LEVEL SETS OF EIKONAL FUNCTIONS ARE JOHN REGULAR

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ABSTRACT. Let u be the unique viscosity solution of $\alpha(x)|\nabla u|= 1$ in the external domain $\mathbb{R}^n \setminus K$ with $u = 0$ on K. In case α is continuous, bounded, and uniformly positive and K is a bounded John domain, we prove that all superlevels of u are John domains, too. Moreover, we give counterexamples showing that John regularity is sharp in this setting.

1. Introduction

This note is concerned with the regularity of the unique viscosity solution $u : \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) of the external problem for the eikonal equation

$$\begin{aligned} \alpha(x)|\nabla u| = 1 \quad &\text{in } \mathbb{R}^n \setminus K, \\ u = 0 \quad &\text{on } K. \end{aligned}$$

where the coefficient $\alpha : \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuous, bounded, and uniformly positive. For all given nonempty compact $K \subset \mathbb{R}^n$, problem (1) admits a unique viscosity solution $u \in C(\mathbb{R}^n)$ [2, 3]. We term such u *eikonal function* in the following. Note that u takes nonpositive values only. More precisely, $u(x) = 0$ if $x \in K$ and $u(x) < 0$ elsewhere.

The aim of this note is to investigate the regularity of the open superlevels

$$U_t := \{x \in \mathbb{R}^n \, : \, u(x) > -t\}$$

of the eikonal function u, where $t > 0$.

To start with, let us record that, in the reference case $\alpha \equiv 1$, eikonal functions have interior-ball regular superlevels. A nonempty open set $U \subset \mathbb{R}^n$ is called *interior-ball regular* if there exists a radius $r > 0$ such that for all $x \in \partial U$ there exists $y \in U$ with $|x-y|= r$ so that $B_r(y) := \{z \in \mathbb{R}^n \, : \, |x-z|< r\} \subset U$. In fact, in case $\alpha \equiv 1$ the eikonal function u in $\mathbb{R}^n \setminus K$ is simply the signed distance from K, namely, $u(x) = -d(x, \partial U) := -\inf_{y\in K}|x - y|$. Correspondingly, for all $t > 0$ one has that $U_t = K + B_t(0)$. These sets are clearly interior-ball regular with respect to the radius t. Note that if $\alpha \equiv 1$ the interior-ball-regularity radius increases as $t \to \infty$.

The interior-ball regularity of superlevels holds in the case of a nonconstant α, too, as long as one assumes the smoothness $\alpha \in C^{1,1}(\mathbb{R}^n)$ and that K is interior-ball regular. Under these assumptions, Lorenz [9] proves that all sets U_t are interior-ball regular. Note however that in the case of a nonconstant α the interior-ball regularity radius may degenerate as $t \to \infty$ in contrast with the case of a constant

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$\alpha \equiv 1$, see [9] and Section 3.2. The reader is also referred to [1, 5, 6] for closely related interior-ball-regularity results for smooth α.

The focus of this note is on a weaker regularity frame, where α is asked to be continuous, bounded, and uniformly positive only, possibly not $C^{1,1}$. If $\nabla\alpha$ is not Lipschitz-continuous, the arguments of [5, 9], which are based on the properties of the optimal control ODE system underlying (1), cannot be applied and the interior ball-regularity of U_t may fail. We present some counterexamples to interior-ball regularity (or even to interior-cone regularity) in Section 3 below.

We recall that a nonempty bounded domain $U \subset \mathbb{R}^n$ is said to be a *John domain* (equivalently, *John regular*) with respect to a fixed point $x_0 \in U$ and a given *John constant* $\kappa > 0$ if it satisfies an internal *twisted cone condition*: for all points $x \in U$ one can find an arc-length parametrized curve $\rho : [0, L_\rho] \to U$ such that $\rho(0) = x$, $\rho(L_\rho) = x_0$, and $d(\rho(s), \partial U) \geq \kappa\, s$ for all $s \in [0, L_\rho]$. Note that John domains are connected.

Our main result states that the superlevels U_t are John domains for all $t > 0$, provided that K is a bounded John domain. In addition, the John constant can be explicitly quantified, independently of t. We have the following.

Theorem 1.1 (John regularity). *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be continuous with $0 < \alpha_* \leq \alpha(x) \leq \alpha^*$ for all $x \in \mathbb{R}^n$, $K \subset\subset \mathbb{R}^n$ be a John domain of John constant κ_0 with respect to the point $x_0 \in \mathring{K}$, and u be the eikonal function from the problem (1). Then, all superlevels U_t are John domains with respect to x_0 with John constant*

$$\kappa := \frac{\alpha_*}{2\alpha^* + \alpha_*} \min\{\kappa_0, 1\}. \tag{2}$$

The proof of Theorem 1.1 is given in Section 2 below.

Compared with the analysis in [9], Theorem 1.1 addresses the case of less regular data α and initial sets K. While in [9] the focus is on the related optimal-control problems, we argue here on a more geometrical level instead. Our quantitive regularity estimate of the John constant from (2), albeit expectedly not optimal, does not degenerate as $t \to \infty$. This differs from [9], where the lower bound on the interior-ball radius goes to 0 as $t \to \infty$. In Section 3, we provide some examples illustrating the sharpness of the result of Theorem 1.1. Note that, in the interior-ball-regular case, sharpness with respect to initial regularity and t-dependence of the interior-ball radius may be difficult to establish, as the different behaviors in the reference cases $\alpha = 1$ and $\alpha \in C^{1,1}(\mathbb{R}^n)$ show.

Before closing this introduction, let us mention that U_t can be seen as *reachable set* for a controlled ODE system, see Section 3. In this connection, we mention [4, 7] where ∂U_t is proved to be negligible with respect to the n-dimensional Lebesgue measure, under different regularity requirements for K.

Our result provides an alternative and sharper take to this fact: By checking that U_t is John regular and by applying [8, Cor. 2.3] one directly obtains that

$$\dim_H(\partial U_t) < n,$$

where $\dim_H$ denotes the classical Hausdorff dimension. This entails that $\mathcal{H}^s(\partial U_t) = 0$ for all $\dim_H(\partial U_t) < s \leq n$, where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. By taking $s = n$ we recover the results of [4, 7].

2. Proof of Theorem 1.1

Let us start by recalling [2] that the eikonal equation from problem (1) turns out to be Lipschitz continuous, hence almost-everywhere differentiable, with bounds

$$0 < \frac{1}{\alpha^*} \leq |\nabla u(x)| \leq \frac{1}{\alpha_*} \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Moreover, u is variationally characterized by the formula

$$u(x) = -\inf \left\{ \int_0^{L_\rho} \frac{\mathrm{d}s}{\alpha(\rho(s))} \; : \rho \in W^{1,\infty}(0, L_\rho), \right.$$
$$\left. |\rho'| = 1 \text{ a.e.}, \;\rho(0) = x, \;\rho(L_\rho) \in K \right\}. \tag{3}$$

The sublevels U_t are open since u is continuous. For convenience of the Reader, we subdivide the remaining part of the proof into three Steps.

Step 1: We first prove that

$$K + B_{t\alpha_*}(0) \subset U_t \subset K + B_{t\alpha^*}(0) \quad \forall t > 0. \tag{4}$$

To check that the sets U_t are bounded for all $t > 0$ one uses the above bounds on ∇u and the characterization in (3) in order to get that

$$-\frac{d(x,K)}{\alpha_*} \leq u(x) \leq -\frac{d(x,K)}{\alpha^*} \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Indeed, for any given $\overline{x} \in K$, by considering the straight curve

$$\overline{\rho}(t) = x - t(x - \overline{x})/|x - \overline{x}|$$

for $t \in [0, |x - \overline{x}|]$ we get

$$-u(x) \leq \int_0^{|x-\overline{x}|} \frac{\mathrm{d}s}{\alpha(\overline{\rho}(s))} \leq \frac{|x - \overline{x}|}{\alpha_*}.$$

Passing to the minimum with respect to $\overline{x} \in K$ (which is possible, since K is compact) one gets the first inequality in (5). On the other hand, for all $\varepsilon > 0$ one can find $\rho_\varepsilon \in W^{1,\infty}(0, L_{\rho_\varepsilon})$ with $|\rho_\varepsilon'| = 1$ a.e., $\rho_\varepsilon(0) = x$, $\rho_\varepsilon(L_{\rho_\varepsilon}) \in K$ and

$$-u(x) + \varepsilon \geq \int_0^{L_{\rho_\varepsilon}} \frac{\mathrm{d}s}{\alpha(\rho_\varepsilon(s))} \geq \frac{L_{\rho_\varepsilon}}{\alpha^*} \geq \frac{d(x,K)}{\alpha^*}$$

so that the second inequality in (5) follows by taking $\varepsilon \to 0$.

In particular, inequalities (5) imply (4). Indeed, if $x \in K + B_{t\alpha_*}(0)$ one uses the first inequality in (5) to get $u(x) \geq -d(x,K)/\alpha_* > -t\alpha_*/\alpha_* = -t$, which implies that $x \in U_t$. On the contrary, if $x \in U_t$ the second inequality in (5) gives $d(x,K) \leq -\alpha^* u(x) < t\alpha^*$, namely, $x \in K + B_{t\alpha^*}(0)$. Note that in case $\alpha \equiv 1$, the inclusions (4) indeed allow to recover $u_t = K + B_t(0)$.

Step 2: We now show that the sets U_t are connected. By contradiction, let $t \in (0, T]$ be given such that U_t has at least two connected components. Call $\widehat{U}$ an open connected component of U_t which does not contain K. At any point $x \in \widehat{U} \subset U_t$ one has $u(x) > -t$. On the other hand, any curve from x to K necessarily contains points on $\partial\widehat{U} \subset \partial U_t$, where $u \equiv -t$ by continuity. Hence, $u(x) \leq -t$

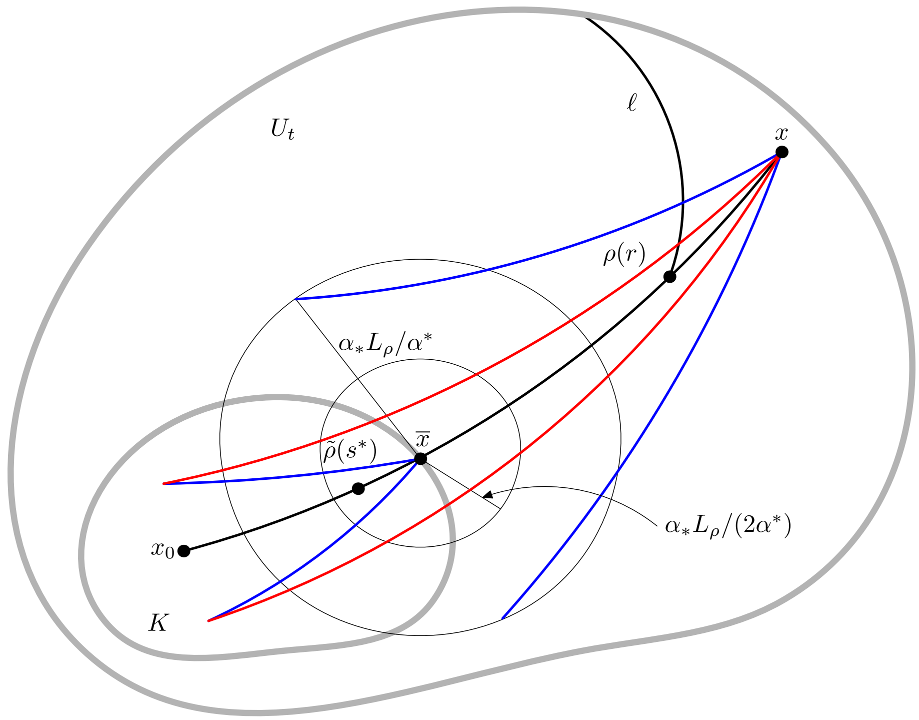

FIGURE 1. The construction for the proof of Theorem 1.1
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by formula (3), a contradiction.

Step 3: We are left with proving that U_t with $t > 0$ is a John domain with respect to x_0 with constant κ from (2). This follows from a geometric construction, which is illustrated in Figure 1. Let $x \in U_t$ be such that

$$0 < \delta < u(x) + t < 2\delta \tag{6}$$

for some small $\delta < t/2$. In view of formula (3) we find an arc-parametrized curve $\rho : [0, L_\rho] \to \mathbb{R}^n$ with $\rho(0) = x$ and $\rho(L_\rho) =: \overline{x} \in K$ such that

$$u(x) \leq -\int_0^{L_\rho} \frac{\mathrm{d}s}{\alpha(\rho(s))} + \delta. \tag{7}$$

Note that, in principle, the curve ρ might not be entirely contained in U_t. Taking any $r \in [0, L_\rho]$, we start by checking that, actually,

$$d(\rho(r), \partial U_t) > \frac{\alpha_*}{\alpha^*} r \quad \forall r \in [0, L_\rho]. \tag{8}$$

In particular, from (8) we infer that the curve ρ cannot intersect ∂U_t. Since $x \in U_t$, this yields that $\rho(r) \in U_t$ for every $r \in [0, L_\rho]$.

In fact, let $\ell : [0, L_\ell] \to \mathbb{R}^n$ be any arc-length parametrized curve connecting $\rho(r)$ to ∂U_t, namely, such that $\ell(0) \in \partial U_t$ and $\ell(L_\ell) = \rho(r)$. The curve resulting from following $\ell(s)$ for $s \in [0, L_\ell]$ and then $\rho(s - L_\ell + r)$ for $s \in (L_\ell, L_\rho + L_\ell - r]$

connects ∂U_t to K. Hence, formula (3) and the fact that $u = -t$ on ∂U_t give

$$t \le \int_0^{L_\ell} \frac{\mathrm{d}s}{\alpha(\ell(s))} + \int_{L_\ell}^{L_\rho + L_\ell - r} \frac{\mathrm{d}s}{\alpha(\rho(s - L_\ell + r))} = \int_0^{L_\ell} \frac{\mathrm{d}s}{\alpha(\ell(s))} + \int_r^{L_\rho} \frac{\mathrm{d}s}{\alpha(\rho(s))}.$$

From (7) and (6) we have

$$\int_0^r \frac{\mathrm{d}s}{\alpha(\rho(s))} + \int_r^{L_\rho} \frac{\mathrm{d}s}{\alpha(\rho(s))} = \int_0^{L_\rho} \frac{\mathrm{d}s}{\alpha(\rho(s))} \le -u(x) + \delta < t.$$

Putting the two inequalities together we obtain

$$\int_0^r \frac{\mathrm{d}s}{\alpha(\rho(s))} < \int_0^{L_\ell} \frac{\mathrm{d}s}{\alpha(\ell(s))}.$$

By using the bounds on α, the latter gives

$$\frac{r}{\alpha^*} < \frac{L_\ell}{\alpha_*}.$$

This proves that the length L_ℓ of any curve connecting $\rho(r)$ to ∂U_t is strictly bounded from below by $\alpha_* r/\alpha^*$. In particular, the lower bound (8) follows by considering a straight curve from $\rho(r)$ to ∂U_t.

Recall now that K is a John domain with respect to x_0 with constant κ_0 and denote by $\rho_0 : [0, L_{\rho_0}] \to K$ an arc-parametrized curve fulfilling $\rho_0(0) = \bar{x}$, $\rho_0(L_{\rho_0}) = x_0$, as well as $d(\rho_0(s), \partial K) \ge \kappa_0 s$ for all $s \in [0, L_{\rho_0}]$. We now concatenate the curves ρ and ρ_0 to define the curve $\tilde{\rho}(s) : [0, L_\rho + L_{\rho_0}] \to \mathbb{R}^n$ as

$$\tilde{\rho}(s) = \begin{cases} \rho(s) & \text{for } s \in [0, L_\rho], \\ \rho_0(s - L_\rho) & \text{for } s \in (L_\rho, L_\rho + L_{\rho_0}]. \end{cases}$$

We aim at showing that, by choosing κ as in (2), one has that

$$d(\tilde{\rho}(s), \partial U_t) \ge \kappa s \quad \text{for all } s \in [0, L_\rho + L_{\rho_0}]. \tag{9}$$

In order to check (9), we distinguish three cases, depending on the possible values of s in the interval $[0, L_\rho + L_{\rho_0}]$:

- Case $s \in [0, L_\rho]$: Property (9) follows from inequality (8) since

$$\kappa \overset{(2)}{\le} \frac{\alpha_*}{2\alpha^* + \alpha_*} < \frac{\alpha_*}{\alpha^*}.$$

- Case $s \in [L_\rho, s^*]$ with

$$s^* := \min\left\{ L_\rho \left(1 + \frac{\alpha_*}{2\alpha^*}\right), L_\rho + L_{\rho_0} \right\}.$$

As $s - L_\rho \le s^* - L_\rho \le \alpha_* L_\rho/(2\alpha^*)$ we have that

$$\begin{aligned} |\overline{x} - \tilde{\rho}(s)| &= |\overline{x} - \rho_0(s - L_\rho)| \le |\overline{x} - \rho_0(s^* - L_\rho)| \\ &= |\rho_0(0) - \rho_0(s^* - L_\rho)| \le \alpha_* L_\rho/(2\alpha^*), \end{aligned}$$

where in the latter inequality we have used that ρ_0 is parametrized by arc-length, see also Figure 1. On the other hand, we have $d(\overline{x}, \partial U_t) \ge \alpha_* L_\rho/\alpha^*$ from (8). We can hence conclude that $d(\tilde{\rho}(s), \partial U_t) \ge \alpha_* L_\rho/(2\alpha^*)$ and property (9) follows as

$$\frac{\alpha_* L_\rho}{2\alpha^*} = \frac{\alpha_*}{2\alpha^* + \alpha_*} L_\rho \left(1 + \frac{\alpha_*}{2\alpha^*}\right) \overset{(2)}{\ge} \kappa\, s^* \ge \kappa\, s.$$

- Case $s \in [s^*, L_\rho + L_{\rho_0}]$. Note that this case is solely relevant in case

$$s^* = L_\rho \left(1 + \frac{\alpha_*}{2\alpha^*}\right) < L_\rho + L_{\rho_0}. \tag{10}$$

We prove the John property (9) at $s \in [s^*, L_\rho + L_\rho]$ by using again that K is a John domain, namely,

$$d(\tilde{\rho}(s), \partial U_t) > d(\tilde{\rho}(s), \partial K) = d(\rho_0(s - L_\rho), \partial K) \geq \kappa_0(s - L_\rho).$$

Property (9) then follows from

$$\kappa_0(s - L_\rho) = \kappa_0 \left(1 - \frac{L_\rho}{s}\right) s \geq \kappa_0 \left(1 - \frac{L_\rho}{s^*}\right) s = \kappa_0 \frac{\alpha_*}{2\alpha^* + \alpha_*} s \overset{(2)}{\geq} \kappa\, s.$$

This concludes the proof of the theorem. □

Before closing this section, let us remark that (10) occurs for small t only. Indeed, (10) corresponds to

$$\frac{\alpha_* L_\rho}{2\alpha^*} \leq L_{\rho_0}. \tag{11}$$

The left-hand side of (11) can be bounded from below as follows

$$\frac{\alpha_* L_\rho}{2\alpha^*} \geq \frac{\alpha_* d(x, K)}{2\alpha^*} \overset{(5)}{\geq} -\frac{\alpha_*^2}{2\alpha_*} u(x) > \frac{\alpha_*^2}{2\alpha_*}(t - 2\delta),$$

where the latter inequality is a consequence of (6). On the other hand, as K is a John domain one has that $B_{\kappa_0 L_{\rho_0}}(x_0) \subset K$. Letting $R = \sup\{r > 0 : B_r(x_0) \subset K\}$ we have that $\kappa_0 L_{\rho_0} \leq R$ and the right-hand side of (11) can be bounded above by R/κ_0. Hence, for t large enough so that $\alpha_*^2(t - 2\delta)/(2\alpha_*) > R/\kappa_0$ inequality (11) does not hold. Hence, $s^* > L_\rho + L_{\rho_0}$ and the case $s \in [s^*, L_\rho + L_{\rho_0}]$ need not be considered.

3. Counterexamples to regularity

We present here some examples illustrating the sharpness of John regularity in the setting of Theorem 1.1.

In all of this section, we discuss different choices for the function α, all of which are Lipschitz continuous. This allows to draw a connection between the eikonal problem (1) and an optimal control problem for differential systems. Up to sign, the eikonal function u turns out to be the value function of the *minimal-time* problem [5, 9] driven by the controlled ODE system $\dot{y} = \alpha(y)v$, where the measurable *control* $t \mapsto v(t)$ is such that $|v| \leq 1$ almost everywhere. Indeed, when α is Lipschitz continuous one has that the latter differential problem admits a unique strong solution for any such controls v and any given initial datum x. One hence has that

$$u(x) = -\min\Big\{t \geq 0 \, : \, y(t) = x, \ \dot{y} = \alpha(y)v \text{ a.e. in } (0,t), \ v : (0,t) \to \mathbb{R}^n \text{ measurable with } |v| \leq 1 \text{ a.e.}, \ y(0) \in K\Big\}. \tag{12}$$

Correspondingly, $\overline{U_t}$ turns out to be the *reachable set* at time t starting from K for the same controlled ODE system, namely,

$$U_t = \bigcup_{s<t} \Big\{ y(s) \,:\, \dot y = \alpha(y) v \text{ a.e. in } (0,t), \text{ for some } v:(0,t)\to\mathbb{R}^n$$
$$\text{measurable with } |v|\le 1 \text{ a.e. and some } y(0)\in K \Big\}. \tag{13}$$

Note that the sets U_t are generally not smooth, regardless of the smoothness of α. In the smooth case one can equivalently qualify the evolution of ∂U_t in time at point x as driven by the normal velocity $\alpha(x)\nu(x)$, where $\nu(x)$ is the outward-pointing normal to ∂U_t at x.

In all examples below, we indicate points $x\in\mathbb{R}^n$ by $x=(x',x_n)$, where $x'\in\mathbb{R}^{n-1}$ and $x_n\in\mathbb{R}$, and use $(e_1,\dots,e_n)$ for the basis of $\mathbb{R}^n$. The initial set K is assumed to be smooth, contained in the half space $\{x_n\le 0\}$, rotationally symmetric with respect to $\{x'=0\}$, and such that the boundary ∂K contains the $(d-1)$-dimensional ball $\{(x',0)\,:\,|x'|< R\}$ for $R>0$ given and large.

3.1. **No lower bound on the interior-ball radius.** We present a first example proving that if $\alpha\notin C^{1,1}$ one cannot expect to provide a lower bound for the interior-ball radius of U_t as $t\to\infty$. Let $0<\alpha_*<\alpha^*$, and consider the Lipschitz continuous function

$$\alpha(x) = \begin{cases} \alpha^* & \text{for } \ |x'|\le \delta, \\ \alpha^*\left(1-\dfrac{|x'|-\delta}{\varepsilon\delta}\right) + \alpha_*\dfrac{|x'|-\delta}{\varepsilon\delta} & \text{for } \ \delta<|x'|<(1+\varepsilon)\delta, \\ \alpha_* & \text{for } \ |x'|\ge(1+\varepsilon)\delta. \end{cases}$$

The parameter $\delta>0$ will be chosen later (in relation with $t>0$), while $\varepsilon>0$ is arbitrarily small. Note that α is Lipschitz continuous with $\|\nabla\alpha\|_\infty = (\alpha^*-\alpha_*)/(\varepsilon\delta)$.

In what follows, we assume that

$$\alpha^* > \sqrt{5}\alpha_* \tag{14}$$

as this somewhat simplifies computations. Note however that such constraint could be removed and one could treat the general case $\alpha^*>\alpha_*$ as well, at the expense of a more involved argument.

As α is Lipschitz continuous and radially symmetric with respect to x', the supersets U_t are well-defined and radially symmetric, as well. In particular, due to radial symmetry and by (3) one has that $u(0,\alpha^* t)=t$ for every $t\ge 0$, namely that the point $(0,\alpha^* t)\in\partial U_t$ for all $t\ge0$, recall that $K\subset U_t$ and see Figure 2. In fact, the infimum in (3) is always attained by taking the straight line from $(0,\alpha^*t)$ to the origin.

We first show that, for all $t>0$ there exists $\delta>0$ large enough, so that the superset U_t does not contain a ball with radius 2δ touching ∂U_t only at $(0,\alpha^*t)$. Note that, by symmetry, such ball is necessarily $B:=B_{2\delta}(0,\alpha^*t-2\delta)$. We argue as follows: we consider a point $x_0=(x_0',\alpha^*t-2\delta)$ with $|x_0'|=2\delta$, which belongs to $\overline{B}$, and prove that $u(x_0)\le -t$. Since we are assuming that $\partial U_t\cap B=(0,\alpha^*t)$, this in particular entails that $x_0\notin\overline{U_t}$, hence $B\not\subset U_t$.

In order to check that $u(x_0)\le -t$ one has to prove that no curve γ can connect K and x_0 in time smaller or equal to t. More precisely, cf. (13), we show that for

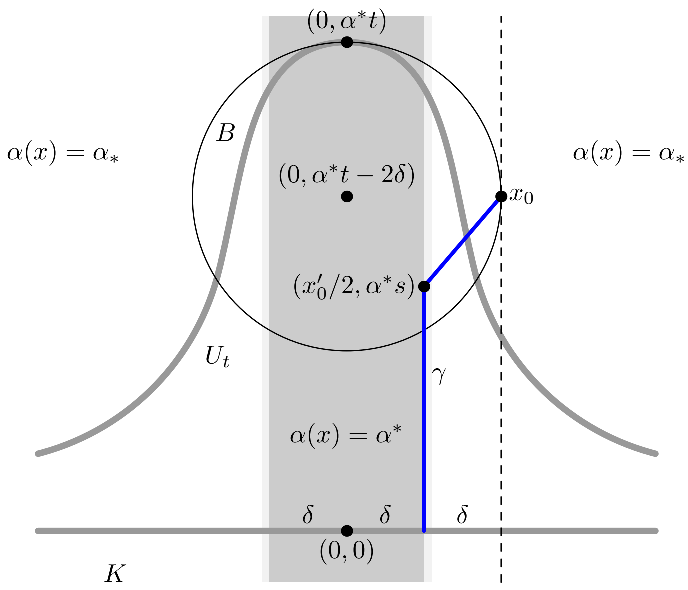

FIGURE 2. The example of Section 3.1

all $\gamma_0 = \gamma(0) \in K$ and all measurable $v : (0,t) \to \mathbb{R}^n$ with $|v|\leq 1$ and $\dot\gamma = \alpha(\gamma)v$ a.e. one has that $\gamma(t) \neq x_0$. Indeed, if γ is such that $|(\gamma(\cdot))'|\geq (1+\varepsilon)\delta$ for all times (recall that the notation γ' refers to the first $n-1$ components of the vector γ), then, in view of (12) the minimal time to connect x_0 with K (recall that $\{(x',0) : |x'|< R\} \subset \partial K$ with R large) is $(\alpha^* t - 2\delta)/\alpha_*$. This is however strictly larger than t for

$$(\alpha^* - \alpha_*)t > 2\delta, \tag{15}$$

that is, for δ large enough, given t. As a consequence, no curve with $|(\gamma(\cdot))'|\geq (1+\varepsilon)\delta$ for all times can connect x_0 to K.

We are left with the case of a curve γ with $|\gamma(\cdot)'|< (1+\varepsilon)\delta$ at least for some times. As ε is assumed to be arbitrarily small, in the simple geometry of this example an optimal curve connecting K and x_0 is arbitrarily close to the curve γ following the line $\{x' = x_0'/2\}$ up to some time $s < t$ and then the straight segment between $(x_0'/2, \alpha^* s)$ and x_0, namely,

$$\gamma(\tau) = \begin{cases} (x_0'/2, \alpha^*\tau) & \text{for } \ 0 \leq \tau < s, \\ \dfrac{t-\tau}{t-s}(x_0'/2, \alpha^* s) + \left(1 - \dfrac{t-\tau}{t-s}\right) x_0 & \text{for } \ s \leq \tau \leq t. \end{cases}$$

Note that the distance between $\gamma(s) = (x_0'/2, \alpha^* s)$ and x_0 is $((\alpha^*(t-s) - 2\delta)^2 + \delta^2)^{1/2}$. In order to cover such segment in time $t-s$ one would need

$$\begin{aligned} \Big((\alpha^*(t-s) - 2\delta)^2 + \delta^2\Big)^{1/2} &= \int_0^{t-s} \alpha(\gamma(\tau))\,\mathrm{d}\tau \\ &= \int_0^{t-s} \max\left\{-\frac{\alpha^* - \alpha_*}{\varepsilon(t-s)}\tau + \alpha^*, \alpha_*\right\} \mathrm{d}\tau < (\varepsilon\alpha^* + (1-\varepsilon)\alpha_*)(t-s). \end{aligned}$$

The inequality between the first and the last term above is equivalent to

$$\left((\alpha^*)^2 - (\varepsilon\alpha^* + (1-\varepsilon)\alpha_*)^2\right)(t-s)^2 + 5\delta^2 < 4\delta\alpha^*(t-s).$$

This is however false, regardless of the value $s < t$, as soon as ε is chosen small enough. Indeed, we can equivalently rewrite the latter as

$$\left((\alpha^*)^2 - (\varepsilon\alpha^* + (1-\varepsilon)\alpha_*)^2 - \frac{4}{5}(\alpha^*)^2\right)(t-s)^2 + \left(\frac{2}{\sqrt{5}}\alpha^*(t-s) - \sqrt{5}\delta\right)^2 < 0.$$

As the second term in the left-hand side above is nonnegative, the inequality is false as soon as the coefficient of $(t-s)^2$ in the first term is positive, namely,

$$(\alpha^*)^2 - (\varepsilon\alpha^* + (1-\varepsilon)\alpha_*)^2 - \frac{4}{5}(\alpha^*)^2 > 0$$

which follows from (14) for ε small enough.

We have eventually proved that $x_0 \in \partial B$ cannot be reached from K in time t, which entails that $x_0 \notin U_t$ and, ultimately, $B \not\subset U_t$. In particular, U_t does not fulfill the interior-ball condition with radius 2δ.

Assume now on the contrary that δ is given. Relation (15) reveals that U_t does not fulfill the interior-ball condition with radius 2δ for $t > 0$ large enough. In particular, this entails that no uniform-in-time lower bound on the interior-ball radius of U_t can be inferred from α_* and α^* only.

3.2. **No interior-ball regularity.** Let us now consider problem (1) in the open set $\mathbb{R}^{n-1} \times \{x_n < \alpha^* T\}$ for some arbitrary but fixed final time $T > 0$. For any $\alpha^* > \alpha_* > 0$ choose $0 < \beta < \eta < \pi/2$ in such a way that

$$\frac{\sin(\eta - \beta)}{\cos\beta} \geq \frac{\alpha_*}{\alpha^*} \tag{16}$$

by possibly taking η large and β small enough. Moreover, let $\varepsilon > 0$ be arbitrarily small with $(1+\varepsilon)\beta < \eta$.

Define the Lipschitz continuous function

$$\alpha(x) = \begin{cases} \alpha^* & \text{if } x \in A^*, \\ (1-\theta(x))\,\alpha^* + \theta(x)\alpha_* & \text{if } x \in A_\varepsilon, \\ \alpha_* & \text{if } x \in A_*, \end{cases} \tag{17}$$

where the regions A^*, A_ε, and A_* are defined as

$$A^* := \left\{(x', x_n) \in \mathbb{R}^n \times \{x_n < \alpha^* T\} \;:\; \frac{|x'|}{\alpha^* T - x_n} \leq \tan\beta\right\},$$

$$A_\varepsilon := \left\{(x', x_n) \in \mathbb{R}^n \times \{x_n < \alpha^* T\} \;:\; \tan\beta < \frac{|x'|}{\alpha^* T - x_n} < \tan((1+\varepsilon)\beta)\right\},$$

$$A_* := \left\{(x', x_n) \in \mathbb{R}^n \times \{x_n < \alpha^* T\} \;:\; \tan((1+\varepsilon)\beta) \leq \frac{|x'|}{\alpha^* T - x_n}\right\},$$

and $\theta(x) \in (0,1)$ for $x \in A_\varepsilon$ is given by

$$\theta(x) = \frac{\dfrac{|x'|}{\alpha^* T - x_n} - \tan\beta}{\tan((1+\varepsilon)\beta) - \tan\beta},$$

see Figure 3.

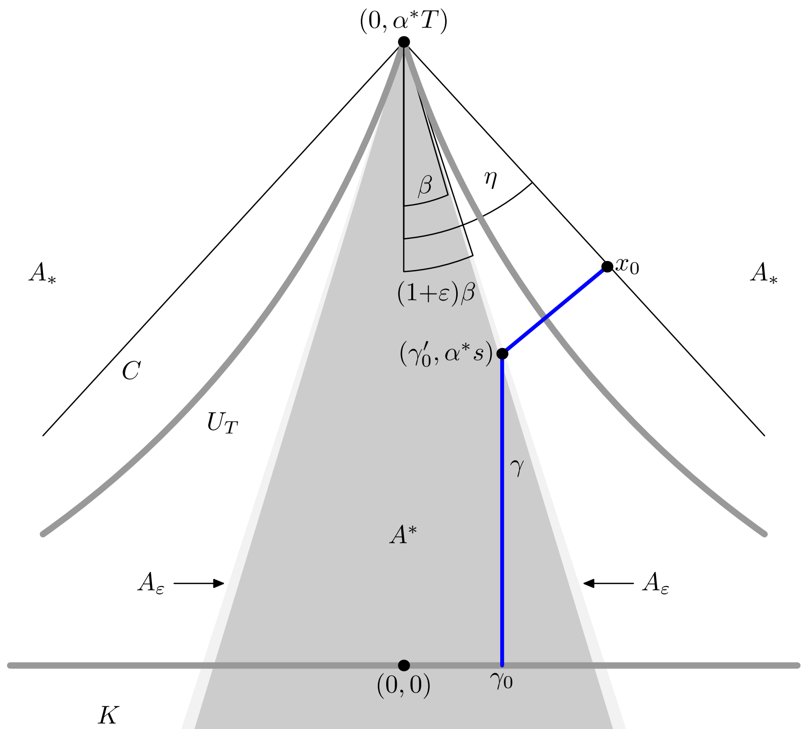

FIGURE 3. The example of Section 3.2

By symmetry, and by (3) one has that $u(0, \alpha^* T) = T$, namely that $(0, \alpha^* T) \in \partial U_T$. Additionally, we have that $(0, \alpha^* t) \in U_T$ for every $0 \le t < T$. We aim at showing that $U_T \subset C$ locally, where C is the cone with vertex at $(0, \alpha^* T)$, axis $-e_n$, and opening η, namely,

$$C = \left\{ (x', x_n) \in \mathbb{R}^n \times \{x_n < \alpha^* T\} \ : \ \frac{|x'|}{\alpha^* T - x_n} < \tan \eta \right\},$$

see Figure 3. In order to prove that $U_T \subset C$ locally, we show that all $x_0 \in \partial C$ with $x_0 \neq (0, \alpha^* T)$ and $(x_0)_n > \alpha_* T$ cannot be reached in time T from K.

Consider a curve γ connecting $\gamma_0 := \gamma(0) \in K$ with x_0. Assume first that $\gamma(t)$ is in region A_* for all times. Then necessarily $(\gamma(t))_n \le \alpha_* t$ for all times $t \in [0, T]$. In particular $(\gamma(T))_n \le \alpha_* T < (x_0)_n$ so that x_0 cannot be reached by γ in time T.

Assume on the contrary that γ enters the region $A^* \cup A_\varepsilon$ for some times. As ε is arbitrarily small, in the geometry of this example, one has that the optimal curve in this class is arbitrary close to the curve γ following the line $\{x' = \gamma_0'\}$ with $\gamma_0' = x_0' \alpha^* (T - s) \sin \beta / |x_0'|$ up to some time $s < T$ and then the straight segment between $(\gamma_0', \alpha^* s)$ and x_0, namely,

$$\gamma(t) = \begin{cases} \gamma_0 + \alpha^* t e_n & \text{for } \ 0 \le t < s, \\ \left(\dfrac{T - t}{T - s} \right) (\gamma_0', \alpha^* s) + \left(1 - \dfrac{T - t}{T - s} \right) x_0 & \text{for } \ s \le t \le T. \end{cases}$$

The distance between $\gamma(s) = (\gamma_0', \alpha^* s)$ and x_0 is

$$\alpha^*(T-s)\frac{\sin(\eta-\beta)}{\cos\beta}.$$

On the other hand, by using condition (16) and choosing ε small enough depending on α_*, α^*, η, and β one finds that the latter quantity is strictly greater than

$$\left(\frac{\sin(\eta-(1+\varepsilon)\beta)}{\sin(\eta-\beta)}\alpha_* + \left(1-\frac{\sin(\eta-(1+\varepsilon)\beta)}{\sin(\eta-\beta)}\right)\alpha^*\right)(T-s),$$

which bounds from above the maximum distance that the curve γ can travel in region $A_* \cup A_\varepsilon$ in time $T-s$. We hence conclude that the curve γ cannot reach x_0 in time T. This proves that U_T is locally contained in the cone C. In particular, U_T is not interior-ball regular.

Before closing this section, let us remark that α is defined in $\mathbb{R}^{n-1}\times\{x_n < \alpha^* T\}$ only and cannot be continuously extended to the whole $\mathbb{R}^n$.

3.3. **No interior-cone regularity.** The example of Section 3.2 is not interior-ball regular but still interior-cone regular. We rework the example in order to prove that even interior-cone regularity may fail. To this aim, we still use the choice of α, β, η, and θ from (17), but redefining the regions A^*, A_ε, and A_* as in Figure 5 (see details below). In this case, one can prove that there exists a time T^* such that U_{T^*} is not interior-cone regular, but merely John regular.

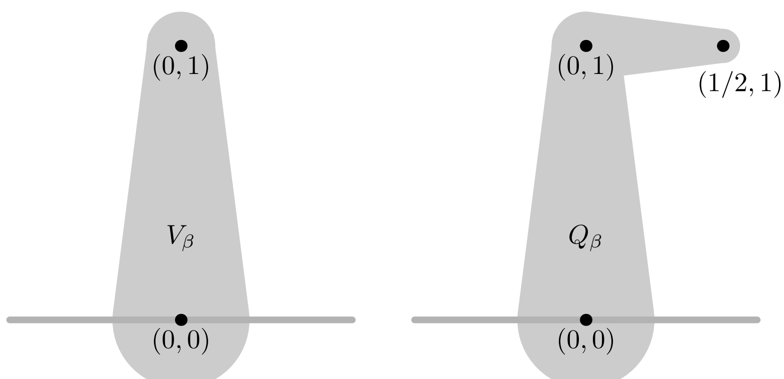

Figure 4. A two-dimensional representation of the sets V_β and Q_β

In order to give some details, let us start by defining the union of balls

$$V_\beta := \bigcup_{s\in[0,1]} B_{(\tan\beta)(2-s)}(se_n)$$

see Figure 4. The parameter $\beta > 0$ plays the role of the opening of a cone, see Section 3.2, and will later be chosen to be small. Correspondingly, for $\beta < \eta < \pi/2$ and $\varepsilon > 0$ arbitrarily small we analogously define V_η and $V_{(1+\varepsilon)\beta}$.

Moreover, we let

$$Q_\beta := V_\beta \cup \left(\frac{1}{2}JV_\beta + (0,\dots,0,1)\right),$$

where J is a rotation, mapping e_n to e_1, see Figure 4. Correspondingly, for $\beta < \eta < \pi/2$ and $\varepsilon > 0$ arbitrarily small we analogously define Q_η and $Q_{(1+\varepsilon)\beta}$ by replacing β by η or $(1+\varepsilon)\beta$, respectively.

Finally, we let A^* and A_ε be given by

$$A^* := \bigcup_{i=0}^{\infty}\left(\frac{1}{4^i}Q_\beta + \sum_{j=1}^{i}\frac{1}{4^{j-1}}\left(\frac{1}{2},0,\dots,0,1\right)\right),$$

$$A_\varepsilon := \left(\bigcup_{i=0}^{\infty}\left(\frac{1}{4^i}Q_{(1+\varepsilon)\beta} + \sum_{j=1}^{i}\frac{1}{4^{j-1}}\left(\frac{1}{2},0,\dots,0,1\right)\right)\right)\setminus A^*.$$

One has that $A^*, A_\varepsilon \subset \mathbb{R}^{n-1}\times\{x_n < 4/3\}$ and defines $A_* = \mathbb{R}^{n-1}\times\{x_n < 7\alpha^*/3\}\setminus(A^*\cup A_\varepsilon)$.

Consider now the point

$$x^* = \sum_{j=1}^{\infty}\frac{1}{4^{j-1}}\left(\frac{1}{2},0,\dots,0,1\right) = \left(\frac{2}{3},0,\dots,0,\frac{4}{3}\right).$$

Clearly, $x^* \in \overline{A^*}$ can be reached from K in finite time T^* by a curve entirely in $\overline{A^*}$, see Figure 5. Assume now that $\alpha^* > \alpha_* > 0$ are given in such a way that one can choose η with $\eta > \beta > 0$ small and still fulfilling (16) (this is, for instance, the case for $\alpha_* << \alpha^*$) and define

$$\alpha(x) = \begin{cases} \alpha_* & \text{for } x\in A_*,\\ \dfrac{\alpha^*\, d(x,A_*) + \alpha_*\, d(x,A^*)}{\max\{d(x,A_*),d(x,A^*)\}} & \text{for } x\in A_\varepsilon,\\ \alpha^* & \text{for } x\in A^*.\end{cases}$$

Note that α is locally Lipschitz continuous in $\mathbb{R}^{n-1}\times\{x_n<4/3\}$ as one has that $\max\{d(x,A_*),d(x,A^*)\}$ is uniformly positive in A_ε.

For ε small one can then argue as in Section 3.2 in order to check that in a neighborhood of x^* the superlevel U_{T^*} is contained in the set

$$A_\eta := \bigcup_{i=0}^{\infty}\left(\frac{1}{4^i}Q_\eta + \sum_{j=1}^{i}\frac{1}{4^{j-1}}\left(\frac{1}{2},0,\dots,0,1\right)\right).$$

Since A_η is (John regular but) not interior-cone regular, so is U_{T^*}, for $x^*\in A_\eta\cap U_{T^*}$.

Once again, such Lipschitz continuous α cannot be continuously extended to $\mathbb{R}^n$.

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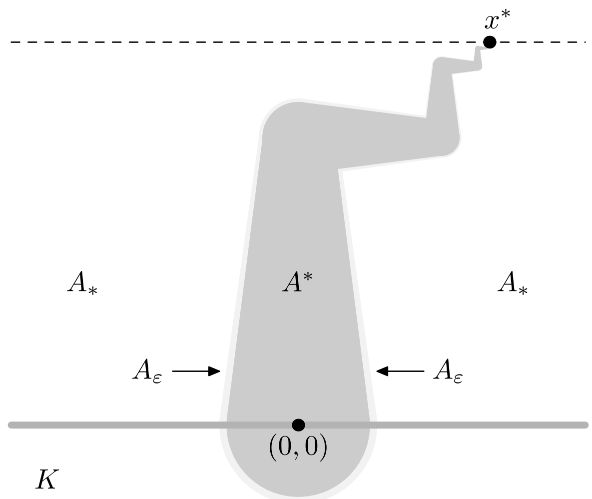

FIGURE 5. The example of Section 3.3 in $\mathbb{R}^2$

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