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# Reparametrizations and approximate values of integrals of the calculus of variations

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**Abstract**

We prove an approximation result, that implies the non-occurrence of the Lavrentiev phenomenon.


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## 1. Introduction

The purpose of this paper is to prove a general theorem on reparametrizations of an interval on to itself which states that, given an absolutely continuous function $x$ on an interval $[a,b]$ and $\varepsilon > 0$, under appropriate conditions on $L$ and $\psi$, there exists a reparametrization $s = s_\varepsilon(t)$ of $[a,b]$ such that the composition $x_\varepsilon = x \circ s_\varepsilon$ is at once Lipschitzian and is such that

$$\int_a^b L(x_\varepsilon(t), x'_\varepsilon(t))\psi(t, x_\varepsilon(t))\,dt \leqslant \int_a^b L(x(t), x'(t))\psi(t, x(t))\,dt + \varepsilon.$$

An application of Theorem 1 is the non-occurrence of the Lavrentiev phenomenon for a class of functionals of the calculus of variations. We recall that in 1926

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Lavrentiev [8] published an example of a functional of the kind

$$\int_a^b L(t, x(t), x'(t))\, dt, \quad x(a) = A, \ x(b) = B,$$

whose infimum taken over the space of absolutely continuous functions was strictly lower than the infimum taken over the space of Lipschitzian functions. The occurrence of this phenomenon in a minimum problem is not a minor nuisance, since de facto it prevents the possibility of computing the true absolute minimum of a variational problem by numerical methods. In the autonomous case, sufficient conditions to prevent the occurrence of this phenomenon were given by several authors, by imposing enough growth conditions on the Lagrangean $L$ to insure that solutions themselves (exist and) are Lipschitzian, as in [4] or [2]; in this case, the question was finally settled in a paper by Alberti and Serra Cassano [1].

Our result applies to non-autonomous problems; it applies to multidimensional rotationally invariant problems, where the measure is $r^D\, dr$, and, even in the simple autonomous case, it applies to problems with obstacles or with other constraints.

In Sections 3 and 4 of this paper we prove our main result. Section 5 is devoted to applications to avoid the Lavrentiev phenomenon.

## 2. The main result

The following is our main result, a reparametrization theorem.

**Theorem 1.** *Let* $x : [a, b] \to \mathbb{R}^N$ *be absolutely continuous and set* $C = \{x(t) : t \in [a, b]\}$. *Let* $L : C \times \mathbb{R}^N \to \mathbb{R}$ *be continuous and such that* $L(x, \cdot)$ *is convex, and let* $\psi$: $[a, b] \times C \to [c, +\infty)$ *be continuous, with* $c > 0$. *Then*:

(i)

$$\int_a^b L(x(t), x'(t))\psi(t, x(t))\, dt$$

*exists, finite or* $+\infty$;

(ii) *given any* $\varepsilon > 0$, *there exists a Lipschitzian function* $x_\varepsilon$, *a reparametrization of* $x$, *such that* $x(a) = x_\varepsilon(a)$, $x(b) = x_\varepsilon(b)$ *and*

$$\int_a^b L(x_\varepsilon(t), x'_\varepsilon(t))\psi(t, x_\varepsilon(t))\, dt \leqslant \int_a^b L(x(t), x'(t))\psi(t, x(t))\, dt + \varepsilon.$$

**Remark 1.** The only technical assumption of Theorem 1 is the hypothesis that $\psi$ is bounded below by a positive constant. However, in the proof of Theorem 1, this assumption is used only to infer that $\int_a^b L(x(t), x'(t))\, dt$ is finite. Hence the theorem holds under the following more general assumption: $\psi(t,x)\geqslant 0$ and $\int_a^b L(x(t),x'(t))\,dt<+\infty$.

To verify how sharp our assumptions are, consider the following example of Manià [3,5,9]. Consider the problem of minimizing the functional

$$\int_0^1 [t-x(t)^3]^2[x'(t)]^6\,dt, \quad x(0)=0,\ x(1)=1.$$

Then the infimum taken over the space of absolutely continuous functions (assumed in $x(t)=\sqrt[3]{t}$) is strictly lower than the infimum taken over the space of Lipschitzian functions.

As a consequence, the result of Theorem 1 cannot hold for the functional of Manià evaluated along $x(t)=\sqrt[3]{t}$.

Setting $\psi(t,x)=[t-x^3]^2$ and $L(x,\xi)=\xi^6$, we see that $\psi\geqslant 0$ (but not $\psi\geqslant c>0$) and that

$$\int_0^1 [x'(t)]^6\,dt=\int_0^1 1/(3^6t^4)\,dt=+\infty.$$

Hence, the assumption $\psi(t,x)\geqslant 0$ and $\int_a^b L(x(s),x'(s))\,ds<+\infty$ cannot possibly be dropped.

## 3. Preliminary results

The proof of Theorem 1 is based on some simple properties of the (set-valued) function $(x,\xi)\to\{L^*(x,p): p\in\partial_\xi L(x,\xi)\}$, where $L^*(x,p)$ is the *polar* of $L$ with respect to its second variable, i.e.

$$L^*(x,p)=\sup_{\xi\in\mathbb{R}^N}[\langle p,\xi\rangle-L(x,\xi)].$$

To establish these properties we shall need some preliminary propositions. In what follows, by $B[\xi,r]$ we shall mean the closed ball centered at $\xi$ and radius $r$.

**Proposition 1.** *Let $f$ be a convex function and $p\in\partial f(\xi)$. Then $f^*$ is finite at $p$ and $f^*(p)=\langle\xi,p\rangle-f(\xi)$.*

**Proof.** See [10, p. 218]. □

**Proposition 2.** (i) *Let $f,f_n:\mathbb{R}^N\to\mathbb{R}$ be convex and let $f_n$ converge pointwise to $f$; let $p_n\in\partial f_n(\xi)$. Then the sequence $\{p_n\}$ admits a subsequence converging to some $p\in\partial f(\xi)$.*

(ii) *Let* $L: C \times \mathbb{R}^N \to \mathbb{R}$ *be continuous and such that* $L(x,\cdot)$ *is convex; let* $x_n \to x \in C$ *and set* $f(\xi) = L(x,\xi)$, $f_n(\xi) = L(x_n,\xi)$. *Then the same conclusion as in* (i) *holds for* $p_n \in \partial f_n(\xi_n) = \partial_\xi L(x_n,\xi_n)$.

**Proof.** We prove (i) and (ii) at once setting, in case (i), $\xi_n = \xi$ and noticing that, in both cases, we have that, for every $z$, $f_n(\xi_n + z) \to f(\xi + z)$.

The sequence $\{p_n\}$ cannot be unbounded; if it were, along a subsequence we would have $|p_n| \to \infty$; choose a further subsequence so that $p_n/|p_n| \to p_0$, where $|p_0| = 1$. We have

$$f(\xi + p_0) = \lim_{n\to\infty} f_n(\xi_n + p_0) \geqslant \limsup_{n\to\infty} [f_n(\xi_n) + \langle p_n, p_0\rangle] = +\infty$$

a contradiction, since $f$ is finite at $\xi + p_0$. Hence, the sequence $\{p_n\}$ is bounded and we can select a subsequence converging to $p_*$. If it were $p_* \notin \partial f(\xi)$ there would exist $\xi'$ such that $f(\xi') < f(\xi) + \langle p_*, \xi' - \xi\rangle$. Since

$$\begin{aligned} f(\xi') &= f(\xi + (\xi' - \xi)) = \lim_{n\to\infty} f_n(\xi_n + (\xi' - \xi)) \\ &\geqslant \limsup_{n\to\infty} [f_n(\xi_n) + \langle p_n, \xi' - \xi\rangle] = f(\xi) + \langle p_*, \xi' - \xi\rangle \end{aligned}$$

we would have a contradiction. □

**Proposition 3.** *Let* $f: \mathbb{R}^N \to \mathbb{R}$ *be convex. The map* $t \to \{\langle \xi t, p\rangle - f(\xi t) : p \in \partial f(\xi t)\}$, *from* $[0, +\infty)$ *to the closed convex subsets of* $\mathbb{R}$, *is monotonically increasing.*

**Proof.** (a) Assume, in addition, that $f$ is smooth; then we have $\nabla(\langle \xi, \nabla f(\xi)\rangle - f(\xi)) = \xi^T H$, where $H$ is the Hessian matrix of $f$. Hence

$$\frac{d}{dt}(\langle \xi t, \nabla f(\xi t)\rangle - f(\xi t)) = t\xi^T H \xi \geqslant 0,$$

so that $t_2 \geqslant t_1$ implies

$$\langle \xi t_2, \nabla f(\xi t_2)\rangle - f(\xi t_2) \geqslant \langle \xi t_1, \nabla f(\xi t_1)\rangle - f(\xi t_1).$$

(b) In general, the map $\phi(t) = f(\xi t)$, being convex, is differentiable for a.e. $t$. Let $t_1^+ > t_1$ and $t_2^- < t_2$ be points where $\phi$ is differentiable. Approximate $f$ by a sequence $\{f_n\}$ of convex smooth maps, converging pointwise to $f$. Set $\phi_n(t) = f_n(\xi t)$: in particular, applying the previous Proposition 2, we have that $\phi_n'(t)$ converges to $\phi'(t)$ both at $t_1^+$ and at $t_2^-$. Applying point (a) to $f_n$ we obtain that

$$t_2^- \phi_n'(t_2^-) - \phi_n(t_2^-) \geqslant t_1^+ \phi_n'(t_1^+) - \phi_n(t_1^+)$$

so that, passing to the limit as $n \to \infty$,

$$t_2^- \phi'(t_2^-) - \phi(t_2^-) \geqslant t_1^+ \phi'(t_1^+) - \phi(t_1^+).$$

By the monotonicity of the subdifferential of $\phi$, for every $a_1 \in \partial\phi(t_1)$ and $a_2 \in \partial\phi(t_2)$, we have

$$t_2 a_2 - \phi(t_2^-) \geqslant t_2^- \phi'(t_2^-) - \phi(t_2^-) \geqslant t_1^+ \phi'(t_1^+) - \phi(t_1^+) \geqslant t_1 a_1 - \phi(t_1^+)$$

and passing to the limit as $t_2^+ \to t_2$ and $t_1^- \to t_1$, by the continuity of $\phi$, one has

$$t_2 a_2 - \phi(t_2) \geqslant t_1 a_1 - \phi(t_1).$$

Since [7, p. 257], $\partial\phi(t) = \{\langle \xi, p \rangle : p \in \partial f(\xi t)\}$, the claim is proved. □

**Proposition 4.** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex. Then the function $f(\xi/(1+\cdot))(1+\cdot)$ is convex in $(-1, \infty)$. Moreover, given $\delta$ there are $\theta$, $0 \leqslant \theta \leqslant 1$ and $p_\theta \in \partial f(\xi/(1+\theta\delta))$, such that*

$$f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) = -\delta f^*(p_\theta).$$

**Proof.** (a) Assume, in addition, that $f$ is $C^2$. Then, computing the derivatives, one obtains

$$\frac{d}{d\alpha}\left(f\left(\frac{\xi}{1+\alpha}\right)(1+\alpha)\right) = \left\langle \nabla f\left(\frac{\xi}{1+\alpha}\right), \frac{-\xi}{1+\alpha} \right\rangle + f\left(\frac{\xi}{1+\alpha}\right) = -f^*\left(\nabla f\left(\frac{\xi}{1+\alpha}\right)\right)$$

and

$$\frac{d^2}{d\alpha^2}\left(f\left(\frac{\xi}{1+\alpha}\right)(1+\alpha)\right) = \frac{1}{(1+\alpha)^3}\,\xi^T H \xi,$$

where $H$ is the Hessian matrix of $f$ computed at $\xi/(1+\alpha)$, so that the second derivative is non-negative, and the map $f(\xi/(1+\cdot))(1+\cdot)$ is convex.

(b) In the general case, approximate the convex map $f$ by a sequence of convex differentiable maps $f_n$ converging pointwise to $f$ to obtain the required convexity and to have:

$$\begin{aligned} f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) &= \lim_{n\to\infty} f_n\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f_n(\xi) \\ &= \lim_{n\to\infty} \delta\left[-\left\langle \frac{\xi}{1+\theta_n\delta}, \nabla f_n\left(\frac{\xi}{1+\theta_n\delta}\right)\right\rangle + f_n\left(\frac{\xi}{1+\theta_n\delta}\right)\right]. \end{aligned}$$

(c) Applying Proposition 2, let $p_\theta \in \partial f(\xi/(1+\theta\delta))$ be the limit of a converging subsequence of $\{\nabla f_n(\xi/(1+\theta_n\delta))\}$. We have

$$f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) = \delta\left[-\left\langle \frac{\xi}{1+\theta\delta}, p\right\rangle + f\left(\frac{\xi}{1+\theta\delta}\right)\right] = -\delta f^*(p). \qquad \square$$

## 4. Proof of Theorem 1

**Proof.** (i) For every $t \in [a,b]$, $L(x(t), x'(t)) \geqslant L(x(t), 0) + \langle p_0(t), x'(t)\rangle$, where $p_0(t)$ is any selection from $\partial_\xi L(x(t), 0)$. Let $E_- = \{t \in [a,b] : (L(x(t), x'(t)))^- > 0\}$, let $\chi_-$ be the characteristic function of $E_-$. Then, in particular, $-(L(x(t), x'(t)))^- = L(x(t), x'(t))\chi_-(t) \geqslant [L(x(t), 0) + \langle p_0(t), x'(t)\rangle]\chi_-(t)$, hence

$$\begin{aligned}\int_a^b -(L(x(t), x'(t)))^- \psi(t, x(t))\, dt &= \int_a^b L(x(t), x'(t))\chi_-(t)\psi(t, x(t))\, dt \\ &\geqslant \int_a^b [L(x(t), 0) + \langle p_0(t), x'(t)\rangle]\chi_-(t)\psi(t, x(t))\, dt.\end{aligned}$$

Since $\psi$ is bounded and, by Proposition 2, $p_0(t)$ is bounded, the claim follows by Hölder's inequality.

(ii) In case

$$\int_a^b L(x(t), x'(t))\psi(t, x(t))\, dt = +\infty,$$

any parametrization $t : [a,b] \to [a,b]$ that would make $x \circ t$ Lipschitzian, is acceptable as $x_\varepsilon$. Hence from now on we shall assume

$$-\infty < \int_a^b L(x(t), x'(t))\psi(t, x(t))\, dt < +\infty.$$

We have also

$$+\infty > \int_a^b |L(x(t), x'(t))|\, \psi(t, x(t))\, dt \geqslant c \int_a^b |L(x(t), x'(t))|\, dt.$$

(a) $C = \{x(t) : t \in [a,b]\}$ is a compact subset of $\mathbb{R}^N$: consider the set

$$V = \{(x,p) : x \in C, p \in \partial_\xi L(x, \xi), |\xi| \leqslant 1\}.$$

By (b) of Proposition 2, arguing by contradiction, we obtain that $V$ is compact. Then, $\min_V L^*(x,p)$ is attained and is finite: let $(x_n, p_n), p_n \in \partial_\xi L(x_n, \xi_n), |\xi_n| \leqslant 1$, be a minimizing sequence; we can assume that $x_n \to x, x \in C, \xi_n \to \xi, p_n \to p, p \in \partial_\xi L(x, \xi)$. By Proposition 1, we have that $L^*(x_n, p_n) = \langle \xi_n, p_n\rangle - L(x_n, \xi_n) \to \langle \xi, p\rangle - L(x, \xi) = L^*(x,p)$.

Set

$$m = \min_V L^*(x,p).$$

Applying Proposition 3, we obtain that $L^*(x,p) \geqslant m$, any $x \in C$ and any $p \in \partial_\xi L(x,\xi)$, for any $\xi \in \mathbb{R}^N$. Hence we have that, for every $x \in C$ and any $p$,

$$L^*(x,p) - m \geqslant 0.$$

Consider $\tilde{L}(x,\xi) = L(x,\xi) + m$. Since $\partial_\xi L(x,\xi) = \partial_\xi \tilde{L}(x,\xi)$, we have that $\tilde{L}^*(x,p) = L^*(x,p) - m$, and we infer that $\tilde{L}^*(x,p) \geqslant 0$.

(b) Set $\ell = \int_a^b |\tilde{L}(x(s),x'(s))|\,ds$ and let $\Psi$ be such that $|\psi(s,x)| \leqslant \Psi$, $\forall (s,x) \in [a,b] \times C$.

From the uniform continuity of $\psi(\cdot,x(\cdot))$ on $[a,b] \times [a,b]$, we infer that we can fix $k \in \mathbb{N}$ such that $\forall (s_1,t_1),(s_2,t_2) \in [a,b] \times [a,b]$, with $|s_2 - s_1| \leqslant (b-a)/2^k$ and $|t_2 - t_1| \leqslant (b-a)/2^k$ we have

$$|\psi(s_2,x(t_2)) - \psi(s_1,x(t_1))| \leqslant \min\left\{\frac{\varepsilon}{4\ell}, \frac{\varepsilon}{2m(b-a)}\right\}.$$

For $i = 0, \ldots, 2^k - 1$ set $I_i = [(b-a)i/2^k, (b-a)(i+1)/2^k]$, $H_i = \int_{I_i} |x'(s)|\,ds$, $\mu = \max\{2^{k+1}H_i/(b-a) : i = 0, \ldots, 2^k - 1\}$ and

$$T_{H_i} = \left\{s \in I_i : |x'(s)| \leqslant \frac{2^{k+1}H_i}{b-a}\right\};$$

it follows that $|T_{H_i}| \geqslant (b-a)/2^{k+1}$. Set also $T = \bigcup_{i=0}^{2^k-1} T_{H_i}$.

Since $\{(x(s),x'(s)) : s \in T\}$ belongs to a compact set and $L$ is continuous, there exists a constant $M$, such that

$$\left|\tilde{L}(x(s),2x'(s))\frac{1}{2} - \tilde{L}(x(s),x'(s))\right| \leqslant M$$

for all $s \in T$.

(c) For every $n \in \mathbb{N}$ set $S_n^i = \{s \in I_i : |x'(s)| > n\}$, $\varepsilon_n^i = \int_{S_n^i} (\frac{|x'(s)|}{n} - 1)\,ds$ and $\varepsilon_n = \sum_{i=0}^{2^k-1} \varepsilon_n^i$. From the integrability of $|x'|$, we have that $\lim_{n\to\infty} \varepsilon_n = 0$.

(d) Having defined $\varepsilon_n^i$, for all $n$ such that $\varepsilon_n^i \leqslant (b-a)/2^{k+2}$, choose $\Sigma_n^i \subset T_{H_i}$ such that $|\Sigma_n^i| = 2\varepsilon_n^i$. This is possible from point (c).

(e) Define the absolutely continuous functions $t_n$ by $t_n(s) = a + \int_a^s t_n'(\tau)\,d\tau$, where

$$t_n'(s) = \begin{cases} 1 + (\frac{|x'(s)|}{n} - 1), & s \in S_n = \bigcup_{i=0}^{2^k-1} S_n^i, \\ 1 - \frac{1}{2}, & s \in \Sigma_n = \bigcup_{i=0}^{2^k-1} \Sigma_n^i, \\ 1, & \text{otherwise}. \end{cases}$$

One verifies that $\forall i = 0, \dots, 2^k - 1$, the restriction of $t_n$ to $I_i$ is an invertible map from $I_i$ onto itself (in particular, each $t_n$ is an invertible map from $[a, b]$ onto itself). It follows that $|t_n(s) - s| \leqslant (b - a)/2^k$.

(e) We have

$$\begin{aligned}
&\int_a^b \tilde{L}\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s) \psi(t_n(s), x(s))\, ds - \int_a^b \tilde{L}(x(s), x'(s)) \psi(s, x(s))\, ds \\
&\quad = \int_a^b \left[\tilde{L}\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s) - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds \\
&\qquad + \int_a^b \tilde{L}(x(s), x'(s))[\psi(t_n(s), x(s)) - \psi(s, x(s))]\, ds,
\end{aligned}$$

and, from the definition of $t_n'$,

$$\begin{aligned}
&\int_a^b \left[\tilde{L}\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s) - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds \\
&\quad = \int_{S_n} \left[\tilde{L}\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds \\
&\qquad + \int_{\Sigma_n} \left[\tilde{L}(x(s), 2x'(s)) \frac{1}{2} - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds.
\end{aligned}$$

We wish to estimate the above integrals. Since $\Sigma_n \subset T$, we obtain

$$\int_{\Sigma_n} \left[\tilde{L}(x(s), 2x'(s)) \frac{1}{2} - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds \leqslant 2M\Psi\varepsilon_n.$$

Moreover, for every $s \in S_n$

$$\begin{aligned}
&\tilde{L}\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}(x(s), x'(s)) \\
&\quad \leqslant -\left(\frac{|x'(s)|}{n} - 1\right) \tilde{L}^*\left(x(s), p\left(x(s), n\frac{x'(s)}{|x'(s)|}\right)\right) \leqslant 0,
\end{aligned}$$

hence

$$\int_{S_n} \left[\tilde{L}\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}(x(s), x'(s))\right] \psi(t_n(s), x(s))\, ds \leqslant 0.$$

(f) The choice of $k$ implies that

$$\int_a^b \tilde{L}(x(s), x'(s))[\psi(t_n(s), x(s)) - \psi(s, x(s))]\, ds \leqslant \frac{\varepsilon}{4}.$$

We have obtained

$$\int_a^b \tilde{L}\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s)\psi(t_n(s), x(s))\, ds - \int_a^b \tilde{L}(x(s), x'(s))\psi(s, x(s))\, ds \leqslant 2M\Psi\varepsilon_n + \frac{\varepsilon}{4}.$$

(g) Fix $n$ such that $2M\Psi\varepsilon_n \leqslant \varepsilon/4$.

Then, the conclusion of (f) proves the Theorem; in fact, defining $x_\varepsilon = x \circ s_n$, where $s_n$ is the inverse of the function $t_n$, we obtain, by the change of variable formula [11], that

$$\begin{aligned}\int_a^b \tilde{L}(x_\varepsilon(t), x_\varepsilon'(t))\psi(t, x_\varepsilon(t))\, dt &= \int_a^b \tilde{L}\left(x_\varepsilon(t_n(s)), \frac{dx_\varepsilon}{dt}(t_n(s))\right) t_n'(s)\psi(t_n(s), x_\varepsilon(t_n(s)))\, ds \\ &= \int_a^b \tilde{L}\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s)\psi(t_n(s), x(s))\, ds \\ &\leqslant \int_a^b \tilde{L}(x(s), x'(s))\psi(s, x(s))\, ds + \frac{\varepsilon}{2}\end{aligned}$$

so that

$$\begin{aligned}&\int_a^b L(x_\varepsilon(t), x_\varepsilon'(t))\psi(t, x_\varepsilon(t))\, dt - \int_a^b L(x(s), x'(s))\psi(s, x(s))\, ds \\ &\leqslant \int_a^b [L(x_\varepsilon(t), x_\varepsilon'(t)) + m]\psi(t, x_\varepsilon(t))\, dt - \int_a^b [L(x(s), x'(s)) + m]\psi(s, x(s))\, ds + \frac{\varepsilon}{2} \\ &= \int_a^b \tilde{L}(x_\varepsilon(t), x_\varepsilon'(t))\psi(t, x_\varepsilon(t))\, dt - \int_a^b \tilde{L}(x(s), x'(s))\psi(s, x(s))\, ds + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Moreover, $x_\varepsilon$ is Lipschitzian. In fact, consider the equality $x_\varepsilon'(t_n(s)) = x'(s)/t_n'(s)$ and fix $s$ where $t_n'(s)$ exists; we obtain

$$\left|\frac{dx_\varepsilon}{dt}(t_n(s))\right| \begin{cases} = n, & s \in S_n, \\ \leqslant \mu, & s \in \Sigma_n, \\ \leqslant n, & \text{otherwise} \end{cases}$$

hence, at almost every $s$, the norm of the derivative of $x_\varepsilon$ is bounded by $n$. This completes the proof. □

## 5. Applications: the non-occurrence of the Lavrentiev phenomenon

The theorems below present some applications of Theorem 1 to prevent the occurrence of the Lavrentiev phenomenon to different classes of minimum problems.

Denote by $\mathrm{Lip}([a,b])$ and by $\mathrm{AC}([a,b])$, respectively, the space of all Lipschitzian and absolutely continuous functions from $[a,b]$ to $\mathbb{R}^N$. Let $E \subset \mathbb{R}^N$ and consider the functional

$$I(x) = \int_a^b L(x(s), x'(s))\psi(s, x(s))\, ds.$$

Call $\inf(P)_\infty$ the infimum of $\{I(x) : x \in \mathrm{Lip}([a,b]), x(t) \in E, x(a) = A, x(b) = B\}$ and $\inf(P)_1$ the infimum of $\{I(x) : x \in \mathrm{AC}([a,b]), x(t) \in E, x(a) = A, x(b) = B\}$.

**Theorem 2.** *Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex and let $\psi : [a,b] \times E \to [c, +\infty)$ be continuous, with $c > 0$; then* $\inf(P)_\infty = \inf(P)_1$.

In the previous Theorem $E$ can be any subset of $\mathbb{R}^N$ such that the set of absolutely continuous functions with values in $E$ and satisfying the boundary conditions is non-empty. In particular, $x \in E$ can describe a problem with an obstacle.

As an application to a problem with a constraint different from an obstacle, let $E = \mathbb{R}^2 \setminus \{0\}$ and call $\inf(P^i)_\infty$ the infimum of $\{I(x) : x \in \mathrm{Lip}, x(t) \in E, x(a) = x(b)\}$ and having prescribed *rotation number* $i(x) = k$. Call $\inf(P^i)_1$ the infimum of the same problem but for $x \in AC$.

**Theorem 3.** *Let $L : E \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex and let $\psi : [a,b] \times E \to [c, +\infty)$ be continuous, with $c > 0$; then* $\inf(P^i)_\infty = \inf(P^i)_1$.

**Proof.** As it is well known the rotation number $i$ is independent of the parametrizations of $x$. □

Theorem 3 applies in particular to the case $L(x, \xi) = |\xi|^2/2 + 1/|x|$, the case of the Newtonian potential generated by a body fixed at the origin. Gordon [6] proved that Keplerian orbits are minima to this problem with $k = 1$.

As a further application, we consider a vectorial case. Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $L(u, \cdot)$ is convex (we shall assume that the Lagrangian is independent of the integration variable). Suppose that $L(u, \cdot)$ has the symmetry of being rotationally invariant, i.e. assuming that there exists a function $h : E \times [0, \infty) \to \mathbb{R}$ such that $L(u, \xi) = h(u, |\xi|)$.

Consider the functional

$$I(u) = \int_{S[a,b]} L(u(x), \nabla u(x))\, dx,$$

where $S[a,b] = \{x \in \mathbb{R}^{D+1} : a \leqslant |x| \leqslant b\}$. Denote by $\inf(P)_\infty$ the infimum of $\{I(u) : u \in \mathrm{Lip}(S[a,b]), u(x) \in E, u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\}$ and $\inf(P)_1$ the infimum of $\{I(u) : u \in W^{1,1}(S[a,b]), u(x) \in E, u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\}$. It is our purpose to prove that $\inf(P)_\infty = \inf(P)_1$.

Observe that if $w:[a,b]\to E$ is such that $u(x)=w(|x|)$ then

$$I(u)=C_D\int_a^b L(w(r),w'(r))r^D\,dr,\quad w(a)=A, w(b)=B,$$

where

$$C_D=\frac{\pi^{(D+1)/2}}{\Gamma((D+3)/2)}(b^{D+1}-a^{D+1}).$$

**Theorem 4.** *Let* $L:E\times\mathbb{R}^N\to\mathbb{R}$ *be continuous and such that* $L(u,\cdot)$ *is convex; then* $\inf(P)_\infty=\inf(P)_1$.

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