CONSISTENCY OF MINIMIZING MOVEMENTS WITH SMOOTH MEAN CURVATURE FLOW OF DROPLETS WITH PRESCRIBED CONTACT-ANGLE IN $\mathbb{R}^3$

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Abstract. In this paper we prove that in $\mathbb{R}^3$ the minimizing movement solutions for mean curvature motion of droplets, obtained in [6], starting from a regular droplets sitting on the horizontal plane with a regular relative adhesion coefficient, coincide with the smooth mean curvature flow of droplets with a prescribed contact-angle.

1. Introduction

Capillary droplets, known for their distinctive behavior resulting from the interplay of surface tension and capillary forces, have attracted considerable interest across a range of scientific and engineering fields, for instance in the study of wetting phenomena, energy minimizing drops and their adhesion properties, as well as because of their connections with minimal surfaces (see e.g. [1, 7, 10, 18, 12, 20]).

In this paper as in [6] we are interested in the mean curvature motion of a droplet sitting on a horizontal hyperplane with a prescribed (possibly nonconstant) relative adhesion coefficient. Such evolution of droplets can be seen as mean curvature flow of hypersurfaces with a prescribed Neumann-type boundary condition. There are quite a few results related to the well-posedness of the classical mean curvature flow with boundary (see e.g. [28, 36] for mean curvature flow with Dirichlet boundary conditions and [3, 25, 31] for mean curvature flow with Neumann-type boundary conditions).

The mean curvature evolution of (bounded) smooth sets even without boundary conditions can produce a singularity in finite time. In the literature, to continue the flow after singularity, several notions of weak solutions have been introduced, see e.g. [2, 5, 9, 11, 19, 21, 29, 32]. These weak solutions are constructed using various qualitative characterizations of the classical smooth flow such as comparison principles, the monotonicity of the area of the evolving hypersurfaces, the level-set formulation, the signed-distance formulation, phase-field approximations etc., and known to coincide with the smooth flow.

Some of those weak solutions has been extended to the case with boundary conditions, see e.g. [22, 37] for Brakke flow with Dirichlet and/or dynamic boundary conditions, [23, 27, 31] and [8] for viscosity flow with Neumann-type and Dirichlet boundary conditions, [6, 26] for BV-distributional solutions with Neumann-type boundary conditions and [6] and [35] for minimizing movements with Neumann-type and Dirichlet boundary conditions. Sometimes, in defining these weak flows, the associated mathematical aproach may require a relaxation of the boundary conditions. Consequently, a natural question arises: do these weak solutions coincide with the smooth flow when the latter exists?

In this paper we study such consistency problem for the minimizing movement solution for mean curvature evolution of droplets. Namely, as in [6] modelling the regions occupied

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by droplets by sets of finite perimeter in $\Omega := \mathbb{R}^2 \times (0, +\infty)$, and we introduce the capillary analogue of the Almgren-Taylor-Wang functional

$$\mathcal{F}_\beta(E; E_0, \tau) := \mathcal{C}_\beta(E, \Omega) + \frac{1}{\tau} \int_{E\Delta E_0} \mathrm{d}_{E_0}(x)dx, \tag{1.1}$$

where E, E_0 are sets of finite perimeter in Ω, $\tau > 0$,

$$\mathcal{C}_\beta(E, \Omega) := P(E, \Omega) + \int_{\partial\Omega} \beta\chi_E d\mathcal{H}^{n-1}$$

is the capillary functional [14, 20] for some $\beta \in L^\infty(\partial\Omega)$, $\mathrm{d}_{E_0}(\cdot) := \mathrm{dist}(x, \Omega \cap \partial^* E_0)$ and $\partial^* E_0$ is the reduced boundary of E_0.

Following De Giorgi [13] we define

Definition 1.1 (**GMM**)**.** Let $\mathcal{S}$ be a topological space, $\mathcal{F} : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+ \to [-\infty, +\infty]$ be a functional and $u_0 \in \mathcal{S}$.

(a) Given $\tau > 0$, a family $\{u(\tau, k)\}_{k\in\mathbb{N}_0}$ is called a (discrete) *flat flow* starting from u_0 provided that $u(\tau, 0) := u_0$,

$$\mathcal{F}(u(\tau, k); u(\tau, k-1), \tau) = \min_{v\in\mathcal{S}} \mathcal{F}(v; u(\tau, k-1), \tau).$$

(b) A family $\{u(t)\}_{t\in[0,+\infty)}$ is called a *generalized minimizing movement* (shortly, GMM) starting from u_0 if there exist a sequence $\tau_i \to 0^+$ and flat flows $\{u(\tau_i, \cdot)\}$ such that

$$\mathcal{S}\text{-}\lim_{i\to+\infty} u(\tau_i, \lfloor t/\tau_i \rfloor) = u(t), \quad t \geq 0,$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$.

The collection of all GMM starting from u_0 and associated to $\mathcal{F}$ will be denoted by $GMM(\mathcal{F}, u_0)$.

In [6] we have applied this definition with the metric space $\mathcal{S} = BV(\Omega; \{0, 1\})$ endowed with the $L^1(\Omega)$-distance $d(E, F) := |E\Delta F|$ and with the functional $\mathcal{F}_\beta$, and provided that $\|\beta\|_\infty < 1$, we have obtained the existence and the 1/2-Hölder continuity in time of GMM starting from any bounded droplet E_0 (see [6, Theorem 7.1] and also Theorem 2.10 below). We call any element of $GMM(\mathcal{F}_\beta, E_0)$ a minimizing movement solution for mean curvature flow of droplets starting from E_0.

Now consider the regular case. Let $\beta \in C^{1+\alpha}(\partial\Omega)$ (for some $\alpha \in (0, 1]$) with $\|\beta\|_\infty < 1$, the initial set E_0 be bounded and the manifold $\Omega \cap \partial E_0$ be a $C^{2+\alpha}$-hypersurface with boundary, satisfying the contact-angle condition (the so-called Young's law [14, 20])

$$\nu_{E_0}(x) \cdot \mathbf{e}_3 = -\beta \quad \text{on } \partial\Omega \cap \overline{\Omega \cap \partial E_0}, \tag{1.2}$$

where ν_E is the outer unit normal to E and $\mathbf{e}_3 = (0, 0, 1) \in \mathbb{R}^3$, then in view of [6, Theorem B.1] there exists a unique family $\{E(t)\}_{t\in[0,T^\dagger)}$, defined up to a maximal time $T^\dagger$, such that $E(0) = E_0$, $E(t)$ satisfies the contact-angle condition (1.2) with $E(t)$ in place of E_0 and the surfaces $\Omega\cap\partial E(t)$ move by their mean curvature (see also Theorem 2.6 below). For simplicity, let us call $\{E(t)\}$ the smooth mean curvature flow starting from E_0 with contact-angle β.

Now we are in position to state the main result of the current paper.

Theorem 1.2 (**Consistency of GMM with smooth mean curvature flow**)**.** *Let $\beta \in C^{1+\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1]$ with $\|\beta\|_\infty < 1$ and E_0 be a bounded set such that $\overline{\Omega \cap \partial E_0}$ is a $C^{2+\alpha}$-manifold with boundary satisfying the contact-angle condition (1.2). Let $\{E(t)\}_{t\in[0,T^\dagger)}$ be the unique mean curvature flow starting from E_0 and with contact-angle β. Then for every $F(\cdot) \in GMM(\mathcal{F}_\beta, E_0)$*

$$E(t) = F(t) \quad \textit{for any } t \in [0, T^\dagger). \tag{1.3}$$

Thus, as in the classical mean curvature flow without boundary [2, 30] the minimizing movement solutions for the mean curvature evolution of droplets in $\mathbb{R}^3$, coincides with the smooth mean curvature flow as long as the latter exists.

To prove Theorem 1.2 we mainly follow the arguments of [2, Theorem 7.3] and construct inner and outer barriers for GMM consisting of small forced perturbations of the smooth mean curvature flow with a slightly perturbed contact-angle (Theorem 2.6). It is worth to notice that due to the presence of boundaries, the methods of [30], which strongly rely on the uniform ball conditions, seem not applicable in our setting.

The paper is organized as follows. In Section 2 we provide some preliminary definitions and results which will be important in the proof of Theorem 1.2. Namely, we study the smooth mean curvature evolution of droplets with prescribed contact-angle β, and its various features such as forced evolution of its small tubular neighborhoods (Theorem 2.6) and comparison principles (Theorem 2.8). Moreover, we recall some properties of the minimizers of $\mathcal{F}_\beta$ from [6] (Theorem 2.9), the existence of GMM (Theorem 2.10) and study the GMM starting from truncated balls (Theorem 2.12). We complete this section with weak comparison properties of inner and outer barriers for minimizers of $\mathcal{F}_\beta$ (Lemma 2.13). These results will be the key arguments in the proof of (1.3) in the concluding Section 3.

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2. Preliminaries

Notation. In this section we introduce the notation and some definitions which will be used throughout the paper. Unless otherwise stated, all sets we consider are Lebesgue measurable subsets of the Euclidean space $\mathbb{R}^3$, in which the coordinates (x_1, x_2, x_3) of $x \in \mathbb{R}^3$ are given with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. By $B_r(x)$ we denote the open ball in $\mathbb{R}^3$ of radius $r > 0$ centered at x. The notion $|F|$ stands for the Lebesgue measure of $F \subset \mathbb{R}^3$.

Throughout the paper we assume

$$\Omega := \mathbb{R}^2 \times (0, +\infty),$$

and by $BV(\Omega; \{0,1\})$ we denote the collection of all sets of finite finite perimeter in Ω.

We represent droplets by elements of $BV(\Omega; \{0,1\})$ and by $\beta \in L^\infty(\partial\Omega)$ we denote a relative adhesion coefficient of the boundary $\partial\Omega = \mathbb{R}^2 \times \{0\}$ of Ω, which satisfies

$$\exists \eta \in (0, 1/2): \quad \|\beta\|_\infty \le 1 - 2\eta. \tag{2.1}$$

Given $E \subset BV(\Omega, \{0,1\})$ we denote by

- $P(E, U)$ the *perimeter* of E in an open set $U \subset \Omega$,
- $\partial^* E$ the reduced boundary of E,
- $\nu_E(x)$ the generalized outer unit normal of E at $x \in \partial^* E$.

In what follows we assume that every E coincides with its points $E^{(1)}$ of density one so that $\partial E = \overline{\partial^* E}$. We refer, for instance, to [4, 24, 33] for a more comprehensive information on sets of finite perimeter.

Given $E \in BV(\Omega; \{0,1\})$, we define the *distance function* from the (reduced) boundary in Ω as

$$\mathrm{d}_E(x) := \mathrm{dist}(x, \Omega \cap \partial^* E), \quad x \in \overline{\Omega}.$$

Similarly, we define the *signed distance* as

$$\mathrm{sd}_E(x) = \begin{cases} \mathrm{dist}(x, \Omega \cap \partial^* E) & x \in \Omega \setminus E, \\ -\mathrm{dist}(x, \Omega \cap \partial^* E) & x \in E \end{cases}$$

for $E \in BV(\Omega; \{0,1\})$. We also write

$$\partial^\Omega E := \Omega \cap \partial E$$

and

$$E \prec F \qquad \Longleftrightarrow \qquad E \subset F \quad \text{and} \quad \mathrm{dist}(\partial^\Omega E, \partial^\Omega F) > 0$$

for $E, F \in BV(\Omega; \{0,1\})$. Note that

$$E \subset F \quad \Longleftrightarrow \quad \mathrm{sd}_E \ge \mathrm{sd}_F \text{ in } \Omega \qquad \text{resp.} \qquad E \prec F \quad \Longleftrightarrow \quad \mathrm{sd}_E > \mathrm{sd}_F \text{ in } \Omega. \tag{2.2}$$

The following proposition shows the connection between the regular surfaces and distance functions.

Proposition 2.1. *Let Γ be a $C^{2+\alpha}$-surface (not necessarily connected, and with or without boundary) in Ω for some $\alpha \in [0,1]$. Then:*

(a) *for any $x \in \Gamma$ there exists $r_x > 0$ such that Γ divides $B_{r_x}(x)$ into two connected components and* $\mathrm{dist}(\cdot, \Gamma) \in C^{2+\alpha}(B_{r_x}(x) \setminus \Gamma)$;
(b) *if Γ is compact and has no boundary, then $\inf_{x\in\Gamma} r_x > 0$, i.e., the radius r_x in (a) can be taken uniform in x*;
(c) *if $\Gamma = \partial^\Omega E$ for some $E \subset \Omega$, then for any $x \in \Gamma$ there exists $r_x > 0$ such that $B_r(x) \subset \Omega$ and $\mathrm{sd}_E \in C^{2+\alpha}(B_{r_x}(x))$.*

These assertions are well-known (see e.g. [17]), and can be proven using the local geometry of Γ, i.e. passing to the local coordinates. In case of Proposition 2.1 (c) we write $\kappa_E := \kappa_\Gamma$ to denote the mean curvature of E along the boundary portion Γ with respect to the unit normal to Γ, outer to E. We also set

$$\|II_E\|_\infty := \sup_{x\in\Gamma} |II_\Gamma(x)|,$$

where II_Γ is the second fundamental form of Γ. In what follows we always assume that the unit normals of $\partial^\Omega E$ are outer to E so that the mean curvature of the boundaries of convex sets are nonnegative.

2.1. **Smooth mean curvature evolution of droplets.** In this section we study mean curvature flow of droplets sitting on an inhomogeneous plane. Since we are mainly interested in droplets with a nonempty contact set on $\partial\Omega$, it is natural to restrict ourselves to the ones without connected components not touching to $\partial\Omega$. Such a restriction leads to the following definition.

Definition 2.2 (**Admissibility**)**.** (a) We say a set $E \subset \Omega$ is *admissible* provided that there exist $\alpha \in (0,1]$, a bounded $C^{2+\alpha}$-open set $\mathcal{U} \subset \mathbb{R}^2$ and a $C^{2+\alpha}$-diffeomorphism $p \in C^{2+\alpha}(\overline{\mathcal{U}}; \mathbb{R}^3)$ satisfying

$$p[\mathcal{U}] = \Gamma, \qquad p[\partial\mathcal{U}] = \partial\Gamma, \qquad p \cdot \mathbf{e}_3 > 0 \text{ in } \mathcal{U} \qquad \text{and} \qquad p \cdot \mathbf{e}_3 = 0 \text{ on } \partial\mathcal{U},$$

where $\Gamma := \partial^\Omega E$. Any such map p is called a *parametrization* of Γ.

(b) Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $\alpha \in (0,1]$, satisfy (2.1). We say E is *admissible with contact angle β* if E is admissible (with the same α) and

$$\nu_E \cdot \mathbf{e}_3 = -\beta \quad \text{on } \partial\Omega \cap \overline{\Gamma}.$$

We call that number

$$h_E := \min_{x\in\overline{\Gamma},\, \nu_E(x) = x + \mathbf{e}_3} x \cdot \mathbf{e}_3 \tag{2.3}$$

the *minimal height* of E. Since E satisfies the contact angle condition, by assumption (2.1) $h_E > 0$.

(c) Let Q be a compact set in $\mathbb{R}^m$ for some $m \geq 1$. We say a family $\{E[q]\}_{q\in Q}$ of subsets of Ω is *admissible* if there exist $\alpha \in (0,1]$, a bounded $C^{2+\alpha}$-open set $\mathcal{U} \subset \mathbb{R}^2$ and a map $p \in C^{2+\alpha,2+\alpha}(Q \times \overline{\mathcal{U}}; \mathbb{R}^3)$ such that $p[q,\cdot]$ is a parametrization of $\partial^\Omega E[q]$.

(d) We say a family $\{E[q,t]\}_{q\in Q, t\in[0,T)}$ of subsets of Ω *admissible* if for any $T' \in (0,T)$ there exist $\alpha \in (0,1]$, a bounded $C^{2+\alpha}$-open set $\mathcal{U} \subset \mathbb{R}^2$ and a map $p \in C^{2+\alpha,1+\frac{\alpha}{2},2+\alpha}(Q \times [0,T'] \times \overline{\mathcal{U}}; \mathbb{R}^3)$ such that $p[q,t,\cdot]$ is a parametrization of $\partial^\Omega E[q,t]$.

Remark 2.3.

(a) *By definition, if E is an admissible set, then the $C^{2+\alpha}$-surface $\Gamma := \partial^\Omega E$ is diffeomorphic to a bounded smooth open set in $\mathbb{R}^2$ and not necessarily connected (clearly, boundaries of two connected components do not touch). In particular, Γ cannot not have "hanging" components $\Subset \Omega$. Moreover, its boundary $\partial\Gamma$ lies on $\partial\Omega$ and the relative interior of Γ does not touch to $\partial\Omega$.*

(b) *We are slightly abusing the notion "contact angle" identifying the (true) contact angle $\theta \in (0,\pi)$ with its cosine $\beta = \cos\theta$.*

(c) *When Q is empty in Definition 2.2 (d), then we simply write $\{E[t]\}_{t\in[0,T)}$ to denote the corresponding admissible family.*

Recall that if $E \subset \mathbb{R}^3$ is a $C^{2+\alpha}$-set without boundary, then for sufficiently small $\rho > 0$ the surfaces $\Gamma_r := \{\mathrm{sdist}(\cdot, \partial E) = r\}$ for $r \in (-\rho,\rho)$ foliates the tubular ρ-neighborhood of $\Gamma_0 := \partial E$, and the map $r \mapsto \Gamma_r$ smoothly varies. In the next lemma we construct a similar "foliation", for admissible sets with a given contact angle.

Lemma 2.4 (**Foliations**). *Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $\alpha \in (0,1]$, satisfy (2.1) and E_0 be an admissible set with contact-angle β. Then there exist positive numbers $\rho \in (0,1)$ and $\sigma \in (0,\eta)$, depending only[1] on $\|II_{E_0}\|_\infty$ and h_{E_0} (see (2.3)), and admissible families $\{G_0^\pm[r,s]\}_{(r,s)\in[0,\rho]\times[0,\sigma]}$ such that $G_0^\pm[0,0] = E_0$ and for all $(r,s) \in [0,\rho]\times[0,\sigma]$:*

(a) $\mathrm{dist}(\partial^\Omega G_0^\pm[r,s], \partial^\Omega E_0) \geq r + s$ *and*

$$\begin{aligned} &G_0^-[r,s] \subset E_0 \subset G_0^+[r,s] \\ &\mathrm{dist}(\partial^\Omega G_0^\pm[r,s], \partial^\Omega G_0^\pm[0,s]) = r, \\ &\mathrm{dist}(\partial^\Omega G_0^\pm[0,s], \partial^\Omega E_0) = s; \end{aligned}$$

(b) $G_0^\pm[r,s]$ *is admissible with contact-angle $\beta \pm s$;*

(c) *for all $r', r'' \in [0, \rho/64]$*

$$G_0^+[3\rho/16 + r', s] \subset G_0^+[\rho/2 - r'', s], \quad G_0^-[3\rho/16 + r', s] \supset G_0^-[\rho/2 - r'', s];$$

and

$$\mathrm{dist}(\partial^\Omega G_0^\pm[\rho,s], \partial^\Omega G_0^\pm[\rho/2 - r', s]) \geq \rho/64.$$

Proof. Without loss of generality we assume that E_0 is admissible with the same $\alpha \in (0,1]$. We divide the proof into three steps.

Step 1. We first construct $\sigma > 0$ and the sets $G^\pm[0,s]$ for $s \in [0,\sigma]$.

Since $\Gamma_0 := \partial^\Omega E_0$ is $C^{2+\alpha}$ up to the boundary, there exists $b > 0$ (depending only on the second fundamental form II_{Γ_0} of Γ_0) and a $C^{2+\alpha}$-surface $\widetilde{\Gamma}_0 \subset \mathbb{R}^2 \times (-b, +\infty)$ with

[1]We ignore the dependence on α and η.

$\partial\Gamma_0 \subset \{x_3 = -b\}$ and $\Gamma_0 \subset \widetilde\Gamma_0$. By Proposition 2.1 (a) there exists $\sigma \in (0, \eta/4)$ (depending only on $\|II_{\widetilde\Gamma}\|_\infty$ and h_{E_0}) such that for any $s \in [-4\sigma, 4\sigma]$ the sets

$$\widetilde\Gamma_s = \begin{cases} \{x \in \Omega \setminus \overline{E_0} :\ \mathrm{dist}(x, \widetilde\Gamma_0) = s\} & s > 0, \\ \{x \in \overline{E_0} :\ \mathrm{dist}(x, \widetilde\Gamma_0) = s\} & s \le 0 \end{cases}$$

are $C^{2+\alpha}$-surfaces with boundary, depending smoothly (at least $C^{2+\alpha}$) on s.

Note that $\gamma_0 := \partial\Gamma_0$ is a finite union of planar $C^{2+\alpha}$-curves. Let $\hat F \subset \partial\Omega$ the bounded planar open set enclosed by γ_0. Decreasing σ is necessary (depending only on the L^∞-norm of the planar curvatures of γ_0 and the minimal height h_{E_0}) such that for any $s \in [-\sigma, \sigma]$ we may assume that the sets

$$\gamma_s := \begin{cases} \{z \in \hat F :\ \mathrm{dist}(z, \gamma_0) = -4s\} & s < 0, \\ \{z \in \partial\Omega \setminus \hat F :\ \mathrm{dist}(z, \gamma_0) = 4s\} & s \ge 0, \end{cases} \qquad \text{and} \qquad \zeta_s := \widetilde\Gamma_s \cap \{x_3 = \sigma\}$$

are a union of $C^{2+\alpha}$-curves, homotopic to γ_0. By the $C^{2+\alpha}$-dependence of γ_s on s we can find a bounded $C^{2+\alpha}$-open set $\mathcal U \subset \mathbb R^2$ and a map $p \in C^{2+\alpha,2+\alpha}([-\sigma,\sigma] \times \partial\mathcal U; \mathbb R^3)$ such that $p[s, \partial\mathcal U] = \gamma_s$ for any $s \in [-\sigma, \sigma]$. Now as in [6, Remark B.2] we can extend each $p[s,\cdot]$ as a diffeomorphism to an ϵ-tubular neighborhood $\mathcal U_r^- := \{u \in \mathcal U :\ \mathrm{dist}(u, \partial\mathcal U) \le \epsilon\}$ of $\partial\mathcal U$ (for small $\epsilon > 0$, still keeping $C^{2+\alpha}$-regularity both in s and in u) such that the surface $p[s, \mathcal U_r^-]$ lies in Ω, satisfies the contact-angle condition with $\beta + s$ along γ_s and the distance to Γ_0 is $\ge 3s$. Let us also parametrize the truncations $\widetilde\Gamma_s \cap \overline{\Omega^\sigma}$ of the surfaces $\widetilde\Gamma_s$ by some diffeomorphism $p[s,\cdot] : \{u \in \mathcal U :\ \mathrm{dist}(u, \partial\mathcal U) \ge 8\epsilon\} \to \mathbb R^3$ (still keeping $C^{2+\alpha}$-regularity in $s \in [-\sigma,\sigma]$), where $\Omega^\sigma := \mathbb R^2 \times (\sigma, +\infty)$. Now we extend p arbitrarily to $[-\sigma,\sigma] \times \{u \in \mathcal U :\ \epsilon \le \mathrm{dist}(u, \partial\mathcal U) \le 8\epsilon\}$ in a way that $p \in C^{2+\alpha,2+\alpha}([-\sigma,\sigma] \times \overline{\mathcal U})$, $p[s,\cdot]$ is a diffeomorphism, $p[0, \mathcal U] = \Gamma_0$ and the distance between surfaces $p[s, \overline{\mathcal U}]$ and Γ_0 is equal to s.

For $s \in [0, \sigma]$ we denote by $G_0^+[0,s]$ and $G_0^-[0,s]$ the bounded sets enclosed by $\partial\Omega$ and the $C^{2+\alpha}$-surfaces $\Gamma_0^+[0,s] := p[s, \overline{\mathcal U}]$ and $\Gamma_0^-[0,s] := p[-s, \overline{\mathcal U}]$, respectively.

Notice that by construction $G_0^\pm[0,0] = E_0$ and $\mathrm{dist}(\partial^\Omega G_0^\pm[0,s], \partial^\Omega E_0) = s$ for any $s \in [0,\sigma]$.

Step 2. Now we construct $\rho > 0$ and $G_0^\pm[r,s]$ for $r \in [0,\rho]$ and $s \in [0,\sigma]$.

Since γ_s is $C^{2+\alpha}$-regular in $s \in [-\sigma,\sigma]$, slightly decreasing σ if necessary, we find $\rho > 0$ depending only on σ, $\|II_{E_0}\|_\infty$ and h_{E_0} such that for any $r \in [0,\rho]$ the sets

$$\gamma_{r,-s}^- := \{z \in \hat F_s :\ \mathrm{dist}(z, \gamma_s) = 4r\}, \quad s \le 0$$

and

$$\gamma_{r,s}^+ := \{z \in \partial\Omega \setminus \hat F_s :\ \mathrm{dist}(z, \gamma_s) = 4r\}, \quad s \ge 0$$

are finite unions of $C^{2+\alpha}$-curves homotopic to γ_s, where $\hat F_s \subset \partial\Omega$ is a bounded set enclosed by γ_s. As above consider the truncations $\Gamma_0^\pm[0,s] \cap \overline{\Omega^\sigma}$. Since these truncations are smooth (at least $C^{2+\alpha}$) family of $C^{2+\alpha}$-surfaces with boundary, using Proposition 2.1 (possibly decreasing ρ and σ depending only on h_{E_0}) we can show that for all $r \in [0,\rho]$ the sets

$$\Sigma_{r,s}^+ := \{x \in \Omega^\sigma \setminus G_0^+[0,s] :\ \mathrm{dist}(x, \Gamma_0^+[0,s]) = r\}$$

and

$$\Sigma_{r,s}^- := \{x \in \Omega^\sigma \cap G_0^-[0,s] :\ \mathrm{dist}(x, \Gamma_0^-[0,s]) = r\}$$

are $C^{2+\alpha}$-families in r and s of $C^{2+\alpha}$-surfaces, whose boundaries are on $\{x_3 = \sigma\}$ and a union of $C^{2+\alpha}$-curves homotopic to γ_0. Now as in step 1 we construct $C^{2+\alpha}$-surfaces $\Gamma_0^\pm[r,s]$ with boundary (still $C^{2+\alpha}$-regular in r and s), first starting from $\gamma_{r,s}^\pm$ satisfying the contact-angle condition with $\beta \pm s$, and then extending until we reach $\Sigma^{\pm}_{r,s}$ such a way that the distance between $\Gamma^{\pm}_0[r,s]$ and $\Gamma^{\pm}_0[0,s]$ is r. Since

$$\mathrm{dist}(\gamma^{\pm}_{\frac{3\rho}{16}+r',s}, \gamma^{\pm}_{\frac{\rho}{2}-r'',s}) \ge \tfrac{9\rho}{32} \quad \text{and} \quad \mathrm{dist}(\gamma^{\pm}_{\rho,s}), \gamma^{\pm}_{\frac{\rho}{2},s}) \ge \tfrac{\rho}{2}$$

for all $r', r'' \in [0, \rho/64]$, we may assume additionally that

$$\overline{\Gamma^{\pm}_0[3\rho/16 + r', s]} \cap \overline{\Gamma^{\pm}_0[\rho/2 - r'', s]} = \emptyset \quad \text{and} \quad \mathrm{dist}(\Gamma^{\pm}_0[\rho, s], \Gamma^{\pm}_0[\rho/2 - r', s]) \ge \rho/64 \qquad (2.4)$$

for any $r', r'' \in [0, \rho/64]$.

Now for any $s \in [0,\sigma]$ and $r \in [0,\rho]$ we denote by $G^{\pm}_0[r,s]$ the bounded set enclosed by $\partial\Omega$ and $\Gamma^{\pm}[r,s]$. By construction and step 1, $G^{\pm}_0[r,s]$ satisfies assertions (a) and (b). Moreover, by (2.4) and assumption $\Omega^\sigma \cap G^-_0[a,s] \Supset \Omega^\sigma \cap G^-_0[b,s]$ resp. $\Omega^\sigma \cap G^+_0[a,s] \Subset \Omega^\sigma \cap G^+_0[b,s]$ for $0 < a < b < \rho$, the sets $G^{\pm}_0[r,s]$ satisfy also assertion (c).

We claim that σ, ρ and $\{G^{\pm}\}$ satisfies the remaining assertions of the lemma. Indeed, σ depends only on $\|II_{E_0}\|_\infty$, h_{E_0}, ρ depends only on σ, h_{E_0} and $\|II_{E_0}\|_\infty$, and $G^{\pm}[\cdot,\cdot]$ admits a parametrization $p^{\pm} \in C^{2+\alpha,2+\alpha}(([0,\rho] \times [0,\sigma]) \times \overline{\mathcal{U}})$, which satisfies the assumptions of Definition 2.2 (c) of admissible family with $Q = [0,\rho] \times [0,\sigma]$. □

Corollary 2.5. *Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $\alpha \in (0,1]$, satisfy (2.1) and $\{E[t]\}_{t\in[0,T)}$ be an admissible family contact angle β. Then for any $T' \in (0,T)$ there exist $\rho \in (0,1)$ and $\sigma \in (0,\eta)$ depending only $\sup_{t\in[0,T']} \|II_{E[t]}\|_\infty$ and $\inf_{t\in[0,T']} h_{E[t]}$, and admissible families $\{G^{\pm}_0[r,s,a]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T']}$ such that $G^{\pm}_0[0,0,a] = E[a]$ and for all $(r,s,a) \in [0,\rho] \times [0,\sigma] \times [0,T']$:*

(a) $\mathrm{dist}(\partial^\Omega G^{\pm}_0[r,s,a], \partial^\Omega E[a]) \ge r + s$ *and*

$$G^-_0[r,s,a] \subset E[a] \subset G^+_0[r,s,a],$$
$$\mathrm{dist}(\partial^\Omega G^{\pm}_0[r,s,a], \partial^\Omega G^{\pm}_0[0,s,a]) = r,$$
$$\mathrm{dist}(\partial^\Omega G^{\pm}_0[0,s,a], \partial^\Omega E[a]) = s;$$

(b) $G^{\pm}_0[r,s,a]$ *is admissible with contact-angle* $\beta \pm s$;

(c) *for all* $r', r'' \in [0, \rho/64]$

$$G^+_0[3\rho/16 + r', s, a] \subset G^+_0[\rho/2 - r'', s, a], \quad G^-_0[3\rho/16 + r', s, a] \supset G^-_0[\rho/2 - r'', s, a];$$

and

$$\mathrm{dist}(\partial^\Omega G^{\pm}_0[\rho, s, a], \partial^\Omega G^{\pm}_0[\rho/2 - r', s, a]) \ge \rho/64.$$

Proof. By the definition of admissibility, $E[\cdot]$ admits a parametrization $p \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T'] \times \overline{\mathcal{U}})$ for any $T' \in (0,T)$. Therefore, repeating the same arguments of Lemma 2.4 we construct the required family $\{G^{\pm}_0[r,s,a]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T']}$. □

Now we study the existence and uniqueness of the mean curvature flow starting from a bounded droplet and its some stability properties.

Theorem 2.6. *Let $\beta \in C^{1+\alpha}(\partial\Omega)$ (for some $\alpha \in (0,1]$) satisfy (2.1) and $E_0 \subset \Omega$ be an admissible set with contact-angle β. Then there exist a maximal time $T^\dagger > 0$ and a unique family $\{E[t]\}_{t\in[0,T^\dagger)}$ of admissible sets in Ω such that $E[0] = E_0$, $E[t]$ is admissible with contact angle β and the hypersurfaces $\partial^\Omega E[t]$ flow by mean curvature, i.e.,*

$$v_{E[t]}(x) = -\kappa_{E[t]}(x) \quad \text{for } t \in [0,T^\dagger) \text{ and } x \in \partial^\Omega E[t], \qquad (2.5)$$

where $v_{E[t]}$ is the normal velocity of $\partial^\Omega E(t)$. Moreover, for $T \in (0,T^\dagger)$, let $\rho \in (0,1)$, $\sigma \in (0,\eta)$ and the families $\{G^{\pm}_0[r,s,a]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T']}$ be given by Corollary 2.5. Then (possibly decreasing ρ and σ slightly, depending only on $\{E(t)\}$) there exist unique admissible families $\{G^{\pm}[r,s,a,t]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T'],t\in[a,T]}$ such that

- $G^{\pm}[r,s,a,a] = G_0^{\pm}[r,s,a]$,
- $G^{\pm}[r,s,a,t]$ *is admissible with contact-angle* $\beta \pm s$,
-
$$v_{G^{\pm}[r,s,a,t]}(x) = -\kappa_{G^{\pm}[r,s,a,t]}(x) \pm s \quad \text{for } t \in (a,T) \text{ and } x \in \partial^{\Omega} G^{\pm}[r,s,a,t]. \tag{2.6}$$

Furthermore,

(a) $G^{\pm}[0,0,a,t] = E[t]$ *for all* $t \in [a,T]$;

(b) *there exists an increasing continuous function* $g : [0,+\infty) \to [0,+\infty)$ *with* $g(0) = 0$ *such that*
$$\max_{x \in \partial^{\Omega} G^{\pm}[0,s,a,t]} \operatorname{dist}(x, \partial^{\Omega} G^{\pm}[0,0,a,t]) \le g(s)$$
for all $s \in [0,\sigma]$, $a \in [0,T]$ *and* $t \in [0,T]$;

(c) *there exists* $t^* \in (0, \rho/64)$ *(independent of* r,s *and* a*) such that*
$$G_0^{+}[\rho/2 - t', s, a] \subset G^{+}[\rho, s, a, a + t'] \quad \text{and} \quad G_0^{-}[\rho/2 - t', s, a] \supset G^{-}[\rho, s, a, a + t'] \tag{2.7}$$
for all $t' \in [0, t^*]$ *with* $a + t' \le T$.

Thus, $\{G^{\pm}[r,s,a,\cdot]\}$ is a mean curvature flow starting from $G_0^{\pm}[r,s,a]$ and with forcing s and contact angle $\beta \pm s$.

Proof. The solvability of (2.5) follows from [6, Theorem B.1] and the solvability of (2.6) follows from the well-posedness of (2.5) together with its smooth dependence on initial datum (see also [2, Theorem 7.1] in the case without boundary). Finally, the assertions (a)-(c) follow from the smooth dependence of $G^{\pm}$ on $[r,s,a,t]$. □

By Proposition 2.1 and the regularity of $G^{\pm}[r,s,a,\cdot]$ in time, (2.6) can be rewritten as
$$\frac{\partial}{\partial t} \operatorname{sd}_{G^{\pm}[r,s,a,t]}(x) = -\kappa_{G^{\pm}[r,s,a,t]}(x) + s \qquad \text{for } t \in (a,T) \text{ and } x \in \partial^{\Omega} G^{\pm}[r,s,a,t]. \tag{2.8}$$

Proposition 2.7. *For any* $s \in (0,\sigma]$ *there exists* $\tau_0(s) > 0$ *such that for any* $r \in [0,\rho]$, $a \in [0,T)$, $\tau \in (0,\tau_0)$ *and* $t \in [a+\tau, T]$
$$\frac{\operatorname{sd}_{G^{+}[r,s,a,t-\tau]}(x)}{\tau} > -\kappa_{G^{+}[r,s,a,t]}(x) + \frac{s}{2}, \quad x \in \partial^{\Omega} G^{+}[r,s,a,t], \tag{2.9}$$
and
$$\frac{\operatorname{sd}_{G^{-}[r,s,a,t-\tau]}(x)}{\tau} < -\kappa_{G^{-}[r,s,a,t]}(x) - \frac{s}{2}, \quad x \in \partial^{\Omega} G^{+}[r,s,a,t].$$

Proof. We prove the assertion only for G^{+}. Let
$$g(r,s,a,t,x) := \operatorname{sd}_{G^{+}[r,s,a,t]}(x), \quad t \in [a,T],\ x \in \Omega.$$
By the C^2-regularity of $\Gamma[r,s,a,t] := \partial^{\Omega} G^{+}[r,s,a,t]$ (up to the boundary) as well as its smooth dependence on r,s,a,t, there exists $R_0 > 0$ such that for any $r \in [0,\rho]$, $s \in [0,\sigma]$, $a \in [0,T]$, $t \in [a,T]$, $x \in \Gamma[r,s,a,t]$ and $y \in \Omega \cap \overline{B_{R_0}(x)}$ the projection $\pi[r,s,a,t,y]$ onto $\overline{\Gamma[r,s,a,t]}$ is a singleton. Note that if $\pi[r,s,a,t,y] \in \Gamma(t)$, then $y - \pi[r,s,a,t,y]$ is parallel to the unit normal $\nu_{\Gamma[r,s,a,t]}(\pi[r,s,a,t,y])$. In particular,
$$g(r,s,a,t,y) = \begin{cases} (y - \pi[r,s,a,t,y]) \cdot \nu_{\Gamma[r,s,a,t]}(\pi[r,s,a,t,y]) & \text{if } \pi[r,s,a,t,y] \in \Gamma[r,s,a,t], \\ |y - \pi[r,s,a,t,y]| & \text{if } \pi[r,s,a,t,y] \in \partial\Gamma[r,s,a,t]. \end{cases}$$
Let $p[r,s,a,t,\cdot] : \mathcal{U} \to \mathbb{R}^3$ be a parametrization of $\{\Gamma[r,s,a,t]\}$ (smoothly depending on r,s,a,t). Then there exists a unique $u_{r,s,a,t,y} \in \overline{\mathcal{U}}$ such that
$$\pi[r,s,a,t,y] = p[r,s,a,t,u_{r,s,a,t,y}], \quad y \in B_{R_0}(x).$$

The uniqueness of $u_{r,s,a,t,y}$ and the regularity of the diffeomorphism p as well as the implicit function theorem at boundary [16] imply that the map $t \mapsto u_{r,s,a,t,y}$ is continuously differentiable in $t \in [a,T]$ uniformly in r,s,a,y. Thus, $t \mapsto g(r,s,a,t,y)$ is also continuously differentiable in $t \in [a,T]$ and the map $t \mapsto g_t(r,s,a,t,y)$ is uniformly continuous. Then in view of (2.8), for any $s \in (0,\sigma]$ there exists $\tau_0(s) > 0$ for which (2.9) holds for any $r \in [0,\rho]$, $a \in [0,T)$, $\tau \in (0,\tau_0)$ and $t \in [a+\tau, T]$. $\square$

As in the standard mean curvature flow of (compact) hypersurfaces without boundary, the smooth mean curvature flow of droplets also enjoys comparison principles, see also [6, Proposition B.4].

Theorem 2.8 (**Strong comparison**). *Let $\{E_1(t)\}_{t\in[0,T^\dagger)}$ and $\{E_2(t)\}_{t\in[0,T^\dagger)}$ be smooth flows with forcing s_1 and s_2 and contact-angles β_1 and β_2, respectively. Assume that $E_1(0) \prec E_2(0)$, $s_1 \le s_2$ and $\beta_1 < \beta_2$ on $\partial\Omega$. Then $E_1(t) \prec E_2(t)$ for all $t \in [0,T^\dagger)$.*

Proof. Let $\bar t \in (0,T^\dagger)$ be the first contact time of $\overline{\partial^\Omega E_1(\cdot)}$ and $\overline{\partial^\Omega E_2(\cdot)}$. By the contact angle condition and the assumption $\beta_1 > \beta_2$, a contact point x_0 cannot be on $\partial\Omega$. Therefore, from the inclusion $E_1(\bar t) \subset E_2(\bar t)$ we find $\kappa_{E_1(\bar t)}(x_0) \ge \kappa_{E_2(\bar t)}(x_0)$, and hence,from the evolution equation and the assumption $s_1 \le s_2$ we get

$$v_{E_1(\bar t)}(\bar t, x_0) - v_{E_2(\bar t)}(\bar t, x_0) = -\kappa_{E_1(\bar t)}(x_0) + s_1 + \kappa_{E_2(\bar t)}(x_0) - s_2 \le 0.$$

Now using the Hamilton trick (see e.g. [34, Chapter 2]) we conclude that the distance between $\partial^\Omega E_1(t)$ and $\partial^\Omega E_2(t)$ is nondecreasing in $(\bar t - \epsilon, \bar t)$ for small $\epsilon > 0$. In particular, $\overline{\partial^\Omega E_1(\bar t)} \cap \overline{\partial^\Omega E_2(\bar t)} = \emptyset$, a contradiction. $\square$

2.2. **GMM for mean curvature flow of droplets.** Notice that the capillary Almgren-Taylor-Wang functional (1.1) can be rewritten as

$$\mathcal{F}_\beta(E; E_0, \tau) = \mathcal{C}_\beta(E,\Omega) + \frac{1}{\tau}\int_E \mathrm{sd}_{E_0}\,dx - \frac{1}{\tau}\int_{E_0} \mathrm{sd}_{E_0}\,dx. \tag{2.10}$$

Let us recall some properties of $\mathcal{F}_\beta$ and its minimizers from [6].

Theorem 2.9. *Let $E_0 \in BV(\Omega;\{0,1\})$ be bounded, $\tau > 0$ and $\beta \in L^\infty(\partial\Omega)$ satisfy (2.1).*

(a) *The functional $\mathcal{F}_\beta(\cdot; E_0,\tau)$ is $L^1(\Omega)$-lower semicontinuous.*
(b) *There exists a minimizer E_τ of $\mathcal{F}_\beta(\cdot; E_0,\tau)$ and every minimizer of $\mathcal{F}_\beta(\cdot; E_0,\tau)$ is bounded.*
(c) *There exists a bounded set E_+ containing E_0 such that for any $F_0 \subset E_+$ the minimizer of $\mathcal{F}_\beta(\cdot; F_0,\tau)$ is a subset of E_+.*
(d) *There exists $\vartheta \in (0,1/2)$ depending only on η such that for any minimizer E_τ of $\mathcal{F}_\beta(\cdot; E_0,\tau)$*
$$\sup_{x\in \overline{E_\tau \Delta E_0}} \mathrm{d}_E(x) \le \tfrac{1}{\vartheta}\sqrt{\tau}. \tag{2.11}$$
Moreover, for any ball $B_r(x)$ centered at $x \in \overline{\Omega}$,
$$P(E_\tau, B_r(x)) \le \tfrac{1}{\vartheta} r^2, \quad r > 0,$$
and for any ball $B_r(x)$ centered at $x \in \partial E_\tau$
$$\vartheta \le \tfrac{|B_r(x)\cap E_\tau|}{|B_r(x)|} \le 1-\vartheta \qquad \text{and} \qquad P(E_\tau, B_r(x)) \ge \vartheta r^2$$
whenever $r \in (0, \vartheta\sqrt{\tau})$. In particular, E_τ can be assumed open and $\mathcal{H}^2(\partial E_\tau \setminus \partial^* E_\tau) = 0$.
(e) *There exist unique minimal and maximal minimizers $E_{\tau *}$ and E_τ^* of $\mathcal{F}_\beta(\cdot; E_0,\tau)$ such that $E_{\tau *} \subset E_\tau \subset E_\tau^*$ for any minimizer E_τ.*
(f) *If $F_0 \subset E_0$, then for any minimizers F_τ and E_τ of $\mathcal{F}_\beta(\cdot; F_0,\tau)$ and $\mathcal{F}_\beta(\cdot; E_0,\tau)$ one has $F_\tau \subset E_\tau^*$ and $F_{\tau *} \subset E_\tau$, where $F_{\tau *}$ and E_τ^* are the minimal and maximal minimizers of $\mathcal{F}_\beta(\cdot; F_0,\tau)$ and $\mathcal{F}_\beta(\cdot; E_0,\tau)$, respectively.*

(g) *Let β be C^1 on $\partial\Omega$, E_τ be a minimizer of $\mathcal{F}_\beta(\cdot;E_0,\tau)$ and $\Gamma := \partial^\Omega E_\tau$. Then by* [15, Theorem 1.5] *and the standard regularity theory for minimizers of the presribed curvature functional, Γ is a $C^{2+\gamma}$-hypersurface with boundary (for some $\gamma\in(0,1]$) and $\nu_{E_\tau}\cdot \mathbf{e}_3 = -\beta$ on $\partial\Gamma$.*

Using the statements (a)-(d) in Theorem 2.9 we can establish

Theorem 2.10 (**Existence of GMM** [6]). *For any bounded $E_0 \in BV(\Omega;\{0,1\})$ the $GMM(\mathcal{F}_\beta,E_0)$ is nonempty. Moreover, there exists $C>0$ such that every $E(\cdot)\in GMM(\mathcal{F}_\beta,E_0)$ is bounded uniformly in time and satisfies*

$$|E(s)\Delta E(t)| \le C|t-s|^{1/2},\quad t,s>0.$$

If, additionally, $|\partial E_0|=0$, then this inequality holds for all $s,t\ge 0$.

2.3. **Evolution of truncated balls.** In this section we study the GMM starting from truncated balls, which generalizes [2, Theorem 5.4]. Notice that due to the presence of Ω and β in $\mathcal{F}_\beta$, we cannot directly apply [2, Theorem 5.4]. In particular, we cannot use a "passage-to-complements" argument.

We start with the following technical lemma.

Lemma 2.11. *Let $\tau\in(0,1)$, $E_0\in BV(\Omega;\{0,1\})$ be bounded, E_τ be any minimizer of $\mathcal{F}_\beta(\cdot;E_0,\tau)$, $p\in\Omega$ and $r>0$. Then*

$$\Omega\cap B_r(p)\subset E_0 \qquad \Longrightarrow \qquad \Omega\cap B_{r-\frac{5}{2r}\tau}(p)\subset E_\tau, \tag{2.12}$$

$$\Omega\cap B_r(p)\cap E_0=\emptyset \qquad \Longrightarrow \qquad \Omega\cap B_{r-\frac{5}{2r}\tau}(p)\cap E_\tau=\emptyset \tag{2.13}$$

whenever $\tau<\min\left\{\frac{2r^2}{5},\frac{\vartheta^2r^2}{25}\right\}$.

Proof. For shortness we write $B_\rho := B_\rho(p)$ for $\rho>0$. Let $F_0 := B_r\cap\Omega$ and $F_{\tau*}$ be the minimal minimizer of $\mathcal{F}_\beta(\cdot;F_0,\tau)$, see Theorem 2.9 (e). By (2.11)

$$\sup_{x\in\overline{F_{\tau*}\Delta F_0}} \operatorname{dist}(x,\partial^\Omega F_0)\le \tfrac{1}{\vartheta}\sqrt{\tau},\qquad \sup_{x\in\overline{E_\tau\Delta E_0}} \operatorname{dist}(x,\partial^\Omega F_0)\le \tfrac{1}{\vartheta}\sqrt{\tau}. \tag{2.14}$$

(a) By Theorem 2.9 (f) $F_{\tau*}\subset E_\tau$. Since $\operatorname{dist}(\partial B_a,\partial B_b)=|a-b|$, by (2.14)

$$\operatorname{dist}(\partial^\Omega F_{\tau*},\partial^\Omega B_{4r/5})\ge \operatorname{dist}(\partial^\Omega B_r,\partial^\Omega B_{4r/5})-\operatorname{dist}(\partial^\Omega F_{\tau*},\partial^\Omega B_r)\ge \tfrac{r}{5}-\tfrac{\sqrt{\tau}}{\vartheta}>0$$

provided that $\tau<\frac{\vartheta^2r^2}{25}$. Further we work only with such τ. Let

$$\rho := \sup\{t\in(0,r]:\ B_t\cap\Omega\subset F_{\tau*}\}.$$

Clearly, $\rho\ge 4r/5$. Note that if $\rho=r$, then $B_r\cap\Omega\subset F_{\tau*}\subset E_\tau$ and we are done. So assume $\rho\in[4r/5,r)$.

Fix $\epsilon\in(0,r-\rho)$. By the definition of ρ

$$|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|\to 0 \qquad \text{as } \epsilon\to 0^+,$$

where $\Omega^c := \mathbb{R}^n\setminus\Omega$. Moreover, by the minimality of $F_{\tau*}$ and (2.10)

$$\begin{aligned}0\le \mathcal{F}_\beta(F_{\tau*}\cup[B_{\rho+\epsilon}\cap\Omega];F_0,\tau)-\mathcal{F}_\beta(F_{\tau*};F_0,\tau)&=P(F_{\tau*}\cup[B_{\rho+\epsilon}\cap\Omega])-P(F_{\tau*})\\&+\int_{\partial\Omega}[\beta-1]\chi_{[\Omega\cap B_{\rho+\epsilon}]\setminus F_{\tau*}}\,d\mathcal{H}^{n-1}+\frac{1}{\tau}\int_{[\Omega\cap B_{\rho+\epsilon}]\setminus F_{\tau*}}\operatorname{sd}_{F_0}\,dx.\end{aligned} \tag{2.15}$$

Then for a.e. $\epsilon\in(0,r-\rho)$ with $\mathcal{H}^{n-1}(\partial\Omega\cap\partial B_{\rho+\epsilon})=0$ and $\mathcal{H}^{n-1}(\partial F_{\tau*}\cap\partial B_{\rho+\epsilon})=0$ one has

$$\begin{aligned}P(F_{\tau*}\cup[B_{\rho+\epsilon}\cap\Omega])-P(F_{\tau*})=&P(B_{\rho+\epsilon})-P([B_{\rho+\epsilon}\cap F_{\tau*}]\cup[\Omega^c\cap B_{\rho+\epsilon}])\\ \le&3|B_1|^{1/3}\Big(|B_{\rho+\epsilon}|^{\frac23}-\big|[B_{\rho+\epsilon}\cap F_{\tau*}]\cup[\Omega^c\cap B_{\rho+\epsilon}]\big|^{\frac23}\Big)\end{aligned}$$

$$
\begin{aligned}
&=3|B_1|^{1/3}|B_{\rho+\epsilon}|^{\frac23}\Big(1-\Big|1-\tfrac{|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|}{|B_{\rho+\epsilon}|}\Big|^{\frac23}\Big)\\
&\le 2|B_1|^{1/3}\,\tfrac{|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|}{|B_{\rho+\epsilon}|^{1/3}}+o(|B_{\rho+\epsilon}\setminus[F_\tau\cup\Omega^c]|)
\end{aligned}
$$

as $\epsilon\to 0^+$. Here in the first inequality we used the isoperimetric inequality. Furthermore, since $B_{\rho+\epsilon}\subset B_r$ so that

$$-\mathrm{sd}_{F_0}=d_{F_0}\ge r-\rho-\epsilon>0$$

in $B_{\rho+\epsilon}$,

$$\int_{[\Omega\cap B_{\rho+\epsilon}]\setminus F_{\tau*}}\mathrm{sd}_{E_0}\,dx\le-\tfrac{r-\rho-\epsilon}{\tau}\,|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|.$$

Finally, since $\beta\le 1$, from (2.15) we get

$$\Big(\tfrac{2|B_1|^{1/3}}{|B_{\rho+\epsilon}|^{1/3}}+o(1)-\tfrac{r-\rho-\epsilon}{\tau}\Big)|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|\ge 0.$$

Since $|B_{\rho+\epsilon}\setminus[F_{\tau*}\cup\Omega^c]|>0$ and $|B_{\rho+\epsilon}|^{1/3}=(\rho+\epsilon)|B_1|^{1/3}$, from the last inequality we deduce

$$\frac{r-\rho-\epsilon}{\tau}\le\frac{2}{\rho+\epsilon}+o(1).$$

Therefore, letting $\epsilon\to0^+$ and recalling $\rho\ge 4r/5$ we deduce

$$\rho\ge r-\tfrac{5}{2r}\tau,$$

which is positive provided that $\tau<\frac{2r^2}{5}$. This implies (2.12).

(b) By (2.14) and assumption $E_0\cap B_r=\emptyset$

$$\mathrm{dist}(\partial^\Omega E_\tau,\partial^\Omega B_{4r/5})\ge\mathrm{dist}(\partial^\Omega E_0,\partial^\Omega B_{4r/5})-\mathrm{dist}(\partial^\Omega E_\tau,\partial^\Omega E_0)\ge\tfrac{r}{5}-\tfrac{\sqrt{\tau}}{\vartheta}>0$$

provided that $\tau<\frac{\vartheta^2r^2}{25}$. In particular, if

$$\rho=\sup\{r\in(0,r]:\ B_t\cap E_\tau=\emptyset\}$$

then $\rho\ge 4r/5$. Without loss of generality we assume that $\rho<r$ and fix any small $\epsilon\in(0,r-\rho)$ such that $\mathcal{H}^{n-1}(\partial B_{\rho+\epsilon}\cap(\partial\Omega\cup\partial E_\tau)=0$. By the maximality of ρ, $|[E_\tau\cap\Omega^\epsilon]\cap B_{\rho+\epsilon}|>0$ for small $\epsilon>0$, where $\Omega^\epsilon:=\mathbb{R}^{n-1}\times(\epsilon,+\infty)$. Let $G_\epsilon:=B_{\rho+\epsilon}\cap\Omega^\epsilon$. Then by the minimality of E_τ and (2.10)

$$0\le\mathcal{F}_\beta(E_\tau\setminus G_\epsilon;E_0,\tau)-\mathcal{F}_\beta(E_\tau;E_0,\tau)=P(E_\tau\setminus G_\epsilon)-P(E_\tau)-\tfrac1\tau\int_{E_\tau\cap G_\epsilon}\mathrm{sd}_{E_0}\,dx.\quad(2.16)$$

Since $B_{\rho+\epsilon}\subset B_r$ and $B_r\cap E_0=\emptyset$, for any $x\in E_\tau\cap G_\epsilon$

$$\mathrm{sd}_{E_0}(x)=d_{E_0}(x)\ge r-\rho-\epsilon>0.$$

Thus, the volume term of (2.16) is estimated as

$$\tfrac1\tau\int_{E_\tau\cap G_\epsilon}\mathrm{sd}_{E_0}\,dx\ge\tfrac{r-\rho-\epsilon}{\tau}\,|E_\tau\cap G_\epsilon|.$$

For the perimeter term, as in (a)

$$P(E_\tau\setminus G_\epsilon)-P(E_\tau)=P(B_{\rho+\epsilon})-P(B_{\rho+\epsilon}\setminus[E_\tau\cap\Omega^\epsilon])\le\Big(\tfrac{2}{\rho+\epsilon}+o(1)\Big)|E_\tau\cap G_\epsilon|.$$

Inserting these estimates in (2.16) and letting $\epsilon\to0^+$ we get $\rho\ge r-\frac{5}{2r}\tau$ and (2.13) follows. □

Applying this lemma inductively we get

Theorem 2.12. *Let $\tau \in (0,1)$, $E_0 \in BV(\Omega;\{0,1\})$ be a bounded set, $\{E(\tau,k)\}$ be any flat flows starting from E_0 and associated to $\mathcal{F}_\beta$, and let $p \in \overline{\Omega}$ and $R>0$. Then*

$$\Omega \cap B_R(p) \subset E_0 \qquad \Longrightarrow \qquad \Omega \cap B_{R-\frac{80}{R}k\tau}(p) \subset E(\tau,k) \tag{2.17}$$

$$B_R(p) \cap E_0 = \emptyset \qquad \Longrightarrow \qquad B_{R-\frac{80}{R}k\tau}(p) \cap E(\tau,k) = \emptyset \tag{2.18}$$

for all integers $k \ge 0$ with $k\tau \le \widehat{T} := \min\Big\{\frac{7R^2}{640}, \frac{\vartheta^2 R^2}{1600}\Big\}$.

Proof. For shortness, let $B_r := B_r(p)$. Let $r_0 := R$ and

$$r_k := r_{k-1} - \tfrac{10}{r_{k-1}}\tau, \quad k \ge 1.$$

By induction we can show that if $m\tau \le \frac{7R^2}{640}$ for some m, then $r_m \ge \frac{R}{8}$. In particular, for such m using $R = r_0 > r_1 > \ldots > r_m \ge R/8$ we get

$$r_m = R - 10\tau \sum_{k=1}^m \tfrac{1}{r_k} \ge R - \tfrac{80}{R}\, m\tau. \tag{2.19}$$

Now we fix any integer $m \ge 1$ and $\tau \in (0,1)$ with $m\tau < \widehat{T}$. Then as we observed earlier, $R = r_0 > r_1 > \ldots > r_{m-1} > r_m \ge \frac{R}{8}$. In particular, for any $k = 1,\ldots,m-1$, from the positivity of r_k we deduce

$$\tau < \tfrac{r_{k-1}^2}{10} \tag{2.20}$$

and also from the estimate $r_{k-1} > R/8$ and the definition of $\widehat{T}$

$$\tau \le \tfrac{\widehat{T}}{m} \le \tfrac{\vartheta^2}{25}\Big(\tfrac{R}{8}\Big)^2 \le \tfrac{\vartheta^2 r_{k-1}^2}{25}. \tag{2.21}$$

In view of (2.20) and (2.21) we can apply Lemma 2.11 with $r = r_{k-1}$ and τ. In particular, by induction

$$\Omega \cap B_R \subset E_0 \qquad \Longrightarrow \qquad \Omega \cap B_{r_{k-1} - \frac{5}{2r_{k-1}}\tau} \subset E(\tau,k)$$

and

$$B_R \cap E_0 = \emptyset \qquad \Longrightarrow \qquad B_{r_{k-1} - \frac{5}{2r_{k-1}}\tau} \cap E(\tau,k) = \emptyset$$

for all $k = 1,\ldots,m$. Now by definition and the estimate (2.19) (which holds with $m = k$) and the inequality $r_k \ge R/8$ we find

$$r_{k-1} - \tfrac{5}{2r_{k-1}}\tau = r_k \ge R - \tfrac{80}{R}k\tau.$$

Inserting this in the last two relations we deduce (2.17) and (2.18). □

2.4. **Smooth barriers for minimizers of $\mathcal{F}_\beta$.** The aim of this section is the following analogue of [2, Lemma 7.3].

Lemma 2.13. *Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $\alpha \in (0,1]$, satisfy (2.1), $E_0 \in BV(\Omega;\{0,1\})$ be a bounded set, $\tau > 0$ and E_τ be a minimizer of $\mathcal{F}_\beta(\cdot;E_0,\tau)$. Let G_0 and G_τ be admissible sets (with the same α).*

(a) *Assume that $E_0 \subset G_0$, $E_\tau \subset G_\tau$, G_τ satisfies the contact-angle condition with $\beta + s$ for some $s \in (0,\eta)$ and*

$$\frac{\mathrm{sd}_{G_0}(x)}{\tau} > -\kappa_{G_\tau}(x), \quad x \in \partial^\Omega G_\tau. \tag{2.22}$$

Then $E_\tau \prec G_\tau$.

(b) *Assume that* $G_0 \subset E_0$, $G_\tau \subset E_\tau$, G_τ *satisfies the contact-angle condition with* $\beta - s$ *for some* $s \in (0,\eta)$ *and*

$$\frac{\mathrm{sd}_{G_0}(x)}{\tau} < -\kappa_{G_\tau}(x), \quad x \in \partial^\Omega G_\tau.$$

Then $G_\tau \prec E_\tau$.

Proof. (a) By the regularity of E_τ (see Theorem 2.9 (f)), $\overline{\partial^\Omega E_\tau}$ is a C^2-hypersurface with boundary, and hence, by the first variation formula,

$$\frac{\mathrm{sd}_{E_0}(x)}{\tau} = -\kappa_{E_\tau}(x), \quad x \in \partial^\Omega E_\tau, \quad \text{and} \quad \nu_{E_\tau}(x)\cdot \mathbf{e}_3 = -\beta(x), \quad x \in \partial\Omega \cap \overline{\partial^\Omega E_\tau}. \tag{2.23}$$

By contradiction, let there exist $x_0 \in \overline{\partial^\Omega E_\tau} \cap \overline{\partial^\Omega G_\tau}$. By assumption $E_\tau \subset G_\tau$ and the contact-angle condition, $x_0 \in \Omega$ and $\kappa_{E_\tau}(x_0) \ge \kappa_{G_\tau}(x_0)$. On the other hand, by assumption $E_0 \subset G_0$ and (2.2) $\mathrm{sd}_{E_0}(x_0) \ge \mathrm{sd}_{G_0}(x_0)$, and therefore, from (2.22) and the first equality in (2.23) it follows that

$$\frac{\mathrm{sd}_{G_0}(x_0)}{\tau} \le \frac{\mathrm{sd}_{E_0}(x_0)}{\tau} = -\kappa_{E_\tau}(x_0) \le -\kappa_{G_\tau}(x) < \frac{\mathrm{sd}_{G_0}(x_0)}{\tau},$$

a contradiction.

(b) is analogous. $\square$

3. Proof of Theorem 1.2

Let $\{E(t)\}_{t\in[0,T^\dagger)}$ be a smooth mean curvature flow starting from E_0 and with contact-angle β, and let $F(\cdot) \in GMM(\mathcal{F}_\beta, E_0)$. Following [2, Theorem 7.4] we fix any $T \in (0,T^\dagger)$ and show

$$E(t) = F(t) \quad \text{for any } 0 < t < T. \tag{3.1}$$

Let $\rho \in (0,1)$, $\sigma \in (0,\eta)$, the smooth flows $\{G^\pm[r,s,a,t] :\ (r,s,a) \in [0,\rho]\times[0,\sigma]\times[0,T],\ t \in [a,T]\}$ starting from $\{G_0^\pm[r,s,a]\}$, and $t^* > 0$ be given by the second part of Theorem 2.6. Let $\tau_j \searrow 0$ and flat flows $\{F(\tau_j,k)\}_{k\ge 0}$ be such that $F(\tau_j,0) = E_0$, $F(\tau_j,k)$ for $k \ge 1$ is a minimizer of $\mathcal{F}_\beta(\cdot;F(\tau_j,k-1),\tau_j)$ and

$$\lim_{j\to+\infty} |F(\tau_j,\lfloor t/\tau_j\rfloor)\Delta F(t)| = 0 \quad \text{for all } t \ge 0. \tag{3.2}$$

For $s \in (0,\sigma]$ let $\tau_0(s) > 0$ be given by Proposition 2.7.

We start with an ancillary technical lemma.

Lemma 3.1. *Assume that* $t_0 \in [0,T)$ *and* $k_0 \in \mathbb{N}_0$ *are such that*

$$G_0^-[0,s,t_0] \subset F(\tau_j,k_0) \subset G_0^+[0,s,t_0]. \tag{3.3}$$

Then there exists $\bar t \in (0,t^*]$ *depending only on* t^* *and* ρ *such that*

$$G^-[0,s,t_0,t_0+k\tau_j] \subset F(\tau_j,k_0+k) \subset G^+[0,s,t_0,t_0+k\tau_j]$$

for all $s \in (0,\sigma]$, $j \ge 1$ *with* $\tau_j \in (0,\tau_0(s))$ *and* $k = 0,1,\dots,\lfloor \bar t/\tau_j\rfloor$ *with* $t_0 + k\tau_j < T$. *Moreover, let* $t_0 + \bar t < T$, *the increasing continuous function* g *be given by Theorem 2.6 (b) and* $\bar\sigma \in (0,\sigma/2)$ *be such that* $4g(2\bar\sigma) < \sigma$. *Then for any* $s \in (0,\bar\sigma)$ *there exists* $\bar j(s) > 1$ *such that*

$$G_0^-[0,4g(2s),t_0+\bar t] \subset F(\tau_j,k_0+\bar k_j) \subset G_0^+[0,4g(2s),t_0+\bar t] \tag{3.4}$$

whenever $j > \bar j(s)$, *where* $\bar k_j := \lfloor \bar t/\tau_j\rfloor$.

Proof. By Corollary 2.5 (a) and (3.3)

$$G_0^-[\rho/64,s,t_0] \prec G_0^-[0,s,t_0] \subset F(\tau_j,k_0) \subset G_0^+[0,s,t_0] \prec G_0^+[\rho/64,s,t_0]. \tag{3.5}$$

Again by Corollary 2.5 (a)

$$\mathrm{dist}(\partial^\Omega G_0^\pm[\rho/64,s,t_0], \partial^\Omega G_0^\pm[0,s,t_0]) = \rho/64,$$

and hence, by (3.5) $B_{\rho/64}(x) \subset F(\tau_j, k_0)$ if $x \in G_0^-[\rho, s, t_0]$ and $B_{\rho/64}(x) \cap F(\tau_j, k_0) = \emptyset$ if $x \in \Omega \setminus G_0^+[\rho, s, t_0]$. Therefore, using Theorem 2.12 and again (3.3) we obtain

$$\begin{cases} B_{\frac{\rho}{64} - \frac{5120}{\rho}\tau_j k}(x) \subset F(\tau_j, k_0 + k) & x \in G_0^-[\rho, s, t_0], \\ B_{\frac{\rho}{64} - \frac{5120}{\rho}\tau_j k}(x) \cap F(\tau_j, k_0 + k) = \emptyset & x \in \Omega \setminus G_0^+[\rho, s, t_0], \end{cases} \quad k = 0, 1, \ldots, \lfloor t^{**}/\tau_j \rfloor, \tag{3.6}$$

where

$$t^{**} := \min\left\{\tfrac{7\rho^2}{64^2\cdot 640}, \tfrac{\vartheta^2\rho^2}{64^2\cdot 1600}\right\}.$$

Since $\mathrm{dist}(\partial^\Omega G_0^\pm[r, s, t_0], \partial^\Omega G_0^\pm[0, s, t_0]) = r$ (Corollary 2.5 (a)), from (3.3) and (3.6) we deduce

$$G_0^-\left[\tfrac{3\rho}{16} + \tfrac{\rho}{64} - \tfrac{5120}{\rho}\, k\tau_j, s, t_0\right] \subset F(\tau_j, k_0 + k) \subset G_0^+\left[\tfrac{3\rho}{16} + \tfrac{\rho}{64} - \tfrac{5120}{\rho}\, k\tau_j, s, t_0\right] \tag{3.7}$$

for all $0 \le k \le \lfloor t^{**}/\tau_j \rfloor$. Now applying Corollary 2.5 (c) with $r' = \frac{\rho}{64} - \frac{5120}{\rho}\, k\tau_j$ and $r'' = k\tau_j \le t^{**} < \rho/64$, we further estimate (3.7) as

$$G_0^-[\tfrac{\rho}{2} - k\tau_j, s, t_0] \subset F(\tau_j, k_0 + k) \subset G_0^+[\tfrac{\rho}{2} - k\tau_j, s, t_0], \quad k = 0, 1, \ldots, \lfloor t^{**}/\tau_j \rfloor. \tag{3.8}$$

Set

$$\bar t := \min\left\{t^*, t^{**}\right\},$$

where t^* is given by Theorem 2.6 (c). Then by (2.7) and (3.8)

$$G_0^-[\rho, s, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k) \subset G_0^+[\rho, s, t_0 + k\tau_j], \quad k = 0, 1, \ldots, \lfloor \bar t/\tau_j \rfloor, \tag{3.9}$$

with $t_0 + k\tau_j < T$. We claim for such k and $j \ge 1$ with $\tau_j \in (0, \tau_0(s))$

$$G^-[0, s, t_0, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k) \subset G^+[0, s, t_0, t_0 + k\tau_j].$$

Indeed, let

$$\bar r := \inf\left\{r \in [0, \rho] : \ F(\tau_j, k_0 + k) \subset G^+[r, s, t_0, t_0 + k\tau_j] \quad k = 0, 1, \ldots, \lfloor \bar t/\tau_j \rfloor, \ t_0 + k\tau_j < T\right\}.$$

By (3.9) the infimum is taken over a nonempty set. By contradiction, assume that $\bar r > 0$. By the continuity of $G^+[r, s, t_0, t_0 + k\tau_j]$ at $r = \bar r$, there exists the smallest integer $k \le \lfloor \bar t/\tau_j \rfloor$ (clearly, $k > 0$ by (3.5)) such that

$$\overline{\partial^\Omega F(\tau_j, k_0 + k)} \cap \overline{\partial^\Omega G^+[\bar r, s, t_0, t_0 + k\tau_j]} \ne \emptyset. \tag{3.10}$$

By the minimality of $k \ge 1$

$$F(\tau_j, k_0 + k - 1) \subset G^+[\bar r, s, t_0, t_0 + (k-1)\tau_j], \qquad F(\tau_j, k_0 + k) \subset G^+[\bar r, s, t_0, t_0 + k\tau_j].$$

Moreover, by construction $G^+[\bar r, s, t_0, t_0 + k\tau_j]$ satisfies the contact angle condition with $\beta + s$ and by Proposition 2.7 applied with $\tau = \tau_j \in (0, \tau_0(s))$

$$\frac{\mathrm{sd}_{G^+[\bar r, s, t_0, t_0 + (k-1)\tau_j]}(x)}{\tau_j} > -\kappa_{G^+[\bar r, s, t_0, t_0 + k\tau_j]}(x) + \frac{s}{2}, \quad x \in \partial^\Omega G^+[\bar r, s, t_0, t_0 + k\tau_j]$$

and

$$\frac{\mathrm{sd}_{G^-[\bar r, s, t_0, t_0 + (k-1)\tau_j]}(x)}{\tau_j} < -\kappa_{G^-[\bar r, s, t_0, t_0 + k\tau_j]}(x) - \frac{s}{2}, \quad x \in \partial^\Omega G^+[\bar r, s, t_0, t_0 + k\tau_j].$$

However, in view of Lemma 2.13 (a), these properties imply $F(\tau_j, k_0 + k) \prec G^+[\bar r, s, t_0, t_0 + k\tau_j]$, which contradicts to (3.10). Thus, $\bar r = 0$. Analogous contradiction argument based on Lemma 2.13 (b) shows $G^-[0, s, t_0, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k)$ for all $0 \le k \le \lfloor \bar t/\tau_j \rfloor$.

Finally, let us prove (3.4). Recall that by construction $G_0^-[0, 2s, t_0] \prec G_0^-[0, s, t_0]$ and $G_0^+[0, s, t_0] \prec G_0^+[0, 2s, t_0]$, therefore, by the strong comparison principle (Theorem 2.8) $G^-[0, 2s, t_0, t] \prec G^-[0, s, t_0, t]$ and $G^+[0, s, t_0, t] \prec G^+[0, 2s, t_0, t]$ for all $t \in [t_0, T]$. Now the continuity of $G^\pm[0, s, t_0, t]$ on its parameters we could find $\bar j = \bar j(s) > 1$ such that for all $j > \bar j$

$$G^-[0,2s,t_0,t+\bar t] \prec G^-[0,s,t_0,t+\bar k_j\tau_j] \subset F(\tau_j,\bar k_j) \subset G^+[0,s,t_0,t+\bar k_j\tau_j] \prec G^+[0,2s,t_0,t+\bar t]. \tag{3.11}$$

By the definition of g,

$$\max_{x\in\partial^\Omega G^\pm[0,2s,t_0,t+\bar t]} \operatorname{dist}(x,\partial^\Omega E(t+\bar t)) \le g(2s) \tag{3.12}$$

and therefore, by construction in Corollary 2.5 (a)

$$\operatorname{dist}(\partial^\Omega G_0^\pm[0,4g(2s),t_0+\bar t],\partial^\Omega E(t+\bar t)) = 4g(2s) > 0.$$

Combining this with (3.12) and the construction of $G_0^\pm$ we deduce

$$G_0^-[0,4g(2s),t_0+\bar t] \prec G_0^-[0,2s,t_0+\bar t] \quad \text{and} \quad G_0^+[0,2s,t_0+\bar t] \prec G_0^+[0,4g(2s),t_0+\bar t].$$

These inclusions together with (3.11) imply (3.4). $\square$

Now we are ready to prove the equality (3.1). Let $\bar t$ be given by Lemma 3.1,

$$N := \lfloor T/\bar t\rfloor + 1$$

and let $\sigma_0 \in (0,\sigma/16)$ be such that the numbers

$$\sigma_l = 4g(2\sigma_{l-1}),\quad l=1,\dots,N,$$

satisfy $\sigma_l \in (0,\sigma/16)$. By the monotonicity and continuity of g together with $g(0)=0$, such choice of σ_0 is possible.

Fix any $s\in(0,\sigma_0)$ and let

$$a_0(s) := s,\quad a_l(s) := 4g(2a_{l-1}(s)),\quad l=1,\dots,N.$$

Note that $a_l(s)\in(0,\sigma_l)$. In particular, the numbers $\bar j_l^s := \bar j(a_l(s))$, given by the last assertion of Lemma 3.1, are well-defined. Let also

$$\widetilde{j}_l^s := \max\{j\ge 1:\ \tau_j\notin(0,\tau_0(a_l(s)))\}$$

and

$$\bar j_s := 1 + \max_{l=0,\dots,N}\max\{\bar j_l^s,\widetilde{j}_l^s\}.$$

By Corollary 2.5 (a)

$$G_0^-[0,s,0]\subset E(0)=E_0=F(\tau_j,0)\subset G_0^+[0,s,0]$$

for all $j>\bar j_s$. Therefore, by Lemma 3.1 applied with $k_0=0$ and $t_0=0$ we find

$$G^-[0,s,0,k\tau_j]\subset F(\tau_j,k)\subset G^+[0,s,0,k\tau_j],\quad k=0,1,\dots,\bar k_j,$$

where $\bar k_j := \lfloor \bar t/\tau_j\rfloor$. Moreover, since $s\in(0,\sigma_0,)$ by the last assertion of Lemma 3.1

$$G_0^-[0,a_1(s),\bar t]\subset F(\tau_j,\bar k_j)\subset G_0^+[0,a_1(s),\bar t]$$

for all $j\ge \bar j_s$. Hence, we can reapply Lemma 3.1 with $s:=a_1(s)$, $t_0=\bar t$ and $k_0=\bar k_j$, to find

$$G^-[0,a_1(s),0,\bar t+k\tau_j]\subset F(\tau_j,\bar k_j+k)\subset G^+[0,a_1(s),0,\bar t+k\tau_j],\quad k=0,1,\dots,\bar k_j.$$

In particular, since $j>\bar j_s>\bar j(a_1(s))$, again by the last assertion of Lemma 3.1 we deduce

$$G_0^-[0,a_2(s),2\bar t]\subset F(\tau_j,2\bar k_j)\subset G_0^+[0,a_2(s),2\bar t].$$

Repeating this argument at most N times, for all $j\ge\bar j_s$ we find

$$G^-[0,a_l(s),0,l\bar t+k\tau_j]\subset F(\tau_j,l\bar k_j+k)\subset G^+[0,a_l(s),0,l\bar t+k\tau_j],\quad k=0,1,\dots,\bar k_j \tag{3.13}$$

whenever $l=0,\dots,N$ and $l\bar t+k\tau_j\le T$.

Now take any $t\in(0,T)$, and let $l:=\lfloor t/\bar t\rfloor$ and $k=\lfloor t/\tau_j\rfloor - l\bar k_j$ so that $l\bar k_j+k=\lfloor t/\tau_j\rfloor$. By means of l and k, as well as the definition of $\bar k_j$ we represent (3.13) as

$$G^-\Big[0,a_l(s),0,l\bar t+\tau_j\Big\lfloor\tfrac{t}{\tau_j}\Big\rfloor-l\tau_j\Big\lfloor\tfrac{\bar t}{\tau_j}\Big\rfloor\Big]$$
$$\subset F\Big(\tau_j,\Big\lfloor\tfrac{t}{\tau_j}\Big\rfloor\Big)\subset G^+\Big[0,a_l(s),0,l\bar t+\tau_j\Big\lfloor\tfrac{t}{\tau_j}\Big\rfloor-l\tau_j\Big\lfloor\tfrac{\bar t}{\tau_j}\Big\rfloor\Big] \tag{3.14}$$

for all $j>\bar j_s$. Since

$$\lim_{j\to+\infty}\Big(l\bar t+\tau_j\Big\lfloor\tfrac{t}{\tau_j}\Big\rfloor-l\tau_j\Big\lfloor\tfrac{\bar t}{\tau_j}\Big\rfloor\Big)=t,$$

by the continuous dependence of $G^\pm$ on its parameters, as well as the convergence (3.2) of the flat flows, letting $j\to+\infty$ in (3.14) we obtain

$$G^-[0,a_l(s),0,t]\subset F(t)\subset G^+[0,a_l(s),0,t], \tag{3.15}$$

where due to the L^1-convergence the inclusions in (3.2) here are up to some negligible sets. Now we let $s\to0^+$ and recalling that $a_l(s)\to0$ (by the continuity of g and assumption $g(0)=0$), from (3.15) we deduce

$$G^-[0,0,0,t]\subset F(t)\subset G^+[0,0,0,t].$$

Then by Theorem 2.6 (a)

$$F(t)=G^\pm[0,0,0,t]=E(t).$$

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