AN INFINITE DOUBLE BUBBLE THEOREM

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Abstract. The classical double bubble theorem characterizes the minimizing partitions of $\mathbb{R}^n$ into three chambers, two of which have prescribed finite volume. In this paper we prove a variant of the double bubble theorem in which two of the chambers have infinite volume. Such a configuration is an example of a *(1,2)-cluster*, or a partition of $\mathbb{R}^n$ into three chambers, two of which have infinite volume and only one of which has finite volume [3]. A $(1,2)$-cluster is locally minimizing with respect to a family of weights $\{c_{jk}\}$ if for any $B_r(0)$, it minimizes the interfacial energy $\sum_{j<k} c_{jk}\mathcal{H}^{n-1}(\partial\mathcal{X}(j)\cap\partial\mathcal{X}(k)\cap B_r(0))$ among all variations with compact support in $B_r(0)$ which preserve the volume of $\mathcal{X}(1)$. For $(1,2)$ clusters, the analogue of the weighted double bubble is the *weighted lens cluster*, and we show that it is locally minimizing. Furthermore, under a symmetry assumption on $\{c_{jk}\}$ that includes the case of equal weights, the weighted lens cluster is the unique local minimizer in $\mathbb{R}^n$ for $n\leq 7$, with the same uniqueness holding in $\mathbb{R}^n$ for $n\geq 8$ under a natural growth assumption. We also obtain a closure theorem for locally minimizing $(N,2)$-clusters.

1. Introduction

The classical cluster problem in $\mathbb{R}^n$ is to find the configuration of N regions of prescribed finite volumes and an exterior region that minimizes the total area of the interfaces between regions [24]. Variants include the immiscible fluid problem, in which the interfaces between pairs of regions are weighted by coefficients that depend on the pair. The existence of minimizers for a general class of problems of this type has been proved by Almgren in [5]. This existence opens the door to the analysis and possible characterization of their shape, an issue which has been extensively studied but still presents many interesting open questions.

When $N\leq n+1$ and the weights on the interfaces are equal, the canonical configuration of N bounded chambers and an exterior region known as the *standard N-bubble* is a natural candidate for minimality [34, Problem 2]. The N-bubble conjecture states that the standard N-bubble is the unique minimizer of the equal weights energy. If $N=1$, this reduces to the isoperimetric problem. For any pair of volumes, the double bubble conjecture was fully settled first in the plane by Foisy-Alfaro-Brock-Hodges-Zimba in [14], next in $\mathbb{R}^3$ by Hutchings-Morgan-Ritoré-Ros [15], then in $\mathbb{R}^4$ by Reichardt-Heilmann-Lai-Spielman [27], and finally in $\mathbb{R}^n$ for arbitrary n by Reichardt [26]. We refer the reader to the references therein for additional results that also contributed to the complete resolution of the double bubble conjecture. The triple bubble conjecture was verified in $\mathbb{R}^2$ by Wichiramala [38]. Recently, Milman and Neeman have shown in [21] that N-bubbles are the unique minimizers when $N\leq\min\{4,n\}$ and in [22] that quintuple bubbles are minimizers when $n\geq 5$. The characterization of minimizers in the immiscible fluids problem with non-equal weights has received less attention, having only been resolved in the case of two chambers by Lawlor [17]; see also [15, Theorem 7.2].

In classical clusters, there is a single chamber with infinite volume. In [3], Alama-Bronsard-Vriend have generalized this concept through the introduction of (N,M) clusters, which partition $\mathbb{R}^n$ into N chambers of finite prescribed volume and M chambers of infinite volume, with M allowed to be strictly greater than one [3]. This requires changing the definition of minimality, since any (N,M)-cluster when $M\geq 2$ automatically has infinite energy on all of space. One possible notion is thus *local minimality* subject to compactly supported perturbations which preserve the volume of the non-infinite chambers. The generalization of the standard weighted double bubble is the *standard*

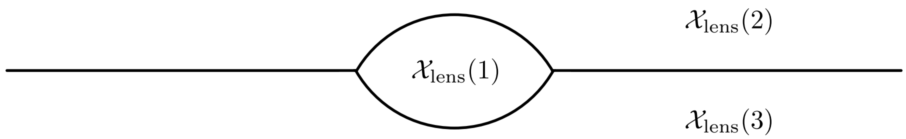

FIGURE 1.1. The standard weighted lens cluster $\mathcal{X}_{\rm lens}$.

weighted lens cluster, which the same authors showed is the unique local minimizer for the equal weights energy in the plane. The equal weights problem was further studied by Novaga-Paolini-Tortorelli [25], who obtained a general closure theorem for local minimizers when the limiting (N, M) cluster has flat interfaces outside some compact set. The closure theorem allows for the construction of several locally minimizing (N, M)-clusters in any dimension. They also proved that any local minimizer in the plane necessarily has at most 3 chambers with infinite volume and completely characterized planar local minimizers when $N + M \leq 4$.

In this paper we study $(1, 2)$-clusters $\{\mathcal{X}(j)\}_{j=1}^3$, where $|\mathcal{X}(1)| < \infty$, and the weighted energy $\sum_{j<k} c_{jk}\mathcal{H}^{n-1}(\partial^*\mathcal{X}(j) \cap \partial^*\mathcal{X}(k))$, where the family of weights satisfies standard positivity and triangularity conditions, and first show that the standard weighted lens cluster is locally minimizing (Theorem 2.8). Our main result (Theorem 2.9) can be summarized as follows:

If $c_{12} = c_{13}$, then, up to rigid motions of $\mathbb{R}^n$, the standard weighed lens cluster is the unique locally minimizing $(1, 2)$-cluster in $\mathbb{R}^n$ for $n \leq 7$. The same uniqueness holds in $\mathbb{R}^n$ for $n \geq 8$ among locally minimizing $(1, 2)$-clusters with planar growth at infinity.

The planar growth assumption means that asymptotically, the two infinite volume chambers have the same energy density as a pair of complementary halfspaces. This assumption is natural in light of the existence of singular perimeter minimizers in higher dimensions; see Remark 2.10. The proof of Theorem 2.9 is based on an energy comparison argument with the standard weighted lens cluster and the geometric rigidity entailed by the resulting energy identity. The comparison is made possible by decay properties of exterior minimal surfaces with planar growth; see Section 2.3 for a more detailed discussion of the proof. We also include a proof of the equivalence between the above notion of local minimality and another natural one (Lemma 4.10) and use it to strengthen the closure theorem from [25] when there are two chambers with infinite volume (Corollary 4.11).

The partitioning problem for (N, M) clusters introduced in [3] comes from the study of triblock copolymer models in the 2D torus by Alama-Bronsard-Lu-Wang in [2]. In [1], these authors study a partitioning problem in the torus with three phases, one of which occupies nearly all of the total area with the other two accounting for only a tiny fraction. In that case, the global minimizers form weighted double-bubble or core-shell patterns depending on the parameters. In [2], they consider this partitioning problem but with two of the phases occupying nearly all of the total area and the third accounting for only a tiny fraction. In that case, we expect minimizers of the nonlocal triblock copolymer energy to form a lamellar pattern with the two majority phases, with tiny droplets of the third phase aligned on each lamellar stripe. By blowing up the droplets in the limit of vanishing area of the third phase we expect to recover the lens shape in the plane, and the characterization of the *vesica piscis* as the unique minimizer (up to symmetries) of this problem is critical to the analysis of the small-area limit of the triblock problem.

The rest of the paper is organized as follows. In Section 2, we give precise definitions and statements of our results, and Section 3 contains some background results. Section 4 establishes various useful properties of locally minimizing (N, M)-clusters. Some of these may be applicable in future investigations of (N, M)-clusters, and so we have opted for generality regarding weights/number of chambers/etc. when possible in this part. Finally, we prove Theorem 2.8 and Theorem 2.9 in Section 5.

2. Results

2.1. **General (N,M)-clusters.** We begin by recalling the notion of an (N, M)-cluster due to Alama-Bronsard-Vriend [3] and some basic terminology for these objects. Definitions 2.1-2.3 can be found in [3, Section 1], and Definitions 2.4-2.5 are the weighted analogues of the energy and local minimizers in [3].

Definition 2.1 ((N, M)-clusters)**.** An (N, M)-*cluster* $(\mathcal{E}, \mathcal{F})$ is a pair of finite families of sets of locally finite perimeter

$$\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N\,, \quad \mathcal{F} = \{\mathcal{F}(i)\}_{i=1}^M$$

such that, denoting by $|\cdot|$ the Lebesgue measure,

(i) $0 < |\mathcal{E}(h)| < \infty$ for $1 \leq h \leq N$,
(ii) $|\mathcal{F}(i)| = \infty$ for $1 \leq i \leq M$,
(iii) any two distinct sets from either family are Lebesgue disjoint, that is

$$\begin{aligned}
&|\mathcal{E}(h) \cap \mathcal{E}(h')| = 0 \quad \text{for all } 1 \leq h < h' \leq N\,,\\
&|\mathcal{F}(i) \cap \mathcal{F}(i')| = 0 \quad \text{for all } 1 \leq i < i' \leq M\,, \quad \text{and}\\
&|\mathcal{E}(h) \cap \mathcal{F}(i)| = 0 \quad \text{for all } 1 \leq h \leq N \text{ and } 1 \leq i \leq M\,, \quad \text{and}
\end{aligned}$$

(iv) $\left|\mathbb{R}^n \setminus (\cup_{h=1}^N \mathcal{E}(h) \cup \cup_{i=1}^M \mathcal{F}(i))\right| = 0$.

The sets with finite Lebesgue measure are called *proper chambers* and the sets with infinite Lebesgue measure are called *improper chambers*.

We will streamline the notation by referring to an (N, M)-cluster $(\mathcal{E}, \mathcal{F})$ by the single family $\mathcal{X} = \{\mathcal{X}(j)\}_{j=1}^{N+M}$, where $\mathcal{X}(j) = \mathcal{E}(j)$ if $1 \leq j \leq N$ and $\mathcal{X}(j + N) = \mathcal{F}(j)$ if $1 \leq j \leq M$.

Definition 2.2 (Volume vector)**.** For an (N, M)-cluster $\mathcal{X}$, the *volume vector* is the $(N+M)$-tuple

$$\mathbf{m}(\mathcal{X}) = (|\mathcal{X}(1)|, \ldots, |\mathcal{X}(N + M)|)$$

with entries in the extended positive real numbers $(0, \infty]$.

Definition 2.3 (Interfaces)**.** For an (N, M)-cluster $\mathcal{X}$, the *interfaces* are the locally $\mathcal{H}^{n-1}$-rectifiable sets formed by the intersection of the reduced boundaries of two distinct chambers, so

$$\mathcal{X}(j, k) = \partial^* \mathcal{X}(j) \cap \partial^* \mathcal{X}(k) \qquad \forall 1 \leq j < k \leq N + M\,.$$

Since an (N, M)-cluster $\mathcal{X}$ corresponds to a Caccioppoli partition of $\mathbb{R}^n$, the Lebesgue points of $\mathcal{X}(j)$ and the interfaces partition $\mathbb{R}^n$ up to a $\mathcal{H}^{n-1}$-null set (see e.g. [7, Theorem 4.17]), that is

$$\mathcal{H}^{n-1}\left(\mathbb{R}^n \setminus \left[\bigcup_{1\leq j\leq N+M} \mathcal{X}(j)^{(1)} \cup \bigcup_{1\leq j<k\leq N+M} \mathcal{X}(j, k)\right]\right) = 0\,, \tag{2.1}$$

and any pair of sets from either union are disjoint.

Definition 2.4 (Relative (weighted) perimeter)**.** For an (N, M)-cluster $\mathcal{X}$ and family of weights $\mathbf{c} := \{c_{jk}\}_{1\leq j<k\leq N+M}$, the $\mathbf{c}$*-perimeter of* $\mathcal{X}$ *relative to a Borel set* $B \subset \mathbb{R}^n$ is

$$P_{\mathbf{c}}(\mathcal{X}; B) = \sum_{1\leq j<k\leq N+M} c_{jk} \mathcal{H}^{n-1}(\mathcal{X}(j, k) \cap B)\,. \tag{2.2}$$

Definition 2.5 ($\mathbf{c}$-locally minimizing clusters)**.** A $\mathbf{c}$*-locally minimizing* (N, M)*-cluster* $\mathcal{X}$ associated to a family $\mathbf{c}$ of weights satisfies $\mathrm{cl}\, \partial^* \mathcal{X}(j) = \partial \mathcal{X}(j)$ for all $1 \leq j \leq N + M$ (following the convention for minimizing clusters in [19, Part IV]) and, for every $r > 0$, the local minimality condition

$$P_{\mathbf{c}}(\mathcal{X}; B_r(0)) \leq P_{\mathbf{c}}(\mathcal{X}'; B_r(0)) \tag{2.3}$$

for every (N,M)-cluster $\mathcal{X}'$ such that $\mathbf{m}(\mathcal{X}) = \mathbf{m}(\mathcal{X}')$ and $\mathcal{X}(j)\Delta\mathcal{X}'(j) \subset\subset B_r(0)$ for each $1 \le j \le N+M$.

2.2. **Statements.** To state the main theorem, we must first discuss the behavior of $\mathbf{c}$-locally minimizing $(1,2)$-clusters at infinity; the proofs of the following statements are included in Section 4. Let us assume that the family of weights $\mathbf{c} = \{c_{12}, c_{13}, c_{23}\}$ satisfies

$$\min\{c_{12}, c_{13}, c_{23}\} > 0, \tag{pos.}$$

$$c_{12} < c_{13} + c_{23}, \quad c_{13} < c_{12} + c_{23}, \quad \text{and} \quad c_{23} < c_{13} + c_{23}\,. \tag{$\triangle$-ineq.}$$

Comparison arguments based on (pos.)-($\triangle$-ineq.) show that if $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing $(1,2)$-cluster, then $\mathcal{X}(1) \subset\subset B_R(0)$ for some $R < \infty$. In particular, the volume constraint is trivially satisfied when testing the minimality (2.3) with $\mathcal{X}'$ such that $\mathcal{X}(1) = \mathcal{X}'(1)$, which is the case for any $\mathcal{X}'$ which only differs from $\mathcal{X}$ on $\mathbb{R}^n \setminus \mathrm{cl}\, B_R(0)$. As a consequence, $\mathcal{X}(2)$ and $\mathcal{X}(3)$ are perimeter minimizers in $\mathbb{R}^n \setminus \mathrm{cl}\, B_R(0)$, with common boundary satisfying the minimal surface equation distributionally, so the monotonicity formula allows us to compute the energy density of $\mathcal{X}$ at infinity.

Definition 2.6 (Density at infinity)**.** If the family $\mathbf{c}$ satisfies (pos.)-($\triangle$-ineq.) and $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing $(1,2)$-cluster, then the *density of $\mathcal{X}$ at infinity* is

$$\Theta_\infty(\mathcal{X}) = \lim_{r\to\infty} \frac{P_{\mathbf{c}}(\mathcal{X}; B_r(0))}{r^{n-1}}\,. \tag{2.4}$$

We say $\mathcal{X}$ has *planar growth at infinity* if $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing $(1,2)$-cluster and $\Theta_\infty(\mathcal{X}) = c_{23}\,\omega_{n-1}$, where ω_{n-1} is the $(n-1)$-dimensional measure of the unit ball in $\mathbb{R}^{n-1}$.

Next we define the standard weighted lens cluster; see Figure 1.1.

Definition 2.7 (Standard weighted lens cluster)**.** The *standard weighted lens cluster* $\mathcal{X}_{\rm lens}$ in $\mathbb{R}^n$ associated to weights $\{c_{12}, c_{13}, c_{23}\}$ satisfying (pos.)-($\triangle$-ineq.) is the $(1,2)$-cluster such that

(i) $|\mathcal{X}_{\rm lens}(1)| = 1$,
(ii) $\partial\mathcal{X}_{\rm lens}(1) = \mathrm{cl}\,\mathcal{X}_{\rm lens}(1,2) \cup \mathrm{cl}\,\mathcal{X}_{\rm lens}(1,3)$, where $\mathcal{X}_{\rm lens}(1,2) \subset \{x_n > 0\}$ and $\mathcal{X}_{\rm lens}(1,3) \subset \{x_n < 0\}$ are spherical caps with boundaries contained in the plane $\{x_n = 0\}$,
(iii) $\mathcal{X}_{\rm lens}(2,3) = \{x_n = 0\} \setminus D$, where $D \subset \{x_n = 0\}$ is the closed $(n-1)$-dimensional disk with boundary (relative to $\{x_n = 0\}$) given by $\partial D = \{x_n = 0\} \cap \partial\mathcal{X}_{\rm lens}(1)$, and
(iv) for each j, the two smooth surfaces forming $\partial\mathcal{X}_{\rm lens}(j)$ and meeting along ∂D form an angle $\theta_j \in (0,\pi)$, and

$$\frac{\sin\theta_1}{c_{23}} = \frac{\sin\theta_2}{c_{13}} = \frac{\sin\theta_3}{c_{12}}\,,$$

where $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

The following theorems establish the $\mathbf{c}$-local minimality of the standard weighted lens cluster as well as its uniqueness under the additional symmetry assumption $c_{12} = c_{13}$ and a natural growth assumption in higher dimensions. By scaling, they may also be applied to $(1,2)$ clusters with volume vector (m,∞,∞) for any $m \in (0,\infty)$.

Theorem 2.8 (Local minimality of the standard weighted lens cluster)**.** *If the family $\mathbf{c}$ of weights satisfies (pos.)-($\triangle$-ineq.), then the standard weighted lens cluster $\mathcal{X}_{\rm lens}$ is a $\mathbf{c}$-locally minimizing $(1,2)$ cluster in $\mathbb{R}^n$.*

Theorem 2.9 (Weighted double bubble theorem for $(1,2)$-clusters)**.** *If $1 \le n \le 7$ and the family $\mathbf{c}$ of weights satisfies (pos.)-($\triangle$-ineq.) and $c_{12} = c_{13}$, then, up to rigid motions of $\mathbb{R}^n$, the standard weighted lens cluster $\mathcal{X}_{\rm lens}$ is the unique $\mathbf{c}$-locally minimizing $(1,2)$-cluster with volume vector $(1,\infty,\infty)$. If $n \ge 8$, the same uniqueness holds among $\mathbf{c}$-locally minimizing $(1,2)$-clusters that also have planar growth at infinity.*

Theorem 2.9 characterizes all $\mathbf{c}$-locally minimizing $(1,2)$-clusters in $\mathbb{R}^n$ when $c_{12} = c_{13}$ and $1 \le n \le 7$. In $\mathbb{R}^n$ for $n \ge 8$, there may be other $\mathbf{c}$-locally minimizing $(1,2)$-clusters.

Remark 2.10 (Other $\mathbf{c}$-locally minimizing clusters in $\mathbb{R}^n$ for $n \ge 8$)**.** The blow-down of any $\mathbf{c}$-locally minimizing $(1,2)$-cluster $\mathcal{X}$ in $\mathbb{R}^n$ corresponds to a perimeter minimizer in all of space. When $n \le 7$, this blow-down necessarily corresponds to a halfspace, as there are no other entire perimeter minimizers. However when $n \ge 8$, the blow-down might be a singular minimizing cone. By Allard's theorem, the blow-down cone is singular and non-planar if and only if $\Theta_\infty(\mathcal{X}) > c_{23}\omega_{n-1}$. It seems possible that locally minimizing $(1,2)$-clusters can be constructed using a cone such as the Simons cone in $\mathbb{R}^8$ to define $\mathcal{X}(2)$ and $\mathcal{X}(3)$ outside $\mathcal{X}(1)$, in which case the planar growth restriction is optimal.

Remark 2.11 (Uniqueness of the lens under a different notion of minimality)**.** An alternative notion of local minimality for a $(1,2)$-cluster $\mathcal{X}$ to Definition 2.5 would be for (2.3) to hold on every $B_r(0)$ among those $\mathcal{X}'$ such that $\mathcal{X}(j)\Delta\mathcal{X}'(j) \subset\subset B_r(0)$ and $|\mathcal{X}(j)\cap B_r(0)| = |\mathcal{X}'(j)\cap B_r(0)|$ for each $1 \le j \le 3$. The lens cluster $\mathcal{X}_{\rm lens}$ is certainly minimal under this definition since it is less restrictive than Definition 2.5, in that it requires minimality against fewer competitors. However, these two notions are actually equivalent; see Lemma 4.10. Thus the symmetric lens cluster is the unique local minimizer under this definition as well by Theorem 2.9. As a corollary to the equivalency of these minimality notions, the closure theorem from [25] can be strengthened in the case of two improper chambers by removing the asymptotic flatness assumption; see Corollary 4.11.

Remark 2.12 (Connection with large-volume exterior isoperimetry)**.** In [20], for a compact set W, the second author and F. Maggi studied the limit as $v \to \infty$ of minimizers for the exterior isoperimetric problem

$$\min\{P(E;\mathbb{R}^n \setminus W) : |E| = v,\ E \subset \mathbb{R}^n \setminus W\}\,.$$

The limiting object as $v \to \infty$ of the boundaries ∂E_v of a sequence $\{E_v\}_{v>0}$ of minimizers is an exterior minimal surface in $\mathbb{R}^n \setminus W$. A key tool in obtaining a sharp geometric description of ∂E_v for large v is the analysis of fine properties of this minimal surface, in particular uniqueness of blow-down cones and asymptotic decay. The properties of the interface $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$, which is an exterior minimal surface, play a similar role in our analysis. This connection is visible in for example Corollary 4.8 and the beginning of the proof of Theorem 2.9, in which asymptotic properties of $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$ are important.

2.3. **Discussion.** The proof of Theorem 2.8 utilizes the minimality of the weighted double bubbles as shown by Lawlor [17]. When $c_{12} = c_{13}$, an alternate proof using the symmetry is also available; see Remark 5.2. The proof of Theorem 2.9 requires showing that given $\mathbf{c}$ with $c_{12} = c_{13}$, any $\mathbf{c}$-locally minimizing $(1,2)$-cluster $\mathcal{X}$ must in fact be equivalent via null sets and rigid motions to $\mathcal{X}_{\rm lens}$. The starting point is using the planar growth and the uniqueness of blow-downs/asymptotic expansion from Maggi-N. [20] to show that outside some compact set, $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$ is the graph, say over $H := \{x \in \mathbb{R}^n : x_n = 0\}$ of an exterior solution u to the minimal surface equation decaying very fast to H. The hyperplane H is thus our candidate for the plane over which we expect $\mathcal{X}$ to be symmetric, in the sense that $\mathcal{X}(1)$ is symmetric over H and $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3) = H \setminus {\rm cl}\,\mathcal{X}(1)$. In fact, if we knew that $\mathcal{X}$ possessed this symmetry, then we could use the uniqueness of minimizers for the classical liquid drop problem in a halfspace and the relationship between $P_{\mathbf{c}}$ and the liquid drop energy (Lemma 3.3) to conclude that, up to translations along H, $\mathcal{X} = \mathcal{X}_{\rm lens}$. So to prove the uniqueness of the lens, it remains to show that $\mathcal{X}$ is symmetric over H. For the proper chamber $\mathcal{X}(1)$, one possible way of obtaining such symmetry is via a rigidity result for the Steiner inequality due to Barchiesi-Cagnetti-Fusco [8]; for the improper chambers, symmetry, or flatness, should come from the fact that the projection of $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$ onto H strictly decreases $\mathcal{H}^{n-1}$ measure if $\nu_{\mathcal{X}(2)}$ deviates from $-e_n$; see Figure 2.1. We therefore prove a local Steiner-type inequality for

FIGURE 2.1. On the left is the original cluster $\mathcal{X}$, and on the right is the "symmetrized" cluster $\mathcal{X}^S$ over the dotted line H. Unless $\mathcal{X}$ is planar outside some compact set, $\mathcal{X}^S$ will not be a compactly supported variation of $\mathcal{X}$. If $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$ decays fast enough to H, then $P_{\mathbf{c}}(\mathcal{X};B_r)$ and $P_{\mathbf{c}}(\mathcal{X}^S;B_r)$ can be compared up to small error.

$(1,2)$ clusters in Lemma 5.1 – it is here we use the symmetry assumption $c_{12}=c_{13}$ – and would like to compare the energy of the "symmetrized" $\mathcal{X}$ to $\mathcal{X}$ itself. This is in general not allowable, since the symmetrization will not be a compactly supported variation of $\mathcal{X}$. However, the decay of $\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)$ to H makes such a comparison possible on $B_r(0)$ up to a small error of order r^{-1}. By sending $r\to\infty$ and making use of a monotonicity property of the energy gap between $\mathcal{X}$ and its symmetrization, we are able to conclude that, up to translating $\mathcal{X}$, $P_{\mathbf{c}}(\mathcal{X};B_r(0))=P_{\mathbf{c}}(\mathcal{X}_{\rm lens};B_r(0))$ for all $r>0$. At this point we can finally show that $\mathcal{X}$ must be symmetric over H due to the aforementioned rigidity considerations.

3. Notation and Preliminaries

3.1. **Notation.** Let C_r be the infinite cylinder

$$C_r=\{x\in\mathbb{R}^n : x_1^2+\cdots+x_{n-1}^2<r^2\}\,.$$

We will write $\overline{x}$ to distinguish points in $\mathbb{R}^{n-1}$ from points in $\mathbb{R}^n$, and set $B_r^{n-1}(\overline{x})$ to be the $(n-1)$-dimensional ball of radius r centered at $\overline{x}\in\mathbb{R}^{n-1}$.

For Borel sets $A,B\subset\mathbb{R}^n$ and $1\le k\le n$, we write $A\overset{\mathcal{H}^k}{=}B$ when $\mathcal{H}^k(A\Delta B)=0$ and $A\overset{\mathcal{H}^k}{\subset}B$ when $\mathcal{H}^k(A\setminus B)=0$.

We will adhere to common notation regarding sets of finite perimeter; see for example the book [19]. For a set $E\subset\mathbb{R}^n$ of locally finite perimeter $P(E;B)$ is the perimeter of E inside B if $B\subset\mathbb{R}^n$ is Borel, $E^{(t)}$ is the set of points of Lebesgue density $t\in[0,1]$, $\partial^e E=\mathbb{R}^n\setminus(E^{(1)}\cup E^{(0)})$ is the essential boundary of E, and ∂^*E is the reduced boundary with outer (measure-theoretic) normal $\nu_E:\partial^*E\to\mathbb{S}^{n-1}$. When we are using these concepts on some k-dimensional set of locally finite perimeter $F\subset\mathbb{R}^k$ where $k\neq n$, we will often write $F^{(1)_{\mathbb{R}^k}}$, $F^{(0)_{\mathbb{R}^k}}$, and $\partial^*_{\mathbb{R}^k}F$ to emphasize that these operations are taken with respect to $\mathbb{R}^k$. The $L^1_{\rm loc}$-convergence of the characteristic functions $\mathbf{1}_{E_m}$ of sets of finite perimeter to some $\mathbf{1}_E$ will be denoted by $E_m\overset{\rm loc}{\to}E$.

To describe the convergence of clusters, we say that $\mathcal{X}_m$ *locally converges to* $\mathcal{X}$, or

$$\mathcal{X}_m\overset{\rm loc}{\to}\mathcal{X}\,,$$

if, for each $1\le j\le N+M$,

$$\big|(\mathcal{X}_m(j)\Delta\mathcal{X}(j))\cap K\big|\to 0\qquad\forall K\subset\subset\mathbb{R}^n\,.$$

Lastly, we remark that given weights $\{c_{12},c_{13},c_{23}\}$ and corresponding c_1, c_2, c_3 defined via the linear system $c_{jk}=c_j+c_k$, ($\triangle$-ineq.) is equivalent to

$$\min\{c_1,c_2,c_3\}>0\,,\tag{3.1}$$

and, for any Borel $B \subset \mathbb{R}^n$,

$$P_{\mathbf{c}}(\mathcal{X}; B) = c_1 P(\mathcal{X}(1); B) + c_2 P(\mathcal{X}(2); B) + c_3 P(\mathcal{X}(3); B) . \tag{3.2}$$

3.2. **Preliminaries.** Here we collect some facts and theorems from the literature that will be used in the proofs.

First is Federer's theorem [7, Theorem 3.61], which says that if $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, then

$$\partial^* E \subset E^{(1/2)} \subset \partial^e E , \quad \partial^e E \overset{\mathcal{H}^{n-1}}{=} \partial^* E , \quad \text{and} \quad \mathbb{R}^n \overset{\mathcal{H}^{n-1}}{=} E^{(1)} \cup E^{(0)} \cup E^{(1/2)} . \tag{3.3}$$

Second are the formulas for perimeters of unions and intersections [19, Theorem 16.3]. For any Borel $G \subset \mathbb{R}^n$ and sets of locally finite perimeter E and F, we have

$$P(E \cup F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(0)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G) \quad \text{and} \tag{3.4}$$

$$P(E \cap F; G) = P(E; F^{(1)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G) . \tag{3.5}$$

As a consequence of (3.4)-(3.5) and (3.3), if E and F are sets of finite perimeter and H is a set of locally finite perimeter such that $\mathcal{H}^{n-1}(\partial^* H \cap (\partial^* E \cup \partial^* F)) = 0$, then for every Borel $G \subset \mathbb{R}^n$,

$$\begin{aligned} P((E \cap H) \cup (F \setminus H); G) = \mathcal{H}^{n-1}(\partial^* E \cap H^{(1)} \cap G) + \mathcal{H}^{n-1}(\partial^* F \cap H^{(0)} \cap G) \\ + \mathcal{H}^{n-1}((E^{(1)} \Delta F^{(1)}) \cap \partial^* H \cap G) . \end{aligned} \tag{3.6}$$

Next, following the presentation in [8, Section 1], we state a rigidity result from symmetrization. Decomposing $\mathbb{R}^n = \{(\overline{x}, y) : \overline{x} \in \mathbb{R}^{n-1},\ y \in \mathbb{R}\}$, the *one-dimensional slices* of a Borel measurable $E \subset \mathbb{R}^n$ (with respect to the subspace $\{(\overline{x}, y) \in \mathbb{R}^n : y = 0\}$) and their accompanying measures are defined for each $\overline{x} \in \mathbb{R}^{n-1}$ as

$$E_{\overline{x}} := \{y \in \mathbb{R} : (\overline{x}, y) \in E\} \quad \text{and} \quad L_E(\overline{x}) = \mathcal{H}^1(E_{\overline{x}}) .$$

We also set

$$\pi(E)^+ := \{\overline{x} \in \mathbb{R}^{n-1} : L_E(\overline{x}) > 0\} .$$

The *Steiner symmetral* of E is

$$E^S := \{(\overline{x}, y) \in \mathbb{R}^n : \overline{x} \in \pi(E)^+,\ |y| < \mathcal{H}^1(E_{\overline{x}})/2\} .$$

The classical Steiner inequality says that if E is a set of finite perimeter, then

$$P(E^S; B \times \mathbb{R}) \leq P(E; B \times \mathbb{R}) \quad \forall \text{ Borel } B \subset \mathbb{R}^{n-1} , \tag{3.7}$$

with equality when $B = \mathbb{R}^{n-1}$ implying that $E_{\overline{x}}$ is an interval for $\mathcal{H}^{n-1}$-a.e. $\overline{x} \in \pi(E)^+$ [8, Theorem 1.1.(a)].

To state the theorem on equality cases in (3.7), we first introduce the condition

$$\mathcal{H}^{n-1}\big(\{x \in \partial^*(E^S) : \nu_{E^S}(x) \cdot e_n = 0\} \cap (\Omega \times \mathbb{R})\big) = 0 , \tag{3.8}$$

for any open $\Omega \subset \mathbb{R}^{n-1}$. Geometrically speaking, (3.8) says that $\partial^*(E^S)$ has no non-negligible "vertical parts" over Ω. Under the assumption (3.8), the function L_E belongs to $W^{1,1}(\mathbb{R}^{n-1})$ [8, Proposition 3.5] (as opposed to the weaker $BV(\mathbb{R}^{n-1})$ when (3.8) fails). Therefore, the Lebesgue average of L_E exists for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in \mathbb{R}^{n-1}$ [13, pages 160 and 156]. We define the *precise representative*

$$L_E^*(\overline{x}) = \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{\omega_{n-1} r^{n-1}} \displaystyle\int_{\{|\overline{z} - \overline{x}| < r\}} L_E(\overline{z}) \, d\mathcal{L}^{n-1}(\overline{z}) & \text{if the limit exists} \\ 0 & \text{otherwise} , \end{cases}$$

so that $L_E^*(\overline{x})$ is equal to its Lebesgue average for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in \mathbb{R}^{n-1}$. The following rigidity result, which we state in co-dimension 1, was proved by Barchiesi-Cagnetti-Fusco in [8] for Steiner symmetrization of arbitrary co-dimension; see also [9] for additional results on rigidity in (3.7).

Theorem 3.1. [8, Theorem 1.2] *If $\Omega \subset \mathbb{R}^{n-1}$ is a connected open set, $E \subset \mathbb{R}^n$ is a set of finite perimeter such that $P(E^S;\Omega\times\mathbb{R}) = P(E;\Omega\times\mathbb{R})$, (3.8) holds, and $L_E^*(\overline{x}) > 0$ for $\mathcal{H}^{n-2}$-a.e. $\overline{x}\in\Omega$, then $E\cap(\Omega\times\mathbb{R})$ is Lebesgue equivalent to a translation along $\{\overline{0}\}\times\mathbb{R}$ of $E^S\cap(\Omega\times\mathbb{R})$.*

We will also use a slicing result for clusters adapted from the corresponding result for slices of sets of finite perimeter by lines from [8, Theorem 2.4], which in turn is based on Vol'pert [35].

Lemma 3.2 (Slicing by lines)**.** *If $\mathcal{X}$ is an (N, M)-cluster in $\mathbb{R}^n$, then there exists Borel measurable $F \subset \mathbb{R}^{n-1}$ such that $\mathcal{H}^{n-1}(\mathbb{R}^{n-1}\setminus F) = 0$, and, if $\overline{x}\in F$, then*

- *(i) $\mathcal{X}(j)_{\overline{x}} \subset \mathbb{R}$ is a set of locally finite perimeter for all $1 \le j \le N+M$,*
- *(ii) $(\mathcal{X}(j)^{(1)})_{\overline{x}} \overset{\mathcal{H}^1}{=} (\mathcal{X}(j)_{\overline{x}})^{(1)_{\mathbb{R}}}$ and $(\mathcal{X}(j)^{(0)})_{\overline{x}} \overset{\mathcal{H}^1}{=} (\mathcal{X}(j)_{\overline{x}})^{(0)_{\mathbb{R}}}$ for $1\le j\le N+M$,*
- *(iii) $[\partial^*\mathcal{X}(j)\cap\partial^*\mathcal{X}(k)]_{\overline{x}} = \partial^*_{\mathbb{R}}(\mathcal{X}(j)_{\overline{x}})\cap\partial^*_{\mathbb{R}}(\mathcal{X}(k)_{\overline{x}})$ for all $1\le j<k\le N+M$, and*
- *(iv) $\nu_{\mathcal{X}(j)}(\overline{x},t)\cdot e_n \neq 0$ and $\nu_{\mathcal{X}(j)}(\overline{x},t)\cdot e_n/|\nu_{\mathcal{X}(j)}(\overline{x},t)\cdot e_n| = \nu_{\mathcal{X}(j)_{\overline{x}}}(t)$ for every $t\in[\partial^*\mathcal{X}(j)]_{\overline{x}}$ and each $1\le j\le N+M$.*

Proof. Items (i), (iii), and (iv) follow from [8, Theorem 2.4.(i)-(iii)] and (2.1), while (ii) is a consequence of Fubini's theorem and the Lebesgue points theorem. □

Finally, we will need a classical result drawn from [19, Chapter 19] on the liquid drop problem in a halfspace. For a halfspace $H\subset\mathbb{R}^n$, set of finite perimeter $E\subset H$, and real number $\beta\in\mathbb{R}$, let

$$\mathcal{F}_\beta(E;H) = P(E;H) - \beta P(E;\partial H)\,. \tag{3.9}$$

Before stating the relevant theorem, we motivate it through a lemma connecting (3.9) to $P_{\mathbf{c}}$ under some symmetry assumptions.

Lemma 3.3. *If $\mathbf{c}$ is a family of weights satisfying $c_{12} = c_{13}$, H is an open halfspace, R is the reflection map over ∂H, and $\mathcal{X}$ is a $(1,2)$-cluster such that $\mathcal{X}(1)\subset\subset B_r(0)$,*

$$\mathcal{X}(1)\overset{\mathcal{L}^n}{=}R(\mathcal{X}(1))\,,\quad \mathcal{X}(2)\overset{\mathcal{L}^n}{=}H\setminus\mathcal{X}(1)\,,\quad \textit{and}\quad \mathcal{X}(3)\overset{\mathcal{L}^n}{=}\mathbb{R}^n\setminus(\mathcal{X}(1)\cup\mathcal{X}(2))\,, \tag{3.10}$$

then

$$\partial H\overset{\mathcal{H}^n}{\subset}\mathcal{X}(1)^{(1)}\cup\mathcal{X}(1)^{(0)}\,,\quad \partial H\cap\partial^*(\mathcal{X}(1)\cap H)\overset{\mathcal{H}^{n-1}}{=}\mathcal{X}(1)^{(1)}\cap\partial H\,, \tag{3.11}$$

$$\partial H\cap(\mathcal{X}(1)\cap H)^{(0)} = \partial H\cap\mathcal{X}(1)^{(0)}\overset{\mathcal{H}^{n-1}}{=}\mathcal{X}(2,3)\,, \tag{3.12}$$

and

$$P_{\mathbf{c}}(\mathcal{X};C_r) = 2c_{13}\mathcal{F}_{c_{23}/(2c_{13})}(\mathcal{X}(1)\cap H;H) + c_{23}\omega_{n-1}r^{n-1}\,. \tag{3.13}$$

Proof. First, by Federer's theorem (3.3) and the fact that the Lebesgue density of $\mathcal{X}(1)\cap H$ along ∂H is at most $1/2$,

$$\partial H\overset{\mathcal{H}^{n-1}}{=}\partial H\cap\left[\partial^*(\mathcal{X}(1)\cap H)\cup(\mathcal{X}(1)\cap H)^{(0)}\right]. \tag{3.14}$$

Through routine manipulations which we omit based on (3.14), (3.3), and (3.5), one concludes (3.11)-(3.12). For (3.13), we start by using the equality $c_{12}=c_{13}$, (2.1), and $\mathcal{X}(1)\subset\subset B_r(0)\subset C_r$ to rewrite

$$P_{\mathbf{c}}(\mathcal{X};C_r) = c_{13}P(\mathcal{X}(1)) + c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3)\cap C_r)\,. \tag{3.15}$$

By the first equivalence in (3.11) and the fact that $\mathcal{X}(1)$ is symmetric over ∂H,

$$c_{13}P(\mathcal{X}(1)) = c_{13}P(\mathcal{X}(1);\mathbb{R}^n\setminus\partial H) = 2c_{13}P(\mathcal{X}(1);H) = 2c_{13}P(\mathcal{X}(1)\cap H;H)\,. \tag{3.16}$$

Also, from (3.11)-(3.12),

$$c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3)\cap C_r) = c_{23}\mathcal{H}^{n-1}(\partial H\cap(\mathcal{X}(1)\cap H)^{(0)}\cap C_r) = c_{23}\mathcal{H}^{n-1}(\partial H\cap\mathcal{X}(1)^{(0)}\cap C_r)$$
$$= c_{23}\big[\mathcal{H}^{n-1}(\partial H\cap C_r) - \mathcal{H}^{n-1}(\partial H\cap\partial^*(\mathcal{X}(1)\cap H)\cap C_r)\big]\,. \qquad (3.17)$$

The equality (3.13) follows by inserting the sum of (3.16) and (3.17) into (3.15). □

The next theorem is a rephrasing of [19, Theorem 19.21] and characterizes minimizers for the liquid drop problem without gravity in a halfspace in terms of standard weighted lens clusters.

Theorem 3.4. *If* $\mathbf{c}$ *is a family of weights satisfying* (pos.)-($\triangle$-ineq.) *and* $c_{12} = c_{13}$, $\mathcal{X}_{\rm lens}$ *is the standard weighted lens cluster corresponding to the weights* $\mathbf{c}$, *and* $H = \{x\in\mathbb{R}^n : x_n > 0\}$, *then* $\mathcal{X}_{\rm lens}(1)\cap H$ *is the unique minimizer up to horizontal translations and* $\mathcal{L}^n$*-null sets for the variational problem*

$$\inf\big\{\mathcal{F}_{c_{23}/(2c_{13})}(E;H) : |E| = 1/2,\, E\subset H,\, P(E)<\infty\big\}\,. \qquad (3.18)$$

Proof. First, the assumptions (pos.)-($\triangle$-ineq.) and $c_{12}=c_{13}$ imply that

$$\beta := c_{23}/(2c_{13}) = c_{23}/(c_{12}+c_{13}) \in (0,1)\,.$$

The only assumption needed to apply the characterization of minimizers in (3.18) from [19, Theorem 19.21] is $|\beta|<1$, and thus the unique minimizer up to horizontal translations and Lebesgue null sets for (3.18) is $B\cap H$, where B is the ball determined by the two conditions $|B\cap H| = 1/2$ and

$$-e_n\cdot\nu_B(x) = -c_{23}/(2c_{13}) \qquad \forall x\in\partial B\cap\partial H\,. \qquad (3.19)$$

The angle θ formed by ν_B and $-e_n$ along ∂H is a 90° counterclockwise rotation of the angle θ_2 formed by $\partial(H\setminus B)$ along its singular set $\partial B\cap\partial H$. So the cluster $\mathcal{X}$ given by

$$\mathcal{X}(1) = (B\cap H)\cup R(B\cap H)\,, \quad \mathcal{X}(2) = H\setminus B\,, \quad \mathcal{X}(3) = H^c\setminus R(B\cap H)$$

is the standard symmetric lens cluster corresponding to the angles $\theta_2=\theta_3$ along the singular sets of $\partial\mathcal{X}(2)$ and $\partial\mathcal{X}(3)$ and $\theta_1 = 2\pi-2\theta_2$ formed by the two spherical caps comprising $\partial\mathcal{X}(1)$. By multiplying (3.19) by $-2\sin\theta_2/c_{23}$ on both sides, we obtain

$$\frac{-2\cos\theta_2\sin\theta_2}{c_{23}} = \frac{\sin\theta_2}{c_{13}}\,.$$

Since $\theta_2 = \pi-\theta_1/2$, the left hand side simplifies to $\sin\theta_1/c_{23}$. By $c_{12}=c_{13}$ and $\theta_2=\theta_3$, the resulting equations for θ_i define the standard weighted lens cluster corresponding to the family $\mathbf{c}$, so $|\mathcal{X}(j)\Delta\mathcal{X}_{\rm lens}(j)| = 0$ for $1\le j\le 3$. Thus $\mathcal{X}_{\rm lens}(1)\cap H$ is Lebesgue equivalent to $B\cap H$, and the theorem is complete. □

We end the preliminaries with a short discussion of weighted double bubbles [17].

Definition 3.5 (Standard weighted double bubble)**.** The *standard weighted double bubble* $\mathcal{X}_{\rm bub}^m$ in $\mathbb{R}^n$ associated to weights $\mathbf{c}$ satisfying (pos.)-($\triangle$-ineq.) and $m>0$ is the $(2,1)$-cluster such that

(i) $|\mathcal{X}_{\rm bub}^m(1)| = 1$, $|\mathcal{X}_{\rm bub}^m(2)| = m$,
(ii) $\partial\mathcal{X}_{\rm bub}^m(j,k)$ are three spherical caps, meeting along an $(n-2)$-dimensional sphere ∂D contained in the plane $\{x_n=0\}$, with $\partial\mathcal{X}_{\rm bub}^m(2)\cap\partial\mathcal{X}_{\rm bub}^m(3)\subset\{x_n\ge 0\}$, and
(iii) for each j, the two smooth surfaces forming $\partial\mathcal{X}_{\rm bub}^m(j)$ and meeting along ∂D form an angle $\theta_j\in(0,\pi)$, and

$$\frac{\sin\theta_1}{c_{23}} = \frac{\sin\theta_2}{c_{13}} = \frac{\sin\theta_3}{c_{12}}\,,$$

where $\theta_1+\theta_2+\theta_3 = 2\pi$.

In [17], Lawlor has proved that

$$P_{\mathbf{c}}(\mathcal{X}^m_{\rm bub}) \le P_{\mathbf{c}}(\mathcal{X}) \tag{3.20}$$

for any $(2,1)$-cluster $\mathcal{X}$ with volume vector $(1,m,\infty)$, and that equality holds if and only if $\mathcal{X} = \mathcal{X}^m_{\rm bub}$ up to Lebesgue null sets and rigid motions of $\mathbb{R}^n$. The last lemma, for which we omit the proof, follows from the definitions of $\mathcal{X}^m_{\rm bub}$ and $\mathcal{X}_{\rm lens}$.

Lemma 3.6 (Convergence of weighted double bubbles to weighted lens clusters)**.** *If* $\mathbf{c}$ *satisfies* (pos.)-(△-ineq.)*, then as* $m \to \infty$*,* $\mathcal{X}^m_{\rm bub}(j)$ *converges locally in the Hausdorff distance to* $\mathcal{X}_{\rm lens}(j)$ *for each* $1 \le j \le 3$*, and for any* R *with* $\mathcal{X}_{\rm lens}(1) \subset\subset B_R(0)$*,* $P_{\mathbf{c}}(\mathcal{X}^m_{\rm bub}; B_R(0)) \to P_{\mathbf{c}}(\mathcal{X}_{\rm lens}; B_R(0))$*.*

4. Properties of c-locally minimizing (N,M)-clusters

We present the relevant properties of the weighted energy $P_{\mathbf{c}}$ and $\mathbf{c}$-locally minimizing (N,M)-clusters. First, by a result of Ambrosio and Braides [6, Example 2.8] (see also [37, Section 7]), (pos.)-(△-ineq.) imply that $P_{\mathbf{c}}$ is lower-semicontinuous, that is, if $\mathcal{X}_m \overset{\rm loc}{\to} \mathcal{X}$ and $U \subset \mathbb{R}^n$ is open, then

$$P_{\mathbf{c}}(\mathcal{X}; U) \le \liminf_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}_m; U)\,. \tag{4.1}$$

The lower-semicontinuity still holds even if some chambers of $\mathcal{X}_m$ vanish in the local limit, so that the limiting object $\mathcal{X}$ is an (N',M')-cluster where $N \le N'$, $M \le M'$ and the energy on the left hand side is $P_{\mathbf{c}'}(\mathcal{X}; U)$, where $\mathbf{c}'$ is the subfamily of indices corresponding to non-vanishing chambers.

Lemma 4.1 (Volume density estimate)**.** *If* $\mathbf{c}$ *is a family of weights satisfying* (pos.)-(△-ineq.) *and* $\mathcal{X}$ *is a* $\mathbf{c}$*-locally minimizing* $(1,2)$*-cluster, then there exists* $\Lambda > 0$*,* $\varepsilon \in (0,1)$*, and* $r_0 > 0$ *such that*

$$P_{\mathbf{c}}(\mathcal{X}; B_{r_0}(x)) \le P_{\mathbf{c}}(\mathcal{X}'; B_{r_0}(x)) + \Lambda |\mathcal{X}(1)\Delta\mathcal{X}'(1)| \tag{4.2}$$

whenever $\mathcal{X}(j)\Delta\mathcal{X}'(j) \subset B_{r_0}(x)$ *for some* $x \in \mathbb{R}^n$ *and each* j*, and*

$$|B_r(y) \cap \mathcal{X}(j)| \ge \varepsilon\omega_n r^n \quad \textit{if } y \in \partial\mathcal{X}(j),\ r < r_0, \textit{ and } 1 \le j \le 3\,. \tag{4.3}$$

As a consequence, each $\mathcal{X}(j)^{(1)}$ *is open and satisfies* $\mathrm{cl}\,\partial^*\mathcal{X}(j) = \partial\mathcal{X}(j)^{(1)}$*, and* $\mathcal{X}(1)$ *is bounded.*

Proof. By the volume fixing variations construction for sets of finite perimeter (e.g. [19, Lemma 17.21]), we may choose $z_1, z_2 \in \partial^*\mathcal{X}(1)$, disjoint balls $B_{|z_1-z_2|/4}(z_\ell)$ for $\ell = 1,2$, $\sigma_0 > 0$, $C > 0$, and a pair of one parameter families of diffeomorphisms $\{f^\ell_t\}_t$, $\ell = 1,2$, such that for every $\sigma \in [-\sigma_0, \sigma_0]$, there exists $t \in [-C\sigma, C\sigma]$ such that for each $1 \le j \le 3$, $f^\ell_t(\mathcal{X}(j))\Delta\mathcal{X}(j) \subset\subset B_{|z_1-z_2|/4}(z_\ell)$,

$$|f^\ell_t(\mathcal{X}(1))| = |\mathcal{X}(1)| + \sigma\,, \quad \text{and} \quad |P(f^\ell_t(\mathcal{X}(j)); B_{|z_1-z_2|/4}(z_\ell)) - P(\mathcal{X}(j); B_{|z_1-z_2|/4}(z_\ell))| \le C|\sigma|\,.$$

If we let $r_0 = \min\{|z_1-z_2|/4, (\sigma_0/\omega_n)^{1/n}\}$, then any $B_{r_0}(x) \subset \mathbb{R}^n$ is disjoint from $B_{|z_1-z_2|/4}(z_{\ell_0})$ for some $\ell_0 \in \{1,2\}$. Therefore, given any $\mathcal{X}'$ and $B_{r_0}(x)$ as in the statement of the lemma, we can modify $\mathcal{X}'$ on $B_{|z_1-z_2|/4}(z_{\ell_0})$ by replacing each $\mathcal{X}'(j) \cap B_{|z_1-z_2|/4}(z_{\ell_0})$ with $f^{\ell_0}_t(\mathcal{X}(j)) \cap B_{|z_1-z_2|/4}(z_{\ell_0})$, where t is chosen based on $\sigma = |\mathcal{X}(1)| - |\mathcal{X}'(1)|$. We then test the minimality of $\mathcal{X}$ against this modified $\mathcal{X}'$, which is admissible by our choice of σ, and (4.2) is verified with $\Lambda = \Lambda(C, \mathbf{c})$.

Moving on to (4.3), fix any $r < r_0$ and $y \in \partial\mathcal{X}(1)$; the cases $y \in \partial\mathcal{X}(j)$ where $j = 2,3$ are proved similarly. Let $m(r) = |\mathcal{X}(1) \cap B_r(y)|$. We claim that we may decrease r_0 and choose $C_1 > 0$, both independently of y, such that, for every $r < r_0$ with

$$\mathcal{H}^{n-1}(\partial^*\mathcal{X}(j) \cap \partial B_r(y)) = 0 \quad \forall 1 \le j \le 3 \quad \text{and} \tag{4.4}$$

$$m'(r) = \mathcal{H}^{n-1}(\mathcal{X}(1)^{(1)} \cap \partial B_r(y))\,, \tag{4.5}$$

we have

$$m(r)^{(n-1)/n} \le C_1 m'(r)\,. \tag{4.6}$$

Note that (4.4) and (4.5) both hold at almost every $r < r_0$ by the local finiteness of $\mathcal{H}^{n-1} \llcorner \partial^* \mathcal{X}(j)$ and differentiating m using the co-area formula, respectively. The derivation of (4.3) from (4.6) follows dividing by $m(r)^{(n-1)/n}$, which is valid since $y \in \partial\mathcal{X}(1) = \operatorname{cl}\partial^*\mathcal{X}(1)$ implies that $m(r) > 0$ for every $r > 0$, and integrating. To prove (4.6) for suitable $r < r_0$, let us suppose

$$\mathcal{H}^{n-1}(\mathcal{X}(1,2) \cap B_r(y)) \leq \mathcal{H}^{n-1}(\mathcal{X}(1,3) \cap B_r(y))\,; \tag{4.7}$$

the case with the opposite inequality is the same. For the subsequent calculation it is convenient to use the rewritten formula for the energy from (3.2). Testing (4.2) on $B_{r_0}(y)$ with $\mathcal{X}'$ where $\mathcal{X}'(1) = \mathcal{X}(1) \setminus B_r(y)$, $\mathcal{X}'(2) = \mathcal{X}(2)$, and $\mathcal{X}'(3) = \mathcal{X}(3) \cup (B_r(y) \cap \mathcal{X}(1))$ and cancelling like terms, we obtain

$$\begin{aligned}(c_1 + c_2)&\mathcal{H}^{n-1}(\mathcal{X}(1,2) \cap B_r(y)) + (c_1 + c_3)\mathcal{H}^{n-1}(\mathcal{X}(1,3) \cap B_r(y)) \\ &\leq (c_2 + c_3)\mathcal{H}^{n-1}(\mathcal{X}(1,2) \cap B_r(y)) + (c_1 + c_3)\mathcal{H}^{n-1}(\mathcal{X}(1)^{(1)} \cap \partial B_r(y)) + \Lambda m(r)\,,\end{aligned}$$

where we used (4.4) and (3.6) to compute $P_{\mathbf{c}}(\mathcal{X}'; \partial B_r(y))$. After adding $c_1\mathcal{H}^{n-1}(\mathcal{X}(1)^{(1)} \cap \partial B_r(y)) = c_1 m'(r)$ to both sides, we simplify using (4.7) and find

$$c_1 P(\mathcal{X}(1) \cap B_r(y)) = c_1 P(\mathcal{X}(1); B_r(y)) + c_1 m'(r) \leq (2c_1 + c_3)m'(r) + \Lambda m(r)\,. \tag{4.8}$$

Applying the isoperimetric inequality to $\mathcal{X}(1) \cap B_r(y)$ then gives

$$c_1 n\, \omega_n^{1/n} m(r)^{(n-1)/n} \leq (2c_1 + \max\{c_2, c_3\})m'(r) + \Lambda m(r)\,.$$

Decreasing r_0 if necessary based on Λ so that we may absorb $\Lambda m(r)$ onto the left hand side, (4.6) follows. To see that each $\mathcal{X}(j)^{(1)}$ is open, if $y \in \mathcal{X}(j)^{(1)}$, then $|B_\rho(y) \cap \mathcal{X}(j)| > (1 - \varepsilon/2^{n+1})\omega_n \rho^n$ for some $\rho < r_0$. Then for any $z \in B_{\rho/2}(y)$ and $k \neq j$,

$$|B_{\rho/2}(z) \cap \mathcal{X}(k)| \leq |B_\rho(y) \cap \mathcal{X}(k)| \leq \omega_n \rho^n - |B_\rho(y) \cap \mathcal{X}(j)| < \varepsilon\omega_n\rho^n/2^{n+1}\,. \tag{4.9}$$

By the density estimate (4.3) on $B_{\rho/2}(z)$ when $z \in \partial\mathcal{X}(k)$, (4.9) prevents z from belonging to $\partial\mathcal{X}(k)$ for any $k \neq j$. Since $z \in B_{\rho/2}(y)$ was arbitrary, $\partial\mathcal{X}(k) \cap B_{\rho/2}(y) = \varnothing$ for each $k \neq j$. Then by $y \in \mathcal{X}(j)^{(1)}$ (which precludes $B_\rho(y) \subset \mathcal{X}(k)$), this implies that

$$|B_{\rho/2}(y) \cap \mathcal{X}(k)| = 0 \quad \text{for } k \neq j\,.$$

Thus $B_{\rho/2}(y) \subset \mathcal{X}(j)^{(1)}$, and $\mathcal{X}(j)^{(1)}$ is open. For the characterization of $\operatorname{cl}\partial^*\mathcal{X}(j)$, first note that by (3.3) and the fact that $x \in \mathcal{X}(j)^{(1/2)}$ entails $B_r(x) \cap \mathcal{X}(j)^t \neq \varnothing$ for all $r > 0$ and $t \in \{0, 1\}$, $\operatorname{cl}\partial^*\mathcal{X}(j) \subset \operatorname{cl}\mathcal{X}(j)^{(1/2)} \subset \partial\mathcal{X}(j)^{(1)}$. The reverse inclusion follows from the equivalence $\operatorname{cl}\partial^*\mathcal{X}(j) = \{x : 0 < |B_r(x) \cap \mathcal{X}(j)| < \omega_n r^n\, \forall r\}$ [19, Eq. (12.12)]. Lastly, (4.3) and $|\mathcal{X}(1)| < \infty$ imply that $\partial\mathcal{X}(1)$ is bounded, which together with $|\mathcal{X}(1)| < \infty$ implies that $\mathcal{X}(1)$ is bounded also. □

Remark 4.2 (Bounded proper chambers for **c**-locally minimizing $(N, 2)$- and $(N, 3 - N)$-clusters)**.** The final conclusion of Lemma 4.1, namely the boundedness of the proper chambers, holds for general **c**-locally minimizing $(N, 2)$-clusters by a different argument. First, following Almgren [5, VI.10-12], one obtains an almost-minimality inequality

$$P_{\mathbf{c}}(\mathcal{X}; B_r(x)) \leq P_{\mathbf{c}}(\mathcal{X}'; B_r(x)) + \sum_j \Lambda|\mathcal{X}(j)\Delta\mathcal{X}'(j)| \tag{4.10}$$

as in [25, Theorem 5.1, Lemma 5.2, and Remark 5.3] in the paper of Novaga-Paolini-Tortorelli. Second, (4.10) allows one to run the elimination argument of Leonardi [18, Theorem 3.1] on any subfamily $\mathcal{I} \subset \{1, \ldots, N + M\}$ such that $\#\mathcal{I} = N + M - 2$, yielding positive η and ρ such that

$$\text{“if } r < \rho \text{ and } |B_r(x) \cap (\cup_{j\in\mathcal{I}}\mathcal{X}(j))| \leq \eta r^n, \text{ then } (\cup_{j\in\mathcal{I}}\mathcal{X}(j)) \cap B_{r/2}(x) = \varnothing\text{''}\,. \tag{4.11}$$

The boundedness of the proper chambers $\cup_{j=1}^N \mathcal{X}(j)$ when $M = 2$ follows from applying (4.11) to $\mathcal{I} = \{1, \ldots, N\}$, although does not appear to be an immediate consequence when $M > 2$ and $N > 1$. When $N + M = 3$, (4.11) applied to $\mathcal{X}(j)$ for each j can be used to deduce lower volume density bounds on every chamber and thus boundedness of proper ones. Although this covers the case of Lemma 4.1, we have included that proof since it is simpler than (4.11). To our knowledge, lower volume density bounds as in (4.3) are not known for locally minimizing clusters with general weights $\mathbf{c}$ satisfying (pos.)-($\triangle$-ineq.) when there are four or more chambers. In the equal weights case, lower density bounds/boundedness of the proper chambers were proved by Novaga-Paolini-Tortorelli [25, Theorem 2.4].

Remark 4.3 (Further properties of $\mathbf{c}$-locally minimizing (N,M)-clusters)**.** The property (4.11) (see also [37, Property P]) implies that $\mathcal{X}(j)^{(1)}$ is open for each j and, combined with a first variation argument, that each $\mathcal{X}(j,k)$ is an analytic hypersurface with constant mean curvature λ_{jk}, which is 0 when $\min\{j,k\}\geq N+1$. In addition, by an unpublished result of White [36, 37], under the assumptions (pos.)-($\triangle$-ineq.), the singular set of the interfaces should have Hausdorff dimension at most $n-2$; see also [11]. We also refer the reader to [16, 10, 23] and the references therein for more on immiscible fluid clusters.

Theorem 4.4 (Exterior monotonicity formula for $\mathbf{c}$-locally minimizing (N,M)-clusters)**.** *If $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing (N,M)-cluster and $R>0$ is such that $\cup_h\mathcal{E}(h)\subset\subset B_R(0)$, then*

$$0=\sum_{N+1\leq j<k\leq N+M}\int_{\mathcal{X}(j,k)\setminus\mathrm{cl}\,B_R(0)}c_{jk}\,\mathrm{div}^{\mathcal{X}(j,k)}\,X(x)\;d\mathcal{H}^{n-1}(x)\quad\forall X\in C^\infty_c(\mathbb{R}^n\setminus\mathrm{cl}\,B_R(0);\mathbb{R}^n)\,. \tag{4.12}$$

If R is also a point of differentiability of the increasing function $r\mapsto P_{\mathbf{c}}(\mathcal{X};B_r(0))$, which includes almost every $r>0$, then there exists $C(R)\in\mathbb{R}$ such that

$$\frac{P_{\mathbf{c}}(\mathcal{X};B_s\setminus B_R)}{s^{n-1}}+\frac{C(R)}{s^{n-1}}\leq\frac{P_{\mathbf{c}}(\mathcal{X};B_r\setminus B_R)}{r^{n-1}}+\frac{C(R)}{r^{n-1}}\qquad\forall R<s<r<\infty\,. \tag{4.13}$$

Proof. It is convenient to use the language of varifolds (see e.g. [30]), so we introduce the locally $\mathcal{H}^{n-1}$-rectifiable set

$$S=\cup_{N+1\leq j<k\leq N+M}\mathcal{X}(j,k)\setminus\mathrm{cl}\,B_R(0)=\cup_{1\leq i<i'\leq M}\big[\partial^*\mathcal{F}(i)\cap\partial^*\mathcal{F}(i')\big]\setminus\mathrm{cl}\,B_R(0)\,,$$

Borel measurable multiplicity function

$$\theta=\sum_{N+1\leq j<k\leq N+M}c_{jk}\mathbf{1}_{\mathcal{X}(j,k)\setminus\mathrm{cl}\,B_R(0)}\,, \tag{4.14}$$

and the rectifiable varifold $V=\mathbf{var}\,(S,\theta)$ induced by S and θ. The assumption that $\cup_h\mathcal{E}(h)\subset\subset B_R(0)$ implies that for any $X\in C^\infty_c(\mathbb{R}^n\setminus\mathrm{cl}\,B_R(0);\mathbb{R}^n)$, every cluster in the one parameter family of (N,M)-clusters induced by $\mathcal{X}$ and the one parameter family of diffeomorphisms with initial velocity X has volume vector equal to $\mathbf{m}(\mathcal{X})$. Testing the $\mathbf{c}$-local minimality of $\mathcal{X}$ against this family gives (4.12). In terms of V, this shows that V is stationary in $\mathbb{R}^n\setminus\mathrm{cl}\,B_R(0)$. Denoting by $\|V\|$ the weight of V, we have $\|V\|(B_R(0))=0$ by construction, and thus

$$\delta V(X)=\int\mathrm{div}^S X(x)\,d\|V\|(x)=0\qquad\forall X\in C^\infty_c(\mathbb{R}^n\setminus\partial B_R(0);\mathbb{R}^n)\,. \tag{4.15}$$

Turning now towards (4.13), we first claim that there exists a Radon measure σ supported on $\partial B_R(0)$ and Borel measurable $\nu_V^{\mathrm{co}}:\partial B_R(0)\to\mathbb{R}^n$ with $|\nu_V^{\mathrm{co}}|=1$ σ-a.e. on $\partial B_R(0)$ such that

$$\delta V(X)=\int X(x)\cdot\nu_V^{\mathrm{co}}(x)\,d\sigma(x)\qquad\forall X\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)\,. \tag{4.16}$$

To prove (4.16), let $\{\varphi_\varepsilon\}_{\varepsilon>0}\subset C^\infty_c(\mathbb{R}^n;[0,1])$ be a family of cut-off functions such that $\varphi_\varepsilon=1$ on $\partial B_R(0)$, $\mathrm{spt}\,\varphi_\varepsilon\subset B_{R+2\varepsilon}(0)\setminus\mathrm{cl}\,B_{R-2\varepsilon}(0)$, and $\|\nabla\varphi\|_{L^\infty}\leq\varepsilon^{-1}$. Then by (4.15) and our differentiability assumption on R,

$$\begin{aligned}\delta V(X) &= \lim_{\varepsilon\to 0}\int \mathrm{div}^S\left[X(1-\varphi_\varepsilon)\right](x)\,d\|V\|(x) + \int \mathrm{div}^S\left[X\varphi_\varepsilon\right](x)\,d\|V\|(x)\\ &\le \limsup_{\varepsilon\to 0}\left[\varepsilon^{-1}\|X\|_{L^\infty} + \|\mathrm{div}^S X\|_{L^\infty}\right]\|V\|(B_{R+2\varepsilon}(0)\setminus \mathrm{cl}\,B_{R-2\varepsilon}(0))\\ &\le C(R,\mathcal{X})\|X\|_{L^\infty}\,. \end{aligned} \tag{4.17}$$

Thus V has bounded first variation in $\mathbb{R}^n$, and so by appealing to the Riesz representation theorem, we obtain a $\mathbb{R}^n$-valued Radon measure μ such that

$$\delta V(X) = \int X(x)\cdot d\mu(x) \qquad \forall X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)\,.$$

But by (4.15), $|\mu|(\mathbb{R}^n\setminus\partial B_R(0)) = 0$, which implies that, setting $\sigma = |\mu|$, σ is supported on $\partial B_R(0)$ and there exists $\nu_V^{\mathrm{co}} : \partial B_R(0)\to\mathbb{R}^n$ with $|\nu_V^{\mathrm{co}}| = 1$ σ-a.e. such that (4.16) holds. The monotonicity formula derived from (4.16) (following the computations in [30, Section 17]) with

$$C(R) = \frac{-1}{(n-1)}\int x\cdot\nu_V^{\mathrm{co}}(x)\,d\sigma(x)$$

can be found in [20, Theorem 2.7.(i)]; see also [32, Equation 1.8]. □

As an immediate corollary of the previous two results, we can justify Definition 2.6.

Corollary 4.5 (Density at infinity)**.** *If $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing (N,M)-cluster and $\cup_h\mathcal{E}(h)\subset B_R(0)$ for some $R>0$, as is the case for $N=1$ and $M=2$, then*

$$\Theta_\infty(\mathcal{X}) = \lim_{r\to\infty}\frac{P_{\mathbf{c}}(\mathcal{X};B_r(0))}{r^{n-1}} \tag{4.18}$$

exists and equals $\lim_{r\to\infty} r^{1-n}P_{\mathbf{c}}(\mathcal{X};B_r(0)\setminus\mathrm{cl}\,B_R(0))$ for every $R>0$.

The next corollary establishes the existence up to subsequences of blow-down clusters.

Corollary 4.6 (Existence of blow-down clusters)**.** *If $\mathbf{c}$ satisfies (pos.)-($\triangle$-ineq.), $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing (N,M)-cluster such that $\cup_h\mathcal{E}(h)\subset B_R(0)$ for some $R>0$, and $R_m\to\infty$ then, up to a subsequence, there exists $M'\le M$, subfamily $\mathcal{I} := \{j_1,\dots,j_{M'}\}\subset\{N+1,\dots,N+M\}$ and a conical $(0,M')$-cluster $\mathcal{X}_\infty$ such that, setting $\mathbf{c}' = \{c_{jk}\}_{j,k\in\mathcal{I}}$, $\mathcal{X}_\infty$ is $\mathbf{c}'$-locally minimizing,*

$$\mathcal{X}(j_\ell)/R_m \overset{\mathrm{loc}}{\to} \mathcal{X}_\infty(\ell) \quad \forall 1\le\ell\le M'\,, \qquad \textit{and} \tag{4.19}$$

$$P_{\mathbf{c}'}(\mathcal{X}_\infty;B_1(0)) = \Theta_\infty(\mathcal{X})\,. \tag{4.20}$$

Proof. By Corollary 4.5, $\sup_m P_{\mathbf{c}}(\mathcal{X}/R_m;B_R)<\infty$ for every $R>0$. Therefore, the existence of a (not relabelled) subsequence of radii $R_m\to\infty$, $M'\le M$, subfamily $\{j_1,\dots,j_{M'}\}\subset\{N+1,\dots,N+M\}$, and $(0,M')$-cluster $\mathcal{X}_\infty$ such that (4.19) holds is a consequence of (pos.) and compactness in BV. Furthermore, by (4.1) and (4.18), we have

$$P_{\mathbf{c}'}(\mathcal{X}_\infty;B_R(0)) \le \liminf_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}/R_m;B_R(0)) = \Theta_\infty(\mathcal{X})R^{n-1} \qquad \forall R>0\,. \tag{4.21}$$

We claim now that for any $(0,M')$-cluster $\mathcal{X}'$ with $\mathcal{X}'(\ell)\Delta\mathcal{X}_\infty(\ell)\subset\subset B_R(0)$ for some $R>0$ and each ℓ,

$$P_{\mathbf{c}'}(\mathcal{X}_\infty;B_R(0)) \le P_{\mathbf{c}'}(\mathcal{X}';B_R(0))\,. \tag{4.22}$$

To prove (4.22), choose $R'<R$ such that $\mathcal{X}'(\ell)\Delta\mathcal{X}_\infty(\ell)\subset\subset B_{R'}(0)$ for each ℓ and

$$\begin{aligned}&\mathcal{H}^{n-1}\big((\mathcal{X}(j)^{(1)}/R_m)\cap\partial B_{R'}(0)\big)\to 0 \quad \forall j\notin\mathcal{I}\,, \quad \text{and}\\ &\mathcal{H}^{n-1}\big(((\mathcal{X}(j_\ell)^{(1)}/R_m)\Delta\mathcal{X}_\infty(\ell)^{(1)})\cap\partial B_{R'}(0)\big)\to 0 \quad \text{for all } 1\le\ell\le M'\,;\end{aligned} \tag{4.23}$$

this is possible by the co-area formula and (4.19). Let $\mathcal{X}'_m$ be the (N,M)-clusters defined by

$$\mathcal{X}'_m(j) = \mathcal{X}(j) \quad \text{if } 1 \le j \le N\,,$$
$$\mathcal{X}'_m(j_\ell) = \big[R_m(\mathcal{X}'(\ell)\cap B_{R'}(0)) \cup (\mathcal{X}(j_\ell)\setminus B_{R_mR'}(0))\big]\setminus \cup_{1\le j\le N}\mathcal{X}(j) \quad \text{if } 1\le \ell\le M'\,,$$
$$\mathcal{X}'_m(j) = \mathcal{X}(j)\setminus B_{R_mR'}(0) \quad \text{if } N+1\le j\le N+M,\, j\notin \mathcal{I}\,.$$

Then by (4.1), (2.3) applied to the $\mathbf{c}$-local minimizers $\mathcal{X}/R_m$, and (3.6),

$$\begin{aligned}
P_{\mathbf{c}'}(\mathcal{X}_\infty; B_{R'}(0)) &\le \liminf_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}/R_m; B_{R'}(0)) \le \liminf_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}'_m/R_m; B_{R'}(0))\\
&\le \liminf_{m\to\infty}\Big[\sup_{c_{jk}} \sum_{1\le j\le N} P(\mathcal{X}(j))/R_m^{n-1}\Big] + P_{\mathbf{c}'}(\mathcal{X}'; B_{R'}(0))\\
&\quad + \liminf_{m\to\infty} \sup c_{jk} \sum_{j\in\{N+1,\dots,N+M\}\setminus\mathcal{I}} \mathcal{H}^{n-1}\big((\mathcal{X}(j)^{(1)}/R_m)\cap \partial B_{R'}(0)\big)\\
&\quad + \liminf_{m\to\infty} \sup c_{jk} \sum_{1\le \ell\le M'} \mathcal{H}^{n-1}\big(((\mathcal{X}(j_\ell)^{(1)}/R_m)\Delta \mathcal{X}_\infty(\ell)^{(1)})\cap\partial B_{R'}(0)\big)\,.
\end{aligned}$$

By the fact that $\sum_{j=1}^N P(\mathcal{X}(j))$ is finite and (4.23), (4.22) follows.

We now choose representatives for each $\mathcal{X}_\infty(\ell)$ such that $\operatorname{cl}\partial^*\mathcal{X}_\infty(\ell) = \partial\mathcal{X}_\infty(\ell)$ (see e.g. [19, Remark 15.3]). Since $\mathcal{X}_\infty$ also satisfies the minimality inequality (4.22), $\mathcal{X}_\infty$ is a $\mathbf{c}'$-locally minimizing $(0,M')$-cluster. It remains to prove (4.20) and that $\mathcal{X}_\infty$ is conical (each chamber is a cone). First, we claim that

$$P_{\mathbf{c}'}(\mathcal{X}_\infty; B_R(0)) = \Theta_\infty(\mathcal{X})R^{n-1} \qquad \text{for a.e. } R>0\,. \tag{4.24}$$

By (4.21), we know that $P_{\mathbf{c}'}(\mathcal{X}_\infty; B_R(0)) \le \Theta_\infty(\mathcal{X})R^{n-1}$ for all $R>0$. The reverse inequality is obtained by choosing R such that (4.23) holds at R (which is a.e. $R>0$) and recalling that

$$\Theta_\infty(\mathcal{X})R^{n-1} = \lim_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}; B_{R_mR}(0))/R_m^{n-1}\,.$$

We then test the minimality of $\mathcal{X}/R_m$ on $B_R(0)$ against the competitor which agrees with $\mathcal{X}_\infty$ inside $B_R(0)$ and $\mathcal{X}/R_m$ outside. By a similar argument as in the preceding paragraph, this yields

$$\Theta_\infty(\mathcal{X})R^{n-1} \le P_{\mathbf{c}'}(\mathcal{X}_\infty; B_R(0)) \quad \text{for a.e. } R>0\,,$$

which finishes the proof of (4.24). Now $\mathcal{X}_\infty$ is a $\mathbf{c}'$-locally minimizing $(0,M')$-cluster on all of space, and so the varifold

$$V = \mathbf{var}\Big(\bigcup_{1\le\ell<\ell'\le M'} \mathcal{X}_\infty(\ell,\ell'),\ \sum_{1\le\ell<\ell'\le M'} c_{j_\ell j_{\ell'}}\mathbf{1}_{\mathcal{X}_\infty(\ell,\ell')}\Big)$$

is stationary in all of space. Since V has constant energy density by (4.24) and is stationary, it is a cone, and thus so is $\mathcal{X}_\infty$. ☐

Remark 4.7 (Uniqueness of the blow-down cluster)**.** The uniqueness of blow-downs of a $\mathbf{c}$-locally minimizing cluster $\mathcal{X}$ is not known in full generality, although it should hold in some specific cases. For example, if a single blow-down has two non-empty chambers and interface which has an isolated singularity at the origin, then versions of the arguments in [4, 29, 12] adapted to blow-downs of exterior minimal surfaces (see also [31, pg 269]) should yield uniqueness. The next corollary proves uniqueness and decay when some blow-down is a pair of halfspaces.

Corollary 4.8 (Asymptotic expansion for $\mathbf{c}$-locally minimizing $(N,2)$-clusters with planar growth)**.** *If $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing $(N,2)$-cluster in $\mathbb{R}^n$ with bounded proper chambers, and in addition $\Theta_\infty(\mathcal{X}) = c_{(N+1)(N+2)}\omega_{n-1}$ if $n\ge 8$, then, up to a rotation, there exists $R_0>0$ such that on*

$\mathbb{R}^n \setminus C_{R_0}$, $\partial\mathcal{X} \setminus C_{R_0} = \mathcal{X}(N+1, N+2) \setminus C_{R_0}$ *coincides with the graph of a solution* $u : \mathbb{R}^{n-1} \setminus B^{n-1}_{R_0}(\overline{x})$ *to the minimal surface equation satisfying the expansion at infinity*

$$u(\overline{x}) = \begin{cases} a & \overline{x} \in \mathbb{R},\ n = 2\,, \\ a + \dfrac{b}{|\overline{x}|^{n-3}} + \dfrac{\overline{c}\cdot\overline{x}}{|\overline{x}|^{n-1}} + \mathrm{O}(|\overline{x}|^{1-n}) & \overline{x} \in \mathbb{R}^{n-1},\ n \geq 3\,, \end{cases} \tag{4.25}$$

for some $a, b \in \mathbb{R}$ *and* $\overline{c} \in \mathbb{R}^{n-1}$*.*

Remark 4.9 (Applications of the asymptotic expansion)**.** In the equal weights case, the proper chambers are bounded [25, Theorem 2.4]. Therefore, the asymptotic expansion (4.25) holds for locally minimizing $(N, 2)$-clusters, and could be used for example in studying the analogue of the N-bubble conjecture for $(N-1, 2)$-clusters.

Proof. Since $\mathcal{X}$ is a $\mathbf{c}$-locally minimizing $(N, 2)$-cluster in $\mathbb{R}^n$ with $\cup_h \mathcal{E}(h) \subset B_R(0)$ for some $R > 0$, $\mathcal{X}(N+1)$ and $\mathcal{X}(N+2)$ are perimeter minimizers in $\mathbb{R}^n \setminus \mathrm{cl}\, B_R(0)$. Since they both have infinite volume, we can choose $R_m \to \infty$ such that $\partial B_{R_m}(0) \cap \mathcal{X}(N+1, N+2) \neq \varnothing$ for all m. Then Corollary 4.6 implies that there exists a blow-down $\mathcal{X}_\infty$ which is a conical $\mathbf{c}$-locally minimizing $(0, M')$-cluster in $\mathbb{R}^n$, for $M' \in \{1, 2\}$. However, since $\Theta_\infty(\mathcal{X}) > 0$, (4.20) implies that we cannot have $M' = 1$. So we have a locally minimizing, conical $(0, 2)$-cluster in all of space: in other words, a pair of complementary entire perimeter minimizing cones. In $\mathbb{R}^n$ for $n < 8$, since all entire perimeter minimizers are halfspaces, $\mathcal{X}_\infty(N+1)$ and $\mathcal{X}_\infty(N+2)$ are complementary halfspaces. By Corollary 4.6, this gives $\Theta_\infty(\mathcal{X}) = c_{(N+1)(N+2)}\omega_{n-1}$ when $n < 8$. In $\mathbb{R}^n$ for $n \geq 8$, the same conclusion is a consequence of our assumption that $\Theta_\infty(\mathcal{X}) = c_{(N+1)(N+2)}\omega_{n-1}$ and Allard's theorem. We finish the proof in cases.

The case $n = 2$: Since $\mathcal{X}(N+1)$ and $\mathcal{X}(N+2)$ are perimeter minimizers in $\mathbb{R}^2 \setminus \cup_h \mathrm{cl}\, \mathcal{E}(h)$, $\mathcal{X}(N+1, N+2) \setminus \mathrm{cl}\, B_R(0)$ is a union of rays. Since $\Theta_\infty(\mathcal{X}) = c_{(N+1)(N+2)}\omega_{n-1}$, there can be only two such rays, and so every blow-down of $\mathcal{X}$ is the same pair of complementary halfspaces. Since $\mathcal{X}(N+1)$ and $\mathcal{X}(N+2)$ blow down to complementary halfspaces, each of them has one unbounded connected component, which we call C_1 and C_2, respectively. Let $\{C_\ell\}_{\ell>2}$ be all the other connected components of the other chambers in addition to any bounded components of $\mathcal{X}(N+1)$, $\mathcal{X}(N+1)$, $\partial\mathcal{X}(N+1)$ and $\partial\mathcal{X}(N+2)$. Then C_1 and C_2 are perimeter minimizers in the open set $\mathbb{R}^2 \setminus \mathrm{cl} \cup_{\ell>2} C_\ell$, and $\partial C_1 \cap \partial C_2$ blows down to a line, which we take to be $\{x_2 = 0\}$ by rotating. Thus we can find r_i, $a_i \in \mathbb{R}$ for $i = 1, 2$ such that $\cup_{\ell>2} C_\ell \subset [r_1, r_2] \times I$ for an interval $I \in \mathbb{R}$ and $\mathcal{X}(N+1, N+2) \setminus ([r_1, r_2] \times I)$ is the graph of a function $u : \mathbb{R} \setminus [r_1, r_2] \to \{a_1, a_2\}$ with

$$u(\overline{x}) = \begin{cases} a_1 & \overline{x} < r_1 \\ a_2 & \overline{x} > r_2 \end{cases}$$

and each $(r_i, a_i) \in \partial(\cup_{\ell>2} C_\ell)$. We must show that $a_1 = a_2$, as in [20, Proof of Theorem 1.1, step five] or [25, Figure 3].

Let s be the segment connecting (r_1, a_1) and (r_2, a_2). Let $r_1' = r_1 - 1$ and $r_2' = r_2 + 1$, and s' be the segment joining (r_1', a_1) and (r_2', a_2). We create a new cluster $\mathcal{X}'$ in the following fashion.

- Rotate $\cup_{\ell>2} C_\ell$ around the midpoint of s via a rotation map R so that $R(r_i, a_i) \in s'$ for $i = 1, 2$.
- Let $C' = R(\mathrm{cl} \cup_{\ell>2} C_\ell) \cup s'$, so that $C' \cup \{(t, a_1) : t \leq r_1'\} \cup \{(t, a_2) : t \geq r_2'\}$ disconnects $\mathbb{R}^2$ into two unbounded open sets. Call these C_1' and C_2' and add them to $\mathcal{X}'(N+1)$ and $\mathcal{X}'(N+2)$, respectively.

Direct computation shows that on any rectangle $\mathcal{R} = [r_1', r_2'] \times I'$ for large interval I',

$$\begin{aligned} &P_{\mathbf{c}}(\mathcal{X}; \mathcal{R}) - P_{\mathbf{c}}(\mathcal{X}'; \mathcal{R}) \\ &\qquad = c_{(N+1)(N+2)}\big[2 - \sqrt{(r_2 - r_1 + 2)^2 + (a_1 - a_2)^2} + \sqrt{(r_2 - r_1)^2 + (a_1 - a_2)^2}\big] \geq 0\,, \end{aligned}$$

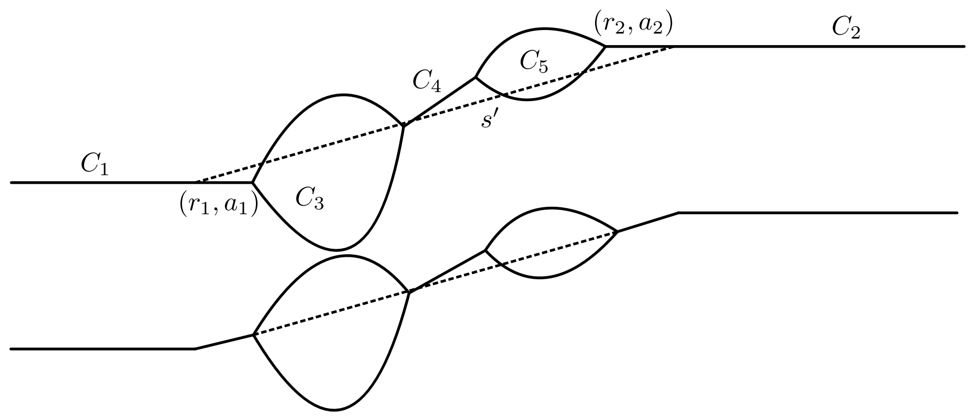

FIGURE 4.1. On top is the original configuration, and below is the rotated configuration with less energy. The dotted segment is s', and C_3, C_4, and C_5 are the two components of $\mathcal{X}(1)$ and the segment connecting them. They are all rotated around the midpoint of s' so that the rotated (r_1, a_1) and (r_2, a_2) lie on s'.

with equality if and only if $a_1 = a_2$ by the triangle inequality. By the $\mathbf{c}$-local minimality of $\mathcal{X}$, we deduce that $a_1 = a_2$.

The case $n \geq 4$: In this case, the rectifiable varifold $V = \mathbf{var}\,(\mathcal{X}(N+1, N+2), 1)$ is stationary in $\mathbb{R}^n \setminus B_R(0)$ for some $R > 0$, has planar growth at infinity, and, for some Radon measure σ on $\partial B_R(0)$ and unit-valued $\nu_V^{\rm co} : \partial B_R(0) \to \mathbb{S}^{n-1}$,

$$\delta V(X) = \int X(x) \cdot \nu_V^{\rm co}(x)\, d\sigma(x) \qquad \forall X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n) \tag{4.26}$$

by (4.16). These are precisely the assumptions necessary to apply [20, Theorem 2.1.(ii)] and obtain uniqueness of the tangent plane at infinity to the varifold V. Up to a rotation, we can assume this tangent plane is $\{x_n = 0\}$. Due to Allard's theorem and the convergence of V/R to $\mathbf{var}\,(\{x_n = 0\}, 1)$, V can be parametrized as a minimal graph over $\{x_n = 0\}$ outside some compact set. By [20, Proposition 4.1], which derives the asymptotic expansion for exterior solutions to the minimal surface equation following [28, Propositions 1 and 3], there exists $R_0 > 0$ such that $\mathcal{X}(N+1, N+2) \setminus C_{R_0}$ is the graph of $u : \mathbb{R}^{n-1} \setminus B_{R_0}^{n-1}(\overline{0}) \to \mathbb{R}$ satisfying the minimal surface equation and

$$u(\overline{x}) = a + \frac{b}{|\overline{x}|^{n-3}} + \frac{\overline{c} \cdot \overline{x}}{|\overline{x}|^{n-1}} + \mathrm{O}(|\overline{x}|^{1-n}) \qquad |\overline{x}| \geq R_0\,. \tag{4.27}$$

The case $n = 3$: The exact same argument as for $n \geq 4$ can be repeated verbatim, except that the expansion for the function u has a logarithmic term in the expansion which we must eliminate (cf. [20, Proposition 4.1]). So

$$u(\overline{x}) = a + b \ln |\overline{x}| + \frac{\overline{c} \cdot \overline{x}}{|\overline{x}|^2} + \mathrm{O}(|\overline{x}|^{-2}) \qquad |\overline{x}| \geq R_0 \tag{4.28}$$

and we would like to show that $b = 0$ using the minimality of $\mathcal{X}$. Assume for contradiction that $b \neq 0$. Then by the catenoidal growth in the expansion (4.28), the energy of $\mathcal{X}$ inside large cylinders can be estimated from below by

$$P_{\mathbf{c}}(\mathcal{X}; C_r) \geq c_{(N+1)(N+2)} \pi r^2 + C \ln r + \mathrm{O}(1) \tag{4.29}$$

for some $C > 0$. To make our comparison cluster, for each r, let Π_r be the plane containing the curve

$$\{(r\cos\theta, r\sin\theta, \overline{c} \cdot (\cos\theta, \sin\theta)/r) : 0 \leq \theta < 2\pi\}\,,$$

and let H_r^+ and H_r^- be the halfspaces with common boundary Π_r such that $\nu_{H_r^+} \cdot e_3 < 0$. Up to relabelling, we may assume that inside C_r, $\mathcal{X}(N+1)$ is "below" $\mathcal{X}(N+2)$, or more precisely, that $\mathcal{X}(N+1) \cap \partial C_r \subset \{(\overline{x}, t) : t \leq u(\overline{x}), |\overline{x}| = r\}$ and $\mathcal{X}(N+2) \cap \partial C_r \subset \{(\overline{x}, t) : t \geq u(\overline{x}), |\overline{x}| = r\}$. We define a new cluster $\mathcal{X}_r$ by setting

$$
\begin{aligned}
\mathcal{X}_r(j) &= \mathcal{X}(j) \quad \text{if } 1 \leq j \leq N \\
\mathcal{X}_r(N+1) &= (H_r^- \cap C_r \setminus \cup_{j=1}^N \mathcal{X}_r(j)) \cup \mathcal{X}(N+1) \setminus C_r \\
\mathcal{X}_r(N+2) &= (H_r^+ \cap C_r \setminus \cup_{j=1}^N \mathcal{X}_r(j)) \cup \mathcal{X}(N+2) \setminus C_r \,.
\end{aligned}
$$

Then for large r, $\mathcal{X}(j) \Delta \mathcal{X}_r(j) \subset\subset B_{r+1}^2(\overline{0}) \times (-r, r)$. Also, by (3.6) and (4.28) (which allow us to estimate the error coming from "gluing" along ∂C_r), we can compute

$$
\begin{aligned}
P_{\mathbf{c}}(\mathcal{X}_r; B_{r+1}^2(\overline{0}) \times (-r, r)) &\leq c_{(N+1)(N+2)} \int_{B_r^2(\overline{0})} \sqrt{1 + |\overline{c}|^2/r^4} \, d\overline{x} + 2\pi r (r^{-2}) \\
&\qquad + \sup c_{jk} \sum_{j=1}^N P(\mathcal{X}(j)) \\
&\leq c_{(N+1)(N+2)} \pi r^2 + \mathrm{O}(1) \,.
\end{aligned} \tag{4.30}
$$

For large r, the combination of (4.30) and (4.29) contradicts the $\mathbf{c}$-local minimality of $\mathcal{X}$. $\square$

Lemma 4.10 (Equivalent notions of minimality)**.** *If* $\mathbf{c}$ *satisfies* (pos.)-($\triangle$-ineq.) *and an* $(N, 2)$-*cluster* $\mathcal{X}$ *satisfies, for every* $r > 0$*,*

$$
P_{\mathbf{c}}(\mathcal{X}; B_r(0)) \leq P_{\mathbf{c}}(\mathcal{X}'; B_r(0)) \tag{4.31}
$$

for any $\mathcal{X}'$ *such that* $\mathcal{X}(j) \Delta \mathcal{X}'(j) \subset\subset B_r(0)$ *and* $|\mathcal{X}(j) \cap B_r(0)| = |\mathcal{X}'(j) \cap B_r(0)|$ *for each* $1 \leq j \leq N+2$*, then* $\mathcal{X}$ *is a* $\mathbf{c}$*-locally minimizing* $(N, 2)$*-cluster in the sense of Definition 2.5.*

Proof. The proof is divided into three main steps. After collecting a preliminary fact in step zero, in step one we prove an energy bound for $\mathcal{X}$ on any $B_r(x)$. Next, we use that bound and the fact that $|\mathcal{X}(N+1)| = \infty$ to locate balls $B_{R_k}(x_k)$ with $R_k \to \infty$ such that the volume fraction of $\mathcal{X}(N+1)$ on $B_{R_k}(x_k)$ is bounded away from 0 and 1. Finally, we use the volume fixing variations construction on $B_{R_k}(x_k)$ to fix the volumes of the improper chambers and prove (4.31).

Step zero: First, the local minimality condition (4.31) (with the volume of all the chambers being preserved on $B_r(0)$) is the same as the notion used in [18, Theorem 3.1] (see [18, pg. 8]), and so we may apply that theorem to find positive η and ρ such that

$$
\text{"if } r < \rho \text{ and } |B_r(x) \cap (\cup_{j \leq N} \mathcal{X}(j))| \leq \eta r^n, \text{ then } (\cup_{j \leq N} \mathcal{X}(j)) \cap B_{r/2}(x) = \varnothing\text{"} \,. \tag{4.32}
$$

As a consequence of (4.32) and the fact that $|\mathcal{X}(j)| < \infty$ for $j \leq N$, $\mathcal{X}(j)$ is bounded if $j \leq N$.

Step one: Here we prove that if $x \in \mathbb{R}^n$, $r > 0$, and $\rho_i \nearrow r$, then for any $1 \leq J \leq N+2$ such that $|\mathcal{X}(J) \cap B_r(x)| > 0$,

$$
P_{\mathbf{c}}(\mathcal{X}; B_r(x)) \leq C(N, n, \mathbf{c}) \big[|B_r(x) \setminus \mathcal{X}(J)|^{(n-1)/n} + \limsup_{i \to \infty} \mathcal{H}^{n-1}(\partial B_{\rho_i}(x) \setminus \mathcal{X}(J)^{(1)}) \big] \,. \tag{4.33}
$$

For later use, we note that (4.33) also implies

$$
P_{\mathbf{c}}(\mathcal{X}; B_r(x)) \leq C r^{n-1} \quad \text{for all } x \in \mathbb{R}^n, \, r > 0 \,. \tag{4.34}
$$

Since $|\mathcal{X}(J) \cap B_r(x)| > 0$ and $\rho_i \nearrow r$, by restricting to a tail of $\{\rho_i\}_i$ we may assume that $|\mathcal{X}(J) \cap B_{\rho_i}(x)| > 0$ for all i. For each i, define a new cluster $\mathcal{X}_i$ by setting

$$
\mathcal{X}_i(j) \cap B_{\rho_i}(x) = A_j^i \,, \qquad \mathcal{X}_i(j) \setminus B_{\rho_i}(x) = \mathcal{X}(j) \setminus B_{\rho_i}(x)
$$

where A_j^i are pairwise disjoint open annuli centered at x satisfying two conditions: first, $|A_j^i \cap B_{\rho_i}(x)| = |\mathcal{X}(j) \cap B_{\rho_i}(x)|$, and second that the outermost annulus is A_J^i. With $\mathcal{X}_i$ defined as such, the largest individual contribution to $P_{\mathbf{c}}(\mathcal{X}; B_{\rho_i}(x))$ comes from the inner boundary of the outermost annulus A^i_J, which in turn depends on $|B_{\rho_i}(x)\setminus\mathcal{X}(J)|$. More precisely, we may estimate

$$\begin{aligned} P_{\mathbf{c}}(\mathcal{X}_i; B_{\rho_i}(x)) &\leq \sup_{j,k} c_{jk} \sum_j \mathcal{H}^{n-1}(\partial A^i_j \cap B_{\rho_i}(x)) \\ &\leq (N+1)(\sup c_{jk}) n\, \omega_n^{1/n} |B_{\rho_i}(x)\setminus \mathcal{X}(J)|^{(n-1)/n} . \end{aligned} \tag{4.35}$$

Now since $A^i_J \neq \varnothing$ and borders $\partial B_{\rho_i}(x)$, the Lebesgue density of A^i_J and thus $\mathcal{X}_i(J)$ is at least $1/2$ for all $y \in \partial B_{\rho_i}(x)$. Therefore by (2.1) and $\partial^*\mathcal{X}_i(J) \overset{\mathcal{H}^{n-1}}{=} \mathcal{X}_i(J)^{(1/2)}$, $\partial B_{\rho_i}(x) \overset{\mathcal{H}^n}{\subset} \mathcal{X}_i(J)^{(1)} \cup \partial^*\mathcal{X}_i(J)$. Furthermore, the Lesbesgue density of $\mathcal{X}_i(J)$ at $y \in \partial B_{\rho_i}(x)$ is at least that of $\mathcal{X}(J)$ at y, since $\mathcal{X}_i(J) = \mathcal{X}(J) \cup A^i_J$ in a neighborhood of $\partial B_{\rho_i}(x)$. Thus $\mathcal{X}(J)^{(1)} \cap B_{\rho_i}(x) \subset \mathcal{X}_i(J)^{(1)} \cap B_{\rho_i}(x)$. Putting these observations together, we estimate

$$P_{\mathbf{c}}(\mathcal{X}_i; \partial B_{\rho_i}(x)) \leq \sup_{j,k} c_{jk} \mathcal{H}^{n-1}(\partial B_{\rho_i}(x) \setminus \mathcal{X}_i(J)^{(1)}) \leq \sup_{j,k} c_{jk} \mathcal{H}^{n-1}(\partial B_{\rho_i}(x) \setminus \mathcal{X}(J)^{(1)}) . \tag{4.36}$$

Then by the minimality (4.31) and (4.35)-(4.36),

$$\begin{aligned} P_{\mathbf{c}}(\mathcal{X}; B_r(x)) \leq P_{\mathbf{c}}(\mathcal{X}_i; B_r(x)) &= P_{\mathbf{c}}(\mathcal{X}_i; B_{\rho_i}(x)) + P_{\mathbf{c}}(\mathcal{X}_i; \partial B_{\rho_i}(x)) + P_{\mathbf{c}}(\mathcal{X}; B_r(x) \setminus \mathrm{cl}\, B_{\rho_i}(x)) \\ &\leq C(N,n,\mathbf{c}) \big[|B_{\rho_i}(x) \setminus \mathcal{X}(J)|^{(n-1)/n} + \mathcal{H}^{n-1}(\partial B_{\rho_i}(x) \setminus \mathcal{X}(J)^{(1)}) \big] \\ &\qquad + P_{\mathbf{c}}(\mathcal{X}; B_r(x) \setminus \mathrm{cl}\, B_{\rho_i}(x)) . \end{aligned}$$

Since $\mathcal{X}(j)$ are sets of locally finite perimeter, $P_{\mathbf{c}}(\mathcal{X}; B_r(x) \setminus \mathrm{cl}\, B_{\rho_i}(x)) \to 0$ as $\rho_i \to r$. Therefore, sending $i \to \infty$ concludes the proof of (4.33).

Step two: In this step we show the following claim, which we use to finish the proof in step three.

Claim: There exists $\delta \in (0, 1/2)$ and sequences $R_k \to \infty$ and $\{x_k\}_k \subset \mathbb{R}^n$ such that

$$\delta \omega_n R_k^n \leq |\mathcal{X}(N+1) \cap B_{R_k}(x_k)| \leq (1-\delta)\omega_n R_k^n \quad \forall k . \tag{4.37}$$

To prove (4.37): If (4.37) holds with $x = 0$ and any sequence R_k diverging to infinity, we are done. So it suffices to consider the case that

$$\min\big\{ |\mathcal{X}(N+1) \cap B_R(0)|/(\omega_n R^n)\ ,\ |B_R(0) \setminus \mathcal{X}(N+1)|/(\omega_n R^n) \big\} \to 0 \quad \text{as } R \to \infty . \tag{4.38}$$

Since $R \mapsto |\mathcal{X}(N+1) \cap B_R(0)|/(\omega_n R^n)$ is continuous in R, if (4.38) holds, then actually only one of $|\mathcal{X}(N+1) \cap B_R(0)|/(\omega_n R^n) \to 0$ or $|B_R(0) \setminus \mathcal{X}(N+1)|/(\omega_n R^n) \to 0$ holds along the whole sequence. Furthermore, since $|\mathcal{X}(j)| < \infty$ for $j \notin \{N+1, N+2\}$, the former case entails $|B_R(0) \setminus \mathcal{X}(N+2)|/(\omega_n R^n) \to 0$, while the latter entails $|\mathcal{X}(N+2) \cap B_R(0)|/(\omega_n R^n) \to 0$. So by interchanging the labels on $\mathcal{X}(N+1)$ and $\mathcal{X}(N+2)$ if necessary, we may as well assume that

$$|\mathcal{X}(N+1) \cap B_R(0)|/(\omega_n R^n) \to 0 \quad \text{as } R \to \infty \tag{4.39}$$

and prove (4.37). The idea is to use the energy bound (4.33) to find $R_k \to \infty$ such that

$$P(\mathcal{X}(N+1) \cap B_{R_k}(0)) \lesssim |\mathcal{X}(N+1) \cap B_{R_k}(0)|^{(n-1)/n} . \tag{4.40}$$

(4.40) implies that the rescaled unit volume sets $E_k := (\mathcal{X}(N+1) \cap B_{R_k}(0))/|\mathcal{X}(N+1) \cap B_{R_k}(0)|^{1/n}$ have uniformly bounded perimeters. Such a uniform perimeter bound is only possible if these rescalings are not too "sparse"; that is, $\max_{x\in\mathbb{R}^n} |E_k \cap B_1(x)|$ is bounded away from 0 uniformly in k. These "good" balls on which the volume fraction of E_k is non-negligible will rescale to be our $B_{R_k}(x_k)$, with $R_k \to \infty$ since $|\mathcal{X}(N+1)| = \infty$.

First, we obtain a rewritten version of (4.33). Let $m(r) = |B_r(0) \cap \mathcal{X}(N+1)|$. By the co-area formula, m is differentiable almost everywhere, with derivative belonging to $L^1_{\mathrm{loc}}(\mathbb{R})$ and given by $m'(r) = \mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_r(0))$ for almost every r. Then almost every r is a Lebesgue point of m'. For such a Lebesgue point r, we can find a sequence $\rho_i \nearrow r$ as $i \to \infty$ such that $m'(\rho_i) \to m'(r)$

as $i \to \infty$. Inserting these ρ_i into the estimate (4.33) with $J = N+2$, we obtain for almost every $r > 0$

$$P_{\mathbf{c}}(\mathcal{X}; B_r(0)) \le C(N, n, \mathbf{c})\big[|B_r(0) \setminus \mathcal{X}(N+2)|^{(n-1)/n} + \mathcal{H}^{n-1}(\partial B_r(0) \setminus \mathcal{X}(N+2)^{(1)})\big]\,. \quad (4.41)$$

Now since $|\mathcal{X}(N+1)| = \infty$ and $|\mathcal{X}(j)| < \infty$ for $1 \le j \le N$, we can choose $R_0 > 0$ such that

$$|B_r(0) \setminus \mathcal{X}(N+2)|^{(n-1)/n} \le 2|B_r(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} \quad \forall r > R_0\,. \quad (4.42)$$

Also, by the boundedness of the proper chambers from step zero and (2.1), there exists $R_1 > R_0$ such that

$$\partial B_r(0) \overset{\mathcal{H}^n}{\subset} \mathcal{X}(N+1)^{(1)} \cup \mathcal{X}(N+2)^{(1)} \cup \mathcal{X}(N+1, N+2) \quad \forall r > R_1\,. \quad (4.43)$$

Since $\mathcal{H}^{n-1} \llcorner \mathcal{X}(N+1, N+2)$ is locally finite, $\mathcal{H}^{n-1}(\mathcal{X}(N+1, N+2) \cap \partial B_r(0)) = 0$ for all but countably many $r > 0$. So

$$\partial B_r(0) \overset{\mathcal{H}^n}{\subset} \mathcal{X}(N+1)^{(1)} \cup \mathcal{X}(N+2)^{(1)} \quad \text{for a.e. } r > R_1\,, \quad (4.44)$$

in which case

$$\mathcal{H}^{n-1}(\partial B_r(0) \setminus \mathcal{X}(N+2)^{(1)}) = \mathcal{H}^{n-1}(\partial B_r(0) \cap \mathcal{X}(N+1)^{(1)}) \quad \text{for a.e. } r > R_1\,. \quad (4.45)$$

By plugging (4.42) and (4.45) into (4.41), for almost every $r > R_1$ we have

$$P_{\mathbf{c}}(\mathcal{X}; B_r(0)) \le C(N, n, \mathbf{c})\big[2|B_r(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} + \mathcal{H}^{n-1}(\partial B_r(0) \cap \mathcal{X}(N+1)^{(1)})\big] \quad (4.46)$$

as well as (4.44). As a last modification of this energy bound, by (pos.) and (2.1) we have

$$P(\mathcal{X}(N+1); B_r(0)) \le \max c_{jk}^{-1} P_{\mathbf{c}}(\mathcal{X}; B_r(0))\,.$$

Thus up to changing the values of the multiplicative constants in (4.46), for almost every $r > R_1$,

$$P(\mathcal{X}(N+1); B_r(0)) \le C\big[|B_r(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} + \mathcal{H}^{n-1}(\partial B_r(0) \cap \mathcal{X}(N+1)^{(1)})\big] \text{ and} \quad (4.47)$$

$$\partial B_r(0) \overset{\mathcal{H}^n}{\subset} \mathcal{X}(N+1)^{(1)} \cup \mathcal{X}(N+2)^{(1)}\,. \quad (4.48)$$

We claim now that we can choose $r_k \to \infty$ such that (4.47)-(4.48) hold for all k and

$$(C + 1/2)\mathcal{H}^{n-1}(\partial B_{r_k}(0) \cap \mathcal{X}(N+1)^{(1)}) \le P(\mathcal{X}(N+1); B_{r_k}(0))/2\,. \quad (4.49)$$

Indeed, if this were not possible, then there would be $R_2 > R_1$ such that for almost every $r > R_2$, $P(\mathcal{X}(N+1); B_r(0))/2 < (C+1/2)\mathcal{H}^{n-1}(\partial B_r(0) \cap \mathcal{X}(N+1)^{(1)})$. Adding $\mathcal{H}^{n-1}(\partial B_r(0) \cap \mathcal{X}(N+1)^{(1)})/2$ to both sides of this inequality and using (3.5) (which applies by (4.48)) gives

$$\begin{aligned} P(\mathcal{X}(N+1) \cap B_r(0))/2 &= P(\mathcal{X}(N+1); B_r(0))/2 + \mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_r(0))/2 \\ &\le (C+1)\mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_r(0)) \quad \text{for a.e. } r > R_2\,. \end{aligned} \quad (4.50)$$

By recalling that $m'(r) = \mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_r(0))$ for almost every r and using the isoperimetric inequality on the left hand side of (4.50), this inequality rewrites as

$$n\omega_n^{1/n} m(r)^{(n-1)/n}/2 \le (C+1)m'(r) \quad \text{for a.e. } r > R_2\,. \quad (4.51)$$

Since $|\mathcal{X}(N+1)| = \infty$, then $m(r) > 0$ on some interval $(R_3, \infty) \subset (R_2, \infty)$, and we may integrate (4.51) on (R_3, ρ) to find

$$C_1\rho + C_2 \le m(\rho)^{1/n} \quad \text{for a.e. } \rho > R_3 \quad (4.52)$$

and some constants C_1 and C_2 with $C_1 > 0$. But (4.52) implies that $m(\rho)$ grows like ρ^n as $\rho \to \infty$, which contradicts (4.39). Concluding our contradiction argument, it must be the case that we can in fact choose $r_k \to \infty$ such that (4.47), (4.48), and (4.49) hold for all r_k.

Now for these $r_k \to \infty$ we have chosen, we add $\mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_{r_k}(0))/2$ to both sides of (4.47), yielding

$$P(\mathcal{X}(N+1); B_{r_k}(0)) + \mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_{r_k}(0))/2 \\ \leq C|B_{r_k}(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} + (C+1/2)\mathcal{H}^{n-1}(\partial B_{r_k}(0) \cap \mathcal{X}(N+1)^{(1)}) .$$

Due to the inequality (4.49), this simplifies to

$$\big[P(\mathcal{X}(N+1); B_{r_k}(0)) + \mathcal{H}^{n-1}(\mathcal{X}(N+1)^{(1)} \cap \partial B_{r_k}(0))\big]/2 \leq C|B_{r_k}(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} .$$

By the containment in (4.48) and (3.5), the left hand side is $P(\mathcal{X}(N+1) \cap B_{r_k}(0))/2$, so we have

$$P(\mathcal{X}(N+1) \cap B_{r_k}(0)) \leq 2C|B_{r_k}(0) \cap \mathcal{X}(N+1)|^{(n-1)/n} . \tag{4.53}$$

Now, setting $s_k = |\mathcal{X}(N+1) \cap B_{r_k}(0))|^{1/n}$ we define the rescaled sets of finite perimeter

$$E_k = (\mathcal{X}(N+1) \cap B_{r_k}(0))/s_k .$$

By the definition of s_k, the isoperimetric inequality and (4.53), and $|\mathcal{X}(N+1)| = \infty$ and (4.39),

$$|E_k| = 1 , \quad E_k \subset B_{r_k/s_k}(0) , \tag{4.54}$$

$$C(n) \leq P(E_k) = s_k^{1-n} P(\mathcal{X}(N+1) \cap B_{r_k}(0)) \leq 2C \quad \text{for some } C(n) , \tag{4.55}$$

$$s_k \to \infty \quad \text{and} \quad s_k/r_k \to 0 . \tag{4.56}$$

By the statement of Almgren's nucleation lemma [5, VI.13] from [19, Lemma 29.10], (4.54)-(4.55) yield the existence of a dimensional constant $c(n)$ such that, setting $\varepsilon = \min\{1, C(n)/(2nc(n))\}/2$, there exist $y_k \in \mathbb{R}^n$ such that

$$|E_k \cap B_1(y_k)| \geq \left(c(n)\frac{\varepsilon}{P(E_k)}\right)^n \geq \left(c(n)\frac{\varepsilon}{2C}\right)^n . \tag{4.57}$$

Set

$$\delta = \min\{(c(n)\varepsilon/(2C))^n/\omega_n, 1/4\} .$$

Rescaling back by s_k and setting $x_k = s_k y_k$, (4.57) says that in terms of $\mathcal{X}(N+1) \cap B_{r_k}$,

$$|\mathcal{X}(N+1) \cap B_{s_k}(x_k)| \geq |\mathcal{X}(N+1) \cap B_{r_k}(0) \cap B_{s_k}(x_k)| \geq \delta\omega_n s_k^n , \tag{4.58}$$

where $\delta \in (0, 1/2)$.

We may now finish the proof of (4.37). If

$$|\mathcal{X}(N+1) \cap B_{s_k}(x_k)| \leq (1-\delta)\omega_n s_k^n , \tag{4.59}$$

then we set $s_k = R_k$, and (4.37) follows from (4.58), (4.59), and (4.56). Suppose instead that

$$|\mathcal{X}(N+1) \cap B_{s_k}(x_k)| > (1-\delta)\omega_n s_k^n . \tag{4.60}$$

We observe that since $\mathcal{X}(N+1) \cap B_{r_k}(0) \cap B_{s_k}(x_k) \neq \varnothing$ and $s_k/r_k \to 0$ (from (4.56)), it must be the case that $x_k \in B_{2r_k}(0)$ for large k. By combining this containment with (4.39), we estimate

$$\frac{|\mathcal{X}(N+1) \cap B_{r_k}(x_k)|}{\omega_n r_k^n} \leq 3^n \frac{|\mathcal{X}(N+1) \cap B_{3r_k}(0)|}{\omega_n (3r_k)^n} \to 0 \quad \text{as } k \to \infty . \tag{4.61}$$

By the continuity of $r \mapsto |\mathcal{X}(N+1) \cap B_r(x_k)|/(\omega_n r^n)$, (4.60) and (4.61) allow us to choose $R_k \in (s_k, r_k)$ for large k such that

$$|\mathcal{X}(N+1) \cap B_{R_k}(x_k)| = \omega_n R_k^n/2 , \tag{4.62}$$

which also implies (4.37).

Step three: By rescaling the density bound (4.37) and the energy bound (4.34), the sets $F_k := (\mathcal{X}(N+1) - x_k)/R_k$ satisfy

$$\delta\omega_n \le |F_k \cap B_1(0)| \le (1-\delta)\omega_n \quad \text{and} \quad \sup_k P(F_k; B_r(0)) \le Cr^{n-1} \quad \forall r > 0\,. \tag{4.63}$$

After restricting to a further subsequence and using the facts that $\cup_{j\le N}\mathcal{X}(j) \subset\subset B_R(0)$ for some $R > 0$ and $R_k \to \infty$, we may choose $z \in \mathrm{cl}\, B_1(0)$ and $t_k \to 0$ such that

$$B_1(0) \cap \big[(\cup_{j\le N}\mathcal{X}(j) - x_k)/R_k\big] \subset B_1(0) \cap \big[(B_R(0) - x_k)/R_k\big] \subset B_{t_k}(z) \quad \text{for large } k\,; \tag{4.64}$$

the left hand side is empty if $\cup_{j\le N}\mathcal{X}(j) \cap B_{R_k}(x_k) = \varnothing$. From the compactness for sets of finite perimeter, there exists a limiting set of locally finite perimeter F that also satisfies the density bound on $B_1(0)$ from (4.63). In particular, $\partial^* F \cap B_1(0) \neq \varnothing$, which means that we can choose $x \in \partial^* F \cap B_1(0)$ and $t > 0$ such that $B_{t_k}(z) \cap B_t(x) = \varnothing$ for large k. Thus for large k,

$$\cup_{j\le N}\mathcal{X}(j) \subset\subset B_R(0) \subset B_{R_k t_k}(R_k z + x_k) \subset B_{R_k t}(R_k x + x_k)^c\,.$$

By the volume fixing variations construction [19, Theorem 29.14] applied to $\{F_k\}_k$, we may obtain $\sigma_0 > 0$, $C > 0$, and $K_0 \in \mathbb{N}$ such that for any $\sigma \in [\sigma_0, \sigma_0]$ and $k \ge K_0$, there exists F_k^σ such that $F_k^\sigma \Delta F_k \subset\subset B_t(x)$,

$$\begin{aligned} &|F_k^\sigma \cap B_t(x)| = |F_k \cap B_t(x)| + \sigma \qquad \text{and} \\ &P(F_k^\sigma; B_t(x)) \le P(F_k; B_t(x)) + C|\sigma|\,. \end{aligned}$$

Now let $\mathcal{X}'$ be an $(N,2)$-cluster such that $\mathcal{X}'(j)\Delta\mathcal{X}(j) \subset B_r(0)$ for some $r > 0$ and $|\mathcal{X}(j)| = |\mathcal{X}'(j)|$ for each j. It suffices to prove (4.31) on $B_R(0)$. Let $\sigma_k = (|\mathcal{X}(N+1) \cap B_R(0)| - |\mathcal{X}'(N+1) \cap B_R(0)|)/R_k^n$, and define the $(N,2)$-clusters $\mathcal{X}_k$ for k large enough such that $B_R(0) \cap B_{R_k t}(R_k x + x_k) = \varnothing$ (in other words, the rescaled and translated volume fixing variations $R_k F_k^\sigma + x_k$ do not disturb the proper chambers) and $|\sigma_k| \le \sigma_0$ by

$$\begin{aligned} \mathcal{X}_k(j) &= \mathcal{X}'(j) \ \text{ for } j \le N\,, \\ \mathcal{X}_k(N+1) &= \big[\mathcal{X}'(N+1) \cap B_R(0)\big] \cup \big[(R_k F_k^{\sigma_k} + x_k) \cap B_{R_k t}(R_k x + x_k)\big] \\ &\quad \cup \big[\mathcal{X}(N+1) \cap B_R(0)^c \cap B_{R_k t}(R_k x + x_k)^c\big] \\ \mathcal{X}_k(N+2) &= \big[\mathcal{X}'(N+2) \cap B_R(0)\big] \cup \big[B_{R_k t}(R_k x + x_k) \setminus (R_k F_k^{\sigma_k} + x_k)\big] \\ &\quad \cup \big[\mathcal{X}(N+2) \cap B_R(0)^c \cap B_{R_k t}(R_k x + x_k)^c\big]\,. \end{aligned}$$

Then $\mathcal{X}_k(j)\Delta\mathcal{X}(j) \subset\subset B_R(0) \cup B_{R_k t}(x_k + R_k x)$ for each j. Let $B_{s_k}(0)$ be large balls containing $B_R(0) \cup B_{R_k t}(x_k + R_k x)$. Due to the facts that for $j \le N$, $\mathcal{X}(j) \subset\subset B_R(0)$, $|\mathcal{X}(j)| = |\mathcal{X}'(j)|$, and $\mathcal{X}'(j)\Delta\mathcal{X}(j) \subset\subset B_R(0)$, we can compute

$$|\mathcal{X}_k(j) \cap B_{s_k}(0)| = |\mathcal{X}_k(j) \cap B_R(0)| = |\mathcal{X}(j) \cap B_R(0)| = |\mathcal{X}(j) \cap B_{s_k}(0)| \quad \forall j \le N\,.$$

Furthermore, using in order $\mathcal{X}_k(N+1)\Delta\mathcal{X}(N+1) \subset\subset B_R(0) \cup B_{R_k t}(x_k + R_k x)$ and the definition of $F_k^{\sigma_k}$ followed by the definition of F_k and our choice of σ_k,

$$\begin{aligned} &|\mathcal{X}_k(N{+}1) \cap B_{s_k}(0)| \\ &\quad = |\mathcal{X}'(N+1) \cap B_R(0)| + |(R_k F_k^{\sigma_k} + x_k) \cap B_{R_k t}(x_k + R_k x)| \\ &\qquad + |\mathcal{X}(N+1) \cap B_{s_k}(0) \cap B_R(0)^c \cap B_{R_k t}(x_k + R_k x)^c| \\ &\quad = |\mathcal{X}'(N+1) \cap B_R(0)| + R_k^n\big(|F_k \cap B_t(x)| + \sigma_k\big) \\ &\qquad + |\mathcal{X}(N+1) \cap B_{s_k}(0) \cap B_R(0)^c \cap B_{R_k t}(x_k + R_k x)^c| \\ &\quad = |\mathcal{X}'(N+1) \cap B_R(0)| + |\mathcal{X}(N+1) \cap B_{R_k t}(x_k + R_k x)| + |\mathcal{X}(N+1) \cap B_R(0)| \\ &\qquad - |\mathcal{X}'(N+1) \cap B_R(0)| + |\mathcal{X}(N+1) \cap B_{s_k}(0) \cap B_R(0)^c \cap B_{R_k t}(x_k + R_k x)^c| \\ &\quad = |\mathcal{X}(N+1) \cap B_{s_k}(0)|\,. \end{aligned}$$

So $|\mathcal{X}_k(j)\cap B_{s_k}(0)| = |\mathcal{X}(j)\cap B_{s_k}(0)|$ for $j \le N+1$, and thus for $j = N+2$ also since $\{\mathcal{X}(j)\cap B_{s_k}(0)\}_j$ partition $B_{s_k}(0)$. We may therefore test the volume-constrained minimality of $\mathcal{X}$ against $\mathcal{X}_k$ on $B_{s_k}(0)$ and cancel like terms, yielding

$$\begin{aligned}P_{\mathbf{c}}(\mathcal{X}; B_R(0)) + c_{(N+1)(N+2)}P(\mathcal{X}(N+1); B_{R_k t}(x_k + R_k x))\\ &\le P_{\mathbf{c}}(\mathcal{X}'; B_R(0)) + c_{(N+1)(N+2)}P(R_k F_k^{\sigma_k} + x_k; B_{R_k t}(x_k + R_k x))\\ &\le P_{\mathbf{c}}(\mathcal{X}'; B_R(0)) + c_{(N+1)(N+2)}P(\mathcal{X}(N+1); B_{R_k t}(x_k + R_k x))\\ &\qquad + R_k^{n-1}C\big||\mathcal{X}(N+1)\cap B_R(0)| - |\mathcal{X}'(N+1)\cap B_R(0)|\big|R_k^{-n}\,.\end{aligned}$$

By subtracting $c_{(N+1)(N+2)}P(\mathcal{X}(N+1); B_{R_k t}(x_k + R_k x))$ from both sides and sending $k \to \infty$, we conclude that $P_{\mathbf{c}}(\mathcal{X}; B_R(0)) \le P_{\mathbf{c}}(\mathcal{X}'; B_R(0))$. Thus $\mathcal{X}$ is $\mathbf{c}$-locally minimizing in the sense of Definition 2.5. $\square$

In [25], the authors prove a closure theorem [25, Theorem 2.17] for $\mathbf{c}$-locally minimizing clusters when $\mathbf{c}$ is a family of equal positive weights under the assumption that for every pair of indices $j \ne k$ corresponding to improper chambers of the limiting cluster $\mathcal{X}$, there exists an $(n-1)$-dimensional halfspace contained in $\mathcal{X}(j,k)$ [25, Definition 2.15]. For the case where there are two improper chambers, the following corollary of Lemma 4.10 removes this assumption on the limiting cluster.

Corollary 4.11 (Closure for locally minimizing clusters)**.** *If $\mathbf{c}_{\mathrm{Id}}$ is the family of equal unit weights, $\{\mathcal{X}_m\}_m$ is a sequence of $(N', N+2-N')$-clusters for some $N' < N+2$ such that, for every open $A \subset\subset \mathbb{R}^n$,*

$$P_{\mathbf{c}_{\mathrm{Id}}}(\mathcal{X}_m; A) \le P_{\mathbf{c}_{\mathrm{Id}}}(\mathcal{X}'; A) \tag{4.65}$$

whenever $\mathcal{X}'(j)\Delta\mathcal{X}_m(j) \subset\subset A$ and $|\mathcal{X}'(j)\cap A| = |\mathcal{X}_m(j)\cap A|$ for $1 \le j \le N+2$, and there exists an $(N,2)$-cluster $\mathcal{X}$ such that $\mathcal{X}_m(j) \overset{\mathrm{loc}}{\to} \mathcal{X}(j)$ for $1 \le j \le N+2$, then $\mathcal{X}$ is a $\mathbf{c}_{\mathrm{Id}}$-locally minimizing cluster in the sense of Definition 2.5.

Proof. In the terminology of Novaga-Paolini-Tortorelli [25, Definition 2.12], the minimality condition (4.65) means that each $\mathcal{X}_m$ is a locally J-isoperimetric $(N+2)$-partition of $\mathbb{R}^n$, where $J = \{1, \dots, N+2\}$ refers to the indices of the chambers subject to volume constraints. By the closure theorem for J-isoperimetric partitions [25, Theorem 2.13], $\mathcal{X}$ is a locally J-isoperimetric $(N+2)$-partition of $\mathbb{R}^n$, or in the language of Lemma 4.10, an $(N,2)$-cluster satisfying (4.31) for the energy $P_{\mathbf{c}_{\mathrm{Id}}}$. By the conclusion of Lemma 4.10, $\mathcal{X}$ is a $\mathbf{c}_{\mathrm{Id}}$-locally minimizing $(N,2)$-cluster. $\square$

The previous corollary should hold for any family of weights $\mathbf{c}$ satisfying (pos.)-($\triangle$-ineq.) as well, although we do not pursue this here.

5. Proof of Theorems 2.8 and 2.9

Proof of Theorem 2.8. Let $\mathcal{X}$ be a $(1,2)$-cluster such that $\mathcal{X}(j)\Delta\mathcal{X}_{\mathrm{lens}}(j) \subset\subset B_r(0)$ for some $r > 0$ and each j, and $|\mathcal{X}(1)| = 1$. We must show that $P_{\mathbf{c}}(\mathcal{X}_{\mathrm{lens}}; B_r(0)) \le P_{\mathbf{c}}(\mathcal{X}; B_r(0))$. Since $\mathcal{X}_{\mathrm{lens}}(1)$ is bounded by its definition, we choose $R > r$ such that $\mathcal{X}_{\mathrm{lens}}(1) \subset\subset B_R(0)$ and claim it is enough to show that

$$P_{\mathbf{c}}(\mathcal{X}_{\mathrm{lens}}; B_R(0)) \le P_{\mathbf{c}}(\mathcal{X}; B_R(0))\,. \tag{5.1}$$

Indeed, by our assumption on $\mathcal{X}$, $P_{\mathbf{c}}(\mathcal{X}_{\mathrm{lens}}; B_R(0)\setminus B_r(0)) = P_{\mathbf{c}}(\mathcal{X}; B_R(0)\setminus B_r(0))$, so (5.1) gives the desired minimality on $B_r(0)$. For each $m > 0$, let $B_{r_m}(y_m)$ be the ball such that $\partial\mathcal{X}^m_{\mathrm{bub}}(2) \subset \partial B_{r_m}(y_m)$, where $\mathcal{X}^m_{\mathrm{bub}}$ is the standard weighted double bubble from Definition 3.5 with volume vector $(1, m, \infty)$. We set

$$\sigma_m = |\mathcal{X}^m_{\mathrm{bub}}(2)\cap B_R(0)| - |\mathcal{X}(2)\cap B_R(0)|\,. \tag{5.2}$$

By the volume fixing variations construction [19, Lemma 17.21], there exists $\Lambda > 0$ and a family A_m of diffeomorphic images of $B_1(0)$ with $A_m \Delta B_1(0) \subset\subset B_{1/2}(e_n)$ for large m satisfying

$$|A_m| = |B_1(0)| + \sigma_m / r_m^n \quad \text{and} \tag{5.3}$$

$$P(A_m; B_{1/2}(e_n)) \leq P(B_1(0); B_{1/2}(e_n)) + \Lambda |\sigma_m| / r_m^n \,. \tag{5.4}$$

By the fact that $B_1(0) \setminus B_{1/2}(e_n) = A_m \setminus B_{1/2}(e_n)$, the sets $r_m A_m + y_m$ satisfy

$$(r_m A_m + y_m) \setminus B_{r_m/2}(y_m + r_m e_n) = B_{r_m}(y_m) \setminus B_{r_m/2}(y_m + r_m e_n) \,. \tag{5.5}$$

We also have, by the definition of $B_{r_m}(y_m)$,

$$B_{r_m}(y_m) \setminus B_R(0) = \mathcal{X}_{\text{bub}}^m(2) \setminus B_R(0) \,. \tag{5.6}$$

Next, we note that $y_m \cdot e_n \to \infty$ as $m \to \infty$ by the definition of $\mathcal{X}_{\text{bub}}^m$, and so

$$\operatorname{cl} B_R(0) \cap B_{r_m/2}(y_m + r_m e_n) = \varnothing \quad \text{for large } m \,. \tag{5.7}$$

As a consequence of (5.5) and (5.7), $B_{r_m}(y_m) \cap B_R(0) = (r_m A_m + y_m) \cap B_R(0)$ for large m. By this equivalence, (5.3) rescaled by r_m^n, and (5.6), we have

$$\begin{aligned} |(r_m A_m + y_m) \setminus B_R(0)| &= |(r_m A_m + y_m)| - |B_{r_m}(y_m) \cap B_R(0)| \\ &= |B_{r_m}(y_m)| + \sigma_m - |B_{r_m}(y_m) \cap B_R(0)| = |\mathcal{X}_{\text{bub}}^m(2) \setminus B_R(0)| + \sigma_m \,. \end{aligned} \tag{5.8}$$

Next, using in order (5.6), (5.7) to split $\operatorname{cl} B_R(0)^c$, (5.5) to cancel equal terms, and (5.4) rescaled by r_m^{n-1}, we estimate

$$\begin{aligned} &P(r_m A_m + y_m; \operatorname{cl} B_R(0)^c) - P(\mathcal{X}_{\text{bub}}^m(2); \operatorname{cl} B_R(0)^c) \\ &\qquad = P(r_m A_m + y_m; \operatorname{cl} B_R(0)^c) - P(B_{r_m}(y_m); \operatorname{cl} B_R(0)^c) \\ &\qquad = P(r_m A_m + y_m; B_{r_m/2}(y_m + r_m e_n)) - P(B_{r_m}(y_m); B_{r_m/2}(y_m + r_m e_n)) \\ &\qquad\quad + P\big(r_m A_m + y_m; B_{r_m/2}(y_m + r_m e_n)^c \cap \operatorname{cl} B_R(0)^c\big) \\ &\qquad\quad - P\big(B_{r_m}(y_m); B_{r_m/2}(y_m + r_m e_n)^c \cap \operatorname{cl} B_R(0)^c\big) \\ &\qquad = P(r_m A_m + y_m; B_{r_m/2}(y_m + r_m e_n)) - P(B_{r_m}(y_m); B_{r_m/2}(y_m + r_m e_n)) \\ &\qquad = r_m^{n-1}\big[P(A_m; B_{1/2}(e_n)) - P(B_1(0); B_{1/2}(e_n))\big] \leq \Lambda |\sigma_m| / r_m \overset{m\to\infty}{\to} 0 \,. \end{aligned} \tag{5.9}$$

Let us now define the $(2,1)$-clusters $\mathcal{X}_m$ via

$$\begin{aligned} &\mathcal{X}_m(1) = \mathcal{X}(1) \,, \quad \mathcal{X}_m(2) = \big(\mathcal{X}(2) \cap B_R(0)\big) \cup \big((r_m A_m + y_m) \setminus B_R(0)\big) \,, \\ &\mathcal{X}_m(3) = \big(\mathcal{X}(3) \cap B_R(0)\big) \cup \big(B_R(0)^c \setminus (r_m A_m + y_m)\big) \,. \end{aligned}$$

Due to the definition of $\mathcal{X}_m$, (5.8), and (5.2), we may compute

$$\begin{aligned} |\mathcal{X}_m(2)| &= |\mathcal{X}(2) \cap B_R(0)| + |(r_m A_m + y_m) \setminus B_R(0)| \\ &= |\mathcal{X}(2) \cap B_R(0)| + |\mathcal{X}_{\text{bub}}^m(2) \setminus B_R(0)| + \sigma_m \\ &= |\mathcal{X}(2) \cap B_R(0)| + |\mathcal{X}_{\text{bub}}^m(2) \setminus B_R(0)| + |\mathcal{X}_{\text{bub}}^m(2) \cap B_R(0)| - |\mathcal{X}(2) \cap B_R(0)| \\ &= |\mathcal{X}_{\text{bub}}^m(2)| \,. \end{aligned}$$

Therefore, $\mathbf{m}(\mathcal{X}_m) = (1, m, \infty)$. By $\mathcal{X}(j) \Delta \mathcal{X}_{\text{lens}}(j) \subset\subset B_R(0)$ for each j and (5.5),

$$(\mathcal{X}(2)^{(1)} \Delta (r_m A_m + y_m)) \cap \partial B_R(0) = (\mathcal{X}_{\text{lens}}(2)^{(1)} \Delta \mathcal{X}_{\text{bub}}^m(2)^{(1)}) \cap \partial B_R(0) \,,$$

so that by the local Hausdorff convergence of $\mathcal{X}_{\text{bub}}^m(2)$ to $\mathcal{X}_{\text{lens}}(2)$ from Lemma 3.6,

$$\mathcal{H}^{n-1}\big((\mathcal{X}(2)^{(1)} \Delta (r_m A_m + y_m)) \cap \partial B_R(0)\big) \overset{m\to\infty}{\to} 0 \,. \tag{5.10}$$

Now, by $P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; B_R(0)) \to P_{\mathbf{c}}(\mathcal{X}_{\rm lens}; B_R(0))$ from Lemma 3.6 and the minimality of $\mathcal{X}_{\rm bub}^m$ tested against $\mathcal{X}_m$ (which is permissible since $\mathbf{m}(\mathcal{X}_m) = (1, m, \infty)$), we may write

$$
\begin{aligned}
P_{\mathbf{c}}(\mathcal{X}_{\rm lens}; B_R(0)) &= \lim_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; B_R(0)) = \lim_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m) - P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; \mathbb{R}^n \setminus B_R(0)) \\
&\leq \liminf_{m\to\infty} P_{\mathbf{c}}(\mathcal{X}_m) - P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; \mathbb{R}^n \setminus B_R(0))\,.
\end{aligned} \tag{5.11}
$$

We compute $P_{\mathbf{c}}(\mathcal{X}_m)$ using (3.6) and estimate $-P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; \mathbb{R}^n \setminus \mathrm{cl}\, B_R(0))$ using (5.9), yielding

$$
\begin{aligned}
P_{\mathbf{c}}(\mathcal{X}_m) - P_{\mathbf{c}}(\mathcal{X}_{\rm bub}^m; \mathbb{R}^n \setminus B_R(0)) &\leq P_{\mathbf{c}}(\mathcal{X}; B_R(0)) + c_{23}\mathcal{H}^{n-1}\big((\mathcal{X}(2)^{(1)} \Delta (r_m A_m + y_m)) \cap \partial B_R(0)\big) \\
&\quad + c_{23} P(r_m A_m + y_m; \mathbb{R}^n \setminus \mathrm{cl}\, B_R(0)) \\
&\quad - c_{23} P(r_m A_m + y_m; \mathbb{R}^n \setminus \mathrm{cl}\, B_R(0)) + c_{23}\Lambda |\sigma_m|/r_m \\
&= P_{\mathbf{c}}(\mathcal{X}; B_R(0)) + \mathrm{o}(1)\,,
\end{aligned} \tag{5.12}
$$

where in the last line we have used (5.10). Combining (5.11)-(5.12) and sending $m \to \infty$ gives the desired minimality inequality (5.1). □

For the remainder of this section we focus on the case where $c_{12} = c_{13}$. It will be convenient to rewrite the energy. Specifically, for a $(1,2)$-cluster $\mathcal{X}$ and energy $P_{\mathbf{c}}(\cdot; B)$ corresponding to weights $c_{12} = c_{13}$ and c_{23}, we have

$$
P_{\mathbf{c}}(\mathcal{X}; B) = c_{13} P(\mathcal{X}(1); B) + c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap B) \quad \text{for every Borel } B \subset \mathbb{R}^n\,. \tag{5.13}
$$

Before presenting the proof of Theorem 2.9, we give a lemma on the "symmetrization" of a $(1,2)$-cluster $\mathcal{X}$ that will be used several times. For a $(1,2)$-cluster $\mathcal{X}$, we define $\mathcal{X}^S$ via

$$
\mathcal{X}^S(1) = \mathcal{X}(1)^S\,, \quad \mathcal{X}^S(2) = \{x \in \mathbb{R}^n : x_n > 0\} \setminus \mathcal{X}^S(1)\,, \quad \mathcal{X}^S(3) = \mathbb{R}^n \setminus (\mathcal{X}^S(1) \cup \mathcal{X}^S(2))\,;
$$

recall that $\mathcal{X}(1)^S$ denotes the Steiner symmetrization of $\mathcal{X}(1)$ over the plane $\{x \in \mathbb{R}^n : x_n = 0\}$.

Lemma 5.1 (Steiner inequality for $(1,2)$-clusters)**.** *If $\mathbf{c}$ is a family of positive weights with $c_{12} = c_{13}$, $\mathcal{X}$ is a $(1,2)$-cluster in $\mathbb{R}^n$, and there exists $a > 0$ such that*

$$
\mathbb{R}^{n-1} \times (a, \infty) \subset \mathcal{X}(2)^{(1)}\,, \quad \mathbb{R}^{n-1} \times (-\infty, -a) \subset \mathcal{X}(3)^{(1)}\,, \tag{5.14}
$$

then, setting $E(\mathcal{X}) := \{\overline{x} \in \mathbb{R}^{n-1} : \mathcal{H}^1(\mathcal{X}(1)_{\overline{x}}) = 0\}$,

(i) $\mathcal{X}^S(2,3) \overset{\mathcal{H}^{n-1}}{=} E(\mathcal{X}) \times \{0\}$, and for any Borel $B \subset \mathbb{R}^n$

$$
P_{\mathbf{c}}(\mathcal{X}^S; B) = c_{13} P(\mathcal{X}(1)^S; B) + c_{23}\mathcal{H}^{n-1}((E(\mathcal{X}) \times \{0\}) \cap B)\,, \qquad \textit{and} \tag{5.15}
$$

(ii) for any $r > 0$,

$$
P(\mathcal{X}(1); C_r) \geq P(\mathcal{X}^S(1); C_r) \qquad \textit{and} \tag{5.16}
$$

$$
\mathcal{H}^{n-1}(\mathcal{X}(2,3); C_r) \geq \mathcal{H}^{n-1}(E(\mathcal{X}) \cap B_r^{n-1}(\overline{0}))\,, \tag{5.17}
$$

with equality in (5.16) only if for $\mathcal{H}^{n-1}$-a.e. $x \in B_r^{n-1}(\overline{0})$, $\mathcal{X}(1)_{\overline{x}}$ is an interval, and equality in (5.17) only if there exists $E_r'(\mathcal{X}) \overset{\mathcal{H}^{n-1}}{=} E(\mathcal{X}) \cap B_r^{n-1}(\overline{0})$ and $t : E_r'(\mathcal{X}) \to [-a, a]$ such that

$$
\mathcal{X}(2,3) \cap C_r \overset{\mathcal{H}^{n-1}}{=} \{(\overline{y}, t(\overline{y})) : \overline{y} \in E_r'(\mathcal{X})\} \quad \textit{and} \tag{5.18}
$$

$$
\nu_{\mathcal{X}(2)}(x) = -e_n \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \mathcal{X}(2,3) \cap C_r\,. \tag{5.19}
$$

Proof. The proof is divided into steps, with item (i) in step one and item (ii) further broken up into steps two through four.

Step one: Here we prove (i). We begin with (5.13) for $\mathcal{X}^S$, which reads

$$
P_{\mathbf{c}}(\mathcal{X}^S; B) = c_{13} P(\mathcal{X}^S(1); B) + c_{23}\mathcal{H}^{n-1}(\partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) \cap B)\,.
$$

Therefore, (i) would follow from the $\mathcal{H}^{n-1}$-equivalence

$$\partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) \overset{\mathcal{H}^{n-1}}{=} E(\mathcal{X}) \times \{0\}\,. \tag{5.20}$$

Towards showing (5.20), we first observe that by $\partial^* \mathcal{X}^S(j) \overset{\mathcal{H}^{n-1}}{=} \mathcal{X}^S(j)^{(1/2)}$ for $j = 2, 3$ (Federer's theorem) and the containments $\mathcal{X}^S(2)^{(1)} \subset \{x_n > 0\}$ and $\mathcal{X}^S(3)^{(1)} \subset \{x_n < 0\}$,

$$\partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) = \partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) \cap \{x \in \mathbb{R}^n : x_n = 0\}\,. \tag{5.21}$$

By appealing to Lemma 3.2, let $F \subset \mathbb{R}^{n-1}$ be a set such that $\mathcal{H}^{n-1}(F^c) = 0$ and if $\overline{x} \in F$, then (i)-(iv) from that lemma hold for both $\mathcal{X}$ and $\mathcal{X}^S$. Applying in order $\mathcal{H}^{n-1}(F^c) = 0$, a rephrasing of the set intersection, and Lemma 3.2.(iii), we have

$$\begin{aligned}\partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) \cap \{x \in \mathbb{R}^n : x_n = 0\} &\overset{\mathcal{H}^{n-1}}{=} \partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) \cap (F \times \{0\}) \\ &= \{(\overline{x}, 0) \in \partial^* \mathcal{X}^S(2) \cap \partial^* \mathcal{X}^S(3) : \overline{x} \in F\} \\ &= \{(\overline{x}, 0) : 0 \in \partial^*_{\mathbb{R}}[\mathcal{X}^S(2)_{\overline{x}}] \cap \partial^*_{\mathbb{R}}[\mathcal{X}^S(3)_{\overline{x}}],\, \overline{x} \in F\}\,.\end{aligned} \tag{5.22}$$

But by the definition of Steiner symmetrization,

$$\partial^*_{\mathbb{R}}[\mathcal{X}^S(2)_{\overline{x}}] = \{\mathcal{H}^1(\mathcal{X}(1)_{\overline{x}})/2\} \quad \text{and} \quad \partial^*_{\mathbb{R}}[\mathcal{X}^S(3)_{\overline{x}}] = \{-\mathcal{H}^1(\mathcal{X}(1)_{\overline{x}})/2\}\,,$$

so 0 can only belong to both if $\mathcal{H}^1(\mathcal{X}(1)_{\overline{x}}) = 0$. Thus the right hand side of (5.22) is

$$\{(\overline{x}, 0) : \mathcal{H}^1(\mathcal{X}(1)_{\overline{x}}) = 0,\, \overline{x} \in F\}\,. \tag{5.23}$$

Combining (5.21)-(5.23) and using $\mathcal{H}^{n-1}(F^c) = 0$ one last time, we arrive at (5.20) as desired.

Step two: In steps two through four we prove (ii). Since (5.16) and the corresponding claim regarding equality cases is simply a restatement of the Steiner inequality (3.7) and the restriction that equality induces on the slices of $\mathcal{X}(1)$, we focus on the remainder of (ii) involving $\mathcal{X}(2,3) \cap C_r$.

As a preliminary, in this step we show that if $G \subset F$ is Borel measurable and $\mathcal{H}^{n-1}(G) = 0$, then

$$\mathcal{H}^{n-1}\big(\mathcal{X}(2,3) \cap (G \times \mathbb{R})\big) = 0\,. \tag{5.24}$$

To prove (5.24), it suffices to construct a sequence $\{G_j\}_j$ of subsets of G with $\cup_j G_j = G$ and show that for any compact $K \subset\subset \mathbb{R}$,

$$\mathcal{H}^{n-1}\big(\mathcal{X}(2,3) \cap (G_j \times K)\big) = 0 \qquad \forall j\,. \tag{5.25}$$

Let $G_j := \{\overline{x} \in G : \mathcal{H}^0(\partial^*_{\mathbb{R}}[\mathcal{X}(2)_{\overline{x}}] \cap \partial^*_{\mathbb{R}}[\mathcal{X}(3)_{\overline{x}}] \cap K) \le j\}$. By $G \subset F$ and Lemma 3.2.(i) (which guarantees the local finiteness of the perimeters of the slices), $\cup_j G_j = G$. To estimate the $\mathcal{H}^{n-1}$-measure of $\mathcal{X}(2,3) \cap (G_j \times K)$, let $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection map $\mathbf{p}(x) = (x_1, \dots, x_{n-1})$. A direct computation shows that the tangential Jacobian of $\mathbf{p}$ with respect to the locally $\mathcal{H}^{n-1}$-rectifiable set $\mathcal{X}(2,3)$ is given by

$$J^{\mathcal{X}(2,3)}\mathbf{p}(x) = \sqrt{\det\big(\nabla^{\mathcal{X}(2,3)}\mathbf{p}(x)^* \nabla^{\mathcal{X}(2,3)}\mathbf{p}(x)\big)} = |\nu_{\mathcal{X}(2)}(x) \cdot e_n| \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{X}(2,3)\,,$$

where $\nu_{\mathcal{X}(2)}(x)$ is the measure-theoretic outer unit normal to $\mathcal{X}(2)$. Using the fact that $\nu_{\mathcal{X}(2)}(x) \cdot e_n \neq 0$ on $G \times \mathbb{R}$ (from Lemma 3.2.(iv) and $G \subset F$), the area formula, and Lemma 3.2.(iii), we compute

$$\begin{aligned}\int_{\mathcal{X}(2,3)\cap(G_j \times K)} 1\, d\mathcal{H}^{n-1}(x) &= \int_{\mathcal{X}(2,3)\cap(G_j \times K)} \frac{J^{\mathcal{X}(2,3)}\mathbf{p}(x)}{|\nu_{\mathcal{X}(2)}(x) \cdot e_n|}\, d\mathcal{H}^{n-1}(x) \\ &= \int_{G_j} \left(\int_{\mathcal{X}(2,3)\cap(\{\overline{y}\} \times K)} \frac{1}{|\nu_{\mathcal{X}(2)}(x) \cdot e_n|}\, d\mathcal{H}^0(x) \right) d\mathcal{H}^{n-1}(\overline{y})\end{aligned}$$

$$\leq \int_{G_j} j \times \sup_{x\in(\{\overline{y}\}\times K)\cap\mathcal{X}(2,3)} \frac{1}{|\nu_{\mathcal{X}(2)}(x)\cdot e_n|}\, d\mathcal{H}^{n-1}(\overline{y}) = 0\,,$$

where in the last equality we have used $\nu_{\mathcal{X}(2)}\cdot e_n \neq 0$ on $G\times\mathbb{R}$ and $\mathcal{H}^{n-1}(G_j)\leq\mathcal{H}^{n-1}(G)=0$ to conclude that the integral vanishes. This completes (5.24).

Step three: In the third step we prove the inequality (5.17), with the equality case being analyzed in step four. Let $\mathcal{X}(2,3)^{\|} = \{x\in\mathcal{X}(2,3) : \nu_{\mathcal{X}(2)}(x)\cdot e_n\neq 0\}$ and $\mathcal{X}(2,3)^{\perp} = \mathcal{X}(2,3)\setminus\mathcal{X}(2,3)^{\|}$. Then by the area formula and the fact that $\nu_{\mathcal{X}(2)}(x)\cdot e_n\neq 0$ on $F\times\mathbb{R}$,

$$\begin{aligned}
\int_{\mathcal{X}(2,3)\cap C_r} 1\, d\mathcal{H}^{n-1}(x) &= \int_{\mathcal{X}(2,3)^{\|}\cap([F\cap E(\mathcal{X})]\times\mathbb{R})\cap C_r} \frac{J^{\mathcal{X}(2,3)}\mathbf{p}(x)}{|\nu_{\mathcal{X}(2)}(x)\cdot e_n|}\, d\mathcal{H}^{n-1}(x)\\
&\quad + \mathcal{H}^{n-1}\big(\big[\mathcal{X}(2,3)^{\perp}\cup((F^c\cup E(\mathcal{X})^c)\times\mathbb{R})\big]\cap C_r\big)\\
&= \int_{F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})} \bigg(\int_{\mathcal{X}(2,3)\cap(\{\overline{y}\}\times\mathbb{R})} \frac{1}{|\nu_{\mathcal{X}(2)}(x)\cdot e_n|}\, d\mathcal{H}^0(x)\bigg)\, d\mathcal{H}^{n-1}(\overline{y})\\
&\quad + \mathcal{H}^{n-1}\big(\big[\mathcal{X}(2,3)^{\perp}\cup((F^c\cup E(\mathcal{X})^c)\times\mathbb{R})\big]\cap C_r\big)\,. \qquad (5.26)
\end{aligned}$$

Now if $\overline{y}\in F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})$, then by the definitions of F and $E(\mathcal{X})$, $\mathcal{X}(2)_{\overline{y}}$ and $\mathcal{X}(3)_{\overline{y}}$ are sets of locally finite perimeter in $\mathbb{R}$ and

$$\mathcal{H}^1(\mathcal{X}(1)_{\overline{y}}) = 0\,. \qquad (5.27)$$

Furthermore, by (5.14) and Lemma 3.2.(*ii*),

$$(a,\infty) \overset{\mathcal{H}^1}{\subset} \mathcal{X}(2)_{\overline{y}}^{(1)_{\mathbb{R}}} \quad\text{and}\quad (-\infty,-a) \overset{\mathcal{H}^1}{\subset} \mathcal{X}(3)_{\overline{y}}^{(1)_{\mathbb{R}}} \quad \forall\, \overline{y}\in F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})\,. \qquad (5.28)$$

The combination of (5.27) and (5.28) implies that $\partial^*_{\mathbb{R}}[\mathcal{X}(2)_{\overline{y}}]\cap\partial^*_{\mathbb{R}}[\mathcal{X}(3)_{\overline{y}}]\neq\varnothing$, which by Lemma 3.2.(*iii*) yields

$$\partial^*\mathcal{X}(2)\cap\partial^*\mathcal{X}(3)\cap(\{y\}\times\mathbb{R})\neq\varnothing \quad \forall y\in F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})\,. \qquad (5.29)$$

Inserting this into (5.26), we conclude that

$$\begin{aligned}
\mathcal{H}^{n-1}(\mathcal{X}(2,3)\cap C_r) &= \int_{F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})} \bigg(\int_{\mathcal{X}(2,3)\cap(\{\overline{y}\}\times\mathbb{R})} \frac{1}{|\nu_{\mathcal{X}(2)}(x)\cdot e_n|}\, d\mathcal{H}^0(x)\bigg)\, d\mathcal{H}^{n-1}(\overline{y}) \qquad (5.30)\\
&\quad + \mathcal{H}^{n-1}\big(\big[\mathcal{X}(2,3)^{\perp}\cup((F^c\cup E(\mathcal{X})^c)\times\mathbb{R})\big]\cap C_r\big)\\
&\geq \int_{F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})} 1\, d\mathcal{H}^{n-1}(\overline{y}) = \mathcal{H}^{n-1}(F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0}))\,. \qquad (5.31)
\end{aligned}$$

The inequality (5.31) is equivalent to (5.17) since $\mathcal{H}^{n-1}(F^c)=0$.

Step four: It remains to demonstrate the two necessary conditions (5.18)-(5.19) for equality in (5.17). If equality holds in (5.17), then it holds in (5.31) since $\mathcal{H}^{n-1}(F^c)=0$. Inspecting (5.31), we see that equality forces several conditions:

(*a*) for $\mathcal{H}^{n-1}$-a.e. $\overline{y}\in F\cap E(\mathcal{X})\cap B_r^{n-1}(\overline{0})$, $\mathcal{X}(2,3)\cap(\{\overline{y}\}\times\mathbb{R})$ is a singleton $\{(\overline{y},t(\overline{y}))\}$,

(*b*) for these $\overline{y}\in F\cap B_r^{n-1}(\overline{0})$ such that $\mathcal{H}^0\big(\mathcal{X}(2,3)\cap(\{\overline{y}\}\times\mathbb{R})\big)=1$, $\nu_{\mathcal{X}(2)}((\overline{y},t(\overline{y})))=\pm e_n$, and

(*c*) $\mathcal{H}^{n-1}\big(\big[\mathcal{X}(2,3)^{\perp}\cup((F^c\cup E(\mathcal{X})^c)\times\mathbb{R})\big]\cap C_r\big)=0$.

By (*c*) and the fact that $\nu_{\mathcal{X}(2)}(x)\cdot e_n\neq 0$ on $F\times\mathbb{R}$,

$$\begin{aligned}
\mathcal{H}^{n-1}(\mathcal{X}(2,3)\cap C_r) &= \mathcal{H}^{n-1}\big(\mathcal{X}(2,3)^{\|}\cap\big([F\cap E(\mathcal{X})]\times\mathbb{R}\big)\cap C_r\big)\\
&= \mathcal{H}^{n-1}\big(\mathcal{X}(2,3)\cap\big([F\cap E(\mathcal{X})]\times\mathbb{R}\big)\cap C_r\big)\,. \qquad (5.32)
\end{aligned}$$

Also, due to (a) and Lemma 3.2.(iii), for $\mathcal{H}^{n-1}$-a.e. $\overline{y} \in F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0})$, $\partial^*_{\mathbb{R}}[\mathcal{X}(2)_{\overline{y}}] \cap \partial^*_{\mathbb{R}}[\mathcal{X}(2)_{\overline{y}}]$ consists of the single point $t(\overline{y})$. In addition, by (5.28), $t(\overline{y}) \in [-a, a]$ and

$$\nu_{\mathcal{X}(2)_{\overline{y}}}(t(\overline{y})) = -1 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \overline{y} \in F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0})\,. \tag{5.33}$$

So then using in order (5.33), Lemma 3.2.(iv), and (b), we have

$$\begin{aligned} -1 = \nu_{\mathcal{X}(2)_{\overline{y}}}(t(\overline{y})) &= \frac{\nu_{\mathcal{X}(2)}(\overline{y}, t(\overline{y})) \cdot e_n}{|\nu_{\mathcal{X}(2)}(\overline{y}, t(\overline{y})) \cdot e_n|} \\ &= \nu_{\mathcal{X}(2)}(\overline{y}, t(\overline{y})) \cdot e_n \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \overline{y} \in F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0})\,. \end{aligned} \tag{5.34}$$

Therefore, by (a) and (5.34), we may choose Borel measurable $E_r'(\mathcal{X}) \subset F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0})$ with

$$\mathcal{H}^{n-1}(F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0}) \setminus E_r'(\mathcal{X})) = 0 \tag{5.35}$$

such that

$$\mathcal{X}(2,3) \cap C_r \cap (E_r'(\mathcal{X}) \times \mathbb{R}) = \{(\overline{y}, t(\overline{y})) : \overline{y} \in E_r'(\mathcal{X})\} \quad \text{and} \tag{5.36}$$

$$\nu_{\mathcal{X}(2)}(x) = -e_n \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{X}(2,3) \cap C_r \cap (E_r'(\mathcal{X}) \times \mathbb{R})\,. \tag{5.37}$$

Since $\mathcal{H}^{n-1}(F \cap E(\mathcal{X}) \cap B_r^{n-1}(\overline{0}) \setminus E_r'(\mathcal{X})) = 0$, by our preliminary estimate (5.24) and (5.36) we find

$$\begin{aligned} \mathcal{H}^{n-1}\big(\mathcal{X}(2,3) \cap \big([F \cap E(\mathcal{X})] \times \mathbb{R}\big) \cap C_r\big) &= \mathcal{H}^{n-1}\big(\mathcal{X}(2,3) \cap (E_r'(\mathcal{X}) \times \mathbb{R}) \cap C_r\big) \\ &= \mathcal{H}^{n-1}\big(\mathcal{X}(2,3) \cap \{(\overline{y}, t(\overline{y})) : \overline{y} \in E_r'(\mathcal{X})\}\big)\,. \end{aligned} \tag{5.38}$$

Finally, (5.38) and (5.32) guarantee that

$$\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap (E_r'(\mathcal{X})^c \times \mathbb{R}) \cap C_r) = 0\,, \tag{5.39}$$

in which case (5.18)-(5.19) follow from (5.36)-(5.37). □

Remark 5.2 (Alternative proof of minimality)**.** When $c_{12} = c_{13}$, the symmetrization inequalities from Lemma 5.1 can be combined with the characterization of minimizing liquid drops in Theorem 3.4 to give an short proof of the **c**-local minimality of $\mathcal{X}_{\rm lens}$.

Proof of Theorem 2.9. The minimality of $\mathcal{X}_{\rm lens}$ has already been proved in Theorem 2.8. The uniqueness proof when $c_{12} = c_{13}$ is divided into steps, and the main outline is to combine the various types of rigidity contained in (5.40) (the asymptotic expansion), Lemma 5.1 (the "flatness" in (5.18)-(5.19)), Theorem 3.4 (characterization of minimizing liquid drops), and Theorem 3.1 (rigidity in the Steiner inequality) to conclude that up to rotation and translation, $\mathcal{X} = \mathcal{X}_{\rm lens}$.

Step one: Let $\mathcal{X}$ be a **c**-locally minimizing $(1,2)$-cluster in $\mathbb{R}^n$, with the added assumption that $\mathcal{X}$ has planar growth at infinity if $n \geq 8$. In this step we collect some basic properties of $\mathcal{X}$.

First, by Lemma 4.1, there exists $R > 0$ such that $\mathcal{X}(1) \subset\subset B_R(0)$. Second, by the boundedness of $\mathcal{X}(1)$ and the planar growth of $\mathcal{X}$ at infinity in $\mathbb{R}^n$ for $n \geq 8$, we may apply Corollary 4.8 to find $R_0 > R$, such that, up to a rotation and translation, $\mathcal{X}(2,3) \setminus C_{R_0}$ coincides with the graph of a solution $u : \mathbb{R}^{n-1} \setminus B_{R_0}^{n-1}(\overline{0})$ to the minimal surface equation satisfying the expansion at infinity

$$u(\overline{x}) = \begin{cases} 0 & \overline{x} \in \mathbb{R}^1,\ n = 2 \\ 0 + \dfrac{b}{|\overline{x}|^{n-3}} + \dfrac{\overline{c} \cdot \overline{x}}{|\overline{x}|^{n-1}} + \mathrm{O}(|\overline{x}|^{1-n}) & \overline{x} \in \mathbb{R}^{n-1},\ n \geq 3 \end{cases} \tag{5.40}$$

for some $b \in \mathbb{R}$, which we assume is 0 when $n = 3$, and $\overline{c} \in \mathbb{R}^{n-1}$.

Next, we claim that there exists $a > 0$ such that,

$$\mathrm{cl}\,\mathcal{X}(1) \cup \mathcal{X}(2,3) \subset\subset \mathbb{R}^{n-1} \times [-a, a]\,. \tag{5.41}$$

To see this, note that for large $r > R_0$, the rectifiable varifold $V_r = \mathbf{var}\,(\mathcal{X}(2,3)\cap C_r, 1)$ is stationary in the open set $\mathbb{R}^n \setminus \big[\mathrm{cl}\,\mathcal{X}(1) \cup \big(\partial B_r^{n-1}(r)(\overline{0}) \times [-1,1]\big)\big]$. Since $\mathrm{cl}\,\mathcal{X}(1) \cup (\partial B_r^{n-1}(r)(\overline{0}) \times [-1,1])$ is compact, we can apply the convex hull property [30, Theorem 19.2] to deduce that

$$\mathrm{spt}\, V_r = \mathrm{cl}\,(\mathcal{X}(2,3) \cap C_r) \subset \mathrm{Conv}\,\big(\mathrm{cl}\,\mathcal{X}(1) \cup \big(\partial B_r^{n-1}(r)(\overline{0}) \times [-1,1]\big)\big)\,, \tag{5.42}$$

where Conv denotes the convex hull. Let $a \geq 1$ be large enough so that $\mathrm{cl}\,\mathcal{X}(1) \subset\subset \mathbb{R}^{n-1} \times [-a,a]$. Then sending $r \to \infty$ in (5.42) yields (5.41). As a corollary, up to interchanging $\mathcal{X}(2)$ and $\mathcal{X}(3)$,

$$\mathbb{R}^{n-1} \times (a,\infty) \subset \mathcal{X}(2)^{(1)}\,, \quad \mathbb{R}^{n-1} \times (-\infty,-a) \subset \mathcal{X}(3)^{(1)}\,. \tag{5.43}$$

This will allow us to freely use Lemma 5.1 on $\mathcal{X}$ and any compactly supported variation of $\mathcal{X}$, since (5.43) is the only assumption on $\mathcal{X}$ in the statement of that lemma.

Step two: In this step we prove via a comparison argument that $\mathcal{X}$ achieves equality in (5.16) and (5.17) for all $r > R_0$. Let us begin by noting that by $\mathcal{X}(1) \subset\subset B_R(0)$ and (5.41),

$$E(\mathcal{X}) := \{\overline{x} \in \mathbb{R}^{n-1} : \mathcal{H}^1(\mathcal{X}(1)_{\overline{x}}) = 0\} \supset \mathbb{R}^{n-1} \setminus B_R^{n-1}(\overline{0})\,. \tag{5.44}$$

Thus by (5.43)-(5.44) and the fact that $\mathcal{H}^{n-1}$-measure decreases upon projection onto $\{x_n = 0\}$,

$$\begin{aligned}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r \setminus C_s) &\geq \omega_{n-1}(r^{n-1} - s^{n-1}) \\ &= \mathcal{H}^{n-1}(E(\mathcal{X}) \cap B_r^{n-1}(\overline{0}) \setminus B_s^{n-1}(\overline{0})) \qquad \forall R < R_0 < s < r\,.\end{aligned} \tag{5.45}$$

Therefore, the function $\mathbf{e} : (R_0, \infty) \to \mathbb{R}$ defined by

$$\mathbf{e}(r) = c_{13}\big[P(\mathcal{X}(1)) - P(\mathcal{X}^S(1))\big] + c_{23}\big[\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) - \mathcal{H}^{n-1}(E(\mathcal{X}) \cap B_r^{n-1}(\overline{0}))\big]$$

satisfies

$$\mathbf{e}(r) \text{ is increasing for } r > R_0\,, \tag{5.46}$$

by (5.45), and, by (5.16)-(5.17),

$$0 \leq \mathbf{e}(r) \quad \forall r > R_0\,. \tag{5.47}$$

We show now that $\mathbf{e}(r) = 0$ for all $r > R_0$. Let us first consider the case that $n \geq 3$. For each $r > R_0$, let Π_r be the plane

$$\Pi_r = \{(\overline{x}, \overline{c} \cdot \overline{x}/r^{n-1}) : \overline{x} \in \mathbb{R}^{n-1}\}\,,$$

and let H_r^+ and H_r^- be the halfspaces with common boundary Π_r such that $\nu_{H_r^+} \cdot e_n < 0$. Π_r is chosen in this fashion because on ∂C_r, it is very close to $\mathcal{X}(2,3)$. More precisely, by the definition of Π_r and since $\mathcal{X}(2,3) \setminus C_R$ is the graph of u, for $r > R_0$,

$$\partial C_r \cap ((H_r^+)^{(1)} \Delta \mathcal{X}(2)^{(1)}) \subset \big\{(\overline{x}, t) : \overline{x} \in \partial B_r^{n-1}(\overline{0}),\ |t - u(\overline{x})| \leq |u(\overline{x}) - \overline{c} \cdot \overline{x}/r^{n-1}|\big\}\,.$$

By the asymptotic expansion (5.40), we may thus estimate

$$\mathcal{H}^{n-1}\big(\partial C_r \cap ((H_r^+)^{(1)} \Delta \mathcal{X}(2)^{(1)})\big) \leq \omega_{n-2} r^{n-2} \mathrm{O}(r^{1-n}) = \mathrm{O}(r^{-1})\,. \tag{5.48}$$

Next, let T_r be a rotation matrix such that $T_r(\{x_n = 0\}) = \Pi_r$, and define a new $(1,2)$-cluster $\mathcal{X}_r$ by replacing $\mathcal{X}(j) \cap C_r$ with $T_r(\mathcal{X}^S(j)) \cap C_r$ for each j, so

$$\begin{aligned}\mathcal{X}_r(1) &= T_r(\mathcal{X}^S(1))\,, \\ \mathcal{X}_r(2) &= \big(T_r(\mathcal{X}^S(2)) \cap C_r\big) \cup (\mathcal{X}(2) \setminus C_r) = \big(H_r^+ \setminus T_r(\mathcal{X}^S(1)) \cap C_r\big) \cup (\mathcal{X}(2) \setminus C_r)\,, \\ \mathcal{X}_r(3) &= \big(T_r(\mathcal{X}^S(3)) \cap C_r\big) \cup (\mathcal{X}(3) \setminus C_r) = \big(H_r^- \setminus T_r(\mathcal{X}^S(1)) \cap C_r\big) \cup (\mathcal{X}(3) \setminus C_r)\,.\end{aligned}$$

Then for large r, $\mathcal{X}(j) \Delta \mathcal{X}_r(j) \subset\subset B_{r+1}^{n-1}(\overline{0}) \times (-a-1, a+1)$. Also, recalling the characterization of $\mathcal{X}^S(2,3)$ from Lemma 5.1.(i), we have

$$\partial^* \mathcal{X}_r(2) \cap \partial^* \mathcal{X}_r(3) \cap C_r = T_r(\mathcal{X}^S(2,3)) \cap C_r = T_r\big(E(\mathcal{X}) \times \{0\}\big) \cap C_r\,. \tag{5.49}$$

By the $\mathbf{c}$-local minimality of $\mathcal{X}$, (5.49), and the fact that $\mathcal{X}$ and $\mathcal{X}_r$ agree outside C_r, we estimate

$$\begin{aligned}
0 &\le P_{\mathbf{c}}(\mathcal{X}_r; B^{n-1}_{r+1}(\bar{0}) \times (-a-1, a+1)) - P_{\mathbf{c}}(\mathcal{X}; B^{n-1}_{r+1}(\bar{0}) \times (-a-1, a+1)) \\
&= c_{13} P(T_r(\mathcal{X}^S(1))) + c_{23}\mathcal{H}^{n-1}\big(T_r\big(E(\mathcal{X}) \times \{0\}\big) \cap C_r\big) + c_{23} P(\mathcal{X}_r(2); \partial C_r) \\
&\quad - c_{13} P(\mathcal{X}(1)) - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) \\
&= c_{13}\big[P(\mathcal{X}^S(1)) - P(\mathcal{X}(1))\big] + c_{23}\mathcal{H}^{n-1}\big(T_r\big(E(\mathcal{X}) \times \{0\}\big) \cap C_r\big) + c_{23} P(\mathcal{X}_r(2); \partial C_r) \\
&\quad - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r)\,. \qquad (5.50)
\end{aligned}$$

By using in order $T_r(\{x_n = 0\}) = \Pi_r$ and the definition of $E(\mathcal{X})$, the definition of Π_r and the "cut-and-paste" formula (3.6) along ∂C_r, $\sqrt{1+t^2} \le 1 + t^2$ and (5.48), and $n \ge 3$, we bound the second, third, and fourth terms from above:

$$\begin{aligned}
&c_{23}\mathcal{H}^{n-1}\big(T_r\big(E(\mathcal{X}) \times \{0\}\big) \cap C_r\big) + c_{23} P(\mathcal{X}_r(2); \partial C_r) - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) \\
&\quad = c_{23}\mathcal{H}^{n-1}(\Pi_r \cap C_r) - c_{23}\mathcal{H}^{n-1}\big(T_r(\{\bar{x} : \mathcal{H}^1(\mathcal{X}(1)_{\bar{x}}) > 0\})\big) + c_{23} P(\mathcal{X}_r(2); \partial C_r) \\
&\qquad - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) \\
&\quad = c_{23}\int_{B^{n-1}_r(\bar{0})} \sqrt{1 + |\bar{c}|^2/r^{2n-2}}\, d\bar{x} - c_{23}\mathcal{H}^{n-1}\big(\{\bar{x} : \mathcal{H}^1(\mathcal{X}(1)_{\bar{x}}) > 0\}\big) \\
&\qquad + c_{23}\mathcal{H}^{n-1}\big((\mathcal{X}(2)^{(1)} \Delta (H_r^+)^{(1)}) \cap \partial C_r\big) - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) \\
&\quad \le c_{23}\omega_{n-1} r^{n-1} + c_{23}|\bar{c}|^2 \omega_{n-1} r^{n-1+2-2n} - c_{23}\mathcal{H}^{n-1}\big(\{\bar{x} : \mathcal{H}^1(\mathcal{X}(1)_{\bar{x}}) > 0\}\big) \\
&\qquad + \mathrm{O}(r^{-1}) - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) \\
&\quad = c_{23}\mathcal{H}^{n-1}(E(\mathcal{X}) \cap B^{n-1}_r(\bar{0})) - c_{23}\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) + \mathrm{O}(r^{-1})\,. \qquad (5.51)
\end{aligned}$$

Plugging the bound (5.51) into (5.50) for $r > R_0$, we arrive at

$$0 \le c_{13}\big[P(\mathcal{X}^S(1)) - P(\mathcal{X}(1))\big] + c_{23}\big[\mathcal{H}^{n-1}(E(\mathcal{X}) \cap B^{n-1}_r(\bar{0})) - \mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r)\big] + \mathrm{O}(r^{-1})\,.$$

After moving the first two terms to the other side, this inequality says in terms of $\mathbf{e}(r)$ that

$$\mathbf{e}(r) \le \mathrm{O}(r^{-1})\,. \qquad (5.52)$$

In summary, by combining (5.46), (5.47), and (5.52), we have shown that for $n \ge 3$, $\mathbf{e}(r)$ is increasing, non-negative, and bounded from above by $\mathrm{O}(r^{-1})$ as $r \to \infty$ – so $\mathbf{e}(r)$ must be 0 for all $r > R_0$ if $n \ge 3$. The same conclusion when $n = 2$ follows by testing the minimality of $\mathcal{X}$ against $\mathcal{X}^S$, which is acceptable by (5.40), to find $\mathbf{e}(r) \le 0$, and recalling that $\mathbf{e}(r) \ge 0$ from (5.47). In fact, since (5.16)-(5.17) guarantee that both the terms in $\mathbf{e}(r)$ are each non-negative for $r > R_0$, $\mathbf{e}(r) \equiv 0$ on (R_0, ∞) actually gives the finer information

$$P(\mathcal{X}(1)) = P(\mathcal{X}^S(1)) \quad \text{and} \qquad (5.53)$$

$$\mathcal{H}^{n-1}(\mathcal{X}(2,3) \cap C_r) = \mathcal{H}^{n-1}(E(\mathcal{X}) \cap B^{n-1}_r(\bar{0})) \qquad \forall r > R_0\,. \qquad (5.54)$$

Step three: Let $r_{\rm lens}$ be the radius of the disk $\mathcal{X}_{\rm lens}(1)^{(1)} \cap \{x_n = 0\}$. Here we show that (5.53)-(5.54) force

$$\mathcal{X}(2,3) \overset{\mathcal{H}^{n-1}}{=} \{x_n = 0\} \setminus \big(\mathrm{cl}\, B^{n-1}_{r_{\rm lens}}(0) \times \{0\}\big) \quad \text{and} \qquad (5.55)$$

$$\mathcal{X}^S = \mathcal{X}_{\rm lens} \quad \text{up to translations of } \mathcal{X}^S(1) \text{ along } \{x_n = 0\} \text{ and } \mathcal{L}^n\text{-null sets}\,. \qquad (5.56)$$

First, according to (5.18)-(5.19) from Lemma 5.1, the equality (5.54) for every $r > R_0$ entails the existence of $E'(\mathcal{X}) \overset{\mathcal{H}^{n-1}}{=} E(\mathcal{X})$ and $t : E'(\mathcal{X}) \to [-a, a]$ such that

$$\mathcal{X}(2,3) \overset{\mathcal{H}^{n-1}}{=} \{(\bar{y}, t(\bar{y})) : \bar{y} \in E'(\mathcal{X})\} \quad \text{and} \qquad (5.57)$$

$$\nu_{\mathcal{X}(2)}(x) = -e_n \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{X}(2,3)\,. \qquad (5.58)$$

By a simple argument using the divergence theorem [33, Lemma 2.2], the fact that $\nu_{\mathcal{X}(2)} = -e_n$ $\mathcal{H}^{n-1}$-a.e. on $\mathcal{X}(2,3)$ implies that for every $x \in \mathcal{X}(2,3)$ and $B_r(x)$ with $|B_r(x) \setminus (\mathcal{X}(2) \cup \mathcal{X}(3))| = 0$, which ensures that $\partial^*\mathcal{X}(2) \cap B_r(x) = \mathcal{X}(2,3)$, we have

$$\partial^*\mathcal{X}(2) \cap B_r(x) = P_x \cap B_r(x) \quad \text{for the plane } P_x \ni x \text{ with normal } -e_n\,.$$

Thus for any compact $K \subset\subset \mathbb{R}^n$ such that $\mathrm{cl}\,\mathcal{X}(1) \subset K$ and $\mathbb{R}^n \setminus K$ is connected,

$$\mathcal{X}(2,3) \setminus K = \{x_n = 0\} \setminus K\,, \tag{5.59}$$

where we have used (5.40) (which says that asymptotically $\mathcal{X}(2,3)$ is $\{x_n = 0\}$) to eliminate the possiblity that $\mathcal{X}(2,3)$ is a vertical translation of $\{x_n = 0\}$. In particular,

$$\mathcal{X}(j)\Delta\mathcal{X}_{\mathrm{lens}}(j) \subset B_{R_0}(0) \quad \forall 1 \le j \le 3\,.$$

Now we can test the minimality of $\mathcal{X}$ against $\mathcal{X}_{\mathrm{lens}}$ on C_r for any $r > R_0$ to obtain

$$P_{\mathbf{c}}(\mathcal{X}; C_r) \le P_{\mathbf{c}}(\mathcal{X}_{\mathrm{lens}}; C_r)\,. \tag{5.60}$$

By (3.13) applied to $\mathcal{X}^S$, (5.53)-(5.54), (5.60), and then (3.13) applied to $\mathcal{X}_{\mathrm{lens}}$, for any $r > R_0$

$$\begin{aligned} 2c_{13}\mathcal{F}_{c_{23}/(2c_{13})}(\mathcal{X}^S(1) \cap H; H) + c_{23}\omega_{n-1}r^{n-1} &= P_{\mathbf{c}}(\mathcal{X}^S; C_r) = P_{\mathbf{c}}(\mathcal{X}; C_r) \\ &\le P_{\mathbf{c}}(\mathcal{X}_{\mathrm{lens}}; C_r) \\ &= 2c_{13}\mathcal{F}_{c_{23}/(2c_{13})}(\mathcal{X}_{\mathrm{lens}}(1) \cap H; H) + c_{23}\omega_{n-1}r^{n-1}\,. \end{aligned}$$

But by Theorem 3.4, $\mathcal{X}_{\mathrm{lens}} \cap \{x_n > 0\}$ is the unique, volume-constrained minimizer for $\mathcal{F}_{c_{23}/(2c_{13})}$ up to horizontal translations and $\mathcal{L}^n$-null sets. This immediately yields (5.56).

Turning now towards (5.55), let us ignore the horizontal translation and null sets, so that $\mathcal{X}_{\mathrm{lens}} = \mathcal{X}^S$. We define the function $L_{\mathcal{X}(1)} : \mathbb{R}^{n-1} \to [0, \infty)$ by

$$L_{\mathcal{X}(1)}(\overline{x}) = \mathcal{H}^1(\mathcal{X}(1)_{\overline{x}}) = \mathcal{H}^1(\mathcal{X}^S(1)_{\overline{x}})\,.$$

Since $\mathcal{X}^S = \mathcal{X}_{\mathrm{lens}}$, the precise representative of $L_{\mathcal{X}(1)}$ (see Subsection 3.2) is given by

$$L^*_{\mathcal{X}(1)}(\overline{x}) = \mathcal{H}^1(\mathcal{X}_{\mathrm{lens}}(1)_{\overline{x}}) =: L_{\mathrm{lens}}(\overline{x})\,. \tag{5.61}$$

Therefore

$$E'(\mathcal{X}) \overset{\mathcal{H}^{n-1}}{=} E(\mathcal{X}) \overset{\mathcal{H}^{n-1}}{=} \mathbb{R}^{n-1} \setminus B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \quad \text{and thus} \quad |\mathcal{X}(1) \setminus C_{r_{\mathrm{lens}}}| = 0\,. \tag{5.62}$$

By the volume density estimate (4.3), this yields $\mathrm{cl}\,\mathcal{X}(1) \setminus \mathrm{cl}\,C_{r_{\mathrm{lens}}} = \varnothing$. We can therefore choose $r' > 0$ such that $\mathrm{cl}\,\mathcal{X}(1) \subset \mathrm{cl}\,B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times [-r', r']$. Since this set is compact and connected, (5.59) says that

$$\mathcal{X}(2,3) \setminus \big(\mathrm{cl}\,B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times [-r', r']\big) = \{x_n = 0\} \setminus \big(\mathrm{cl}\,B^{n-1}_{r_{\mathrm{lens}}}(0) \times \{0\}\big)\,. \tag{5.63}$$

Combined with (5.62) and (5.57), (5.63) implies (5.55).

Step four: Finally we conclude the proof of uniqueness by showing that $\mathcal{X}_{\mathrm{lens}} = \mathcal{X}$ up to the rotations, translations, and null sets we have accrued thus far. By the fact that $P(\mathcal{X}^S(1); \partial C_r) = P(\mathcal{X}_{\mathrm{lens}}(1); \partial C_r) = 0$, (3.7), and (5.53),

$$\begin{aligned} P(\mathcal{X}^S(1)) &= P(\mathcal{X}^S(1); B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R}) \le P(\mathcal{X}(1); B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R}) \\ &\le P(\mathcal{X}(1); B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R}) + P(\mathcal{X}(1); \partial B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R}) = P(\mathcal{X}(1)) = P(\mathcal{X}^S(1))\,. \end{aligned}$$

Thus these are all equalities, and so

$$P(\mathcal{X}^S(1); B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R}) = P(\mathcal{X}(1); B^{n-1}_{r_{\mathrm{lens}}}(\overline{0}) \times \mathbb{R})\,. \tag{5.64}$$

Furthermore, by (5.61)

$$L^*_{\mathcal{X}(1)}(\overline{x}) = L_{\mathrm{lens}}(\overline{x}) > 0 \quad \forall \overline{x} \in B^{n-1}_{r_{\mathrm{lens}}}(\overline{0})\,. \tag{5.65}$$

Lastly, by $\mathcal{X}^S(1) = \mathcal{X}_{\rm lens}(1)$ and the explicit description of $\partial\mathcal{X}_{\rm lens}(1)$ as two spherical caps,

$$\mathcal{H}^{n-2}\big(\{x \in \partial^*(\mathcal{X}(1)^S) : \nu_{E^S}(x)\cdot e_n = 0\} \cap (B^{n-1}_{r_{\rm lens}}(\bar{0}) \times \mathbb{R})\big) = 0 \tag{5.66}$$

(in fact it is empty). We therefore have a connected $(n-1)$-dimensional set $\Omega = B^{n-1}_{r_{\rm lens}}(\bar{0})$, a set of finite perimeter $\mathcal{X}(1)$ achieving equality in the Steiner inequality (3.7) over Ω (by (5.64)), and a distribution function $L^*_{\mathcal{X}(1)}$ that is strictly positive $\mathcal{H}^{n-2}$-a.e. on Ω (by (5.65)) and satisfies (3.8) (by (5.66)). These are exactly the assumptions of Theorem 3.1, which we apply then to conclude that $\mathcal{X}(1) \overset{\mathcal{L}^n}{=} \mathcal{X}^S(1)$ up to a vertical translation. However, we know that $\mathcal{X}(2,3) \overset{\mathcal{H}^{n-1}}{=} \{x_n = 0\} \setminus \big(\mathrm{cl}\, B^{n-1}_{r_{\rm lens}}(0) \times \{0\}\big)$ by (5.55), and it is impossible that the $(1,2)$-cluster $\mathcal{X}$ could have interfaces given by

$$\mathcal{X}(1,2) \cup \mathcal{X}(2,3) \overset{\mathcal{H}^{n-1}}{=} \partial^*\mathcal{X}_{\rm lens}(1) + te_n \text{ for } t \neq 0\,, \quad \mathcal{X}(2,3) \overset{\mathcal{H}^{n-1}}{=} \{x_n = 0\} \setminus \big(\mathrm{cl}\, B^{n-1}_{r_{\rm lens}}(0) \times \{0\}\big)\,.$$

So $\mathcal{X}(1) \overset{\mathcal{L}^n}{=} \mathcal{X}^S(1)$, which together with (5.55)-(5.56) gives

$$\mathcal{X}(j) \overset{\mathcal{L}^n}{=} \mathcal{X}_{\rm lens}(j) \quad \forall\, 1 \le j \le 3\,.$$

The proof of Theorem 2.9 is complete. □

Remark 5.3. As mentioned in Remark 2.12, the exterior minimal surface $\mathcal{X}(2,3)$ is similar to the exterior minimal surfaces arising from large volume isoperimetry in [20]. The usage of the convex hull property in step one, the gluing onto tilted planes in step two, and the monotonicity of the "cylindrical energy gap" have natural analogues in [20, Steps 2-3, proof of Theorem 1.6] and [20, Step 2, proof of Theorem 1.1].

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