omega_3^2(\omega_2^2 - 2) - 2\omega_3(\omega_2 - 1) = 0,$$

and hence,

$$2\omega_3^2\omega_2 + \omega_3(\omega_2^2 - 2) - 2(\omega_2 - 1) = 0. \tag{6.19}$$

Now multiplying (6.18) by $\omega_2 - 1$ and adding to (6.19) we get

$$\omega_3^2(\omega_2^3 - \omega_2^2 + 2\omega_2) + \omega_3(\omega_2^2 - 2\omega_2) = 0,$$

and thus,

$$\omega_3 = \frac{2 - \omega_2}{\omega_2^2 - \omega_2 + 2}.$$

Inserting this expression of ω_3 in (6.13) and simplifying the similar terms we get

$$\frac{(2 - \omega_2)^2}{\omega_2^2 - \omega_2 + 2} + 2 = 0,$$

which does not have real solutions.

Thus, we have a unique solution

$$\omega_1 = \overline{\omega}_1 = 1, \quad \omega_2 = \overline{\omega}_2 = 1, \quad \omega_3 = \overline{\omega}_3 = \frac{1}{2}, \quad \omega_4 = \overline{\omega}_4 = 1.$$

Then the corresponding angles are

$$\theta_1 = \theta_2 = \theta_4 = \overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}_4 = 60^o, \quad \theta_3 = \overline{\theta}_3 = 90^o$$

and the sidelengths are

$$a_1 = a_3 = \overline{a}_1 = \overline{a}_3 = a, \quad a_2 = \overline{a}_2 = 2a.$$

Thus, $\mathbf{A}$ is that hexagon, symmetric with respect to horizontal axis, whose three consecutive segments in the upper half-plane have lengths $a, 2a$ and a. Note that the homothety center (the origin) is not the midpoint of $[A_1A_4]$, rather, it divides this segment in proportion $1:2$ (starting from A_1).

Assuming initially $\mathcal{S}^0$ have lengths $a_o := \mathcal{H}^1(S_1^0)$ and $2a_o = \mathcal{H}^1(S_2^0)$, let us find the function $r(\cdot)$ satisfying $\mathcal{S}(t) = r(t)\mathcal{S}^0$. Since the triangle A_1OA_2 is equilateral, $h(t) = \frac{\sqrt{3}}{2}(a_o - a(t))$ and $\kappa^\phi_{S_1(t)} = \frac{1}{\sqrt{3}a(t)}$. Thus, the ϕ-curvature equation $h'(t) = -\kappa^\phi_{S_1(t)}$ is expressed as

$$\frac{\sqrt{3}}{2}a' = \frac{1}{\sqrt{3}a} \quad \text{so that} \quad a(t) = \sqrt{a_o^2 - \frac{2}{3}t}, \quad t \in [0, \tfrac{3a_o^2}{2}).$$

Whence the function $r(t) = \sqrt{1 - \frac{2t}{3a_o^2}}$ satisfies $\mathcal{S}(t) = r(t)\mathcal{S}^0$.

Two half-lines: case 1. Let us check whether the network $\mathcal{S}^0$ in Figure 20 (b) is a self-shrinker. With the notation of Figure 21, the half-lines of $\mathcal{S}^0$ start from A_1 and A_2, so that $\theta_1 = \overline{\theta}_1 = \theta_2 = 60^o$. Thus, (6.4) is represented as

$$\omega_1 = \overline{\omega}_1 = \omega_2 = 1, \quad \overline{\omega}_4\omega_4 = 1, \quad \overline{\omega}_3\overline{\omega}_2 = \omega_4\omega_3.$$

In this case

$$a_1 = a, \quad a_2 = \frac{a}{\omega_3}, \quad a_3 = \frac{\omega_4\omega_3 - \omega_4 + 1}{\omega_4\omega_3}a,$$

and

$$\overline{a}_1 = \frac{a}{\overline{\omega}_2}, \quad \overline{a}_2 = \frac{\omega_4\omega_3 - \overline{\omega}_3 + 1}{\omega_3\omega_4}, \quad \overline{a}_3 = \frac{\omega_4 + \overline{\omega}_3 - 1}{\omega_4\omega_3},$$

and hence, by the definition of c_i in (6.6) and (6.8),

$$\gamma_1 = \gamma_2 = \gamma_6 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2\omega_3}{\overline{\omega}_2\omega_3 + \omega_3 + \overline{\omega}_2}, \quad \gamma_3 = \frac{2}{\sqrt{3}}\frac{\omega_4\omega_3^2}{\omega_4\omega_3 - \omega_4 + 1},$$

and

$$\gamma_4 = \frac{2}{\sqrt{3}}\frac{\omega_4\omega_3^2}{\overline{\omega}_4\overline{\omega}_3 - \overline{\omega}_4 + 1}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2\omega_3\omega_4}{\overline{\omega}_4\overline{\omega}_3 - \overline{\omega}_3 + 1}.$$

From the equalities $\gamma_1 = \gamma_3$ and $\gamma_3 = \gamma_4$ as well as $\overline{\omega}_4 = \frac{1}{\omega_4}$ we obtain

$$\overline{\omega}_2 = \frac{\omega_3^2\omega_4}{1 - (1 + \omega_3^2)\omega_4} \quad \text{and} \quad \omega_3 = 1 - (1 - \omega_3)\omega_4^2.$$

Inserting these relations in the equality $\gamma_3 = \gamma_5$ we get

$$\frac{1}{\omega_3\omega_4 - \omega_4 + 1} = \frac{\omega_3^2\omega_4}{(1 - \omega_4 - \omega_3^2\omega_4)(\omega_4 + \omega_3 - \omega_3\omega_4)},$$

and hence

$$\omega_4 = \frac{\omega_3^3 + \omega_3^2 + \omega_3 - 1}{\omega_3^3 - \omega_3^2 + \omega_3 - 1}.$$

Inserting the expressions of $\overline{\omega}_2$, $\overline{\omega}_3$ and ω_4 in the identity $\overline{\omega}_2\overline{\omega}_3 = \omega_3\omega_4$ gives

$$(\omega_3^3 + \omega_3^2 + \omega_3 - 1)^2 + (\omega_3^2 + 2\omega_3 - 1)(\omega_3^2 + 1)^2 = 0.$$

After simplification (recalling that $\omega_3 > 0$), this equation reduces to

$$\omega_3^4 + 2\omega_3^2 + 2\omega_3^2 + 2\omega_3 - 1 = 0$$

which admits a unique positive solution $\omega_3 \approx 0.33925$. Then the sidelengths of $\mathbf{A}$ are defined as

$$a_1 = a, \quad a_2 \approx 2.94771a, \quad a_3 \approx 2.33925a, \quad a_4 \approx 1.60847a, \quad a_5 \approx 2.33924a, \quad a_6 \approx 2.94772a.$$

Moreover, the corresponding angles θ_i and $\overline{\theta}_i$ are

$$\theta_1 = 60^o, \quad \theta_2 = 60^o, \quad \theta_3 \approx 100.51566^o, \quad \theta_4 \approx 77.77741^o,$$

$$\overline{\theta}_1 = 60^o, \quad \overline{\theta}_2 \approx 100.51576^o, \quad \overline{\theta}_3 \approx 77.77726^o, \quad \overline{\theta}_4 \approx 42.22259^o.$$

In particular, $\mathbf{A}$ has no symmetry. Assuming $a_1 = a_o$ for the initial hexagon, let us look for a function $r(\cdot)$ satisfying $\mathcal{S}(t) = r(t)\mathcal{S}^0$. Since the triangle A_1OA_2 is equilateral, $h(t) = \frac{\sqrt{3}}{2}(a_o - a(t))$ and $\kappa^\phi_{S_1(t)} = \frac{1}{\sqrt{3}a(t)}$. Thus, the ϕ-curvature equation $h'(t) = -\kappa^\phi_{S_1(t)}$ is expressed as

$$\frac{\sqrt{3}}{2}a' = \frac{1}{\sqrt{3}a} \quad \text{so that} \quad a(t) = \sqrt{a_o^2 - \frac{2}{3}t}, \quad t \in [0, \tfrac{3a_o^2}{2}).$$

Whence the function $r(t) = \sqrt{1 - \frac{2t}{3a_o^2}}$ satisfies $\mathcal{S}(t) = r(t)\mathcal{S}^0$.

Two half-lines: case 2. Let $\mathcal{S}^0$ be as in Figure 20 (c) so that with the notation of Figure 21 (d) the half-lines of $\mathcal{S}^0$ start from A_1 and A_3. Then $\theta_1 = \theta_3 = \overline{\theta}_1 = 60^o$ so that

$$\omega_1 = \omega_3 = \overline{\omega}_1 = 1, \quad \overline{\omega}_3\overline{\omega}_2 = \omega_4\omega_2.$$

Then

$$a_1 = a, \quad a_2 = \omega_2 a, \quad a_3 = \frac{a}{\omega_4}, \quad \overline{a}_1 = \frac{\omega_2}{\overline{\omega}_2}a, \quad \overline{a}_2 = \frac{\omega_4\omega_2 - \overline{\omega}_3 + 1}{\omega_4}a, \quad \overline{a}_3 = \frac{\overline{\omega}_3\overline{\omega}_4 - \overline{\omega}_3 + 1}{\omega_3}a.$$

Now recalling the definitions of c_i in (6.6) and (6.7) we compute

$$\gamma_1 = \gamma_6 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2}{\omega_2 + \overline{\omega}_2}, \quad \gamma_2 = \gamma_3 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_4}{\omega_2\omega_4 + 1},$$

and

$$\gamma_4 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_4}{\overline{\omega}_3\overline{\omega}_4 - \overline{\omega}_4 + 1}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2\omega_4}{\omega_4\omega_2 - \overline{\omega}_3 + 1}.$$

From the equality $\gamma_1 = \gamma_2$ we get $\overline{\omega}_2 = \omega_2^2\omega_3$; hence, inserting this in the equality $\gamma_4 = \gamma_5$ we obtain $\overline{\omega}_3 = \frac{1}{\omega_2}$. Since $\overline{\omega}_4 = \frac{1}{\omega_4}$, the system (6.12) is reduced to three equalities

$$\frac{1}{\omega_2\omega_4 + 1} = \frac{\omega_2\omega_4}{\omega_2\omega_4 - \omega_2 + 1} = \frac{\omega_2^2\omega_4}{\omega_2^2\omega_4 + \omega_2 - 1}.$$

From the second equality (recalling that $\omega_2 > 0$) we get $\omega_2 = 1$. Then the first equality in implies $\omega_4 = 0$, which contradicts the positivity of ω_4. Thus, $\mathcal{S}^0$ cannot be a self-shrinker.

Two half-lines: case 3. Let $\mathcal{S}^0$ be as in Figure 20 (d) so that with the notation of Figure 21 the half-lines of $\mathcal{S}^0$ start from A_1 and A_4. Then $\theta_1 = \overline{\theta}_1 = \theta_4 = \overline{\theta}_4 = 60^o$ so that

$$\omega_1 = \overline{\omega}_1 = \omega_4 = \overline{\omega}_4 = 1, \quad \omega_3\omega_2 = \overline{\omega}_3\overline{\omega}_2$$

and

$$a_1 = a_3 = a, \quad \overline{a}_1 = \overline{a}_3 = \frac{\omega_2}{\overline{\omega}_2}a, \quad a_2 = \frac{\omega_2\omega_3 - \omega_3 + 1}{\omega_3}a, \quad \overline{a}_2 = \frac{\omega_2\omega_3 - \overline{\omega}_3 + 1}{\omega_3}a.$$

Now using the definitions of c_i in (6.6) and (6.7) we find

$$\gamma_1 = \gamma_6 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2}{\omega_2 + \overline{\omega}_2}, \quad \gamma_2 = \frac{2}{\sqrt{3}}\frac{\omega_2\omega_3}{\omega_2\omega_3 - \omega_3 + 1}$$

and

$$\gamma_3 = \gamma_4 = \frac{2}{\sqrt{3}}\frac{\omega_2\overline{\omega}_2\omega_3}{\omega_2 + \overline{\omega}_2}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2\omega_3}{\omega_2\omega_3 - \overline{\omega}_3 + 1}.$$

From the equality $\gamma_1 = \gamma_3$ we get $\omega_2\omega_3 = \overline{\omega}_2\overline{\omega}_3 = 1$. Thus, inserting the relations $F_3 = \frac{1}{\omega_2}$ and $\overline{\omega}_3 = \frac{1}{\overline{\omega}_2}$ in (6.12) we deduce

$$\frac{\overline{\omega}_2}{\omega_2 + \overline{\omega}_2} = \frac{\omega_2}{2\omega_2 - 1} = \frac{\overline{\omega}_2^2}{2\overline{\omega}_2\omega_2 - \omega_2}.$$

From the second equality it follows $\overline{\omega}_2 = \omega_2$ or $\overline{\omega}_2 = \frac{\omega_2}{2\omega_2 - 1}$. In the former case, the first equality reduces to $\frac{1}{2\omega_2} = \frac{1}{2\omega_2 - 1}$, which is impossible. In the latter case, from the first equality we get the quadratic equation $2\omega_2^2 - 2\omega_2 + 1 = 0$, which has no positive solutions. Therefore, $\mathcal{S}^0$ is not a self-shrinker.

Remark 6.3. In the Euclidean case there exists a unique convex self-shrinking lense-shaped network [36]. As we have seen Example 5.3, at the maximal time the hexagon **A** shrinks to a point, but not self-similarly.

Three half-lines: case 1. Let $\mathcal{S}^0$ be as in Figure 20 (e) so that with the notation of Figure 21 the half-lines of $\mathcal{S}^0$ start from A_1, A_2 and A_6. Then $\theta_1 = \theta_2 = \overline{\theta}_1 = \overline{\theta}_2 = \theta_4 = \overline{\theta}_4 = 60^o$ so that $\omega_2 = \overline{\omega}_2 = 1$, and (6.4) is rewritten as

$$\omega_4\overline{\omega}_4 = 1, \quad \omega_4\omega_3 = \overline{\omega}_3, \quad \overline{\omega}_3\overline{\omega}_4 = \omega_3.$$

In this case,

$$a_1 = \overline{a}_1 = a, \quad a_2 = \frac{a}{\omega_3}, \quad a_3 = \overline{a}_3 = \frac{\omega_4\omega_3 - \omega_4 + 1}{\omega_4\omega_3}a, \quad \overline{a}_2 = \frac{a}{\omega_4\omega_3},$$

and hence, using the above discussions for the definition of c_i in (6.6) and (6.9) we get

$$\gamma_1 = \gamma_2 = \gamma_6 = \gamma_5 = \frac{2}{\sqrt{3}}\,\frac{\omega_4\omega_3}{2\omega_4\omega_3 + \omega_4 + 1}, \quad \gamma_3 = \frac{2}{\sqrt{3}}\,\frac{\omega_4\omega_3^2}{\omega_4\omega_3 - \omega_4 + 1}, \quad \gamma_4 = \frac{2}{\sqrt{3}}\,\frac{\omega_4^2\omega_3^2}{\omega_4\omega_3 + \omega_4 - 1}.$$

Thus, (6.12) reduced to the following system:

$$\frac{2}{\sqrt{3}}\,\frac{\omega_4\omega_3}{2\omega_4\omega_3 + \omega_4 + 1} = \frac{2}{\sqrt{3}}\,\frac{\omega_4\omega_3^2}{\omega_4\omega_3 - \omega_4 + 1} = \frac{2}{\sqrt{3}}\,\frac{\omega_4^2\omega_3^2}{\omega_4\omega_3 + \omega_4 - 1}. \tag{6.20}$$

From the second equality it follows that $\omega_4 = 1$ or $\omega_4 = \frac{1}{\omega_3 - 1}$. If $\omega_4 = 1$, inserting this in the second equality we find $\frac{1}{\omega_3+1} = 1$, i.e., $\omega_3 = 0$, which contradicts the positivity of ω_3. In case $\omega_4 = \frac{1}{\omega_3-1}$, inserting this in the first equation in (6.20) we get $\frac{1}{\frac{2\omega_3+1}{\omega_3-1}+1} = \frac{\omega_3}{2}$, which does not admit any positive solution. Thus, $\mathcal{S}^0$ is not a self-shrinker.

6.2. **Three half-lines: case 2.** Let $\mathcal{S}^0$ be as in Figure 20 (f) so that with the notation of Figure 21 the half-lines of $\mathcal{S}$ start from A_1, A_2 and A_4. Then $\theta_1 = \theta_2 = \theta_4 = \overline{\theta}_1 = \overline{\theta}_4 = 60^o$ so that

$$\omega_2 = \omega_4 = \overline{\omega}_4 = 1, \quad \omega_3 = \overline{\omega}_3\overline{\omega}_2.$$

Whence

$$a_1 = a_3 = a, \quad a_2 = \frac{a}{\overline{\omega}_3\overline{\omega}_2}, \quad \overline{a}_1 = \overline{a}_3 = \frac{a}{\overline{\omega}_2}, \quad \overline{a}_2 = \frac{\overline{\omega}_2\overline{\omega}_3 - \overline{\omega}_3 + 1}{\overline{\omega}_2\overline{\omega}_3}a$$

and hence, using the computations of c_i in (6.6)-(6.8) above we deduce

$$\gamma_1 = \gamma_2 = \gamma_6 = \frac{2}{\sqrt{3}}\,\frac{\overline{\omega}_3\overline{\omega}_2}{\overline{\omega}_3\overline{\omega}_2 + \overline{\omega}_3 + 1}, \quad \gamma_3 = \gamma_4 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_3\overline{\omega}_2^2}{\overline{\omega}_2 + 1}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_3\overline{\omega}_2^2}{\overline{\omega}_3\overline{\omega}_2 - \overline{\omega}_3 + 1}.$$

Thus, (6.12) reduces to

$$\frac{2}{\sqrt{3}}\,\frac{\overline{\omega}_3\overline{\omega}_2}{\overline{\omega}_3\overline{\omega}_2 + \overline{\omega}_3 + 1} = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_3\overline{\omega}_2^2}{\overline{\omega}_2 + 1} = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_3\overline{\omega}_2^2}{\overline{\omega}_3\overline{\omega}_2 - \overline{\omega}_3 + 1}.$$

Then from the first equality we have $\overline{\omega}_3 = \frac{1}{\overline{\omega}_2(\overline{\omega}_2+1)}$, and from the second equality $\overline{\omega}_3 = \frac{\overline{\omega}_2}{\overline{\omega}_2-1}$. Therefore, $\overline{\omega}_2^3 + \overline{\omega}_2^2 - \overline{\omega}_2 + 1 = 0$, which has no positive roots. Thus, $\mathcal{S}^0$ is not a self-shrinker.

Three half-lines: case 3. Let $\mathcal{S}^0$ be as in Figure 20 (g) so that with the notation of Figure 21 the half-lines of $\mathcal{S}^0$ start from A_1, A_3 and A_5. Since the quadrangles $A_1OA_3A_2$, $A_3OA_5A_4$ and $A_1OA_5A_6$ are rhombi with the same sidelength and one 60^o interior angle, $\theta_1 = \theta_3 = \theta_4 = \overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}_4 = 60^o$ and $\theta_2 = \overline{\theta}_2$. Then

$$\omega_1 = \overline{\omega}_1 = \omega_3 = \overline{\omega}_3 = \omega_4 = \overline{\omega}_4 = 1, \quad \omega_2 = \overline{\omega}_2,$$

and

$$a_1 = a_3 = \overline{a}_1 = \overline{a}_3 = a, \quad a_2 = \overline{a}_2 = \omega_2 a.$$

Thus, using (6.7) in the definitions of γ_i we find

$$\gamma_1 = \gamma_6 = \frac{1}{\sqrt{3}}, \quad \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \frac{2}{\sqrt{3}}\frac{\omega_2}{\omega_2 + 1}.$$

Therefore (6.12) reads as

$$\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\frac{\omega_2}{\omega_2+1},$$

which has a unique solution $\omega_2 = 1$. By the definition of ω_2, $\sin\theta_2 = \sin(60^o + \theta_2)$, which has a unique (admissible) solution $\theta_2 = 60^o$. Then $\mathbf{A}$ is a homothetic Wulff shape of radius a and the homothety center – the origin of $\mathbf{A}$ – is located at the center. In this case, clearly, all γ_i are equal to 1, so that $\mathcal{S}^0$ is a self-shrinker.

Let us seek the function $r(\cdot)$ satisfying $\mathcal{S}(t) = r(t)\mathcal{S}^0$ with $a_o := \mathcal{H}^1(S_1^0)$. Since $h(t) = \frac{\sqrt{3}}{2}(a_o - a(t))$ and $\kappa^\phi_{S_i(t)} = \frac{1}{\sqrt{3}a(t)}$, the equation $h'(t) = -\kappa^\phi_{S_i(t)}$ is equivalent to

$$\frac{\sqrt{3}}{2}a' = \frac{1}{\sqrt{3}a} \quad \text{so that} \quad a(t) = \sqrt{a_o^2 - \frac{2}{3}t}, \quad t\in[0,\tfrac{3a_o^2}{2}).$$

Then $r(t) = \sqrt{1-\frac{2t}{3a_o^2}}$ satisfies $\mathcal{S}(t) = r(t)\mathcal{S}^0$.

Four half-lines: case 1. Let $\mathcal{S}^0$ be as in Figure 20 (h) so that with the notation of Figure 21 the half-lines of $\mathcal{S}^0$ start from A_1, A_2, A_3 and A_4. Then $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \overline{\theta}_1 = \overline{\theta}_4 = 60^o$ so that

$$\omega_1 = \overline{\omega}_1 = \omega_2 = \omega_3 = \omega_4 = \overline{\omega}_4 = 1, \quad \overline{\omega}_3\overline{\omega}_2 = 1.$$

Then

$$a_1 = a_2 = a_3 = a, \quad \overline{a}_1 = \overline{a}_3 = \frac{a}{\overline{\omega}_2} = \overline{\omega}_3 a, \quad \overline{a}_2 = (2-\overline{\omega}_3)a$$

and recalling the definitions of c_i in (6.6) and (6.10)

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_6 = \frac{2}{\sqrt{3}}\frac{1}{3+2\overline{\omega}_3}, \quad \gamma_5 = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2}{2-\overline{\omega}_3}.$$

Therefore, (6.12) is equivalent to

$$\frac{2}{\sqrt{3}}\frac{1}{3+2\overline{\omega}_3} = \frac{2}{\sqrt{3}}\frac{\overline{\omega}_2}{2-\overline{\omega}_3}.$$

This equation implies $\overline{\omega}_3^2 = -3$, which is impossible. Thus, $\mathcal{S}^0$ is not a self-shrinker.

Four half-lines: case 2. Let $\mathcal{S}^0$ be as in Figure 20 (i) so that with the notation of Figure 21 (d) the half-lines of $\mathcal{S}^0$ start from A_1, A_2, A_6 and A_4. Then $\theta_1 = \theta_2 = \overline{\theta}_1 = \overline{\theta}_2 = \theta_4 = \overline{\theta}_4 = 60^o$ and $\theta_3 = \overline{\theta}_3$ so that

$$\omega_1 = \overline{\omega}_1 = \omega_2 = \overline{\omega}_2 = \omega_4 = \overline{\omega}_4 = 1, \quad \overline{\omega}_3 = \omega_3.$$

Then

$$a_1 = \overline{a}_1 = a_3 = \overline{a}_3 = a, \quad a_2 = \overline{a}_2 = \frac{a}{\omega_3}.$$

Thus, by the definition of c_i in (6.6) and (6.9)

$$\gamma_1 = \gamma_6 = \gamma_2 = \gamma_5 = \frac{2}{\sqrt{3}}\frac{\omega_3}{2+2\omega_3}, \quad \gamma_3 = \gamma_4 = \frac{\omega_3}{\sqrt{3}}.$$

Therefore, by (6.12) $\frac{2}{\sqrt{3}}\frac{\omega_3}{2+2\omega_3} = \frac{\omega_3}{\sqrt{3}}$ which implies $\omega_3 = 0$, a contradiction. Thus, $\mathcal{S}^0$ is not a self-shrinker.

Four half-lines: case 3. Let $\mathcal{S}^0$ be as in Figure 20 (j). Then $\mathbf{A}$ is a homothetic Wulff shape and the homothety center is located at the center of $\mathbf{A}$. With the notation of Figure 21 (d), the half-lines of $\mathcal{S}$ start from A_1, A_6, A_3 and A_4. Then $\theta_i = \overline{\theta}_i = 60^o$, $a_i = \overline{a}_i = a$ and by (6.9) $c_i = \frac{2}{3\sqrt{3}}$ for all possible i. Hence, all γ_i equal to $\frac{1}{\sqrt{3}}$ and $\mathcal{S}^0$ is a self-shrinker.

Let us define $r(\cdot)$ satisfying $\mathcal{S}(t) = r(t)\mathcal{S}^0$ with $a_o := \mathcal{H}^1(S_1^0)$. Since $h(t) = \frac{\sqrt{3}}{2}(a_o - a(t))$ and $\kappa^\phi_{S_i(t)} = -\frac{2}{3\sqrt{3}a(t)}$, the equation $h'(t) = -\kappa^\phi_{S_i(t)}$ is equivalent to

$$\frac{\sqrt{3}}{2}a' = -\frac{2}{3\sqrt{3}a} \quad \text{so that} \quad a(t) = \sqrt{a_o^2 - \frac{4}{9}t}, \quad t \in [0, \tfrac{9a_o^2}{4}).$$

Then $r(t) := \sqrt{1 - \frac{4t}{9a_o^2}}$ satisfies $\mathcal{S}(t) = r(t)\mathcal{S}^0$.

Five half-lines. Let $\mathcal{S}^0$ be as in Figure 20 (k). Then $\mathbf{A}$ is a homothetic Wulff shape and the homothety center is located at the center of $\mathbf{A}$. Let $\mathcal{S}(t) = r(t)\mathcal{S}^0$ for some $r(\cdot)$ to be defined later and let $R(t)$ be the sidelength of $\mathbf{A}$. In this case all heights of segments of $\mathcal{S}(t)$ from $\mathcal{S}^0$ are equal to $h(t) := \frac{\sqrt{3}}{2}(R_0 - R(t))$, and by (6.11) the ϕ-curvatures of all segments are equal to $-\frac{2}{6\sqrt{3}R(t)}$. Thus, ϕ-curvature flow equation is represented as

$$-\frac{\sqrt{3}}{2}R'(t) = \frac{1}{3\sqrt{3}R(t)}, \quad \text{hence,} \quad R(t) = \sqrt{R_0^2 - \frac{4}{9}t}, \quad t \in [0, \tfrac{9}{4R_0^2}).$$

Hence, the function $r(t) := \sqrt{1 - \frac{4t}{9R_0^2}}$ satisfies $\mathcal{S}(t) = r(t)\mathcal{S}^0$.

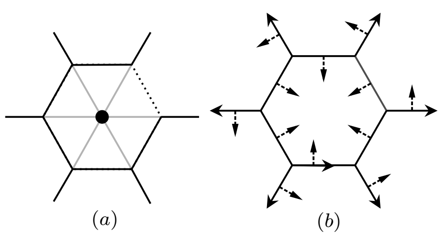

Fig. 22.

Six half-lines. Let $\mathcal{S}^0$ be as in Figure 20 (l). Then $\mathbf{A}$ is a homothetic Wulff shape and the homothety center is located at the center of $\mathbf{A}$. $\mathcal{S}^0$ admits a locally constant CH field; one example of such a field is drawn in Figure 22 (b). Thus, $\mathcal{S}^0$ is critical, and therefore, it stays still. Notice that $\mathcal{S}^0$ is not a local minimizer of ℓ_ϕ, since removing one facet of the Wulff shape decreases the length of $\mathcal{S}$ in every disc D_R compactly containing $\mathbf{A}$.

6.3. **Homothety center on $\partial\mathbf{A}$.** In this short section we assume that more than one half-line of a self-shrinker $\mathcal{S}^0$ start from the same vertex of $\mathbf{A}$ or a half-line is collinear with a segment of $\mathbf{A}$ having a common vertex. In this situation the homothety center is necessarily located at this vertex. In particular, the segments of $\mathbf{A}$ ending at this vertex should have zero ϕ-curvature. For instance, in Figure 20 (a) two half-lines of $\mathcal{S}^0$ start from the same vertex A_1 (coinciding with the origin) of the hexagon $\mathbf{A}$ of sidelength $a_o > 0$ and one more half-line bisects the angle at A_4; let $\mathcal{S}(t) := r(t)\mathcal{S}^0$ be the family of homotheties of $\mathcal{S}^0$, with $r(t)$ to be defined.

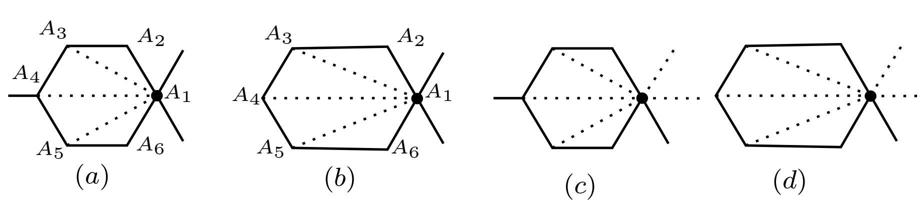

Fig. 23.

Repeating the same arguments of Section 6.1, we can show that:

- for the network $\mathcal{S}^0$ in Figure 23 (a), $\mathbf{A}$ is homothetic to the Wulff shape. Let $a(t)$ be the sidelength of $\mathbf{A}(t)$ in $\mathcal{S}(t) = \cup_i S_i(t)$. Assuming that $\partial\mathbf{A}(t)$ is oriented counterclockwise, so that its unit normal vector field points inside $\mathbf{A}(t)$, we compute

$$\kappa^\phi_{S_1(t)} = \kappa^\phi_{S_6(t)} = 0, \quad \kappa^\phi_{S_2(t)} = \kappa^\phi_{S_5(t)} = -\frac{2}{\sqrt{3}a(t)}, \quad \kappa^\phi_{S_3(t)} = \kappa^\phi_{S_4(t)} = -\frac{4}{\sqrt{3}a(t)},$$

where $a(t) > 0$ is the length of $\mathbf{A}(t)$. Note that A_1A_3 is orthogonal to A_3A_4 so that $\mathcal{H}^1([A_1A_3]) = \sqrt{3}a(t)$, and $[A_1Q]$ is orthogonal to the straight line containing $[A_2A_3]$, and therefore $\mathcal{H}^1([A_1Q]) = \frac{\sqrt{3}a(t)}{2}$. Now consider the heights between the corresponding segments of $\partial\mathbf{A}$ and $\partial\mathbf{A}(t)$: we have

$$h_1(t) = h_6(t) = 0, \quad h_2(t) = h_5(t) = \frac{\sqrt{3}}{2}(a_o - a(t)), \quad h_3(t) = h_4(t) = \sqrt{3}(a_o - a(t)).$$

Then the ϕ-curvature equation $h_i'(t) = -\kappa^\phi_{S_i(t)}$ is equivalent to $-a'(t) = \frac{4}{3a(t)}$, which admits the unique solution

$$a(t) = \sqrt{a_o^2 - \frac{8}{3}t}, \quad t \in [0, \tfrac{3a_o^2}{8}).$$

Thus, $\mathcal{S}(\cdot)$ is the homothetically shrinking solution starting from $\mathcal{S}^0$, with $r(t) = \sqrt{1 - \frac{8t}{3a_o^2}}$.
- One checks that the network $\mathcal{S}^0$ in Figure 23 (b) is not a self-shrinker. The same holds for the network in Figure 23 (d).
- The network in Figure 23 (c) is obtained from (a) adding one or two dotted half-lines. Clearly, this does not affect to the self-similarity.

Finally, the networks in Figure 23 (a) and (c) are examples of simple self-shrinkers with multiple junctions.

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