NON-RIGIDITY OF THE ABSOLUTELY CONTINUOUS PART OF $\mathcal{A}$-FREE MEASURES

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Abstract. We generalize a result by Alberti, showing that, if a first-order linear differential operator $\mathcal{A}$ belongs to a certain class, then any L^1 function is the absolutely continuous part of a measure μ satisfying $\mathcal{A}\mu = 0$. When $\mathcal{A}$ is scalar valued, we provide a necessary and sufficient condition for the above property to hold true and we prove dimensional estimates on the singular part of μ. Finally, we show that operators in the above class satisfy a Lusin-type property.

1. Introduction and main results

In [Alb91, Theorem 3] it was shown that any L^1 vector field is the absolutely continuous component of the distributional gradient of some SBV function. In other words, given $f \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, there exists $u \in L^1(\mathbb{R}^d)$ and a $(d-1)$-rectifiable finite measure σ with values in $\mathbb{R}^d$ such that

$$Du = f + \sigma, \qquad |\sigma|(\mathbb{R}^d) \leq C \|f\|_{L^1(\mathbb{R}^d)},$$

where the equation holds in the sense of distributions and the constant C depends only on d.

Since any distributional gradient Du satisfies $\operatorname{curl} Du = 0$ in the sense of distributions,[1] the condition $Du = f + \sigma$ can be written as $\operatorname{curl}(f + \sigma) = 0$.

The question we address in this paper is whether one can prove a similar result if curl is replaced by some other differential operator:

Question 1.1. *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^d, \mathbb{R}^m) \to C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ be a first-order linear differential operator and $k \in \{1, \dots, d-1\}$. Is it true that for any $f \in L^1(\mathbb{R}^d, \mathbb{R}^m)$ there exists a k-rectifiable measure σ with values in $\mathbb{R}^m$ such that*

$$\mathcal{A}(f + \sigma) = 0, \qquad |\sigma|(\mathbb{R}^d) \leq C \|f\|_{L^1(\mathbb{R}^d)},$$

where the equation holds in the sense of distributions and the constant C depends only on $\mathcal{A}$?

We call *k-balanceable* those operators for which Question 1.1 has a positive answer. While [Alb91, Theorem 3] gives a positive answer to the above question with $k = d - 1$ when $\mathcal{A} = \operatorname{curl}$, it is clear that the answer is negative for at least some first-order linear differential operator. For instance, in the cases $\mathcal{A}\varphi = D\varphi$ or $\mathcal{A}\varphi = (\operatorname{curl}\varphi, \operatorname{div}\varphi)$, it is well-known that a measure satisfying $\mathcal{A}\mu = 0$ is the distribution induced by a smooth function f that satisfies $\mathcal{A}f = 0$ in the classical sense: in these cases, Question 1.1 has a negative answer. In general, the structure of the singular part of a *$\mathcal{A}$-free measure* largely depends on the wave cone associated to $\mathcal{A}$, as was noted in [DR16] (see also [DR19; Arr23; Arr21; ADH+23]).

The first main result of the present work characterizes scalar-valued first-order linear differential operators for which Question 1.1 has a positive answer, namely those operators $\mathcal{A}\colon C^\infty(\mathbb{R}^d, \mathbb{R}^m) \to C^\infty(\mathbb{R}^d, \mathbb{R})$ of the form

$$\mathcal{A}f(x) = \sum_{i=1}^{d} \sum_{j=1}^{m} A_{ij} \partial_i f_j(x) \tag{1.1}$$

for every $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^m)$, where $A = (A_{ij}) \in \mathbb{R}^{d \times m}$. For instance, the divergence operator is represented as above with the choice $A = \mathrm{Id}$.

Theorem 1.2. *Let $\mathcal{A}$ be a scalar-valued first order linear differential operator as in (1.1) with $r \coloneqq \operatorname{rank} A > 0$. The following hold:*

a) $\mathcal{A}$ is k-balanceable for every $k \geq d + 1 - r$;

b) there exists $f \in L^1(\mathbb{R}^d, \mathbb{R}^m)$ such that $\mathcal{A}(f + \sigma) \neq 0$ for every Radon measure σ satisfying[2] *$|\sigma| \perp \mathcal{H}^{d+1-r}$.*

In particular, if $\operatorname{rank} A = 1$, then Question 1.1 has a negative answer for any $k \in \{1, \dots, d-1\}$.

A quick summary of the proof of Theorem 1.2 is given at the beginning of Section 3.

Key words and phrases. $\mathcal{A}$-free measures, Lusin property.

[1] Where $(\operatorname{curl} Du)_{ij} = \partial_j u_i - \partial_i u_j$ for $1 \leq i < j \leq d$.

[2] Here $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure on $\mathbb{R}^d$ and $\sigma \perp \mu$ means that there is some $E \subset \mathbb{R}^d$ such that $\sigma(\mathbb{R}^d \setminus E) = 0$ and $\mu(E) = 0$.

Remark 1.3. Notice that $\mathcal{A}\mu = 0$ does not imply $|\mu| \ll \mathcal{H}^{d+1-r}$, as shown by Example 3.3. Hence one cannot replace part b) of Theorem 1.2 with the stronger

$$\mathcal{A}(f+\sigma) = 0 \implies |\sigma| \ll \mathcal{H}^{d+1-r},$$

because $\mathcal{A}(f+\sigma) = 0$ is equivalent to $\mathcal{A}(f+\sigma+\mu) = 0$.

The second part of the present work concerns vector-valued first order operators, namely those operators of the form $\mathcal{A} = (\mathcal{A}^1, \dots, \mathcal{A}^n)$, where each $\mathcal{A}^i$ is a scalar-valued operators, as for $\mathcal{A} = \operatorname{curl}$. In this framework, answering Question 1.1 becomes substantially harder. The reason is that, in this case, the condition $\mathcal{A}(f+\sigma) = 0$ is a *system* of partial differential equations. Although Theorem 1.2 provides a solution for each equation, in general these measures are not solutions for the other equations of the system.

However, it is possible to build a solution of the system $\mathcal{A}(f+\sigma) = 0$, provided the matrices A^i which define each scalar-valued component $\mathcal{A}^i$ of $\mathcal{A}$ satisfy some algebraic properties which relate each other, see Condition 4.1. Since its formulation is quite technical, we refer the reader to Section 4; here we mention that key parts are a "shared rank-2" property (4.1a) and a "combined antisymmetry" (4.1c) of matrices A^i. Exploiting these algebraic properties, we can produce $(d-1)$-rectifiable measures which balance $\mathcal{A}$. This is precisely the content of the second main result of this paper:

Theorem 1.4. *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^d, \mathbb{R}^m) \to C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ be a fist-order linear differential operator satisfying Condition 4.1; then $\mathcal{A}$ is $(d-1)$-balanceable.*

We show in Proposition 4.7 that $\mathcal{A} = \operatorname{curl}$ satisfies Condition 4.1. In particular, Theorem 1.4 generalizes Alberti's original result. Moreover, it is easily checked (see Proposition 4.8) that a scalar operator of the form (1.1) satisfies Condition 4.1 if and only if $\operatorname{rank} A \geq 2$, consistently with Theorem 1.2.

The third main result of the present paper is a Lusin-type property which generalizes [Alb91, Theorem 1]:

Theorem 1.5. *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^d, \mathbb{R}^m) \to C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ be a fist-order linear differential operator satisfying Condition 4.1 and let $f \in L^1(\Omega, \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^d$ is an open set with finite measure; for every $\varepsilon > 0$ there exist an open set $U \subset \Omega$ and a function $h \in C^\infty(\Omega, \mathbb{R}^m)$ with the following properties:*

$$\mathcal{L}^d(U) < \varepsilon \mathcal{L}(\Omega);$$
$$f = h \qquad \text{in } \Omega \setminus U;$$
$$\mathcal{A}h = 0 \qquad \text{in } \Omega;$$
$$\|h\|_{L^p(\Omega)} \leq C\varepsilon^{\frac{1}{p}-1} \|f\|_{L^p(\Omega)} \qquad \forall p \in [1, +\infty],$$

where the constant C depends only on $\mathcal{A}$.

As above, it is worth remarking that $\mathcal{A} = \operatorname{curl}$ and any scalar operator of the form (1.1) with $\operatorname{rank} A \geq 2$ satisfy Condition 4.1. The function h in the statement of Theorem 1.5 is built, in our proof, via an appropriate mollification of the measure σ given by Theorem 1.4. This choice is clearly not unique.

As previously stated, our result is related to the line of work initiated in [DR16] of determining the structure of $\mathcal{A}$-free measures, that is measures μ satisfying $\mathcal{A}\mu = 0$ in the sense of distributions. While most works in this direction (for instance, [DR16; ADH+19]) focus on determining what singular measures are admissible as singular parts of $\mathcal{A}$-free measures, our results address rigidity and non-rigidity properties of the absolutely continuous part of $\mathcal{A}$-free measures.

A similar question has been investigated in [Arr23]: given any differential operator $\mathcal{B}$, one asks what functions are the absolutely continuous part of $\mathcal{B}u$ for some function u, thus dealing with measures in the *image* of the operator $\mathcal{B}$, while in our work we study the non-rigidity of the absolutely continuous part of measures in the *kernel* of an operator $\mathcal{A}$. By the above mentioned connection between the range of the gradient operator and the kernel of curl, [Arr23] also generalizes [Alb91, Theorem 3], although with different techniques. We moreover mention that, while the approach of [Arr23] is intrinsically limited to adding a $(d-1)$-rectifiable measure σ to a function $f \in L^1$ in order to solve $f + \sigma = \mathcal{B}u$ for some u, our setting allows for more singular rectifiable measures. Compare for instance [Arr23, Corollary 6], which states that divergence is $(d-1)$-balanceable, with Theorem 1.2, which proves that divergence is k-balanceable for every $k \in \{1, \dots, d-1\}$. One should also see [Rai19] for more connections between our point of view and that of [Arr23].

In [AM22], a problem related to Question 1.1 is studied in the framework of *flat* and rectifiable currents: in [AM22, Proposition 3.3] the authors prove that every k-dimensional flat chain with finite mass has the same boundary of a k-rectifiable current.

Finally, we remark that Question 1.1 is worth-studying also if one simply requires $\sigma \perp \mathcal{L}^d$, without any dimensional constraint on σ.

The rest of the paper is structured as follows. In Section 2, we collect most notation used thoughout the paper and we state and prove some simple results concerning first-order linear differential operators. Sections 3, 4 and 5 are dedicated to the proofs of Theorems 1.2, 1.4 and 1.5, respectively.

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2. Notation and reduction of the problem

Here we collect in a brief and mostly schematic form the notation and definitions used throughout the paper.

General notation

$B_r(x)$	open ball of radius r centered at x; when $x=0$, we omit its indication;
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$, namely the set $\{x\in\mathbb{R}^d\colon \lvert x\rvert=1\}$;
$\nu_E(x)$	outer unit normal to a set E at $x\in\partial E$;
$\operatorname{div}_\Sigma X$	tangential divergence of $X\in C^1(\mathbb{R}^d,\mathbb{R}^d)$ with respect to $T_x\Sigma$, where $\Sigma\subset\mathbb{R}^d$ is a k-dimensional manifold of class C^1, namely $\operatorname{div}_\Sigma X(x)=\sum_{j=1}^k \partial_{\tau_j}X(x)\cdot\tau_j$, where $\{\tau_j\}_{j=1}^k$ is an orthonormal basis of $T_x\Sigma$.
$\mu\perp\sigma$	mutually singular Borel measures μ,σ, namely for which there exists $E\subseteq X$ such that $\lvert\mu\rvert(E)=\lvert\sigma\rvert(X\setminus E)=0$;
$\mu\ll\sigma$	the measure μ is absolutely continuous with respect to σ, i.e. $\mu(E)=0$ if $\sigma(E)=0$;
$\mathcal{M}(U,\mathbb{R}^m)$	the space of finite Radon measures on $U\subset\mathbb{R}^d$ with values in $\mathbb{R}^m$;
$\mathcal{M}^k(U,\mathbb{R}^m)$	Space of k-rectifiable finite Radon measures on U with values in $\mathbb{R}^m$, namely the set of measures $\mu\in\mathcal{M}(U,\mathbb{R}^m)$ for which there exist a k-rectifiable set $E\subseteq U$ and $\theta\in L^1_{\mathcal{H}^k}(E,\mathbb{R}^m)$ such that $\mu=\theta\mathcal{H}^k\llcorner E$.

Cubes

$q(Q)$	center of the cube Q;
$\mathcal{P}_\ell(Q)$	dyadic decomposition of $Q=Q_r(x)$ at level ℓ, namely the collection of $2^{d\ell}$ cubes with faces parallel to those of Q, with side length $2^{-\ell}r$;
$\Gamma_{i\pm}$	the face of $Q=(-1,1)^d$ where the exterior unit normal is $\nu_Q=\pm e_i$, namely $\{x\colon x_i=\pm1 \text{ and } \lvert x_j\rvert<1 \text{ for } j\neq i\}$

Operators

$\mathscr{O}(d,m,n)$	space of first order linear differential operators $\mathcal{A}: C^\infty(\mathbb{R}^d,\mathbb{R}^m)\to C^\infty(\mathbb{R}^d,\mathbb{R}^n)$;
$(A^1,\dots,A^n)$	operator $\mathcal{A}\in\mathscr{O}(d,m,n)$ represented by $A^1,\dots,A^n\in\mathbb{R}^{d\times m}$, namely $\mathcal{A}f(x)=\sum_{i,j}\big(A^1_{ij}\partial_i f_j(x),\dots,A^n_{ij}\partial_i f_j(x)\big)$;
$\mathcal{A}\cong\mathcal{B}$	operators $\mathcal{A}=(A^1,\dots,A^n)$ and $\mathcal{B}=(B^1,\dots,B^h)$ such that $\operatorname{span}(A^1,\dots,A^n)=\operatorname{span}(B^1,\dots,B^h)$.

We say that $\mathcal{A}\in\mathscr{O}(d,m,n)$ is *scalar-valued* if $n=1$, otherwise that it is *vector-valued.* We moreover recall the following definition, which we already stated in the introduction.

Definition 2.1 (k-balanceable operator)**.** Given $k\in\mathbb{N}$, we say that a first order linear differential operator $\mathcal{A}\colon C^\infty(\mathbb{R}^d,\mathbb{R}^m)\to C^\infty(\mathbb{R}^d,\mathbb{R}^n)$ is *k-balanceable* if there is $C>0$ such that, for every $f\in L^1(\mathbb{R}^d,\mathbb{R}^m)$, there is a k-rectifiable measure σ in $\mathbb{R}^d$ with values in $\mathbb{R}^m$ such that

$$\mathcal{A}(f+\sigma)=0 \qquad\text{and}\qquad \lvert\sigma\rvert(\mathbb{R}^d)\le C\,\|f\|_{L^1(\mathbb{R}^d)}\,.$$

We now turn our attention to Question 1.1. The main result of this section is the fact that, in order to prove that an operator $\mathcal{A}$ is k-balanceable, it is sufficient to show that one can balance $v_j\mathbf{1}_{E_j}$, for some basis $\{v_j\}_j$ of $\mathbb{R}^m$ and some bounded "almost closed" sets $E_j\subset\mathbb{R}^d$ of positive measure.

Proposition 2.2. *Let $\mathcal{A}\in\mathscr{O}(d,m,n)$ and let $\mathcal{F}=\{v_1,\dots,v_m\}$ be a basis of $\mathbb{R}^m$; let us assume that, for each $v\in\mathcal{F}$, there exists a bounded Borel set $E\subset\mathbb{R}^d$ and $\sigma\in\mathcal{M}^k(\mathbb{R}^d,\mathbb{R}^m)$ such that*

$$\mathcal{L}^d(E)>0,\qquad \mathcal{L}^d(\bar E\setminus E)=0,\qquad \mathcal{A}(v\mathbf{1}_E+\sigma)=0.$$

Then $\mathcal{A}$ is k-balanceable.

Proof. The proof is inspired to the one of [Alb91, Theorem 3]. We need to prove that, for any $f\in L^1(\mathbb{R}^d,\mathbb{R}^m)$, there exists $\mu\in\mathcal{M}^k(\mathbb{R}^d,\mathbb{R}^m)$ such that

$$\mathcal{A}(f+\mu)=0,\qquad \lvert\mu\rvert(\mathbb{R}^d)\le C\,\|f\|_{L^1(\mathbb{R}^d)}\,, \tag{2.1}$$

with C independent of f. Arguing component-wise, we can fix $v \in \mathcal{F}$ and prove the validity of (2.1) for any $f \in L^1(\mathbb{R}^d, \mathrm{span}(v))$.

By $\mathcal{L}^d(\bar{E} \setminus E) = 0$, it follows $\mathbf{1}_{\bar{E}} = \mathbf{1}_E$ $\mathcal{L}^d$-a.e., thus the measure σ in the statement satisfies $\mathcal{A}(v\mathbf{1}_{\bar{E}} + \sigma) = 0$ as well. Therefore we can assume that E is closed. Moreover, up to translations and rescaling, we can assume that $0 \in E$ and that $E \subset B_1$.

Towards (2.1), we aim at defining following sequences:

- a sequence of functions $\{f_\ell\}_{\ell \in \mathbb{N}}$ such that

$$\|f - f_\ell\|_{L^1} \leq 2^{-\ell} \|f\|_{L^1}\,; \tag{2.2}$$

- a sequence of k-rectifiable measures $\{\mu_\ell\}_{\ell \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$ satisfying

$$\mathcal{A}\left(f_\ell + \sum_{i=1}^{\ell} \mu_i\right) = 0 \qquad \text{and} \qquad |\mu_\ell|(\mathbb{R}^d) \leq 2^{-\ell} \tilde{C} \|f\|_{L^1}\,, \tag{2.3}$$

  where $\tilde{C} > 0$ is a constant independent of f.

By (2.2) and (2.3), $\mu \coloneqq \sum_{i=1}^{+\infty} \mu_i$ satisfies (2.1).

- **Definition of f_0, μ_0.**
  We set $f_0 \equiv 0$ and $\mu_0 = 0$. Clearly f_0, μ_0 satisfy (2.2) and (2.3).
- **Definition of f_ℓ, μ_ℓ satisfying (2.2) and (2.3).**
  Let us assume that $f_{\ell-1}$ and $\mu_1, \dots, \mu_{\ell-1}$ are defined and satisfy (2.2) and (2.3). Let us fix an open ball B such that

$$\|f - f_{\ell-1}\|_{L^1(\mathbb{R}^d \setminus B)} \leq 2^{-\ell-1} \|f\|_{L^1(\mathbb{R}^d)}\,. \tag{2.4}$$

If $x \in \mathbb{R}^d$ is a Lebesgue point of $u \coloneqq f - f_{\ell-1}$, since $x + rE \subset B_r(x)$, it holds

$$\limsup_{r \to 0} \fint_{x+rE} |u(y) - u(x)| \,\mathrm{d}\mathcal{L}^d(y) \leq \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x))}{\mathcal{L}^d(x + rE)} \fint_{B_r(x)} |u(y) - u(x)| \,\mathrm{d}\mathcal{L}^d(y) = 0.$$

Thus, for every Lebesgue point $x \in B$ of u, there exists $\rho_x \in \big(0, \mathrm{dist}(x, \partial B)\big)$ such that

$$\frac{1}{\mathcal{L}^d(x + rE)} \int_{x+rE} |u(y) - u(x)| \,\mathrm{d}\mathcal{L}^d(y) < \frac{2^{-\ell-1} \|f\|_{L^1(\mathbb{R}^d)}}{\mathcal{L}^d(B)} \qquad \forall r \in (0, \rho_x). \tag{2.5}$$

Now let us call $\tilde{B}$ the set of Lebesgue points of u in B and let us consider the covering $\mathcal{F}$ of $\tilde{B}$ given by

$$\mathcal{F} \coloneqq \left\{x + rE \colon x \in \tilde{B},\ r \in (0, \rho_x)\right\}.$$

By the version of Vitali covering theorem in [Fed14, Theorem 2.8.17] applied to

$$V = \{(x, x + rE) \colon x + rE \in \mathcal{F}\}, \qquad \delta = \mathrm{diam}, \qquad \phi = \mathcal{L}^d,$$

there exists a countable sub-family $\mathcal{F}_\ell \coloneqq \{F_i = x_i + r_i E\}_{i \in \mathbb{N}} \subset \mathcal{F}$ where the F_i are mutually disjoint, such that

$$0 = \mathcal{L}^d\left(\tilde{B} \setminus \bigcup_{i \in \mathbb{N}} F_i\right) = \mathcal{L}^d\left(B \setminus \bigcup_{i \in \mathbb{N}} F_i\right), \tag{2.6}$$

where the last equality follows from the Lebesgue theorem. We now define

$$\varphi_\ell(x) = \sum_{i \in \mathbb{N}} \big(f(x_i) - f_{\ell-1}(x_i)\big) \mathbf{1}_{F_i}(x).$$

By the definition of $\mathcal{F}$, it holds $\varphi_\ell = 0$ on $\mathbb{R}^d \setminus B$, thus

$$\begin{aligned} \|f - f_{\ell-1} - \varphi_\ell\|_{L^1(\mathbb{R}^d)} &= \|f - f_{\ell-1}\|_{L^1(\mathbb{R}^d \setminus B)} + \|f - f_{\ell-1} - \varphi_\ell\|_{L^1(B)} \\ &\overset{(2.4),(2.6)}{\leq} 2^{-\ell-1} \|f\|_{L^1(\mathbb{R}^d)} + \sum_{i \in \mathbb{N}} \int_{F_i} |u(x) - u(x_i)| \,\mathrm{d}x \\ &\overset{(2.5)}{\leq} 2^{-\ell-1} \|f\|_{L^1(\mathbb{R}^d)} \left(1 + \frac{1}{\mathcal{L}^d(B)} \sum_{i \in \mathbb{N}} \mathcal{L}^d(F_i)\right) \\ &= 2^{-\ell} \|f\|_{L^1(\mathbb{R}^d)}\,, \end{aligned} \tag{2.7}$$

where the last equality is a consequence of (2.6) together with the fact that the sets F_i are mutually disjoint and that $F_i \subseteq B$. Defining $f_\ell \coloneqq f_{\ell-1} + \varphi_\ell$, by (2.7) we obtain immediately $\|f - f_\ell\|_{L^1} \leq 2^{-\ell} \|f\|_{L^1}$, proving (2.2).

In order to show the existence of μ_ℓ satisfying (2.3), we first estimate

$$\|\varphi_\ell\|_{L^1(\mathbb{R}^d)} \leq \|f - f_{\ell-1} - \varphi_\ell\|_{L^1(\mathbb{R}^d)} + \|f - f_{\ell-1}\|_{L^1(\mathbb{R}^d)} \leq 2^{-\ell+2} \|f\|_{L^1(\mathbb{R}^d)}\,, \tag{2.8}$$

where the last inequality follows by (2.7) and by inductive hypothesis.

Fix now $i \in \mathbb{N}$ and let $\bar{\varphi}_i := f(x_i) - f_{\ell-1}(x_i)$. Let also $\tau(x) = x_i + r_i x$ be the affine map such that $\tau(E) = F_i$. We define

$$\mu_{F_i} := r_i^d (\bar{\varphi}_i \cdot v)(\tau)_\# \sigma$$

which, by $\mathcal{A}(v\mathbf{1}_E + \sigma) = 0$ and by linearity, satisfies

$$\mathcal{A}\Big(\bar{\varphi}_i \mathbf{1}_{F_i} + \mu_{F_i}\Big) = r_i^d (\bar{\varphi}_i \cdot v)\, \mathcal{A}\Big((\tau)_\#(v\mathbf{1}_E + \sigma)\Big) = 0. \tag{2.9}$$

Moreover,

$$|\mu_{F_i}|(\mathbb{R}^d) = r_i^d |\bar{\varphi}_i||\sigma|(\mathbb{R}^d) \le C_1 |\bar{\varphi}_i| \mathcal{L}^d(F_i), \tag{2.10}$$

where C_1 depends only on $\mathcal{A}$.

Therefore, defining $\mu_\ell := \sum_{i\in\mathbb{N}} \mu_{F_i}$ and summing (2.9) and (2.10) for $i \in \mathbb{N}$, we obtain

$$\mathcal{A}(\varphi_\ell + \mu_\ell) = 0,$$

$$|\mu_\ell|(\mathbb{R}^d) \le \sum_{i\in\mathbb{N}} |\mu_{F_i}|(\mathbb{R}^d) \le C_1 \sum_{i\in\mathbb{N}} \mathcal{L}^d(F_i)|\bar{\varphi}_i| = C_1 ||\varphi_\ell||_{L^1(\mathbb{R}^d)} \overset{(2.8)}{\le} \tilde{C} 2^{-\ell} \|f\|_{L^1},$$

where $\tilde{C} = 4C_1$ depends only on $\mathcal{A}$. Thus, since $f_\ell = f_{\ell-1} + \varphi_\ell$ and, by inductive hypothesis, it holds $\mathcal{A}(f_{\ell-1} + \sum_{i=1}^{\ell-1} \mu_i) = 0$, we have

$$\mathcal{A}\left(f_\ell + \sum_{i=1}^{\ell} \mu_i\right) = 0.$$

- **Convergence.**
  By (2.3), the sequence of the partial sums of $\sum_{j\in\mathbb{N}} \mu_j$ is a Cauchy sequence in the strong topology of $\mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$. Hence it converges, in the same topology, to a measure $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$ which is k-rectifiable (by strong convergence) and which satisfies, by (2.3),

$$\mathcal{A}(f + \mu) = 0, \qquad |\mu|(\mathbb{R}^d) \le \tilde{C} \|f\|_{L^1}.$$

□

Remark 2.3. We underline the fact that the same proof works in case one merely requires $\sigma \perp \mathcal{L}^d$, again by strong convergence of the series $\sum_{j\in\mathbb{N}} \mu_j$ where each $\mu_j \perp \mathcal{L}^d$.

We conclude this section with a simple remark, whose proof we omit, which we will use in the rest of the paper.

Lemma 2.4. *Let $\mathcal{A} = (A^1, \dots, A^n) \in \mathscr{O}(d, m, n)$, where $A^k \in \mathbb{R}^{d\times m}$ and let $\sigma \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$ be such that $\sigma = g|\sigma|$, where $g \colon \mathbb{R}^d \to \mathbb{R}^m$. Then*

$$\mathcal{A}\sigma = \big(\operatorname{div}(A^1 g|\sigma|), \dots, \operatorname{div}(A^n g|\sigma|)\big).$$

3. Balancing scalar operators

In this section we prove Theorem 1.2. The proof is split in two parts. Both parts rest on the fact that any scalar valued operator $\mathcal{A}$ can be written, up to choosing appropriate bases in $\mathbb{R}^d$ and $\mathbb{R}^m$, as a "lower dimensional" divergence, namely in the form $\operatorname{div}_r f := \partial_1 f_1 + \dots \partial_r f_r$, with $r = \operatorname{rank} A$. Geometrically, given $f \in \mathbb{R}^m$ and a cube Q with a pair $\Gamma_\pm$ of faces perpendicular to Af, the equation $\mathcal{A}(f_1\mathbf{1}_Q + \sigma) = 0$ means that $A\sigma$ "transports" the mass of Γ_+ into that of Γ_-.

In order to prove part a) of Theorem 1.2, given $k \ge d + 1 - r$, we first produce a 1-rectifiable measure μ that transports the mass of a face of a $(d + 1 - k)$-dimensional cube to the mass of the opposite face. This is done in Lemma 3.1, see also Figure 1. Then, the desired measure σ is an appropriate rescaling of $\sigma = \mu \times \mathcal{H}^{k-1} \llcorner (-1, 1)^{k-1}$.

In order to prove part b) of Theorem 1.2, we show that any σ satisfying

$$\operatorname{div}_r(f_1\mathbf{1}_Q + \sigma) = 0$$

admits a disintegration $\lambda \otimes \sigma_{x''}$ where, $\lambda \not\ll \mathcal{H}^{d-r}$ and, for λ-almost every $x'' \in \mathbb{R}^{d-r}$, $\operatorname{div}_r \sigma_{x''}$ is a measure. By [ADH+19], this yields that $\sigma_{x''} \not\ll \mathcal{H}^1$. A coarea-type inequality (Lemma 3.2) provides $|\sigma| \not\ll \mathcal{H}^{d-r+1}$, as desired.

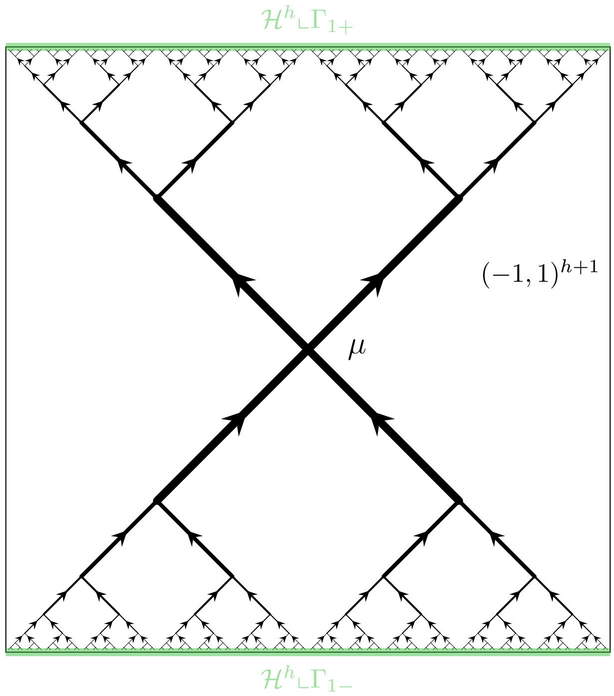

FIGURE 1. The measure μ given by Lemma 3.1.

3.1. **Proof of part a) of Theorem 1.2.** We report the following well-known lemma, illustrated in Figure 1, which we are going to use in the proof of Theorem 1.2.

Lemma 3.1. *Let $h \in \mathbb{N}$. There exists a finite 1-rectifiable measure $\mu \in \mathcal{M}^1((-1,1)^{h+1}, \mathbb{R}^{h+1})$ such that*

$$\operatorname{div} \mu = \mathcal{H}^h \llcorner \Gamma_{1+} - \mathcal{H}^h \llcorner \Gamma_{1-},$$

where $\Gamma_{1\pm} = \{\pm 1\} \times (-1,1)^h$.

Proof. We start by fixing some notation. Throughout the proof, for any two points $p, q \in \mathbb{R}^{h+1}$, we define the measure

$$\mu[p,q] = \frac{q-p}{|q-p|} \mathcal{H}^1 \llcorner [p,q],$$

where $[p,q]$ denotes the segment in $\mathbb{R}^{h+1}$ that joins p and q. Notice that

$$\operatorname{div} \mu[p,q] = \delta_q - \delta_p \qquad \text{and} \qquad \big|\mu[p,q]\big|(\mathbb{R}^{h+1}) = |q-p|. \tag{3.1}$$

We claim there exists $\gamma \in \mathcal{M}^1((-1,1)^{h+1}, \mathbb{R}^{h+1})$ such that

$$\operatorname{div} \gamma = \frac{1}{\mathcal{H}^h(\Gamma_{1+})} \mathcal{H}^h \llcorner \Gamma_{1+} - \delta_0. \tag{3.2}$$

Given γ as above, the measure μ satisfying the conclusion of the Lemma is just $\mu = \mathcal{H}^h(\Gamma_{1+})\big(\gamma - \gamma'\big)$, where γ' is the reflection of γ across the origin, that is the push-forward of γ through the map $\xi(x) \coloneqq -x$.

Let now $Q = (-1,1)^h \subset \mathbb{R}^h$. In order to define γ as in the above claim, we inductively build $\{\gamma_\ell\}_{\ell \in \mathbb{N}}$ as follows.

- We define

$$\gamma_1 \coloneqq \frac{1}{2^h} \sum_{P \in \mathcal{P}_1(Q)} \mu\left[0, \left(\frac{1}{2}, q(P)\right)\right],$$

where $\mathcal{P}_1(Q)$ is the dyadic decomposition of Q at level 1 defined in section 2, $q(P) \in Q$ is the center of P and the notation $(1/2, q(P)) = \{1/2\} \times \{q(P)\} \in (-1,1)^{h+1}$ was used. By (3.1) it holds

$$\operatorname{div} \gamma_1 = -\delta_0 + \frac{1}{2^h} \sum_{P \in \mathcal{P}_1(Q)} \delta_{\left(\frac{1}{2}, q(P)\right)}.$$

- Assume γ_ℓ is defined and satisfies

$$\operatorname{div} \gamma_\ell = -\delta_0 + \frac{1}{2^{\ell h}} \sum_{P \in \mathcal{P}_\ell(Q)} \delta_{(1-2^{-\ell}, q(P))}. \tag{3.3}$$

We define $\gamma_{\ell+1}$ as

$$\gamma_{\ell+1} = \gamma_\ell + \frac{1}{2^{(\ell+1)h}} \sum_{P\in\mathcal{P}_\ell(Q)} \sum_{T\in\mathcal{P}_1(P)} \mu\left[\left(1-2^{-\ell}, q(P)\right), \left(1-2^{-\ell-1}, q(T)\right)\right].$$

It then holds

$$\begin{aligned}
\operatorname{div}\gamma_{\ell+1} &\overset{(3.1)}{=} \operatorname{div}\gamma_\ell + \frac{1}{2^{(\ell+1)h}} \sum_{P\in\mathcal{P}_\ell(Q)} \sum_{T\in\mathcal{P}_1(P)} \left(\delta_{(1-2^{-\ell-1},q(T))} - \delta_{(1-2^{-\ell},q(P))}\right) \\
&\overset{(3.3)}{=} -\delta_0 + \frac{2^h}{2^{(\ell+1)h}} \sum_{P\in\mathcal{P}_\ell(Q)} \delta_{(1-2^{-\ell},q(P))} - \frac{2^h}{2^{(\ell+1)h}} \sum_{P\in\mathcal{P}_\ell(Q)} \delta_{(1-2^{-\ell},q(P))} \\
&\quad + \frac{1}{2^{(\ell+1)h}} \sum_{T\in\mathcal{P}_{\ell+1}(Q)} \delta_{(1-2^{-\ell-1},q(T))} \\
&= -\delta_0 + \frac{1}{2^{(\ell+1)h}} \sum_{T\in\mathcal{P}_{\ell+1}(Q)} \delta_{(1-2^{-\ell-1},q(T))}.
\end{aligned}$$

Notice that, by definition of $\gamma_{\ell+1}$, it holds

$$\begin{aligned}
|\gamma_{\ell+1} - \gamma_\ell|(\mathbb{R}^{h+1}) &\overset{(3.1)}{=} \frac{1}{2^{(\ell+1)h}} \sum_{P\in\mathcal{P}_\ell(Q)} \sum_{T\in\mathcal{P}_1(P)} \left|\left(1-2^{-\ell}, q(P)\right) - \left(1-2^{-\ell-1}, q(T)\right)\right| \\
&= \frac{\sqrt{h+1}}{2^{\ell+1}}.
\end{aligned}$$

Therefore $\{\gamma_\ell\}_{\ell\in\mathbb{N}}$ is a Cauchy sequence in the strong topology of $\mathcal{M}((-1,1)^{h+1},\mathbb{R}^{h+1})$, hence it strongly converges to some $\gamma \in \mathcal{M}^1((-1,1)^{h+1},\mathbb{R}^{h+1})$. Moreover, since

$$\frac{1}{2^{(\ell+1)h}} \sum_{T\in\mathcal{P}_{\ell+1}(Q)} \delta_{(1-2^{-\ell-1},q(T))} \rightharpoonup \frac{1}{\mathcal{H}^h(\Gamma_{1+})} \mathcal{H}^h \llcorner \Gamma_{1+},$$

(3.2) holds true. $\square$

We can now pass to the proof of part a) of Theorem 1.2.

Proof of part a) of Theorem 1.2. Let $\mathcal{A} = (A)$, and let $r := \operatorname{rank} A \geq 1$. The case $r = 1$ (hence $k = 1$) is trivial, since one can choose $\sigma = -f\mathcal{L}^d$. Therefore, we assume $r \geq 2$. Without loss of generality, we may also assume that

$$A = \mathrm{Id}_{r,d,m} := \begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{d\times m}.$$

Indeed, one can always find (for instance, using the singular value decomposition of A) two invertible matrices $U \in \mathbb{R}^{d\times d}$ and $V \in \mathbb{R}^{m\times m}$ such that

$$A = U\,\mathrm{Id}_{r,d,m}\,V^T.$$

Then, letting $\tilde f(x) = (\det U)^{-1} V^T f(Ux)$, the following implication holds:

$$(\mathrm{Id}_{r,d,m})(\tilde f + \tilde\sigma) = 0 \quad\Longrightarrow\quad \mathcal{A}(f+\sigma) = 0,$$

where

$$\sigma := (V^{-1})^T(u_\#\sigma) \quad\text{and}\quad u(x) := Ux.$$

In the following, we denote by $\{f_1,\ldots,f_m\}$ the standard orthonormal basis of $\mathbb{R}^m$ and by $\{e_1,\ldots,e_d\}$ the standard orthonormal basis of $\mathbb{R}^d$. We also let $Q := (-1,1)^d$ be the cube in $\mathbb{R}^d$ with faces parallel to $e_1,\ldots,e_d$. By Proposition 2.2, it is sufficient to prove that, for any $j \in \{1,\ldots,m\}$, there exists $\sigma \in \mathcal{M}^k(\mathbb{R}^d,\mathbb{R}^m)$ such that

$$\mathcal{A}(f_j \mathbf{1}_Q + \sigma) = 0. \tag{3.4}$$

It suffices to consider the case $j = 1$, since the cases $j \in \{2,\ldots,r\}$ are analogous and the cases $j \in \{r+1,\ldots,m\}$ are trivial, as $\sigma = 0$ satisfies (3.4). We remark that, since $D\mathbf{1}_Q = \nu_{\partial Q}\mathcal{H}^{d-1}\llcorner\partial Q$, in the case $j = 1$ (3.4) is equivalent to

$$\mathcal{A}\sigma = \mathcal{H}^{d-1}\llcorner\Gamma_{1+} - \mathcal{H}^{d-1}\llcorner\Gamma_{1-}, \tag{3.5}$$

where Γ_{1+} and Γ_{1-} are the two faces of Q whose exterior unit normal vectors are e_1 and $-e_1$ respectively, as defined in Section 2.

In the rest of the proof, we use the following notation:

$$x = (x', x'') \in Q, \qquad \text{where} \quad x' \in Q' := (-1,1)^{d+1-k}, \quad x'' \in Q'' := (-1,1)^{k-1}.$$

We build σ as follows. By Lemma 3.1, there exists a 1-rectifiable measure $\mu' = \eta'|\mu'| \in \mathcal{M}^1(Q',\mathbb{R}^{d+1-k})$ satisfying

$$\operatorname{div}\mu' = \mathcal{H}^{d-k}\llcorner\Gamma'_{1+} - \mathcal{H}^{d-k}\llcorner\Gamma'_{1-}, \tag{3.6}$$

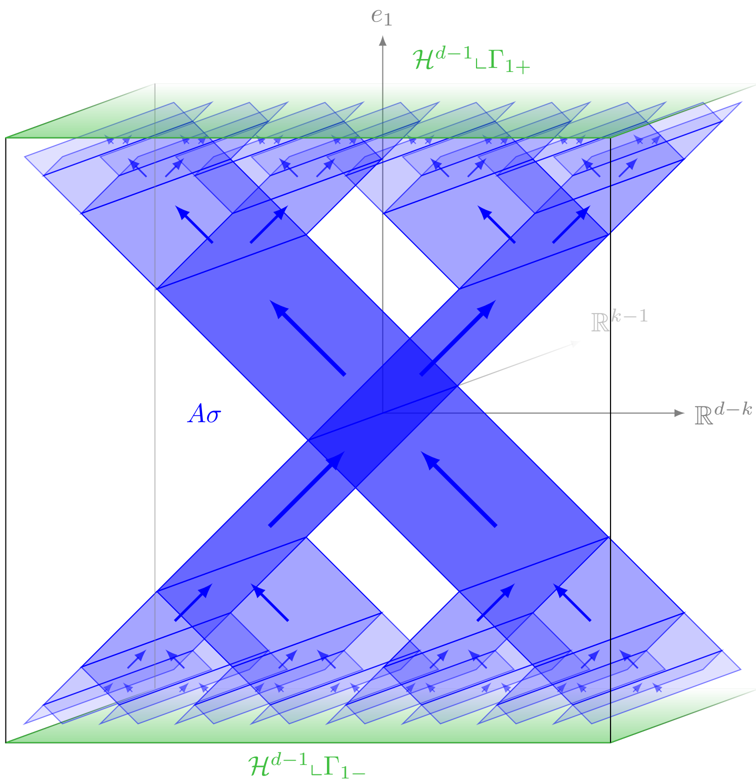

FIGURE 2. The measure $A\sigma$.

where $\Gamma'_{1+}, \Gamma'_{1-}$ are the faces of Q' whose exterior unit normal vectors are e_1 and $-e_1$ respectively. Notice that by assumptions $m \geq r \geq d+1-k$, thus the vector

$$\eta(x) \coloneqq \left(\eta'(x'), 0_{m-(d+1-k)}\right) \in \mathbb{R}^m$$

is well-defined at $|\mu'| \times \mathcal{H}^{k-1}$-almost every x. We may therefore define the measure σ by

$$\sigma \coloneqq \eta(x) \left(|\mu'| \times \mathcal{H}^{k-1} \llcorner Q''\right). \tag{3.7}$$

Notice that $\sigma \in \mathcal{M}^k(Q, \mathbb{R}^m)$ by construction, since it is the product of the 1-rectifiable measure μ' on Q' with the $(k-1)$-rectifiable measure $\mathcal{H}^{k-1} \llcorner Q''$. Moreover, by Lemma 3.1, $|\mu'|(Q')$ depends only on $d-k$, hence there exists $C_2 > 0$ depending only on d such that

$$|\sigma|(Q) \leq C_2$$

for any choice of k. Since $A\eta = (\eta'(x), 0_{k-1})$, it holds

$$A\sigma = (\eta'(x), 0_{k-1}) \left(|\mu'| \times \mathcal{H}^{k-1} \llcorner Q''\right) \tag{3.8}$$

as represented in Figure 2.

We claim that σ satisfies (3.5), which would conclude the proof. In order to do so, let us consider a test function $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{R})$. We have

$$\begin{aligned}
\langle \mathcal{A}\sigma; \varphi \rangle &= -\left\langle \sigma; A^T \nabla\varphi \right\rangle \\
&\overset{(3.7)}{=} -\int_{Q''} \left(\int_{Q'} A\eta(x', x'') \cdot \nabla\varphi(x', x'') \,\mathrm{d}|\mu'|(x') \right) \mathrm{d}\mathcal{H}^{k-1}(x'') \\
&\overset{(3.8)}{=} -\int_{Q''} \left(\int_{Q'} \eta'(x') \cdot \nabla'\varphi(x', x'') \,\mathrm{d}|\mu'|(x') \right) \mathrm{d}\mathcal{H}^{k-1}(x'') \\
&= \int_{Q''} \left\langle \operatorname{div}' \mu'; \varphi(\cdot, x'') \right\rangle \mathrm{d}\mathcal{H}^{k-1}(x'') \\
&\overset{(3.6)}{=} \int_{Q''} \left(\int_{\Gamma'_{1+}} \varphi(x', x'') \,\mathrm{d}\mathcal{H}^{d-k}(x') - \int_{\Gamma'_{1-}} \varphi(x', x'') \,\mathrm{d}\mathcal{H}^{d-k}(x') \right) \mathrm{d}\mathcal{H}^{k-1}(x'') \\
&= \int_{\Gamma_{1+}} \varphi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{\Gamma_{1-}} \varphi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x),
\end{aligned}$$

as claimed. □

3.2. **Proof of part b) of Theorem 1.2.** In this part we use the following result, which may be thought of as a "coarea inequality" for Hausdorff measures. The same proposition, in the case $d = 2$, is stated and proved in [Fal04, Proposition 7.9]. It is easily seen that the same argument shows the following generalization.

Lemma 3.2. *Let $k \in \{1, \dots, d\}$, let $s \in [k, d]$ and let $F \subseteq \mathbb{R}^d$ be a Borel set. Then*

$$\mathcal{H}^s(F) \geq \int_{\mathbb{R}^k} \mathcal{H}^{s-k}(F \cap T_{x''}) \, \mathrm{d}\mathcal{H}^k(x''),$$

where $T_{x''} := \{x = (x', x'') \in \mathbb{R}^d : x' \in \mathbb{R}^{d-k}\}$ for every $x'' \in \mathbb{R}^k$.

Proof of part b) of Theorem 1.2. By the same argument used in the proof of part a) of Theorem 1.2, we may assume that

$$A = \begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{d \times m},$$

where Id_r denotes the r-dimensional identity matrix. We also let $\{f_1, \dots, f_m\}$ and $\{e_1, \dots, e_d\}$ denote the canonical orthonormal basis of $\mathbb{R}^m$ and $\mathbb{R}^d$, respectively. Let us fix $\sigma = \eta|\sigma| \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$ satisfying

$$\mathcal{A}(f_1 \mathbf{1}_Q + \sigma) = 0,$$

where $Q = (-1, 1)^d$. We are going to show that $|\sigma| \not\ll \mathcal{H}^{d+1-r}$, that is

$$\mathcal{H}^{d+1-r}(E) \neq 0 \qquad \text{for every } E \subseteq \mathbb{R}^d \colon |\sigma|(E^c) = 0.$$

As in the first part of the proof, by Lemma 2.4 the equation above means that

$$\operatorname{div}(A\sigma) = \mathcal{H}^{d-1} \llcorner \Gamma_{1+} - \mathcal{H}^{d-1} \llcorner \Gamma_{1-}. \tag{3.9}$$

(3.9) in particular implies $\sigma \neq 0$. In the rest of the proof, we use the notations

$$x = (x', x'') \in \mathbb{R}^d \qquad \text{where} \qquad x' \in \mathbb{R}^r \quad \text{and} \quad x'' \in \mathbb{R}^{d-r},$$

$$\Gamma'_{1\pm} := \{\pm 1\} \times (-1, 1)^{r-1}, \qquad Q'' := (-1, 1)^{d-r}$$

so that $\Gamma_{1\pm} = \Gamma'_{1\pm} \times Q''$. By disintegrating $|\sigma|$ with respect to x'' (see, for instance, [AGS04, Theorem 5.3.1]) we can write

$$\int \varphi(x) \, \mathrm{d}|\sigma|(x) = \int_{\mathbb{R}^{d-r}} \left(\int_{\mathbb{R}^r} \varphi(x', x'') \cdot \big(\eta'(x', x''), \eta''(x', x'')\big) \, \mathrm{d}\sigma_{x''}(x') \right) \mathrm{d}\lambda(x'') \qquad \forall \varphi \in C_c(\mathbb{R}^d, \mathbb{R}^d),$$

where

$$\eta'(x', x'') \in \mathbb{R}^r, \qquad \eta''(x', x'') \in \mathbb{R}^{d-r}, \qquad \lambda \in \mathcal{M}(\mathbb{R}^{d-r}, [0, +\infty]), \qquad \sigma_{x''}(\mathbb{R}^r) = 1 \quad \text{for } \lambda\text{-a.e. } x'' \in \mathbb{R}^{d-r}.$$

We now test (3.9) with $\zeta(x) := \varphi(x')\psi(x'')$, where $\varphi \in C_c^\infty(\mathbb{R}^r, \mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^{d-r}, \mathbb{R})$: taking into account that

$$\nabla \zeta(x) = \big(\nabla' \varphi, 0''\big) \psi(x'') + \big(0', \nabla'' \psi\big) \varphi(x'),$$

we obtain

$$\begin{aligned} \int_{Q''} \psi(x'') \left(\int_{\Gamma'_{1+}} \varphi(x') \, \mathrm{d}\mathcal{H}^{r-1} - \int_{\Gamma'_{1-}} \varphi(x') \, \mathrm{d}\mathcal{H}^{r-1} \right) \mathrm{d}\mathcal{H}^{d-r}(x'') &= \int_{\Gamma_{1+}} \varphi\psi \, \mathrm{d}\mathcal{H}^{d-1} - \int_{\Gamma_{1-}} \varphi\psi \, \mathrm{d}\mathcal{H}^{d-1} \\ &= \int A\eta \cdot \nabla\zeta \, \mathrm{d}|\sigma| \\ &= \int_{\mathbb{R}^{d-r}} \psi(x'') \left(\int_{\mathbb{R}^r} \eta'(x', x'') \cdot \nabla' \varphi(x') \, \mathrm{d}\sigma_{x''}(x') \right) \mathrm{d}\lambda(x''). \end{aligned} \tag{3.10}$$

Choosing $\bar\varphi$ such that $\bar\varphi_{|\Gamma'_{+1}} \equiv 2^{-r+1}$ and $\bar\varphi_{|\Gamma'_{1-}} \equiv 0$, the arbitrariness of ψ in (3.10) implies

$$\mathcal{H}^{d-r} \llcorner Q'' = \beta\lambda, \tag{3.11}$$

where $\beta(x'') := \int_{\mathbb{R}^r} \eta'(x', x'') \cdot \nabla' \bar\varphi(x') \, \mathrm{d}\sigma_{x''}(x')$. (3.11) yields

$$\beta(x'') > 0 \qquad \text{for } \mathcal{H}^{d-r}\text{-a.e. } x'' \in Q''. \tag{3.12}$$

Next, inserting (3.11) in (3.10) and using the arbitrariness of ψ again we deduce that, for any choice of φ, it holds

$$\beta(x'') \left(\int_{\Gamma'_{1+}} \varphi(x') \, \mathrm{d}\mathcal{H}^{r-1} - \int_{\Gamma'_{1-}} \varphi(x') \, \mathrm{d}\mathcal{H}^{r-1} \right) = \int_{\mathbb{R}^r} \eta'(x', x'') \cdot \nabla \varphi(x') \, \mathrm{d}\sigma_{x''}(x') \qquad \text{for } \lambda\text{-a.e. } x'' \in \mathbb{R}^{d-r}.$$

The set of full λ-measure where the above equality holds may in principle depend on the choice of φ. Nevertheless, since $C_c^1(\mathbb{R}^d, \mathbb{R})$ is separable, fixing a countable dense subset X, there exists a set of full λ-measure where the equality is true for any $\varphi \in X$. By density of X, on the same set the equality holds for every $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{R})$, hence

$$\operatorname{div}'\left(\eta' \sigma_{x''}\right) = \beta(x'') \left(\mathcal{H}^{r-1} \llcorner \Gamma'_{1+} - \mathcal{H}^{r-1} \llcorner \Gamma'_{1-}\right) \qquad \text{for } \lambda\text{-a.e. } x'' \in \mathbb{R}^{d-r}.$$

We can read this relation as

$$\operatorname{div}'\left(\eta'(\cdot,x'')\sigma_{x''}+\beta(x'')e_1'\mathbf{1}_{Q'}\right)=0 \qquad \text{for } \lambda\text{-a.e. } x''\in\mathbb{R}^{d-r},$$

where $e_1'=(1,0,\dots,0)\in\mathbb{R}^r$ and $Q'=(-1,1)^r\subset\mathbb{R}^r$. Using notations of [ADH+19], since the 1-dimensional wave cone Λ^1_{div} of the divergence operator satisfies $\Lambda^1_{\mathrm{div}}=\{0\}$, by [ADH+19, Corollary 1.4] it holds

$$\sigma_{x''}\ll\mathcal{H}^1 \qquad \text{for } \lambda\text{-a.e. } x''\in\mathbb{R}^{d-r}. \tag{3.13}$$

Let us now choose $E\subseteq\mathbb{R}^d$ such that $|\sigma|(\mathbb{R}^d\setminus E)=0$; we are going to show that $\mathcal{H}^{d-r+1}(E)\neq 0$, proving that $|\sigma|\not\ll\mathcal{H}^{d-r+1}$.

Let us fix any $F''\subseteq Q''$ with $\lambda(F'')>0$ and call $F:=\mathbb{R}^r\times F''$. Since $\lambda(F'')>0$, by (3.12) we have

$$\mathcal{H}^{d-r}(F'')>0 \tag{3.14}$$

as well. Letting $T_{x''}:=\{x=(x',x'')\in\mathbb{R}^d\colon x'\in\mathbb{R}^r\}$, it holds

$$0<\lambda(F'')=|\sigma|(F)=|\sigma|(E\cap F)=\int_{F''}\sigma_{x''}(E\cap T_{x''})\,\mathrm{d}\lambda(x'')\le\lambda(F''),$$

where the first equality and the last inequality follow by the fact that $\sigma_{x''}$ are probability measures and the second equality is a consequence of $|\sigma|(\mathbb{R}^d\setminus E)=0$. The above relation yields

$$\sigma_{x''}(E\cap T_{x''})=1 \qquad \text{for } \lambda\text{-a.e. } x''\in F''$$

which, by (3.13), produces

$$\mathcal{H}^1(E\cap T_{x''})>0 \qquad \text{for } \lambda\text{-a.e. } x''\in F''.$$

Hence, by (3.11) and (3.12), we infer

$$\mathcal{H}^1(E\cap T_{x''})>0 \qquad \text{for } \mathcal{H}^{d-r}\text{-a.e. } x''\in F''. \tag{3.15}$$

Thus we have

$$\mathcal{H}^{d-r+1}(E)\ge\mathcal{H}^{d-r+1}(E\cap F)\ge\int_{F''}\mathcal{H}^1(E\cap T_{x''})\,\mathrm{d}\mathcal{H}^{d-r}(x'')>0,$$

where the second inequality follows by Lemma 3.2 with $k=d-r$ and $s=d-r+1$, while the last inequality follows by (3.14) and (3.15). ☐

We conclude this section with an example of an $\mathcal{A}$-free measure which is not absolutely continuous with respect to $\mathcal{H}^{d+1-\operatorname{rank}A}$, thus proving the assertion of Remark 1.3.

Example 3.3. Let us fix $2\le r\le d-1$ and $\mathcal{A}=(A)$, where

$$A=\begin{pmatrix}\mathrm{Id}_r & 0\\ 0 & 0\end{pmatrix}\in\mathbb{R}^{d\times m},$$

and let $\mu=\gamma'\mathcal{H}^1\llcorner\gamma$, where γ is a parametrization of a closed regular curve lying in $\operatorname{span}(e_1,e_2)$. It holds

$$\mathcal{A}\mu=\operatorname{div}(A\mu)=\operatorname{div}\mu=0,$$

although $|\mu|\not\ll\mathcal{H}^{d+1-r}$, since $d+1-r\ge 2$.

4. A sufficient condition for balancing vector-valued operators

In this section we state Condition 4.1 and we prove Theorem 1.4. We refer the reader to Section 2 for the relevant notation.

Condition 4.1. There exists a basis $\mathcal{F}$ of $\mathbb{R}^m$ such that, for every $f\in\mathcal{F}$, there exist $\mathcal{B}=(B^1,\dots,B^n)\cong\mathcal{A}$, $g_1,\dots,g_n\in\mathbb{R}^m$, $e\in\mathbb{S}^{d-1}$ and $p\in\{0,\dots,d\}$ with the following properties.

(1) $(B^1f,\dots,B^nf)=(e_1,e_2,\dots,e_p,0,\dots,0)$, where $(e_1,e_2,\dots,e_p)$ is an orthonormal system in $\mathbb{R}^d$.

(2) The following relations are satisfied:

$$B^kg_k=e \quad \forall k\in\{1,\dots,p\}; \tag{4.1a}$$

$$B^\ell g_k\in\operatorname{span}(e_1,\dots,e_p) \qquad \forall k,\ell\in\{1,\dots,n\},\ell\neq k; \tag{4.1b}$$

$$B^\ell g_k\cdot B^hf=-B^\ell g_h\cdot B^kf \qquad \forall \ell,h,k\in\{1,\dots,n\}. \tag{4.1c}$$

Remark 4.2. It clearly follows $p\le n$ as well. Moreover, the conditions (4.1) are not restrictive for $h,k\in\{p+1,\dots,n\}$, since one can choose $g_k=0$ for such k.

Since $\mathcal{A}\cong\mathcal{B}$ yields $\mathcal{A}\mu=0$ if and only if $\mathcal{B}\mu=0$ and by virtue of Proposition 2.2, the following statement imples Theorem 1.4.

Proposition 4.3. *Let $\mathcal{A}$, $f\in\mathcal{F}$ and $\mathcal{B}$ be as in Condition 4.1 and let $B=B_1(0)\subset\mathbb{R}^d$. Then there exists $g\in C^1(\partial B,\mathbb{R}^m)$ such that*

$$\mathcal{B}\big(f\mathbf{1}_B+g\mathcal{H}^{d-1}\llcorner\partial B\big)=0.$$

Therefore, the remaining part of the section is devoted to the proof of Proposition 4.3. We begin with the following algebraic consequences of Condition 4.1.

Remark 4.4 (consequences of Condition 4.1). Taking $h = k$ in (4.1c), one obtains

$$B^\ell g_k \perp B^k f \qquad \forall \ell, k \in \{1, \dots, n\}. \tag{4.2}$$

In particular, choosing $k = \ell$ in the above equation gives

$$e \perp B^k f \qquad \forall k \in \{1, \dots, n\}. \tag{4.3}$$

We moreover record

$$B^k g_h \perp e_k \qquad \forall k \in \{1, \dots, p\},\ \forall h \in \{1, \dots, n\}.$$

Indeed, (4.1c) with $\ell = k \in \{1, \dots, p\}$ yields

$$0 \overset{(4.3)}{=} e \cdot B^h f \overset{(4.1a)}{=} B^k g_k \cdot B^h f \overset{(4.1c)}{=} -B^k g_h \cdot B^k f \overset{(1)}{=} -B^k g_h \cdot e_k \qquad \forall h \in \{1, \dots, n\}.$$

The fact that the measure $\sigma = g\mathcal{H}^{d-1} \llcorner \partial B$ whose existence is claimed by Proposition 4.3 is defined on a smooth surface implies the following characterization of being $\mathcal{A}$-free, which we use in the proof of Proposition 4.3.

Lemma 4.5. *Let $f \in \mathbb{R}^m$ and let $B = B_1(0) \subset \mathbb{R}^d$. Then a vector field $g \in L^1_{\mathcal{H}^{d-1}}(\partial B, \mathbb{R}^m)$ satisfies*

$$\mathcal{A}\big(f\mathbf{1}_B + g\mathcal{H}^{d-1} \llcorner \partial B\big) = 0 \tag{4.4}$$

if and only if, for every $k \in \{1, \dots, n\}$, the following conditions hold:

$$A^k g \perp \nu_{\partial B} \qquad \text{on } \partial B; \tag{4.5a}$$

$$\operatorname{div}_{\partial B} A^k g = -(A^k f) \cdot \nu_{\partial B} \qquad \text{on } \partial B. \tag{4.5b}$$

Proof. Since we can argue component-wise on $\mathcal{A}$, we assume that $\mathcal{A} = (A)$ for some $A \in \mathbb{R}^{d \times m}$.

- **Step 1: if $\mathcal{A}\big(g\mathcal{H}^{d-1} \llcorner \partial B\big)$ is a measure, then (4.5a) holds.**
  Let $\mu := \mathcal{A}\big(g\mathcal{H}^{d-1} \llcorner \partial B\big)$. By Lemma 2.4, we have

$$\mu = \operatorname{div}(Ag\mathcal{H}^{d-1} \llcorner \partial B). \tag{4.6}$$

  Notice that $\operatorname{supp} \mu \subset \operatorname{supp}(g\mathcal{H}^{d-1} \llcorner \partial B) \subset \partial B$. Given $\varphi \in C^\infty(\partial B, \mathbb{R})$ and $\gamma \in C^\infty_c((0,2), \mathbb{R})$ such that $\gamma \equiv 1$ in a neighborhood of 1, let

$$\psi(x) = (|x| - 1)\gamma(|x|)\varphi\left(\frac{x}{|x|}\right).$$

  Since $0 \notin \operatorname{supp} \gamma$, $\psi \in C^\infty_c(\mathbb{R}^d)$ and it holds

$$\nabla\psi(x) = \frac{x}{|x|}\gamma(|x|)\varphi\left(\frac{x}{|x|}\right) + (|x| - 1)\gamma'(|x|)\frac{x}{|x|}\varphi\left(\frac{x}{|x|}\right) + (|x| - 1)\gamma(|x|)\nabla_{\partial B}\varphi\left(\frac{x}{|x|}\right)\frac{1}{|x|}.$$

  On the other hand, $\psi \equiv 0$ on ∂B. Therefore, by (4.6), we have

$$0 = \langle \mu; \psi \rangle = \int_{\partial B} Ag(x) \cdot \nabla\psi(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial B} Ag(x) \cdot \frac{x}{|x|}\varphi\left(\frac{x}{|x|}\right) \mathrm{d}\mathcal{H}^{d-1}(x).$$

  Since φ is arbitrary, we deduce $Ag(x) \cdot \frac{x}{|x|} = 0$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial B$.
- **Step 2: given (4.5a), (4.4) and (4.5b) are equivalent.**
  Since (4.5a) necessarily holds, to finish the proof it is sufficient to assume $Ag(x) \perp \nu_{\partial B}(x)$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial B$ and prove that (4.4) and (4.5b) are equivalent. We have

$$\mathcal{A}(f\mathbf{1}_B) = -\langle Af, \nu_{\partial B} \rangle \mathcal{H}^{d-1} \llcorner \partial B.$$

  Hence, by Lemma 2.4, (4.4) is equivalent to

$$-\langle Af, \nu_{\partial B} \rangle \mathcal{H}^{d-1} \llcorner \partial B = \operatorname{div}(Ag\mathcal{H}^{d-1} \llcorner \partial B).$$

  On the other hand we have

$$\begin{aligned} \operatorname{div}(Ag\mathcal{H}^{d-1} \llcorner \partial B) &= \operatorname{div}_{\partial B}(Ag\mathcal{H}^{d-1} \llcorner \partial B) + \partial_{\nu_{\partial B}}\big(Ag \cdot \nu_{\partial B}\mathcal{H}^{d-1} \llcorner \partial B\big) \\ &\overset{(4.5a)}{=} \operatorname{div}_{\partial B}(Ag)\mathcal{H}^{d-1} \llcorner \partial B, \end{aligned}$$

  obtaining the conclusion. □

Proof of Proposition 4.3. Since by (4.3) we have $e \perp e_k$ for $k \in \{1, \dots, p\}$, we call $e_{p+1} := e$ and (if $p+1 < d$) we choose $\{e_{p+2}, \dots, e_d\}$ so that $\{e_1, \dots, e_d\}$ is an orthonormal basis of $\mathbb{R}^d$; up to a change of coordinates, we may assume $\{e_1, \dots, e_d\}$ is the standard orthonormal basis of $\mathbb{R}^d$.

We define $g \colon \partial B \to \mathbb{R}^m$ as

$$g(x) = -\left(x_{p+1}\sum_{h=1}^{p} x_h g_h\right) + x_{p+1}^2 f \qquad \forall x \in \partial B.$$

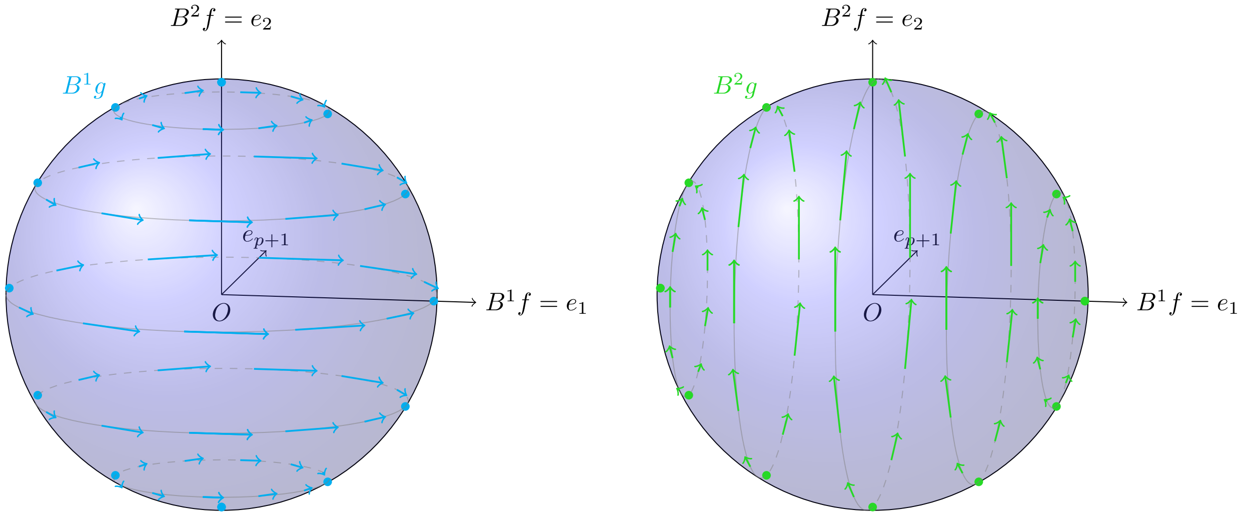

FIGURE 3. illustration of $B^\ell g$ for $d = 3$ and $n = p = 2$. Condition 4.1 implies that $B^1 f \perp B^2 f$ and both these vectors are perpendicular to $e = e_{p+1}$. Notice that, since $d = 3$, Condition 4.1 implies $B^\ell g_h = 0$ for $\ell \neq k$, thus $B^\ell g \in \operatorname{span}(e_\ell, e_{p+1})$.

where $g_1, \dots, g_p$ are given by Condition 4.1. $B^\ell g$ is illustrated in Figure 3.

By Lemma 4.5 it suffices to show that g satisfies (4.5a) and (4.5b) for every $k \in \{1, \dots, n\}$.

a) Since $\nu_{\partial B}(x) = x$, for every $\ell \in \{1, \dots, n\}$ we have to show that

$$B^\ell g(x) \cdot x = 0 \qquad \forall x \in \partial B.$$

If $\ell \in \{1, \dots, p\}$, then (4.1a) and (4.1b) yield $B^\ell g(x) \in \operatorname{span}(e_1, \dots, e_{p+1})$, hence

$$\begin{aligned} B^\ell g(x) \cdot x &= -x_{p+1} \left(\sum_{k=1}^{p} x_k B^\ell g_k + x_{p+1} B^\ell f \right) \cdot \left(\sum_{h=1}^{p+1} x_h e_h \right) \\ &= -x_{p+1} \left(\sum_{k=1}^{p} x_k B^\ell g_k + x_{p+1} B^\ell f \right) \cdot \left(\sum_{h=1}^{p} x_h B^h f + x_{p+1} B^\ell g_\ell \right) \\ &= 0, \end{aligned}$$

where the last equality is obtained by computing scalar products and by exploiting point 1 of Condition 4.1, (4.1b), (4.1c) and (4.2).

If $\ell \in \{p+1, \dots, n\}$, then $B^\ell f = 0$ and $B^\ell g(x) \in \operatorname{span}(e_1, \dots, e_p)$ by (4.1b). Similar (but easier) computations as above produce $B^\ell g(x) \cdot x = 0$.

b) Now we have to prove (4.5b), namely that for every $\ell \in \{1, \dots, n\}$ it holds

$$-\operatorname{div}_{\partial B} B^\ell g(x) = B^\ell f \cdot x \qquad \forall x \in \partial B.$$

We first assume $\ell \in \{1, \dots, p\}$ and, for simplicity of notations, that $\ell = 1$. Hence $B^1 f \cdot x = x_1$ for every $x \in \partial B$. We compute

$$\begin{aligned} -\operatorname{div}_{\partial B} B^1 g(x) &= P_{T_x \partial B} \nabla x_{p+1} \cdot \left(x_1 B^1 g_1 + x_2 B^1 g_2 + \cdots + x_p B^1 g_p - x_p B^1 f \right) \\ &\quad + x_{p+1} \operatorname{div}_{\partial B} \left(x_1 B^1 g_1 + x_2 B^1 g_2 + \cdots + x_p B^1 g_p - x_p B^1 f \right) \\ &\overset{(4.5a)}{=} e_{p+1} \cdot \left(x_1 e_{p+1} + x_2 B^1 g_2 + \cdots + x_p B^1 g_p - x_p e_1 \right) \\ &\quad + x_{p+1} \operatorname{div} \left(x_1 e_{p+1} + x_2 B^1 g_2 + \cdots + x_p B^1 g_p - x_p e_1 \right) \\ &\overset{(4.1b)}{=} x_1 + x_{p+1} \Big(e_1 \cdot e_{p+1} + e_2 \cdot B^1 g_2 + \cdots + e_p \cdot B^1 g_p - e_{p+1} \cdot e_1 \Big) \\ &\overset{(4.2)}{=} x_1, \end{aligned}$$

as desired. On the other hand, if $\ell \in \{p+1,\dots,n\}$, then $B^\ell f = 0$, hence $B^\ell f \cdot x = 0$. Moreover, with similar computations as above,

$$\begin{aligned}\operatorname{div}_{\partial B} B^\ell g(x) &= e_{p+1} \cdot \sum_{k=1}^{p} x_k B^\ell g_k + x_{p+1} \operatorname{div} \sum_{k=1}^{p} x_k B^\ell g_k \\ &\overset{(4.1b)}{=} 0 + x_{p+1} \sum_{k=1}^{p} e_k \cdot B^\ell g_k \\ &\overset{(4.2)}{=} 0. \end{aligned}$$

□

Remark 4.6. With more cumbersome computations, one can repeat the proof of Proposition 4.3 to produce a measure supported on the boundary of a cube whose orientation depends on $\mathcal{B}$.

4.1. $\mathcal{A} = \operatorname{curl}$ **satisfies Condition 4.1.** In this section we prove that $\mathcal{A} = \operatorname{curl}$ satisfies Condition 4.1, thus obtaining a proof of [Alb91, Theorem 3]. For $\mathcal{A} = \operatorname{curl}$, we have $d = m$ and $\operatorname{curl} = (A^{ij})_{1\le i<j\le d}$, where the matrices A^{ij} form the standard basis of antisymmetric matrices, that is

$$(A^{ij})_{k\ell} = \begin{cases} -1 & \text{if } (k,\ell) = (i,j) \\ 1 & \text{if } (k,\ell) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $n = \frac{d(d-1)}{2}$ and, for $i < j$, A^{ij} satisfies

$$A^{ij} f_j = -f_i, \qquad A^{ij} f_i = f_j, \qquad A^{ij} f_\ell = 0 \quad \forall \ell \in \{1,\dots,m\} \setminus \{i,j\}. \tag{4.7}$$

Proposition 4.7. $\mathcal{A} = \operatorname{curl}$ *satisfies Condition 4.1 with* $\mathcal{B} = \mathcal{A}$ *and* $\mathcal{F}$ *being the standard basis of* $\mathbb{R}^d$.

Proof. Let $\mathcal{F} = \{f_1,\dots,f_d\}$ be the standard basis of $\mathbb{R}^d$. Let us fix $j \in \{1,\dots,d\}$ and let $f = f_j$. For simplicity of notations, we assume that $j = d$, but the argument is exactly the same for other indices. With the above notations, we set

$$p = d-1, \qquad \{B^k\}_{k=1}^{d-1} \coloneqq \{A^{kd}\}_{k=1}^{d-1}, \qquad \{B^k\}_{k=p+1}^{n} \coloneqq \{A^{ih}\}_{1\le i<h\le d-1},$$

$$e = f_d, \qquad g_k = \begin{cases} f_k & \text{if } k \in \{1,\dots,d-1\} \\ 0 & \text{if } k \in \{d,\dots,n\}. \end{cases}$$

We are going to show that these choices satisfy Condition 4.1.

(1) The set of matrices $\{B^k\}_{k=1}^{d-1}$ are such that

$$B^k f_d = A^{kd} f_d = -f_k \qquad \forall k \in \{1,\dots,d-1\}.$$

Moreover

$$A^{ih} f_d = 0 \qquad \forall 1 \le i < h \le d-1.$$

Hence the matrices $\{B^k\}_{k=1}^n$ satisfy point 1 of Hypothesis 4.1.

(2) We now prove that the conditions of point 2 are satisfied.
   (a) Since

$$B^k g_k = A^{kd} f_k = f_d = e \qquad \forall k \in \{1,\dots,d-1\},$$

   (4.1a) is satisfied.
   (b) Since $g_k = 0$ for $k \ge d$, we only have to check the validity of (4.1b) for $k \le d-1$. In this case, $B^k = A^{kd}$ and $g_k = f_k$. By (4.7) it holds

$$B^\ell g_k = B^\ell f_k \notin \operatorname{span}(f_1,\dots,f_{d-1}) \iff B^\ell f_k = f_d \iff B^\ell = A^{kd} \iff \ell = k.$$

   Thus (4.1b) holds true.
   (c) It remains to show (4.1c). In order to do so, we first notice that, if $k \ge d$ (or $h \ge d$), then $g_k = 0$ and $B^k f_d = 0$, thus

$$B^\ell g_k \cdot B^h f_d = 0 = -B^\ell g_h \cdot B^k f_d.$$

   If $h \le k \le d-1$, by (4.7) it holds $B^h f_d = A^{hd} f_d = -f_k$ and $B^k f_d = A^{kd} f_d = -g_k$, thus

$$B^\ell g_k \cdot B^h f_d = B^\ell f_k \cdot (-f_h) = -B^\ell f_h \cdot (-f_k) = -B^\ell g_h \cdot B^k f_d,$$

   where the third equality follows by the fact that B^ℓ is antisymmetric. □

4.2. **Comparing scalar and vector case.** It is worth spending a few words on the relation between Theorem 1.2 and Theorem 1.4. We start with the following remarks.

- Theorem 1.2 is sharp in the sense that the condition $\operatorname{rank} A \geq 2$ is necessary and sufficient for positively answer Question 1.1 for scalar operators; moreover Theorem 1.2 sharply characterizes "how singular" the measure σ can be in order to have $\mathcal{A}(f+\sigma) = 0$, depending on $\operatorname{rank} A$.
- On the other hand, Condition 4.1 is sufficient to positively answer Question 1.1 and the measure σ built in the proof of Theorem 1.4 is $(d-1)$-rectifiable.

As mentioned in Section 1, Condition 4.1 is satisfied by any balanceable scalar-valued operator:

Proposition 4.8. *Let $\mathcal{A} = (A) \in \mathscr{O}(d, m, 1)$ be a scalar operator. Then $\mathcal{A}$ satisfies Condition 4.1 if and only if* $\operatorname{rank} A \geq 2$.

Proof. Let us assume $\operatorname{rank} A \geq 2$, let us fix any basis $\mathcal{F} = \{f_1, \ldots, f_m\}$ of $\mathbb{R}^m$ and $f \in \mathcal{F}$. If $Af = 0$, then the properties defined in Condition 4.1 are trivially satisfied by choosing any $g_1 \in \mathbb{R}^m$ such that $|Ag_1| = 1$. Thus we can assume that $Af \neq 0$; then, for some $\lambda \in \mathbb{R} \setminus \{0\}$,

$$(\lambda A)\, f =: e_1 \in \mathbb{S}^{d-1}.$$

Moreover, since $\operatorname{rank} A \geq 2$, there exists $g_1 \in \mathbb{R}^m$ such that

$$e_1 \perp (\lambda A)\, g_1 =: e \in \mathbb{S}^{d-1}.$$

Setting $B \coloneqq \lambda A$, these are precisely part 1 and (4.1a) of Condition 4.1. Since $n = 1$, (4.1b) is clearly void, while (4.1c) follows from $e_1 \cdot e = 0$.

Conversely, if $\mathcal{A} = (A)$ satisfies Condition 4.1, then $\operatorname{rank} A \geq 1$ since $e \in \mathbb{S}^{d-1} \cap \operatorname{Im} A$. Therefore there exists $f \in \mathcal{F}$ such that $Af \neq 0$. For this f, there exists $B = \lambda A$ which satisfies the properties of Condition 4.1. In particular, $Bf =: e_1 \in \mathbb{S}^{d-1}$; moreover there exists $g_1 \in \mathbb{R}^m$ such that $Bg_1 =: e \in \mathbb{S}^{d-1} \cap (Bf)^\perp$ by (4.1a) and (4.1c). This implies $\operatorname{rank} A \geq 2$. □

Remark 4.9. By virtue of Proposition 4.8, Theorem 1.4 provides a positive answer to Question 1.1 for scalar operators $\mathcal{A} = (A)$ with $\operatorname{rank} A \geq 2$: every $f \in L^1(\mathbb{R}^d, \mathbb{R}^m)$ is $\mathcal{A}$-balanceable with a $(d-1)$-rectifiable measure. Since $\operatorname{rank} A \geq 2$, it is also possible to choose $k = d + 1 - 2 = d - 1$ in Theorem 1.2, which states that we can balance any L^1 function with a $(d-1)$-rectifiable measure.

It is worth noting that the $(d-1)$-rectifiable measure produced in the proof of Theorem 1.2 and the one whose existence is stated in Remark 4.6 (which is produced as in the proof of Proposition 4.3) are different, in general. Indeed, let us consider $v \in \mathbb{R}^m$ and the characteristic function of a cube $v\mathbf{1}_Q$. The measure built in the proof of Theorem 1.2 charges the interior of Q, while the one given in Remark 4.6 is supported on ∂Q.

5. A Lusin-type property

In this section we prove Theorem 1.5. The proof is based on Lemma 5.1 below. With this result at hand, the proof of [Alb91, Theorem 1] can be repeated without any modification, except from replacing Du with h such that $\mathcal{A}h = 0$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set with finite measure. For every continuous and bounded function $f : \Omega \to \mathbb{R}^m$ and every $\varepsilon, \eta > 0$, there exist a compact set K and $h \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that*

$$\mathcal{L}^d(\Omega \setminus K) < \varepsilon \mathcal{L}^d(\Omega); \tag{5.1a}$$

$$|f - h| < \eta \qquad \text{on } K; \tag{5.1b}$$

$$\mathcal{A}h = 0 \qquad \text{in } \Omega; \tag{5.1c}$$

$$\|h\|_{L^p(\Omega)} \leq C_1 \varepsilon^{\frac{1}{p}-1} \|f\|_{L^p(\Omega)} \qquad \forall p \in [1, +\infty], \tag{5.1d}$$

where the constant C_1 depends only on $\mathcal{A}$.

Proof. Let $\mathcal{F} \coloneqq \{f_1, \ldots, f_m\}$, $B^1, \ldots, B^n$, $e_1, \ldots, e_p$ and $e \coloneqq e_{p+1}$ be as in Condition 4.1. Up to changing coordinates, we may assume that $e_1, \ldots, e_{p+1}$ are the first $p + 1$ vectors of the standard orthonormal basis of $\mathbb{R}^d$ and that $|f_i| = 1$ for all $f_i \in \mathcal{F}$.

- **Step 1.**

We first show that, given any $v \in \mathcal{F}$ and $\beta > 0$, we may find $u \in C_c^\infty(B_1, \mathbb{R}^m)$ such that

$$u \equiv v \text{ in } B_{1-\beta},$$

$$\mathcal{A}u = 0, \tag{5.2}$$

$$||u||_{L^p(B_1)} \leq C_1 \beta^{1/p-1} \qquad \forall p \in [1, +\infty], \tag{5.3}$$

where C_1 is a constant depending only on $\mathcal{A}$.

We proceed as follows. By Proposition 4.3, there exists $g : \partial B_1 \to \mathbb{R}^m$ such that

$$\mathcal{A}(v\mathbf{1}_{B_1} + g\mathcal{H}^{d-1} \llcorner \partial B_1) = 0. \tag{5.4}$$

We define

$$u(x) := \varphi(|x|)v - |x|\varphi'(|x|)g\left(\frac{x}{|x|}\right),$$

where $\varphi \in C_c^\infty([0,1))$ is such that $\varphi \equiv 1$ on $[0, 1-\beta]$ and $||\varphi'||_{L^\infty} \le \frac{2}{\beta}$. Notice that $u \equiv v$ on $B_{1-\beta}$ and $u \in C_c^\infty(B_1, \mathbb{R}^m)$. In order to prove (5.2), by Lemma 2.4 we need to show that $\operatorname{div}(B^k u) = 0$ for all $k \in \{1, \dots, n\}$. We first consider any test function $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and we compute

$$\begin{aligned}
\left\langle \operatorname{div}\left(|x|B^k g\left(\frac{x}{|x|}\right)\right); \chi\right\rangle &= -\left\langle |x|B^k g\left(\frac{x}{|x|}\right); \nabla\chi\right\rangle \\
&= -\int_0^{+\infty} t\int_{\partial B_t} \nabla\chi(x)\cdot B^k g\left(\frac{x}{t}\right) \mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}t \\
&= -\int_0^{+\infty} t^d\int_{\partial B_1} \nabla\chi(tx)\cdot B^k g(x) \,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}t \\
&= \int_0^{+\infty} t^{d-1}\left\langle \operatorname{div}\left(B^k g\mathcal{H}^{d-1}\llcorner\partial B_1\right); \chi(t\,\cdot)\right\rangle \mathrm{d}t \\
&\overset{(5.4)}{=} \int_0^{+\infty} t^{d-1}\int_{\partial B_1} \left(B^k v\cdot\nu_{B_1}(x)\right)\chi(tx)\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}t \\
&= \int_0^{+\infty} \int_{\partial B_t} \left(B^k v\cdot\nu_{B_t}(x)\right)\chi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}t \\
&= \left\langle B^k v\cdot\frac{x}{|x|}; \chi\right\rangle.
\end{aligned}$$

We therefore have

$$\operatorname{div}(B^k u(x)) = \varphi'(|x|)\frac{x}{|x|}\cdot B^k v - |x|\varphi''(|x|)\frac{x}{|x|}\cdot B^k g\left(\frac{x}{|x|}\right) - \varphi'(|x|)B^k v\cdot\frac{x}{|x|} = 0,$$

where in the second equality the fact that $B^k g \perp \nu_{B_1}$ (by (4.5a)) was used. Thus $\mathcal{A}u = 0$ by Lemma 2.4 and (5.2) is proven. In order to prove (5.3), we use the fact that $\|g\|_\infty \le C$ for some C depending only on $\mathcal{A}$, hence on $B_1 \setminus B_{1-\beta}$ it holds

$$|u(x)| \le C'(1 + |\varphi'(x)|) \le \frac{C''}{\beta}$$

for some C', C'' large enough depending only on $\mathcal{A}$. In particular,

$$\|u\|_{L^p(B_1)} \le C_1\beta^{1/p-1} \qquad \forall p \in [1, +\infty],$$

where C_1 depends only on $\mathcal{A}$.

- **Step 2.**
We now proceed with the proof of the result. Arguing component-wise, we may assume without loss of generality that there exists $v \in \mathcal{F}$ such that $f(x) \in \operatorname{span} v$ for every x. Let us fix a compact set $K' \subset \Omega$ such that $\mathcal{L}^d(\Omega \setminus K') < \frac{\varepsilon}{2}\mathcal{L}^d(\Omega)$. By uniform continuity of f on compact subsets of Ω there exists $\delta > 0$ such that, if $x \in K'$, then $B_{2\delta}(x) \subset \Omega$ and

$$|f(y) - f(z)| < \eta \qquad \forall y, z \in B_\delta(x). \tag{5.5}$$

By Vitali's covering theorem ([Fed14, Theorem 2.8.17]), there exists a countable collection of mutually disjoint open balls $\mathcal{F} := \{B_i := B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ with $r_i \in (0, \delta]$ and $x_i \in K'$ for every i such that

$$\mathcal{L}^d\left(K' \setminus \bigcup_{B_i\in\mathcal{F}} B_i\right) = 0.$$

Given $B_i = B_{r_i}(x_i) \in \mathcal{F}$, we let $\widehat{B}_i := \overline{B_{(1-\beta)r_i}(x_i)}$, where $\beta := \frac{\varepsilon}{2d} \in (0,1)$, and we define

$$K := \bigcup_{i\in\mathbb{N}} \widehat{B}_i.$$

Notice that, by definition of β, it holds

$$\mathcal{L}^d\left(B_1 \setminus \overline{B_{1-\beta}}\right) \le d\beta\mathcal{L}^d(B_1) = \frac{\varepsilon}{2}\mathcal{L}^d\left(B_1\right),$$

therefore

$$\begin{aligned}\mathcal{L}^d(\Omega\setminus K) &\le \mathcal{L}^d(\Omega\setminus K') + \mathcal{L}^d(K'\setminus K)\\ &\le \frac{\varepsilon}{2}\mathcal{L}^d(\Omega) + \mathcal{L}^d\left(\bigcup_{i\in\mathbb{N}}\left(B_i\setminus\widehat{B}_i\right)\right)\\ &\le \frac{\varepsilon}{2}\mathcal{L}^d(\Omega) + \frac{\varepsilon}{2}\mathcal{L}^d\left(\bigcup_{i\in\mathbb{N}} B_i\right)\\ &\le \varepsilon\mathcal{L}^d(\Omega).\end{aligned} \tag{5.6}$$

Next, for all $B_i\in\mathcal{F}$, define

$$a_i := \fint_{B_i} f\,\mathrm{d}\mathcal{L}^d;$$

by (5.5), for every $B_i\in\mathcal{F}$ and every $y\in B_i$ it holds

$$|a_u - f(y)| < \eta. \tag{5.7}$$

Let now

$$h(x) := \sum_{i\in\mathbb{N}} (a_i\cdot v)\, u\left(\frac{x-x_i}{r_i}\right)\mathbf{1}_{B_i}(x), \tag{5.8}$$

where u was defined in Step 1. Then (5.1a) and (5.1b) follow from (5.6) and (5.7), respectively. Moreover, since $\operatorname{supp} u\subset B_1$, by linearity it holds

$$\mathcal{A}h = 0.$$

Lastly, (5.1d) follows immediately from (5.8) and (5.3) in the case $p=+\infty$. For $p\in[1,+\infty)$, we have

$$\begin{aligned}\|h\|_{L^p(\Omega)} &\overset{(5.8)}{\le} \sum_{i\in\mathbb{N}} |a_i|\,\|u\|_{L^p}\left(\mathcal{L}^d(B_i)\right)^{\frac{1}{p}}\\ &\le \|u\|_{L^p}\sum_{i\in\mathbb{N}}\left(\mathcal{L}^d(B_i)\right)^{\frac{1}{p}-1}\int_{B_i}|f|\\ &\le \|u\|_{L^p}\sum_{i\in\mathbb{N}}\|f\|_{L^p(B_i)}\\ &\le C_1\varepsilon^{\frac{1}{p}-1}\|f\|_{L^p(\Omega)}\end{aligned}$$

where in the third inequality we applied Hölder inequality and in the last one we used $\beta=\frac{\varepsilon}{2d}$ and (5.3). ☐

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