ON EIGENVALUES AND EIGENFUNCTIONS OF THE OPERATORS DEFINING MULTIDIMENSIONAL SCALING ON SOME SYMMETRIC SPACES

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Abstract. We study asymptotics of the eigenvalues and eigenfunctions of the operators used for constructing multidimensional scaling (MDS) on closed connected symmetric spaces. They are the limits of eigenvalues and eigenvectors of squared distance matrices of an increasing sequence of finite subsets covering the space densely in the limit. We show that for products of spheres and real projective spaces, the numbers of positive and negative eigenvalues of these operators are both infinite. We also find a class of spaces (namely $\mathbb{RP}^n$ with odd $n > 1$) whose MDS defining operators are not trace class, and original distances cannot be reconstructed from the eigenvalues and eigenfunctions of these operators.

1. Introduction

A problem frequently encountered in the modern data science is that of reconstructing a metric space (X, d) and the Borel probability measure μ on it just from the information on the distances between points of a sufficiently large finite subset $\Sigma_k := \{x_1^k, \dots, x_k^k\} \subset X$. Here we require the subset Σ_k to cover X "almost densely" and with a density approximately μ. Of course, unless X is finite itself, no finite set of points will be sufficient to reconstruct the triple (X, d, μ) and one can only hope to do this in the limit as $k \to \infty$. To be more precise, we suppose we know the distances between points of each set Σ_k of some chosen sequence of finite subsets of X, and would like to recover from it the information on (X, d, μ). This is known as the *learning* problem (or *manifold learning*, when X is a priori supposed to be some smooth, say, Riemannian manifold, and d to be its geodesic distance).

One of the basic algorithms aiming to solve the learning problem and widely used in applications is *multidimensional scaling* (MDS) [21]. Although the latter has been originally proposed only for intrinsically Euclidean data (i.e., when X is a subset of a Euclidean space $\mathbb{R}^n$ and the distance d is Euclidean), it has been extended to generic metric spaces. Moreover, in applied science "folklore", it is often used not only when the distance d is non-Euclidean, but also when d is merely some symmetric function not necessarily satisfying the triangle inequality (the so-called *dissimilarity* function). Whether this application of MDS is justified, i.e., what will be reconstructed by MDS when d is a non-Euclidean distance has been recently posed and solved in [1] with quite an astonishing answer. Namely, take $X := \mathbb{S}^1$ as a unit circle endowed with its geodesic distance, and let the points of $\Sigma_k \subset X$

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to be uniformly spaced so that in the limit as $k \to \infty$ they cover X uniformly. Then according to [1], MDS yields in the limit as $k \to \infty$ a closed curve in an infinite-dimensional space, which is far from being a circle. An easy calculation shows that it is a fractal object, namely, a *snowflake* embedding [18] of a circle in an infinite-dimensional Hilbert space [15]. It becomes an isometric embedding if $\mathbb{S}^1$ is endowed with the geodesic distance raised to some power $\alpha = 1/2$. Although this may be unexpected in view of various commonly used applications of MDS, an explanation of this fact may be also traced back to the classical work [20] by Neumann and Schoenberg. In their paper, all the invariant metrics on the circle that embed isometrically into a Hilbert space are classified, including of course the $1/2$-snowflake re-obtained via MDS, which is actually credited to the earlier work [22], see also the discussion in [9, section 7.3].

1.1. **Asymptotics of MDS embeddings.** In the study of asymptotical behaviour of the spectra of matrices of squared distances between points of finite samples $\Sigma_k \subset X$, as well as of the embedding maps $\mathcal{M}_k\colon \Sigma_k \to \mathbb{R}^k$ produced by MDS, the linear operator T over the space $L^2(X,\mu)$ defined by the following formulae plays an important role.

$$
\begin{aligned}
K(x,y) &:= -\frac{1}{2}d^2(x,y),\\
(\mathcal{K}u)(x) &:= \int_X K(x,y)u(y)\,d\mu(y),\\
T &:= P\mathcal{K}P,
\end{aligned}
\tag{1.1}
$$

where P is the projector operator to the orthogonal complement of constant functions in $L^2(X,\mu)$. Both $\mathcal{K}$ and T are well-defined under just a mild assumption that μ has a finite 4-th order moment, i.e.

$$
\int_X d^4(x_0,y)\,d\mu(y) < \infty \tag{1.2}
$$

for some $x_0 \in X$ (which holds for instance when μ is finite and X is bounded). Moreover, in this case, they are self-adjoint Hilbert–Schmidt (and hence compact) operators. What is more important is the following: Under the same assumption (1.2), suppose the empirical measures μ_k of finite samples $\Sigma_k := \{x_1^k, \dots, x_k^k\}$ defined by

$$
\mu_k := \frac{1}{k}\sum_{i=1}^k \delta_{x_i^k},
$$

δ_y standing for the Dirac mass concentrated in $y \in X$, converge to μ as $k \to \infty$ in the Kantorovich[1] 4-distance W_4, i.e. $\lim_k W_4(\mu_k,\mu) = 0$. Then according to [10, theorem 5.8], the maps $\mathcal{M}_k$ viewed as functions from X to $\mathbb{R}^\infty$ (with $\mathbb{R}^k$ canonically identified with the subspace of $\mathbb{R}^\infty$ having all zero coordinates except the first k coordinates) converge to some map $\mathcal{M}\colon X \to \mathbb{R}^\infty$, called further *infinite MDS map*, in measure μ with respect to the product topology on $\mathbb{R}^\infty$ (see also [12]). The latter is given by the formula

$$
\mathcal{M}(x) := \left(\sqrt{\lambda_1^+}\phi_1^+(x), \sqrt{\lambda_2^+}\phi_2^+(x).\dots, \sqrt{\lambda_j^+}\phi_j^+(x), \dots\right), \tag{1.3}
$$

[1] Usually, though historically incorrectly, the distances W_p among probability measures are called Wassertsein distances.

where $\lambda_1^+ \geq \lambda_2^+ \geq \cdots > 0$ are positive eigenvalues of T (counting multiplicity), and $\{\phi_j^+\}_j \in L^2(X,\mu)$ is an orthonormal system in $L^2(X,\mu)$ made of the respective eigenfunctions, i.e. $T\phi_j^+ = \lambda_j^+\phi_j^+$. Note that the definition of $\mathcal{M}$ depends on the choice of ϕ_j^+. Here we silently assume that if the set of positive eigenvalues of T contains $N < \infty$ elements, then $(\mathcal{M}(x))_j := 0$ for $j > N$.

By calculating explicitly the eigenvalues and eigenfunctions of T and using (1.3), one shows in [10] that $\mathcal{M}$ gives a snowflake (Assouad-type) embedding of any m-dimensional sphere $\mathbb{S}^m$ or any m-dimensional flat torus $(\mathbb{S}^1)^m$ into the Hilbert space ℓ^2 of square summable sequences (in the calculations one assumes μ to be the respective volume measure in all these cases).

Another important observation is the following: With the stronger assumption that T is a trace-class operator, i.e.,

$$\sum_i |\lambda_i| < +\infty,$$

where $\{\lambda_i\}$ stands for the sequence of *all* eigenvalues of T, let the metric measure space (X,d,μ) be, say, infinitesimally doubling (which includes any smooth Riemannian manifold equipped with geodesic distance and volume measure, see [5, theorem 3.4.3]). Then by [10, theorem 5.8] (see also [12]), in a sense, distances between almost every pair of points can be recovered from the spectrum of T and the set of the respective eigenfunctions. Namely, in this case, we have

$$\sum_{i=1}^{\infty} \lambda_i \left(\phi_i(x) - \phi_i(y)\right)^2 = d^2(x,y) \quad \text{for } \mu\otimes\mu\text{-a.e. } (x,y), \tag{1.4}$$

where $\{\phi_i\}$ stands for an orthonormal basis in $L^2(X,\mu)$ made of eigenfunctions of T with $T\phi_i = \lambda_i\phi_i$. The importance of the trace class condition on the operators T and $\mathcal{K}$ for the asymptotics of the spectra of distance matrices has been also studied recently in [19]. This condition is discussed a lot in [12] as well, the metric measure spaces for which it holds being called *traceable* in the latter paper.

1.2. **Questions and results.** The above-cited results raise a series of curious questions. Namely, one asks whether there are natural examples of spaces (X,d,μ) such that

- (Q1) no infinite MDS map (i.e., independently on the choice of eigenfunctions of T) gives a topological embedding of X into a separable Hilbert space (which of course without loss of generality may be considered ℓ^2),
- (Q2) the operator T is not trace-class (i.e. in terms of [12], the metric measure space is not traceable) and/or the distance reconstruction formula (1.4) is not valid.

We find both examples among just compact Riemannian manifolds with volume measure (and even more, among compact symmetric spaces), namely, both $(Q1)$ and $(Q2)$ are satisfied by are satisfied by odd-dimensional projective spaces of sufficiently high dimension. In particular, this answers (negatively) the open Question 1 from [12]. We do so by studying the operator T, its eigenvalues and eigenfunctions for symmetric compact Riemannian manifolds with volume measure. Note that in general, there seems to be no easy way to find either the spectrum or eigenfunctions of T. However, in this particular case the situation greatly simplifies since we are able to show that T commutes with the Laplace-Beltrami operator, which allows us to search for its eigenfunctions among the eigenfunctions of the latter. We are able then to show that if X is a finite product of spheres of any dimensions, the infinite MDS map gives a snowflake embedding of X into ℓ^2 thus generalizing the results of [10], while if X is a projective space with sufficiently high dimension, then M does not send X to ℓ^2 at all, and in particular the distance reconstruction formula (1.4) is not valid. Curiously however, as long as X is a finite product of spheres and projective spaces, the spectrum of T contains infinitely many positive and negative eigenvalues. This contrasts with the case that (X, d) is isometrically embeddable in a Hilbert space, in which all the eigenvalues of T are positive.

2. Notation and preliminaries

For vectors x and y in the Euclidean space $\mathbb{R}^n$, we denote by $x \cdot y$ their Euclidean scalar product. The Euclidean norm is denoted by $|\cdot|$. Let ℓ^2 be the usual Banach space of square summable sequences equipped with its usual norm $\|\cdot\|_2$. The space $\mathbb{R}^\infty$ stands for the linear space of all real-valued sequences (sometimes denoted by $\mathbb{R}^\mathbb{N}$ in the literature), equipped with its product topology. This space is metrizable and in fact a Polish space. The norm $\|\cdot\|_2$ can be extended to a pseudo-distance on $\mathbb{R}^\infty$ taking values in $[0, +\infty]$ which will be used in Lemma 2.1 below (note that this pseudo-distance does not induce the product topology on $\mathbb{R}^\infty$). If X is a smooth Riemannian manifold, we denote by $C^\infty(X)$ the set of infinitely smooth functions over X.

Throughout the paper we sometimes use the big Theta notation by D. Knuth.

For a metric measure space (X, d, μ), we will assume μ to be a Borel probability measure. By $\langle x, y\rangle$ we denote both the standard scalar product in the Hilbert space $L^2(X, \mu)$. For a $u \in L^2(X, \mu)$ we let $u^\perp$ stand for its orthogonal complement in $L^2(X, \mu)$. The spectrum of a linear operator T counting multiplicity is denoted as $Spec(T)$. Its signature $sgn(T)$ is written in the form (a, b, c) where the three numbers in the parentheses are the numbers of zero, positive and negative eigenvalues respectively counting multiplicities.

Recall for a metric measure space (X, d, μ), the MDS map $\mathcal{M} : X \to \mathbb{R}^\infty$ defined as in (1.3) are obtained from the positive eigenvalues and their corresponding eigenfunction the operator T. Such maps are not unique, since a different choice of the orthonormal set $\{\phi_j^+\}$ yields a different map. However, they all have the common property given by the following statement.

Lemma 2.1. *If the operator T is Hilbert-Schmidt, then for any MDS maps $\mathcal{M}^1, \mathcal{M}^2$, we have*

$$\|\mathcal{M}^1(x) - \mathcal{M}^1(y)\|_2 = \|\mathcal{M}^2(x) - \mathcal{M}^2(y)\|_2, \quad \text{for all } x, y \in X.$$

Moreover, in this case the right-hand side of (1.4) is independent of the choice of eigenfunctions ϕ_j of $\mathcal{K}$.

Proof. Since T is compact, every non-zero eigenspace of T is finite-dimensional. A new choice of the orthonormal set of $\{\phi_j^+\}$ is obtained from $L^2(X, \mu)$-orthogonal transformation of each eigenspace $E_{\lambda_\alpha^+}$, corresponding to the eigenvalue $\lambda_\alpha^+ > 0$. Let $\{\phi_j^\alpha\}$ and $\{\tilde{\phi}_j^\alpha\}$ be two orthonormal bases of $E_{\lambda_\alpha^+}$. The space $E_{\lambda_\alpha^+}$ with L^2-norm can be identified as a Euclidean space with the standard norm, and orthogonal transformation on Euclidean spaces preserves Euclidean distances. Thus, we have

$$\sum_j \left(\phi_j^\alpha(x) - \phi_j^\alpha(y)\right)^2 = \sum_j \left(\tilde{\phi}_j^\alpha(x) - \tilde{\phi}_j^\alpha(y)\right)^2$$

Combining the equation above with the defining formula of MDS maps in (1.3), we obtain the desired equality. The independence of the right-hand side of (1.4) on the choice of eigenfunctions ϕ_j of $\mathcal{K}$ is shown in the same way. □

2.1. **Review on Riemannian symmetric spaces.** We give a brief review of the basic properties of Riemannian symmetric spaces in this section. We begin by recollecting the basic facts on Riemannian manifolds which will be used in this paper. The metric g of a connected Riemannian manifold (M^n, g) defines a volume measure μ and a distance function d on M, where

$$d(x, y) = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve from } x \textit{ to } y\}.$$

This yields a metric measure space (M, d, μ). By the Hopf-Rinow theorem, (M, d) is a complete metric space if (M, g) is geodesically complete. If (M, g) is complete and connected, then for any $x, y \in M$, there exists a distance minimizing geodesic. For any $x \in M$, the tangential cut locus at x is the set of $v \in T_xM$ such that $\exp(tv)$ is a minimizing geodesic for $t \in [0, 1]$, but fails to be a minimizing geodesic for any $t > 1$. The cut locus $\mathcal{C}_x$ at x is the image of this set under the exponential map at x. If $y \in \mathcal{C}_x$, we have either y is conjugate to x or there is more than one distance minimizing geodesic from x to y [14, lemma 8.2]. Therefore, we have $y \in \mathcal{C}_x$ if and only if $x \in \mathcal{C}_y$. Define the symmetric subset of $M \times M$ by

$$\mathcal{C} = \{(x, y) \in M \times M : x \in \mathcal{C}_y\}$$

It is a well-known fact that the function $d^2(x, \cdot)$ is smooth outside $\mathcal{C}_x$.

By a Laplacian (operator) on the Riemannian manifold, we always mean the **Laplace-Beltrami** operator.

Let (M, g) be a Riemannian manifold. Recall that a local geodesic symmetry at $p \in M$ is a local diffeomorphism r_p on a neighbourhood of p such that for all geodesics $\gamma(t)$ with $\gamma(0) = p$, we have $r_p(\gamma(t)) = \gamma(-t)$.

Definition 2.2. *A Riemannian manifold (M, g) is called a symmetric space if for every $p \in M$, the local geodesic symmetry r_p can be extended to a global isometry on M fixing p.*

It is obvious from the definition that all symmetric spaces are complete, and all connected symmetric spaces are homogeneous, i.e., the isometry group G acts transitively on (M, g). In fact, every connected symmetric space is a reductive homogeneous space as follows. Denote G and K the isometry group and the isotropy of some $p \in M$, respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras of G and K. The geodesic symmetry $r_p \in G$ at p satisfies $r_p^2 = \text{Id}$, where Id stands for the identity map. Denote by $Ad(r_p)$ the adjoint action of the element r_p in the Lie group G. We see that $Ad(r_p)$ is an involutive Lie algebra automorphism of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ such that $\mathfrak{k}$ and $\mathfrak{m}$ are the eigenspaces of $Ad(r_p)$ corresponding to the eigenvalues 1 and -1, respectively.

We close this section with some geodesic properties of symmetric spaces which will be used later. Let $\gamma : \mathbb{R} \to M$ be a complete geodesic for the symmetric space (M, g). Then the following family of composed maps $\tau_s = r_{\gamma(s)} \circ r_{\gamma(0)}$ is a 1-parameter subgroup of the isometry group G. Easily we have

$$\tau_s(\gamma(t)) = \gamma(t+s).$$

Such maps are called *geodesic transvections* along γ. Although determining the geometry of cut loci of a general Riemannian manifold can be difficult, the cut loci of symmetric spaces have been well studied, see [6, 16, 17]. In particular, for a compact symmetric space (M^n, g), the cut locus at any $x \in M$ is a finite disjoint union of regular submanifolds with possible different dimensions [17, theorem 3.3]. Taking the union of the submanifolds of dimension $n-1$ in this decomposition of $\mathcal{C}_x$, the Riemannian volume density μ together with the perpendicular unit vector fields define a measure $\underline{\mu}_x$ on $\mathcal{C}_x$. Hence, we may view $\mathcal{C}_x$ as a "piecewise smooth" manifold of dimension $n-1$. Note that $\underline{\mu}_x$ can be zero if the decomposition of $\mathcal{C}_x$ has no components of dimension $n-1$, for example when M is the standard sphere.

3. The MDS map for closed connected symmetric spaces

From now on, we focus on the MDS maps of closed connected symmetric spaces. Let (S, g) be a closed connected symmetric space. Denote d and μ the distance function and Borel measure induced by g as before (i.e., the geodesic distance and the Riemannian volume measure). Since S is compact, it is well known that there is an orthonormal basis of $L^2(S)$ contained in $C^\infty(S)$ consisting of Laplacian eigenfunctions, see e.g., [23, p. 2] and [11, theorem 2.2.17]. In addition, each eigenvalue of the Laplace-Beltrami operator Δ has finite multiplicity, and eigenspaces of distinct eigenvalues are mutually orthogonal.

Several easy consequences follow from our assumption. Since S is compact, the integral kernel of $\mathcal{K}$ is bounded and uniformly continuous on $S \times S$. Thus, each eigenfunction of $\mathcal{K}$ (also for T) is continuous. In addition, for each non-zero eigenvalue of $\mathcal{K}$, the corresponding eigenspace is finite-dimensional.

As both the Laplacian operator and the integral kernel $K(x, y)$ are closely related to the distance function, we would like to establish a relation between them. We begin with the following lemma leading to the symmetric property of the integral kernel of $\mathcal{K}$ for symmetric spaces.

Lemma 3.1. *Let μ be the volume measure on S. For a compact symmetric space S, the integral kernel of $\mathcal{K}$ satisfies*

$$\Delta_x K(x,y) = \Delta_y K(x,y) \text{ for } \mu \otimes \mu\text{−a.e. } (x,y) \in S \times S \tag{3.1}$$

Here Δ_x and Δ_y are the Beltrami-Laplace operators with respect to the x (i.e. first) and y (second) coordinates, respectively.

Proof. First note that the integral kernel $K(x,y) = -\dfrac{1}{2}d^2(x,y)$ is always symmetric, i.e. $K(x,y) = K(y,x)$. For a fixed $x \in S$, the function $K(x,\cdot)$ is smooth on $S \setminus \mathcal{C}_x$, which is an open set of full measure in S. Since $x \in C_y$ if and only if $y \in \mathcal{C}_x$, the functions $\Delta_x K(x,y)$ and $\Delta_y K(x,y)$ are well defined a.e. on $S \times S$ with respect to $\mu \otimes \mu$.

Suppose that $x \notin C_y$, and let $\gamma : [-1,1] \to S$ be the unique minimizing geodesic from x to y. The geodesic symmetry r_0 at the median $\gamma(0)$ is an isometry interchanging x and y. Denote K_y the function $K(\cdot, y)$. Since r_0 is an isometry, in a neighbourhood U of y we have

$$(\Delta K_y)(r_0(z)) = \Delta(K_y \circ r_0)(z), \quad \text{for all } z \in U. \tag{3.2}$$

Denote Δ_1, Δ_2 the Laplace-Beltrami operator with respect to the first and second coordinates in $S \times S$, respectively. We also have

(3.3)

$$\Delta(K_y \circ r_0)(z) = \Delta_1 K(r_0(z), y) = \Delta_1 K(z, r_0(y)) = \Delta_1 K(z, x) = \Delta_2 K(x, z).$$

Combining (3.2) and (3.3) we get

$$(\Delta_1 K)(r_0(z), x) = (\Delta K_y)(r_0(z)) = \Delta_2 K(x, z),$$

and taking $z := y$, so that $r_0(z) = r_0(y) = x$, we get the desired equality. □

The following statement is valid.

Proposition 3.2. *Let S be a closed connected symmetric space and μ is its volume measure. The operator T commutes with the self-adjoint extension Δ^D to $L^2(S,\mu)$ of the Laplace-Beltrami operator Δ on D, in the sense*

$$\langle Tf, \Delta h\rangle = \langle T\Delta f, h\rangle, \quad \textit{for all } f, h \in C^\infty(S). \tag{3.4}$$

In other words. $\Delta^D(Tf) = T(\Delta^D f)$ for every $f \in C^\infty(S)$. Thus in particular T preserves each eigenspace of the Laplace-Beltrami operator.

Proof. Since the only harmonic functions on closed manifolds are constants, the Laplace-Beltrami operator Δ commutes with the projection P in (1.1) when acting on smooth functions. Since $T = P\mathcal{K}P$, it suffices to show $\mathcal{K}$ commutes with Δ by the following equality.

$$\langle \mathcal{K}f, \Delta h\rangle = \langle \mathcal{K}\Delta f, h\rangle, \quad \text{for all } f, h \in C^\infty(S). \tag{3.5}$$

From Lemma 3.1, we have

$$\Delta_x K(x,y) = \Delta_y K(x,y) \text{for } \mu \otimes \mu\text{-a.e. } (x,y) \in S \times S$$

For any $x \in S$, denote $\mathcal{C}_x$ the cut locus of the point x as before. We cut the manifold S from $\mathcal{C}_x$, and obtain a manifold M_x with boundary ∂M_x. Let $i_x : \partial M_x \to \mathcal{C}_x$ be the canonical projection on the boundary. Using the decomposition theorem of $\mathcal{C}_x$ mentioned in Section 2.1 (see [17] for details), we obtain a decomposition of $\mathcal{C}_x$ (therefore, also of ∂M_x) as a finite union of disjoint regular submanifolds. Let $\mathcal{C}_x(l)$ for $1 \le l \le l_0$ be the $n-1$-dimensional components in this decomposition of $\mathcal{C}_x$. Each $\mathcal{C}_x(l)$ is a regular submanifold of dimension $n-1$. Thus, for any $y \in \mathcal{C}_x(l)$, there is chart $\psi_y : U_y \to \mathbb{R}^n$ such that $\psi_y(y) = 0 \in \mathbb{R}^n$ and the pre-image $\psi_y^{-1}(H_{n-1})$ is exactly $\mathcal{C}_x(l) \cap U_y$, where H_{n-1} is the hyperplane with vanishing last coordinate in $\mathbb{R}^n$. When we cut along $\mathcal{C}_x$ to obtain M_x, we see from the chart ψ_y that the pre-image of $\mathcal{C}_x(l)$ under $(i_x)^{-1}$ is a double cover $\partial M_x^0(l) \sqcup \partial M_x^1(l)$ (boundaries of M_x corresponding to half spaces of with positive and negative last coordinate in the chart ψ_y). Thus, by restricting only to components of ∂M_x of dimension $n-1$, the unit outer normal N_x defines a measure $\underline{\mu}^x$ on ∂M_x. The space $(\partial M_x, \underline{\mu}^x)$ is isomorphic to $(\mathcal{C}_x, \underline{\mu}_x) \oplus (\mathcal{C}_x, \underline{\mu}_x)$ as measure spaces. For a compact symmetric space (S^n, g), the conjugate locus at any $x \in S$ has dimension strictly less than $n-1$ [6, theorem 7.3.3]. Since $\exp_x$ is non-singular outside the tangential conjugate locus, $\nabla_y K(x,y)$ is well-defined a.e. on ∂M_x.

For all $f, h \in C^\infty(S)$, we can apply the divergence theorem to obtain

$$\begin{aligned}\langle \Delta^D \mathcal{K} f, h\rangle &= \int_S f(y) \int_S \Delta_x h(x) K(x,y) d\mu(x) d\mu(y)\\ &= \int_S f(y) \int_{\partial M_y} K(x,y) \langle \nabla_x h, N^y(x)\rangle d\underline{\mu}^y(x) d\mu(y)\\ &\quad - \int_S f(y) \int_S \langle \nabla_x h, \nabla_x K(x,y)\rangle d\mu(x) d\mu(y)\end{aligned}$$

Note that for each $y \in S$, the integral

$$\int_{\partial M_y} K(x,y)\langle \nabla_x h, N^y(x)\rangle d\underline{\mu}^y(x) = 0$$

because h is smooth on S (in fact, the integrals over opposite sides of the boundary cancel out). Therefore, we get

$$\langle \Delta^D \mathcal{K} f, h\rangle = -\int_S f(y) \int_S \langle \nabla_x h, \nabla_x K(x,y)\rangle d\mu(x) d\mu(y). \tag{3.6}$$

Analogously, we obtain

$$\langle \mathcal{K} \Delta f, h\rangle = -\int_S h(x) \int_S \langle \nabla_y f, \nabla_y K(x,y)\rangle d\mu(y) d\mu(x) \tag{3.7}$$

On the other hand, from (3.1) and the divergence theorem, we get

$$\begin{aligned}&\int_S h(x) \left(\int_{\partial M_x} f(y) \langle \nabla_y K, N^x_y\rangle d\underline{\mu}^x(y) - \int_S \langle \nabla_y f, \nabla_y K\rangle d\mu(y)\right) d\mu(x)\\ =& \int_S f(y) \left(\int_{\partial M_y} h(x) \langle \nabla_x K, N^y_x\rangle d\underline{\mu}^y(x) - \int_S \langle \nabla_x h, \nabla_x K\rangle d\mu(x)\right) d\mu(y)\end{aligned}$$

Comparing this with (3.6),(3.7), we need to prove

$$\begin{aligned}&\int_S h(x) \int_{\partial M_x} f(y) \langle \nabla_y K, N^x_y\rangle d\underline{\mu}^x(y) d\mu(x)\\ =& \int_S f(y) \int_{\partial M_y} h(x) \langle \nabla_x K, N^y_x\rangle d\underline{\mu}^y(x) d\mu(y)\end{aligned} \tag{3.8}$$

To this aim, for any $y \in \mathcal{C}_x(l)$ not conjugate to x, let $y_j \in M^j_x(l)$ be the pre-image of y for $j = 0, 1$. Since distance minimizing geodesic segments starting from x cannot intersect $\mathcal{C}_x$ except for the end-points, there exist distance minimizing unit speed geodesics γ_j in M_x from x to y_j for $j = 0, 1$. For simplicity, denote the geodesic in S corresponding to γ_j by the same notation. The geodesic transvection τ_j along γ_j sending x to y is an isometry. It maps $\mathcal{C}_x$ to $\mathcal{C}_y$, and the curve γ_j into itself with $\tau_j \circ r_x(y) = x$. Thus, the component of $\mathcal{C}_y$ containing x is a regular submanifold of dimension $n-1$, and the preimage $i_y^{-1}(x)$ also contains exactly two points in ∂M_y. Let x_j be induced by the reversed curve of γ_j. Because y_j is not conjugate to x and x_j is not conjugate to y, we have $\nabla_y K(x, y_j)$ and $\nabla_x K(y, x_j)$ are well-defined. If we can show

$$\sum_{j=0}^{1} \langle \nabla_y K(x, y_j), N^x_{y_j}\rangle = \sum_{j=0}^{1} \langle \nabla_x K(y, x_j), N^y_{x_j}\rangle, \tag{3.9}$$

then (3.8) is just an application of the Fubini theorem and hence the proof would be concluded. Since γ_j is a distance minimizing geodesic segment from x to y_j, we have

$$\nabla_y K(x, y_j) = -d(x,y)\dot{\gamma_j}(d(x,y))$$

Combining the equality above with the fact $\tau_j \circ r_x$ is an isometry mapping y to x, we obtain

$$(\tau_j \circ r_x)_*(\nabla_y K(x, y_j)) = \nabla_x K(x_j, y),\ (\tau_j \circ r_x)_*(N^x_{y_j}) = N^y_{x_j}, \quad \text{for all } j = 0, 1$$

Thus (3.9) holds which proves (3.8) and therefore (3.5) which as explained concludes the proof of the fact that T and Δ^D commute as claimed..

To prove the last claim, i.e. that T preserves each eigenspace of Δ, note that for all $f_1, f_2 \in C^\infty(S)$ such that $\Delta f_i = \lambda_i f$ with $i = 1$ and 2, we have

$$\langle Tf_1, \lambda_2 f_2\rangle = \langle Tf, \Delta f_2\rangle = \langle T\Delta f_1, f_2\rangle = \langle \lambda_1 T f_1, f_2\rangle.$$

Therefore, Tf_1 is perpendicular to all eigenspaces of Δ corresponding to eigenvalues distinct from λ_1. Since there is an orthonormal basis of $L^2(S)$ consisting of Δ-eigenfunctions, we see Tf_1 is contained in the eigenspace of λ_1 for Δ. This completes the proof. □

4. MDS for elementary symmetric spaces

4.1. **Products of spheres.** We can derive the spectrum of T for product spaces based on the components in the decomposition. More precisely, we can obtain the spectrum of T for a product of finite number of spaces based on the spectra of all the T_i which are the spectra of the MDS associated operators for component spaces in the product. The discussion of a product of two spaces has already been given in [10, section 6.2] (see also [12, sections 4.1, 4.3]). Here we follow their approach to derive the general formula of a product of N spaces, which serves both as a review of the above cited results, amd also gives us a possibility to compute explicitly the signature of the operator T associated with the product space of spheres. These results are summarized in Proposition 4.1 in the sequel.

To this aim, let (X, d, μ) be the metric measure space induced by the product manifold

$$X = \prod_{i=1}^{N} X_i,\ g = \sum_{i=1}^{N} g_i;$$

where each g_i is the Riemannian metric on a compact connected manifold X_i. Denote d_i and μ_i the Riemannian distance and the normalized Riemannian volume on (X_i, g_i). It follows that for $x = (x_i) \in X$ and $y = (y_i) \in X$, we have

$$\mu = \prod_{i=1}^{N} \mu_i,\ d^2(x,y) = \sum_{i=1}^{N} d_i^2(x_i, y_i).$$

Since the space (X_i, g_i) are all connected, the Hilbert spaces $L^2(X, \mu)$ and $L^2(X_i, \mu_i)$ are all separable. One would naturally expect the MDS maps and the associated operator to be in the form of Cartesian products of those for each component. This is exactly the case as in [10, section 6.2]. Suppose $\{\phi_i^j\}$ is an orthonormal basis diagonalizing the associated operator T_i for (X_i, d_i, μ_i). Then $\{\phi_{i_1}^{j_1}, \dots, \phi_{i_N}^{j_N}\}$ is an orthonormal basis for $L^2(X, \mu)$ by the Fubini theorem. According to [10, proposition 6.2], the associated operator T is diagonalizable with respect to this basis. Since the constant functions are in the kernel of T_i for all $1 \leq i \leq N$, we can choose $\{\phi_i^j\}$ such that the non-constant elements are in $1^\perp$. Then for $\Phi = \phi_{i_1}^{j_1} \cdots \phi_{i_N}^{j_N}$, we have $T(\Phi) = 0$ unless the components of Φ contain exactly 1 non-constant function. To see this, we first compute

$$-2K(\Phi)(x) = \sum_{i=1}^{N} \int_X d_i^2(x_i, y_i)\phi_{i_1}^{j_1}(y_1) \cdots \phi_{i_n}^{j_N}(y_N) d\mu(y)$$

If there is more than one non-constant component in Φ, apply the Fubini theorem to the integrals above. We can re-order the integration so that inside each integral, the innermost term satisfy

$$\int_{X_l} \phi_l^{j_l} d\mu_l = 0$$

Therefore, if $N > 1$, the kernel of T is infinite dimensional.

If all components of Φ are constant, clearly $T(\Phi) = P\mathcal{K}P(\Phi) = 0$. Assume $\Phi = \phi_1^{j_1}$ is non-constant and $T_1(\phi_1^{j_1}) = \lambda_1^{j_1}\phi_1^{j_1}$. We obtain $T(\Phi) = \lambda_1^{j_1}\Phi$. As a result, for the non-zero spectra of T and T_i, we have

$$Spec(T) \setminus \{0\} = \sqcup_{i=1}^{N} \left(Spec(T_i) \setminus \{0\}\right), \tag{4.1}$$

where both sides of the relation above are eigenvalues counted with multiplicity. This implies T is a trace-class operator if and only if each T_i is trace-class.

Since the map T for any compact space has at least one positive eigenvalue, the product of N Riemannian manifolds shall have at least N positive eigenvalues counting multiplicity.

Now let X be a product of N spheres with $N > 1$. We know from [10, section 6.1] that the operator T_i for each component sphere is trace-class and has signature $(1, \infty, \infty)$, where the three numbers in the parentheses denote the number of zero, positive and negative eigenvalues counting multiplicity. The condition $N > 1$ implies T has an infinite-dimensional kernel. From (4.1), we see T has infinitely many strictly positive and negative eigenvalues counting multiplicity. Therefore T has signature (∞, ∞, ∞). Summing up, we obtain the following statements.

Proposition 4.1. *Let $X = \prod_{i=1}^N X_i$ and $g = \sum_{i=1}^N g_i$ be the product space of compact connected Riemannian manifolds. Denote by d and μ the geodesic distance and the volume measure induced by g respectively. Then the map T for (X, d, μ) has at least N positive eigenvalues counting multiplicity, and T is a trace-class operator if and only if each T_i is trace-class.*

In particular, if X is a finite product of N spheres (possibly just circles), then the operator T is trace-class and has signature (∞, ∞, ∞) for $N > 1$, i.e. T has infinitely many zero, positive, and negative eigenvalues counting multiplicity.

Remark 4.2. Suppose X is a product of spheres (in particular just circles). Since X is closed symmetric, by Proposition 3.2 the operator T preserves all eigenspaces of Δ. Let $E_{\lambda_i^\alpha}$ be an eigenspace of the Laplace-Beltrami operator for (X_i, g_i) and denote $\pi_i : X \to X_i$ the canonical projection. The map π_i pulls back the space of functions $E_{\lambda_i^\alpha}$ on X_i to a space of functions $\pi_i^*\left(E_{\lambda_i^\alpha}\right)$ on X. The operator T preserves the space $\pi_i^*\left(E_{\lambda_i^\alpha}\right)$ and is self-adjoint on it with respect to the $L^2(X, \mu)$-norm. In fact, the operator T is simply a constant scaling on $\pi_i^*\left(E_{\lambda_i^\alpha}\right)$. To show this, denote by G the isometry group of g, then for any $s \in G$ we have

$$\begin{aligned}\mathcal{K}(s \cdot f)(x) &= \int_X K(x,y) f(s \cdot y) d\mu(y) \\ &= \int_X K(s \cdot x, s \cdot y) f(s \cdot y) d\mu(y) \\ &= \int_X K(s \cdot x, z) f(z) d\mu(z) = \mathcal{K}(f)(s \cdot x).\end{aligned}$$

Therefore, the natural action of G commutes with the operator $\mathcal{K}$ (hence T). Let G_i be the isometry group of (X_i, g_i). Since each component (X_i, g_i) is a sphere, the isometry group G_i acts irreducibly on $E_{\lambda_i^\alpha}$ [7, theorem 3.1], and preserves the eigenspaces of T when viewed as a subgroup of G. Thus, T can have only one real eigenvalue on each $\pi_i^* \left(E_{\lambda_i^\alpha}\right)$. In particular, the map T has only one real eigenvalue on each Laplacian eigenspace of standard spheres.

4.2. **Projective spaces.**

4.2.1. *Signature of T for projective spaces.* Here we consider the case $X := \mathbb{RP}^n$, an n-dimensional real projective space equipped with its geodesic distance d and Riemannian volume measure μ. Since $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$, we expect the MDS for projective spaces to behave similarly compared to spheres. On the other hand, we will see in this section how the global topology makes a difference in MDS maps on projective spaces compared to the MDS on spheres. We start by determining the signature of T for projective spaces $\mathbb{RP}^n$.

The Laplacian eigenfunctions on $\mathbb{RP}^n$ are well-defined projections of spherical harmonics. Hence, they are projections of spherical harmonics of even degree. Clearly as for the spheres, the group $SO(n)$ acts irreducibly on each Laplacian eigenspace of $\mathbb{RP}^n$. Thus, each eigenspace of T is a direct sum of eigenspace of Δ on $\mathbb{RP}^n$.

For $x, y \in \mathbb{S}^n$, the distance between the lines $[x]$ and $[y]$ on $\mathbb{RP}^n$ is $\arccos(|x \cdot y|)$. We know that the operators T and $\mathcal{K}$ share all eigenvalues and eigenfunctions except for those corresponding to constant functions. To compute the spectrum of $\mathcal{K}$ for projective spaces, let $2k$ be even. Then by the Funk-Hecke theorem [4, p. 98], the eigenvalues λ_{2k}^n of $\mathcal{K}$ for $\mathbb{RP}^n$ corresponding to spherical harmonics of degree $2k$ are given by

$$\lambda_{2k}^n = \sigma_n \int_0^1 \arccos^2(t) P_{2k}^n(t)(1-t^2)^{(n-2)/2} dt, \tag{4.2}$$

where $P_{2k}^n(t)$ is the Legendre polynomials for $\mathbb{S}^n$ of degree $2k$, and the numbers

$$\sigma_n = -\frac{\mathrm{vol}(\mathbb{S}^{n-1})}{\mathrm{vol}(\mathbb{S}^n)}$$

are negative constants depending only on n. To avoid the confusion, let us emphasize the subscript $2k$ in the notation λ_{2k}^n does not stand for the consecutive number of the eigenvalue in the increasing order. Inside the integral in (4.2), only the term

$$F_{2k}^n(t) = P_{2k}^n(t)(1-t^2)^{(n-2)/2} = R_{2k}^n \left(\frac{d}{dt}\right)^{2k} (1-t^2)^{2k+(n-2)/2} \tag{4.3}$$

is affected by the dimension n and even degree $2k$. For $2k$ even, the Rodrigues constants R^n_{2k} are given by [13, p. 22]

$$R^n_{2k} = \frac{1}{4^k}\frac{\Gamma(n/2)}{\Gamma(n/2+2k)}. \tag{4.4}$$

These constants are all positive. Note that the equations (4.2) and (4.3)) are still valid for the special case $n=1$, where P^1_{2k} are given by Chebyshev polynomials of the first kind. For $n \geq 1$ and $2k \geq 2$, a substitution $t = \cos\theta$ using $\frac{d}{d\theta} = -\sin\theta \cdot \frac{d}{dt}$ yields

$$\begin{aligned}
&F^n_{2k}(\cos\theta)\\
=&R^n_{2k}\left(\frac{-1}{\sin\theta}\frac{d}{d\theta}\right)^{2k}\left[(\sin\theta)^{4k+n-2}\right]\\
=&(4k+n-2)R^n_{2k}\left(\frac{-1}{\sin\theta}\frac{d}{d\theta}\right)^{2k-2}\Big[(2(2k-1)+n-2)(\sin\theta)^{2(2k-2)+n-2}\\
&-(2(2k-1)+n-1)(\sin\theta)^{2(2k-2)+n}\Big]\\
=&(4k+n-2)\Big[(2(2k-1)+n-2)\frac{R^n_{2k}}{R^n_{2k-2}}F^n_{2k-2}(\cos\theta)\\
&-(2(2k-1)+n-1)\frac{R^n_{2k}}{R^{n+2}_{2k-2}}F^{n+2}_{2k-2}(\cos\theta)\Big]
\end{aligned}$$

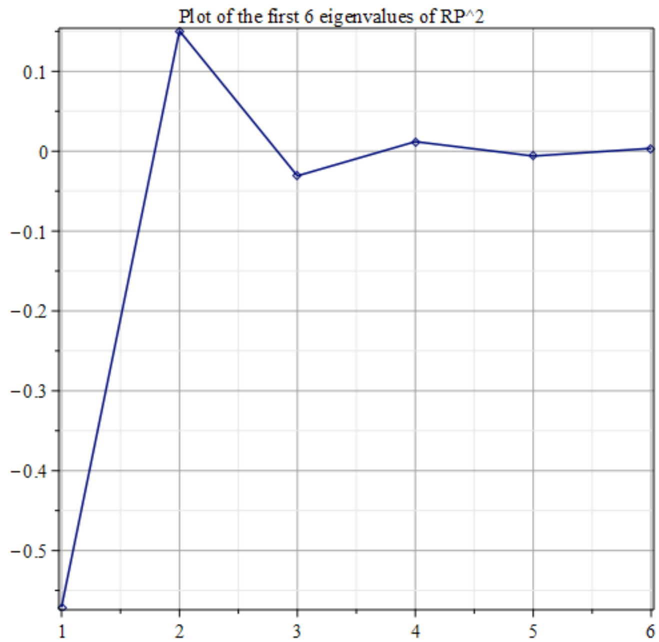
By linearity of the integration, we combine the equation above and (4.2) to obtain

$$\begin{aligned}
\frac{1}{\sigma_n}\lambda^n_{2k} =&(4k+n-2)\Bigg((4k+n-4)\frac{R^n_{2k}}{\sigma_n R^n_{2k-2}}\lambda^n_{2k-2}\\
&-(4k+n-3)\frac{R^n_{2k}}{\sigma_{n+2}R^{n+2}_{2k-2}}\lambda^{n+2}_{2k-2}\Bigg)
\end{aligned} \tag{4.5}$$

Note for $n \geq 1$ and $2k \geq 2$, we always have $4k+n-4 > 0$. From the signs of the coefficients in the equation above, we see the following: If $\{\lambda^n_{2k}\}^\infty_{k=1}$ is a sequence of alternating signs indexed by even numbers $2k$, the signs of numbers in the sequence $\{\lambda^{n+2}_{2k}\}$ will also be alternating. For the basic case $n=1$, the Riemannian manifold $\mathbb{RP}^1$ is isomorphic to $\mathbb{S}^1$ by doubling the angles between lines. For $\mathbb{S}^1$, the signs of eigenvalues of $\mathcal{K}$ corresponding to the eigenfunctions $\cos(m\theta)$ are alternating, depending on whether m is odd or even [2, section 2]. Thus, the signs of the sequence $\{\lambda^1_{2k}\}$ indexed by even $2k$ are also alternating. Combined with the argument above, we obtain the following result.

Lemma 4.3. *For n odd, the MDS defining operator T on $\mathbb{RP}^n$ has both infinitely many positive and negative eigenvalues.*

We then turn to the case of even-dimensional projective spaces, starting from the basic case of $\mathbb{RP}^2$. We expect that the signature of T for even-dimensional projective spaces is similar to the odd dimension cases. A numerical computation for the first 6 eigenvalues (corresponding to even degree spherical harmonics up to $2k=10$) shows they have alternating signs, see the pictures below.



Although computing the spectrum of T for projective spaces can be difficult, we can easily obtain some information on $sgn(T)$ for $\mathbb{RP}^2$ using Mercer's theorem.

Lemma 4.4. *The operator T on $\mathbb{RP}^2$ has infinitely many strictly positive and negative eigenvalues counting multiplicity.*

Proof. Since $\mathbb{RP}^2$ is homogeneous, we only need to prove this statement for $\mathcal{K}$. Suppose $\mathcal{K}$ has only finitely many strictly negative eigenvalues counting multiplicity. Let λ_{2k} be the eigenvalue of $\mathcal{K}$ for $\mathbb{RP}^2$ corresponding to the degree $2k$ spherical harmonics. Let $\{\phi_{2k}^m\}$ with $-2k \leq m \leq 2k$ denote the standard L^2-orthonormal basis with respect to μ of the Laplacian eigenspace $\mathcal{H}_{2k}$ of $\mathbb{RP}^2$ corresponding to degree $2k$ spherical harmonics. The function

$$N(x,y) = \sum_{\lambda_{2k}<0} \lambda_{2k} \sum_{m=-2k}^{2k} \phi_{2k}^m(x)\phi_{2k}^m(y) \tag{4.6}$$

is smooth and bounded, since the summation above is finite. We can decompose the integral kernel as

$$K(x,y) = N(x,y) + H(x,y),$$

where $N(x,y)$ is a negative integral kernel and $H(x,y)$ is a positive integral kernel. Because the positive operator

$$H(f)(x) = \int_{\mathbb{RP}^2} H(x,y) f(y) d\mu(y)$$

has a continuous bounded kernel, Mercer's theorem [3, theorem 4.10] implies the following convergence with respect to summation of $\lambda_{2k}\phi_{2k}^m(x)\phi_{2k}^m(y)$ is absolute and uniform:

$$H(x,y) = \sum_{\lambda_{2k}\geq 0} \sum_{m=-2k}^{2k} \lambda_{2k}\phi_{2k}^m(x)\phi_{2k}^m(y). \tag{4.7}$$

From the basic properties of spherical harmonics [4, theorem 3.3.3] we obtain

$$\begin{aligned}\sum_{m=-2k}^{2k} \phi_{2k}^m(x)\phi_{2k}^m(y) =& \dim(\mathcal{H}_k)P_{2k}(\cos(d(x,y)))\\ =&(4k+1)P_{2k}(x\cdot y).\end{aligned}$$

Here we use x,y to denote both points in $\mathbb{RP}^2$ and their lifts to $\mathbb{S}^2$ (hence also to $\mathbb{R}^3$). Since P_{2k} is even, the value of $P_{2k}(x\cdot y)$ is always well-defined. Therefore, we obtain the absolute and uniform convergence of the series

$$H(x,y) = \sum_{\lambda_{2k}\geq 0} \lambda_{2k}(4k+1)P_{2k}(x\cdot y),$$

and, taking io account that in (4.6) we have only a finite sum, it follows that on $(\mathbb{RP}^2)^2$ the series

$$-\frac{1}{2}d^2(x,y) = -\frac{1}{2}\arccos^2(|x\cdot y|) = K(x,y) = \sum_{k=0}^{\infty} \lambda_{2k}(4k+1)P_{2k}(x\cdot y) \tag{4.8}$$

converge absolutely and uniformly. This implies for $t\in[0,1]$, the convergence

$$\arccos^2(t) = -2\sum_{k=0}^{\infty} \lambda_{2k}(4k+1)P_{2k}(t) \tag{4.9}$$

is also absolute and uniform over $[0,1]$.

On the other hand, from (4.2) we know the eigenvalues of $\mathcal{K}$ for $\mathbb{RP}^1$ are given by

$$\lambda_{2j}^1 = -\frac{1}{\pi}\int_0^1 \arccos^2(t)C_{2j}(t)\frac{1}{\sqrt{1-t^2}}dt, \tag{4.10}$$

where the C_{2j} are the Chebyshev polynomials of the first kind of degree $2j$. Since the convergence in (4.9) is absolute and uniform, the Lebesgue dominated convergence theorem implies

$$\lambda_{2j}^1 = \frac{2}{\pi}\sum_{k=0}^{\infty}\lambda_{2k}(4k+1)\int_0^1 P_{2k}(t)C_{2j}(t)\frac{1}{\sqrt{1-t^2}}dt \tag{4.11}$$

According to [8, p. 96], the even degree Legendre polynomials can be expanded by Chebyshev polynomials as

$$\begin{aligned}P_{2k}(t) = &\left(\frac{\Gamma(1/2+k)}{\Gamma(1/2)\Gamma(k+1)}\right)^2 C_0+\\ &\frac{2}{\Gamma^2(1/2)}\sum_{i=1}^{k}\left(\frac{\Gamma(k-i+1/2)\Gamma(k+i+1/2)}{\Gamma(k-i+1)\Gamma(k+i+1)}\right)C_{2i}(t)\end{aligned}$$

The functions $C_{2j}(t)$ are even, and we have

$$\int_0^1 C_{2i}(t)C_{2j}(t)\frac{1}{\sqrt{1-t^2}}dt = \frac{\pi}{4}\left(\delta_j^i+\delta_i^0\delta_j^0\right)$$

Therefore, we obtain

$$\int_0^1 P_{2k} C_{2j} \frac{1}{\sqrt{1-t^2}} dt = 0,\ 0 \le k < j, \tag{4.12}$$

$$\int_0^1 P_{2k} C_{2j} \frac{1}{\sqrt{1-t^2}} dt > 0,\ k \ge j. \tag{4.13}$$

These equalities and (4.11) imply λ^1_{2j} is always non-negative for sufficiently large j. This leads to a contradiction. Thus, the operator $\mathcal{K}$ (hence also T) for $\mathbb{RP}^2$ has to admit infinitely many negative eigenvalues counting multiplicity. Similarly, we can prove that T on $\mathbb{RP}^2$ also has infinitely many positive eigenvalues counting multiplicity, concluding the proof. □

We can get now the result analogous to Lemma 4.3 but for even dimensional projective spaces.

Lemma 4.5. *For n even, the MDS defining operator T on $\mathbb{RP}^n$ has both infinitely many positive and negative eigenvalues.*

Proof. For $n = 2$ the statement is given by Lemma 4.4. For higher but even dimensions of the projective space one uses the following induction argument on the dimension: Suppose that the operator $\mathcal{K}$ for $\mathbb{RP}^n$ has infinitely many strictly positive and negative eigenvalues counting multiplicity. It follows that for arbitrarily large $k_0 \ge 1$, we can find $k \ge k_0$ such that $\lambda^n_{2k} \le 0$ and $\lambda^n_{2k-2} > 0$. The sign of the coefficients in (4.5) tells that $\lambda^{n+2}_{2k-2} > 0$. Therefore $\mathcal{K}$ (hence also T) for $\mathbb{RP}^{n+2}$ has infinitely many strictly positive eigenalues. Similarly, we see $\mathcal{K}$ and T have infinitely many negative eigenvalues. □

Combining Lemma 4.3 with Lemma 4.5 gives the following result.

Theorem 4.6. *For any projective space $\mathbb{RP}^n$, the operator T always has both infinitely many positive and negative eigenvalues counting multiplicity.*

4.2.2. *Spectral asymptotics of T on odd-dimensional projective spaces.* Now we estimate the norm of the eigenvalues of T for odd-dimensional projective spaces. This will show how the MDS maps of $\mathbb{RP}^n$ differ from the MDS maps of $\mathbb{S}^n$ for odd n. To estimate the asymptotics of the eigenvalues $\{\lambda^n_{2k}\}$, we need the following lemma.

Lemma 4.7. *For odd-dimensional projective space $\mathbb{RP}^n$, we have*

$$\lambda^n_{2k} = \Theta(k^{-(n+3)/2}).$$

Proof. We prove this statement by induction on n. For the base case $n = 1$, a simple integration by parts shows

$$\lambda^1_{2k} = \Theta(k^{-2}).$$

Suppose $\lambda^n_{2k} = \Theta(k^{-(n+3)/2})$ holds. All coefficients for the eigenvalues in (4.5) are non-zero. Solving this equation for λ^n_{2k-2}, we get

$$|\lambda^{n+2}_{2k-2}| \le C_1(n,k) \left(\frac{R^{n+2}_{2k-2}}{R^n_{2k}} \right) |\lambda^n_{2k}| + C_2(n,k) \left(\frac{R^{n+2}_{2k-2}}{R^n_{2k-2}} \right) |\lambda^n_{2k-2}|$$

$$|\lambda^{n+2}_{2k-2}| C_2(n,k) \ge C_2(n,k) \left(\frac{R^{n+2}_{2k-2}}{R^n_{2k-2}} \right) |\lambda^n_{2k-2}|,$$

where the second inequality comes from the fact that the sequence $\{\lambda_{2k}^n\}$ indexed by $2k$ has alternating signs. Here $C_1(n,k)$ and $C_2(n,k)$ are positive functions such that $C_1(n,k) = \Theta(k^{-2})$ and $C_2(n,k) = \Theta(1)$ when n is fixed. Moreover, from the formula of Rodrigues constants, we get

$$0 < \frac{R_{2k-2}^{n+2}}{R_{2k}^n} = \Theta(k) \text{ and } 0 < \frac{R_{2k-2}^{n+2}}{R_{2k-2}^n} = \Theta(k^{-1}). \tag{4.14}$$

Then these inequalities together imply $\lambda_{2k}^{n+2} = \Theta(k^{-1-(n+3)/2})$. □

Recall that by Lemma 2.1, for any $x, y \in X$, the value of $\|\mathcal{M}(x) - \mathcal{M}(y)\|_2$ is independent of the choice of the eigenfunctions of $\mathcal{K}$ defining $\mathcal{M}$. We are able to state the following result regarding the MDS maps for odd-dimensional projective spaces.

Theorem 4.8. *For every odd $n > 1$, the operator T for $\mathbb{RP}^n$ is not trace-class. Furthermore, given any $x \in \mathbb{RP}^n$, there exists a positive volume measure set U_x such that*

$$\|\mathcal{M}(x) - \mathcal{M}(y)\|_2 = +\infty, \quad \textit{for all } y \in U_x, \tag{4.15}$$

and the distance reconstruction formula (1.4) does not hold $\mu\otimes\mu$ almost everywhere.

Proof. According to [4, theorem 3.1.4], the dimension of the space of degree $2k$ spherical harmonics on $\mathbb{RP}^n$ is given by

$$dim(\mathcal{H}_{2k}^n) = \frac{4k+n-1}{2k+n-1}\binom{2k+n-1}{n-1}. \tag{4.16}$$

Using the relation of binomial coefficients, we easily deduce that $dim(\mathcal{H}_{2k}^n)$ is a polynomial in k of degree $n-1$. Then for all odd n, we have

$$|\lambda_{2k}^n| dim(\mathcal{H}_{2k}^n) = \Theta(k^{(n-5)/2}) \tag{4.17}$$

Therefore, for all odd n with $n > 1$, the operator T on $\mathbb{RP}^n$ is not a trace-class operator.

First we show $\|\mathcal{M}(x) - \mathcal{M}(y)\|_2 = +\infty$ when $x \cdot y = 0$. Let $\{\phi_{2k}^i\}$ with $i \in I_{2k}^n$ be an orthonormal basis with respect to μ of $\mathcal{H}_{2k}^n$. For $x, y \in \mathbb{RP}^n$, we have

$$\begin{aligned}
\|\mathcal{M}(x) - \mathcal{M}(y)\|_2^2 &= \sum_{\lambda_{2k}>0} \lambda_{2k} \sum_{i \in I_{2k}^n} (\phi_{2k}^i(x) - \phi_{2k}^i(y))^2 \\
&= \sum_{k=0}^{\infty} \lambda_{4k+2} \sum_{i \in I_{4k+2}^n} (\phi_{4k+2}^i(x) - \phi_{4k+2}^i(y))^2 \\
&= \sum_{k=0}^{\infty} \lambda_{4k+2} \cdot dim(\mathcal{H}_{4k+2}^n) \cdot 2(1 - P_{4k+2}^n(x \cdot y))
\end{aligned} \tag{4.18}$$

The Legendre polynomials have the following recurrence relation (see [4, proposition 3.3.11])

$$(k+n-1)P_{k+1}^n(t) - (2k+n-1)tP_k^n(t) + kP_{k-1}^n(t) = 0. \tag{4.19}$$

Therefore, for fixed n we can see $\{|P_{4k+2}^n(0)|\}$ for $k \geq 0$ is a decreasing sequence uniformly bounded away from 1. Since $x \cdot y = 0$, for any $n > 1$, there exists some

$\delta_n > 0$ such that

$$\|\mathcal{M}(x) - \mathcal{M}(y)\|_2^2 \geq 2\delta_n \sum_{k=0}^{\infty} \lambda_{4k+2} \cdot dim(\mathcal{H}^n_{4k+2}).$$

Using (4.17), we can see $\|\mathcal{M}(x) - \mathcal{M}(y)\|_2^2 = +\infty$ for odd $n > 1$.

From (4.18), we see that $\|\mathcal{M}(x) - \mathcal{M}(y)\|_2$ depends only on $|x \cdot y|$. Moreover, if $x_0 \cdot x_1 = 0$, the triangle inequality implies there is a set of positive measure U_{x_i} such that $\|\mathcal{M}(y) - \mathcal{M}(x_i)\| = +\infty$ for all $y \in U_{x_i}$ and for some $i = 0$ or 1. The operator T commutes with the action of the isometry group G, so the distance $\|\mathcal{M}(x) - \mathcal{M}(y)\|_2$ is invariant under the action by G. Since G acts transitively on $\mathbb{RP}^n$, it follows that for every $x \in \mathbb{RP}^n$ with odd $n > 1$, there exists a positive measure set U_x such that (4.15) holds.

Finally, by the Fubini theorem one has that the series on the left-hand side of the distance reconstruction formula (1.4) does not converge on a set of couples (x, y) of positive $\mu \otimes \mu$ measure, and hence (1.4) does not hold $\mu \otimes \mu$ almost everywhere, which concludes the proof. □

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