GLOBAL L^∞ GRADIENT ESTIMATES FOR SOLUTIONS TO A CERTAIN DEGENERATE ELLIPTIC EQUATION

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Abstract. In view of applications to the study of regularity properties of minimizers for a continuous model of transportation, which is a kind of divergence-constrained optimization problem, we prove a global L^∞ gradient estimate for solutions of an elliptic equation, whose ellipticity constants degenerate at every point where $|\nabla u| \leq \delta$, with $\delta > 0$. The exposition is as self-contained as possible.

1. Introduction

In this paper we provide a complete and self-contained proof of the global Lipschitz regularity of weak solutions to the Neumann boundary value problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\nabla \mathcal{H}^*(\nabla u)) & = & f, & \text{in } \Omega, \\ \langle \nabla \mathcal{H}^*(\nabla u), \nu \rangle & = & 0, & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\mathcal{H}^*$ is a convex (but not necessarily strictly convex) increasing function having $q-$growth, with $1 < q < \infty$, and such that its Hessian matrix $D^2\mathcal{H}^*$ has degenerating eigenvalues in the region $\{x\,:\,|\nabla u(x)| \leq \delta\}$, while ν is the outer normal versor to $\partial\Omega$. Problem (1.1) has a clear variational nature, as far as it can be considered as the Euler-Lagrange equation of the functional

$$\mathfrak{H}(u) = \int_\Omega \mathcal{H}^*(\nabla u(x))\,dx - \int_\Omega f(x)\,u(x)\,dx,\ \ u \in W^{1,q}_\diamond(\Omega),$$

the space $W^{1,q}_\diamond(\Omega)$ being a subspace of the standard Sobolev space $W^{1,q}(\Omega)$, consisting of functions with zero-mean on Ω (any other subspace obtained quotienting modulo non-zero constants would do the same job). In particular, solutions to (1.1) are always intended in the usual weak sense, i.e.

$$\int_\Omega \langle \nabla \mathcal{H}^*(\nabla u(x)), \nabla\varphi(x)\rangle\,dx = \int_\Omega f(x)\,\varphi(x)\,dx, \quad \text{for every } \varphi \in W^{1,q}(\Omega),$$

and they are minimum points of the aforementioned functional $\mathfrak{H}$. We warn the reader that, for the sake of clearness of the exposition, we avoid the temptation of pursuing too far our investigation: in particular, our function $\mathcal{H}^*$ has a special, but significant structure, given by

$$\mathcal{H}^*(z) = \frac{1}{q}(|z| - \delta)_+^q,\ \ z \in \mathbb{R}^N,$$

where $\delta > 0$, $1 < q < \infty$ and $(\,\cdot\,)_+$ stands for the positive part, so that the corresponding equation is given by

$$-\mathrm{div}\left((|\nabla u| - \delta)_+^{q-1} \frac{\nabla u}{|\nabla u|} \right) = f. \tag{1.2}$$

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As it will be apparent from the arguments below, the same computations will work to cover more general situation (for example, the case of $\mathcal{H}^*$ depending also on the x variable). On the datum f we will make a minimal (optimal on the scale of Lebesgue spaces) regularity assumption: namely

$$f \in L^{N+\alpha}(\Omega),$$

where α could be any number strictly greater than 0 (with $\alpha = 0$ the result would be clearly false). We remark that for the equation (1.2), in the case of dimension $N = 2$ an interesting (local) continuity result for the whole block $\nabla\mathcal{H}^*(\nabla u)$ has been proven by Santambrogio and Vespri in [15].

First of all, we want to give some motivations for addressing regularity issues of an equation with this type of degeneracy:

for example, the authors of [2] have arrived to this regularity problem, starting from a model of optimal transportation with congestion effects, proposed by M. J. Beckmann in the 50s (see [1]). This is expressed as a divergence-constrained minimization problem of the kind

$$\min \left\{ \int_\Omega \mathcal{H}(\phi(x))\, dx \,:\, \operatorname{div}\phi = \mu_0 - \mu_1,\ \langle \phi, \nu\rangle = 0 \text{ on } \partial\Omega \right\}, \tag{1.3}$$

with $\mathcal{H}$ strictly convex, which is exactly the dual problem of maximizing the energy $-\mathfrak{H}$ over $W^{1,q}_\diamond(\Omega)$, provided that $\mathcal{H}^*$ is the Legendre-Fenchel conjugate of $\mathcal{H}$ (indeed, the $*$ symbol was not by chance...). In this model, μ_0 and μ_1 are positive measures with the same mass (usually normalized to be probability measures), standing for the centers of production and consumption of a given commodity, while ϕ describes the transportation activities in the economically balanced (with respect to the commodity under consideration) region Ω. Then in [2] some regularity results were needed on the optimizer ϕ_0 for (1.3), in order to apply the DiPerna-Lions theory of flows (see [5]): indeed, the integral curves of the vector field

$$\widehat{\phi}_0 = \frac{\phi_0}{(1-t)\mu_0 + t\mu_1}, \tag{1.4}$$

give rise to a so-called *Wardrop equilirium* for a given congestion function g, provided that $g(x) = |\nabla\mathcal{H}(x)|$ (see [2], Section 3, for more details). Roughly speaking, a Wardrop equilibrium is nothing but a probability measure Q over the space of absolutely continuous paths in Ω, which gives full mass to the set of geodesics for the congested metric d_Q induced by the traffic distribution (here represented by Q) itself and given by

$$d_Q(x,y) = \inf_{\sigma\in \mathrm{Lip}([0,1];\Omega)} \left\{ \int_0^1 g(i_Q(\sigma(t)))\, |\sigma'(t)|\, dt \,:\, \sigma(0) = x,\ \sigma(1) = y \right\},\ x, y \in \Omega,$$

where i_Q is the *traffic intensity* arising from Q, so that for every $x \in \Omega$ the quantity $i_Q(x)$ stands for the total cumulated traffic at the point x. One of the main issue solved by [2] is exactly the fact that one can prove the existence of a Wardrop equilibrium Q_0 just by looking at the convex optimization problem (1.3) and then taking Q_0 to be the probability measure concentrated on the flow of (1.4) starting at μ_0. Clearly, in order to be able to define this flow, one should prove that ϕ_0 is regular enough (Sobolev regularity and boundedness of ϕ_0 was what we needed in [2]): looking at primal-dual optimality conditions, it is a straightforward fact to see that there must hold

$$\phi_0 = \nabla\mathcal{H}^*(\nabla u_0), \tag{1.5}$$

for some potential u_0 minimizing $\mathfrak{H}$, so that regularity issues on ϕ_0 immediately translate into regularity properties of the potential u_0 (observe that the potential u_0 can be seen as a Lagrange multiplier for the constraint on the divergence). Clearly, when $\mathcal{H}(\phi) = 1/p\,|\phi|^p$, then

$$-\mathrm{div}\,\nabla\mathcal{H}^*(\nabla u) = -\Delta_q\, u \ + \ \text{ homogeneous Neumann boundary conditions},$$

that is the familiar $q-$Laplacian operator, where p and q are conjugated exponent, for which regularity results are nowadays widely established (see [4, 11, 12]). Anyway, as explained in [2], for modeling reasons the assumption "$\mathcal{H}$ differentiable at 0" is not a realistic one and a more correct choice for the cost function $\mathcal{H}$ should be

$$\mathcal{H}(z) = \frac{1}{p}|z|^p + \delta|z|,\ z \in \mathbb{R}^N,$$

so that the congested metric d_Q is non degenerate, i.e. passing from a point with no traffic should have a cost (or, in other terms, you are not allowed to go at infinite speed in a region where there is no traffic). In this way the corresponding Beckmann's problem (1.3) consists in minimizing a combination of the L^p and the L^1 norm of vector fields with prescribed divergence: its dual formulation is precisely the problem whose regularity is under investigation in this paper, with $f = \mu_1 - \mu_0$. We remark that divergence-constrained optimization problems are a wide class of important problems in Mathematics (both Pure and Applied): we just cite Optimal Transportation and Shape Optimization as leading examples of fields where they frequently occur. This means that then it is not so uncommon that non-specialists of Regularity Theory (as the authors of [2] were) land on this hard shore with some concrete problems at hand (*is this optimizer regular? can I prove well-posedness of my problem in a smoother class?*) and the needing for *ad hoc* and precise results, which sometimes could be quite hard to trace back in the literature.

In this sense, it is worth pointing out that in a first version of [2] the regularity up to the boundary has not been dealt with in an adequate manner. Subsequently, in the actual version of [2] we have been able to fix the gap by using a shorter strategy, which heavily relies on some regularity results, contained in the paper [7] by Fonseca, Fusco and Marcellini, for minimizers of non convex variational problems (see also the paper [6] by Esposito, Mingione and Trombetti for some improvements and generalizations of these results). This shorter strategy provides a satisfactory result for the scopes of [2], where for other reasons quite strong assumptions were needed on μ_0 and μ_1: namely, $\mu_i = \rho_i \cdot \mathscr{L}^N$ for $i = 0, 1$, with ρ_0, ρ_1 Lipschitz functions, $\mathscr{L}^N$ denoting the $N-$dimensional Lebesgue measure. Anyway, it is a matter of fact that it does not give an optimal result in terms of the right-hand side of (1.2): indeed, the results contained in [7], if on the one hand are valid for minimizers of

$$\int_\Omega G(x, \nabla u(x))\, dx,$$

with $G(x, z)$ having a $q-$growth with respect to z and uniformly convex *at infinity* (i.e. outside a fixed ball, like in the case of our function $\mathcal{H}^*$), on the other hand are only of local character and, more important, they are not directly applicable to our case $G(x, u(x), \nabla u(x)) = \mathcal{H}^*(\nabla u(x)) - f(x)u(x)$, with f in a Lebesgue space, because they are valid under the assumptions (see [7], Theorem 2.7)

$$|G(x, z)| \le L(1 + |z|^q) \ \text{ and } \ |\nabla_x\nabla_z G(x, z)| \le L(1 + |z|^{q-1}),\ x \in \Omega,\ z \in \mathbb{R}^N \setminus B_M(0),$$

which do not fit in our framework (see also the final Remark in [2]).

So we believe that the distinguished features (regularity up to the boundary, Neumann boundary condition, right-hand side f in an optimal Lebesgue space) of a full result as that presented in this paper is of interest.

As we briefly mentioned, another interesting issue which requires regularity results for minimizers of an energy of the kind $\mathfrak{H}$, that is convex but not necessarily strictly convex, is the study of variational problems with lack of convexity, as in the problem of minimizing an energy given by a double-well potential. Indeed, it can be easily seen that the function $\mathcal{H}^*$ is the convex envelope of a function of the type

$$\mathcal{F}(z) = \frac{1}{q}\big||z| - \delta\big|^q, \ z \in \mathbb{R}^N,$$

whose corresponding energy

$$\mathfrak{F} = \int_\Omega \mathcal{F}(\nabla u(x))\, dx - \int_\Omega f(x)\, u(x)\, dx, \ u \in W^{1,q}(\Omega),$$

fails to be lower semicontinuous on $W^{1,q}(\Omega)$, due to the non-convexity of $\mathcal{F}$. Then $\mathfrak{H}$ can be seen as the relaxation of $\mathfrak{F}$, so that

$$\min \mathfrak{H} = \inf \mathfrak{F},$$

and the study of regularity properties of minimizers of $\mathfrak{H}$ can be of interest in order to study the asymptotic behaviour of infimizing sequences for the original problem: this is the point of view of Carstensen and Müller in [3], in which anyway the main interest is in Sobolev regularity results for the term $\nabla\mathcal{H}^*(\nabla u)$ (the *stress field*, in the terminology of [3]). Indeed, it can be shown that a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ for the problem relative to $\mathfrak{F}$ is bounded in $W^{1,q}(\Omega)$ and thus weakly converging to a function u: this in general will not be a minimizer of the original problem, due to the lack of semicontinuity. Anyway, the related stress fields $\nabla\mathcal{F}(\nabla u_n)$ still weakly converge, to a limit stress field of the form $\nabla\mathcal{H}^*(\nabla u_0)$, with u_0 minimizer of $\mathfrak{H}$ (see [3] and the references therein for more details).

Before going into the details of the regularity result, some words on the method of proof are in order: the desired estimate is achieved by means of considering more regular (we would say *more elliptic* or *more convex*, depending on the point of view) approximating problems, for which one can provide robust a priori estimates, which in the end depend only on the behaviour at infinity of the operator. This strategy is nowadays classical: in order to apply it, we have benefited of a careful reading of the seminal papers [4] and [11] by Di Benedetto and Lewis, respectively. In particular, we underline that the variational nature of the approximating problems seems to play a crucial role in our proof, like in [11], while this was unnecessary in [4]: this is due to the fact that the convergence we obtain on the solutions u_ε of the approximating problem is not strong enough to permit to work only with the weak formulations of the equations, while in [4] this is possible thanks to uniform $C^{1,\alpha}$ estimates, from which one can infer uniform convergence (on compact sets) of u_ε and of their gradients. This is clearly intimately linked to the type of degeneracy of our problem, which allows only for $C^{0,1}$ estimate (observe that every $\delta-$Lipschitz function is a solution of (1.2) with $f \equiv 0$).

2. Main result, strategy and basic tools

As in the Introduction, let us take $\mathcal{H}^*(z) = 1/q\,(|z| - \delta)_+^q, \ z \in \mathbb{R}^N$, where $\delta > 0$, $1 < q < \infty$ and $(\,\cdot\,)_+$ stands for the positive part. Observe that we have

$$\nabla\mathcal{H}^*(z) = (|z| - \delta)_+^{q-1}\frac{z}{|z|}, \ z \in \mathbb{R}^N,$$

and

$$\langle\nabla\mathcal{H}^*(z), z\rangle \ge c_q\,|z|^q - \delta^q, \ z \in \mathbb{R}^N,$$

with $c_q > 0$ being a constant depending on q only, while the Hessian matrix of $\mathcal{H}^*$ satisfies

$$\frac{(|z| - \delta)_+^{q-1}}{|z|}|\xi|^2 \le \langle D^2\mathcal{H}^*(z)\,\xi, \xi\rangle \le (q-1)(|z| - \delta)_+^{q-2}|\xi|^2, \ \xi \in \mathbb{R}^N,$$

that is the ellipticity constants degenerate for $|z| \le 1$.

Remark 1. Let us spend some words more on the type of degeneracy of our equation. For example, when $q \ge 2$, using the following monotonicity inequality (see [13], Chapter 10)

$$\langle |z|^{q-2}z - |w|^{q-2}w, z-w\rangle \ge C_q\,|z-w|^2(|z|^2+|w|^2)^{\frac{q-2}{2}},\ z,w\in\mathbb{R}^N,$$

it is not difficult to see that the operator $\nabla\mathcal{H}^*$ satisfies (it is enough to adapt the argument in [2], Lemma 4.1)

$$\begin{aligned}\langle \nabla\mathcal{H}^*(z) - \nabla\mathcal{H}^*(w), z-w\rangle \ge & C_q\left|(|z|-\delta)_+\frac{z}{|z|} - (|w|-\delta)_+\frac{w}{|w|}\right|^2 \\ & \times\left((|z|-\delta)_+^2 + (|w|-\delta)_+^2\right)^{\frac{q-2}{2}},\ z,w\in\mathbb{R}^N.\end{aligned}$$

In particular, inside the ball $B_\delta(0)$ the operator $\nabla\mathcal{H}^*$ does not satisfy any monotonicity property, thus considerably differing from the $q-$Laplacian operator. This in turn implies that no higher differentiability results are possible for the solutions of (1.2), not even in a fractional sense (while this is the case for the $q-$Laplacian, see [14]). On the contrary, as one can expect, higher regularity results can be obtained for the quantity

$$(|\nabla u|-\delta)_+\frac{\nabla u}{|\nabla u|},$$

as shown in [2], Theorem 4.2 (Sobolev regularity) and [15], Theorem 11 (continuity, when $N=2$). The same considerations apply to the case $1<q<2$.

The main result of this paper is the following.

Theorem 2.1. *Let $\Omega\subset\mathbb{R}^N$ be a $C^{2,1}$ domain. Given $f\in L^{N+\alpha}$ with zero-mean, every solution $u\in W^{1,q}_\diamond(\Omega)$ of the following Neumann boundary value problem*

$$\begin{cases} -\mathrm{div}\,(\nabla\mathcal{H}^*(\nabla u)) = f, & \text{in } \Omega,\\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu = 0, & \text{on } \partial\Omega,\end{cases} \tag{2.1}$$

is such that $u\in W^{1,\infty}(\Omega)$.

The proof is post-poned to the next three Sections, here we have to spend some words on the strategy: we think of (2.1) as the Euler-Lagrange equation of the original optimization problem

$$\min_{\varphi\in W^{1,q}_\diamond(\Omega)}\int_\Omega \mathcal{H}^*(\nabla\varphi)\,dx - \int_\Omega f\,\varphi\,dx,$$

and then we approximate the latter with a more regular one, depending on a small parameter $\varepsilon\in(0,\varepsilon_0]$, i.e.

$$\min_{\varphi\in W^{1,q}_\diamond(\Omega)}\int_\Omega \mathcal{H}^*_\varepsilon(\nabla\varphi)\,dx - \int_\Omega f_\varepsilon\,\varphi\,dx,$$

possessing a unique solution u_ε such that $u_\varepsilon\rightharpoonup u$ in $W^{1,q}(\Omega)$, with u solution of the original problem (see Proposition 2.2 below). In particular we will suppose that

$$\|u_\varepsilon\|_{W^{1,q}(\Omega)}\le C,\quad \text{for every } 0<\varepsilon\le\varepsilon_0, \tag{2.2}$$

then we aim to prove that the solutions u_ε satisfy uniform L^∞ gradient estimates, independent of ε, which consequently will pass to the limit, showing the required regularity on the original solution u.

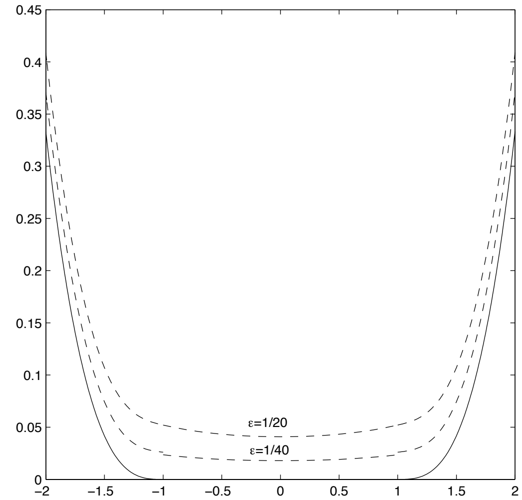

FIGURE 1. The approximating functions $\mathcal{H}^*_\varepsilon$.

Remark 2. As far as the original problem is convex but not strictly convex, in general we can not expect any kind of uniqueness for the minimizers of $\mathfrak{H}$. Anyway, for our scope it is important to stress the fact that, given two distinct minimizers u_1, u_2 of $\mathfrak{H}$ over $W^{1,q}_\diamond(\Omega)$, we have

$$\nabla\mathcal{H}^*(\nabla u_1(x)) = \nabla\mathcal{H}^*(\nabla u_2(x)), \quad \text{for } \mathscr{L}^N-\text{a.e. } x \in \Omega,$$

which is a consequence of the primal-dual optimality condition (1.5) and of the uniqueness for the solution of the dual problem (1.3) (this uniqueness comes from the strict convexity of $\mathcal{H}$). This in particular implies that, once for a particular minimizer u it is proven that $\nabla u \in L^\infty$, the same must be true for any other minimizer.

2.1. **Approximation.** For every $\varepsilon \in (0, \varepsilon_0]$ let us consider a smooth function $\mathcal{H}^*_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ with the following basic properties:

(C1) $\mathcal{H}^*_\varepsilon$ is strictly convex and depends only on the modulus, that is $\mathcal{H}^*_\varepsilon(z) = H^*_\varepsilon(|z|)$;
(C2) $1/q\,(t-\delta)^q_+ \le H^*_\varepsilon(t) \le A\,t^q + 1$, for some A independent of ε;
(C3) for every $\varepsilon_1 > \varepsilon_2$ we have $H^*_{\varepsilon_1} \ge H^*_{\varepsilon_2}$ and H^*_ε converges to $1/q\,(t-\delta)^q_+$ as ε goes to 0.

Moreover we require that the functions $\mathcal{H}^*_\varepsilon$ further satisfy the following ellipticity and growth conditions:

(G1) there exists a positive constant $\mu = \mu(\varepsilon)$ such that

$$\frac{1}{\mu}(1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \le \langle D^2\mathcal{H}^*_\varepsilon(z)\,\xi, \xi\rangle \le \mu(1+|z|^2)^{\frac{q-2}{2}}|\xi|^2;$$

(G2) there exist two constants $\lambda > 0$ and $M >> \delta$ *independent of* ε such that

$$\frac{1}{\lambda}(1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \le \langle D^2\mathcal{H}^*_\varepsilon(z)\,\xi, \xi\rangle \le \lambda(1+|z|^2)^{\frac{q-2}{2}}|\xi|^2, \quad \text{for every } |z| \ge M; \tag{2.3}$$

(G3) there exists a constant κ, independent of ε, such that

$$|\nabla\mathcal{H}^*_\varepsilon(z)| \le \kappa(1+|z|^2)^{\frac{q-1}{2}}, \quad \text{for every } z \in \mathbb{R}^N.$$

Remark 3. For example, in the case $q > 2$, we could simply take $\mathcal{H}^*_\varepsilon(z) = \mathcal{H}^*(z) + \varepsilon(1+|z|^2)^{\frac{q}{2}}$, while for $q \in (1,2]$ the same choice would be feasible, modulo a smoothing of $\mathcal{H}^*$ around $|z| = \delta$.

With the previous assumptions, we have that the equation

$$-\mathrm{div}\, \nabla \mathcal{H}^*_\varepsilon(\nabla u) = f_\varepsilon,$$

f_ε being a smooth approximation of f, is uniformly elliptic outside a ball of radius M, with ellipticity constants independent of ε. Moreover, for every fixed $\varepsilon > 0$, this is also a uniformly elliptic equation (but now with ellipticity constants degenerating with ε), so that every solution u_ε is regular enough to compute first derivatives of $\nabla \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon)$ (see [10]): indeed, roughly speaking, our scope will be that of deriving the equation, thus obtaining a linear equation for the gradient which can be used to establish estimates for it.

With the aid of hypotheses (C1)–(C3), it is quite easy to prove the following basic result, granting the convergence of the minimizers of the approximating problem to a minimizer of the original problem.

Proposition 2.2. *Let $f \in L^p(\Omega)$, with $p = q/(q-1)$ and $\int_\Omega f\,dx = 0$. We then consider $\{f_\varepsilon\}_{\varepsilon>0} \subset L^p(\Omega)$ such that $f_\varepsilon \rightharpoonup f$ in $L^p(\Omega)$ and having zero-mean. Then every functional*

$$\mathfrak{H}_\varepsilon(u) = \int_\Omega \mathcal{H}^*_\varepsilon(\nabla v(x))\,dx - \int_\Omega f_\varepsilon(x)\,v(x)\,dx,\ \ v \in W^{1,q}_\diamond(\Omega),$$

admits a unique minimizer $u_\varepsilon \in W^{1,q}_\diamond(\Omega)$. Moreover, we get that $\{u_\varepsilon\}_{\varepsilon>0}$ weakly converges in $W^{1,q}$ to a minimizer of

$$\mathfrak{H}(u) = \int_\Omega \mathcal{H}^*(\nabla u(x))\,dx - \int_\Omega f(x)\,u(x)\,dx,\ \ u \in W^{1,q}_\diamond(\Omega).$$

Proof. We are not concerned here with existence and uniqueness of the minimizers, which follow in a standard way, from to the assumptions on $\mathcal{H}^*_\varepsilon$. We directly go to the second part of the statement, so first of all, we show that the sequence of minimizers $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies estimate (2.2): we can clearly suppose that

$$\|f_\varepsilon\|_{L^p(\Omega)} \le C, \quad \text{for every } 0 < \varepsilon \le \varepsilon_0,$$

then by means of the minimality of u_ε we get

$$\int_\Omega \langle \nabla \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon), \nabla u_\varepsilon\rangle\,dx = \int_\Omega f_\varepsilon u_\varepsilon\,dx,$$

so that

$$\begin{aligned} c_1 \int_\Omega |\nabla u_\varepsilon|^q\,dx - c_2 \le \int_\Omega \langle \nabla \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon), \nabla u_\varepsilon\rangle\,dx &= \int_\Omega f_\varepsilon u_\varepsilon\,dx \\ &\le \|f_\varepsilon\|_{L^p(\Omega)} \|u_\varepsilon\|_{L^q(\Omega)} \\ &\le C\,\|f_\varepsilon\|_{L^p(\Omega)} \|\nabla u_\varepsilon\|_{L^q(\Omega)} \\ &\le \frac{C}{\tau}\|f_\varepsilon\|^p_{L^p(\Omega)} + C\tau\|\nabla u_\varepsilon\|^q_{L^q(\Omega)}, \end{aligned}$$

having used Poincaré's inequality and Young's inequality, where the constants c_1, c_2 do not depend on ε. From the latter we can easily infer the desired uniform estimate (2.2), thus giving the weak compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $W^{1,q}(\Omega)$.

We now call u the weak limit (up to a subsequence) of u_ε and we show that this is indeed a minimizer of $\mathfrak{H}$. By minimality of every u_ε we know that

$$\int_\Omega \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon)\,dx - \int_\Omega f_\varepsilon\, u_\varepsilon\,dx \le \int_\Omega \mathcal{H}^*_\varepsilon(\nabla v)\,dx - \int_\Omega f_\varepsilon\, v\,dx, \quad \text{for every } v\in W^{1,q}_\diamond(\Omega),$$

then we observe that

$$\int_\Omega \mathcal{H}^*(\nabla u)\,dx \le \liminf_{\varepsilon\to 0}\int_\Omega \mathcal{H}^*(\nabla u_\varepsilon)\,dx \le \liminf_{\varepsilon\to 0}\int_\Omega \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon)\,dx$$

having used the fact that $\mathcal{H}^*_\varepsilon\ge\mathcal{H}^*$ and the semicontinuity of the term

$$W^{1,q}(\Omega)\ni v\mapsto \int_\Omega \mathcal{H}^*(\nabla v)\,dx.$$

Moreover, thanks to the fact that $u_\varepsilon\to u$ in L^q and $f_\varepsilon\rightharpoonup f$ in L^p, we get

$$\int_\Omega f\,u\,dx = \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon\,u_\varepsilon\,dx.$$

So in the end we have obtained

$$\begin{aligned}\int_\Omega \mathcal{H}^*(\nabla u)\,dx - \int_\Omega f\,u\,dx &\le \liminf_{\varepsilon\to 0}\int_\Omega \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon)\,dx - \int_\Omega f_\varepsilon\,u_\varepsilon\,dx\\ &\le \lim_{\varepsilon\to 0}\int_\Omega \mathcal{H}^*_\varepsilon(\nabla v)\,dx - \int_\Omega f_\varepsilon\,v\,dx\\ &\le \int_\Omega \mathcal{H}^*(\nabla v)\,dx - \int_\Omega f\,v\,dx, \text{ for every } v\in W^{1,q}_\diamond(\Omega).\end{aligned}$$

having used (in last inequality) the monotone convergence of $\mathcal{H}^*_\varepsilon$ to $\mathcal{H}^*$. This clearly proves the minimality of u, thus concluding the proof. □

We set $w_\varepsilon=(1+|\nabla u_\varepsilon|^2)^{\frac{q}{2}}$ and this function will play a fundamental role in the whole discussion: in particular, we will prove that $w_\varepsilon\in L^\infty$, which in turn will imply that ∇u_ε itself is in L^∞. We will also set

$$k_0=(M^2+1)^{\frac{q}{2}}, \tag{2.4}$$

so that $\{w_\varepsilon>k_0\}=\{|\nabla u_\varepsilon|>M\}$, which is the *good* region, that is the region in which the approximating equations are uniformly elliptic, with ellipticity constants independent of ε.

2.2. **Reduction to the boundary.** For the solutions u_ε of the approximating problems we will confine ourselves to prove a uniform L^∞ gradient estimate near the boundary, the interior estimate being simpler and easily deducible from the former. In order to do this, we proceed as follows (the same idea can be found in [3], Theorem 4.3):

we fix a boundary point $x_0\in\partial\Omega$ and a neighborhood ω containing it, then we apply a diffeomorphism Ψ taking the half-ball

$$B^+=\{(x',x_N)\in\mathbb{R}^N\,:\,|(x',x_N)|<1,\ x_N\ge 0\},$$

into $\omega\cap\Omega$ and the straight part of ∂B into $\omega\cap\partial\Omega$. We then define $\widehat{u}_\varepsilon=u_\varepsilon\circ\Psi$ and we extend it to the whole ball just by reflection (let us still call it $\widehat{u}_\varepsilon$), doing the same with the datum f_ε. Thanks to the $C^{2,1}$ regularity assumption on the boundary $\partial\Omega$, to the homogeneous Neumann boundary condition on u_ε and to the fact that $\mathcal{H}^*_\varepsilon(z) = h_\varepsilon(|z|)$, we can construct Ψ of a special form in such a way that the function $\widehat{u}_\varepsilon$ is a local weak solution in the ball B of an equation of the type

$$-\text{div}\, H_\varepsilon(x, \nabla u) = \widehat{f}_\varepsilon, \tag{2.5}$$

with $z \mapsto H_\varepsilon(x,z)$ satisfying the ellipticity conditions (G1)–(G3) uniformly in x and with $x \mapsto H_\varepsilon(x,z)$ being Lipschitz, more precisely we have

$$|H_\varepsilon(x,z) - H_\varepsilon(y,z)| \leq L\,(1+|z|^2)^{\frac{q-1}{2}}\,|x-y|. \tag{2.6}$$

Let us be a little bit more precise about this procedure: the function H_ε so constructed has the form

$$H_\varepsilon(x,z) = |\det \widehat{M}(x)|^{-1}\,\nabla\mathcal{H}^*_\varepsilon(z\,\widehat{M}(x))\,\widehat{M}(x)^t,\ \ (x,z) \in B\times\mathbb{R}^N,$$

where the matrix $\widehat{M}$ is linked to the Jacobian matrix $D\Psi$ and is defined by

$$\widehat{M}(y) = \begin{cases} [D\Psi(y)]^{-1}, & \text{if } y \in B^+, \\ \mathcal{R}[D\Psi(\mathcal{R}y)]^{-1}\mathcal{O}(\mathcal{R}y), & \text{if } y \in B^- := \mathcal{R}B^+, \end{cases}$$

with $\mathcal{R}$ being the reflection matrix with respect to the hyperplane $\{x_N = 0\}$ and $\mathcal{O}$ a matrix field of class $C^{1,1}$ defined in B^+, such that for every $x \in B^+$, $\mathcal{O}(x)$ is an orthogonal matrix verifying the commuting property

$$[D\psi(x',0)]^{-1} = \mathcal{R}[D\psi(x',0)]^{-1}\mathcal{O}(y',0), \quad \text{for every } |y'| < 1.$$

It is precisely here, in showing the existence of such a matrix field $\mathcal{O}$, that the diffeomorphism has to be properly constructed and the $C^{2,1}$ regularity assumption on $\partial\Omega$ plays a fundamental role (see [3], Lemmas 4.4 and 4.5).

Then, in order to prove Theorem 2.1, it will be enough to prove local uniform L^∞ gradient estimates for solutions of (2.5) in the ball $B = \{x \in \mathbb{R}^N : |x| < 1\}$.

2.3. **Basic tools.** Finally, before starting the proof of our main result, we need to recall two fundamental tools in Elliptic Regularity: the first is a higher integrability result for the gradient, which can be found in [16], Theorem 3.3.6 (which anyway deals with rather more general situations). This is based on an amended version (proven by Stredulinsky in [17]) of the well-known Gehring Lemma ([8]). Here we give a slightly simplified version of the statement of Theorem 3.3.6 of [16], adapted to our needing.

Theorem A. *Let $1 < q < N$ and $F \in L^{(q^*)'+\alpha}(\Omega)$ for some $\alpha > 0$, where $q^* = Nq/(N-q)$. Let $\mathcal{A} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be a Borel function satisfying the conditions*

(i) $|\mathcal{A}(x,\nabla u)| \leq c_0|\nabla u|^{q-1} + a$;
(ii) $\langle\mathcal{A}(x,\nabla u), \nabla u\rangle \geq c_1|\nabla u|^q - \gamma$;

where c_0, c_1 are positive constants and the nonnegative Borel measurable functions a, γ are such that $a \in L^{p+\alpha}(\Omega)$ and $\gamma \in L^{1+\alpha}(\Omega)$, where $p = q/(q-1)$. Then if $u \in W^{1,q}_{\rm loc}(\Omega)$ is a local weak solution of

$$-\text{div}\,\mathcal{A}(x,\nabla u) = F(x),$$

there exists $\theta > 0$ such that $\nabla u \in L^{q+\theta}_{\rm loc}(\Omega)$, with θ depending only on N, q, c_0 and α. Moreover, for every pair of concentric cubes $Q_\varrho(x_0) \subset Q_{3\varrho}(x_0) \subset\subset \Omega$ the following estimate holds

$$\left(\fint_{Q_\varrho(x_0)} |\nabla u(x)|^{q+\theta}\,dx\right)^{\frac{q}{q+\theta}} \leq C\left[\fint_{Q_{3\varrho}(x_0)} |\nabla u(x)|^q\,dx + \mathcal{M}(Q_{3\varrho(x_0)})\right], \tag{2.7}$$

with the constant C depending only on N, q, θ and the distance of x_0 from $\partial\Omega$, while $\mathcal{M}$ is given by

$$\mathcal{M}(Q_{3\varrho(x_0)}) = \left(\fint_{Q_{3\varrho}(x_0)} |\gamma|^{1+\alpha}\, dx \right)^{\frac{1}{(1+\alpha)}} + \left(\fint_{Q_{3\varrho}(x_0)} |a|^{p+\alpha}\, dx \right)^{\frac{1}{(p+\alpha)}}$$
$$+ \left(\fint_{Q_{3\varrho}(x_0)} |u - \overline{u}_{Q_{3\varrho}(x_0)}|^{q^*}\, dx \right)^{\frac{t}{q^*}} + \left(\fint_{Q_{3\varrho}(x_0)} |F|^{(q^*)'+\alpha}\, dx \right)^{\frac{t'}{(q^*)'+\alpha}},$$

and $\overline{v}_E$ standing for the average of v over a generic set E. Here t is a suitable exponent such that $t < q^*$ and $t' < (q^*)' + \alpha$.

Remark 4. The content of Theorem 3.3.6 of [16] in indeed more general and it also comprises the case $q \geq N$. In this situation, the thesis of Theorem A above is still true taking $F \in L^{1+\alpha}(\Omega)$ and replacing the L^{p^*} norm of $u - \overline{u}_{Q_{3\varrho}}$ in estimate (2.7) with any L^r norm (critical case $q = N$) or its L^∞ norm (super critical case $q > N$).

The second result is the following, giving an L^∞ bound for a class of functions which sometimes are called *De Giorgi classes* (but the terminology could not be standard, see also [10], Theorem 5.2, for a similar definition and related results). The proof of the following fact can be found in [9], Theorem 7.2.

Theorem B. *Let $v \in W^{1,p}_{\rm loc}(\Omega)$ be a positive function and suppose that there exist constants $C, \chi > 0$, an exponent $\vartheta > 0$ and a radius $R_0 > 0$, such that for every couple of concentric balls $B_\varrho(x_0)$ and $B_R(x_0)$ with $\varrho < R \leq R_0$, we get*

$$\int_{B_\varrho(x_0)} |\nabla (v-k)_+|^p\, dx \leq \frac{C}{(R-\varrho)^p} \int_{B_R(x_0)} (v-k)_+^p\, dx + C(\chi^p + k^p R^{-N\vartheta}) |\{v > k\} \cap B_R(x_0)|^{1-\frac{p}{N}+\vartheta},$$

for every $k \geq k_0$. Then $v \in L^\infty_{\rm loc}(\Omega)$ and for every $x_0 \in \Omega$ and $R \leq \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)\}$ we get the estimate

$$\sup_{B_{R/2}(x_0)} v \leq C \left[\left(\fint_{B_R(x_0)} v^p\, dx \right)^{\frac{1}{p}} + k_0 + \chi R^{\frac{N\vartheta}{p}} \right]. \tag{2.8}$$

Having declared our strategy and introduced all the required tools, we will dedicate the next three sections to the proof of Theorem 2.1.

3. Proof of Theorem 2.1: Step 1 – Integrability gain

We now want to work with equation (2.5), which is defined in the ball B. The first step is to show that $w \in L^2_{\rm loc}(B)$ uniformly in ε, with an estimate on $\|w\|_{L^2}$ depending only on the data and $\|\nabla u\|_{L^q}$: in order to guarantee this gain of integrability on w, we wish to use a Moser-type argument, applied to the equation (3.1). In the sequel we will drop the subscript ε for the solutions u_ε and for the approximated data f_ε, just for notational convenience: the only important fact is the uniform assumption

$$\|f_\varepsilon\|_{L^{N+\alpha}} \leq C, \quad \text{for every } 0 < \varepsilon \leq \varepsilon_0.$$

The weak formulation of (2.5) is given by

$$\int \langle H_\varepsilon(x,\nabla u),\nabla\varphi\rangle\,dx=\int f\,\varphi\,dx,\quad \text{for every } \varphi\in W_0^{1,q}(B),$$

and deriving this equation with respect to x_i, we arrive at

$$\int \langle \nabla_z H_\varepsilon(x,\nabla u)D_i^2u,\nabla\varphi\rangle\,dx+\int\langle \partial_{x_i}H_\varepsilon(x,\nabla u),\nabla\varphi\rangle\,dx=-\int f\,\varphi_{x_i}\,dx, \tag{3.1}$$

where D_i^2u is the $i-$th column of the Hessian matrix D^2u.

First of all, observe that by means of Theorem A, we already have obtained the gain of integrability $w\in L^{1+\theta/q}$, for some $\theta>0$ (the H_ε can be easily constructed so to satisfy the mild hypothesis of Theorem A uniformly in ε). Then, keeping in mind the definition (2.4) of k_0, let us choose

$$\varphi=u_{x_i}(w^s-k_0^s)_+\zeta^2,$$

with $s\geq\theta/q$, where ζ is a C_0^∞ cut-off function supported on some ball $B_R(x_0)\subset\subset B$, equal to 1 on a smaller concentric ball $B\varrho(x_0)$ and such that its maximal slope is of order $(R-\varrho)^{-1}$.

Inserting φ into (3.1) and summing over $i=1,\dots,N$, we obtain

$$\sum_{i=1}^N\int_{\{w>k_0\}}\langle\nabla_z H_\varepsilon(x,\nabla u)D_i^2u\,u_{x_i},\nabla w\rangle\,w^{s-1}\,\zeta^2\,dx \tag{3.2}$$

$$+\sum_{i=1}^N\int\langle\nabla_z H_\varepsilon(x,\nabla u)D_i^2u,D_i^2u\rangle\,(w^s-k_0^s)_+\,\zeta^2dx \tag{3.3}$$

$$+2\sum_{i=1}^N\int\langle\nabla_z H_\varepsilon(x,\nabla u)D_i^2u\,u_{x_i},\nabla\zeta\rangle\,\zeta\,(w^s-k_0^s)_+u_{x_i}\,dx \tag{3.4}$$

$$+\sum_{i=1}^N\int_{\{w>k_0\}}\langle\partial_{x_i}H_\varepsilon(x,\nabla u),\nabla w\rangle\,w^{s-1}\,u_{x_i}\,\zeta^2dx \tag{3.5}$$

$$+\sum_{i=1}^N\int\langle\partial_{x_i}H_\varepsilon(x,\nabla u),D_i^2u\rangle\,(w^s-k_0^s)_+\,\zeta^2dx \tag{3.6}$$

$$+2\sum_{i=1}^N\int\langle\partial_{x_i}H_\varepsilon(x,\nabla u),\nabla\zeta\rangle\,u_{x_i}\,(w^s-k_0^s)_+\,\zeta dx \tag{3.7}$$

$$=-\sum_{i=1}^N\int f\,u_{x_ix_i}\,(w^s-k_0^s)_+\zeta^2 \tag{3.8}$$

$$-\sum_{i=1}^N\int f\,u_{x_i}\,((w^s-k_0^s)_+)_{x_i}\,\zeta^2 \tag{3.9}$$

$$-\sum_{i=1}^N 2\int f\,\zeta_{x_i}(w^s-k_0^s)_+u_{x_i}\zeta. \tag{3.10}$$

We start by saying that the two main terms are given by (3.2) and (3.4), which are the corner-stones that in the end will give a Caccioppoli-type inequality. In the sequel, in order to provide a cleaner and easier to follow description of the estimates, we divide the integrals in the previous equation in three groups: the *main terms* (3.2), (3.3) and (3.4), *terms containing* f, i.e. (3.8), (3.9) and (3.10) and *terms containing* $\partial_{x_i} H_\varepsilon$, which are (3.5), (3.6) and (3.7). We aim to analyze each group separately, starting from the basic ones.

3.1. **The main terms.** So first of all, we proceed to estimate (3.2) and (3.4): using the fact

$$\nabla w = q\, w^{\frac{q-2}{q}}\, D^2 u \nabla u,$$

the first term can be written as

$$s \int_{\{w>k_0\}} \langle \nabla_z H_\varepsilon(x,\nabla u) D^2 u \nabla u, \nabla w\rangle\, w^{s-1}\, \zeta^2\, dx = \frac{s}{q} \int_{\{w>k_0\}} \langle \nabla_z H_\varepsilon(x,\nabla u)\nabla w, \nabla w\rangle\, \zeta^2\, w^{s-2+\frac{2}{q}} dx$$

and recalling the fact that $\{w > k_0\} = \{|\nabla u| > M\}$ and taking into account (2.3), we can estimate this integral from below with

$$\begin{aligned} \frac{s}{\lambda q} \int_{\{w>k_0\}} |\nabla w|^2\, w^{s-1}\, \zeta^2\, dx &= \frac{4s}{\lambda q (s+1)^2} \int_{\{w>k_0\}} \left|\nabla (w^{\frac{s+1}{2}})\right|^2 \zeta^2\, dx \\ &= \frac{4s}{\lambda q (s+1)^2} \int \left|\nabla \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}} \right)_+ \right|^2 \zeta^2\, dx := \mathcal{J}. \end{aligned}$$

Concerning the other term, we observe

$$\begin{aligned} 2\int \langle \nabla_z H_\varepsilon(x,\nabla u) D^2 u \nabla u, \nabla \zeta\rangle\, (w^s - k_0^s)_+ \zeta\, dx &\le \frac{2}{q} \int |\nabla_z H_\varepsilon(x,\nabla u)| |\nabla w|\, |\nabla \zeta|\, \zeta (w^s - k_0^s)_+ w^{\frac{2-q}{q}}\, dx \\ &\le \frac{2\lambda}{q} \int |\nabla w|\, |\nabla \zeta|\, \zeta\, (w^s - k_0^s)_+\, dx, \\ &\le \frac{2\lambda}{q} \int_{\{w>k\}} |\nabla w|\, |\nabla \zeta|\, \zeta\, w^s\, dx. \end{aligned}$$

Finally, using Young's inequality the last integral can be treated as

$$\int_{\{w>k\}} |\nabla w|\, |\nabla \zeta|\, \zeta\, w^s\, dx \le \tau \int_{\{w>k\}} |\nabla w|^2\, w^{s-1}\, \zeta^2\, dx + \frac{1}{\tau} \int_{\{w>k\}} w^{s+1}\, |\nabla \zeta|^2,\ dx,$$

and the first term can be absorbed in $\mathcal{J}$, taking $\tau > 0$ small enough. Before going on, we observe that the first group contains also the term

$$\sum_{i=1}^{N} \int \langle \nabla_z H_\varepsilon(\nabla u) D_i^2 u, D_i^2 u\rangle\, (w^s - k_0^s)_+\, \zeta^2 dx,$$

which has positive sign and we could consequently be tempted to drop it: on the contrary, it will be crucial to keep it, in order to absorb similar terms appearing on the right-hand side (for this reason, we will call it *sponge term*, see below). So, it is important to give an estimation from below for it: indeed, we get

$$\int \langle \nabla_z H_\varepsilon(\nabla u) D^2 u, D^2 u\rangle\, (w^s - k_0^s)_+\, \zeta^2 dx \ge \frac{1}{\lambda} \int w^{\frac{q-2}{q}}\, |D^2 u|^2\, (w^s - k_0^s)_+\, \zeta^2\, dx := \frac{1}{\lambda}\, \mathcal{S}(B_R), \tag{3.11}$$

and sometimes, for simplicity, we will call this the *sponge term*. Up to now, we have obtained

$$\int \left|\nabla \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+\right|^2 \zeta^2 \, dx + \mathcal{S}(B_R) \le C \int_{\{w>k_0\}} \left|w^{\frac{s+1}{2}}\right|^2 |\nabla \zeta^2| \, dx + \text{ Estimates for } ((3.5) - (3.10)),$$

with the constant C depending on q, s and λ, then using the following simple observation

$$w^{s+1} 1_{\{w>k_0\}} \le 2 \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 + 2\, k_0^{s+1}, \tag{3.12}$$

the previous can be recast into

$$\int_{B_\varrho(x_0)} \left|\nabla \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+\right|^2 \zeta^2 \, dx + \mathcal{S}(B_R) \le C \int \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 |\nabla \zeta|^2 \, dx + \frac{k_0^{2\beta} C}{(R-\varrho)^2} |B_R| + \text{ Estimates for } ((3.5) - (3.10)), \tag{3.13}$$

which is the kind of estimate which will provide the desired gain of integrability.

3.2. **Terms containing the datum f.** Using Young's inequality and the fact that $w \ge 1$, we get

$$\begin{aligned} -\sum_i \int f\, u_{x_i x_i} (w^s - k_0^s)_+ \zeta^2 \, dx &\le \int |f|\, |D^2 u|\, (w^s - k_0^s)_+ \, \zeta^2 \, dx \\ &\le \tau \int |D^2 u|^2 \, w^{\frac{q-2}{q}} \, (w^s - k_0^s)_+ \, \zeta^2 \, dx \\ &\quad + \frac{1}{\tau} \int |f|^2 \, (w^s - k_0^s)_+ \, \zeta^2 \, dx, \end{aligned}$$

and the first integral can be absorbed in the left-hand side, taking $\tau > 0$ small enough (for example $\tau = 1/(2\lambda)$) by means of the sponge term $\mathcal{S}(B_R)$. Concerning the other term, we can proceed as follows, noticing that

$$(w^s - k_0^s)_+ \le w^{s+1} 1_{\{w>k_0\}} \le 2 \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 + 2\, k_0^{s+1}, \tag{3.14}$$

so that using Hölder's inequality and Sobolev inequality we get

$$\begin{aligned} \int |f|^2 \, (w^s - k_0^s)_+ \, \zeta^2 \, dx &\le 2 \int |f|^2 \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 \zeta^2 \, dx + 2\, k_0^{s+1} \int |f|^2 \, \zeta^2 \, dx \\ &\le \left(\int_{B_R(x_0)} |f|^N \, dx\right)^{\frac{2}{N}} \left(\int \left(\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+ \zeta\right)^{2^*} dx\right)^{\frac{2}{2^*}} \\ &\quad + 2\, k_0^{s+1} \int_{B_R(x_0)} |f|^2 \, dx \\ &\le c\, \|f\|_{L^{N+\alpha}}^2 \, R^{\frac{2\alpha}{N+\alpha}} \int \left|\nabla \left(\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+ \zeta\right)\right|^2 dx \\ &\quad + c\, k_0^{s+1} \|f\|_{L^{N+\alpha}}^2 |B_R|^{1-\frac{2}{N+\alpha}}, \end{aligned}$$

and then we simply observe that

$$\int \left|\nabla\left(\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+ \zeta\right)\right|^2 dx \le 2\int \left|\nabla\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+\right|^2 \zeta^2\, dx + 2\int |\nabla\zeta|^2 \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 dx,$$

so that the term involving the gradient of $(w^{(s+1)/2} - k_0^{(s+1)/2})_+$ will be absorbed in the left-hand side of (3.13), choosing the radius R small enough (so that all the estimates will be true for every R smaller than a suitable R_0), while the other term can be kept in the right-hand side of (3.13).

Let us go on, with the second term involving f in the derived equation: we use the fact that $|\nabla u| \le w^{\frac{1}{q}}$ and that $w^{\frac{1}{q}+s-1} \le w^s$, so to obtain

$$\begin{aligned}-\sum_i \int f\, u_{x_i}((w^s - k_0^s)_+)_{x_i}\, \zeta^2\, dx &\le s\int_{\{w>k_0\}} |f|\, w^{\frac{1}{q}+s-1}\, |\nabla w|\, \zeta^2\, dx \\ &\le \frac{s}{\tau}\int_{\{w>k_0\}} |f|^2\, w^{s+1}\, \zeta^2\, dx + s\tau \int_{\{w>k_0\}} |\nabla w|^2\, w^{s-1}\, \zeta^2\, dx,\end{aligned}$$

and the second term can be absorbed by the left-hand side of (3.13), while the first can be estimated as before, using (3.12). Finally, the last integral: we use the simple fact that $w^{1/q+s} \le w^{s+1}$ and Young's inequality, so that

$$\begin{aligned}-2\sum_i \int f\, u_{x_i}\, \zeta_{x_i}\, (w^s - k_0^s)_+\, \zeta\, dx &\le 2\int_{\{w>k_0\}} |f|\, w^{s+1}\, |\nabla\zeta|\, \zeta\, dx \\ &\le \int_{\{w>k_0\}} |f|^2\, w^{s+1}\, \zeta^2\, dx + \int_{\{w>k_0\}} |\nabla\zeta|^2 \left|w^{\frac{s+1}{2}}\right|^2 dx,\end{aligned}$$

and again the first term as already been estimated using (3.12), while the second can be recast into

$$\int_{\{w>k_0\}} |\nabla\zeta|^2 \left|w^{\frac{s+1}{2}}\right|^2 dx \le 2\int |\nabla\zeta|^2 \left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+^2 dx + \frac{C\, k_0^{s+1}}{(R-\varrho)^2}\, |B_R|,$$

again using (3.12), the latter being exactly the same term of the right-hand side in (3.13).

Putting all together, after the estimation of the first two groups of terms we have obtained

$$\begin{aligned}\int \left|\nabla\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+\right|^2 \zeta^2\, dx + \mathcal{S}(B_R) &\le C_1 \int_{\{w>k_0\}} \left|\left(w^{\frac{s+1}{2}} - k_0^{\frac{s+1}{2}}\right)_+\right|^2 |\nabla\zeta^2|\, dx \\ &\quad + \frac{C\, k_0^{s+1}}{(R-\varrho)^2}\, |B_R| + C\, k_0^{s+1} \|f\|^2_{L^{N+\alpha}}\, |B_R|^{1-\frac{2}{N+\alpha}} \\ &\quad + \text{ Estimates for } ((3.5)-(3.7)),\end{aligned} \tag{3.15}$$

with C_1 depending on q, λ, s, $\|f\|_{L^{N+\alpha}}$.

3.3. **Terms involving derivatives of H.**

We are left with the handling of the terms

$$\text{Estimates for } ((3.5)-(3.7)),$$

and clearly we aim to obtain an inequality of the type (3.15). Let us start with the (3.5): we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{w>k_0\}}\langle\partial_{x_i}H_\varepsilon(x,\nabla u),\nabla w\rangle\, w^{s-1}\,u_{x_i}\,\zeta^2dx &\le C\int_{\{w>k_0\}}|\nabla w|\,w^{\frac{q-1}{q}}\,w^{s-1}\,w^{\frac{1}{q}}\zeta^2\,dx\\ &\le C\,\tau\int_{\{w>k_0\}}|\nabla w|^2\,w^{s-1}\,\zeta^2\,dx\\ &+\frac{C}{\tau}\int_{\{w>k_0\}}w^{s+1}\,\zeta^2\,dx,\end{aligned}$$

and the first term can be absorbed by the left-hand side of (3.15), while for the other we can simply proceed as before, using (3.12) in combination with Hölder and Sobolev inequality to get

$$\begin{aligned}\int_{\{w>k_0\}}w^{s+1}\,\zeta^2\,dx &\le C\,|B_R|^{\frac{2}{N}}\int\left|\nabla\left(w^{\frac{s+1}{2}}-k_0^{\frac{s+1}{2}}\right)_+\right|^2\zeta^2\,dx\\ &+C\,|B_R|^{\frac{2}{N}}\int|\nabla\zeta|^2\left(w^{\frac{s+1}{2}}-k_0^{\frac{s+1}{2}}\right)_+^2dx\\ &+C\,k_0^{s+1}|B_R|,\end{aligned}$$

the term containing the gradient of $(w^{\frac{s+1}{2}}-k_0^{\frac{s+1}{2}})_+$ being absorbed, taking R small enough.

We go on with (3.6), that is

$$\begin{aligned}\sum_{i=1}^{N}\int\langle\partial_{x_i}H_\varepsilon(x,\nabla u),D_i^2u\rangle\,(w^s-k_0^s)_+\,\zeta^2dx &\le C\int|D^2u|\,w^{\frac{q-1}{q}}\,(w^s-k_0^s)_+\,\zeta^2\,dx\\ &\le C\tau\int|D^2u|^2\,w^{\frac{q-2}{q}}\,(w^s-k_0^s)_+\,\zeta^2\,dx\\ &+\frac{C}{\tau}\int w\,(w^s-k_0^s)_+\,\zeta^2\,dx,\end{aligned}$$

and the first integral will be absorbed by $\mathcal{S}(B_R)$ in the left-hand side, while the other can be estimated as before.

We are only left with the last integral (3.7), for which we can easily derive the following estimate

$$\begin{aligned}2\sum_{i=1}^{N}\int\langle\partial_{x_i}H_\varepsilon(x,\nabla u),\nabla\zeta\rangle\,u_{x_i}\,(w^s-k_0^s)_+\,\zeta dx &\le C\int w\,(w^s-k_0^s)_+\,|\nabla\zeta|\,\zeta\,dx\\ &\le C\int_{\{w>k_0\}}|\nabla\zeta|^2\,w^{s+1}\,dx+C\int_{\{w>k_0\}}\zeta^2\,w^{s+1}\,dx,\end{aligned}$$

and these latter two terms have already been treated.

3.4. **Caccioppoli-type inequality.** Finally, inserting the latter informations into (3.15) and setting $\beta=(s+1)/2$, we can drop the sponge term $\mathcal{S}(B_R)$, add the term $\int(w^\beta-k_0^\beta)_+\,|\nabla\zeta|^2\,dx$ on both sides and then apply Sobolev inequality, in order to get

$$\left(\int_{B_\varrho(x_0)} \left(w^\beta - k_0^\beta\right)_+^{2^*} dx\right)^{\frac{2}{2^*}} \le \frac{C}{(R-\varrho)^2} \int_{B_R(x_0)} \left(w^\beta - k_0^\beta\right)_+^{2} dx + \frac{k_0^{2\beta} C}{(R-\varrho)^2} |B_R| + C\, k_0^{2\beta} \|f\|_{L^{N+\alpha}}^2 \, |B_R|^{1-\frac{2}{N+\alpha}}, \tag{3.16}$$

for a constant C that can be chosen so to depend only on $q, \lambda, \|f\|_{L^{N+\alpha}}$ and β, but not on ε. Starting from $\beta = \frac{1+\theta/q}{2}$, choosing a sequence of concentric balls and iterating a suitable number of times, we finally obtain $w \in L^2_{\rm loc}$, uniformly in ε. Moreover, for every $B_R \Subset \Omega$ we get an estimate of the type

$$\|w\|_{L^2(B_\varrho)} \le C = C(\lambda, N, \delta, L, \|f\|_{L^{N+\alpha}}, R-\varrho),$$

for every $0 < \varrho < R$.

4. Proof of Theorem 2.1: Step 2 – Boundedness of the gradient

The next step is to show that w is in a suitable De Giorgi class, i.e. it satisfies an estimate of the type (2.8) (with $p = 2$, as we will see): then using Theorem B we will obtain that $w \in L^\infty_{\rm loc}$, with an estimate on $\|w\|_\infty$ depending on the L^2 norm of w. Using the fact that $w \in L^2_{\rm loc}$ uniformly in ε (as shown in the previous section), we will finally obtain that $\nabla u_\varepsilon \in L^\infty_{\rm loc}$ uniformly in ε.

So, in order to conclude, we proceed exactly as in the last section, but with a different choice for the test function, as far as we now want to show that $(w-k)_+$ satisfies a suitable Caccioppoli inequality, for every $k \ge k_0$.

Keeping this in mind, the right choice for the test function is given by $\varphi = u_{x_i}(w-k)_+\zeta^2$, then inserting this into (3.1) and summing over $i = 1, \dots, N$, we obtain

$$\sum_{i=1}^N \int \langle \nabla_z H_\varepsilon(x, \nabla u) D_i^2 u \, u_{x_i}, \nabla (w-k)_+ \rangle \, \zeta^2 \, dx \tag{4.1}$$

$$+ \sum_{i=1}^N \int \langle \nabla_z H_\varepsilon(x, \nabla u) D_i^2 u, D_i^2 u \rangle \, (w-k)_+ \, \zeta^2 dx \tag{4.2}$$

$$+2 \sum_{i=1}^N \int \langle \nabla_z H_\varepsilon(x, \nabla u) D_i^2 u \, u_{x_i}, \nabla \zeta \rangle \, \zeta \, (w-k)_+ \, u_{x_i} \, dx \tag{4.3}$$

$$+ \sum_{i=1}^N \int \langle \partial_{x_i} H_\varepsilon(x, \nabla u), \nabla (w-k)_+ \rangle \, u_{x_i} \, \zeta^2 dx \tag{4.4}$$

$$+ \sum_{i=1}^N \int \langle \partial_{x_i} H_\varepsilon(x, \nabla u), D_i^2 u \rangle \, (w-k)_+ \, \zeta^2 dx \tag{4.5}$$

$$+2 \sum_{i=1}^N \int \langle \partial_{x_i} H_\varepsilon(x, \nabla u), \nabla \zeta \rangle \, u_{x_i} \, (w-k)_+ \, \zeta dx \tag{4.6}$$

$$= - \sum_{i=1}^N \int f \, u_{x_i x_i} \, (w^s - k_0^s)_+ \zeta^2 \, dx \tag{4.7}$$

$$- \sum_{i=1}^N \int f \, u_{x_i} \, ((w^s - k_0^s)_+)_{x_i} \, \zeta^2 \, dx \tag{4.8}$$

$$- \sum_{i=1}^N 2 \int f \, \zeta_{x_i} (w^s - k_0^s)_+ u_{x_i} \zeta \, dx. \tag{4.9}$$

As before, we divide the estimates into three groups.

4.1. **The main terms.** We observe that, using the fact that $\nabla(w-k)_+ = q \, w^{\frac{q-2}{q}} \, D^2 u \nabla u \, 1_{\{w>k\}}$, the integral (4.1) can be written as

$$\int \langle \nabla_z H_\varepsilon(x, \nabla u) D^2 u \nabla u, \nabla (w-k)_+ \rangle \, \zeta^2 \, dx = \frac{1}{q} \int \langle \nabla_z H_\varepsilon(x, \nabla u) \nabla (w-k)_+, \nabla (w-k)_+ \rangle \, \zeta^2 \, w^{\frac{2-q}{q}} dx$$

and the previous integral is restricted to the set $\{w > k\}$, so that taking $k \geq k_0$, with k_0 given by (2.4), and taking into account (2.3) we get

$$\int \langle \nabla_z H_\varepsilon(x, \nabla u) \nabla (w-k)_+, \nabla (w-k)_+ \rangle \, w^{\frac{2-q}{2}} \zeta^2 \, dx \geq \frac{1}{q\lambda} \int |\nabla (w-k)_+|^2 \zeta^2.$$

Then we observe that the term (4.2) can be treated as follows (this is the sponge term, as before)

$$\sum_{i=1}^N \int \langle \nabla_z H_\varepsilon(x, \nabla u) D_i^2 u, D_i^2 u \rangle \, (w-k)_+ \zeta^2 \, dx \geq \frac{1}{\lambda} \int |D^2 u|^2 \, w^{\frac{q-2}{q}} \, (w-k)_+ \, \zeta^2 \, dx := \frac{1}{\lambda} \mathcal{S}'(B_R),$$

while concerning the third integral (4.3), we get

$$\begin{aligned}2\int \langle \nabla_z H_\varepsilon(x,\nabla u) D^2 u \nabla u, \nabla\zeta\rangle\,(w-k)_+\zeta\,dx &= \frac{2}{q}\int \langle \nabla_z H_\varepsilon(x,\nabla u)\nabla(w-k)_+,\nabla\zeta\rangle \\ &\quad\times (w-k)_+\, w^{\frac{2-q}{q}}\,\zeta\,dx \\ &\le \frac{2}{q}\int |\nabla_z H_\varepsilon(x,\nabla u)||\nabla(w-k)_+|\,|\nabla\zeta| \\ &\quad\times \zeta\,(w-k)_+\,w^{\frac{2-q}{q}}\,dx \\ &\le \frac{2\lambda}{q}\int |\nabla(w-k)_+|\,|\nabla\zeta|\,\zeta(w-k)_+\,dx,\end{aligned}$$

where we have used again (2.3) and the fact that we are integrating over a region where $|\nabla u|\ge M$. Summarizing, we have obtained

$$\begin{aligned}\int |\nabla(w-k)_+|^2\,\zeta^2\,dx + \int |D^2 u|^2\,w^{\frac{q-2}{q}}\,(w-k)_+\,\zeta^2\,dx &\le C\int |\nabla(w-k)_+|\,|\nabla\zeta|\,\zeta(w-k)_+\,dx \\ &\quad + \text{Estimates for } (4.4)-(4.9),\end{aligned}$$

and with standard calculations we can then obtain the inequality

$$\begin{aligned}\int |\nabla(w-k)_+|^2\,\zeta^2\,dx + \int |D^2 u|^2\,w^{\frac{q-2}{q}}\,(w-k)_+\,\zeta^2\,dx &\le C\int |\nabla\zeta|^2 (w-k)_+^2\,dx, \\ &\quad + \text{Estimates for } (4.4)-(4.9),\end{aligned}\tag{4.10}$$

for every $k\ge k_0$.

4.2. **Terms containing f_ε.** Using the fact that $w\ge 1$ and Young's inequality, we get

$$\begin{aligned}\sum_i \int f\,u_{x_i x_i}(w-k)_+\zeta^2\,dx &\le \int |f|\left(\sum_i |D_i^2 u|^2\right)^{\frac12}(w-k)_+\,\zeta^2\,dx \\ &\le \tau\int |D^2u|^2 w^{\frac{q-2}{q}}(w-k)_+\zeta^2\,dx + \frac{1}{\tau}\int |f|^2(w-k)_+\zeta^2\,dx\end{aligned}$$

and observe that the first integral can be absorbed, choosing τ small enough, by the left-hand side. Concerning the second integral, we observe that we can get

$$\begin{aligned}\int |f|^2\,(w-k)_+\,\zeta^2\,dx &\le \frac12\int |f|^2\,(w-k)_+^2\,\zeta^2 + \frac12\int_{\{w\ge k\}\cap B_R} |f|^2 \\ &\le \frac12\left(\int |f|^N\,dx\right)^{\frac{2}{N}}\left(\int (w-k)_+^{2^*}\zeta^{2^*}\,dx\right)^{\frac{2}{2^*}} \\ &\quad + \frac12\|f\|^2_{L^{N+\alpha}}\,|\{w\ge k\}\cap B_R|^{1-\frac{2}{N+\alpha}} \\ &\le \frac{c}{2}\|f\|^2_{L^{N+\alpha}}\,R^{\frac{2\alpha}{N+\alpha}}\int |\nabla((w-k)_+\zeta)|^2\,dx \\ &\quad + \frac12\|f\|^2_{L^{N+\alpha}}\,|\{w\ge k\}\cap B_R|^{1-\frac{2}{N+\alpha}},\end{aligned}$$

just as in the previous section.

Let us then consider (4.8): we first observe that $|\nabla u| \le w^{\frac{1}{q}}$, so that

$$\sum_i \int f\, u_{x_i}((w-k)_+)_{x_i}\, \zeta^2\, dx \le \int |f|\, w^{\frac{1}{q}}\, |\nabla (w-k)_+|\, \zeta^2\, dx$$
$$\le \frac{1}{\tau}\int |f|^2\, w^{\frac{2}{q}}\, \zeta^2\, dx + \tau \int |\nabla (w-k)_+|^2\, \zeta^2\, dx,$$

and observe that the second term will be absorbed in the left-hand side.

As before, we write $w^{\frac{1}{q}}$ in place of ∇u and then we use Young's inequality, so (4.9) can be estimated as follows

$$2\sum_i \int f\, u_{x_i}\, \zeta_{x_i}\, (w-k)_+\, \zeta\, dx \le \int |f|^2\, w^{\frac{2}{q}} \zeta^2\, dx + \int |\nabla \zeta|^2\, (w-k)_+^2\, dx.$$

Before putting all the estimates together, we observe that the last two integral have a common term, which can be treated as follows

$$\int |f|^2\, w^{\frac{2}{q}}\, \zeta^2\, dx \le \int |f|^2\, w\, \zeta^2\, dx \le \int |f|^2\, (w-k)_+\, \zeta^2\, dx + k \int_{\{w>k\}\cap B_R} |f|^2\, dx,$$
$$\le \int |f|^2\, (w-k)_+\, \zeta^2\, dx + k\, \|f\|^2_{L^{N+\alpha}}\, |\{w \ge k\} \cap B_R|^{1-\frac{2}{N+\alpha}},$$

and the first integral above has already been estimated.

All in all, we have obtained the following:

$$\begin{aligned}\int |\nabla (w-k)_+|^2\, \zeta^2\, dx + \mathcal{S}'(B_R) dx &\le C_1 \int (w-k)_+^2\, |\nabla \zeta|^2\, dx \\ &\quad + C\,(1+k) \|f\|^2_{L^{N+\alpha}}\, |\{w \ge k\} \cap B_R|^{1-\frac{2}{N+\alpha}} \\ &\quad + \text{Estimates for } (4.4) - (4.6),\end{aligned} \tag{4.11}$$

with C_1 depending on $\|f\|_{L^{N+\alpha}}$.

4.3. **Terms involving derivatives of H_ε.** Using the growth conditions on $\partial_{x_i} \nabla \mathcal{H}^*_\varepsilon(x, \nabla u)$ and the fact that all the integrals are restricted to a region where $\{|\nabla u| > M\}$, we get

$$(4.4) \le c \int w^{\frac{q-1}{q}}\, |D^2 u|\, (w-k)_+\, \zeta^2\, dx \le c\tau \int w^{\frac{q-2}{q}}\, |D^2 u|^2\, (w-k)_+\, \zeta^2\, dx$$
$$+ \frac{c}{\tau} \int w\, (w-k)_+\, \zeta^2\, dx,$$

and we observe that the first term can be absorbed by the sponge term $\mathcal{S}'(B_R)$ as before, while using the simple inequality

$$w(w-k)_+ \le (w-k)_+^2 + k(w-k)_+ \le \frac{3}{2}(w-k)_+^2 + \frac{k^2}{2},$$

and the fact that

$$|\{w>k\} \cap B_R(x_0)| \le |B_R(x_0)|^{\frac{2}{N+\alpha}}\, |\{w>k\} \cap B_R(x_0)|^{1-\frac{2}{N+\alpha}} \le c\, |\{w>k\} \cap B_R(x_0)|^{1-\frac{2}{N+\alpha}},$$

the second term can be written as

$$\begin{aligned}\int w\,(w-k)_+\,\zeta^2\,dx &\le c\int (w-k)_+^2\,\zeta^2\,dx + ck^2|\{w>k\}\cap B_R(x_0)|\\ &\le cR^2\int |\nabla(w-k)_+|^2\,\zeta^2 dx + cR^2\int |\nabla\zeta|^2\,(w-k)_+^2\,dx\\ &+\overline{c}\,k^2|\{w>k\}\cap B_R(x_0)|^{1-\frac{2}{N+\alpha}},\end{aligned}$$

which is a kind of term that we have already treated (the first can be absorbed, taking R small enough, while the second and the third are good).

Concerning (4.5), we easily get

$$\begin{aligned}\sum_{i=1}^N\int \langle \partial_{x_i} H_\varepsilon(x,\nabla u), D_i^2 u\rangle\,(w-k)_+\,\zeta^2 dx &\le c\int w^{\frac{q-1}{q}}\,|\nabla(w-k)_+|\,|\nabla u|\,\zeta^2\,dx\\ &\le \int w\,|\nabla(w-k)_+|\,\zeta^2\,dx,\\ &\le c\tau\int |\nabla(w-k)_+|^2\,\zeta^2\,dx + \frac{c}{\tau}\int_{\{w>k\}} w^2\,\zeta^2\,dx,\end{aligned}$$

and the first can be absorbed, while the second has been already estimated in the case of (4.4): we can simply use $w^2\le 2(w-k)_+^2+2k^2$ and proceed as before.

Finally, for the last term, we have

$$(4.6)\le c\int w\,(w-k)_+\,|\nabla\zeta|\,\zeta\,dx \le \frac{c}{2}\int_{\{w>k\}} w^2\,\zeta^2\,dx + \frac{c}{2}\int (w-k)_+^2\,|\nabla\zeta|^2\,dx,$$

both being terms already estimated.

4.4. **Caccioppoli-type inequality.** All in all, putting all these estimates together into (4.11), after having added the term $\int |\nabla\zeta|^2\,(w-k)_+^2\,dx$ and applying Sobolev inequality, we can finally obtain the inequality

$$\int_{B_\varrho(x_0)} |\nabla(w-k)_+|^2\,dx \le \frac{C}{(R-\varrho)^2}\int_{B_R(x_0)} (w-k)_+^2\,dx + C\,k^2\,|\{w\ge k\}\cap B_R(x_0)|^{1-\frac{2}{N+\alpha}}, \tag{4.12}$$

which is valid for every $k\ge k_0$, with C depending on $\lambda, N, L, \delta, \|f\|_{L^{N+\alpha}}$, but not on ε. In particular, inequality (4.12) implies that w is in a De Giorgi class, so that thanks to Theorem B we get for every $B_R\Subset\Omega$

$$\sup_{B_{R/2}(x_0)} w\le C = C(R,\|w\|_{L^2(B_R)},k_0),$$

which gives the desired conclusion, thanks to the fact that $w\in L^2_{\rm loc}$, uniformly in ε as already proven in the previous section.

5. Proof of Theorem 2.1: Conclusion

Let us now take $u\in W^{1,q}_\diamond(\Omega)$ weak solution of (2.1), under the hypotheses of Theorem 2.1. What we have proven so far implies in particular that the minimizers u_ε of $\mathfrak{F}_\varepsilon$, which are equi-bounded in $W^{1,q}$, are also equi-bounded in $W^{1,\infty}$, so that $u_\varepsilon \overset{*}{\rightharpoonup} \widetilde{u}$ in $W^{1,\infty}$, up to a subsequence. It is only left to observe that this limit $\widetilde{u} \in W^{1,\infty}(\Omega)$ is a minimizer of $\mathfrak{F}$ by means of Proposition 2.2 and thus another weak solution of (2.1): this and the fact that (see Remark 2)

$$\nabla \mathcal{H}^*(\nabla \widetilde{u}(x)) = \nabla \mathcal{H}^*(\nabla u(x)), \quad \text{for } \mathscr{L}^N\text{−a.e. } x \in \Omega,$$

finally imply $\nabla u \in L^\infty(\Omega)$ as desired.

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