A CHARACTERIZATION OF HORIZONTALLY TOTALLY GEODESIC HYPERSURFACES IN HEISENBERG GROUPS

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Abstract. In this paper we achieve a first concrete step towards a better understanding of the so called *Bernstein problem* in higher dimensional Heisenberg groups. Indeed, in the sub-Riemannian Heisenberg group $\mathbb{H}^n$, with $n \geqslant 2$, we show that the only entire hypersurfaces with vanishing horizontal symmetric second fundamental form and countable characteristic set are hyperplanes. This result relies on a sub-Riemannian characterization of a higher dimensional ruling property, as well as on the study of sub-Riemannian geodesics on Heisenberg hypersurfaces.

1. Introduction

In the present paper we achieve a first concrete step towards a better understanding of the so called *Bernstein problem* in higher dimensional sub-Riemannian Heisenberg groups. Namely, we reduce the solution of the latter to validity of suitable sub-Riemannian curvature estimates. The characterization of entire minimal hypersurfaces in higher dimensional sub-Riemannian Heisenberg groups is an intriguing open problem in the framework of sub-Riemannian geometry. This issue, which is typically known as *Bernstein problem* in view of its Euclidean counterpart (cf. [28] for a complete survey of the topic), fits into the broader context of minimal hypersurfaces in sub-Riemannian structures (cf. [9, 10, 11, 16, 19, 25, 26, 29, 39, 40, 41, 48] and references therein). The study of this and related issues is particularly relevant in the sub-Riemannian Heisenberg group $\mathbb{H}^n$, since the latter constitutes a prototypical model in the setting of Carnot groups (cf. [21, 6]), sub-Riemannian manifolds (cf. [2]), CR manifolds (cf. [7]) and Carnot-Carathéodory spaces (cf. [30]). We briefly recall that the n-th Heisenberg group $(\mathbb{H}^n, \cdot)$ is $\mathbb{R}^{2n+1}$ endowed with the group law

$$p \cdot p' = (\bar{x}, \bar{y}, t) \cdot (\bar{x}', \bar{y}', t') = (\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}'))),$$

where

$$Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}')) = \sum_{j=1}^{n} \left(x_j' y_j - x_j y_j'\right)$$

and where we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (z, t) = (\bar{x}, \bar{y}, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$. With this operation, $\mathbb{H}^n$ is a Carnot group, whose associated *horizontal distribution*, which we denote by $\mathcal{H}$, is generated by the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \qquad \text{and} \qquad Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t}$$

for $j = 1, \ldots, n$. The standard sub-Riemannian structure

$$(\mathbb{H}^n, \mathcal{H}, \langle \cdot, \cdot \rangle)$$

is given by a suitable scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. One of the key differences between the Euclidean and the Heisenberg setting is that, as pointed out in [3], the classical Federer's notion

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of rectifiability in metric spaces (cf. [20]) is not suitable for the Heisenberg group. To solve this issue, the authors of [22] introduced the intrinsic notion of $\mathbb{H}$-*regular hypersurface*, together with a related notion of *intrinsic rectifiability*. We recall that an $\mathbb{H}$-regular hypersurface is a subset of $\mathbb{H}^n$ which can be described locally as the zero locus of a $C^1_{\mathbb{H}}$-function (cf. [22] for more precise definitions). A special class of $\mathbb{H}$-regular hypersurfaces is that of *non-characteristic hypersurfaces*. In this setting, given a hypersurface S of class C^1, we say that a point $p \in S$ is *characteristic* as soon as

$$\mathcal{H}_p = T_p S,$$

and otherwise we say that p is *non-characteristic*. In this last case, the *horizontal tangent space*

$$\mathcal{H}T_p S = \mathcal{H}_p \cap T_p S$$

is a $(2n-1)$-dimensional vector space. The set of characteristic points of S is denoted by S_0 and is called the *characteristic set* of S. After [22], it was clear that the importance of $\mathbb{H}$-regular hypersurfaces went beyond rectifiability, as proved for instance from the striking differences between characteristic and non-characteristic setting in the approach to the sub-Riemannian Bernstein problem in $\mathbb{H}^1$. A first study of the latter was carried out by [9, 44] in the class of t-*graphs* of class C^2. We recall that a hypersurface S is a t-*graph* whenever there exists $u : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ such that

$$\operatorname{graph}(u) := S = \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \,:\, (\bar{x}, \bar{y}) \in \mathbb{R}^{2n}\}.$$

In the previous set of papers, the authors classified minimal t-graphs of class C^2 in the first Heisenberg group $\mathbb{H}^1$, finding examples of minimal smooth t-graphs which are not planes. These results were generalized in [33, 16, 17] to more general embedded C^2-hypersurfaces in $\mathbb{H}^1$. Moreover, as pointed out in [41], if one consider hypersurfaces with low regularity, the examples of minimal hypersurfaces which are not hyperplanes increase considerably. However, the situation is different when considering non-characteristic hypersurfaces. In this context, a meaningful counterpart of hyperplanes in the Euclidean setting is the class of *vertical hyperplanes*. Let us recall that a vertical hyperplane is a set S of the form

$$S = \{p \in \mathbb{H}^n \,:\, \langle (\bar{x}, \bar{y}), (\bar{a}, \bar{b}) \rangle = c\},$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. An easy computation (cf. Section 4) shows that S is non-characteristic. Moreover, every hyperplane which is not vertical is characteristic (cf. again Section 4). A first result in this direction was achieved in [5] in the class of *intrinsic graphs* (cf. [5]). Indeed, the authors showed that the only minimal intrinsic graphs defined by a C^2 function in $\mathbb{H}^1$ are vertical hyperplanes. This result was generalized in [23] to the class of non-characteristic minimal C^1-hypersurfaces of $\mathbb{H}^1$, in [38] to the class of minimal intrinsic graphs defined by an Euclidean Lipschitz function in $\mathbb{H}^1$, and in [27] to the class of (X, Y)-Lipschitz surfaces in the sub-Finsler Heisenberg group $\mathbb{H}^1$. We point out that, as shown in [37], weakening the regularity of the defining function allows to find examples of minimal hypersurfaces which are not vertical planes even in the class of intrinsic graphs. While the Bernstein problem is well understood in $\mathbb{H}^1$, very few results are known in higher dimensions. On one hand, as in $\mathbb{H}^1$, there is no rigidity in the class of smooth t-graphs ([47]). On the other hand, when $n \geqslant 5$, there are counterexamples even in the class of smooth intrinsic graphs (cf. [5, 16]). The Bernstein problem for non-characteristic hypersurfaces is still open when $n = 2, 3, 4$. In $\mathbb{H}^1$, a key step consists in understanding that the non-characteristic part $S \setminus S_0$ of an area-stationary surface S is foliated by *horizontal line* segments in the following sense.

Ruling Property. [10, 23] Let S be an area-stationary surface of class C^1 in $\mathbb{H}^1$. Then, S is foliated by horizontal line segments with endpoints in S_0.

Here, by horizontal line, we mean an Euclidean line γ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{n} a_j X_j(\gamma(t)) + \sum_{j=1}^{n} b_j Y_j(\gamma(t)),$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ and for any $t \in \mathbb{R}$. The importance of this ruling property became even more evident in [49], where the author showed a Bernstein Theorem in the class of those minimal intrinsic graphs which present the aforementioned ruling property, thus without assuming any regularity on the surfaces. The importance of this merely differential property can be appreciated even by a sub-Riemannian viewpoint. Indeed, in analogy with the Riemannian setting, the sub-Riemannian structure of $\mathbb{H}^n$ allows to associate to the non-characteristic part of S various notions of *horizontal second fundamental forms*, which act on the horizontal tangent distribution $\mathcal{H}TS$. Despite these forms have been introduced and studied by many authors in different ways (cf. [32, 15, 12, 43, 32]), and differently from the Riemannian framework, it is possible to distinguish a *symmetric* form $\tilde{h}$ and a *non-symmetric* form h. These forms has shown to be important in various settings, for instance to introduce the so-called *horizontal mean curvature* H (cf. [9, 39, 15]), a suitable notion of *horizontal umbilicity* (cf. [8]) and for the study of rigid motions (cf. [12]). In the particular case of $\mathbb{H}^1$, the vanishing of the form h, which coincides both with the symmetric form $\tilde{h}$ and with the horizontal mean curvature H, is equivalent to the aforementioned ruling property. In the higher dimensional case, however, h and $\tilde{h}$ may differ, although it is in general true that the norm of $\tilde{h}$ is controlled by the norm of h (cf. Section 6).

The aim of the present paper is twofold. On one hand, we propose a generalization of the ruling property to higher dimensional Heisenberg groups, relating this new notion with the vanishing of the symmetric form $\tilde{h}$. More precisely, we will call *horizontally totally geodesic* an hypersurface such that $\tilde{h} \equiv 0$ on its non-characteristic part. We stress that hypersurfaces for which $h \equiv 0$ are particular instances of horizontally totally geodesic hypersurfaces. On the other hand, it is not always the case that horizontally totally geodesic hypersurfaces satisfy $h \equiv 0$ (cf. Section 6). In the Riemannian framework, this name is motivated by the fact that every geodesic of a totally geodesic hypersurface is a geodesic of the ambient manifold. This last characterization allows to deduce that, in $\mathbb{R}^n$, the only totally geodesic hypersurfaces are hyperplanes. The second aim of this paper is to provide an analogous result in the Heisenberg group. We stress that, at least in the non-characteristic case, hypersurfaces with $h \equiv 0$ are easily vertical hyperplanes (cf. Section 6). Surprisingly, the same phenomenon continues to hold under the weaker requirement $\tilde{h} \equiv 0$. The main achievements of this paper can be then summarized in the following result.

Theorem 1.1 (Main Theorem). *Let $S \subseteq \mathbb{H}^n$ be an hypersurface without boundary of class C^2. The following are equivalent.*

(i) S is horizontally totally geodesic.
(ii) S is ruled.

If in addition $n \geqslant 2$, S is (topologically) closed and S_0 is countable, then (i) and (i) hold if and only if S is a hyperplane.

In particular, in the non-characteristic setting, Theorem 1.1 constitutes an important tool in order to approach the resolution of the Bernstein problem in the higher dimensional case.

Corollary 1.2. *Let $S \subseteq \mathbb{H}^n$ be a non-characteristic hypersurface without boundary of class C^2. Assume that $n \geqslant 2$ and that S is (topologically) closed. If S is horizontally totally geodesic, then S is a vertical hyperplane.*

Indeed, Corollary 1.2 allows to reduce the complexity of the problem to the estimate of the norm of the horizontal second fundamental form $\tilde{h}$ associated to a minimal hypersurface. We point out that an approach based on curvature estimates for minimal hypersurfaces is already available in $\mathbb{R}^n$, in view of the celebrated paper [45]. Our approach to Theorem 1.1 can be summarized in the following steps.

Introduction of the higher dimensional ruling property. The starting point consists in generalizing the ruling property to the higher dimensional case, which is done in Section 3 in two equivalent ways (cf. Definition 3.1, Definition 3.1 and Proposition 3.2). After discussing the connection between the characteristic set and this new notion (cf. Proposition 3.2), we show that the latter is well-behaved with respect to the intrinsic geometry of $\mathbb{H}^n$. Namely, we prove that the class of ruled hypersurfaces is closed under the action of intrinsic dilations (cf. Proposition 3.6), and the action of the so-called *pseudohermitian transformations* (cf. Theorem 3.10).

Rigidity results for ruled hypersurfaces. Subsequently, we exploit the ruling property to provide rigidity results for some classes of hypersurfaces. Basically, we show that under some constraints on the size of the characteristic set, the higher dimensional ruling property is more rigid than the corresponding one in $\mathbb{H}^1$.

Theorem 1.3. *Let S be an hypersurface of class C^1. Assume that $n \geqslant 2$ and that S is closed and without boundary. Assume that S_0 is countable and that S is ruled. Then S is an hyperplane.*

This result constitutes a first remarkable difference with $\mathbb{H}^1$, where there are instance of smooth ruled non-characteristic hypersurfaces which are not planes (cf. Section 4). Another striking difference with the first Heisenberg group can be appreciated studying the ruling property in the class of *intrinsic cones*.

Theorem 1.4. *Let $n \geqslant 2$ and let S be a ruled conical closed hypersurface. If $S_0 \neq \emptyset$, then S is the horizontal hyperplane $\mathcal{H}_0$.*

As a corollary of the previous characterization, it is easy to provide examples of entire minimal characteristic hypersurfaces which are not ruled.

Theorem 1.5. *Let $n \geqslant 2$ and let $S := \operatorname{graph}(u)$, where $u(\bar{x}, \bar{y}) = \frac{1}{2}x_1^2 - \frac{1}{2}y_1^2$. Then S is a minimal smooth hypersurface which is not ruled.*

Introduction of the horizontal second fundamental forms. In Section 6 we begin building a bridge between the aforementioned results, which are only differential in spirit, with the sub-Riemannian structure of $\mathbb{H}^n$. To this aim, we formally introduce the two aforementioned second fundamental forms $\tilde{h}$ and h. highlighting the main differences between $\mathbb{H}^1$ and higher dimensional Heisenberg groups. Moreover, we prove some simple formulas for the norms of h and $\tilde{h}$ (cf. Proposition 6.1 and Proposition 6.4), which allows to relate in a quantitative way the two quantities.

Ruled if and only if horizontally totally geodesic. In view of Theorem 1.3, the main remaining obstacle to prove Theorem 1.1 is to show the equivalence between the property of being horizontally totally geodesic and the ruling property.

Theorem 1.6. *Let S be a hypersurface without boundary of class C^2. Then S is ruled if and only if S is horizontally totally geodesic.*

This result strongly relies on a local existence and uniqueness result for a particular geodesic-type initial value problem on the hypersurface (cf. Theorem 7.5). Sub-Riemannian geodesics have been extensively studied in the last years (cf. e.g. [35, 34, 1, 14, 36] and references therein).

Although local existence results for sub-Riemannian geodesics are available (cf. e.g. [34]), it is not always the case that sub-Riemannian geodesics satisfies the standard geodesic equation

$$\nabla_{\dot{\Gamma}}\dot{\Gamma} = 0,$$

being ∇ a suitable sub-Riemannian connection (cf. [44]). Therefore, we devote Section 7 to the study of the initial value problem associated to this kind of equations on hypersurfaces. The proof of Theorem 7.5 can be reduced to the study of curves in domains of suitable *intrinsic graphs*, and its main difficulty lies in the fact that that the initial value problem that we need to consider is *a priori* overdetermined. Once Theorem 7.5 is achieved, we are then in position to prove Theorem 1.6, and so, in view of the previous considerations, to conclude the proof of Theorem 1.1.

We point out that, in view of Theorem 1.6, we can read Theorem 1.4 and Theorem 1.5 from the sub-Riemannian standpoint. Namely, when $n \geqslant 2$, the only horizontally totally geodesic sufficiently smooth cone is the horizontal hyperplane, and there exist minimal characteristic conical hypersurfaces which are not horizontally totally geodesic.

This set of results and considerations both provides a direct way to approach the Bernstein problem via curvature estimates, and highlights once more many interesting differences between $\mathbb{H}^1$ and higher dimensional Heisenberg groups. According to the authors' hope, it may give a burst in the grasp of such an interesting open problem as the Bernstein problem in this anisotropic setting.

Plan of the paper. In Section 2 we collect some basic preliminaries about the sub-Riemannian Heisenberg group. In Section 3 we introduce the higher dimensional ruling property and we study some of its properties. In Section 4 we prove Theorem 1.3, while in Section 5 we prove Theorem 1.4 and Theorem 1.5. In Section 6 we recall the main definition and properties of the horizontal forms h and $\tilde{h}$, and we introduce the notion of horizontally totally geodesic hypersurface. In Section 7 we introduce the relevant geodesic-type initial value problem (cf. (7.1)) and we show a local existence and uniqueness result (cf. Theorem 7.5). Moreover, we prove Theorem 1.6 and Theorem 1.1.

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2. Preliminaries

2.1. **The Heisenberg group $\mathbb{H}^n$.** In the following we denote by T the left-invariant vector field $\frac{\partial}{\partial t}$. In this way $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ is a basis of left-invariant vector fields. Moreover, the only nontrivial commutation relations are

$$[X_j, Y_j] = -[Y_j, X_j] = -2T$$

for any $j = 1, \ldots, n$. A vector field which is tangent to $\mathcal{H}$ at every point is called *horizontal*. For given $q \in \mathbb{H}^n$ and $\lambda > 0$, we define the *left-translation* $\tau_q : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ and the *intrinsic dilation* $\delta_\lambda : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\tau_q(p) := q \cdot p \qquad \text{and} \qquad \delta_\lambda(p) := (\lambda z, \lambda^2 t)$$

for any $p = (z, t) \in \mathbb{H}^n$ respectively. It is well known that both τ_q and δ_λ are global diffeomorphisms, and that δ_λ is a Lie group isomorphism. Moreover, we define the *complex structure* J by letting

$$J(X_i) = Y_i, \qquad J(Y_i) = -X_i \qquad \text{and} \qquad J(T) = 0$$

for any $i = 1, \dots, n$, and extending it by linearity for general vector fields. Given $p \in \mathbb{H}^n$ we will often identify the vector $\sum_{j=1}^n v_j X_j|_p + v_{n+j} Y_j|_p$ with the point $(v_1, \dots, v_{2n}, 0) \in \mathbb{H}^n$. The Haar measure of $\mathbb{H}^n$ coincides with the $(2n+1)$-dimensional Lebesgue measure $\mathcal{L}^{2n+1}$. The homogeneity property $\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^{2n+1}(E)$ holds for any measurable set $E \subseteq \mathbb{H}^n$, where $Q = 2n+2$ is the *homogeneous dimension* of $\mathbb{H}^n$ (cf. [46]).

2.2. **Sub-Riemannian structure on $\mathbb{H}^n$.** We let g be the unique Riemannian metric on $\mathbb{H}^n$ which makes $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ orthonormal. For the sake of notational simplicity, we let

$$Z_j = X_j, \qquad Z_{n+j} = Y_j \qquad \text{and} \qquad Z_{2n+1} = T$$

for any $j = 1, \dots, n$, and we recall for the sake of completeness that the horizontal distribution $\mathcal{H}$ is defined by

$$\mathcal{H}_p = \operatorname{span}\{Z_1|_p, \dots, Z_{2n}|_p\}$$

for any $p \in \mathbb{H}^n$. If we restrict g to the horizontal distribution $\mathcal{H}$, and we denote this restriction by $\langle \cdot, \cdot \rangle$, then $\mathbb{H}^n$ inherits a *sub-Riemannian structure* which realizes it as a *sub-Riemannian manifold.* We denote by ∇ the so-called *pseudohermitian connection* (cf. e.g. [43]), i.e. the unique *metric connection* (cf. [18]) with torsion tensor given by

$$\nabla_X Y - \nabla_Y X - [X, Y] = 2\langle J(X), Y \rangle T \tag{2.1}$$

for any pair of vector fields X and Y. The most relevant feature of ∇ (cf. [42]) is the following property:

$$\nabla_{Z_i} Z_j = 0 \tag{2.2}$$

for any $i, j = 1, \dots, 2n+1$, and so can be seen as a flat connection on $\mathbb{H}^n$.

2.3. **Carnot-Carathéodory structure on $\mathbb{H}^n$.** If $\Gamma : [a, b] \longrightarrow \mathbb{H}^n$ is an absolutely continuous curve, we say that it is *horizontal* whenever

$$\dot{\Gamma}(t) \in \mathcal{H}_{\Gamma(t)} \tag{2.3}$$

for almost every $t \in [a, b]$, and we say that it is *sub-unit* whenever it is horizontal with $|\dot{\Gamma}(t)| = 1$ for almost every $t \in [a, b]$. Moreover, we define

$$d(p, p') := \inf\{T \ : \ \Gamma : [0, T] \longrightarrow \mathbb{H}^n \text{ is sub-unit, } \Gamma(0) = p \text{ and } \Gamma(T) = p'\}$$

which, by the Chow-Rashevskii theorem (cf. [13]), defines a distance on $\mathbb{H}^n$, called *Carnot-Carathéodory distance.* The metric space $(\mathbb{H}^n, d)$ is then a prototype of *Carnot-Carathéodory space* (cf. [30]).

2.4. **Horizontal perimeter and horizontal gradient.** If $\Omega \subseteq \mathbb{H}^n$ is open and $E \subseteq \mathbb{H}^n$ is measurable with $\chi_E \in L^1_{loc}(\Omega)$, we recall (cf. e.g. [22, 24]) that the $\mathbb{H}$-*perimeter* of E in Ω is defined by

$$P_{\mathbb{H}}(E, \Omega) := \sup\left\{ \int_E \operatorname{div}_{\mathbb{H}}(\bar{\varphi})\, d\mathcal{L}^{2n+1} \ : \ \bar{\varphi} \in C^1_c(\Omega, \mathcal{H}),\ |\bar{\varphi}|_p \leqslant 1 \text{ for any } p \in \Omega \right\},$$

where by $C^1_c(\Omega, \mathcal{H})$ we denote the class of compactly supported C^1 sections of the horizontal distribution $\mathcal{H}$, and $\operatorname{div}_{\mathbb{H}}$ is the so called *horizontal divergence*, defined by

$$\operatorname{div}_{\mathbb{H}}\left(\sum_{j=1}^n (\varphi_j X_j + \varphi_{n+j} Y_j) \right) := \sum_{j=1}^n (X_j \varphi_j + Y_j \varphi_{n+j})$$

for any $\sum_{j=1}^n (\varphi_j X_j + \varphi_{n+j} Y_j) \in C^1(\Omega, \mathcal{H})$. Moreover, we say that a set E as above is an $\mathbb{H}$-*Caccioppoli set* whenever $P_{\mathbb{H}}(E, \Omega) < +\infty$ for any bounded open set $\Omega \subseteq \mathbb{H}^n$. Finally, we recall (cf. e.g. [46]) that an $\mathbb{H}$-Caccioppoli set E is an $\mathbb{H}$-*perimeter minimizer* whenever

$$P_{\mathbb{H}}(E, \Omega) \leqslant P_{\mathbb{H}}(F, \Omega)$$

for any $\Omega \Subset \mathbb{H}^n$ and for any $\mathbb{H}$-Caccioppoli set F such that $E\Delta F \Subset \Omega$. The sub-Riemannian structure of $\mathbb{H}^n$ allows to define a distributional notion of *horizontal gradient* (cf. [22]). More precisely, if $f \in L^1_{loc}(\Omega)$, we let

$$\langle \nabla_{\mathbb{H}} f, \varphi \rangle := -\int_\Omega f \operatorname{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1}$$

for any $\varphi \in C^\infty_c(\Omega, \mathbb{R}^{2n})$. When f is continuous and $\nabla_{\mathbb{H}} f$ is represented by a continuous vector field, then we say that $f \in C^1_{\mathbb{H}}(\Omega)$. Moreover, in this case,

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^n X_j f X_j + \sum_{j=1}^n Y_j f Y_j.$$

2.5. **Hypersurfaces in $\mathbb{H}^n$.** We say that $S \subseteq \mathbb{H}^n$ is an $\mathbb{H}$-*regular hypersurface* if, for any $p \in S$, there exists an open neighborhood U of p and a function $f \in C^1_{\mathbb{H}}(U)$ such that

$$S \cap U = \{q \in \mathbb{H}^n \,:\, f(q) = 0\} \qquad \text{and} \qquad \nabla_{\mathbb{H}} f \neq 0 \text{ on } U.$$

Here and in the rest of the paper, whether not specified, a hypersurface will be always at least of class C^1 and without boundary. If S is an hypersurface of class C^1, we define

$$S_0 := \{p \in S : \mathcal{H}_p = T_p S\}$$

and we call it the *characteristic set* of S. Notice that, since S is of class C^1 and $\mathcal{H}$ is a smooth distribution, then S_0 is closed in S. Moreover, let us define

$$\mathcal{H}T_p S := \mathcal{H}_p \cap T_p S.$$

When $p \in S_0$, then $\dim(\mathcal{H}T_p S) = 2n$. On the contrary, when $p \in S \setminus S_0$, we have $\dim(\mathcal{H}T_p S) = 2n-1$. In this case, the *horizontal normal* to S at p is defined by

$$v^{\mathbb{H}}(p) := \frac{N^{\mathbb{H}}(p)}{|N^{\mathbb{H}}(p)|_p} \tag{2.4}$$

for any $p \in S \setminus S_0$, where $N_{\mathbb{H}}(p)$ is the a section of the horizontal bundle defined by

$$N^{\mathbb{H}}(p) := \sum_{j=1}^n (\langle N(p), X_j|_p \rangle_{\mathbb{R}^{2n+1}} X_j|_p + \sum_{j=1^n} \langle N(p), Y_j|_p \rangle_{\mathbb{R}^{2n+1}} Y_j|_p,$$

being $N(p)$ the Euclidean unit normal to S at p. It is clear that a hypersurface of class C^1 with empty characteristic set is $\mathbb{H}$-regular (cf. e.g. [47]). When S is of class C^2, it is easy to check that

$$\sum_{h=1}^{2n} v_h^{\mathbb{H}} Z_k(v_h^{\mathbb{H}}) = 0 \tag{2.5}$$

for any $k = 1, \ldots, 2n$, where by $v^{\mathbb{H}}$ we mean any C^2 extension of $v^{\mathbb{H}}|_S$ in a neighborhood of S such that $|v^{\mathbb{H}}| = 1$. There is a particular choice of such an extension which allows to derive further relations. Indeed, if we let $d^{\mathbb{H}}$ be the signed Carnot-Carathéodory distance from S with respect to $(Z_1, \ldots, Z_{2n})$, then it is well known (cf. [43]) that $d^{\mathbb{H}}$ inherits the same regularity of S in a neighborhood of any non-characteristic point $p \in S$. Moreover, since $d^{\mathbb{H}}$ satisfies the horizontal Eikonal equation in a neighborhood of S, then $v^{\mathbb{H}}|_S$ can be extended by letting

$$v^{\mathbb{H}} = \sum_{j=1}^n X_j d^{\mathbb{H}} X_j + \sum_{j=1}^n Y_j d^{\mathbb{H}} Y_j. \tag{2.6}$$

With this particular extension,

$$Z_k(v_h^{\mathbb{H}}) = Z_h(v_k^{\mathbb{H}}) \tag{2.7}$$

for any $h, k = 1, \dots, 2n$ such that $|h-k| \neq n$. Moreover,

$$X_k(v^{\mathbb{H}}_{n+k}) = Y_k(v^{\mathbb{H}}_k) - 2Td^{\mathbb{H}} \qquad \text{and} \qquad Y_k(v^{\mathbb{H}}_k) = X_k(v^{\mathbb{H}}_{n+k}) + 2Td^{\mathbb{H}} \tag{2.8}$$

for any $k = 1, \dots, n$. Finally, thanks to (2.5), (2.7) and (2.8), we see that

$$\sum_{h=1}^{2n} v^{\mathbb{H}}_h Z_h(v^{\mathbb{H}}_k) = -2Td^{\mathbb{H}} J(v^{\mathbb{H}})_k \tag{2.9}$$

for any $k = 1, \dots, 2n$. Moreover, a simple computation shows that

$$Td^{\mathbb{H}} = \frac{N_{2n+1}}{|N^{\mathbb{H}}|} \tag{2.10}$$

According to [15], we define the *horizontal tangential derivatives*

$$\delta_i \xi = Z_i \bar{\xi} - g_{\mathbb{H}}\left(\nabla_{\mathbb{H}}, \bar{\xi}\right) v^{\mathbb{H}}_i$$

for any $i = 1, \dots, 2n$, where ξ is a C^1 function on an open subset of S and $\bar{\xi}$ is any C^1 extension of ξ. As customary, the horizontal tangential derivatives do not depend on the chosen extension (cf. [15]). We recall that the *tangent pseudohermitian connection* ∇^S is defined in the non-characteristic part of S by

$$\nabla^S_X Y = \nabla_X Y - \langle \nabla_X Y, v^{\mathbb{H}} \rangle v^{\mathbb{H}}$$

for any pair of tangent horizontal vector fields X and Y. Notice that ∇^S is easily a metric connection with respect to the metric $\langle \cdot, \cdot \rangle$ restricted to $\mathcal{H}TS$. We say that an hypersurface of class C^1 is $\mathbb{H}$-*minimal* whenever it coincides with the boundary of an $\mathbb{H}$-perimeter minimizer.

2.6. **Intrinsic graphs.** Let us denote points $q \in \mathbb{R}^{2n}$ by $q = (\xi_1, \dots, \xi_n, \eta_2, \dots, \eta_n, \tau) = (\bar{\xi}, \tilde{\eta}, \tau)$. We wish to ientify $\mathbb{R}^{2n}$ with $\{p \in \mathbb{H}^n : y_1 = 0\}$. To this aim, we introduce the immersion map $i : \mathbb{R}^{2n} \longrightarrow \mathbb{H}^n$ defined by

$$i(\bar{\xi}, \tilde{\eta}, \tau) = (\bar{\xi}, 0, \tilde{\eta}, \tau)$$

for any $(\bar{\xi}, \tilde{\eta}, \tau) \in \mathbb{R}^{2n}$. Moreover, we identify $\mathbb{R}$ with $\{(\bar{0}, y_1, \tilde{0}, 0) : y_1 \in \mathbb{R}\}$ by means of the inclusion $j : \mathbb{R} \longrightarrow \mathbb{H}^n$ defined by

$$j(y_1) = (\bar{0}, y_1, \tilde{0}, 0)$$

for any $y_1 \in \mathbb{R}$. The maps i and j are clearly smooth, injective and open. For a given open set $\Omega \subseteq \mathbb{R}^{2n}$ and a function $\varphi : \Omega \longrightarrow \mathbb{R}$, we recall that the Y_1-*graph of* φ *on* Ω is defined by

$$\mathrm{graph}_{Y_1}(\varphi, \Omega) = \{(\bar{\xi}, \varphi(\bar{\xi}, \tilde{\eta}, \tau), \tilde{\eta}, \tau - \xi_1 \varphi(\bar{\xi}, \tilde{\eta}, \tau)) : (\bar{\xi}, \tilde{\eta}, \tau) \in \Omega\}.$$

Moreover, we define its parametrization map $\Psi : \Omega \longrightarrow \mathbb{H}^n$ by

$$\Psi(\bar{\xi}, \tilde{\eta}, \tau) = (\bar{\xi}, \varphi(\bar{\xi}, \tilde{\eta}, \tau), \tilde{\eta}, \tau - \xi_1 \varphi(\bar{\xi}, \tilde{\eta}, \tau))$$

for any $(\bar{\xi}, \tilde{\eta}, \tau) \in \Omega$. We introduce also the *intrinsic projection map* $\Pi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ by

$$\Pi(\bar{x}, \bar{y}, t) = (\bar{x}, \tilde{y}, t + x_1 y_1)$$

for any $(\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. It is easy to check that

$$\Pi(\Psi(q)) = q \qquad \text{and} \qquad \Psi(\Pi(p)) = p$$

for any $q \in \Omega$ and any $p \in \mathrm{graph}_{Y_1}(\varphi, \Omega)$. If $\varphi \in C^1(\Omega)$ and $S = \mathrm{graph}_{Y_1}(\varphi, \Omega)$, then

$$T_{\Psi(q)} S = \mathrm{span}\left(\left.\frac{\partial \Psi}{\partial \xi_1}\right|_q, \dots, \left.\frac{\partial \Psi}{\partial \xi_n}\right|_q, \left.\frac{\partial \Psi}{\partial \eta_2}\right|_q, \dots, \left.\frac{\partial \Psi}{\partial \eta_n}\right|_q, \left.\frac{\partial \Psi}{\partial \tau}\right|_q \right).$$

Letting $D\varphi = (\varphi_{\xi_1}, \dots \varphi_{\xi_n}, \varphi_{\eta_2}, \dots, \varphi_{\eta_n}, \varphi_\tau)$, an easy computation shows that

$$\left.\frac{\partial \Psi}{\partial \xi_1}\right|_q = X_1|_{\Psi(q)} + \varphi_{\xi_1}(q) Y_1|_{\Psi(q)} - 2\varphi(q) T|_{\Psi(q)},$$

$$\frac{\partial \Psi}{\partial \xi_j}\Big|_q = X_j|_{\Psi(q)} + \varphi_{\xi_j}(q) Y_1|_{\Psi(q)} - \eta_j T|_{\Psi(q)}, \qquad \frac{\partial \Psi}{\partial \eta_j}\Big|_q = Y_j|_{\Psi(q)} + \varphi_{\eta_j}(q) Y_1|_{\Psi(q)} + \xi_j T|_{\Psi(q)}$$

for any $j = 2, \dots, n$ and

$$\frac{\partial \Psi}{\partial \tau}\Big|_q = \varphi_\tau(q) Y_1|_{\Psi(q)} + T|_{\Psi(q)}.$$

It is easy to check that $(E_1, \dots, E_n, F_2, \dots, F_n)$ constitutes a global frame of HTS, where

$$E_1 = X_1 + W^\varphi \varphi Y_1, \quad E_j = X_j + \tilde{X}_j \varphi Y_1 \quad \text{and} \quad F_j = Y_j + \tilde{Y}_j \varphi Y_1$$

for any $j = 2, \dots, n$, and where the family of vector fields $\nabla^\varphi = (W^\varphi \varphi, \tilde{X}_2, \dots \tilde{X}_n, \tilde{Y}_2, \dots, \tilde{Y}_n)$ is defined by

$$W^\varphi = \frac{\partial}{\partial \xi_1} + 2\varphi \tilde{T}, \quad \tilde{X}_j = \frac{\partial}{\partial \xi_j} + \eta_j \tilde{T} \quad \text{and} \quad \tilde{Y}_j = \frac{\partial}{\partial \eta_j} - \xi_j \tilde{T}$$

for any $j = 2, \dots, n$, where we have set $\tilde{T} = \frac{\partial}{\partial \tau}$. Therefore, a quick computation implies that

$$v^{\mathbb{H}} = W^{-\frac{1}{2}} \left(W^\varphi \varphi X_1 + \sum_{j=2}^{n} \tilde{X}_j \varphi X_j - Y_1 + \sum_{j=2}^{n} \tilde{Y}_j \varphi Y_j \right), \tag{2.11}$$

where we have set

$$W = 1 + |\nabla^\varphi \varphi|^2.$$

Notice that, since

$$[\tilde{X}_j, \tilde{Y}_j] = -2\tilde{T}$$

for any $j = 2, \dots, n$, then (Ω, d_φ) is a *Carnot-Carathéodory space* (cf. [30]) where Ω is any domain of $\mathbb{R}^{2n}$ and d_φ is the *Carnot-Carathéodory distance* induced by ∇^φ.

3. Higher dimensional ruled hypersurfaces

As already mentioned, a key step in the study of minimal surfaces in $\mathbb{H}^1$ consists in showing that the non-characteristic part of an area-stationary surface is foliated by horizontal line segments. This property extends to the higher dimensional case as follows.

Definition 3.1 (Local ruling property)**.** *Let S be an hypersurface of class C^1. We say that S is* locally ruled at $p \in S \setminus S_0$ *if there exists a neighborhood U of p such that*

$$p \cdot \mathcal{H}T_pS \cap U \subseteq S.$$

Moreover, we say that S is locally ruled *if it is locally ruled at $p \in S \setminus S_0$ for any $p \in S \setminus S_0$.*

Beside this local definition, we propose a global one, which will be useful in the following.

Definition 3.1 (Global ruling property)**.** *Let S be a hypersurface of class C^1. We say that S is* ruled *if for any $p \in S \setminus S_0$, for any $v \in \mathcal{H}T_pS$ and for any $s \geqslant 0$, the following property holds. If s is maximal with the property that*

$$p \cdot \delta_\tau(v) \in S$$

for any $\tau \in [0, s]$, then

$$p \cdot \delta_s(v) \in S_0.$$

The previous two definitions are actually equivalent.

Proposition 3.2. *Let S be an hypersurface of class C^1. Then the following are equivalent.*

(i) S is locally ruled.
(ii) S is ruled.

Proof. Assume that S is ruled. Assume by contradiction that there exists $p \in S \setminus S_0$ and a sequence $(p_h)_h \subseteq p \cdot \mathcal{H}T_pS \setminus S$ converging to p as $h \to +\infty$. Then, for any $h \in \mathbb{N}$, there exists $\lambda_h > 0$ and $v_h \in \mathcal{H}T_pS$ such that $p_h = p \cdot \delta_{\lambda_h}(v_h)$. If, up to a subsequence, for any h there exists $0 < \mu_h \leqslant \lambda_h$ such that $p \cdot \delta_{\mu_h}(v_h)$ belongs to the manifold boundary of S, then, being the latter closed, so does p, a contradiction with $p \in S$. Therefore, since $p_h \notin S$ and up to a subsequence, we can assume that for any h there exists $s_h \geqslant 0$ maximal such that $p \cdot \delta_\tau(v_h) \in S$ for any $\tau \in [0, s_h]$. Clearly $s_h \leqslant \lambda_h$. Therefore, being S ruled, then $q_h := p \cdot \delta_{s_h}(v_h) \in S_0$. But then, by construction, $(q_h)_h$ converges to p as $h \to +\infty$, and so, being S_0 closed, we conclude that $p \in S_0$, a contradiction. On the contrary, assume that S is locally ruled. Assume by contradiction that that there exists $p \in S \setminus S_0$, $w \in \mathcal{H}T_pS$ and s maximal with the property that

$$p \cdot \delta_\tau(w) \in S \tag{3.1}$$

for any $\tau \in [0, s]$ and

$$p \cdot (sw, 0) \notin S_0.$$

Set $\bar{p} := p \cdot (sw, 0)$. Consider the left-invariant vector field

$$W = \sum_{j=1}^{2n} w_j Z_j.$$

Recalling that left-translations preserve the horizontal distribution, and being W left invariant, we conclude that

$$d\tau_{(sw,0)}|_p(W_p) = W|_{\bar{p}} \in \mathcal{H}_{\bar{p}}.$$

Moreover, by construction, W is clearly tangent to S at $\bar{p}$. We conclude that $w \in \mathcal{H}T_{\bar{p}}S$. Since S is locally ruled and $\bar{p} \in S \setminus S_0$, there exists $\tilde{s} > 0$ such that $\bar{p} \cdot (\tilde{s}w, 0) \in S$, which implies, recalling the definition of $\bar{p}$, that

$$\bar{p} \cdot (\tilde{s}w, 0) = p \cdot (\bar{s}w, 0) \cdot (\tilde{s}w, 0) = p \cdot ((\bar{s} + \tilde{s})w, 0) \in S,$$

a contradiction with (3.1). □

Proposition 3.2. *Let S be a ruled hypersurface of class C^1. Assume that S is (topologically) closed. Let $p \in S \setminus S_0$ and $v \in \mathcal{H}T_pS$ be such that*

$$\{p \cdot \delta_s(v, 0) \,:\, s \geqslant 0\} \cap S_0 = \emptyset.$$

Then

$$\{p \cdot \delta_s(v, 0) \,:\, s \geqslant 0\} \subseteq S.$$

In particular, if

$$p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset,$$

then

$$p \cdot \mathcal{H}T_pS \subseteq S. \tag{3.2}$$

Proof. Let $p \in S \setminus S_0$ and $v \in \mathcal{H}T_pS$ be as in the statement, and assume by contradiction that there exists $\lambda > 0$ such that $q = p \cdot \delta_\lambda(v) \notin S$. Being S closed, there exists $s \geqslant 0$ maximal as in Definition 3.1. Then we can argue as in the proof of Proposition 3.2 to find $s \geqslant 0$ such that $p \cdot \delta_s(v) \in S_0$, which is a contradiction. The second claim clearly follows from the first one. □

Notice that, in view of Proposition 3.2, the notion of ruled hypersurface becomes much more simpler in the case of non-characteristic hypersurfaces. Indeed, if S is a closed, non-characteristic ruled hypersurface of class C^1 and $p \in S$, then clearly $p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset$. Therefore a closed non-characteristic hypersurface of class C^1 is ruled if and only if it satisfies (3.2). Now let us discuss some instances of ruled hypersurfaces. We begin with the simplest non-characteristic smooth hypersurface.

Example 3.3 (Vertical Hyperplanes)**.** Let S be a vertical hyperplane of the form
$$S = \{p \in \mathbb{H}^n \, : \, \langle (\bar{x}, \bar{y}), (\bar{a}, \bar{b}) \rangle = c\}$$
for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. Without loss of generality, we assume that $a_1 \neq 0$. It is easy to see that
$$T_pS = \operatorname{span}\{(a_2, -a_1, 0, \dots, 0), (a_3, 0, -a_1, 0, \dots, 0), \dots, (b_n, 0, \dots, 0, -a_1, 0), T\}$$
for any $p \in S$. Notice that $S_0 = \emptyset$. We show that S is ruled. Indeed, noticing that $T \in T_pS$ for any $p \in S$, it follows that
$$\mathcal{H}T_pS = \operatorname{span}\{Z_2|_p, \dots, Z_n|_p, W_1|_p, \dots, W_n|_p\}$$
for any $p \in S$, where
$$Z_i = a_i X_1 - a_1 X_i \qquad \text{and} \qquad W_j = b_j X_1 - a_1 Y_j \tag{3.3}$$
for any $i = 2, \dots, n$ and $j = 1, \dots, n$. Let now $p = (\bar{x}, \bar{y}, t) \in S$, and let $w = (\bar{x}', \bar{y}', 0) \in \mathcal{H}T_pS$. Then there exists $\alpha_j, \beta_j \in \mathbb{R}$ such that
$$w = \left(\sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j, -\alpha_2 a_1, \dots, -\beta_n a_1, 0 \right).$$
We conclude noticing that
$$\langle (\bar{x}', \bar{y}'), (\bar{a}, \bar{b}) \rangle = a_1 \sum_{j=2}^{n} \alpha_j a_j + a_1 \sum_{j=1}^{n} \beta_j b_j - \sum_{j=2}^{n} \alpha_j a_1 a_j - \sum_{j=1}^{n} \beta_j a_1 b_j = 0.$$

Next we consider an instance in the characteristic case.

Example 3.4 (Horizontal Hyperplane)**.** Let S be the horizontal hyperplane $\mathcal{H}_0$. Notice that
$$T_pS = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial y_n} \right\} = \operatorname{span}\{X_1 - y_1 T, \dots, X_n - y_n T, Y_1 + x_1 T, \dots, Y_n + x_n T\}$$
for any $p \in S$. This in particular implies that $S_0 = \{0\}$. Therefore, let $p = (\bar{x}, \bar{y}, t) \neq 0$, and assume without loss of generality that $y_1 \neq 0$. This implies that
$$\mathcal{H}T_pS = \operatorname{span}\{y_2 X_1 - y_1 X_2, \dots, y_n X_1 - y_1 X_n, x_1 X_1 + y_1 Y_1, \dots, x_n X_1 + y_1 Y_n\}.$$
Therefore, let $w = (z, 0) \in \mathcal{H}T_pS$, and let $\alpha_j, \beta_j \in \mathbb{R}$ be such that
$$z = \left(\sum_{j=2}^{n} \alpha_j y_j + \sum_{j=1}^{n} \beta_j x_j, -\alpha_2 y_1, \dots, -\alpha_n y_1, \beta_1 y_1, \dots, \beta_n y_1 \right).$$
Hence it follows that
$$Q((\bar{x}, \bar{y}), z) = y_1 \sum_{j=2}^{n} \alpha_j y_j + y_1 \sum_{j=1}^{n} \beta_j x_j - \sum_{j=2}^{n} \alpha_j y_1 y_j - \sum_{j=1}^{n} \beta_j y_1 x_j = 0.$$

With the next couple of propositions we show that the class of ruled C^1-hypersurfaces is closed under the action of left translations and intrinsic dilations.

Proposition 3.5. *Let S be a ruled hypersurface of class C^1. Then $\tau_q(S)$ is ruled for any $q \in \mathbb{H}^n$.*

Proof. Fix $q = (\bar{x}^q, \bar{y}^q, t) \in \mathbb{H}^n$, define $\tilde{S} := \tau_q(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p \in S$ be such that $\tilde{p} = \tau_q(p)$. Being $\tau_q : S \longrightarrow \tilde{S}$ a diffeomorphism, then $d\tau_q|_p : T_pS \longrightarrow T_{\tilde{p}}\tilde{S}$ is an isomorphism. Therefore we have that
$$d\tau_q|_p(T_pS) = T_{\tilde{p}}\tilde{S}.$$

Moreover, by definition of $\mathcal{H}$, it is also the case that

$$d\tau_q|_p(\mathcal{H}_p) = \mathcal{H}_{\tilde{p}}.$$

Hence we infer that

$$d\tau_q|_p(\mathcal{H}T_pS) = d\tau_q|_p(\mathcal{H}_p \cap T_pS) = d\tau_q|_p(\mathcal{H}_p) \cap d\tau_q|_p(T_pS) = \mathcal{H}_{\tilde{p}} \cap T_{\tilde{p}}\tilde{S} = \mathcal{H}T_{\tilde{p}}\tilde{S}.$$

In particular, notice that $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot \mathcal{H}T_{\tilde{p}}S$ and assume that there exists $s \geqslant 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in \mathcal{H}T_pS$ be such that $d\tau_q|_p(v) = w$. By the left-invariance of the horizontal distribution, it follows that $w = (\bar{a}, \bar{b}, 0)$. Therefore s is maximal with the property that $p \cdot \delta_\tau(v) \in S$ for any $\tau \in [0, s]$. Hence $p \cdot \delta_s(v) \in S_0$, and so, since

$$\tilde{p} \cdot \delta_s(w) = \tilde{p} \cdot (s\bar{a}, s\bar{b}, 0) = q \cdot p \cdot (s\bar{a}, \bar{b}, 0) = q \cdot (p \cdot \delta_s(v))$$

and observing that $\tau_q(S_0) = \tilde{S}_0$, we conclude that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. □

Proposition 3.6. *Let S be a ruled hypersurface. Then $\delta_\lambda(S)$ is ruled for any $\lambda > 0$.*

Proof. Fix $\lambda > 0$, define $\tilde{S} := \delta_\lambda(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p = (\bar{x}, \bar{y}, t) \in S$ be such that $\tilde{p} = \delta_\lambda(p)$. Arguing as in the proof of Proposition 3.5, we get that

$$d\delta_\lambda|_p(\mathcal{H}T_pS) = \mathcal{H}T_{\tilde{p}}\tilde{S}. \tag{3.4}$$

Therefore, again, $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot \mathcal{H}T_{\tilde{p}}S$ and assume that there exists $s \geqslant 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in \mathcal{H}T_pS$ be such that $d\delta_\lambda|_p(v) = w$. We claim that that $w = \delta_\lambda(v)$. Indeed, recalling that the Jacobian matrix of δ_λ is a diagonal matrix with diagonal $(\lambda, \ldots, \lambda, \lambda^2)$, then

$$\begin{aligned} w(f)(q) &= \sum_{j=1}^n a_j \frac{\partial (f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^n b_j \frac{\partial (f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^n (a_j y_j - b_j x_j)\, T(f \circ \delta_\lambda)(p) \\ &= \sum_{j=1}^n \lambda a_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^n \lambda b_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^n ((\lambda a_j)(\lambda y_j) - (\lambda b_j)(\lambda x_j)) T f(\tilde{p}). \end{aligned} \tag{3.5}$$

The conclusion then follows as in the previous proof, just noticing that

$$\delta_\lambda(p \cdot \delta_\tau(v)) = \delta_\lambda(p) \cdot \delta_\lambda(\delta_\tau(v)) = \tilde{p} \cdot \delta_{\lambda\tau}(v) = \tilde{p} \cdot \delta_\tau(\delta_\lambda(v)) = \tilde{p} \cdot \delta_\tau(w)$$

for any $\tau \in \mathbb{R}$, and that $\delta_\lambda(S_0) = \tilde{S}_0$. □

In view of Proposition 3.5, we can enlarge the class of examples of ruled hypersurfaces.

Example 3.7 (Non-Vertical Hyperplanes)**.** We already know that $\mathcal{H}_0$ is a characteristic ruled smooth hypersurface. For any fixed $q = (\bar{x}_q, \bar{y}_q, t_q) \in \mathbb{H}^n$, we know from Proposition 3.5 that $\tau_q(\mathcal{H}_0)$ is a characteristic ruled smooth hypersurface. Moreover, an easy computation shows that

$$\tau_q(\mathcal{H}_0) = \{(\bar{x}, \bar{y}, t) \in \mathbb{H}^n \,:\, \langle (\bar{a}, \bar{b}), (\bar{x}, \bar{y}) \rangle + t + d = 0\},$$

where $(\bar{a}, \bar{b}) = (-\bar{y}_q, \bar{x}_q)$ and $d = -t_q$. Finally, notice that any hyperplane which is not vertical can be obtained as left-translation of the horizontal hyperplane $\mathcal{H}_0$. Hence we conclude that every hyperplane of $\mathbb{H}^n$ is ruled, and it is non-characteristic if and only if it is vertical. Finally, notice that we cannot exploit Proposition 3.6 to obtain more ruled hypersurfaces, since dilations of hyperplanes are hyperplanes.

To conclude this section, we show that the class of ruled hypersurfaces is closed under the action of the so-called *pseudohermitian transformations* of $\mathbb{H}^n$. To introduce this notion, we define the map $\mathcal{J} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\mathcal{J}(\bar{x}, \bar{y}, t) := (-\bar{y}, \bar{x}, t)$$

for any $p = (\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. The map $\mathcal{J}$ is a global diffeomorphism which preserves the horizontal distribution, related to the CR structure J by

$$d\mathcal{J}|_{\mathcal{H}} = J|_{\mathcal{H}}.$$

A global diffeomorphism $\varphi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is said to be a pseudohermitian transformation of $\mathbb{H}^n$ if it preserves the horizontal distribution and it commutes with $\mathcal{J}$, that is

$$d\varphi(\mathcal{H}) \subseteq \mathcal{H} \qquad \text{and} \qquad \varphi \circ \mathcal{J} = \mathcal{J} \circ \varphi.$$

Let us begin by considering a special subclass of pseudohermitian transformations. To this aim, let us define the map $\varphi_R : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\varphi_R(\bar{x}, \bar{y}, t) := (R(\bar{x}, \bar{y}), t), \tag{3.6}$$

where R is an orthogonal matrix of the form

$$R = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

where A and B are real-valued $n \times n$ matrices.

Proposition 3.8. *Let φ_R be as in (3.6). Then φ_R is a pseudohermitian transformation. Moreover, it holds that*

$$d\varphi_R|_p(\bar{a}, \bar{b}, 0) = (R(\bar{a}, \bar{b}), 0)$$

for any $p \in \mathbb{H}^n$ and any $(\bar{a}, \bar{b}, 0) \in H_p$.

Proof. Let $p = (\bar{x}, \bar{y}, t)$ and $(\bar{a}, \bar{b}, 0)$ as in the statement, and let $\tilde{p} := \varphi_R(p) = (\bar{\tilde{x}}, \bar{\tilde{y}}, t)$. We first claim that

$$d\varphi_R|_p(X_j|_p) = \sum_{k=1}^{n} \left(R_{kj} X_k|_{\tilde{p}} + R_{(n+k)j} Y_k|_{\tilde{p}} \right)$$

and

$$d\varphi_R|_p(Y_j|_p) = \sum_{k=1}^{n} \left(R_{k(n+j)} X_k|_{\tilde{p}} + R_{n+k)(n+j)} Y_k|_{\tilde{p}} \right)$$

for any $j = 1, \ldots, n$. Indeed, let ψ be a C^1 function defined in a neighborhood of $\tilde{p}$. Let us recall that, since $(\bar{\tilde{x}}, \bar{\tilde{y}}) = R(\bar{x}, \bar{y})$ and R is orthogonal, then $(\bar{x}, \bar{y}) = R^T(\bar{\tilde{x}}, \bar{\tilde{y}})$, which means, recalling also the special block shape of R, that

$$-x_j = \sum_{k=1}^{n} \left(-R_{kj} \tilde{x}_k - R_{(n+k)j} \tilde{y}_k \right) = \sum_{k=1}^{n} \left(-R_{(n+k)(n+j)} \tilde{x}_k + R_{k(n+j)} \tilde{y}_k \right)$$

and

$$y_j = \sum_{k=1}^{n} \left(R_{k(n+j)} \tilde{x}_k + R_{(n+k)(n+j)} \tilde{y}_k \right) = \sum_{k=1}^{n} \left(-R_{(n+k)j} \tilde{x}_k + R_{kj} \tilde{y}_k \right).$$

for any $j = 1, \dots, n$. Then it holds that

$$\begin{aligned}
d\varphi_R|_p(X_j|_p)(\psi)(\tilde{p}) &= X_j|_p(\psi \circ \varphi_R)(p) \\
&= \frac{\partial}{\partial x_j}(\psi \circ \varphi_R)(p) + y_j T(\psi \circ \varphi_R)(p) \\
&= \sum_{k=1}^{n} \left(R_{kj} \frac{\partial \psi}{\partial x_k}(\tilde{p}) + R_{(n+k)j} \frac{\partial \psi}{\partial y_k}(\tilde{p}) \right) + y_j T(\psi)(\tilde{p}) \\
&= \sum_{k=1}^{n} \left(R_{kj} \left(\frac{\partial \psi}{\partial x_k}(\tilde{p}) + \tilde{y}_k T(\psi(\tilde{p}) \right) + R_{(n+k)j} \left(\frac{\partial \psi}{\partial y_k}(\tilde{p}) - \tilde{x}_k T(\psi)(\tilde{p}) \right) \right) \\
&= \sum_{k=1}^{n} \left(R_{kj} X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)j} Y_k|_{\tilde{p}}(\psi)(\tilde{p}) \right)
\end{aligned}$$

and, similarly,

$$d\varphi_R|_p(Y_j|_p)(\psi)(\tilde{p}) = \sum_{k=1}^{n} \left(R_{k(n+j)} X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)(n+j)} Y_k|_{\tilde{p}}(\psi)(\tilde{p}) \right)$$

for any $j = 1, \dots, n$. Hence we conclude that

$$\begin{aligned}
d\varphi_R|_p(\bar{a}, \bar{b}, 0) &= \sum_{j=1}^{n} (a_j d\varphi_R|_p(X_j|_p) + b_j d\varphi_R|_p(Y_j|_p)) \\
&= \sum_{j,k=1}^{n} \left(a_j \left(R_{kj} X_k|_{\tilde{p}} + R_{(n+k)j} Y_k|_{\tilde{p}} \right) + b_j \left(R_{k(n+j)} X_k|_{\tilde{p}} + R_{n+k)(n+j)} Y_k|_{\tilde{p}} \right) \right) \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \left(R_{kj} a_j + R_{k(n+j)} b_j \right) X_k|_{\tilde{p}} + \sum_{j=1}^{n} \left(R_{(n+k)j} a_j + R_{(n+k)(n+j)} b_j \right) Y_k|_{\tilde{p}} \right) \\
&= (R(\bar{a}, \bar{b}), 0).
\end{aligned}$$

□

As a consequence of the previous result, it is easy to see that the class of ruled hypersurfaces is closed under the action of maps of the form (3.6).

Proposition 3.9. *Let S be a ruled hypersurface. Then $\varphi_R(S)$ is ruled for any φ_R as in* (3.6).

Proof. The proof of this result, with the help of Proposition 3.8, follows as the proof of Proposition 3.5 and Proposition 3.6, noticing that $\varphi_R(S_0) = (\varphi_R(S))_0$ and that, for a given $p = (z, t) \in S \setminus S_0$, $(v, 0) \in \mathcal{H}T_pS$ and $s \in \mathbb{R}$, it holds that

$$\begin{aligned}
\varphi_R(p \cdot \delta_s(v, 0)) &= \varphi_R(z + sv, t + Q(z, sv)) \\
&= (R(z + sv), t + sQ(z, v)) \\
&= (Rz + sRv, t + sQ(Rz, Rv)) \\
&= (Rz, t) \cdot (sRv, 0)) \\
&= \varphi_R(p) \cdot \delta_s(Rv, 0).
\end{aligned}$$

□

As a corollary of Proposition 3.8, we can conclude our initial statement.

Theorem 3.10. *If S is ruled, then $\varphi(S)$ is ruled for any pseudohermitian transformation φ.*

Proof. It follows combining Proposition 3.5, Proposition 3.9 and [**?**, Theorem 4.1]. □

4. Ruled hypersurfaces with countable characteristic set

The aim of this section is to characterise ruled hypersurfaces of $\mathbb{H}^n$ with countable characteristic set, when $n \geqslant 2$. In the first Heisenberg group $\mathbb{H}^1$ there are examples of ruled, non-characteristic, smooth surfaces which are not vertical planes. As an instance, let us consider the surface S parametrized by the map $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{H}^1$ defined by

$$\varphi(t,\theta) := (t\cos\theta, t\sin\theta, \theta).$$

Notice that φ is smooth and injective. Moreover,

$$\frac{\partial\varphi}{\partial t}(t,\theta) = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} = \cos\theta X|_{\varphi(t,\theta)} + \sin\theta Y|_{\varphi(t,\theta)}$$

and

$$\frac{\partial\varphi}{\partial\theta}(t,\theta) = -t\sin\theta\frac{\partial}{\partial x} + t\cos\theta\frac{\partial}{\partial y} + T = -t\sin\theta X|_{\varphi(t,\theta)} + t\cos\theta Y|_{\varphi(t,\theta)} + (1+t^2)T.$$

This implies that S is a smooth, non-characteristic surface, and moreover

$$\mathcal{H}T_{\varphi(t,\theta)}S = \operatorname{span}\left\{\frac{\partial\varphi}{\partial t}(t,\theta)\right\}$$

for any $(t,\theta) \in \mathbb{R}^2$. Finally, for given $t,\theta,s \in \mathbb{R}$, it holds that

$$(t\cos\theta, t\sin\theta, \theta)\cdot(s\cos\theta, s\sin\theta, 0) = ((t+s)\cos\theta, (t+s)\sin\theta, \theta) \in S,$$

and so S is ruled. However, the situation in higher dimensional Heisenberg groups is quite different, and the ruling condition turns out to be more restrictive. Indeed, we are going to prove that the only closed, ruled hypersurfaces with countable characteristic set in $\mathbb{H}^n$, with $n \geqslant 2$, are hyperplanes. To this aim, we already know that vertical hyperplanes are non-characteristic and ruled, and that every non-vertical hyperplane

$$P := \left\{(\bar{x},\bar{y},t) \in \mathbb{H}^n \ : \ \sum_{j=1}^n a_j x_j + \sum_{j=1}^n b_j y_j + ct + d = 0\right\},$$

where clearly $c \neq 0$, is ruled and satisfies

$$P_0 = \left\{\left(\frac{b_1}{c},\dots,\frac{b_n}{c},-\frac{a_1}{c},\dots,-\frac{a_n}{c},-\frac{d}{c}\right)\right\}.$$

Before proving Theorem 1.3 we establish some preliminary results.

Proposition 4.1. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Then*

$$p\cdot\mathcal{H}T_pS \subseteq S$$

for any $p \in S \setminus S_0$.

Proof. Let S and p as in the statement. Assume by contradiction that there exists $q \in p\cdot\mathcal{H}T_pS \setminus S$. Combining Proposition 3.2 with the fact that S_0 is countable and that S is ruled, it is easy to construct a sequence $(q_h)_h \subseteq S$ converging to q as $h \to \infty$. Being S closed, then $q \in S$, a contradiction. □

Proposition 4.2. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Assume that $0 \in S \setminus S_0$. Then*

$$S \cap \mathcal{H}_0 = \mathcal{H}T_0S.$$

Proof. First, since $0 \in S \setminus S_0$ and in view of Proposition 4.1, then $\mathcal{H}T_0S \subseteq S \cap \mathcal{H}_0$. Assume by contradiction that there exists $q = (z_q, 0) \in (S \cap \mathcal{H}_0) \setminus \mathcal{H}T_0S$. If S is tangent to $\mathcal{H}_0$ at q, then $q \in S \setminus S_0$. Otherwise, since $\mathcal{H}T_0S$ is closed and S_0 is countable, it is possible to find another point in $(S \setminus S_0) \setminus \mathcal{H}T_0S$. In the end, we can assume that $q \in S \setminus S_0$. Again, thanks to Proposition 4.1, $q \cdot \mathcal{H}T_qS \subseteq S$, and so $q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0 \subseteq S \cap \mathcal{H}_0$. Note that both $\mathcal{H}T_0S$ and $q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0$ are affine subspaces of $\mathcal{H}_0$. Moreover, $\dim(\mathcal{H}T_0S) = 2n-1$ and $\dim(q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) \geqslant 2n-2$. Therefore we conclude that

$$\dim(\mathcal{H}T_0S \cap (q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0)) \geqslant \dim(\mathcal{H}T_0S) + \dim(q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) - 2n = 2n-3 \geqslant 1,$$

since $n \geqslant 2$. Therefore $(\mathcal{H}T_0S) \cap (q \cdot \mathcal{H}T_qS \cap H_0)$ contains a one-dimensional affine subspace of H_0. In particular, being S_0 countable, there exists $p = (z_p, 0) \in \mathcal{H}T_0S \cap (q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) \cap (S \setminus S_0)$. Let $v \in \mathcal{H}T_pS$ be such that $p \cdot tv = q$ for some $t \in \mathbb{R}$, and let $\gamma_p(t) := (tz_p, 0)$. Notice that, by construction, then $\gamma_p(t) \in S$ for any $t \in \mathbb{R}$. Moreover, $\dot{\gamma}_p(1) = (z_p, 0) \in \mathcal{H}_p$, and so $w := (z_p, 0) \in \mathcal{H}T_pS$. Again, since $p \in S \setminus S_0$ and in view of Proposition 4.1, then $p \cdot \mathcal{H}T_pS \subseteq S$. Therefore, in particular, it holds that

$$p \cdot (\alpha v + \beta w) \in S$$

for any $\alpha, \beta \in \mathbb{R}$. Hence, if we let $\gamma_q(t) := (tz_q, 0)$, we conclude that $\gamma(t) \in S \cap \mathcal{H}_0$ for any $t \in \mathbb{R}$, and so $\dot{\gamma}_q(0) = (z_q, 0) \in T_0S$. Since clearly $(z_q, 0) \in \mathcal{H}_0$, then $q \in \mathcal{H}T_0S$, which is a contradiction. $\square$

Proposition 4.3. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Then either S is a t-graph or S is a vertical hyperplane.*

Proof. If S is a t-graph we are done. If S is not a t-graph, being S_0 countable, there exists $p \in S \setminus S_0$ such that $T|_p \in T_pS$. Up to a left-translation, recalling Proposition 3.5, we assume that $p = 0$. We show that S is a vertical hyperplane, dividing the proof into some steps.
Step 1. Thanks to Proposition 4.2, we know that there exists $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ such that

$$\mathcal{H}T_0S = \mathcal{H}_0 \cap S = \{(\bar{x}, \bar{y}, 0) \in \mathbb{H}^n \,:\, \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle = 0\}.$$

We assume without loss of generality that $a_1 \neq 0$, and we let $f(\bar{x}, \bar{y}) := \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle$. We claim that

$$\pi(p \cdot \mathcal{H}T_pS) \subseteq \pi(\mathcal{H}T_0S)$$

for any $p \in \mathcal{H}T_0S \cap (S \setminus S_0)$, where here and in the following the map $\pi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ is defined by

$$\pi(\bar{x}, \bar{y}, t) := (\bar{x}, \bar{y}).$$

Assume by contradiction that there exists $p = (z_p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v = (v, 0) \in \mathcal{H}T_pS$ such that $z_p + v \notin \pi(\mathcal{H}T_0S)$. This is equivalent to say that $f(z_p + v) \neq 0$. Let us define $q := p \cdot v = (z_p + v, Q(z_p, v))$. Since $p \in S \setminus S_0$ and by Proposition 4.1, then $q \in S$. Moreover, $Q(z_p, v) \neq 0$, since otherwise $q \in \mathcal{H}T_0S$ and consequently $f(z_p + v) = 0$. Moreover, since $z_p \in \mathcal{H}T_0S$, then, letting $\gamma(t) := (tz_p, 0)$, it holds that $\gamma(t) \in S$ for any $t \in \mathbb{R}$, and so $(z_p, 0) \in \mathcal{H}T_pS$. Hence, since $p \cdot \mathcal{H}T_pS \subseteq S$, we conclude in particular that

$$P := \{(z_p, 0) + \alpha(z_p, 0) + \beta(v, Q(z_p, v)) \,:\, \alpha, \beta \in \mathbb{R}\} \subseteq S.$$

Notice that P is a vector subspace of $\mathbb{R}^{2n+1}$. Then in particular $0 \in P$ and $(v, Q(z_p, v)) \in T_0S$. Therefore, as $T \in T_0S$, then $(v, 0) \in T_0S$, and so, since $(v, 0) \in \mathcal{H}_0$, we conclude that $(v, 0) \in \mathcal{H}T_0S$. Then $f(v) = 0$, and so, as $p \in \mathcal{H}T_0S$, $f(z_p + v) = f(z_p) + f(v) = 0$, a contradiction.
Step 2. Let $p = (z_p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$. Thanks to Step 1, we know that $\pi(p \cdot \mathcal{H}T_pS) \subseteq \pi(\mathcal{H}T_0S)$. Therefore, if $v \in \mathcal{H}T_pS$, then $f(z_p + v) = 0$. Since $f(z_p) = 0$, we conclude that $f(v) = 0$, which implies that

$$\mathcal{H}T_pS = \mathcal{H}T_0S \tag{4.1}$$

for any $p \in \mathcal{H}T_0S$. Moreover, an easy computation shows that

$$\mathcal{H}T_0S = \operatorname{span}\{Z_2|_0, \dots, Z_n|_0, W_1|_0, \dots, W_n|_0\},$$

where

$$Z_i = a_iX_1 - a_1X_i \qquad \text{and} \qquad W_j = b_jX_1 - a_1Y_j \tag{4.2}$$

for any $i = 2, \dots, n$ and $j = 1, \dots, n$. Then (4.1) allows to conclude that

$$\mathcal{H}T_pS = \operatorname{span}\{Z_2|_p, \dots, Z_n|_p, W_1|_p, \dots, W_n|_p\}. \tag{4.3}$$

Step 3. Let us define

$$\mathcal{Z} := \{z \in \pi(\mathcal{H}T_0S) \,:\, Q(z, w) = 0 \text{ for any } w \in \pi(\mathcal{H}T_0S)\}.$$

Notice that, being Q a bilinear map, then $\mathcal{Z}$ is a vector subspace of $\pi(\mathcal{H}T_0S)$. We claim that $\dim(\mathcal{Z}) \leqslant 2n - 2$. Indeed, assume by contradiction that $\dim(\mathcal{Z}) \geqslant 2n - 1$. Then, since $\mathcal{Z} \subseteq \pi(\mathcal{H}T_0S)$ and $\dim(\pi(\mathcal{H}T_0S)) = 2n - 1$, we conclude that $\mathcal{Z} = \pi(\mathcal{H}T_0S)$. We show that this leads to a contradiction. Assume first that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, and set $z_1 = (0, -1, 0 \dots, 0)$ and $z_2 = (\bar{0}, 0, 1, 0, \dots, 0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1, z_2) = 1 \neq 0$, which implies that $z_1, z_2 \notin \mathcal{Z}$. If it is not the case that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, then assume without loss of generality that $a_2 \neq 0$. Let $z_1 = (-a_2, a_1, 0, \dots, 0)$ and $z_2 = (-b_1, 0, \dots, 0, a_1, 0, \dots, 0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1, z_2) = a_1a_2 \neq 0$, which implies again that $z_1, z_2 \notin \mathcal{Z}$. Therefore we conclude that $\dim(\mathcal{Z}) \leqslant 2n - 2$, and so in particular

$$\overline{\pi(\mathcal{H}T_0S) \setminus \mathcal{K}} = \pi(\mathcal{H}T_0S). \tag{4.4}$$

Step 4. We claim that for any $q = (z_q, t_q) = (x_1^q, \dots, x_n^q, y_1^q, \dots, y_n^q, t_q)$ such that $z_q \in \pi(\mathcal{H}T_0S) \setminus \mathcal{Z}$ there exists $p = (z_p, 0) = (x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v \in \mathcal{H}T_pS$ such that

$$q = p \cdot v. \tag{4.5}$$

Indeed, let q as above, and let $p \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v \in \mathcal{H}T_pS$ to be chosen later. In view of (4.3), we can express v as

$$v = \sum_{j=2}^{n} \alpha_j Z_j|_p + \sum_{j=1}^{n} \beta_j W_j|_p = \left(\sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j \right) X_1|_p - \sum_{j=2}^{n} \alpha_j a_1 X_j|_p - \sum_{j=1}^{n} \beta_j a_1 Y_j|_p.$$

for some $\alpha_2, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$. Therefore, we infer that

$$p \cdot v = \left(x_1^p + \sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j, x_2^p - \alpha_2 a_1, \dots, y_n^p - \beta_n a_1, Q(z_p, v) \right).$$

Let us choose

$$\alpha_i = \frac{x_i^p - x_i^q}{a_1} \qquad \textit{and} \qquad \beta_j = \frac{y_j^p - y_j^q}{a_1}$$

for any $i = 2, \dots, n$ and any $j = 1, \dots, n$. This choice implies that $(p \cdot v)_i = x_i^q$ and $(p \cdot v)_j = y_{j-n}^q$ for any $i = 2, \dots, n$ and any $j = n + 1, \dots, 2n$. Moreover, since $f(z_p) = f(z_q) = 0$, it holds that

$$(p \cdot v)_1 = x_1^p + \sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j = \frac{1}{a_1} \left(\sum_{j=1}^{n} (a_j x_j^p + b_j y_j^p) - \sum_{j=2}^{m} a_j x_j^q + \sum_{j=1}^{n} b_j y_j^q \right) = x_1^q.$$

Finally, notice that

$$\begin{aligned}
Q(z_p,v) &= \left(\sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j\right) y_1^p - \sum_{j=2}^n \alpha_j a_1 y_j^p + \sum_{j=1}^n \beta_j a_1 x_j^p \\
&= \frac{1}{a_1}\Big(\sum_{j=2}^n a_j x_j^p y_1^p - \sum_{j=2}^n a_j x_j^q y_1^p + \sum_{j=1}^n b_j y_j^p y_1^p - \sum_{j=1}^n b_j y_j^q y_1^p \\
&\quad - \sum_{j=2}^n a_1 x_j^p y_j^p + \sum_{j=2}^n a_1 x_j^q y_j^p + a_1 x_1^p y_1^p + \sum_{j=2}^n a_1 x_j^p y_j^p - \sum_{J=1}^n a_1 x_j^p y_j^q\Big) \\
&= \frac{1}{a_1}\Big(-\sum_{j=1}^n a_j x_j^q y_1^p - \sum_{j=1}^n b_j y_j^q y_1^p + \sum_{j=1}^n a_1 x_j^q y_j^p - \sum_{j=1}^n a_1 x_j^p y_j^q\Big) \\
&= Q(z_p,z_q),
\end{aligned}$$

where in the third equality we exploited the fact that $f(z_p)=0$, while the fourth equality follows from $f(z_q)=0$. Since we assumed $z_q \notin \mathcal{Z}$, then there exists uncountably many $w \in \pi(\mathcal{H}T_0S)$ such that $Q(w,z_q)\neq 0$. Therefore, since S_0 is countable, it is possible to choose $w \in \pi(\mathcal{H}T_0S)$ such that, setting

$$z_p = \frac{t_q}{Q(w,z_q)} w,$$

then $p \in (S\setminus S_0)$. We conclude that $p \in \mathcal{H}T_0S \cap (S\setminus S_0)$ and $Q(z_p,z_q)=t_q$.
Step 5. We are now able to conclude. Indeed, thanks to (4.5) we infer that

$$\pi(HT_0S\setminus\mathcal{K})\times\mathbb{R}\subseteq S.$$

But then, being S closed and recalling (4.4), we conclude that

$$\pi(HT_0S)\times\mathbb{R} = \overline{\pi(HT_0S)\setminus\mathcal{K}}\times\mathbb{R} = \overline{\pi(HT_0S)\times\mathbb{R}} \subseteq \overline{S} = S.$$

Therefore S contains the vertical hyperplane $\pi(HT_0S)\times\mathbb{R}$. The thesis then follows in view of the topological assumptions on S. □

Proof of Theorem 1.3. Let S be as in the statement. If S is a vertical hyperplane, we are done. If not, in view of Proposition 4.3, S is a t-graph. Being S_0 countable, and recalling Proposition 3.5, up to a left translation we may assume that $0 \in S\setminus S_0$ and that $T|_0 \notin T_0S$. Since $0\in S\setminus S_0$, we infer by Proposition 4.2 that

$$\mathcal{H}T_0S = \mathcal{H}_0\cap S = \{(\bar{x},\bar{y},0)\in\mathbb{H}^n \,:\, \langle(\bar{x},\bar{y}),(\bar{a},\bar{b})\rangle = 0\}$$

for some $0\neq(\bar{a},\bar{b})\in\mathbb{R}^{2n}$. Again, by Proposition 3.8 we may assume that $a_1\neq 0$. Being S an entire t-graph, and since $T|_0\notin T_0S$ and $0\in S\setminus S_0$, there exists $v=(z_v,t_v)\in T_0S$ such that $f(z_v)\neq 0$ and $t_v\neq 0$. Let us set $c := -\frac{f(z_v)}{t_v}$. We claim that

$$S = \{(z,t)\in\mathbb{H}^n \,:\, f(z)+ct=0\} =: S_c.$$

Indeed, let $p=(z_p,t_p)\in S\setminus S_0$. If $t_p=0$, then $f(z_p)=0$, and so $p\in S_c$. Assume then $t_p\neq 0$. Then, being S a t-graph, we infer that $f(z_p)\neq 0$.. Let $v_1,\dots,v_{2n-1}$ be a basis of $\mathcal{H}T_pS$. Since $p\in S\setminus S_0$ and thanks to Proposition 4.1, then $p\cdot\mathcal{H}T_pS\subseteq S$. We claim that there exists $j=1,\dots,2n-1$ such that $Q(z_p,v_j)\neq 0$. Indeed, assume by contradiction that $Q(z_p,v_1)=\ldots=Q(z_p,v_{2n-1})=0$. In this case, recalling that S is ruled, it holds that

$$p\cdot\mathcal{H}T_pS = \left\{\left(z_p+\sum_{j=1}^{2n-1}\alpha_j v_j, t_p\right) \,:\, \alpha_1,\dots,\alpha_{2n-1}\in\mathbb{R}\right\}\subseteq S. \tag{4.6}$$

We claim that

$$\operatorname{span}\{(v_1,0),\dots,(v_{2n-1},0)\} = \operatorname{span}\{Z_2|_0,\dots,Z_n|_0,W_1|_0,\dots,W_n|_0\}, \tag{4.7}$$

where $Z_2,\dots,Z_n,W_1,\dots,W_n$ are defined as (4.2). Indeed, if it was not the case, then (4.6) would imply the existence of $q=(z_q,t_p)\in S$ such that $z_q\in\pi(\mathcal{H}T_0S)$. But since $t_p\neq 0$ and since $(z_q,0)\in S$, we would contradict the fact that S is a t-graph. Notice that (4.7) implies that

$$Z_2|_0,\dots,Z_n|_0,W_1|_0,\dots,W_n|_0\in H_p$$

and so, observing that

$$Z_j|_0 = a_j\frac{\partial}{\partial x_1} - a_1\frac{\partial}{\partial x_j} = a_jX_1|_p - a_1X_j|_p + (a_1y_j - a_jy_1)T$$

for any $j=2,\dots,n$ and

$$W_j|_0 = b_j\frac{\partial}{\partial x_1} - a_1\frac{\partial}{\partial y_j} = b_jX_1|_p - a_1Y_j|_p + (-a_1x_j - b_jy_1)T$$

for any $j=1,\dots,n$, we conclude that

$$z_p = \frac{y_1}{a_1}(-b_1,\dots,-b_n,a_1,\dots,a_n),$$

which implies in particular that $f(z_p)=0$, a contradiction. In this case, it holds that $p\cdot HT_pS\cap\mathcal{H}_0\cap S = p\cdot\mathcal{H}T_pS\cap\mathcal{H}T_0S\neq 0$. Since $n\geqslant 2$, a dimensional argument as in the proof of Proposition 4.2 implies that $\dim(p\cdot\mathcal{H}T_pS\cap\mathcal{H}_0\cap S)\geqslant 1$. Therefore, being S_0 countable, there exists $q=(z_q,0)\in(p\cdot\mathcal{H}T_pS)\cap\mathcal{H}T_0S\setminus S_0$. Let then $w\in\mathcal{H}T_pS$ be such that

$$(z_q,0) = (z_p+w, t_p+Q(z_pw)). \tag{4.8}$$

Arguing as in the proof of Proposition 4.2, recalling that $q\in S\setminus S_0$ and Proposition 4.1, we see that

$$P := \{(z_q,0)+\alpha(z_q,0)+\beta(w,Q(z_q,w))\,:\,\alpha,\beta\in\mathbb{R}\}\subseteq S,$$

and so we conclude as above that $(w.Q(z_q,w))\in T_0S$. This means that there exists $\tilde w\in\pi(\mathcal{H}T_0S)$ and $\alpha\in\mathbb{R}$ such that

$$(w.Q(z_q,w)) = (\tilde w+\alpha z_v,\alpha t_v).$$

Therefore, recalling (4.8), we get that

$$(z_p,t_p) = (z_q,0)-(w,Q(z_p,w)) = (z_q-\tilde w-\alpha z_v,-\alpha t_v)\,.$$

Therefore, since $z_q,\tilde w\in\pi(\mathcal{H}T_0S)$ we conclude that

$$f(z_p)+ct_p = -\alpha(f(z_v)+ct_v) = 0,$$

which implies that $p\in S_c$. Therefore we proved that $S\setminus S_0\subseteq S_c$, and so, being S_0 countable and S and S_c closed, we conclude that $S\subseteq S_c$. The thesis then follows by the topological assumptions on S. □

5. Ruled intrinsic cones

In this section we study ruled hypersurfaces among the class of hypersurfaces which are invariant under intrinsic dilations, that is the class of intrinsic cones. A set $C\subseteq\mathbb{H}^n$ is a *cone* if

$$\delta_\lambda(C)\subseteq C$$

for any $\lambda>0$. It is easy to see that, if C is a cone, then $0\in\overline{C}$, $\delta_\lambda(C)=C$ and $\delta_\lambda(\partial C)=\partial C$ for any $\lambda>0$. We say that S is a *conical hypersurface* if it is both a cone and a hypersurface. Notice that, in view of the aforementioned properties, if C is a cone with boundary of class C^k, then ∂C is a conical hypersurface of class C^k. The simplest instance of non-characteristic conical hypersurfaces is given by vertical hyperplanes passing through the origin. Another simple instance is given by the horizontal plane $\mathcal{H}_0$. In this case we already know that $(\mathcal{H}_0)_0 = \{0\}$. Finally, if u is an homogeneous quadratic polynomial, then graph(u) is a conical smooth hypersurface. Moreover, in this last case, S_0 may be an infinite set. As an instance, consider the graph associated to $u(\bar{x},\bar{y}) = \sum_{j=1}^n x_j y_j$. It is easy to see that

$$T_pS = \operatorname{span}\{X_1,\ldots,X_n, Y_1 + 2x_1T,\ldots,Y_n + 2x_nT\}$$

for any $p = (\bar{x},\bar{y},u(\bar{x},\bar{y})) \in \operatorname{graph}(u)$. Therefore in this case

$$S_0 = \{(\bar{x},\bar{y},u(\bar{x},\bar{y})) \in \operatorname{graph}(u) \,:\, x_1 = \ldots = x_n = 0\}.$$

When a hypersurface is a cone, we can say more about the structure of S_0.

Proposition 5.1. *Let S be a conical hypersurface of class C^1. Then S_0 is a cone.*

Proof. Let $p \in S_0$ and $\lambda > 0$. We prove that $q := \delta_\lambda(p) \in S_0$. If $p = 0$ the thesis is trivial. Assume that $p \neq 0$. We prove that $H_q = T_qS$. Since S is a cone, then $\delta_\lambda : S \longrightarrow S$ is a diffeomorphism, and consequently, recalling (3.4), $d\delta_\lambda|_p : \mathcal{H}T_pS \longrightarrow \mathcal{H}T_qS$ is an isomorphism. we conclude that $\dim(\mathcal{H}T_pS) = \dim(\mathcal{H}T_qS)$, which means that $q \in S_0$. □

Proposition 5.2. *Let S be a conical hypersurface of class C^1. Then $S_0 \subseteq H_0$. Moreover, for any $p \in S_0$ there is a horizontal half line $\gamma : [0,+\infty) \longrightarrow S_0$ such that $\gamma(0) = 0$ and $\gamma(1) = p$.*

Proof. Let $p = (\bar{x},\bar{y},t) \in S_0 \setminus \{0\}$, and set $\gamma(0) = 0$ and $\gamma(\lambda) := \delta_\lambda(p)$. Then γ is a smooth curve with

$$\dot{\gamma}(\lambda) = (\bar{x},\bar{y},2\lambda t) = \sum_{j=1}^n x_jX_j + \sum_{j=1}^n y_jY_j + 2\lambda tT.$$

Moreover, thanks to Proposition 5.1, then $\gamma([0,+\infty)) \subseteq S_0$. Finally, since $\gamma(1) = p$, S is a cone and $p \in S_0$, then $\dot{\gamma}(1) \in T_pS = \mathcal{H}_p$, and so $t = 0$. □

The shape of conical hypersurfaces strongly depends on the size of the associated characteristic set. Exploiting [22, Theorem 4.1], it is easy to see that vertical hyperplanes passing through the origin are the only possible examples of non-characteristic conical hypersurfaces of class C^1. Hence, in the rest of this section we assume that $S_0 \neq 0$. In this case it suffices to reduce to the analysis of t-graphs (cf. [29, Lemma 4.4]).

Proposition 5.3. *Let S be a conical hypersurface of class C^1. If $S_0 \neq \emptyset$, then $S = \operatorname{graph}(u)$ for some $u \in C^1(\mathbb{R}^{2n})$.*

In Section 4 we exhibited two examples of ruled conical smooth hypersurfaces, namely the horizontal hyperplane H_0 and the vertical hyperplanes passing through the origin. The aim of the rest of this section is to show that, in the class of conical hypersurfaces of class C^2, these are the only possible examples. The following characterization follows at once by Theorem 1.3, but we give here a different proof in the spirit of [29, Lemma 4.4].

Theorem 5.4. *Let S be a conical hypersurface of class C^1. Assume that $S_0 = \{0\}$. Then S is ruled if and only if S is the horizontal plane H_0.*

Proof. For sake of notational simplicity, we prove the statement when $n = 2$, being the other cases completely analogous. We already know that $\mathcal{H}_0$ is ruled. Conversely, let S be ruled, and assume by contradiction that there exists $p = (z,t) \in S$ with $t \neq 0$. Then, thanks to Proposition 5.2, $p \in S \setminus S_0$. Moreover, $p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset$, since otherwise there would be an horizontal line joining p and 0, which contradicts the fact that horizontal lines passing through 0 lie in $\mathcal{H}_0$. Therefore, being S ruled and thanks to Proposition 3.2, we infer that $p \cdot \mathcal{H}T_pS \subseteq S$. It is well known (cf. e.g. [12]) that there exists an orthonormal basis u, v, w of $\mathcal{H}T_pS$ such that

$$J(u) = w \qquad \text{and} \qquad J(v) = \nu_S(p). \tag{5.1}$$

Let us set

$$M := \begin{bmatrix} u & v & J(u) & J(v) \end{bmatrix}^T.$$

Then, defining φ_R as in (3.6) and thanks to Proposition 3.8, we can assume that $u = X_1$, $v = X_2$ and $w = Y_1$. Let us define $\varphi : (0, +\infty) \times \mathbb{R}^3 \longrightarrow S$ by

$$\varphi(\lambda, \alpha, \beta, \gamma) := \delta_\lambda \left(p \cdot \left(\frac{\alpha}{\lambda} u + \frac{\beta}{\lambda} v + \frac{\gamma}{\lambda} w \right) \right).$$

Being S a ruled cone, the map φ is well-defined. Moreover, notice that

$$\begin{aligned}
\varphi(\lambda, \alpha, \beta, \gamma) &= \delta_\lambda \left(z + \frac{\alpha u + \beta v + \gamma w}{\lambda}, t + \frac{\alpha Q(z,u) + \beta Q(z,v) + \gamma Q(z,w)}{\lambda} \right) \\
&= (\lambda z + \alpha u + \beta v + \gamma w, \lambda^2 t + \lambda \alpha Q(z,u) + \lambda \beta Q(z,v) + \lambda \gamma Q(z,w)) \\
&= (\lambda x_1 + \alpha, \lambda x_2 + \beta, \lambda y_1 + \gamma, \lambda y_2, \lambda^2 t + \lambda \alpha y_1 + \lambda \beta y_2 - \lambda \gamma x_1).
\end{aligned}$$

Therefore, an easy computation shows that

$$D\varphi(\lambda, \alpha, \beta, \gamma) = \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ y_1 & 0 & 0 & 1 \\ y_2 & 0 & 0 & 0 \\ 2\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1 & \lambda y_1 & \lambda y_2 & -\lambda x_1 \end{bmatrix}$$

We claim that $y_2 \neq 0$. Otherwise, recalling that $p \cdot \mathcal{H}T_p S \subseteq S$, we would have that

$$(x_1, x_2, y_1, 0, t) \cdot (\alpha, \beta, \gamma, 0, 0) = (x_1 + \alpha, x_2 + \beta, y_1 + \gamma, 0, t + \alpha y_1 - \gamma x_1) \in S$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$. Therefore, choosing $\alpha = -x_1$, $\beta = -x_2$ and $\gamma = -y_1$, we conclude that $(0,0,0,0,t) \in S$, which is a contradiction, since $0 \in S$ and S, thanks to Proposition 5.3, is a t-graph. Hence $y_2 \neq 0$, and so, since $\varphi(0,0,0,0) = p$, $D\varphi$ has maximum rank in a neighborhood of $(0,0,0,0)$. In particular,

$$T_{\varphi(q)} S = \operatorname{span} \left\{ \frac{\partial \varphi}{\partial \lambda}(q), \frac{\partial \varphi}{\partial \alpha}(q), \frac{\partial \varphi}{\partial \beta}(q), \frac{\partial \varphi}{\partial \gamma}(q) \right\}$$

for any q sufficiently close to $(0,0,0,0)$. Notice that, if we define the 1-form ω by

$$\omega = dt - \sum_{j=1}^n y_j dx_j + \sum_{j=1}^n x_j dy_j,$$

then $v \in \mathcal{H}$ if and only if $\omega(v) = 0$ for any $v \in T\mathbb{H}^n$. Fix $q = (\lambda, \alpha, \beta, \gamma)$ close to $(0,0,0,0)$. Then

$$\omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \lambda}(q) \right) = 2(\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1),$$

and moreover

$$\omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \alpha}(q) \right) = -\gamma, \qquad \omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \beta}(q) \right) = 0, \qquad \omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \gamma}(q) \right) = \alpha.$$

Therefore, if we choose $\alpha = \gamma = 0$, we conclude that

$$\operatorname{span} \left\{ \frac{\partial \varphi}{\partial \alpha}(q), \frac{\partial \varphi}{\partial \beta}(q), \frac{\partial \varphi}{\partial \gamma}(q) \right\} \subseteq \mathcal{H}T_{\varphi(q)}(S).$$

Moreover, since $y_2 \neq 0$ we can choose $\beta = -\frac{\lambda t}{y_2}$ to conclude that

$$\frac{\partial \varphi}{\partial \lambda}(q) \in \mathcal{H}T_{\varphi(q)} S.$$

Since $\operatorname{rank}(D\varphi(q)) = 4$, we proved that

$$\varphi\left(\lambda, 0, -\frac{\lambda t}{y_2}, 0\right) = \left(\lambda x_1, \lambda x_2 - \frac{\lambda t}{y_2}, \lambda y_1, \lambda y_2, 0\right) \in S_0$$

for any $\lambda > 0$ small enough. Since $y_2 \neq 0$, we proved that there exists $\tilde{p} \neq 0$ such that $\tilde{p} \in S_0$. This is a contradiction with the assumption $S_0 = \{0\}$. $\square$

We are left with the analysis of ruled conical hypersurfaces S with infinite characteristic set. In this case we limit ourselves to consider conical hypersurfaces of class C^2. In this simpler situation, it suffices to consider graphs of quadratic polynomials.

Proposition 5.5. *Let S be a conical hypersurfaces of class C^2. Assume that $S_0 \neq \emptyset$. Then $S = \operatorname{graph}(u)$ for some homogeneous quadratic polynomial u.*

Proof. We already know from Proposition 5.3 that $S = \operatorname{graph}(u)$, where $u \in C^1(\mathbb{R}^{2n})$. Moreover, since S is a hypersurface of class C^2, then $u \in C^2(\mathbb{R}^{2n})$. Finally, since $0 \in S_0$, then $Du(0) = 0$. Therefore

$$u(p) = P_2(p) + o(|p|^2),$$

where P_2 is an homogeneous quadratic polynomial. We show that $u = P_2$. Let $p \in \mathbb{R}^{2n}$, and let $\alpha > 0$. Then it holds that

$$|u(p) - P_2(p)| = \frac{|u(\alpha p) - P_2(\alpha p)|}{\alpha^2} = |p|^2 \frac{o(\alpha^2|p|^2)}{\alpha^2|p|^2}$$

as $\alpha \to +\infty$. The thesis then follows letting $\alpha \to +\infty$. $\square$

Proof of Theorem 1.4. For sake of notational simplicity, we assume again that $n = 2$, being the other cases completely analogous. We divide the proof into some steps.

Step 1. Thanks to Proposition 5.5, we assume that $S = \operatorname{graph}(u)$, where

$$u(\bar{x}, \bar{y}) = ax_1^2 + bx_2^2 + cy_1^2 + dy_2^2 + ex_1x_2 + fx_1y_1 + gx_1y_2 + hx_2y_1 + mx_2y_2 + py_1y_2,$$

for some $a, b, \dots, m, p \in \mathbb{R}$. Let us define $\varphi : \mathbb{R}^4 \longrightarrow \operatorname{graph}(u)$ by

$$\varphi(\bar{x}, \bar{y}) = (\bar{x}, \bar{y}, u(\bar{x}, \bar{y})).$$

Then φ is a global C^2 parametrization of S. Therefore, for any $p = (\bar{x}, \bar{y}) \in \mathbb{R}^4$, $T_{\varphi(p)}S$ is generated by

$$\frac{\partial \varphi}{\partial x_1}(p) = X_1 + (2ax_1 + ex_2 + (f-1)y_1 + gy_2)T,$$
$$\frac{\partial \varphi}{\partial x_2}(p) = X_2 + (ex_1 + 2bx_2 + hy_1 + (m-1)y_2)T,$$
$$\frac{\partial \varphi}{\partial x_2}(p) = Y_1 + ((f+1)x_1 + hx_2 + 2cy_1 + py_2)T$$

and

$$\frac{\partial \varphi}{\partial x_2}(p) = Y_2 + (gx_1 + (m+1)x_2 + py_1 + 2dy_2)T.$$

Let us define the 4×4 real-valued matrix M by

$$M = \begin{bmatrix} 2a & e & f-1 & g \\ e & 2b & h & m-1 \\ f+1 & h & 2c & p \\ g & m+1 & p & 2d \end{bmatrix},$$

and, for any $j = 1, \dots, 4$, we let v_j be the j-th row of M. Notice that $p = (z, t) \in S$ is a characteristic point of S if and only if $M \cdot z = 0$.

Step 2. We prove that $\operatorname{rank}(M) \in \{2, 3\}$. Since we are assuming that S_0 is infinite, then $\operatorname{rank}(M) \leqslant 3$, and so in particular S_0 is a linear subspace of $\mathbb{R}^4$ with $\dim(S_0) \geqslant 1$. Moreover,

$\operatorname{rank}(M) \neq 0$, since otherwise we would have that $S = S_0 \subseteq \mathcal{H}_0$, and so $S = S_0 = \mathcal{H}_0$, which is impossible since 0 is the only characteristic point of H_0. Moreover, we claim that $\operatorname{rank}(M) \geqslant 2$. Otherwise, if $\operatorname{rank}(M) = 1$, then we can assume without loss of generality that $v_1 \neq 0$ and that there exist $A, B, C \in \mathbb{R}$ such that $v_2 = Av_1$, $v_3 = Bv_1$ and $v_4 = Cv_1$. Therefore in particular $e = 2Aa$, $f = 2Ba - 1$ and $g = 2Ca$. Moreover, since $h = Be$ and $h = A(f-1)$, we infer that $0 = Be - A(f-1) = 2ABa - 2ABa + 2A = 2A$, and so $A = 0$. Moreover, since $p = Bg$ and $p = C(f-1)$, we conclude as above that $C = 0$. But this is impossible, since it would imply that $m - 1 = m + 1 = 0$. Therefore we conclude that $\operatorname{rank}(M) \in \{2, 3\}$.

Step 3. Let now $p = (z, p) \in S \setminus S_0$. Since then $M \cdot z \neq 0$, we can assume that $\langle v_1, z\rangle \neq 0$. Hence, there exists an open neighborhood $\tilde{U}$ of p such that $\langle v_1, z_q\rangle \neq 0$ for any $q = (z_q, t_q) \in \tilde{U}$. This implies in particular that $M \cdot z_q \neq 0$ for any $q \in \tilde{U}$, and so $\tilde{U} \cap S \subseteq S \setminus S_0$. Let now U be an open neighborhood of p such that $U \Subset \tilde{U}$. We are going to show that there exists an open neighborhood W of 0 such that

$$\mathcal{H}T_pS \cap W \subseteq \{(\bar{x}, \bar{y}) \in \mathbb{R}^4 \,:\, u(\bar{x}, \bar{y}) = 0\} =: G. \tag{5.2}$$

Let us define

$$A = \frac{\langle v_2, z\rangle}{\langle v_1, z\rangle}, \qquad B = \frac{\langle v_3, z\rangle}{\langle v_1, z\rangle}, \qquad C = \frac{\langle v_4, z\rangle}{\langle v_1, z\rangle}.$$

Recalling the computations of the first step, it is clear that

$$\mathcal{H}T_pS = \operatorname{span}\{X_2 - AX_1, Y_1 - BX_1, Y_2 - CX_1\}.$$

Therefore, being S ruled and $p \in S \setminus S_0$, it follows that

$$\begin{aligned}
&(x_1, x_2, y_1, y_2, u(\bar{x}, \bar{y})) \cdot (-\alpha A, -\beta B, -\gamma C, \alpha, \beta, \gamma, 0) \\
&\qquad = (x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma, \\
&\qquad\quad u(\bar{x}, \bar{y}) - \alpha A y_1 - \beta B y_1 - \gamma C y_1 + \alpha y_2 - \beta x_1 - \gamma x_2) \in S
\end{aligned}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, noticing that

$$\begin{aligned}
u(&x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma) = \\
&ax_1^2 + a\alpha^2A^2 + a\beta^2B^2 + a\gamma^2C^2 - 2a\alpha Ax_1 - 2a\beta Bx_1 - 2a\gamma Cx_1 + 2a\alpha\beta AB \\
&+ 2A\alpha\gamma AC + 2a\beta\gamma BC + bx_2^2 + 2b\alpha x_2 + b\alpha^2 + cy_1^2 + 2c\beta y_1 + c\beta^2 + dy_2^2 + 2d\gamma y_2 \\
&+ d\gamma^2 + ex_1x_2 + e\alpha x_1 - e\alpha Ax_2 - e\alpha^2A - e\beta Bx_2 - e\alpha\beta B - e\gamma Cx_2 - e\alpha\gamma C \\
&+ fx_1y_1 + f\beta x_1 - f\alpha Ay_1 - f\alpha\beta A - f\beta By_1 - f\beta^2B - f\gamma Cy_1 - f\beta\gamma C \\
&+ gx_2y_2 + g\gamma x_1 - g\alpha Ay_2 - g\alpha\gamma A - g\beta By_2 - g\beta\gamma B - g\gamma Cy_2 - g\gamma^2C \\
&+ hx_2y_1 + h\beta x_2 + h\alpha y_1 + h\alpha\beta + mx_2y_2 + m\gamma x_2 \\
&+ m\alpha y_2 + m\alpha\beta + py_1y_2 + p\gamma y_1 + p\beta y_2 + p\beta\gamma,
\end{aligned}$$

we infer that

$$\begin{aligned}
&a\alpha^2A^2 + a\beta^2B^2 + a\gamma^2C^2 - 2a\alpha Ax_1 - 2a\beta Bx_1 - 2a\gamma Cx_1 + 2a\alpha\beta AB \\
&+ 2A\alpha\gamma AC + 2a\beta\gamma BC + 2b\alpha x_2 + b\alpha^2 + 2c\beta y_1 + c\beta^2 + 2d\gamma y_2 \\
&+ d\gamma^2 + e\alpha x_1 - e\alpha Ax_2 - e\alpha^2A - e\beta Bx_2 - e\alpha\beta B - e\gamma Cx_2 - e\alpha\gamma C \\
&+ (f+1)\beta x_1 - (f-1)\alpha Ay_1 - f\alpha\beta A - (f-1)\beta By_1 - f\beta^2B - (f-1)\gamma Cy_1 - f\beta\gamma C \\
&+ g\gamma x_1 - g\alpha Ay_2 - g\alpha\gamma A - g\beta By_2 - g\beta\gamma B - g\gamma Cy_2 - g\gamma^2C \\
&+ h\beta x_2 + h\alpha y_1 + h\alpha\beta + (m+1)\gamma x_2 \\
&+ (m-1)\alpha y_2 + m\alpha\beta + p\gamma y_1 + p\beta y_2 + p\beta\gamma = 0
\end{aligned}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, recalling the definition of A, B and C, we conclude that

$$\begin{aligned}&+ a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 + 2a\alpha\beta AB + 2A\alpha\gamma AC + 2a\beta\gamma BC + b\alpha^2\\&+ c\beta^2 + d\gamma^2 - e\alpha^2 A - e\alpha\beta B - e\alpha\gamma C - f\alpha\beta A - f\beta^2 B - f\beta\gamma C\\&- g\alpha\gamma A - g\beta\gamma B - g\gamma^2 C + h\alpha\beta + m\alpha\beta + p\beta\gamma = 0\end{aligned}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough, which is equivalent to (5.2).
Step 4. Let us define

$$P_p := \operatorname{span}\{(-A, 1, 0, 0), (-B, 0, 1, 0), (-C, 0, 0, 1)\}.$$

Then (5.2) implies that $P_p \cap \pi(W) \subseteq G$. Moreover, it is easy to see that $N := (1, A, B, C)$ is the Euclidean normal to P_p in $\mathbb{R}^4$. Let us define $V = \pi(U)$. Since π is open, then V is an open neighborhood of z. Moreover, being S a t-graph, then $\pi|_S$ is invertible, $V = \pi(U \cap S) = \pi(U \cap (S \setminus S_0))$ and $U \cap S = \pi^{-1}(V)$. Therefore, if $\tilde{z} \in V$, we let $\tilde{z} = z_q$, where q is the unique point in $U \cap S$ such that $\pi(q) = z_q$. For any $z_q \in V$, we define

$$A_q = \frac{\langle v_2, z_q\rangle}{\langle v_1, z_q\rangle}, \qquad B_q = \frac{\langle v_3, z_q\rangle}{\langle v_1, z_Q\rangle}, \qquad C_q = \frac{\langle v_4, z_q\rangle}{\langle v_1, z_Q\rangle},$$

and we let

$$P_q := \operatorname{span}\{(-A_q, 1, 0, 0), (-B_q, 0, 1, 0), (-C_q, 0, 0, 1)\}.$$

Again, $N_q := (1, A_q, B_q, C_q)$ is the Euclidean normal to P_q in $\mathbb{R}^4$. Notice in particular that $A_p = A$, $B_p = B$, $C_p = C$ and $P_p = P$, and that, since $U \Subset \tilde{U} \subseteq S \setminus S_0$, W can be chosen in such a way that $P_q \cap \pi(W) \subseteq G$ for any $z_q \in V$. Moreover, thanks to the choice of U, A_q, B_q and C_q are smooth functions on V.
Step 5. We claim that one between A_q, B_q, C_q is not constant in any neighborhood of z. Indeed, let Z be a neighborhood of z, let $a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbb{R}$ be such that $b_1 x_1' + b_2 x_2' + b_3 y_1' + b_4 y_2' \neq 0$ for any $(\bar{x}', \bar{y}') \in Z$, and define

$$f(\bar{x}', \bar{y}') := \frac{a_1 x_1' + a_2 x_2' + a_3 y_1' + a_4 y_2'}{b_1 x_1' + b_2 x_2' + b_3 y_1' + b_4 y_2'}.$$

If f is constant on Z, then $\nabla f \equiv 0$ on Z. A simple computation shows that this is equivalent to

$$a_1 b_2 - a_2 b_1 = a_1 b_3 - a_3 b_1 = a_1 b_4 - a_4 b_2 = a_2 b_3 - a_3 b_2 = a_2 b_4 - a_4 b_2 = a_3 b_4 - a_4 b_3 = 0.$$

This implies that the matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

has rank one. Therefore, if A_q, B_q and C_q were all constant functions on Z, then we would have that $\operatorname{rank}(M) \leqslant 1$, which contradicts the fact that $\operatorname{rank}(M) > 1$. Therefore without loss of generality, we assume that A_q is not constant in any neighborhood of z.
Step 6. Since A_q is not constant in any neighborhood of z, there exists $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$ such that $A \in (s_1, s_2)$ and for any $s \in (s_1, s_2)$ there exists $q_s \in U$ such that $N_{q_s} = (1, s, B_{q_s}, C_{q_s})$. This implies that $\bigcup_{s \in (s_1, s_2)} P_{q_s} \cap \pi(W)$ has non-empty interior. But then, since $P_q \cap \pi(W) \subseteq G$ for any $q \in U$, G has non-empty interior. Being u a polynomial, the only possibility is that $u \equiv 0$, and thus $S = H_0$. $\square$

Proof of Theorem 1.5. S is clearly a conical smooth hypersurface. Let $p \in S \setminus S_0$. It is well-known that

$$N(p) = \frac{1}{\sqrt{1 + |Du(z)|^2_{\mathbb{R}^{2n}}}}(Du(z), -1) = \frac{1}{\sqrt{1 + x_1^2 + y_1^2}}(x_1, 0, \dots, 0, -y_1, 0, \dots, 0, -1),$$

and so

$$\nu_{\mathbb{H}}(p) = \nu_{\mathbb{H}}(z) = \frac{1}{\sqrt{2(x_1 - y_1)^2 + \sum_{j=2}^{n}(x_j^2 + y_j^2)}}(x_1 - y_1, -y_2, \dots, -y_n, x_1 - y_1, x_2, \dots, x_n).$$

Since in this case $\nu_{\mathbb{H}}$ does not depend on t, an easy computation shows that

$$\operatorname{div}_{\mathbb{H}} \nu_{\mathbb{H}}(p) = \operatorname{div}_{\mathbb{R}^{2n}} \nu_{\mathbb{H}}(z) = 0 \tag{5.3}$$

for any $p \in S \setminus S_0$. Since $n \geqslant 2$, (5.3) allows us to apply [11, Corollary F] and [5, Theorem 2.3], which, together with [46, Example 5.29], imply that S is minimal. We conclude noticing that, in view of Theorem 1.4, S is not ruled. □

6. Horizontal second fundamental forms on $\mathbb{H}^n$

In the current literature, different kinds of second fundamental form are available in the sub-Riemannian setting. We recall that the *horizontal second fundamental form of S at $p \in S \setminus S_0$* (cf. [32, 15, 12]) is the map $h_p : \mathcal{H}T_pS \times \mathcal{H}T_pS \longrightarrow \mathbb{R}$ defined by

$$h_p(X, Y) = -\langle \nabla_X Y, v^{\mathbb{H}} \rangle = \langle \nabla_X v^{\mathbb{H}}, Y \rangle$$

for any $X, Y \in \mathcal{H}T_pS$, the second equality following being ∇ a metric connection. We recall that its norm is defined by

$$|h_p|^2 = \sum_{i,j=1}^{2n-1} h_p(e_i, e_j)^2$$

for any $p \in S$, being $e_1, \dots, e_{2n-1}$ any orthonormal basis of $\mathcal{H}T_pS$. Notice that, in view of (2.1), h may not be symmetric. The *horizontal mean curvature H_p of S at $p \in S \setminus S_0$* is defined by

$$H_p = \tau(h_p) = \sum_{j=1}^{2n-1} h_p(e_j, e_j),$$

where here and in the following τ denote the trace operator. In analogy with the Riemannian case, the horizontal mean curvature coincides with the divergence of the horizontal normal (cf. [15]), meaning that

$$H = \operatorname{div}_{\mathbb{H}} v^{\mathbb{H}}(p)$$

for any $p \in S$. Accordingly, the following characterization of $|h|^2$ holds.

Proposition 6.1. *Let p be a non-characteristic point of S. Let $v^{\mathbb{H}}$ be any unitary C^2 extension of $v^{\mathbb{H}}|_S$. Then*

$$|h_p|^2 = \sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}}) Z_k(v_h^{\mathbb{H}}) + 4(n-1)(Td^{\mathbb{H}})^2.$$

Moreover, if $v^{\mathbb{H}}|_S$ is extended as in (2.6), then

$$|h_p|^2 = \sum_{i,j=1}^{2n} \left(Z_i v_j^{\mathbb{H}}(p)\right)^2 - 4(Td^{\mathbb{H}})^2.$$

Proof. First, we show that the quantity $\sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}})Z_k(v_h^{\mathbb{H}})$ does not depend on the chosen unitary C^2 extension of $v^{\mathbb{H}}|_S$. Indeed, in view of (2.5), we have that

$$\begin{aligned}\sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}})Z_k(v_h^{\mathbb{H}}) &= \sum_{h,k=1}^{2n} \delta_h(v_k^{\mathbb{H}})Z_k(v_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n} v_h^{\mathbb{H}}\langle\nabla_{\mathbb{H}}v_k^{\mathbb{H}}, v^{\mathbb{H}}\rangle Z_k(v_h^{\mathbb{H}}) \\ &= \sum_{h,k=1}^{2n} \delta_h(v_k^{\mathbb{H}})\delta_k(v_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n} \delta_h(v_k^{\mathbb{H}})v_k^{\mathbb{H}}\langle\nabla_{\mathbb{H}}v_h^{\mathbb{H}}, v^{\mathbb{H}}\rangle \\ &= \sum_{h,k=1}^{2n} \delta_h(v_k^{\mathbb{H}})\delta_k(v_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}})v_k^{\mathbb{H}}\langle\nabla_{\mathbb{H}}v_h^{\mathbb{H}}, v^{\mathbb{H}}\rangle - \left(\sum_{h,k=1}^{2n} v_h^{\mathbb{H}}\langle\nabla_{\mathbb{H}}v_h^{\mathbb{H}}, v^{\mathbb{H}}\rangle\right)^2 \\ &= \sum_{h,k=1}^{2n} \delta_h(v_k^{\mathbb{H}})\delta_k(v_h^{\mathbb{H}}).\end{aligned}$$

The claim then follows recalling that the horizontal tangential derivatives do not depend on the chosen extension. Let us extend $v^{\mathbb{H}}$ as in (2.6). Let $e_1, \dots, e_{2n-1}$ be an hortonormal basis of $\mathcal{H}T_pS$. For any $i = 1, \dots, 2n-1$, we let $a_i^1, \dots, a_i^{2n}$ be such that

$$e_i = \sum_{j=1}^{2n} a_i^j Z_j.$$

Then, by construction,

$$\sum_{k=1}^{2n} a_i^k a_j^k = \delta_{ij}, \qquad \sum_{k=1}^{2n} a_i^k v_k^{\mathbb{H}} = 0 \qquad \text{and} \qquad \sum_{l=1}^{2n-1} e_k^l e_k^m = \delta_{km} - v_k^{\mathbb{H}}v_m^{\mathbb{H}} \tag{6.1}$$

for any $i, j = 1, \dots, 2n-1$ and any $l, m = 1, \dots, 2n$. Hence, recalling (2.5) and (2.9),

$$\begin{aligned}|h_p|^2 &= \sum_{i,j=1}^{2n-1} \sum_{h,k,l,m=1}^{2n} a_i^h Z_h(v_k^{\mathbb{H}})a_j^k a_i^l Z_l(v_m^{\mathbb{H}})a_j^m \\ &= \sum_{h,k,l,m=1}^{2n} Z_h(v_k^{\mathbb{H}})Z_l(v_m^{\mathbb{H}})(\delta_{hl} - v_h^{\mathbb{H}}v_l^{\mathbb{H}})(\delta_{km} - v_k^{\mathbb{H}}v_m^{\mathbb{H}}) \\ &= \sum_{h,k=1}^{2n} \left(Z_h(v_k^{\mathbb{H}})\right)^2 - \sum_{k=1}^{2n}\left(\sum_{h=1}^{2n} Z_h(v_k^{\mathbb{H}})v_h^{\mathbb{H}}\right)^2 \\ &= \sum_{i,j=1}^{2n} \left(Z_i v_j^{\mathbb{H}}(p)\right)^2 - 4(Td^{\mathbb{H}})^2.\end{aligned}$$

To prove the second identity, notice that

$$\begin{aligned}\sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}})Z_k(v_h^{\mathbb{H}}) &= \sum_{h,k=1}^{2n} \left(Z_h(v_k^{\mathbb{H}})\right)^2 + 2Td^{\mathbb{H}}\sum_{h,k=1}^{n} X_h(v_{n+k}^{\mathbb{H}}) - 2Td^{\mathbb{H}}\sum_{h,k=1}^{n} Y_h(v_k^{\mathbb{H}}) \\ &= \sum_{h,k=1}^{2n} \left(Z_h(v_k^{\mathbb{H}})\right)^2 + 2Td^{\mathbb{H}}\sum_{k=1}^{n}[X_k, Y_k]d^{\mathbb{H}} \\ &= \sum_{h,k=1}^{2n} \left(Z_h(v_k^{\mathbb{H}})\right)^2 - 4n(Td^{\mathbb{H}})^2\end{aligned}$$

$\square$

In the first Heisenberg group $\mathbb{H}^1$, $\mathcal{H}TS$ is a one dimensional distribution generated by $J(v^{\mathbb{H}})$. In particular (cf. [44]), h completely determines the behavior of $\nabla_{J(v^{\mathbb{H}})}J(v^{\mathbb{H}})$, meaning that

$$\nabla_{J(v^{\mathbb{H}})}J(v^{\mathbb{H}}) = -h(J(v^{\mathbb{H}}), J(v^{\mathbb{H}}))v^{\mathbb{H}}.$$

This consideration is a first crucial step in the study of minimal surfaces, since it allows to infer that, when $H = 0$, then S is ruled by horizontal lines. A horizontal line is a horizontal curve $\Gamma : I \longrightarrow \mathbb{H}^n$ such that

$$\langle \ddot{\Gamma}(s), Z_j|_{\Gamma(s)} \rangle = 0$$

for any $s \in I$ and any $j = 1, \dots, 2n$, where here and in the following I is a domain of $\mathbb{R}$ containing 0. Indeed the following simple characterization holds.

Proposition 6.2. *Let $\Gamma : I \longrightarrow \mathbb{H}^n$ be a horizontal curve. The following are equivalent.*

(i) $\nabla_{\dot{\Gamma}}\dot{\Gamma} = 0$ *along* Γ.

(ii) Γ *is a horizontal line.*

Proof. Let $A = \sum_{j=1}^{2n} A_j Z_j$ be any C^2 extension of $\dot{\Gamma}$. Γ is a horizontal line if and only if $t \mapsto A_j(\Gamma(t))$ is constant on I for any $j = 1, \dots, 2n$. Notice that

$$\nabla_A A\big|_{\Gamma(s)} = \sum_{k=1}^{2n} A(A_k)|_{\Gamma(s)} Z_k|_{\Gamma(s)} = \sum_{k=1}^{2n} \dot{\Gamma}(A_k)|_{\Gamma(s)} Z_k|_{\Gamma(s)} = \sum_{k=1}^{2n} \frac{d(A_k(\Gamma(t))}{dt}\Big|_s Z_k|_{\Gamma(s)}$$

for any $s \in I$. The thesis then follows. $\square$

The higher dimensional case is typically more involved, since it is not always true that

$$\nabla_X X = -h(X, X)v^{\mathbb{H}}.$$

Nevertheless, there is a particular situation in which the second fundamental form provides global information.

Definition 6.3. *Let S be an hypersurface of class C^2. We say that S is* horizontally totally geodesic *when*

$$h(X, X) = 0 \tag{6.2}$$

for any $X \in C^1(S, \mathcal{H}TS)$, that is when h is an alternating *bilinear form.*

Notice that (6.2) is equivalent to require that the symmetric part of h is identically vanishing. Let us denote the latter by $\tilde{h}$, that is

$$\tilde{h}(X, Y) = \frac{h(X, Y) + h(Y, X)}{2}$$

For any $X, Y \in C^1(S, \mathcal{H}TS)$. This kind of symmetric second fundamental form has already been considered, although through different but equivalent definitions, by several authors (cf. e.g. [15, 43]). It is clear that

$$h_p = 0 \implies \tilde{h}_p = 0$$

for any non-characteristic point $p \in S$, while the converse implication may be false in general. More precisely, $|h|$ and $|\tilde{h}|$ can be related in the following way.

Proposition 6.4. *Let p be a non-characteristic point of S. Let $v^{\mathbb{H}}$ be any unitary C^2 extension of $v^{\mathbb{H}}|_S$. Then*

$$|\tilde{h}_p|^2 = \sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}}) Z_k(v_h^{\mathbb{H}}) + 2(n-1)\left(Td^{\mathbb{H}}(p)\right)^2.$$

Moreover,

$$|h_p|^2 = |\tilde{h}_p|^2 + 2(n-1)\left(Td^{\mathbb{H}}(p)\right)^2.$$

Finally, if $v^{\mathbb{H}}$ is extended as in (2.6), then

$$|\tilde{h}_p|^2 = \sum_{h,k=1}^{2n}\left(Z_h v_k^{\mathbb{H}}(p)\right)^2 - 2(n+1)\left(Td^{\mathbb{H}}(p)\right)^2$$

Proof. Let $e_1, \ldots, e_{2n-1}$ be as in the proof of Proposition 6.1. Notice that

$$|\tilde{h}_p|^2 = \tau\left(\tilde{h}_p \cdot \tilde{h}_p^T\right) = \tau\left(\frac{(h_p + h_p^T)^2}{4}\right) = \frac{1}{2}|h_p|^2 + \frac{1}{2}\tau\left(h_p^2\right).$$

Arguing as in the proof of Proposition 6.1,

$$\begin{aligned}
\tau\left(h_p^2\right) &= \sum_{i,j=1}^{2n-1}\sum_{h,k,l,m=1}^{2n} a_i^h Z_h(v_k^{\mathbb{H}}) a_j^k a_j^l Z_l(v_m^{\mathbb{H}}) a_i^m \\
&= \sum_{h,k,l,m=1}^{2n} Z_h(v_k^{\mathbb{H}}) Z_l(v_m^{\mathbb{H}}) \sum_{i=1}^{2n-1} a_i^h a_i^m \sum_{j=1}^{2n-1} a_j^k a_j^l \\
&= \sum_{h,k,l,m=1}^{2n} Z_h(v_k^{\mathbb{H}}) Z_l(v_m^{\mathbb{H}})(\delta_{hm} - v_h^{\mathbb{H}} v_m^{\mathbb{H}})(\delta_{kl} - v_k^{\mathbb{H}} v_l^{\mathbb{H}}) \\
&= \sum_{h,k=1}^{2n} Z_h(v_k^{\mathbb{H}}) Z_k(v_h^{\mathbb{H}}).
\end{aligned}$$

Exploiting Proposition 6.1, the thesis follows. □

In view of Proposition 6.1 and Proposition 6.4, non-characteristic hypersurfaces of class C^2 with $h \equiv 0$ are trivially vertical hyperplanes, provided that $n \geqslant 2$. Indeed, if S is such an hypersurface, N is its Euclidean unit normal and $v^{\mathbb{H}}$ its horizontal unit normal, then Proposition 6.4 and (2.10) imply that $\tilde{h} \equiv 0$, $N_{2n+1} \equiv 0$, $N = N(\bar{x}, \bar{y})$ and $v^{\mathbb{H}} = (N_1, \ldots, N_{2n})$. Hence

$$0 = |\tilde{h}|^2 = \sum_{i,j=1}^{2n} Z_i v_j^{\mathbb{H}} Z_j v_i^{\mathbb{H}} = \sum_{i,j=1}^{2n+1} \frac{\partial N_j}{\partial z_i}\frac{\partial N_i}{\partial z_j},$$

where the last term coincides with the squared norm of the Euclidean second fundamental form of S. Hence S is an hyperplane, which is vertical since $N_{2n+1} \equiv 0$. As already mentioned, when $n \geqslant 2$ it is not in general true that $\tilde{h} = 0$ implies $h = 0$.

Example 6.5. As an instance, consider in $\mathbb{H}^2$ the non-vertical hyperplane

$$S := \{(\bar{x}, \bar{y}, t) \in \mathbb{H}^2 \,:\, a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 + t + d = 0\}$$

for some $a_1, a_2, b_1, b_2, d \in \mathbb{R}$. An easy computation shows that

$$N(p) = \frac{(a_1, a_2, b_1, b_2, 1)}{\sqrt{1 + a_1^2 + a_2^2 + b_1^2 + b_2^2}} \qquad \text{and} \qquad N^{\mathbb{H}}(p) = \frac{(a_1 + y_1, a_2 + y_2, b_1 - x_1, b_2 - x_2)}{\sqrt{1 + a_1^2 + a_2^2 + b_1^2 + b_2^2}}$$

for any $p \in S$. Therefore, S has a unique characteristic point $p_0 = (b_1, b_2, -a_1, -a_2, -d)$. Far from p_0, $v^{\mathbb{H}}$ can be expressed by

$$v^{\mathbb{H}}(p) = \frac{(a_1 + y_1, a_2 + y_2, b_1 - x_1, b_2 - x_2)}{\sqrt{(a_1 + y_1)^2 + (a_2 + y_2)^2 + (b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

for any $p \in S \setminus S_0$.

Recalling (2.10), a tedious but simple computations shows that

$$\sum_{h,k=1}^{4} Z_h(v_k^{\mathbb{H}})Z_k(v_h^{\mathbb{H}}) = -\frac{2}{(a_1+y_1)^2+(a_2+y_2)^2+(b_1-x_1)^2+(b_2-x_2)^2} = -2(Td^{\mathbb{H}})^2.$$

Hence, Proposition 6.4 implies that $\tilde{h} \equiv 0$ on $S \setminus S_0$. Nevertheless, in view of the previous computation and Proposition 6.1, we conclude that

$$|h_p|^2 = \frac{2}{(a_1+y_1)^2+(a_2+y_2)^2+(b_1-x_1)^2+(b_2-x_2)^2}$$

for any $p \in S \setminus S_0$.

7. Local existence of geodesics on hypersurfaces

Let S be an hypersurface of class C^2. Let $p \in S \setminus S_0$ and $w \in \mathcal{H}T_pS$. We wish to find a curve $\Gamma \in C^2(I, S)$ solving the differential problem

$$\begin{cases} \Gamma \text{ is horizontal} & \\ \nabla^S_{\dot\Gamma}\dot\Gamma = 0 & \text{on } I \\ \Gamma(0) = p & \\ \dot\Gamma(0) = w & \end{cases} \tag{7.1}$$

Arguing for instance as in [33], it is not difficult to show that solutions to (7.1) are geodesics in the Carnot-Carathéodory space associated with the sub-Riemannian structure $(S, \langle\cdot,\cdot\rangle_S)$. First, notice that, by means of [22, Theorem 6.5] and [4, Theorem 1.2] and without loss of generality, there exists $\Omega \subseteq \mathbb{R}^{2n}$ and $\varphi \in C(\Omega)$ such that $\nabla^\varphi\varphi \in C(\Omega, \mathbb{R}^{2n-1})$, $U = i(\Omega)\cdot j(\mathbb{R})$ is an open neighborhood of p and

$$S \cap U = \mathrm{graph}_{Y_1}(\varphi, \Omega) \cap U.$$

We need the following lemma.

Proposition 7.1. *$\varphi \in C^2(\Omega)$.*

Proof. Let us consider the map $g : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ defined by

$$g(\bar{x}, \bar{y}, t) = (\bar{x}, \bar{y}, t - x_1y_1)$$

for any $(\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. Notice that g is smooth, bijective and and $\det(Dg) \equiv 1$. Hence g is a smooth diffeomorphism. Let us set $\hat{S} := g(S)$. Notice that $\hat{S}$ is of class C^2. It is easy to check that

$$\hat{S} \cap g(U) = \{(\bar{\xi}, \varphi(\bar{\xi}, \tilde{\eta}, \tau), \tilde{\eta}, \tau) \,:\, (\tilde{\xi}, \bar{\eta}, \tau) \in \Omega\}.$$

Therefore the thesis follows provided that $(\hat{N}(\hat{p}))_{n+1} \neq 0$ for any $\hat{p} \in \hat{S} \cap g(U)$, being $\hat{N}(\hat{p})$ the Euclidean normal to $\hat{S}$ at $\hat{p}$. Assume by contradiction that there exists $\hat{p} \in \hat{S} \cap g(U)$ such that $(\hat{N}(\hat{p}))_{n+1} = 0$. This implies that $(\bar{0}, 1, \tilde{0}, 0) \in T_{\hat{p}}\hat{S}$. Let $p \in S$ be such that $g(p) = \hat{p}$. Noticing that

$$(dg)|_p(Y_1|_p) = (\bar{0}, 1, \tilde{0}, 0) \in T_{\hat{p}}\hat{S},$$

we infer that $Y_1|_p \in T_pS$. Since S is non-characteristic, (2.4) implies that $(v^{\mathbb{H}}(p))_{n+1} = 0$. On the other hand, we know from [4, Theorem 1.2] that $(v^{\mathbb{H}}(p))_{n+1} \neq 0$, a contradiction. □

Therefore we reduce (7.1) to a differential problem for curves in Ω. To this aim, fix $q \in \Omega$ such that $\Psi(q) = p$, and let $\gamma(s) = (\bar{\xi}(s), \tilde{\eta}(s), \tau(s)) : I \longrightarrow \Omega$. If we lift γ to a curve $\Gamma : I \longrightarrow \mathbb{H}^n$ by letting

$$\Gamma(s) = \Psi(\gamma(s)) = (\bar{\xi}(s), \varphi(\gamma(s)), \tilde{\eta}(s)), \tau(s) - \xi_1(s)\varphi(\gamma(s)))$$

for any $s \in I$, then by construction $\Gamma(I) \subseteq S$. From now on, we fix the notation $\alpha(s) := \varphi(\gamma(s))$. To give a meaning to (7.1) we need that $\dot\Gamma$ is horizontal. Notice that

$$\begin{aligned}\dot\Gamma &= (\dot\xi_1, \dots, \dot\xi_n, \dot\alpha, \dot\eta_2, \dots, \dot\eta_n, \dot\tau - \dot\xi_1\alpha - \xi_1\dot\alpha)\\ &= \sum_{j=1}^n \dot\xi_j X_j + \dot\alpha Y_1 + \sum_{2=1}^n \dot\eta_j Y_j + \left(\dot\tau - 2\alpha\dot\xi_1 - \sum_{j=2}^n \eta_j\dot\xi_j + \sum_{j=2}^n \xi_j\dot\eta_j\right) T.\end{aligned}$$

Therefore $\dot\Gamma$ admits a C^1 extension to the whole $\mathcal{H}TS$ if and only if

$$\dot\tau = 2\alpha\dot\xi_1 + \sum_{j=2}^n \eta_j\dot\xi_j - \sum_{j=2}^n \xi_j\dot\eta_j, \tag{7.2}$$

that is if and only if γ is horizontal in (Ω, d_φ). Let us denote such an extension by $A = \sum_{j=1}^{2n}\psi_j Z_j$. This means that $A \in C^1(S, \mathcal{H}TS)$ and

$$\psi_j(\Gamma(s)) = \dot\Gamma_j(s)$$

for any $s \in I$ and any $j = 1, \dots, 2n$. Thanks to the aforementioned properties of ∇^S and recalling (2.2), then

$$\begin{aligned}\nabla^S_{\dot\Gamma}\dot\Gamma\big|_{\Gamma(s)} &= \nabla_{\dot\Gamma}\dot\Gamma\big|_{\Gamma(s)} - \left\langle \nabla_{\dot\Gamma}\dot\Gamma\big|_{\Gamma(s)}, v^{\mathbb{H}}\big|_{\Gamma(s)}\right\rangle v^{\mathbb{H}}\big|_{\Gamma(s)}\\ &= \sum_{j=1}^{2n}\langle\dot\Gamma(s), \nabla_H\psi_j(\Gamma(s))\rangle Z_j\big|_{\Gamma(s)} - \left(\sum_{k=1}^{2n}\langle\dot\Gamma(s), \nabla_H\psi_j(\Gamma(s))\rangle v_k^{\mathbb{H}}\big|_{\Gamma(s)}\right) v^{\mathbb{H}}\big|_{\Gamma(s)}\\ &= \sum_{j=1}^{2n}\ddot\Gamma_j(s) Z_j\big|_{\Gamma(s)} - \left(\sum_{k=1}^{2n}\ddot\Gamma_k(s) v_k^{\mathbb{H}}\big|_{\Gamma(s)}\right) v^{\mathbb{H}}\big|_{\Gamma(s)}\end{aligned}$$

for any $s \in I$. Hence $\nabla^S_{\dot\Gamma}\dot\Gamma = 0$ if and only if

$$\ddot\Gamma_j - v_j^{\mathbb{H}}\langle\ddot\Gamma, v^{\mathbb{H}}\rangle = 0 \tag{7.3}$$

for any $j = 1, \dots, 2n$. We need to traduce (7.3) in terms of γ. To this aim, recalling (7.2), notice that

$$\ddot\Gamma = \sum_{j=1}^n \ddot\xi_j X_j + \ddot\alpha Y_1 + \sum_{2=1}^n \ddot\eta_j Y_j.$$

Lemma 7.2. *It holds that*

$$\langle\ddot\Gamma, v^{\mathbb{H}}\rangle = -W^{-\frac{1}{2}}\left(2\tilde T\varphi\dot\alpha\dot\xi_1 + \langle D^2\varphi\dot\gamma, \dot\gamma\rangle\right).$$

Proof. Notice that

$$\dot\alpha(s) = \langle\dot\gamma, D\varphi(\gamma(s))\rangle \qquad \text{and} \qquad \ddot\alpha(s) = \langle\ddot\gamma(s), D\varphi(\gamma(s))\rangle + \langle D^2\varphi(\gamma(s))\dot\gamma(s), \dot\gamma(s)\rangle \tag{7.4}$$

for any $s \in I$. Moreover, taking derivatives in (7.2), we see that

$$\ddot\tau = 2\dot\alpha\dot\xi_1 + 2\alpha\ddot\xi_1 + \sum_{j=2}^n \eta_j\ddot\xi_j - \sum_{j=2}^n \xi_j\ddot\eta_j. \tag{7.5}$$

Exploiting (2.11), (7.4) and (7.5), we see that

$$
\begin{aligned}
W^{\frac{1}{2}}\langle \ddot{\Gamma}, v^{\mathbb{H}}\rangle &= W^{\varphi}\varphi\ddot{\xi}_1 + \sum_{j=2}^{n}\tilde{X}_j\varphi\ddot{\xi}_j + \sum_{j=2}^{n}\tilde{Y}_j\varphi\ddot{\eta}_j - \ddot{\alpha}\\
&= \ddot{\xi}_1\varphi_{\xi_1} + 2\ddot{\xi}_1\alpha\varphi_\tau + \sum_{j=2}^{n}\ddot{\xi}_j\varphi_{\xi_j} + \sum_{j=2}^{n}\eta_j\ddot{\xi}_j\varphi_\tau + \sum_{j=2}^{n}\ddot{\eta}_j\varphi_{\eta_j} - \sum_{j=2}^{n}\xi_j\ddot{\eta}_j\varphi_\tau\\
&\quad - \ddot{\xi}_1\varphi_{\xi_1} - \sum_{j=2}^{n}\ddot{\xi}_j\varphi_{\xi_j} - \sum_{j=2}^{n}\ddot{\eta}_j\varphi_{\eta_j} - \ddot{\tau}\varphi_\tau - \langle D^2\varphi\dot{\gamma},\dot{\gamma}\rangle\\
&= \tilde{T}\varphi\left(2\alpha\ddot{\xi}_1 + \sum_{j=2}^{n}\eta_j\ddot{\xi}_j - \sum_{j=2}^{n}\ddot{\eta}_j\xi_j - \ddot{\tau}\right) - \langle D^2\varphi\dot{\gamma},\dot{\gamma}\rangle\\
&= -2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 - \langle D^2\varphi\dot{\gamma},\dot{\gamma}\rangle.
\end{aligned}
$$

□

In the following, we let $M = 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 + \langle D^2\varphi\dot{\gamma},\dot{\gamma}\rangle$. Notice that, by Lemma 7.2, the term $\langle \ddot{\Gamma}, v^{\mathbb{H}}\rangle$ does not involve second derivatives of γ. Therefore (7.1) is equivalent to the following differential problem.

(7.6)

$$
\begin{cases}
\ddot{\xi}_1 + W^{-1}W^{\varphi}\varphi M = 0 & \text{on } I, \quad \xi_1(0) = \xi_1^0, \quad \dot{\xi}_1(0) = w_1\\
\ddot{\xi}_j + W^{-1}\tilde{X}_j\varphi M = 0 & \text{on } I, \quad \xi_j(0) = \xi_j^0, \quad \dot{\xi}_j(0) = w_j \quad j = 2,\dots,n\\
\ddot{\alpha} - W^{-1}M = 0 & \text{on } I, \quad \alpha(0) = y_1, \quad \dot{\alpha}(0) = w_{n+1}\\
\ddot{\eta}_j + W^{-1}\tilde{Y}_j\varphi M = 0 & \text{on } I, \quad \eta_j(0) = \eta_j^0, \quad \dot{\eta}_j(0) = w_{n+j} \quad j = 2,\dots,n\\
\dot{\tau} = 2\alpha\dot{\xi}_1 + \sum_{j=2}^{n}\eta_j\dot{\xi}_j - \sum_{j=2}^{n}\xi_j\dot{\eta}_j & \text{on } I, \quad \tau(0) = t + \xi_1^0\varphi(q)
\end{cases}
$$

A key step consist in showing that the third line of (7.6) is redundant.

Lemma 7.3. *A curve $\gamma \in C^2(I,\Omega)$ solves (7.6) if and only if it solves the following differential system.*

(7.7)

$$
\begin{cases}
\ddot{\xi}_1 + W^{-1}W^{\varphi}\varphi M = 0 & \textit{on } I, \quad \xi_1(0) = \xi_1^0, \quad \dot{\xi}_1(0) = w_1\\
\ddot{\xi}_j + W^{-1}\tilde{X}_j\varphi M = 0 & \textit{on } I, \quad \xi_j(0) = \xi_j^0, \quad \dot{\xi}_j(0) = w_j \quad j = 2,\dots,n\\
\ddot{\eta}_j + W^{-1}\tilde{Y}_j\varphi M = 0 & \textit{on } I, \quad \eta_j(0) = \eta_j^0, \quad \dot{\eta}_j(0) = w_{n+j} \quad j = 2,\dots,n\\
\dot{\tau} = 2\alpha\dot{\xi}_1 + \sum_{j=2}^{n}\eta_j\dot{\xi}_j - \sum_{j=2}^{n}\xi_j\dot{\eta}_j & \textit{on } I, \quad \tau(0) = t + \xi_1^0\varphi(q)
\end{cases}
$$

Proof. If $\gamma \in C^2(I,\Omega)$ solves (7.6), then clearly solves (7.7). Conversely, assume that $\gamma \in C^2(I,\Omega)$ solves (7.7). Since $y_1 = \varphi(q)$, then $\alpha(0) = \varphi(\gamma(0)) = \varphi(q) = y_1$. Moreover, notice that

$$\begin{aligned}\dot\alpha(0) &= \langle \dot\gamma(0), D\varphi(q)\rangle \\ &= \dot\xi_1(0)\varphi_{\xi_1}(q) + \sum_{j=2}^n \dot\xi_j(0)\varphi_{\xi_j}(q) + \sum_{j=2}^n \dot\eta_j(0)\varphi_{\eta_j}(q) + \dot\tau(0)\varphi_\tau(q) \\ &= w_1 W^\varphi\varphi(q) + \sum_{j=2}^n w_j\tilde X_j\varphi(q) + \sum_{j=2}^n w_{n+j}\tilde Y_j\varphi(q) \\ &= w_{n+1},\end{aligned}$$

where the last equality follows from (2.11) and the fact that $w \in \mathcal{H}T_pS$. Observe that, recalling (7.5) and exploiting all the second-order equations in (7.7),

$$\begin{aligned}\langle \ddot\gamma, D\varphi\rangle &= \ddot\xi_1\varphi_{\xi_1} + \sum_{j=2}^n \ddot\xi_j\varphi_{\xi_j} + \sum_{j=2}^n \ddot\eta_j\varphi_{\eta_j} + \ddot\tau\tilde T\varphi \\ &= \ddot\xi_1 W^\varphi\varphi + \sum_{j=2}^n \ddot\xi_j\tilde X_j\varphi + \sum_{j=2}^n \ddot\eta_j\tilde Y_j\varphi + 2\tilde T\varphi\dot\alpha\dot\xi_1 \\ &= -W^{-1}M|\nabla^\varphi\varphi|^2 + 2\tilde T\varphi\dot\alpha\dot\xi_1.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\ddot\alpha - W^{-1}M &= W^{-1}(W\langle\ddot\gamma, D\varphi\rangle + W\langle D^2\varphi\dot\gamma,\dot\gamma\rangle - 2\tilde T\varphi\dot\alpha\dot\xi_1 - \langle D^2\varphi\dot\gamma,\dot\gamma\rangle) \\ &= W^{-1}(\langle\ddot\gamma, D\varphi\rangle + |\nabla^\varphi\varphi|^2\langle\ddot\gamma, D\varphi\rangle + |\nabla^\varphi\varphi|^2\langle D^2\varphi\dot\gamma,\dot\gamma\rangle - 2\tilde T\varphi\dot\alpha\dot\xi_1) \\ &= W^{-1}(-W^{-1}M|\nabla^\varphi\varphi|^2 + |\nabla^\varphi\varphi|^2\langle\ddot\gamma, D\varphi\rangle + |\nabla^\varphi\varphi|^2\langle D^2\varphi\dot\gamma,\dot\gamma\rangle) \\ &= \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}(\ddot\alpha - W^{-1}M),\end{aligned}$$

which is equivalent to say that

$$W^{-1}(\ddot\alpha - W^{-1}M) = 0.$$

Being $W^{-1} \neq 0$, the thesis follows. □

We can summarize the previous achievements in the following statement.

Proposition 7.4. *The following properties hold.*

(*i*) *If $\Gamma \in C^2(I,S)$ solves (7.1), then $\gamma : I \longrightarrow \Omega$ defined by*

$$\gamma(s) := \Pi(\Gamma(s))$$

for any $s \in I$ solves (7.7).

(*ii*) *If $\gamma \in C^2(\Omega)$ solves (7.7), then $\Gamma : I \longrightarrow \Omega$ defined by*

$$\Gamma(s) := \Psi(\gamma(s))$$

for any $s \in I$ solves (7.1).

Proof. (*ii*) follows thanks to Lemma 7.3. To prove (*i*), notice that, if Γ is as in the statement, then $\Gamma = \Psi(\sigma)$, where $\sigma = \Pi(\Gamma)$, and so (*i*) easily follows. □

Theorem 7.5. *The initial value problem (7.1) admits a unique local solution $\Gamma \in C^2(I,S)$.*

Proof. In view of Proposition 7.4, it suffices to show that the initial value problem (7.7) admits locally a unique solution. Notice that (7.7) can be seen as a fist-order initial value problem by means of a standard doubling variable argument. More precisely, let us introduce the equations

$$\dot{\xi}_1 = \Xi_1, \qquad \dot{\xi}_j = \Xi_j \qquad \text{and} \qquad \dot{\eta}_j = H_j \tag{7.8}$$

for any $j = 2, \dots, n$, let us define the curve $\tilde{\Gamma} : I \longrightarrow \mathbb{R}^{4n-1}$ by

$$\tilde{\Gamma} = (\xi_1, \dots, \xi_n, \eta_2, \dots, \eta_n, \tau, \Xi_1, \dots, \Xi_n, H_2, \dots, H_n),$$

and let $\tilde{q} = (x_1, \dots, x_n, y_2, \dots, y_n, t - x_1 y_1, w_1, \dots, w_n, w_{n+2}, \dots, w_{2n})$. Then (7.7) is equivalent to the first-order initial value problem

$$\begin{cases} \dot{\tilde{\Gamma}}(s) = F(s, \Gamma(s)) & \text{on } I \\ \tilde{\Gamma}(0) = \tilde{q} \end{cases} \tag{7.9}$$

where $F : I \times \mathbb{R}^{4n-1} \longrightarrow \mathbb{R}^{4n-1}$ is defined in the obvious way taking into account (7.7) and (7.8). Thanks to Proposition 7.1, F is of class C^1 in a neighborhood of $(0, \tilde{q})$. Hence the thesis follows by means of the classical Picard-Lindelöf Theorem (cf. e.g. [31]). $\square$

Proof of Theorem 1.6. Fix $p = (\bar{x}, \bar{y}, t) \in S \setminus S_0$. Assume first that there exists an open neighborhood U of p such that $\tilde{h} \equiv 0$ on U. Fix $w \in \mathcal{H}T_p S$. As before, recalling also Proposition 7.1, we can assume that there exists $\Omega \subseteq \mathbb{R}^{2n}$ and $\varphi \in C^2(\Omega)$ such that

$$S \cap V = \mathrm{graph}_{Y_1}(\varphi, \Omega) \cap V,$$

where $V = i(\Omega) \cdot j(\mathbb{R})$. In view of Theorem 7.5, there exists a small domain $I \subseteq \mathbb{R}$ such that $0 \in I$ and a curve $\Gamma \in C^2(I, S)$ solving (7.1) with initial data $\Gamma(0) = p$ and $\dot{\Gamma}(0) = w$. Since $\tilde{h} \equiv 0$, and recalling Proposition 6.2, we conclude that $\Gamma(s) := p \cdot (sw, 0)$. Hence S is locally ruled at p. Conversely, assume that S is locally ruled in a suitable neighborhood U of p. Assume also that $U \cap S_0 = \emptyset$. Fix $\bar{p} \in U$ and $w \in \mathcal{H}T_{\bar{p}} S$. Since $\Gamma(s) := \bar{p} \cdot (sw, 0)$ lies locally in S, then $h_{\bar{p}}(w, w) = 0$, and so $\tilde{h}_{\bar{p}} = 0$. $\square$

Proof of Theorem 1.1. It follows combining Theorem 1.3 and Theorem 1.6 $\square$

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