# ALMOST SHARP RATES OF CONVERGENCE FOR THE AVERAGE COST AND DISPLACEMENT IN THE OPTIMAL MATCHING PROBLEM

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Abstract. In this note we prove estimates for the average cost in the quadratic optimal transport problem on the two-dimensional flat torus which are optimal up to a double logarithm. We also prove sharp estimates on the displacement. This is based on the combination of a post-processing of our quantitative linearization result together with a quasi-orthogonality property.

## 1. Introduction

The aim of this note is to improve the currently best known rates of convergence of the average cost in the quadratic optimal transport problem on the two-dimensional flat torus from [2]. Compared with the conjecture from Caracciolo and al. in [9], our rate is optimal up to a double logarithm. We first use our quantitative linearization result from [13] (in its post-processed version of [12]) to improve the estimates from [3] on the optimal transport map. We then combine this with a quasi-orthogonality property first observed in [14, (3.28)] and relatively standard heat kernel estimates to conclude.

To state our main result let us set some notation. We work on the 2 dimensional flat torus $\mathbb{T}_1 = (\mathbb{R}/\mathbb{Z})^2$ and consider $(X_i)_{i\geq 1}$ a family of i.i.d. uniformly distributed random variables on $\mathbb{T}_1$. For $n \geq 1$ we define the empirical measure as

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}.$$

The (quadratic) optimal matching problem on the torus is then

$$\mathbb{E}\left[\inf_{\pi\in \mathsf{Cpl}(\mu_n,1)} \int_{\mathbb{T}_1\times\mathbb{T}_1} |x-y|^2 d\pi(x,y)\right] =: \mathbb{E}\left[W_2^2(\mu_n,1)\right].$$

Here, we identify the Lebesgue measure on $\mathbb{T}_1$ with the constant density 1, $\mathsf{Cpl}(\mu_n,1)$ denotes the couplings between $\mu_n$ and the Lebesgue measure on $\mathbb{T}_1$, and $W_2$ is the $L^2$ Kantorovich Wasserstein distance on $\mathbb{T}_1$.

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While it is known since the seminal article [1] that $n\mathbb{E}\left[W_2^2(\mu_n,1)\right] \sim \log n$, it was recently conjectured in [9] that

$$\lim_{n\to\infty}\left(n\mathbb{E}\left[W_2^2(\mu_n,1)\right]-\frac{\log n}{4\pi}\right)\in\mathbb{R}. \tag{1.1}$$

Our main result in this direction is the following theorem.

**Theorem 1.1.** *We have*[i]

$$\left|n\mathbb{E}\left[W_2^2(\mu_n,1)\right]-\frac{\log n}{4\pi}\right|\lesssim \log\log n. \tag{1.2}$$

In light of conjecture (1.1), the error in (1.2) is optimal up to a double logarithm. This essentially improves by $\sqrt{\log n}$ the rate obtained in [2] which built on the approach from [5] where the leading order term in (1.1) was identified. Let us however point out that our result currently holds only in the case of the flat torus while [2, 5] covers any Riemannian manifold without boundary as well as the case of the unit cube. Moreover, since we mostly rely on [13], we are currently not able to treat the bi-partite problem. We refer to [20, 19, 7, 6, 11, 15, 4, 16, 8] for related results.

As in [5, 2] our proof of (1.2) is based on the linearization ansatz proposed in this context by [9] and which we now recall. If $\pi_n = (T_n, \mathrm{Id})_{\#}1$ is the optimal (random) coupling between $\mu_n$ and 1, i.e.

$$W_2^2(\mu_n,1)=\int_{\mathbb{T}_1\times\mathbb{T}_1}|x-y|^2 d\pi_n,$$

this ansatz postulate that $T_n(y) - y$ is well approximated by $\nabla f_n(y)$ where $f_n$ solves the Poisson equation

$$-\Delta f_n = \mu_n - 1.$$

As understood since [5], this ansatz can only make sense after some regularization. For $t > 0$, let $p_t$ be the heat kernel at time $t$ on $\mathbb{T}_1$ and set $f_{n,t} = p_t * f_n$ be a solution of

$$-\Delta f_{n,t} = p_t * (\mu_n - 1).$$

As a consequence of the trace formula[ii], see e.g. [2, Lemma 3.14],

$$n\mathbb{E}\left[\int_{\mathbb{T}_1}|\nabla f_{n,t}|^2\right]=\frac{|\log t|}{4\pi}+O(\sqrt{t}). \tag{1.3}$$

In order to prove (1.2), it is thus 'enough' to prove that for some $t_n$ with $|\log t_n| = \log n + O(\log\log n)$,

$$n\left|\mathbb{E}\left[W_2^2(\mu_n,1)\right]-\mathbb{E}\left[\int_{\mathbb{T}_1}|\nabla f_{n,t}|^2\right]\right|\lesssim\log\log n.$$

The main step to prove this is our second main result.

[i] The notation $A \lesssim B$, which we use in output statements, means that there exists a universal constant $C > 0$ such that $A \le CB$.

[ii] when integrating with respect to the Lebesgue measure we drop the factor $dy$

**Theorem 1.2.** *For any* $t \geq \frac{1}{n} = r_n^2$ *we have*

$$n\mathbb{E}\left[\int_{\mathbb{T}_1} |x - y - \nabla f_{n,t}(x)|^2 \, d\pi_n\right] \lesssim 1 + \log\left(\frac{t}{r_n^2}\right). \tag{1.4}$$

*Moreover, for* $t \geq t_n = n^{-1}\log^3 n$ *we have*

$$n\mathbb{E}\left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t}(y)|^2\right] \lesssim \log\left(\frac{t}{r_n^2}\right). \tag{1.5}$$

Notice that the difference between (1.4) and (1.5) is that we replaced $\nabla f_{n,t}(x)$ in the former by $\nabla f_{n,t}(y)$ in the latter. While both have sharp dependence in $t$, we can go down to the microscopic scale $t = r_n^2 = n^{-1}$ in (1.4) but are restricted to mesoscopic scales $t \geq t_n \gg n^{-1}$ in (1.5). This is most likely an artefact from our proof. Indeed, we derive (1.5) from (1.4) combined with an $L^\infty$ bound on $\nabla^2 f_{n,t_n}$. This imposes the choice $t_n \gg n^{-1}$ (see (2.2) of Lemma 2.1). Still, (1.5) improves by $\sqrt{\log n}$ a similar bound from [3] (see also the recent generalization [10]).

The proof of (1.4) is mostly based on the quantitative linearization result from [13] in its post-processed version from [12]. Let us list the differences between [12, Proposition 4.7] and (1.4). A first point is to pass from a compactly supported convolution kernel as in [12] to the heat kernel as in (1.4). This is done using Lemma 2.4 which relies on relatively standard heat/Green kernel estimates. A second difficulty is to pass from a quenched and localized estimate in [12] to an annealed and global one, see (3.1). This is obtained appealing to stationarity. The argument here is a bit more delicate than its counterpart in [12].

From this sketch of proof it is clear that the 'only' obstacle to obtain (1.5) down to $t = n^{-1}$ is the fact that [13] is currently only known for constant target measures. An extension of this result to arbitrary measures should also allow to extend our results to the bi-partite case.

Let us notice that combining (1.5), (1.3) and $n\mathbb{E}\left[W_2^2(\mu_n, 1)\right] \sim \log n$ together with Cauchy-Schwarz inequality, it is not hard to obtain

$$\left| n\mathbb{E}\left[W_2^2(\mu_n, 1)\right] - \frac{\log n}{4\pi} \right| \lesssim (\log n \log\log n)^{\frac{1}{2}}$$

which gives an alternative proof of the estimate in [2]. Similar sub-optimal error terms coming from the application of Cauchy-Schwarz inequality can be seen in [2, Theorem 1.2] for example. In order to obtain the sharper estimate (1.2) we rely instead on the quasi-orthogonality property

$$\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y)\right] \right| \lesssim 1.$$

This type of estimates, first noticed in [14, (3.28)], see also [13, Lemma 1.7], are a central ingredient in the variational approach to the regularity theory for optimal transport maps (see also [18, 17, 21]).

## 2. Preliminaries

In this section we gather a few technical results which follow from relatively standard heat-kernel estimates. To simplify notation and presentation we provide estimates only for moments of order four but similar bounds can be obtained for moments of arbitrary order. As in [2], for $t \geq 0$ we let

$$q_t(x) = \int_t^\infty (p_s(x) - 1) ds$$

so that

$$\nabla f_{n,t}(y) = \frac{1}{n} \sum_{i=1}^n \nabla q_t(X_i - y). \tag{2.1}$$

**Lemma 2.1.** *Let $t_n = n^{-1} \log^3 n$. Then,*

$$\mathbb{E}[\|\nabla^2 f_{n,t_n}\|_\infty^4]^{\frac{1}{4}} \lesssim \frac{1}{\log n}. \tag{2.2}$$

*For every $n^{-1} \leq s < t < 1$ we have*

$$n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,s} - \nabla f_{n,t}|^4\right]^{\frac{1}{2}} \lesssim 1 + \log\left(\frac{t}{s}\right). \tag{2.3}$$

*Proof.* We first prove (2.2). For $\xi > 0$ define the event $A_\xi^{n,t} = \{\|\nabla^2 f_{n,t}\|_\infty \leq \xi\}$. By [2, Theorem 3.3], there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $0 < t < 1$ we have

$$\mathbb{P}[\left(A_\xi^{n,t}\right)^c] \lesssim \begin{cases} \frac{1}{t^2\xi^3} e^{-Cnt\xi^2} & \text{if } 0 < \xi \leq 1 \\ \frac{1}{t^2\xi^3} e^{-Cnt\xi} & \text{if } \xi \geq 1. \end{cases} \tag{2.4}$$

The estimate for $\xi \geq 1$ is not explicitly contained in the statement of [2, Theorem 3.3] but follows by the exact same argument.

Since for a non-negative random variable $X$ and $a > 0$, $\mathbb{E}[X^p] \lesssim a^p + \int_a^\infty \xi^{p-1} \mathbb{P}[X \geq \xi] d\xi$, (2.4) implies that for every $1 > a > 0$,

$$\mathbb{E}\left[\|\nabla^2 f_{n,t_n}\|_\infty^4\right] \lesssim a^4 + \int_a^1 \frac{1}{t_n^2} e^{-Cnt_n\xi^2} d\xi + \int_1^\infty \frac{1}{t_n^2} e^{-Cnt_n\xi} d\xi. \tag{2.5}$$

Recalling that $t_n = n^{-1} \log^3 n$, the last integral can be estimated as

$$\frac{1}{t_n^2} \int_1^\infty e^{-Cnt_n\xi} d\xi \lesssim \frac{n^2}{\log^9 n} \exp(-C \log^3 n) \ll \frac{1}{\log^4 n}.$$

The other integral can be estimated by

$$\begin{aligned} \int_a^1 \frac{1}{t_n^2} e^{-Cnt_n\xi^2} d\xi &\lesssim \frac{n^2}{\log^6 n} \int_a^\infty \exp(-C\xi^2 \log^3 n) d\xi \\ &\overset{\xi = s\log^{-3/2} n}{\lesssim} \frac{n^2}{\log^{\frac{15}{2}} n} \int_{a \log^{3/2} n}^\infty e^{-Cs^2} ds \lesssim \frac{n^2}{\log^9 n} e^{-Ca^2 \log^3 n}. \end{aligned}$$

Choosing $a = \gamma \log^{-1} n$ with $C\gamma^2 > 2$ we get

$$e^{-Ca^2 \log^3 n} \le n^{-2}$$

and thus

$$\int_a^1 \frac{1}{t_n^2} e^{-Cnt_n\xi^2} d\xi \lesssim \frac{1}{\log^4 n}.$$

Plugging this in (2.5) concludes the proof of (2.2).

We now turn to the proof of (2.3). By stationarity we have

$$\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,s} - \nabla f_{n,t}|^4\right] = \mathbb{E}[|\nabla f_{n,s}(0) - \nabla f_{n,t}(0)|^4].$$

Since $\nabla f_{n,s}(0) = n^{-1}\sum_i \nabla q_s(X_i)$ we have by Rosenthal inequality,

$$\begin{aligned}\mathbb{E}[|\nabla f_{n,s}(0) - \nabla f_{n,t}(0)|^4] &\lesssim n^{-4}\left(n\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 + \left(n\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2\right)^2\right)\\ &= n^{-3}\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 + n^{-2}\left(\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2\right)^2.\end{aligned}$$

To estimate the first right-hand side term we recall from [2, Corollary 3.13] that for $0 < t < 1$,

$$\int_{\mathbb{T}_1} |\nabla q_t|^4 \lesssim t^{-1}. \tag{2.6}$$

Using triangle inequality we thus conclude that

$$\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 \lesssim \int_{\mathbb{T}_1} |\nabla q_s|^4 + \int_{\mathbb{T}_1} |\nabla q_t|^4 \lesssim t^{-1} + s^{-1} \lesssim n. \tag{2.7}$$

For the second right-hand side term we can argue as in [2, Proposition 3.11] using that $q_s = \int_s^\infty (p_r - 1)dr$ and $-\Delta q_s = p_s - 1$ to obtain after integration by parts,

$$\begin{aligned}\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2 &= \int_{\mathbb{T}_1} (q_t - q_s)(p_t - p_s) = \int_{\mathbb{T}_1}\int_s^t p_r(p_t - p_s)drdx\\ &= \int_s^t\int_{\mathbb{T}_1} (p_r p_t - 1)dxdr - \int_s^t\int_{\mathbb{T}_1} (p_r p_s - 1)dxdr.\end{aligned}$$

Using the semi-group property of the heat kernel together with the trace formula (see e.g. [2, Theorem 3.7]) we have

$$\int_s^t\int_{\mathbb{T}_1} (p_r p_t - 1)dxdr = \int_s^t (p_{r+t}(0) - 1)dr = \int_{s+t}^{2t} (p_r(0) - 1)dr = \frac{1}{4\pi}\log\left(\frac{2t}{s+t}\right) + O(1).$$

Arguing similarly we have

$$\int_s^t \int_{\mathbb{T}_1} (p_r p_s - 1) dx dr = \frac{1}{4\pi} \log\left(\frac{s+t}{2s}\right) + O(1)$$

so that

$$\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2\right] = \frac{1}{4\pi} \log\left(\frac{t}{s}\right) + O(1).$$

Combining this with (2.7) concludes the proof of (2.3). □

**Lemma 2.2.** *For $t \geq r_n^2 = \frac{1}{n}$ we have*

$$n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}} |\nabla f_{n,t}(0) - \nabla f_{n,t}(x)|^2 \, d\mu_n\right] \lesssim \frac{1}{(nt)^{1/2}} \lesssim 1. \tag{2.8}$$

*Proof.* Put $E = \mathbb{E}\left[\int_{B_{r_n}} |\nabla f_{n,t}(0) - \nabla f_{n,t}(x)|^2 \, d\mu_n\right]$ and write $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ to get

$$\begin{aligned} E = \mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(0) - \nabla f_{n,t}(X_1)|^2\right] &= \mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(0)|^2\right] \\ &- 2\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla f_{n,t}(0)\cdot\nabla f_{n,t}(X_1)\right] + \mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(X_1)|^2\right]. \end{aligned}$$

We now estimate each term separately using (2.1). For the first one we have

$$\mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(0)|^2\right] = \frac{1}{n^2}\sum_{i,j}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla q_t(X_i)\cdot\nabla q_t(X_j)\right].$$

By independence of the $X_i$ and the fact that $\int_{\mathbb{T}} \nabla q_t(x) = 0$ we obtain that for $i \neq j$ the expectation is 0. Hence,

$$\begin{aligned} \mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(0)|^2\right] &= \frac{n-1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla q_t(X_2)|^2\right] + \frac{1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla q_t(X_1)|^2\right] \\ &= \frac{n-1}{n^2}|B_{r_n}|\int_{\mathbb{T}}|\nabla q_t|^2 + \frac{1}{n^2}\int_{B_{r_n}}|\nabla q_t|^2. \end{aligned}$$

For the last term we have similarly,

$$\begin{aligned} \mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla f_{n,t}(X_1)|^2\right] &= \frac{1}{n^2}\sum_{i,j}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla q_t(X_i - X_1)\cdot\nabla q_t(X_j - X_1)\right] \\ &= \frac{n-1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla q_t(X_2 - X_1)|^2\right] + \frac{1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)|\nabla q_t(0)|^2\right] \\ &= \frac{n-1}{n^2}|B_{r_n}|\int_{\mathbb{T}}|\nabla q_t|^2 + \frac{1}{n^2}|B_{r_n}||\nabla q_t(0)|^2. \end{aligned}$$

Regarding the middle term we have

$$\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla f_{n,t}(0)\cdot\nabla f_{n,t}(X_1)\right]=\sum_{i,j}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla q_t(X_i)\cdot\nabla q_t(X_j-X_1)\right]$$
$$=\frac{n-1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla q_t(X_2)\cdot\nabla q_t(X_2-X_1)\right]+\frac{1}{n^2}\mathbb{E}\left[\chi_{B_{r_n}}(X_1)\nabla q_t(X_1)\cdot\nabla q_t(0)\right]$$
$$=\frac{n-1}{n^2}\int_{B_{r_n}}\int_{\mathbb{T}}\nabla q_t(z)\cdot\nabla q_t(z-x)+\frac{1}{n^2}\int_{B_{r_n}}\nabla q_t(x)\cdot\nabla q_t(0).$$

Since by [2, Proposition 3.12]

$$\sup_{B_{r_n}}|\nabla q_t|\lesssim(r_n+t^{1/2})^{-1}\lesssim t^{-1/2}\tag{2.9}$$

we have

$$\frac{1}{n^2}\int_{B_{r_n}}\nabla q_t(x)\cdot\nabla q_t(0)\lesssim\frac{|B_{r_n}|}{n^2}t^{-1}$$

and similarly for the two other terms with prefactor $n^{-2}$. Therefore, for some $C\gg 1$,

$$nE-C\frac{|B_{r_n}|}{nt}\lesssim\left(|B_{r_n}|\int_{\mathbb{T}}|\nabla q_t|^2-\int_{B_{r_n}}\int_{\mathbb{T}}\nabla q_t(z)\cdot\nabla q_t(z-x)\right)$$
$$=\int_{B_{r_n}}\int_{\mathbb{T}}\nabla q_t(z)\cdot(\nabla q_t(z)-\nabla q_t(z-x))$$
$$=\int_{B_{r_n}}\int_{\mathbb{T}}-\Delta q_t(z)\cdot(q_t(z)-q_t(z-x))$$
$$=\int_{B_{r_n}}\int_{\mathbb{T}}(p_t(z)-1)\cdot(q_t(z)-q_t(z-x))$$
$$=\int_{B_{r_n}}(q_{2t}(0)-q_{2t}(x))\leq|B_{r_n}|r_n\sup_{B_{r_n}}|\nabla q_{2t}|.$$

Here we used that $-\Delta q_t=p_t-1$ together with the semi-group property of $p_t$. Using (2.9) again we conclude the proof of (2.8). □

**Remark 2.3.** *For $t\geq t_n=n^{-1}\log^3 n$, the proof of* (2.8) *can be significantly simplified with the help of the Hessian bounds* (2.2).

Finally, in order to translate the results from [13, 12] to the setting of [5, 2, 3] we will need to be able to switch from convolutions against compactly supported kernels to convolutions against the heat kernel.

**Lemma 2.4.** *Let $\eta\in C_c^\infty(B_1)$ be a smooth convolution kernel. For $r>0$, set $\eta_r=r^{-2}\eta(\cdot/r)$ and let then $\varphi_n^r$ the mean-zero solution of*

$$-\Delta\varphi_n^r=\eta_r*(\mu_n-1).$$

*For every $t\geq n^{-1}$ we have*

$$n\mathbb{E}[|\nabla f_{n,t}(0)-\nabla\varphi_n^{\sqrt{t}}(0)|^4]^{\frac{1}{2}}\lesssim 1.\tag{2.10}$$

*Proof.* We start by noting that $\nabla\varphi_n^{\sqrt{t}}(0) = n^{-1}\sum_i(\eta_{\sqrt{t}} * \nabla q_0)(X_i)$. Since $\nabla f_{n,t}(0) = n^{-1}\sum_i \nabla q_t(X_i)$ we may apply as above Rosenthal inequality to obtain

$$\begin{aligned}
&\mathbb{E}[|\nabla f_{n,t}(0) - \nabla\varphi_n^{\sqrt{t}}(0)|^4] \\
&\lesssim n^{-4}\left(\sum_i \mathbb{E}[|\nabla q_t(X_i) - \nabla(\eta_{\sqrt{t}} * q_0)(X_i)|^4] + (\sum_i \mathbb{E}[|\nabla q_t(X_i) - \nabla(\eta_{\sqrt{t}} * q_0)(X_i)|^2])^2\right) \\
&\qquad = n^{-4}\left(n\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^4 + (n\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2)^2\right) \\
&\qquad \lesssim n^{-3}\left(\int_{\mathbb{T}_1} |\nabla q_t|^4 + \int_{\mathbb{T}_1} |\nabla(\eta_{\sqrt{t}} * q_0)|^4\right) + n^{-2}\left(\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2\right)^2.
\end{aligned}$$

The first right-hand side term is estimated by (2.6). Arguing exactly as in the proof of [12, (3.5)], we may similarly estimate the second right-hand side term by

$$\int_{\mathbb{T}_1} |\nabla(\eta_{\sqrt{t}} * q_0)|^4 \lesssim t^{-1}.$$

In order to conclude the proof of (2.10) we are left with proving

$$\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1. \tag{2.11}$$

To this aim we write

$$\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim \int_{B_{3\sqrt{t}}} |\nabla q_t|^2 + \int_{B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0)|^2 + \int_{\mathbb{T}_1\setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \tag{2.12}$$

For the first right-hand side term we notice that by [2, Proposition 3.12] in $B_{3\sqrt{t}}$ we have

$$|\nabla q_t| \lesssim \frac{1}{\sqrt{t}}$$

so that

$$\int_{B_{3\sqrt{t}}} |\nabla q_t|^2 \lesssim \int_{B_{3\sqrt{t}}} \frac{1}{t} \lesssim 1. \tag{2.13}$$

The second right-hand side term in (2.12) is treated similarly. Indeed, arguing as in [12, (3.7)] we get that in $B_{3\sqrt{t}}$,

$$|\nabla(\eta_{\sqrt{t}} * q_0)| \lesssim \frac{1}{\sqrt{t}}$$

so that

$$\int_{B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1. \tag{2.14}$$

We finally turn to the last right-hand side term in (2.12). We further decompose it as

$$\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 + \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0) - \nabla q_0|^2. \tag{2.15}$$

For the first right-hand side term we have by definition of $q_t$,

$$\nabla q_t - \nabla q_0 = -\int_0^t \nabla p_s ds.$$

Moreover, by [2, Theorem 3.8&3.9], we have in $\mathbb{T}_1\backslash B_{3\sqrt{t}}$

$$|\nabla p_s|^2 \lesssim \frac{|x|^2}{s^4} \exp(-c\frac{|x|^2}{s}).$$

Using Cauchy-Schwarz, we find

$$\begin{aligned}\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 \lesssim t\int_0^t \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla p_s|^2 &\lesssim t\int_0^t \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} \frac{|x|^2}{s^4} \exp(-c\frac{|x|^2}{s})\\ &\overset{x=\sqrt{s}y}{\lesssim} t\int_0^t \int_{3\sqrt{t/s}}^\infty \frac{r^3}{s^2} \exp(-cr^2) drds.\end{aligned}$$

Using that for $A \geq 3$,

$$\int_A^\infty r^3 \exp(-cr^2) \lesssim A^2 \exp(-cA^2)$$

we find

$$\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 \lesssim t^2 \int_0^t s^{-3} \exp(-c\frac{t}{s}) ds \overset{s=tu}{=} \int_0^1 u^{-3} \exp(-\frac{1}{u}) du \lesssim 1.$$

We now estimate the second right-hand side term in (2.15). For $x \in \mathbb{T}_1\backslash B_{3\sqrt{t}}$ and $y \in B_{\sqrt{t}}$ we have arguing exactly as in [12, Lemma 3.1]

$$|\nabla q_0(x-y) - \nabla q_0(x)| \lesssim \frac{|y|}{|x|^2}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} \left|\nabla(\eta_{\sqrt{t}} * q_0) - \nabla q_0\right|^2 &= \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} \left|\int_{B_{\sqrt{t}}} \eta_{\sqrt{t}}(y)(\nabla q_0(x-y) - \nabla q_0(x))dy\right|^2 dx\\ &\leq \int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} \left(\int_{B_{\sqrt{t}}} \eta_{\sqrt{t}}(y)|y|\right)^2 |x|^{-4} dx \leq t\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |x|^{-4} \lesssim 1.\end{aligned}$$

We conclude that

$$\int_{\mathbb{T}_1\backslash B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1. \tag{2.16}$$

Injecting (2.13), (2.14) and (2.16) in (2.12) we obtain the desired (2.11). □

We recall that from the concentration properties of $W_2^2(\mu_n, 1)$ (see [5] or [12, Theorem 4.5 & Remark 4.6]) we have the following moment bound:

**Lemma 2.5.** *We have*

$$n\mathbb{E}\left[(W_2^2(\mu_n, 1))^2\right]^{\frac{1}{2}} \lesssim \log n. \tag{2.17}$$

## 3. Proof of the main results

In this section we prove Theorem 1.2 and then Theorem 1.1.

*Proof of Theorem 1.2.* We start with (1.4). Let $t \geq r_n^2$. As in [12], we first rely on stationarity to infer that

$$\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1} |x-y-\nabla f_{n,t}(x)|^2\, d\pi_n\right] = \mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1} |x-y-\nabla f_{n,t_n}(x)|^2\, d\pi_n\right].$$

Let $\varphi_n^r$ be defined as in Lemma 2.4. We now claim that in order to prove (1.4), it is enough to prove

$$n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1} |x-y-\nabla\varphi_n^{r_n}(0)|^2\, d\pi_n\right] \lesssim 1. \tag{3.1}$$

To this aim we use for $(x, y) \in B_{r_n} \times \mathbb{T}_1$ the triangle inequality in the form

$$\begin{aligned}|x-y-\nabla f_{n,t}(x)|^2 &\lesssim |x-y-\nabla\varphi_n^{r_n}(0)|^2\\ &\quad + \left|\nabla\varphi_n^{r_n}(0)-\nabla f_{n,r_n^2}(0)\right|^2 + \left|\nabla f_{n,r_n^2}(0)-\nabla f_{n,t}(0)\right|^2 + |\nabla f_{n,t}(0)-\nabla f_{n,t}(x)|^2.\end{aligned}$$

In order to prove the claim we need to show that the contributions coming from the last three terms on the right-hand side are controlled. Observe, that $n\mu_n(B_{r_n})$ is a Binomial random variable with all moments of order 1. Hence, we can estimate

$$\begin{aligned}&n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1}\left|\nabla\varphi_n^{r_n}(0)-\nabla f_{n,r_n^2}(0)\right|^2 d\pi_n\right]\\ &\qquad= \frac{n}{|B_{r_n}|}\mathbb{E}\left[\mu_n(B_{r_n})\left|\nabla\varphi_n^{r_n}(0)-\nabla f_{n,r_n^2}(0)\right|^2\right]\\ &\qquad\leq \frac{n}{|B_{r_n}|}\mathbb{E}\left[(\mu_n(B_{r_n}))^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\nabla\varphi_n^{r_n}(0)-\nabla f_{n,r_n^2}(0)\right|^4\right]^{\frac{1}{2}} \overset{(2.10)}{\lesssim} 1.\end{aligned}$$

Second,

$$\begin{aligned}&n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1}\left|\nabla f_{n,r_n^2}(0)-\nabla f_{n,t}(0)\right|^2 d\pi_n\right]\\ &\qquad\leq \frac{n}{|B_{r_n}|}\mathbb{E}\left[(\mu_n(B_{r_n}))^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\nabla f_{n,r_n^2}(0)-\nabla f_{n,t}(0)\right|^4\right]^{\frac{1}{2}} \overset{(2.3)}{\lesssim} \left(1+\log\left(\frac{t}{r_n^2}\right)\right).\end{aligned}$$

Finally, by (2.8) of Lemma 2.2 we have

$$n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1}|\nabla f_{n,t_n}(0)-\nabla f_{n,t_n}(x)|^2\,d\pi_n\right]$$
$$=n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}}|\nabla f_{n,t_n}(0)-\nabla f_{n,t_n}(x)|^2\,d\mu_n\right]\lesssim 1.$$

We now establish (3.1). After rescaling and changing $\varphi_n^{r_n}$ into $-\varphi_n^{r_n}$, [12, Proposition 4.6] yields the existence of a random radius $r_{*,n}\geq r_n$ with moments of every order i.e. for every $p\geq 1$,

$$\mathbb{E}\left[\left(\frac{r_{*,n}}{r_n}\right)^p\right]\lesssim_p 1 \tag{3.2}$$

such that

$$\sup\{|x-y-\nabla\varphi_n^{r_{*,n}}(0)|\ :\ (x,y)\in\mathsf{Spt}\,\pi_n\cap(B_{r_n}\times\mathbb{T}_1)\}\lesssim r_{*,n}.$$

Moreover, combining [12, Lemma 4.3 & Theorem 4.5] together with [13, Lemma 2.10] we may further assume that $r_{*,n}$ is such that for some $\alpha\in(0,1)$,

$$|\nabla\varphi_n^{r_{*,n}}(0)-\nabla\varphi_n^{r_n}(0)|\lesssim r_{*,n}\left(\frac{r_{*,n}}{r_n}\right)^{2+\alpha}.$$

We can thus estimate using triangle inequality,

$$n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1}|x-y-\nabla\varphi_n^{r_n}(0)|^2\,d\pi_n\right]$$
$$\lesssim n\mathbb{E}\left[\frac{1}{|B_{r_n}|}\int_{B_{r_n}\times\mathbb{T}_1}|x-y-\nabla\varphi_n^{r_{*,n}}(0)|^2\,d\pi_n\right]+n\mathbb{E}\left[\frac{\mu(B_{r_n})}{|B_{r_n}|}|\nabla\varphi_n^{r_{*,n}}(0)-\nabla\varphi_n^{r_n}(0)|^2\right]$$
$$\lesssim n\mathbb{E}\left[r_{*,n}^4\right]^{\frac12}+n\mathbb{E}\left[r_{*,n}^4\left(\frac{r_{*,n}}{r_n}\right)^{4(2+\alpha)}\right]^{\frac12}\overset{(3.2)}{\lesssim}1.$$

This concludes the proof of (3.1) and, hence, of (1.4).

To show (1.5) we use the triangle inequality to estimate for $t\geq t_n$,

$$n\mathbb{E}\left[\int_{\mathbb{T}_1}|T_n(y)-y-\nabla f_{n,t}(y)|^2\right]$$
$$\lesssim n\mathbb{E}\left[\int_{\mathbb{T}_1}|\nabla f_{n,t}-\nabla f_{n,t_n}|^2\right]+n\mathbb{E}\left[\int_{\mathbb{T}_1}|T_n(y)-y-\nabla f_{n,t_n}(y)|^2\right].$$

Using Hölder inequality and (2.3) we see that it is enough to prove (1.5) for $t = t_n$. We then use triangle inequality again to write

$$
\begin{aligned}
n\mathbb{E}\left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t_n}(y)|^2\right] &= n\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1} |x - y - \nabla f_{n,t_n}(y)|^2 \, d\pi_n\right] \\
&\lesssim n\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1} |x - y - \nabla f_{n,t_n}(x)|^2 \, d\pi_n\right] + n\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1} |\nabla f_{n,t_n}(x) - \nabla f_{n,t_n}(y)|^2 \, d\pi_n\right].
\end{aligned}
$$

Since the second right-hand side term is estimated as

$$
\begin{aligned}
n\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1} |\nabla f_{n,t_n}(x) - \nabla f_{n,t_n}(y)|^2 \, d\pi_n\right] &\lesssim n\mathbb{E}[\|\nabla^2 f_{n,t_n}\|_\infty^4]^{\frac{1}{2}} \mathbb{E}[(W_2^2(\mu_n, 1))^2]^{\frac{1}{2}} \\
&\overset{(2.2)\&(2.17)}{\lesssim} \frac{\log n}{\log^2 n} = \frac{1}{\log n},
\end{aligned}
$$

we conclude by (1.4). □

We finally prove Theorem 1.1.

*Proof of Theorem 1.1.* We start by writing

$$
|T_n(y) - y|^2 = |T_n(y) - y - \nabla f_{n,t_n}(y)|^2 + |\nabla f_{n,t_n}(y)|^2 + 2(T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y).
$$

After integration and using (1.5) with $t = t_n$, we thus get

$$
\begin{aligned}
\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} |T_n(y) - y|^2\right] - \frac{\log n}{4\pi} \right| &\lesssim \log\log n + \left| n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2\right] - \frac{\log n}{4\pi} \right| \\
&\quad + \left| n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y)\right] \right|.
\end{aligned}
$$

Since by the trace formula, see (1.3)

$$
\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2\right] - \frac{\log n}{4\pi} \right| \lesssim \log\log n,
$$

we are left with the proof of the quasi-orthogonality property

$$
\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y)\right] \right| \lesssim 1. \tag{3.3}
$$

For this we first split the left-hand side as

$$
\begin{aligned}
n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y)\right] &= n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y)\right] \\
&\quad - n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2\right].
\end{aligned} \tag{3.4}
$$

To estimate the first term we argue in the spirit of [18] and introduce the following notation: for $y \in \mathbb{T}_1$ let $X_s = X_s(y)$ be the constant speed geodesic with $X_0 = y$ and $X_1 = T_n(y)$. We thus have

$$\int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y) = \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_0)$$
$$= \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_s) + \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot (\nabla f_{n,t_n}(X_0) - \nabla f_{n,t_n}(X_s)).$$

For the second term we can estimate

$$n\mathbb{E}\left[\left|\int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot (\nabla f_{n,t_n}(X_0) - \nabla f_{n,t_n}(X_s))\right|\right] \lesssim n\mathbb{E}\left[\left\|\nabla^2 f_{n,t_n}\right\|_\infty W_2^2(\mu_n, 1)\right] \overset{(2.2)\&(2.17)}{\lesssim} 1. \tag{3.5}$$

For the first term we use that $\dot{X}_s \cdot \nabla f_{n,t_n}(X_s) = \frac{d}{ds}[f_{n,t_n}(X_s)]$ to write (recall that $X_1 = T_n(y)$ and $X_0 = y$)

$$\int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_s) = \int_{\mathbb{T}_1} (f_{n,t_n}(X_1) - f_{n,t_n}(X_0))$$
$$= \int_{\mathbb{T}_1} f_{n,t_n} d(\mu_n - 1) = \int_{\mathbb{T}_1} f_{n,t_n}(-\Delta f_{n,0}) = \int_{\mathbb{T}_1} \nabla f_{n,t_n} \cdot \nabla f_{n,0}.$$

Using the semi-group property of the heat kernel we get

$$\int_{\mathbb{T}_1} \nabla f_{n,t_n} \cdot \nabla f_{n,0} = \int_{\mathbb{T}_1} |\nabla f_{n,\frac{t_n}{2}}|^2.$$

We thus conclude that

$$\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y)\right] - n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,\frac{t_n}{2}}|^2\right]\right| \lesssim 1.$$

Plugging this back into (3.4) yields

$$\left| n\mathbb{E}\left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y)\right]\right|$$
$$\lesssim 1 + n\left|\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,\frac{t_n}{2}}|^2\right] - \mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}|^2\right]\right| \lesssim 1$$

where in the last line we used once more (1.3). This concludes the proof of (3.3). □

**Remark 3.1.** *Let us point out that the only source of suboptimality in* (1.2) *lies in* (3.5) *where we use in a crucial way the bound on* $\nabla^2 f_{n,t_n}$ *from Lemma 2.1. Indeed, if we knew (using the notation from the proof of Theorem 1.1) that analogously to* (1.4)

$$n\mathbb{E}\left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} \int_0^1 \left|\dot{X}_s - \nabla f_{n,r_n^2}(X_s)\right|^2 ds d\pi_n\right] \lesssim 1$$

*and analogously to* (1.3),

$$n\mathbb{E}\left[\int_{\mathbb{T}_1\times\mathbb{T}_1}\int_0^1 |\nabla f_{n,t}(X_s)|^2 ds d\pi_n\right] = \frac{|\log t|}{4\pi} + O(\sqrt{t}) \tag{3.6}$$

*then the exact same proof would yield an error term which is of order one.*

**Remark 3.2.** *In a similar direction, let us also observe that arguing as in the proof of Lemma 2.2 we can obtain the analog of* (1.3) *(this explicit computation cannot be done for* (3.6)*)*

$$n\mathbb{E}\left[\int_{\mathbb{T}_1} |\nabla f_{n,t}|^2 d\mu_n\right] = \frac{|\log t|}{4\pi} + O(\sqrt{t}).$$

*Instead of relying on* (1.5) *in the proof of* (1.2)*, we could have thus used* (1.4) *(with* $t = t_n$*). This would however make the proof slightly heavier without affecting the final result.*

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