Strict BV relaxed area of Sobolev maps into the circle: the high dimension case

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Abstract

We deal with the relaxed area functional in the strict BV-convergence of non-smooth maps defined in domains of generic dimension and taking values into the unit circle. In case of Sobolev maps, a complete explicit formula is obtained. Our proof is based on tools from Geometric Measure Theory and Cartesian currents. We then discuss the possible extension to the wider class of maps with bounded variation. Finally, we show a counterexample to the locality property in case of both dimension and codimension larger than two.

Key words: Area functional, relaxation, Cartesian currents, strict convergence, $\mathbb{S}^1$-valued singular maps, distributional Jacobian.

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Introduction

In this paper, we continue the analysis of the explicit formula for the *relaxed area* functional with respect to the *strict convergence* for non-smooth vector-valued functions $u : B^n \to \mathbb{R}^2$.

For a smooth function $u : B^n \to \mathbb{R}^2$, we denote by $\mathcal{A}(u, B^n)$ the area of the graph of u, given by

$$\mathcal{A}(u, B^n) := \int_{B^n} \sqrt{1 + |\nabla u|^2 + |M_2(\nabla u)|^2}\, dx$$

where $|M_2(\nabla u)|^2$ is the sum of the square of all 2×2 minors of the gradient matrix ∇u, so that $|M_2(\nabla u)| = |\det \nabla u|$ if $n = 2$. In the sequel we shall write simply $\mathcal{A}(u) = \mathcal{A}(u, B^n)$.

Working with the natural L^1-convergence, the relaxed area functional is defined for every summable function $u \in L^1(B^n, \mathbb{R}^2)$ by

$$\overline{\mathcal{A}}_{L^1}(u) := \inf\Big\{ \liminf_{k\to\infty} \mathcal{A}(u_k) \mid \{u_k\} \subset C^1(B^n, \mathbb{R}^2),\ u_k \to u \text{ strongly in } L^1 \Big\}. \tag{0.1}$$

If $\overline{\mathcal{A}}_{L^1}(u) < \infty$, then necessarily u is a function of *bounded variation*. However, even in low dimension $n = 2$, it turns out that the localized functional $A \mapsto \overline{\mathcal{A}}_{L^1}(u, A)$ fails to be subadditive, and hence it cannot be extended to a Borel measure. This behavior was conjectured by De Giorgi in [15] and proved by Acerbi and Dal Maso in [1], where it is shown that non-subadditivity phenomena arise even for very simple cases like the vortex map $u_V(x) = x/|x|$ and the symmetric triple junction map u_T. A precise computation of the values $\overline{\mathcal{A}}_{L^1}(u_V)$ and $\overline{\mathcal{A}}_{L^1}(u_T)$ can be found in [7](see also [8]) and [9, 35] respectively. Moreover, for the analysis of the triple junction map without symmetry assumptions, we refer to [6], where the authors provide an upper bound for the respective L^1-relaxed area (0.1), conjectured to be optimal. Other interesting upper bounds were recently obtained in [12] for Sobolev maps valued in $\mathbb{S}^1$ and in [36] for piecewise constant maps taking three values.

The non-locality feature previously outlined makes quite challenging the relaxation analysis of $\mathcal{A}$. For this reason, it is interesting to consider some variants of (0.1), for example by strengthening the topology

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of the convergence of u_k to u (see [10, 11, 17, 21]). In recent years it has been proposed to impose the strict BV-convergence. Referring to e.g. [2] for the notation adopted in this paper, we only recall here that a sequence $\{u_k\} \subset BV(B^n, \mathbb{R}^2)$ is said to converge to $u \in BV(B^n, \mathbb{R}^2)$ strictly in the BV-sense, say $u_k \overset{BV}{\to} u$, if $u_k \to u$ in $L^1(B^n, \mathbb{R}^2)$ and $|Du_k|(B^n) \to |Du|(B^n)$, where $|Du|$ denoted the total variation of the distributional derivative Du.

For $u \in BV(B^n, \mathbb{R}^2)$, we thus denote

$$\overline{\mathcal{A}}_{BV}(u) := \inf\Big\{\liminf_{k\to\infty} \mathcal{A}(u_k) \mid \{u_k\} \subset C^1(B^n, \mathbb{R}^2),\ u_k \overset{BV}{\to} u \,\text{strictly in}\, BV\Big\}. \tag{0.2}$$

The reason for this choice is that we expect that whenever $\overline{\mathcal{A}}_{BV}(u) < \infty$, then the localized functional $A \mapsto \overline{\mathcal{A}}_{BV}(u, A)$ gives rise to a Borel measure. In case of low dimension $n = 2$, partial results concerning the explicit formula of $\overline{\mathcal{A}}_{BV}(u)$ have been obtained in [4, 5, 14].

In this paper, we focus on the class of BV-maps taking values into the unit circle. More precisely, we denote by

$$\mathbb{S}^1 := \{y \in \mathbb{R}^2 \,:\, |y| = 1\}, \quad D^2 := \{y \in \mathbb{R}^2 \,:\, |y| < 1\}$$

the unit circle and disk in the target space, and for $X = BV$ or $W^{1,1}$, we let

$$X(B^n, \mathbb{S}^1) := \{u \in X(B^n, \mathbb{R}^2) \mid |u(x)| = 1 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n\}$$

where $\mathcal{L}^n$ is the Lebesgue measure, and we focus on the class of Sobolev maps $W^{1,1}(B^n, \mathbb{S}^1)$.

In low dimension $n = 2$, the following result was obtained in [4].

Theorem 0.1. *Let $u \in W^{1,1}(B^2, \mathbb{S}^1)$, and let* $\operatorname{Det} \nabla u$ *denote the* distributional determinant *of u. Then,*

$$\overline{\mathcal{A}}_{BV}(u) < \infty \iff |\operatorname{Det} \nabla u|(B^2) < \infty\,.$$

In that case, moreover, one has:

$$\overline{\mathcal{A}}_{BV}(u) = \int_{B^2} \sqrt{1 + |\nabla u|^2}\, dx + |\operatorname{Det} \nabla u|(B^2)\,.$$

The previous result says that the *energy gap* is detected by the distributional determinant. Referring to the next section for the notation adopted here, we recall that in any dimension $n \geq 2$, the *current carried by the graph* of a Sobolev map $u \in W^{1,1}(B^n, \mathbb{S}^1)$ is an integer multiplicity rectifiable n-current G_u in $B^n \times \mathbb{S}^1$, with finite mass

$$\mathbf{M}(G_u) = \int_{B^n} \sqrt{1 + |\nabla u|^2}\, dx < \infty\,.$$

Moreover, the relevant singularities of u are detected by the current G_u, i.e., they can be described by homological arguments.

More precisely, denoting by ω_2 the closed 1-form in $\mathbb{S}^1$

$$\omega_2 := \frac{1}{2}\left(y^1\, dy^2 - y^2\, dy^1\right)$$

the singularities are read by the $(n-2)$-dimensional current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ defined by

$$\mathbb{P}(u)(\eta) := -\frac{1}{\pi} G_u(d\eta \wedge \omega_2)\,, \quad \eta \in \mathcal{D}^{n-2}(B^n)\,.$$

Therefore, in low dimension $n = 2$ one has

$$\pi \cdot \mathbb{P}(u)(\eta) = \langle \operatorname{Det} \nabla u, \eta\rangle \quad \forall\, \eta \in C_c^\infty(B^2)\,.$$

If e.g. $u(x) = x/|x|$, one gets $\mathbb{P}(u) = \delta_{0_{\mathbb{R}^2}}$, the unit Dirac mass at the origin.

In our Main Result, we extend the previous explicit formula to any high dimension n.

Theorem 0.2. *Let $n \geq 2$ and $u \in W^{1,1}(B^n, \mathbb{S}^1)$. Then, $\overline{\mathcal{A}}_{BV}(u) < \infty$ if and only if the $(n-2)$-current $\mathbb{P}(u)$ is i.m. rectifiable and with finite mass, $\mathbf{M}(\mathbb{P}(u)) < \infty$. In that case, moreover, one has:*

$$\overline{\mathcal{A}}_{BV}(u) = \int_{B^n} \sqrt{1 + |\nabla u|^2}\, dx + \pi\, \mathbf{M}(\mathbb{P}(u))\,.$$

For our purposes, we exploit in our context a result taken from [30]. It says that if $u \in W^{1,1}(B^n, \mathbb{S}^1)$ satisfies $\overline{\mathcal{A}}_{BV}(u) < \infty$, then there exists a unique optimal *Cartesian current* T_u that encloses the graph of u, and it is given by

$$T_u = G_u + (-1)^{n-2}\mathbb{P}(u) \times [\![D^2]\!] .$$

Therefore, the proof of the energy lower bound readily follows: by Federer's closure-compactness theorem, for every smooth sequence $\{u_h\}$ strictly converging to u, the graph G_{u_h} weakly converges to T_u (up to extracting a subsequence), and one concludes from the semicontinuity of the mass. On the other hand, the energy upper bound holds true as a consequence of the following approximation result:

Theorem 0.3. *Let $n \geq 2$ and $u \in W^{1,1}(B^n, \mathbb{S}^1)$ be a Sobolev map with finite relaxed energy* (0.2)*. Then, there exists a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{R}^2)$ such that $G_{u_h} \rightharpoonup T_u$ weakly in $\mathcal{D}_n(B^n \times \mathbb{R}^2)$ and $\mathbf{M}(G_{u_h}) \to \mathbf{M}(T_u)$ as $h \to \infty$.*

In Section 1, we collect some notation and preliminary results. In Section 2, we give an explicit example, showing the strategy we follow in the proof of the relaxed formula. In Section 3 we prove our Main Result, Theorem 0.2. The proof of the approximation theorem 0.3 is based on several technical results, the proof of which is postponed to Section 4, for the sake of clarity. The fundamental step at the base of Theorem 0.3 is contained in Theorem 3.4 and consists to reduce the proof to the case u is smooth outside a "nice" singular set, precisely $\mathbb{P}(u)$ is a polyhedral chain. This reduction can be done provided that the mass of the singularities current $\mathbb{P}(u_k)$ of the modified map u_k converges to the mass of $\mathbb{P}(u)$. We point out that by a direct application of Bethuel's approximation theorem [13, Thm. 2] and Hardt-Pitts results in [25], one obtains the flat norm convergence of $\mathbb{P}(u_k)$ to $\mathbb{P}(u)$, which is not enough for our purpose. The actual proof requires a deeper use of Bethuel's result in a more involved construction argument, based on Federer's strong polyhedral approximation theorem. Once u can be supposed to be smooth out of the support of the polyhedral $(n-2)$-chain $\mathbb{P}(u)$, by a standard argument based on the dipole construction idea, we can build a recovery sequence for the energy (Theorem 3.5), taking care of removing higher codimension singularities generated in the dipole construction (Theorems 3.6 and 3.7).

Finally, in Section 5 we briefly discuss some related open questions, mainly concerning the validity of an explicit formula of the relaxed energy in the wider class of maps in $BV(B^n, \mathbb{S}^1)$. Moreover, we show the non-locality of $\overline{\mathcal{A}}_{BV}$ in dimension and codimension greater than 2. Precisely, the set function $A \to \overline{\mathcal{A}}_{BV}(u, A)$ fails to be subadditive for $u : B^3 \to \mathbb{R}^3$, even in the Sobolev case, as provided by the vortex map $u_V(x) = x/|x|$.

1 Notation and preliminary results

In this section, we collect some background material and preliminary results.

1.1 Functions of bounded variation

Referring to [2] for the notation on BV-functions, we recall that u belongs to $BV(B^n, \mathbb{R}^2)$ if $u \in L^1(B^n, \mathbb{R}^2)$ and the distributional derivative Du is an $\mathbb{R}^{2\times n}$-valued Borel measure with finite total variation. The usual decomposition

$$Du = D^a u + D^C u + D^J u$$

into the mutually singular absolutely continuous, Cantor, and Jump components is adopted. In particular, $D^a u = \nabla u \, d\mathcal{L}^n$, where $\nabla u \in L^1(B^n, \mathbb{R}^{n\times 2})$ is the approximate gradient and $\mathcal{L}^n$ the Lebesgue measure. The Jump component is given by $D^J u = (u^+ - u^-) \otimes \nu \, \mathcal{H}^{n-1} \llcorner J_u$, where $\mathcal{H}^{n-1}$ is the Hausdorff measure, J_u is the Jump set of u, a countably $(n-1)$-rectifiable set of B^n, ν a unit normal to J_u, and $u^\pm$ the approximate limits of u at points in J_u w.r.t. the given unit normal. The Cantor component satisfies $|D^C u|(B) = 0$ for each Borel set $B \subset B^n$ such that $\mathcal{H}^{n-1}(B) = 0$. Therefore, if $D^J u = D^C u = 0$, actually u is a Sobolev function in $W^{1,1}(B^n, \mathbb{R}^2)$. Finally, we recall that the *strict convergence* $u_k \overset{BV}{\to} u$ in $BV(B^n, \mathbb{R}^2)$ is given by the strong convergence $u_k \to u$ in $L^1(B^n, \mathbb{R}^2)$ joined with the total variation convergence $|Du_k|(B^n) \to |Du|(B^n)$, as $k \to \infty$.

1.2 Rectifiable currents

For a given open set $U \subset \mathbb{R}^N$, the space $\mathcal{D}_k(U)$ of k-dimensional currents in U is the strong dual of the space $\mathcal{D}^k(U)$ of compactly supported smooth k-forms in U, for $k = 0, \ldots, N$. For any $T \in \mathcal{D}_k(U)$, we

define its *mass* $\mathbf{M}(T)$ as
$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \le 1\}\,,$$
where $\|\omega\|$ is the *comass norm.*
The *weak convergence* $T_h \rightharpoonup T$ in $\mathcal{D}_k(U)$ is defined by the convergence
$$\lim_{h\to\infty} T_h(\omega) = T(\omega) \quad \forall\, \omega \in \mathcal{D}^k(U)$$
and in that case one has
$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h)\,.$$
For $k \ge 1$, the *boundary* of $T \in \mathcal{D}_k(U)$ is the $(k-1)$-current ∂T defined by relation
$$\partial T(\eta) := T(d\eta)\,, \qquad \eta \in \mathcal{D}^{k-1}(U)$$
where $d\eta$ is the differential of η, and we set $\partial T = 0$ if $k = 0$. For $k \ge 1$, a k-current T with finite mass is called *rectifiable* if there exist a k-rectifiable set $\mathcal{M}$ in U, an $\mathcal{H}^k \llcorner \mathcal{M}$-measurable function $\xi : \mathcal{M} \to \Lambda^k\mathbb{R}^m$ such that $\xi(x)$ is a simple unit k-vector orienting the approximate tangent space to $\mathcal{M}$ at $\mathcal{H}^k$-a.e. $x \in \mathcal{M}$, and an $\mathcal{H}^k \llcorner \mathcal{M}$-summable and non-negative function $\theta : \mathcal{M} \to [0, +\infty)$ such that
$$T(\omega) = \int_{\mathcal{M}} \theta\, \langle \omega, \xi\rangle\, d\mathcal{H}^k \qquad \forall\, \omega \in \mathcal{D}^k(U)\,.$$
We thus get $\mathbf{M}(T) = \int_{\mathcal{M}} \theta\, d\mathcal{H}^k < \infty$. In addition, if the multiplicity function θ is integer-valued, the current T is called *i.m. rectifiable* and the corresponding class is denoted by $\mathcal{R}_k(U)$. If e.g. $\mathcal{M}$ is a smooth k-manifold in U with $\mathcal{H}^k(\mathcal{M}) < \infty$, taking $\theta = 1$ we obtain the current $[\![\mathcal{M}]\!] \in \mathcal{R}_k(U)$ whose action on k-forms agrees with the classical notation from Differential Geometry. In particular, for $k = 0$, a current T in $\mathcal{R}_0(U)$ is given by
$$T = \sum_{i=1}^m d_i\, \delta_{a_i}$$
where $m \in \mathbb{N}^+$, $d_i \in \mathbb{Z}$, $a_i \in U$ for $i = 1, \dots, m$, and δ_a is the unit Dirac mass at a point $a \in U$.
Finally, a current T is called *integral* if both T and ∂T are i.m. rectifiable currents. By the boundary rectifiability theorem (cf. [34, 30.3]), if T is i.m. rectifiable and $\mathbf{M}(\partial T) < \infty$, then T is integral. We refer to [34, Ch. 6] and [20, Ch. 2] for further details.

1.3 Graph currents

If $u \in C^1(\overline{B}^n, \mathbb{R}^2)$, the graph current G_u in $\mathcal{R}_n(B^n \times \mathbb{R}^2)$ is given by integration on the oriented graph n-manifold $\mathcal{G}_u$. Therefore, by the area formula we equivalently have
$$G_u(\omega) := \int_{B^n} (\mathrm{Id} \bowtie u)^{\#}\omega\,, \qquad \omega \in \mathcal{D}^n(B^n \times \mathbb{R}^2) \tag{1.1}$$
where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ is the graph map, and its mass satisfies
$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = \mathcal{A}(u) = \int_{B^n} \sqrt{1 + |\nabla u|^2 + |M_2(\nabla u)|}\, dx\,. \tag{1.2}$$

To every Sobolev map $u \in BV(B^n, \mathbb{S}^1)$, we associate the n-current G_u in $\mathcal{R}_n(B^n \times \mathbb{R}^2)$ carried by the "graph" of u. It is given again by (1.1), where this time the pull-back makes sense in terms of the approximate gradient ∇u of u. Every n-form $\omega \in \mathcal{D}^n(B^n \times \mathbb{R}^2)$ splits as $\omega^{(0)} + \omega^{(1)} + \omega^{(2)}$ according to the number of "vertica" differentials. Writing $\omega^{(0)} = \phi(x, y)\, dx$ for some $\phi \in C_c^\infty(B^n \times \mathbb{R}^2)$, where $dx := dx^1 \wedge \cdots \wedge dx^n$, we have
$$G_u(\phi(x, y)\, dx) = \int_{B^n} \phi(x, u(x))\, dx\,.$$
Setting moreover $\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$, we may write
$$\omega^{(1)} = \sum_{i=1}^n \sum_{j=1}^2 (-1)^{n-i} \phi_i^j(x, y)\, \widehat{dx^i} \wedge dy^j \tag{1.3}$$

for some $\phi_i^j \in C_c^\infty(B^n \times \mathbb{R}^2)$, and we obtain

$$G_u(\omega^{(1)}) = \sum_{j=1}^{2}\sum_{i=1}^{n} \int_{B^n} \nabla_i u^j(x)\phi_i^j(x,u(x))\,dx\,.$$

Finally, by the area formula we have

$$G_u(\omega^{(2)}) = 0 \quad \forall\,\omega \in \mathcal{D}^n(B^n \times \mathbb{R}^2)\,.$$

In particular, we get

$$\mathbf{M}(G_u) = \int_{B^n} \sqrt{1+|\nabla u|^2}\,dx\,.$$

In general, the graph current G_u of a Sobolev map $u \in W^{1,1}(B^n,\mathbb{S}^1)$ has a non zero boundary in $B^n \times \mathbb{R}^2$. Taking for example $n=2$ and $u(x) = x/|x|$, we have:

$$\partial G_u \llcorner B^2 \times \mathbb{R}^2 = -\delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^1 \,]\!]\,.$$

However, a density argument shows that the boundary current ∂G_u is null on every $(n-1)$-form in $B^n \times \mathbb{R}^2$ which has no "vertical" differentials. Moreover, G_u is an integral flat chain in $B^n \times \mathbb{R}^2$ with support contained in $\overline{B}^n \times \mathbb{S}^1$. Therefore, by Federer's flatness theorem we can see G_u as a current in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$, and actually

$$(\partial G_u) \llcorner B^n \times \mathbb{R}^2 = (\partial G_u) \llcorner B^n \times \mathbb{S}^1\,. \tag{1.4}$$

1.4 Singularities

If $u \in W^{1,1}(B^n,\mathbb{R}^2) \cap L^\infty(B^n,\mathbb{R}^2)$, it is well defined the distribution

$$\mathrm{Div}_{\bar\alpha}\,\mathbf{m}_u := \frac{1}{2}\sum_{j=1}^{2}\sum_{i\in\bar\alpha} \frac{\partial}{\partial x_i}\big(u^j(x)\,((\mathrm{adj}\,\nabla u)_{\overline{\alpha}})_i^j\big) \tag{1.5}$$

for each ordered multi-index α of length $n-2$ in $\{1,\dots,n\}$, where $\bar\alpha$ is the complementary ordered index of length two. For $n=2$, the right hand side of definition (1.5) reduces to the distributional determinant $\mathrm{Det}\,\nabla u$. In high dimension $n \geq 3$, instead, we obtain the $\bar\alpha$-component of the distributional Jacobian $J(u)$, which can be viewed as an $\mathbb{R}^{d(n)}$-valued distribution, with $d(n) = n(n-1)/2$. The notion of distributional Jacobian was first introduced in [27] (see also [3, 31, 33]) to analyse singularities of non-smooth maps and has been widely studied in the literature, together with the related notion of relaxed Jacobian total variation [16, 17, 19, 28, 29, 32]. Notice that

$$\mathrm{Div}_{\bar\alpha}\,\mathbf{m}_u = M_2(\nabla u)_{\bar\alpha} \qquad \text{if } u \text{ is smooth,}$$

where $M_2(\nabla u)_{\bar\alpha}$ is the 2×2 minor of the gradient matrix $\nabla u \in \mathbb{R}^{2\times n}$ with columns detected by $\bar\alpha$.
The measure $\mathrm{Div}_{\bar\alpha}\,\mathbf{m}_u$ can be defined also for any BV-map u with finite relaxed energy (0.2), by considering Du in place of ∇u, see [30] for further details.
Now suppose that $u \in W^{1,1}(B^n,\mathbb{S}^1)$. We can easily relate the distributional Jacobian $J(u)$ to an i.m. rectifiable current $\mathbb{P}(u)$ defined as follows. Let $\pi : B^n \times \mathbb{R}^2 \to B^n$ and $\widehat{\pi} : B^n \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the orthogonal projections onto the first and second factor, respectively. The current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ of the singularities of u is given by

$$\mathbb{P}(u)(\eta) := -\frac{1}{\pi} G_u(\pi^{\#} d\eta \wedge \widehat{\pi}^{\#}\omega_2)\,, \quad \eta \in \mathcal{D}^{n-2}(B^n), \tag{1.6}$$

where ω_2 denote the closed 1-form in $\mathbb{S}^1$

$$\omega_2 := \frac{1}{2}\left(y^1\,dy^2 - y^2\,dy^1\right), \tag{1.7}$$

so that ω_2 is a generator of the first cohomology group of $\mathbb{S}^1$, and $d\omega_2 = dy := dy^1 \wedge dy^2$, as a form in $\mathbb{R}^2$. In the sequel, when it is clear from the context we omit to write the action of the projection maps. Since $d(\eta \wedge \omega_2) = d\eta \wedge \omega_2 + (-1)^{n-2}\eta \wedge dy$, whereas

$$G_u(d\eta \wedge \omega_2) = -\pi \cdot \mathbb{P}(u)(\eta)\,, \quad G_u(\eta \wedge dy) = 0,$$

on account of (1.4) we obtain:

$$(\partial G_u) \llcorner B^n \times \mathbb{R}^2 = (\partial G_u) \llcorner B^n \times \mathbb{S}^1 = -\mathbb{P}(u) \times [\![\mathbb{S}^1]\!] \,. \tag{1.8}$$

Moreover, by the very definition it turns out that

$$\partial \mathbb{P}(u) \llcorner B^n = 0 \,. \tag{1.9}$$

This property is trivial when $n = 2$, whereas in high dimension $n \geq 3$, for every $\varphi \in \mathcal{D}^{n-3}(B^n)$ we get

$$\partial \mathbb{P}(u)(\varphi) = \mathbb{P}(u)(d\varphi) = -\frac{1}{\pi} G_u(\pi^{\#} d(d\varphi) \wedge \widehat{\pi}^{\#} \omega_2) = 0,$$

since $d(d\varphi) = 0$.

For future use, we recall that a Cartesian current T in $B^n \times \mathbb{S}^1$ with underlying map u in $W^{1,1}(B^n, \mathbb{S}^1)$ is given by

$$T = G_u + L \times [\![\mathbb{S}^1]\!] \tag{1.10}$$

for some i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$, with finite mass, satisfying the boundary condition $(\partial L) \llcorner B^n = \mathbb{P}(u)$, compare [23, Sec. 1.5] or [21, Sec. 6.2.2].

Example 1.1. If $u \in W^{1,1}(B^2, \mathbb{S}^1)$, we have $\pi\, \mathbb{P}(u) = \mathrm{Det}\, \nabla u$. In particular, if u is smooth outside a finite set of points $\Sigma = \{a_1, \ldots, a_m\}$, we obtain

$$\mathbb{P}(u) = \sum_{i=1}^{m} \deg(u, a_i)\, \delta_{a_i} \,, \tag{1.11}$$

where $\deg(u, a_i) \in \mathbb{Z}$ is the Brouwer degree[1] of u around the point a_i. For example, with $u(x) = x/|x|$, we get $\mathbb{P}(u) = \delta_{0_{\mathbb{R}^2}}$. In high dimension $n \geq 3$, for any $u \in W^{1,1}(B^n, \mathbb{S}^1)$ we get $\pi\, \mathbb{P}(u) = J(u)$. In Section 2, we deal with the map

$$u(x) = \frac{\widetilde{x}}{|\widetilde{x}|} \,, \qquad x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \,,$$

so that $\mathbb{P}(u) = (-1)^{n-2} [\![\Delta^{n-2}]\!]$, where $[\![\Delta^{n-2}]\!]$ is the $(n-2)$-current given by integration on the naturally oriented $(n-2)$-disk

$$\Delta^{n-2} := \{(0, 0, \widehat{x}) \in \mathbb{R}^n \,:\, |\widehat{x}| \leq 1\} \,.$$

1.5 Stratification

If $n \geq 3$, a current $T \in \mathcal{R}_n(B^n \times \mathbb{R}^2)$ is identified by the measures

$$\begin{gathered} \mu_h[T] := T \llcorner dx \,, \qquad \mu_i^j[T] := T \llcorner (-1)^{i-1} dy^j \wedge \widehat{dx^i} \,, \\ \mu_v^{\bar{\alpha}}[T] := T \llcorner \sigma(\alpha, \bar{\alpha})\, dx^\alpha \wedge dy \,, \qquad dy := dy^1 \wedge dy^2 \end{gathered}$$

for each $i = 1, \ldots, n$, $j = 1, 2$, and each ordered multi-index α of length $n-2$ in $\{1, \ldots, n\}$, where the sign $\sigma(\alpha, \bar{\alpha}) = \pm 1$ is such that $dx^\alpha \wedge dx^{\bar{\alpha}} = \sigma(\alpha, \bar{\alpha})\, dx$. We also fix an order on the set of the $d(n) := n(n-1)/2$ multi-indexes $\bar{\alpha}$ of length two in $\{1, \ldots, n\}$, and we correspondingly denote by $\mu_v[T]$ the $\mathbb{R}^{d(n)}$-valued measure in $B^n \times \mathbb{R}^2$ with components $\mu_v^{\bar{\alpha}}[T]$. If $n = 2$, then $\mu_v[T] := T \llcorner dy$.

Notice that if $T = G_u$ for some smooth function $u \in C^1(\overline{B}^n, \mathbb{R}^2)$, by (1.1) we readily obtain $\mu_h[G_u] = (\mathrm{Id} \bowtie u)_{\#}(\mathcal{L}^n \llcorner B^n)$, $\mu_i^j[G_u] = (\mathrm{Id} \bowtie u)_{\#}(\nabla_i u^j\, \mathcal{L}^n \llcorner B^n)$, and also

$$\mu_v^{\bar{\alpha}}[G_u] = (\mathrm{Id} \bowtie u)_{\#}(M_2(\nabla u)_{\bar{\alpha}}\, \mathcal{L}^n \llcorner B^n) \qquad \forall\, \alpha.$$

1.6 The optimal lifting Cartesian current

Assume now that $u \in W^{1,1}(B^n, \mathbb{S}^1)$ has finite relaxed energy (0.2). Then, viewing G_u as a current in $B^2 \times \mathbb{R}^2$, by the results from [30], it turns out that there exists a unique i.m. rectifiable current $T_u \in \mathcal{R}_n(B^n \times \mathbb{R}^2)$ satisfying the following properties:

i) $\mathbf{M}(T_u) < \infty$ and $(\partial T_u) \llcorner B^n \times \mathbb{R}^2 = 0$;

[1] We refer to [20, Sec. 3.1] for the definition of Brouwer degree of a Sobolev map, see also [4, Sec. 2.3].

ii) if $S_u := T_u - G_u$, then S_u is completely vertical, i.e., $S_u(\omega) = 0$ for every $\omega \in \mathcal{D}^n(B^n \times \mathbb{R}^2)$ such that $\omega^{(2)} = 0$.

In particular, T_u is a Cartesian current in $\mathrm{cart}(B^n \times \mathbb{R}^2)$, see [20, Ch. 4], and

$$\mathbf{M}(T_u) = \mathbf{M}(G_u) + \mathbf{M}(S_u)\,.$$

More precisely, the horizontal component of T_u satisfying $\mu_h[T_u] = (\mathrm{Id} \bowtie u)_\#(\mathcal{L}^n \llcorner B^n)$, we require that the intermediate components only depend on u through formulas

$$\mu_i^j[T_u] = \mu_i^j[u] \quad \forall\, i,j \tag{1.12}$$

where $\mu_i^j[u]$ is the minimal lifting measure in the sense of Jerrard-Jung [26]. Therefore,

$$\mu_i^j[u] = (\mathrm{Id} \bowtie u)_\#(\nabla_i u^j\, \mathcal{L}^n \llcorner B^n)\,.$$

For each multi-index α of length $n-2$ as above, we thus get

$$\int_{B^n} g(x)\, d\mu_v^{\bar\alpha}[T_u] = \langle \mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u, g\rangle \qquad \forall\, g \in C_c^\infty(B^n), \tag{1.13}$$

where $\mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u$ is defined in (1.5), so that actually

$$\mathbb{P}(u)(g(x)\, dx^\alpha) = \frac{1}{\pi}\,(-1)^{n-2}\sigma(\alpha,\bar\alpha)\,\langle \mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u, g\rangle \quad \forall\, g \in C_c^\infty(B^n)\,. \tag{1.14}$$

We are now in position to prove the following

Theorem 1.2. *Let $n \geq 2$ and $u \in W^{1,1}(B^n, \mathbb{S}^1)$ be a Sobolev map with finite relaxed energy* (0.2)*. Then*

$$S_u = (-1)^{n-2}\mathbb{P}(u) \times [\![\, D^2 \,]\!] \tag{1.15}$$

where $\mathbb{P}(u)$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ with finite mass and no inner boundary, see (1.9).

Proof. By (1.13), for every α we get the total variation bound:

$$|\,\mathrm{Div}_{\bar\alpha}\, \mathbf{m}_u|(B^n) \leq |\mu_v^{\bar\alpha}[T_u]|(B^n \times \mathbb{R}^2) < \infty\,.$$

As a consequence, equation (1.14) implies that the current $\mathbb{P}(u)$ has finite mass.

On the other hand, by [22] we already know that the relaxed total variation energy of u as a map in $BV(B^2, \mathbb{S}^1)$ is finite, whence the class of Cartesian currents in $B^n \times \mathbb{S}^1$ with underlying map equal to u is non-empty, see (1.10). Therefore, there exists $L \in \mathcal{R}_{n-1}(B^n)$ such that $(\partial L) \llcorner B^n = \mathbb{P}(u)$, i.e., it turns out that $\mathbb{P}(u)$ is an integral flat chain. As a consequence, by the boundary rectifiability theorem, see [34, Sec. 30], we infer that $\mathbb{P}(u)$ is i.m. rectifiable in $\mathcal{R}_{n-2}(B^n)$. Furthermore, we already know that $\mathbb{P}(u)$ has no inner boundary, see (1.9).

Setting now

$$T = G_u + S_u$$

where S_u is the n-current given by (1.15), it suffices to show that T is a Cartesian current. Since in fact S_u is completely vertical, by uniqueness of the optimal lifting Cartesian current we readily obtain that $T = T_u$. By the structure theorem, see [20, Ch. 4], since we have just obtained that S_u is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{R}^2)$, it suffices to show that T satisfies the null-boundary condition

$$(\partial T) \llcorner B^n \times \mathbb{R}^2 = 0\,. \tag{1.16}$$

In fact, we have:

$$(\partial T) \llcorner B^n \times \mathbb{R}^2 = (\partial G_u) \llcorner B^n \times \mathbb{R}^2 + (-1)^{n-2}(\partial(\mathbb{P}(u) \times [\![\, D^2 \,]\!])) \llcorner B^n \times \mathbb{R}^2$$

where by the definition of boundary of a product of currents

$$(\partial(\mathbb{P}(u) \times [\![\, D^2 \,]\!])) \llcorner B^n \times \mathbb{R}^2 = (\partial \mathbb{P}(u)) \llcorner B^n \times [\![\, D^2 \,]\!] + (-1)^{n-2}\mathbb{P}(u) \times \partial [\![\, D^2 \,]\!]$$

so that (1.16) follows from (1.8), (1.9), and property $\partial [\![\, D^2 \,]\!] = [\![\, \mathbb{S}^1 \,]\!]$. □

Remark 1.3. As a consequence, in high dimension $n \geq 3$, by the previous result we infer that the distributional Jacobian $J(u)$ can be viewed as an $\mathbb{R}^{d(n)}$-valued measure, with $d(n) = n(n-1)/2$, that is concentrated on the $(n-2)$-rectifiable set of points of positive multiplicity of the current $\mathbb{P}(u)$, and actually

$$|J(u)|(B^n) = \pi \cdot \mathbf{M}(\mathbb{P}(u)) < \infty\,.$$

2 Examples

In this section, we give an easier example showing the strategy in our proof. We then show the existence of Sobolev maps in $W^{1,1}(B^n,\mathbb{S}^1)$ for which the relaxed energy is not finite.

2.1 A model example

Let $u\in W^{1,1}(B^n,\mathbb{S}^1)$ be defined as

$$u(x)=\frac{\widetilde{x}}{|\widetilde{x}|}\,,\qquad x=(\widetilde{x},\widehat{x})\in\mathbb{R}^2\times\mathbb{R}^{n-2}.$$

If $n\geq 3$, the singular set of u is the $(n-2)$-disk

$$\Delta^{n-2}:=\{(0,0,\widehat{x})\in\mathbb{R}^n\,:\,|\widehat{x}|\leq 1\}.$$

With the previous notation we get $\mathbb{P}(u)=(-1)^{n-2}[\![\,\Delta^{n-2}\,]\!]$, so that by (1.8)

$$(\partial G_u)\llcorner B^n\times\mathbb{R}^2=(-1)^{n-1}[\![\,\Delta^{n-2}\,]\!]\times[\![\,\mathbb{S}^1\,]\!].$$

This way we can equivalently write the lower bound for the relaxed energy as

$$\int_{B^n}\sqrt{1+|\nabla u|^2}\,dx+|J(u)|(B^n)$$

where in low dimension $n=2$ we clearly have $J(u)=\operatorname{Det}\nabla u$.

For the upper bound estimate, we define a recovery sequence by constructing for each $\varepsilon>0$ small a suitable cone shaped neighborhood U_ε of Δ^{n-2} in the following way:

$$U_\varepsilon:=\{x\in B^n:|\widetilde{x}|\leq\varepsilon(1-|\widehat{x}|)\}$$

and by defining $u_\varepsilon\in C^1(B^n,\mathbb{R}^2)$ as

$$u_\varepsilon(x):=\begin{cases}u(x) & \text{if } x\in B^n\setminus U_\varepsilon,\\ \dfrac{|\widetilde{x}|}{\varepsilon(1-|\widehat{x}|)}u(x) & \text{if } x\in U_\varepsilon.\end{cases}\tag{2.1}$$

In the case $n=3$, U_ε is a (double) cone of basis the disk $\widetilde{B}_\varepsilon:=\{x\in B^3:x=(\widetilde{x},0),|\widetilde{x}|\leq\varepsilon\}$ and of vertices the North and South Poles of B^3 (see Fig. 1).

Let us check that $u_\varepsilon\to u$ in $W^{1,1}(B^n,\mathbb{R}^2)$ as $\varepsilon\to 0$. Clearly, $\int_{U_\varepsilon}|u_\varepsilon|\,dx\to 0$, since $|u_\varepsilon|\leq|u|=1$. Therefore, it is enough to prove that

$$\lim_{\varepsilon\to 0}\int_{U_\varepsilon}|\nabla u_\varepsilon(x)|dx=0\,.\tag{2.2}$$

In cylindrical coordinates

$$\bar{u}_\varepsilon(\rho,\theta,\widehat{x}):=u_\varepsilon(\rho\cos\theta,\rho\sin\theta,\widehat{x}),\ \rho\in[0,1],\ \theta\in[0,2\pi),\ \widehat{x}\in\mathbb{R}^{n-2}\,,$$

where $|\widehat{x}|\leq 1$, we have

$$\bar{u}_\varepsilon(\rho,\theta,\widehat{x})=\frac{\rho}{\varepsilon(1-|\widehat{x}|)}(\cos\theta,\sin\theta)\quad\text{if }\rho\leq\varepsilon(1-|\widehat{x}|)\,.$$

Compute the partial derivatives of $\bar{u}_\varepsilon$:

$$\begin{aligned}\partial_\rho\bar{u}_\varepsilon(\rho,\theta,\widehat{x})&=\frac{1}{\varepsilon(1-|\widehat{x}|)}(\cos\theta,\sin\theta),\\ \partial_\theta\bar{u}_\varepsilon(\rho,\theta,\widehat{x})&=\frac{\rho}{\varepsilon(1-|\widehat{x}|)}(-\sin\theta,\cos\theta),\\ \partial_{\widehat{x}}\bar{u}_\varepsilon(\rho,\theta,\widehat{x})&=\frac{\rho}{\varepsilon(1-|\widehat{x}|)^2}(\cos\theta,\sin\theta)\otimes\frac{\widehat{x}}{|\widehat{x}|}.\end{aligned}$$

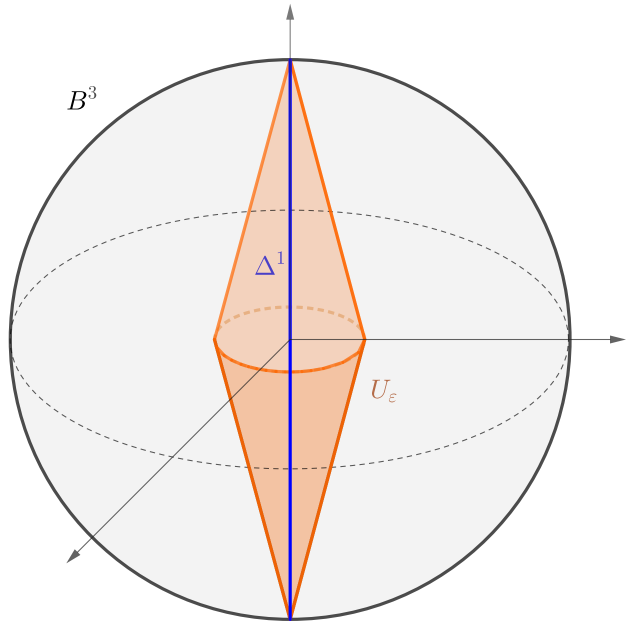

Figure 1: The cone shaped neighborhood U_ε depicted in dimension $n = 3$.

Moreover, by identifying the set $\{\widehat{x} \in \mathbb{R}^{n-2} : |\widehat{x}| \le 1\}$ with Δ^{n-2}, we have

$$\begin{aligned}
&\int_{U_\varepsilon} |\nabla u_\varepsilon(x)| dx = \\
&= \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \rho \sqrt{|\partial_\rho \bar{u}_\varepsilon|^2 + \frac{|\partial_\theta \bar{u}_\varepsilon|^2}{\rho^2} + |\partial_{\widehat{x}} \bar{u}_\varepsilon|^2}\, d\rho d\theta d\widehat{x} \\
&= \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \rho \sqrt{\frac{2}{\varepsilon^2(1-|\widehat{x}|)^2} + \frac{\rho^2}{\varepsilon^2(1-|\widehat{x}|)^4}}\, d\rho d\theta d\widehat{x} \\
&\le \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \left[\frac{2\rho}{\varepsilon(1-|\widehat{x}|)} + \frac{\rho^2}{\varepsilon(1-|\widehat{x}|)^2}\right] d\rho d\theta d\widehat{x} \\
&\le \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} [2+\varepsilon]\, d\rho d\theta d\widehat{x} \to 0 \qquad \text{as } \varepsilon \to 0^+,
\end{aligned}$$

where we used that $\rho = |\widetilde{x}| \le \varepsilon(1 - |\widehat{x}|)$ in U_ε. Therefore, (2.2) holds, and by dominated convergence

$$\lim_{\varepsilon \to 0} \int_{B^n} \sqrt{1 + |\nabla u_\varepsilon|^2}\, dx = \int_{B^n} \sqrt{1 + |\nabla u|^2}\, dx\,. \tag{2.3}$$

It remains to check that

$$\limsup_{\varepsilon \to 0} \int_{B^n} |M_2(\nabla u_\varepsilon)| dx \le \pi\, \mathcal{H}^{n-2}(\Delta^{n-2}) = \pi\, \mathbf{M}(\mathbb{P}(u))\,.$$

We have $|M_2(\nabla u_\varepsilon)| = |M_2(\nabla \bar{u}_\varepsilon)|$, where we compute the components of $M_2(\nabla \bar{u}_\varepsilon)$ w.r.t. the basis in cylindrical coordinates:

$$\begin{aligned}
M_2(\nabla \bar{u}_\varepsilon)_{12} &= \frac{1}{\rho} \partial_\rho \bar{u}_\varepsilon \wedge \partial_\theta \bar{u}_\varepsilon = \frac{1}{\varepsilon^2(1-|\widehat{x}|)^2}\,, \\
M_2(\nabla \bar{u}_\varepsilon)_{1j} &= \partial_\rho \bar{u}_\varepsilon \wedge \partial_{x_j} \bar{u}_\varepsilon = 0 && \forall j = 3, \dots, n\,, \\
M_2(\nabla \bar{u}_\varepsilon)_{2j} &= \frac{1}{\rho} \partial_\theta \bar{u}_\varepsilon \wedge \partial_{x_j} \bar{u}_\varepsilon = -\frac{\rho}{\varepsilon^2(1-|\widehat{x}|)^3} \frac{x_j}{|\widehat{x}|} && \forall\, j = 3, \dots, n\,, \\
M_2(\nabla \bar{u}_\varepsilon)_{ij} &= \partial_{x_i} \bar{u}_\varepsilon \wedge \partial_{x_j} \bar{u}_\varepsilon = 0 && \forall\, i, j = 3, \dots, n,\ i \ne j\,.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{B^n} |M_2(\nabla u_\varepsilon)| dx = \\
&= \int_{U_\varepsilon} |M_2(\nabla u_\varepsilon)| dx = \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \rho \, |M_2(\nabla \overline{u}_\varepsilon)| \, d\rho d\theta d\widehat{x} \\
&\le \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \left[\frac{\rho}{\varepsilon^2(1-|\widehat{x}|)^2} + \frac{\rho^2}{\varepsilon^2(1-|\widehat{x}|)^3} \right] d\rho d\theta d\widehat{x} \\
&= \int_{\Delta^{n-2}} \int_0^{2\pi} \frac{1}{2} d\theta d\widehat{x} + \int_{\Delta^{n-2}} \int_0^{2\pi} \int_0^{\varepsilon(1-|\widehat{x}|)} \frac{\rho^2}{\varepsilon^2(1-|\widehat{x}|)^3} \, d\rho d\theta d\widehat{x} \\
&= \pi \, \mathcal{H}^{n-2}(\Delta^{n-2}) + O(\varepsilon) \to \pi \, \mathcal{H}^{n-2}(\Delta^{n-2}) \quad \text{as } \varepsilon \to 0^+ .
\end{aligned}
$$

Using (2.3), we conclude

$$
\begin{aligned}
\limsup_{\varepsilon \to 0} \mathbf{M}(G_{u_\varepsilon}) &\le \lim_{\varepsilon \to 0} \int_{B^n} \sqrt{1 + |\nabla u_\varepsilon|^2} dx + \limsup_{\varepsilon \to 0} \int_{B^n} |M_2(\nabla u_\varepsilon)| dx \\
&\le \int_{B^n} \sqrt{1 + |\nabla u|^2} dx + \pi \mathcal{H}^{n-2}(\Delta^{n-2}) .
\end{aligned}
$$

Remark 2.1. In the previous example, the choice of the cone shaped neighborhood is not crucial for the computation of the upper bound estimate. We could have followed essentially the same argument also by taking U_ε of cylindrical shape, i.e. by defining $U_\varepsilon := (\widetilde{B}_\varepsilon \times \Delta^{n-2}) \cap B^n$. The advantage of the cone shaped construction is that the width of U_ε shrinks at the boundary of Δ^{n-2}, which will be useful in the case the singular set of u is polyhedral.

2.2 Sobolev maps with unbounded relaxed energy

We show the existence of Sobolev maps $u \in W^{1,1}(B^n, \mathbb{S}^1)$ which do not have finite relaxed energy.

In low dimension $n = 2$, it suffices to find a sequence $\{B_j\}$ of pairwise disjoint balls contained in B^2 such that the restriction $u_{|B_j}$ behaves like a vortex map around the center of B_j. Therefore, by the superadditivity of the set function corresponding to the localization of the relaxed energy, we obtain a contribution equal to π around each singular point. In particular, $|\operatorname{Det} \nabla u|(B^2) = \infty$.

The counterexample in high dimension $n \geq 3$ is trivially obtained by setting $\overline{u}(x) = u(x_1, x_2)$, for $x \in B^n$. In that case, we clearly have $|J(u)|(B^n) = \infty$.

Following an example by [28], we set $B_j := B^2(c_j, 2^{-(j+1)})$, where

$$
c_j = (1 - 2^{1-j}, 0) , \qquad j = 1, 2, \dots
$$

Moreover we define $u_{|B_j} := u^{(j)} : B_j \to \mathbb{R}^2$ by

$$
u^{(j)}(x) := \begin{cases} \dfrac{x - c_j}{|x - c_j|} & \text{if} \quad j = 1, 3, 5, \dots \\ \psi\left(\dfrac{x - c_j}{|x - c_j|} \right) & \text{if} \quad j = 2, 4, 6, \dots \end{cases}
$$

where $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ is defined (in terms of the angle function θ on $\mathbb{S}^1$) by

$$
\psi(\theta) := -\theta + \pi .
$$

If $Q_j := c_j + [-2^{-(j+1)}, 2^{-(j+1)}]^2$ denotes the square circumscribing B_j, we extend $u_{|B_j}$ to Q_j as the continuous map which is constant in the x_1-variable (note that $Q_j \subset B^2$ for every $j \geq 1$, see Fig. 2). Then $u \equiv (1, 0)$ and $u \equiv (-1, 0)$ over all the upper and respectively lower sides of the boundary of the Q_j's which are parallel to the x_1-axis, whereas on the sides parallel to the x_2-axis,

$$
L_j^k := c_j + \{((-1)^k \, 2^{-(j+1)}, x_2) \mid -2^{-(j+1)} \le x_2 \le 2^{-(j+1)}\} , \qquad k = 1, 2 ,
$$

both $u_{|L_j^2}$ and $u_{|L_{j+1}^1}$ parameterize the same half of the circle $\mathbb{S}^1$ with the same orientation. We can thus define u over the convex hull of L_j^2 and L_{j+1}^1, the right-hand side of ∂Q_j and the left-hand side of ∂Q_{j+1}, as the continuous map which is constant along the straight lines connecting the corresponding points in

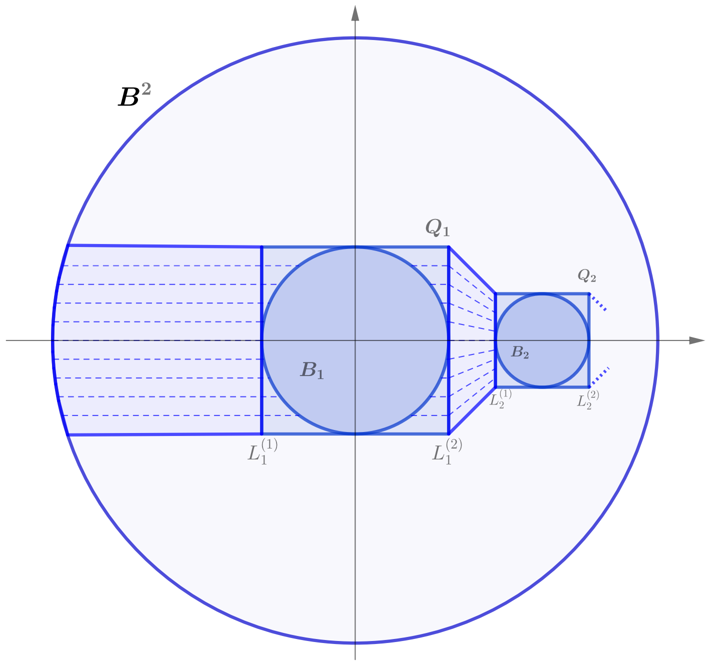

Figure 2: The construction in the source disk B^2. On each disk B_j the vortex map is replicated with alternating orientation.

L_j^2 and L_{j+1}^1 (points on which u takes the same value). We finally define u in the strip connecting L_1^1 to the boundary of B^2 as the continuous map constant in the x_1-variable, and set $u \equiv (1,0)$ or $u \equiv (-1,0)$ in the two remaining components of B^2. Then, it is not difficult to show that $u \in W^{1,1}(B^2, \mathbb{R}^2)$. On the other hand, by the result from [4] we know that for each j, the relaxed energy of $u^{(j)}$ on B_j is greater than π. Therefore, by the superadditivity of the (localized) relaxed functional it turns out that the map u does not have a finite relaxed energy.

3 The explicit formula

The Main Result of this paper is the following

Theorem 3.1. *Let $n \geq 2$ and $u \in W^{1,1}(B^2, \mathbb{S}^1)$. Then, $\overline{\mathcal{A}}_{BV}(u) < \infty$ if and only if the $(n-2)$-current $\mathbb{P}(u)$ is i.m. rectifiable and with finite mass, $\mathbf{M}(\mathbb{P}(u)) < \infty$. In that case, moreover, one has:*

$$\overline{\mathcal{A}}_{BV}(u) = \int_{B^n} \sqrt{1 + |\nabla u|^2}\, dx + \pi\, \mathbf{M}(\mathbb{P}(u))\,.$$

In dimension $n = 2$, recalling that $\pi\, \mathbf{M}(\mathbb{P}(u)) = |\operatorname{Det} \nabla u|(B^n)$, the latter result was proved in [4]. In high dimension $n \geq 3$, the *energy gap*, $\pi\, \mathbf{M}(\mathbb{P}(u))$, agrees with the total variation of the distributional Jacobian $J(u)$.

3.1 Energy lower bound

By the previous results, we readily obtain the energy lower bound:

Proposition 3.2. *If $u \in W^{1,1}(B^n, \mathbb{S}^1)$ has finite relaxed energy* (0.2)*, then*

$$\overline{\mathcal{A}}_{BV}(u) \geq \int_{B^n} \sqrt{1 + |\nabla u|^2}\, dx + \pi\, \mathbf{M}(\mathbb{P}(u))\,.$$

Proof. Choose any smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{R}^2)$ such that $u_h \to u$ in $L^1(B^n, \mathbb{R}^2)$ and $\int_{B^n} |\nabla u_h|\, dx \to \int_{B^n} |\nabla u|\, dx$ as $h \to \infty$, and such that

$$\sup_h \int_{B^2} |M_2(\nabla u_h)|\, dx < \infty\,.$$

Possibly taking a not relabeled subsequence, we may and do assume that

$$\liminf_{h\to\infty} \mathcal{A}(u_h) = \lim_{h\to\infty} \mathcal{A}(u_h) < \infty\,.$$

Since $(\partial G_{u_h}) \llcorner B^n \times \mathbb{R}^2 = 0$ and the mass of G_{u_h} is given by (1.2), with $u = u_h$, by applying Federer-Fleming's closure theorem, see [34, Sec. 32], and on account of the strict convergence $u_h \overset{BV}{\to} u$, it turns out that possibly passing to a subsequence, $G_{u_h} \rightharpoonup T_u$ weakly in $\mathcal{D}_n(B^n \times \mathbb{R}^2)$ to the unique optimal lifting Cartesian current T_u, so that by lower semicontinuity of the mass

$$\mathbf{M}(T_u) \le \liminf_{h\to\infty} \mathbf{M}(G_{u_h}) = \lim_{h\to\infty} \mathcal{A}(u_h)\,.$$

Since we already know that

$$\mathbf{M}(T_u) = \mathbf{M}(G_u) + \mathbf{M}(S_u) = \int_{B^n} \sqrt{1+|\nabla u|^2}\,dx + \pi\,\mathbf{M}(\mathbb{P}(u))$$

the energy lower bound readily follows. □

3.2 The approximation theorem

The energy upper bound, which yields to the validity of Theorem 3.1, is an immediate consequence of the following approximation result:

Theorem 3.3. *Let $n \ge 2$ and $u \in W^{1,1}(B^n, \mathbb{S}^1)$ be a Sobolev map with finite relaxed energy* (0.2)*. Then, there exists a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{R}^2)$ such that $G_{u_h} \rightharpoonup T_u$ weakly in $\mathcal{D}_n(B^n \times \mathbb{R}^2)$ and $\mathbf{M}(G_{u_h}) \to \mathbf{M}(T_u)$ as $h \to \infty$.*

In fact, the weak convergence with the mass implies the strict BV-convergence[2] and the energy limit

$$\lim_{h\to\infty} \mathcal{A}(u_h) = \int_{B^n} \sqrt{1+|\nabla u|^2}\,dx + \pi\,\mathbf{M}(\mathbb{P}(u))\,.$$

Therefore, if $\overline{\mathcal{A}}_{BV}(u) < \infty$, by the explicit formula we obtain that $\mathbf{M}(\mathbb{P}(u)) < \infty$. On the other hand, when $\mathbf{M}(\mathbb{P}(u)) < \infty$, the approximation theorem 3.3 continues to hold, yielding to the optimal upper bound and hence to condition $\overline{\mathcal{A}}_{BV}(u) < \infty$. Therefore, Theorem 3.1 holds true.

In low dimension, the approximating sequence is readily obtained:

Proof of Theorem 3.3, case $n = 2$. By Bethuel's results in [13], we can find a sequence $\{u_h\} \subset W^{1,1}(B^2, \mathbb{S}^1)$ strongly converging to u in $W^{1,1}(B^2, \mathbb{R}^2)$ and such that each u_h is smooth outside a finite set of points. Furthermore, we have

$$\lim_{h\to\infty} \mathbf{M}(\mathbb{P}(u_h)) = \mathbf{M}(\mathbb{P}(u))\,. \tag{3.1}$$

In fact, for any square F of the grid in Bethuel's proof, the restriction of u to the boundary of F is a continuous function with Brouwer degree $d_F \in \mathbb{Z}$ satisfying $|d_F| \le \mathbf{M}(\mathbb{P}(u) \llcorner \operatorname{int}(F))$. As a consequence, it turns out that $\mathbf{M}(\mathbb{P}(u_h)) \le \mathbf{M}(\mathbb{P}(u))$ for each h. Therefore, by lower semicontinuity we obtain (3.1).

We now show that for each h we can find a smooth sequence $\{u_k^{(h)}\}$ in $C^\infty(B^2, \mathbb{R}^2)$ strongly converging to u in $W^{1,1}(B^2, \mathbb{R}^2)$ and such that

$$\lim_{k\to\infty} \mathcal{A}(u_k^{(h)}) = \int_{B^2} \sqrt{1+|\nabla u_h|^2}\,dx + \pi\,\mathbf{M}(\mathbb{P}(u_h))\,.$$

Since we make use of a local argument, without loss of generality we may and do assume that $v = u_h$ is smooth outside the origin and $\mathbb{P}(v) = d\,\delta_{0_{\mathbb{R}^2}}$ for some $d \in \mathbb{Z}$.

For every $\varepsilon > 0$ small, the restriction $v_{|\partial B^2_\varepsilon}$ is a smooth map of degree d. Therefore, we can find a smooth homotopy map $H : [0,1] \times [0, 2\pi] \to \mathbb{S}^1$ such that $H(0, \theta) = (\cos(d\theta), \sin(d\theta))$ and $H(1, \theta) = v(\varepsilon\cos\theta, \varepsilon\sin\theta)$, where we have introduced the standard polar coordinates $x = \rho(\cos\theta, \sin\theta)$. Define now $v_\varepsilon : B^2_\varepsilon \to \mathbb{R}^2$ as

$$v_\varepsilon(\rho\cos\theta, \rho\sin\theta) := \begin{cases} H\Big(2\rho/\varepsilon - 1, \theta\Big) & \text{if } \varepsilon/2 \le \rho \le \varepsilon \\ (2\rho/\varepsilon)\,\big(\cos(d\theta), \sin(d\theta)\big) & \text{if } \rho \le \varepsilon/2\,. \end{cases}$$

[2]The strata of G_{u_h} must converge to the corresponding ones of T_u in measure and in total variation.

It is readily checked that

$$\mathcal{A}(v_\varepsilon) \le \int_{B^2} \sqrt{1+|\nabla v|^2}\, dx + \pi\, |d| + O(\varepsilon)$$

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since moreover the graph currents G_{v_ε} weakly converge to the Cartesian current $T := G_v + d\,\delta_0 \times [\![D^2]\!]$ along a sequence $\varepsilon_h \to 0$, by lower semicontinuity we obtain

$$\lim_{h\to\infty} \mathcal{A}(v_{\varepsilon_h}) = \lim_{h\to\infty} \mathbf{M}(G_{v_{\varepsilon_h}}) = \mathbf{M}(T) = \int_{B^2} \sqrt{1+|\nabla v|^2}\, dx + \pi\, |d|$$

where $|d| = \mathbf{M}(\mathbb{P}(u))$. Further details are omitted. □

In the sequel we therefore assume $n \ge 3$. Theorem 3.3 is obtained by applying the following technical results, the proof of which is collected in the next section.

3.3 Reduction to maps with a nice singular set

Firstly, we find an approximating sequence which is smooth outside a singular set given by (the support of) a polyhedral chain, in such a way that we have mass convergence of the current of the singularities.

Let $Q^n =]-1,1[^n$ denote the open n-cube in $\mathbb{R}^n$ of side two, and let $u \in W^{1,1}(Q^n, \mathbb{S}^1)$. Denote by $R^\infty(Q^n, \mathbb{S}^1)$ the subclass of maps u in $W^{1,1}(Q^n, \mathbb{S}^1)$ which are smooth outside a nice singular set $\operatorname{sing} u$ of codimension two. This means that $\operatorname{sing} u$ is given by the support of some polyhedral $(n-2)$-chain $\mathbb{P}$ in Q^n, and actually $\mathbb{P}(u) = \mathbb{P}$. More precisely, we have

$$\mathbb{P} = \sum_{i=1}^m d_i [\![\Delta_i]\!] , \quad \mathbf{M}(\mathbb{P}) = \sum_{i=1}^m |d_i|\, \mathcal{H}^{n-2}(\Delta_i) < \infty \tag{3.2}$$

for some $m \in \mathbb{N}^+$, where $d_i \in \mathbb{Z}$ and Δ_i is an oriented $(n-2)$-simplex contained in the closure of Q^n, for each i. Notice that d_i coincides with the degree of u around Δ_i up to a sign, precisely $\deg(u, \Delta_i) = (-1)^{n-2} d_i$. The support $\operatorname{spt} \mathbb{P}$ of $\mathbb{P}$ is the union of the closures of the simplices Δ_i. Moreover, after a subdivision we may and do assume that $\operatorname{int}(\Delta_i) \cap \operatorname{int}(\Delta_j) = \emptyset$ for $1 \le i < j \le m$, so that the simplices Δ_i and Δ_j possibly intersect at points in the common $(n-3)$-skeleton.

By Bethuel's theorem, the class $R^\infty(Q^n, \mathbb{S}^1)$ is dense in $W^{1,1}(Q^n, \mathbb{S}^1)$ strongly in $W^{1,1}(Q^n, \mathbb{R}^2)$. To our purposes, we shall see that for maps with finite relaxed energy, something more can be said.

Theorem 3.4. *Assume that $u \in W^{1,1}(Q^n, \mathbb{S}^1)$ has finite relaxed energy. Then, we can find a sequence $\{u_k\} \subset R^\infty(Q^n, \mathbb{S}^1)$ strongly converging to u in $W^{1,1}(Q^n, \mathbb{R}^2)$ and such that*

$$\lim_{k\to\infty} \mathbf{M}(\mathbb{P}(u_k)) = \mathbf{M}(\mathbb{P}(u)) .$$

3.4 Energy approximation at the singular set

Assume now that $n \ge 3$ and $u \in R^\infty(Q^n, \mathbb{S}^1)$ satisfies $\mathbf{M}(\mathbb{P}(u)) < \infty$, with $\mathbb{P}(u) = \mathbb{P}$ as in (3.2). Without loss of generality, we assume that for e.g. $i = 1$ we have $\Delta_1 = \{0_{\mathbb{R}^2}\} \times \widehat{\Delta}$. Let

$$\Delta_\varepsilon := \{(\widetilde{x}, \widehat{x}) \in \mathbb{R}^2 \times \widehat{\Delta} \,:\, |\widetilde{x}| \le \varepsilon y(\widehat{x})\} , \quad \varepsilon > 0$$

where we have denoted

$$y(\widehat{x}) := \operatorname{dist}(\widehat{x}, \partial\widehat{\Delta}) , \tag{3.3}$$

so that for $\varepsilon > 0$ small, the cone Δ_ε intersects the other simplices Δ_i only at points in $\partial\Delta_i$, for $i = 2, \ldots, m$. Since moreover $u \in W^{1,1}(Q^n, \mathbb{S}^1)$ is smooth outside the support of $\mathbb{P}(u)$, for a.e. $\varepsilon > 0$ the restriction of u to the boundary of Δ_ε is in $W^{1,1}$. Furthermore, recalling the definition of the current $\mathbb{P}(u)$, it turns out that for all $\widehat{x} \in \widehat{\Delta}$ the degree of $u(\cdot, \widehat{x})$ around $0_{\mathbb{R}^2}$ is constantly equal to $d = d_i \in \mathbb{Z}$.

The following result allows to remove the dipole $\Delta := \Delta_1$, by paying an amount of energy essentially equal to $\pi\, |d|\, \mathcal{H}^{n-2}(\Delta)$. Of course, the argument will be applied to each simplex Δ_i in the proof of Theorem 3.3.

Theorem 3.5. *For a.e. $\varepsilon > 0$ small, there exists a smooth map $v_\varepsilon : \Delta_\varepsilon \to \mathbb{R}^2$ such that $v_{\varepsilon|\partial\Delta_\varepsilon} = u_{|\partial\Delta_\varepsilon}$ in the sense of the traces, and*

$$\mathbf{M}(G_{v_\varepsilon} \llcorner \Delta_\varepsilon \times \mathbb{R}^2) \le \pi\, |d|\, \mathcal{H}^{n-2}(\Delta) + O(\varepsilon) . \tag{3.4}$$

3.5 Removal of point singularities

In dimension $n = 3$, we also need the following argument that allows to remove point singularities by paying a small amount of energy.

Theorem 3.6. *Let $n \geq 3$ and $u \in W^{1,1}(Q^n, \mathbb{R}^2)$ be smooth outside a discrete set. Then there exists a sequence $\{u_k\} \subset C^\infty(B^n, \mathbb{R}^2)$ such that $u_k \stackrel{BV}{\to} u$ strictly, and*

$$\lim_{k\to\infty} \int_{Q^n} \sqrt{1 + |\nabla u_k|^2 + |M_2(\nabla u_k)|^2}\, dx = \int_{Q^n} \sqrt{1 + |\nabla u|^2 + |M_2(\nabla u)|^2}\, dx\,.$$

3.6 Removal of high codimension singularities

In high dimension $n \geq 4$, instead, we first have to remove singularities of codimension greater than two. More precisely, let $k = 3, \ldots n-1$, integer, and let Δ denote an $(n-k)$-dimensional simplex contained in the closure of Q^n. Without loss of generality, assume $\Delta = \{0_{\mathbb{R}^k}\} \times \widehat{\Delta}$. For $\varepsilon > 0$ small, define again

$$\Delta_\varepsilon := \{(\widetilde{x}, \widehat{x}) \in \mathbb{R}^k \times \widehat{\Delta} \,:\, |\widetilde{x}| \leq \varepsilon y(\widehat{x})\}$$

where $y(\widehat{x})$ is the distance function in (3.3).

Theorem 3.7. *Let $u \in W^{1,1}(B^n, \mathbb{R}^2)$ be smooth in $\operatorname{int}(\Delta_{\varepsilon_0}) \setminus \Delta$ for some $\varepsilon_0 > 0$ small. Then, for a.e. $\varepsilon > 0$ small, there exists a smooth map $v_\varepsilon : \operatorname{int}(\Delta_\varepsilon) \to \mathbb{R}^2$ such that $v_{\varepsilon|\partial\Delta_\varepsilon} = u_{|\partial\Delta_\varepsilon}$ in the sense of the traces, and*

$$\mathbf{M}(G_{v_\varepsilon} \llcorner \Delta_\varepsilon \times \mathbb{R}^2) \leq O(\varepsilon)\,. \tag{3.5}$$

3.7 Proof of the approximation theorem

We are now in position to give the:

Proof of Theorem 3.3, case $n \geq 3$. Since the weak convergence of currents with supports contained in the closure of $B^n \times D^2$ is metrizable, compare [34, Sec. 31], we can apply a diagonal argument. Moreover, since B^n is bilipschitz homeomorphic to Q^n, we may and do assume $u : Q^n \to \mathbb{S}^1$.
Step 1. By Theorem 3.4, we reduce to the case in which $u \in R^\infty(Q^n, \mathbb{S}^1)$ and u is smooth outside the support of $\mathbb{P}(u)$, a polyhedral $(n-2)$-chain in Q^n.
Step 2. By applying iteratively Theorem 3.5, for each $\varepsilon > 0$ we find a Sobolev map $u_\varepsilon \in W^{1,1}(Q^n, \mathbb{R}^2)$ that is smooth outside an $(n-3)$-dimensional polyhedral chain Σ_ε and such that

$$\int_{Q^n} \sqrt{1 + |\nabla u_\varepsilon|^2 + |M_2(\nabla u_\varepsilon)|^2}\, dx \leq \int_{Q^n} \sqrt{1 + |\nabla u|^2}\, dx + \pi\, \mathbf{M}(\mathbb{P}(u)) + \varepsilon$$

with $u_\varepsilon \stackrel{BV}{\to} u$ strictly and $\mathbf{M}(\mathbb{P}(u_\varepsilon)) \to \mathbf{M}(\mathbb{P}(u))$, as $\varepsilon \to 0$.
Step 3. If $n = 3$, the finite set Σ_ε of point singularities of u_ε is removed by means of Theorem 3.6. In high dimension $n \geq 4$, we first apply Theorem 3.7, with $k = 3$, to each $(n-3)$-simplex of Σ_ε, and reduce to the case of a map that is smooth outside an $(n-4)$-dimensional polyhedral chain, given by the union of the faces F of the $(n-3)$-simplices of Σ_ε that lie inside Q^n. If $n \geq 5$, we then iteratively repeat the same argument, by applying Theorem 3.7 for $k = 4, \ldots, n-1$. Finally, we apply Theorem 3.6 in order to remove the finite set of point singularities.
Step 4. By a diagonal argument, we find a good approximating sequence given by Lipschitz-continuous functions, where the weak convergence as currents readily follows. By a standard convolution argument, the proof is complete. □

4 Proofs

In this section we collect the proofs of the technical results leading to Theorem 3.3. Recall that we assume $n \geq 3$.

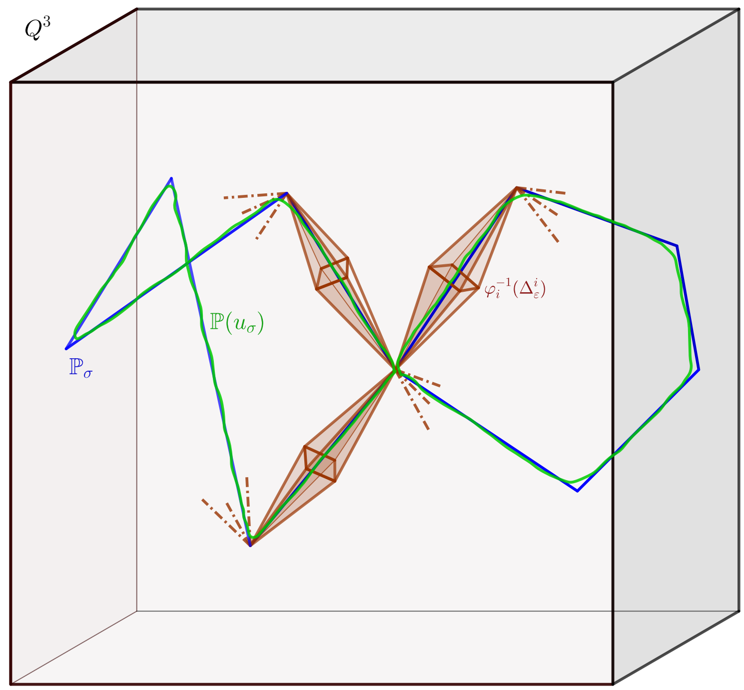

Figure 3: The polyhedral chain $\mathbb{P}_\sigma$ (in blue), the current $\mathbb{P}(u_\sigma)$ (in green) and the pyramidal neighborhoods ${\varphi_i}^{-1}(\Delta^i_\varepsilon)$, depicted in dimension $n = 3$. Notice that both $\mathbb{P}_\sigma$ and $\mathbb{P}(u_\sigma)$ are boundaryless and have support contained in the open cube Q^3, since we are assuming that u is smooth near the boundary of Q^3.

4.1 Reduction to maps with a nice singular set

Proof of Theorem 3.4. We first consider the case when u is smooth in a neighborhood of the boundary of Q^n. Therefore, $\mathbb{P}(u)$ can be viewed as a current in $\mathcal{R}_{n-2}(\mathbb{R}^n)$ satisfying $\partial\mathbb{P}(u) = 0$, see Theorem 1.2, and with support a closed set contained in the open cube Q^n. In the sequel, $c(n)$ will denote a real positive constant only depending on the dimension n, possibly varying from line to line.

Let $\sigma > 0$ small. By Federer's strong polyhedral approximation theorem, see [18, 4.2.2], see also [20, Sec. 2.2.6], there exists a diffeomorphism φ_σ of Q^n onto itself and an $(n-2)$-dimensional polyhedral chain $\mathbb{P}_\sigma$ with support in Q^n, such that $\varphi_{\sigma\#}\mathbb{P}(u) - \mathbb{P}_\sigma = \partial R_\sigma$ for some current $R_\sigma \in \mathcal{R}_{n-1}(Q^n)$ with $\mathbf{M}(R_\sigma) + \mathbf{M}(\partial R_\sigma) < \sigma$. Moreover, $\operatorname{Lip}\varphi_\sigma \le 1+\sigma$, $\operatorname{Lip}{\varphi_\sigma}^{-1} < 1+\sigma$, and $\varphi_\sigma(x) = x$ if the distance of x to the support of $\mathbb{P}(u)$ is greater than σ.

Letting $u_\sigma := u \circ {\varphi_\sigma}^{-1}$, then $u_\sigma \in W^{1,1}(Q^n, \mathbb{S}^1)$, $u_\sigma \to u$ strongly in $W^{1,1}(Q^n, \mathbb{R}^2)$ as $\sigma \to 0$, and $\mathbb{P}(u_\sigma) = \varphi_{\sigma\#}\mathbb{P}(u)$ (see Fig. 3), so that

$$\mathbb{P}(u_\sigma) - \mathbb{P}_\sigma = \partial R_\sigma\,, \quad \mathbf{M}(R_\sigma) + \mathbf{M}(\partial R_\sigma) < \sigma\,. \tag{4.1}$$

As a consequence, the open set

$$U_\sigma = Q^n \setminus \operatorname{spt}\mathbb{P}_\sigma$$

has full measure, and

$$\mathbf{M}(\mathbb{P}(u_\sigma) \llcorner U_\sigma) = \mathbf{M}((\partial R_\sigma) \llcorner U_\sigma) \le \mathbf{M}(\partial R_\sigma) < \sigma\,. \tag{4.2}$$

For any $\sigma > 0$ small, we now write $u = u_\sigma$, $\mathbb{P} = \mathbb{P}_\sigma$, and $U = U_\sigma$, for the sake of simplicity, and we write $\mathbb{P}$ as in (3.2). After a rigid motion φ_i in $\mathbb{R}^n$ we have

$$\varphi_i(\Delta_i) = \{0_{\mathbb{R}^2}\} \times \widehat{\Delta}_i \quad \forall\, i = 1, \dots, m\,.$$

If $\widetilde{x} = (x_1, x_2) \in \mathbb{R}^2$, we let $\|\widetilde{x}\| := |x_1| + |x_2|$, and for $\varepsilon > 0$ small

$$\Delta^i_\varepsilon := \{(\widetilde{x}, \widehat{x}) \in \mathbb{R}^2 \times \widehat{\Delta}_i \,:\, \|\widetilde{x}\| \le \varepsilon y_i(\widehat{x})\}$$

where we have denoted $y_i(\widehat{x}) := \mathrm{dist}(\widehat{x}, \partial\widehat{\Delta}_i)$. Therefore, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in]0, \varepsilon_0[$, for any $1 \le i < j \le m$ the cones Δ^i_ε and Δ^j_ε are interiorly disjoint, and only intersect at points in the $(n-2)$-dimensional set $\Delta^i \cap \Delta^j$.

Denote by $\Sigma^i_\varepsilon(\ell)$ the ℓ-dimensional skeleton of ${\varphi_i}^{-1}(\Delta^i_\varepsilon)$ (see Fig. 3). By a slicing argument, it turns out that for a.e. $\varepsilon \in]0, \varepsilon_0[$ the restriction $u_{|F}$ of u to any ℓ-face F of $\Sigma^i_\varepsilon(\ell)$ is a Sobolev map in $W^{1,1}(F, \mathbb{S}^1)$, for each $\ell = 1, \dots, n-1$ and $i = 1, \dots, m$. In the sequel, we shall tacitly assume that $\varepsilon \in]0, \varepsilon_0[$ is chosen as above.

Claim: For $x \in \mathrm{int}(\Delta_i)$, let $F^i_\varepsilon(x)$ denote the square obtained by the intersection of ${\varphi_i}^{-1}(\Delta^i_\varepsilon)$ with the affine plane orthogonal to Δ_i and containing x. Then, for each $i = 1, \dots, m$ there exists a Borel set $\Sigma_i \subset \Delta_i$, with $\sum_{i=1}^m \mathcal{H}^{n-2}(\Sigma_i) < c(n)\,\sigma$ for some absolute real constant $c(n) > 0$, such that for every $x \in \Delta_i \setminus \Sigma_i$, the 2-dimensional restriction of u to the square $F^i_\varepsilon(x)$ is a Sobolev map with values into $\mathbb{S}^1$ and with no homological singularities outside the center x of the square. The validity of this claim can be checked as a consequence of the mass estimate (4.2) and of a slicing argument.

We now modify the map u as follows. For each $i = 1, \dots, m$, let $v_{i,\varepsilon} : \Delta^i_\varepsilon \to \mathbb{S}^1$ be given by

$$v_{i,\varepsilon}(\widetilde{x}, \widehat{x}) := u\Big(\varepsilon\, y_i(\widehat{x})\, \frac{\widetilde{x}}{\|\widetilde{x}\|}, \widehat{x}\Big) \tag{4.3}$$

and define $u_\varepsilon : Q^n \to \mathbb{S}^1$ by

$$u_\varepsilon(x) = \begin{cases} v_{i,\varepsilon}(\varphi_i(x)) & \text{if } x \in {\varphi_i}^{-1}(\Delta^i_\varepsilon)\,, \quad i = 1, \dots, m \\ u(x) & \text{elsewhere in } Q^n\,. \end{cases}$$

We can find a sequence $\{\varepsilon_h\} \searrow 0$ such that $u_h := u_{\varepsilon_h} \in W^{1,1}(Q^n, \mathbb{S}^1)$ for each h, and $\{u_h\}$ converges to u strongly in $W^{1,1}$. The proof of this fact is omitted, since it follows by using arguments as in the next sections. To this purpose, we only observe that in Theorem 3.7, the computation in (4.15) holds true even in case $k = 2$. Therefore, setting

$$V_h := Q^n \setminus \bigcup_{i=1}^m {\varphi_i}^{-1}(\Delta^i_{\varepsilon_h})\,, \quad h \in \mathbb{N}\,,$$

then V_h is an open subset of U with Lipschitz boundary (except at the points of the $(n-3)$-skeleton of $\mathbb{P}$), and $\mathcal{L}^n(V_h) \to \mathcal{L}^n(U) = \mathcal{L}^n(Q^n)$, as $h \to \infty$.

Denote by $\mathbb{P}_{u,h}$ the slice of the current $\mathbb{P}(u)$ to the $(n-1)$-dimensional boundary ∂V_h. Without loss of generality, we may and do choose the sequence $\{\varepsilon_h\}$ in such a way that $\mathbb{P}_{u,h}$ is an $(n-3)$-rectifiable current satisfying

$$\varepsilon_h\, \mathbf{M}(\mathbb{P}_{u,h}) \le a_h \quad \forall\, h \tag{4.4}$$

where $a_h \to 0$ as $h \to \infty$.

We now apply the approximation theorem by Bethuel [13, Thm. 2], see also [24, Thm. 1.3], to the Sobolev map $u_{|V_h} : V_h \to \mathbb{S}^1$ where, we recall, $u_h = u$ on V_h. This way, for each h we find a sequence $\{v^{(h)}_k\}_k \subset R^\infty(V_h, \mathbb{S}^1)$ strongly converging to $u_{|V_h}$ in $W^{1,1}$.

Denote by $\mathbb{P}(v^{(h)}_k)$ the $(n-2)$-current of the singularities of the Sobolev map $v^{(h)}_k$, so that $\mathrm{spt}\,\mathbb{P}(v^{(h)}_k) \subset \bar{V}_h$ and $(\partial \mathbb{P}(v^{(h)}_k)) \llcorner V_h = 0$ for each k. By an inspection to the construction of the approximating sequence from [13, 24], it turns out that

$$\sup_k \mathbf{M}(\mathbb{P}(v^{(h)}_k)) \le c(n)\, \mathbf{M}(\mathbb{P}(u_h) \llcorner V_h)\,.$$

In fact, the construction makes use of a slicing argument, where the degree of $v^{(h)}_k$ at the boundary of the 2-faces of the grid is bounded (up to an absolute constant) in terms of mass of the current of the singularities times δ^{1-n}, where δ is the mesh of the grid. Therefore, the map $v^{(h)}_k$ being given by homogeneous extension on "bad" sets, one obtains the inequality in the last centered formula.

Therefore, since by (4.2) we can estimate

$$\mathbf{M}(\mathbb{P}(u_h) \llcorner V_{\varepsilon_h}) \le \mathbf{M}(\mathbb{P}(u) \llcorner U) \le \sigma \quad \forall\, h$$

we infer that

$$\sup_k \mathbf{M}(\mathbb{P}(v^{(h)}_k)) \le c(n)\, \sigma \quad \forall\, h\,. \tag{4.5}$$

Moreover, viewing $\mathbb{P}(v_k^{(h)})$ as a current in Q^n, since the mass of the restriction of $\mathbb{P}(v_k^{(h)})$ to the boundary of V_h is bounded (up to an absolute constant factor) in terms of the mass of the restriction of $\mathbb{P}(u)$ to ∂V_h, we also have:

$$\sup_k \mathbf{M}(\partial\mathbb{P}(v_k^{(h)})) \le c(n)\,\mathbf{M}(\mathbb{P}_{u,h}),$$

so that by (4.4) we can estimate

$$\sup_k \mathbf{M}(\partial\mathbb{P}(v_k^{(h)})) \le c(n)\,\frac{a_h}{\varepsilon_h} \quad \forall\, h\,. \tag{4.6}$$

In a way similar to definition (4.3), we now take for each i the zero-homogeneous extension of $v_k^{(h)}$ in ${\varphi_i}^{-1}(\Delta^i_{\varepsilon_h})$ with respect to the coordinates $\widetilde{x}$ orthogonal to the $(n-2)$-simplex Δ_i. We thus find a sequence $\{w_k^{(h)}\} \subset R^\infty(Q^n,\mathbb{S}^1)$ strongly converging to u in $W^{1,1}(Q^n,\mathbb{R}^2)$ as k tends to ∞. Since

$$\mathbf{M}(\mathbb{P}(w_k^{(h)}))\llcorner (U\setminus V_h) \le c(n)\,\varepsilon_h\,\mathbf{M}(\partial\mathbb{P}(v_k^{(h)}))\,,$$

by (4.5) and (4.6) we obtain the mass estimate

$$\sup_k \mathbf{M}\big((\mathbb{P}(u)-\mathbb{P}(w_k^{(h)}))\llcorner U\big) \le c(n)\,(\sigma + a_h) \quad \forall\, h\,. \tag{4.7}$$

By a diagonal argument, we thus find a sequence $\{w_h\} \subset R^\infty(Q^n,\mathbb{S}^1)$ strongly converging to $u=u_\sigma$ in $W^{1,1}(Q^n,\mathbb{R}^2)$ and such that by (4.7) and (4.2)

$$\mathbf{M}\big(\mathbb{P}(w_h)\llcorner U\big) \le c(n)\,(\sigma + a_h) \quad \forall\, h\,. \tag{4.8}$$

Now, recall that $\mathbb{P}=\mathbb{P}_\sigma$ satisfies (3.2). By means of a slicing argument, we deduce that for $\mathcal{H}^{n-2}$-almost every $x\in \operatorname{int}\Delta_i$, where $i=1,\dots,m$, the degree of w_h around x is a well-defined integer, that we denote by $d_h^i(x)$. Therefore, we have:

$$\mathbb{P}(w_h) = \mathbb{P}(w_h)\llcorner U + P_h \tag{4.9}$$

where P_h is an integral polyhedral chain with $\operatorname{spt} P_h \subset \operatorname{spt}\mathbb{P}$, whose action is given by

$$P_h(\eta) = \sum_{i=1}^m \int_{\Delta_i} d_h^i\,\langle \eta,\xi_i\rangle\, d\mathcal{H}^{n-2} \qquad \forall\,\eta\in\mathcal{D}^{n-2}(Q^n)$$

where ξ_i is a unit $(n-2)$-vector orienting Δ_i, for $i=1,\dots,m$.

Denote by θ the multiplicity of the current $\mathbb{P}(u)$. By the previous Claim, we find a measurable set K_σ contained in $\operatorname{spt}\mathbb{P}$ such that $\mathcal{H}^{n-2}(K_\sigma)\le c(n)\,\sigma$ and

$$\sup_h |d_h^i(x)| \le \theta(x)$$

for each $i=1,\dots,m$ and $x\in \operatorname{int}(\Delta_i)\setminus K_\sigma$. Moreover, we also estimate

$$\sup_h \int_{K_\sigma} |d_h^i(x)|\,d\mathcal{H}^{n-2}(x) \le c(n)\,E(\sigma)\,, \quad E(\sigma):=\mathbf{M}(\mathbb{P}(u_\sigma)\llcorner U)$$

so that $E(\sigma)\le\sigma$. Therefore, we obtain

$$\sup_h \mathbf{M}(P_h) \le \mathbf{M}(\mathbb{P}_\sigma) + c(n)\,\sigma$$

and hence, on account of (4.8) and (4.9),

$$\mathbf{M}(\mathbb{P}(w_h)) \le c(n)\,(\sigma+a_h) + \mathbf{M}(\mathbb{P}_\sigma) \quad \forall\, h\,,$$

where, we recall, $a_h\to 0$ as $h\to\infty$.

Letting $\sigma\searrow 0$ along a sequence, recalling that $\mathbf{M}(\mathbb{P}_\sigma)\to\mathbf{M}(\mathbb{P}(u))$, by a further diagonal argument we find a sequence $\{u_k\}\subset W^{1,1}(Q^n,\mathbb{S}^1)$ strongly converging to u in $W^{1,1}$ and such that $\limsup_k \mathbf{M}(\mathbb{P}(u_k)) \le$

$\mathbf{M}(\mathbb{P}(u))$. Since by lower semicontinuity $\mathbf{M}(\mathbb{P}(u)) \le \liminf_k \mathbf{M}(\mathbb{P}(u_k))$, we have proved Theorem 3.4 under the assumption that u is smooth near the boundary of Q^n.

It the general case, we make use of a slicing argument as follows. Let $\|x\| := \sup_{1\le i\le n} |x_i|$ and $Q^n_\lambda = \{x \in \mathbb{R}^n : \|x\| < \lambda\}$, so that $Q^n = Q^n_1$. For a.e. $0 < \lambda < 1$ the restriction $\mathbb{P}(u) \llcorner Q^n_\lambda$ satisfies $\mathbf{M}(\partial(\mathbb{P}(u) \llcorner Q^n_\lambda)) < \infty$. Therefore, the boundary rectifiability theorem (cf. [34, 30.3]) implies that $\mathbb{P}(u) \llcorner Q^n_\lambda$ is an integral $(n-2)$-current in Q^n, with support a closed set contained in the open cube Q^n. We then apply again Federer's strong polyhedral approximation theorem, obtaining for each $\sigma > 0$ small a diffeomorphism φ_σ of Q^n onto itself and an $(n-2)$-dimensional polyhedral chain $\mathbb{P}_\sigma$ with support in Q^n such that $\varphi_{\sigma\#}(\mathbb{P}(u) \llcorner Q^n_\lambda) - \mathbb{P}_\sigma = \partial R_\sigma$ for some current $R_\sigma \in \mathcal{R}_{n-1}(Q^n)$ with $\mathbf{M}(R_\sigma) + \mathbf{M}(\partial R_\sigma) < \sigma$. Setting now $u_{\lambda,\sigma}(x) := u \circ {\varphi_\sigma}^{-1}(x)$, $x \in \varphi_\sigma(Q^n_\lambda)$, we have $\mathbb{P}(u_{\lambda,\sigma}) = \varphi_{\sigma\#}(\mathbb{P}(u) \llcorner Q^n_\lambda)$. Therefore, arguing as before, we find a sequence $\{w_k\} \subset R^\infty(\varphi_\sigma(Q^n_\lambda), \mathbb{S}^1)$ strongly converging to $u_{\lambda,\sigma}$ in $W^{1,1}(\varphi_\sigma(Q^n_\lambda), \mathbb{R}^2)$ and such that $\mathbf{M}(\mathbb{P}(w_k)) \to \mathbf{M}(\mathbb{P}(u_{\lambda,\sigma}))$. Setting for $v = w_k$ or $v = u_{\lambda,\sigma}$

$$\overline{v}(x) := v(\varphi_\sigma(\lambda\, x))\,, \quad x \in Q^n\,,$$

and taking $\lambda \nearrow 1$, the claim follows through a diagonal argument. □

4.2 Energy approximation at the singular set

Proof of Theorem 3.5. Due to the condition on the degree around Δ, setting for simplicity

$$r_\varepsilon(\widehat{x}) := \varepsilon y(\widehat{x}), \tag{4.10}$$

we can find a smooth homotopy map $H : [0,1] \times [0, 2\pi] \times \widehat{\Delta} \to \mathbb{S}^1$ such that $H(0, \theta, \widehat{x}) = (\cos(d\theta), \sin(d\theta))$ and $H(1, \theta, \widehat{x}) = u(r_\varepsilon(\widehat{x})(\cos\theta, \sin\theta), \widehat{x})$. Since $u_{|\partial\Delta_\varepsilon} \in W^{1,1}$, we can assume that $H \in W^{1,1}([0,1] \times [0, 2\pi] \times \widehat{\Delta}, \mathbb{S}^1)$. Define now in cylindrical coordinates $\widetilde{x} = \rho(\cos\theta, \sin\theta)$ the map $v_\varepsilon : \Delta_\varepsilon \to \mathbb{R}^2$ as

$$\overline{v}_\varepsilon(\rho, \theta, \widehat{x}) := \begin{cases} H\Big(2\rho/r_\varepsilon(\widehat{x}) - 1, \theta, \widehat{x}\Big) & \text{if } r_\varepsilon(\widehat{x})/2 \le |\widetilde{x}| \le r_\varepsilon(\widehat{x}) \\ 2\rho/r_\varepsilon(\widehat{x})\,(\cos(d\theta), \sin(d\theta)) & \text{if } |\widetilde{x}| \le r_\varepsilon(\widehat{x})/2\,. \end{cases}$$

Notice that v_ε is smooth on $\partial\Delta_\varepsilon$ and $v_\varepsilon = u$ on $\partial\Delta_\varepsilon$. We claim that (3.4) holds.

In fact, for $r_\varepsilon(\widehat{x})/2 < \rho < r_\varepsilon(\widehat{x})$, setting $t(\rho) = 2\rho/r_\varepsilon(\widehat{x}) - 1$, we compute

$$\partial_\rho \overline{v}_\varepsilon(\rho, \theta, \widehat{x}) = \partial_t H(t(\rho), \theta, \widehat{x}) \cdot \frac{2}{r_\varepsilon(\widehat{x})}\,, \qquad \partial_\theta \overline{v}_\varepsilon(\rho, \theta, \widehat{x}) = \partial_\theta H(t(\rho), \theta, \widehat{x})$$

whereas

$$\nabla_{\widehat{x}} \overline{v}_\varepsilon(\rho, \theta, \widehat{x}) = \nabla_{\widehat{x}} H(t(\rho), \theta, \widehat{x}) - \frac{2\rho}{r_\varepsilon(\widehat{x})^2} \partial_t H(t(\rho), \theta, \widehat{x}) \otimes \nabla r_\varepsilon(\widehat{x})\,.$$

We have

$$\begin{aligned} &\int\limits_{\Delta_\varepsilon \cap \{r_\varepsilon(\widehat{x})/2 < \rho < r_\varepsilon(\widehat{x})\}} |\nabla v_\varepsilon|\, d\mathcal{L}^n \\ &= \int_{\widehat{\Delta}} \int_0^{2\pi} \int_{r_\varepsilon(\widehat{x})/2}^{r_\varepsilon(\widehat{x})} \rho \sqrt{|\partial_\rho \overline{v}_\varepsilon|^2 + \frac{|\partial_\theta \overline{v}_\varepsilon|^2}{\rho^2} + |\nabla_{\widehat{x}} \overline{v}_\varepsilon|^2}\, d\rho d\theta d\widehat{x} \\ &\le \int_{\widehat{\Delta}} \int_0^{2\pi} \int_{r_\varepsilon(\widehat{x})/2}^{r_\varepsilon(\widehat{x})} \rho \Big[\frac{2}{r_\varepsilon(\widehat{x})} |\partial_t H| + \frac{|\partial_\theta H|}{\rho} + |\nabla_{\widehat{x}} H| \\ &\qquad + \frac{2\rho}{r_\varepsilon(\widehat{x})^2} |\partial_t H| |\nabla r_\varepsilon| + \frac{2\sqrt{\rho}}{r_\varepsilon(\widehat{x})} \sqrt{|\nabla_{\widehat{x}} H| |\partial_t H| |\nabla r_\varepsilon|} \Big] d\rho d\theta d\widehat{x}, \end{aligned}$$

where all the partial derivatives of H are computed at $(t(\rho), \theta, \widehat{x})$ and ∇r_ε is computed at $\widehat{x}$. Using that $\rho \le r_\varepsilon(\widehat{x})$ on Δ_ε and $|\nabla r_\varepsilon(\widehat{x})| = \varepsilon$, for some absolute real constant C, we get

$$\begin{aligned} \int\limits_{\Delta_\varepsilon \cap \{r_\varepsilon(\widehat{x})/2 < \rho < r_\varepsilon(\widehat{x})\}} |\nabla v_\varepsilon|\, d\mathcal{L}^n &\le C \int_{\widehat{\Delta}} \int_0^{2\pi} \int_{r_\varepsilon(\widehat{x})/2}^{r_\varepsilon(\widehat{x})} |\nabla H(t(\rho), \theta, \widehat{x})| d\rho d\theta d\widehat{x} \\ &= C\varepsilon \int\limits_{[0,1]\times[0,2\pi]\times\widehat{\Delta}} |\nabla H(t, \theta, \widehat{x})| dt d\theta d\widehat{x} = O(\varepsilon), \end{aligned}$$

where we performed the change of variable $t = t(\rho)$ and we used that $r_\varepsilon(\widehat{x}) \le \varepsilon$.

On the other hand, by the area formula

$$\int_{\Delta_\varepsilon \cap \{r_\varepsilon(\widehat{x})/2 < \rho < r_\varepsilon(\widehat{x})\}} |M_2(\nabla v_\varepsilon)| \, d\mathcal{L}^n = 0\,.$$

Moreover, for $\rho \le r_\varepsilon(\widehat{x})/2$ we get

$$\begin{aligned}
\partial_\rho \bar{v}_\varepsilon(\rho, \theta, \widehat{x}) &= \frac{2}{r_\varepsilon(\widehat{x})}(\cos(d\theta), \sin(d\theta)), \\
\partial_\theta \bar{v}_\varepsilon(\rho, \theta, \widehat{x}) &= \frac{2d\rho}{r_\varepsilon(\widehat{x})}(-\sin(d\theta), \cos(d\theta)), \\
\nabla_{\widehat{x}} \bar{v}_\varepsilon(\rho, \theta, \widehat{x}) &= -\frac{4\rho}{r_\varepsilon(\widehat{x})^2}(\cos(d\theta), \sin(d\theta)) \otimes \nabla r_\varepsilon(\widehat{x}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\Delta_\varepsilon \cap \{\rho < r_\varepsilon(\widehat{x})/2\}} |\nabla v_\varepsilon| \, d\mathcal{L}^n &\le \int_{\widehat{\Delta}} \int_0^{2\pi} \int_0^{r_\varepsilon(\widehat{x})/2} \rho \left[\frac{2}{r_\varepsilon(\widehat{x})} + \frac{2|d|}{r_\varepsilon(\widehat{x})} + \frac{4\rho\varepsilon}{r_\varepsilon(\widehat{x})^2} \right] d\rho d\theta d\widehat{x} \\
&\le 2\pi \int_{\widehat{\Delta}} \int_0^{r_\varepsilon(\widehat{x})/2} \left[1 + |d| + \frac{\varepsilon}{r_\varepsilon(\widehat{x})} \right] = O(\varepsilon),
\end{aligned}$$

where we used that $|\nabla r_\varepsilon(\widehat{x})| = \varepsilon$ and $\rho \le r_\varepsilon(\widehat{x})/2$. Finally we get

$$\begin{aligned}
\int_{\Delta_\varepsilon \cap \{\rho < r_\varepsilon(\widehat{x})/2\}} |M_2(\nabla v_\varepsilon)| \, d\mathcal{L}^n &= \int_{\widehat{\Delta}} \int_0^{2\pi} \int_0^{r_\varepsilon(\widehat{x})/2} \rho |M_2(\nabla \bar{v}_\varepsilon)| d\rho d\theta d\widehat{x} \\
&\le \int_{\widehat{\Delta}} \int_0^{2\pi} \int_0^{r_\varepsilon(\widehat{x})/2} \left[\frac{4|d|\rho}{r_\varepsilon(\widehat{x})^2} + \frac{8|d|\rho^2\varepsilon}{r_\varepsilon(\widehat{x})^2} \right] d\rho d\theta d\widehat{x} \\
&= \int_{\widehat{\Delta}} \int_0^{2\pi} \frac{|d|}{2} d\theta d\widehat{x} + O(\varepsilon) \\
&= \pi \, |d| \, \mathcal{H}^{n-2}(\Delta) + O(\varepsilon),
\end{aligned}$$

so that (3.4) readily follows. □

4.3 Removal of point singularities

Proof of Theorem 3.6. The argument being local, we may and do assume $\Sigma = \{0_{\mathbb{R}^n}\}$. For $r > 0$ small, we choose $v : B^n_r \to \mathbb{R}^2$ smooth and such that $v = u$ on ∂B^n_r. We then define $w : Q^n \to \mathbb{R}^2$ by taking

$$w(x) = \begin{cases} u(x) & \text{if } |x| \ge r \\ u(rx/|x|) & \text{if } \delta < |x| < r \\ v(rx/\delta) & \text{if } |x| \le \delta \end{cases}$$

where $\delta \in (0, r)$ is small, and we first estimate the energy of w on $B^n_r \setminus B^n_\delta$.

Denoting by ∇_τ the tangential component of the derivative at $x \in \partial B^n_\rho$, we get

$$|\nabla w\,(x)| = \frac{r}{\rho} \left| \nabla_\tau u \left(r \, \frac{x}{|x|} \right) \right| , \quad x \in B^n_r \setminus B^n_\delta$$

and also

$$|M_2(\nabla w\,(x))\,| = \left(\frac{r}{\rho}\right)^2 \left| M_2 \left(\nabla_\tau u \left(r \, \frac{x}{|x|} \right) \right) \right| , \quad x \in B^n_r \setminus B^n_\delta$$

so that

$$
\begin{aligned}
\mathbf{M}(G_w \llcorner (B^n_r \setminus B^n_\delta) \times \mathbb{R}^2) = & \int_{B^n_r \setminus B^n_\delta} \sqrt{1+|\nabla w|^2 + |M_2(\nabla w)|^2} dx \\
\le & |B^n_r| + \int_{B^n_r} |\nabla w|\, dx + \int_{B^n_r} |M_2(\nabla w)|\, dx \\
= & |B^n_r| + \int_0^r \frac{r}{\rho} \int_{\partial B^n_\rho} \left| \nabla_\tau u \left(r\, \frac{x}{|x|} \right) \right| \, d\mathcal{H}^{n-1} d\rho \\
& + \int_0^r \Big(\frac{r}{\rho}\Big)^2 \int_{\partial B^n_\rho} \left| M_2\Big(\nabla_\tau u\Big(r\, \frac{x}{|x|}\Big)\Big) \right| \, d\mathcal{H}^{n-1} d\rho \\
= & |B^n_r| + \int_0^r \left(\frac{\rho}{r}\right)^{n-2} d\rho \int_{\partial B^n_r} |\nabla_\tau u(y)| \, d\mathcal{H}^{n-1} \\
& + \int_0^r \left(\frac{\rho}{r}\right)^{n-3} d\rho \int_{\partial B^n_r} |M_2(\nabla_\tau u(y))| \, d\mathcal{H}^{n-1} \\
= & |B^n_r| + \frac{r}{n-1} \int_{\partial B^n_r} |\nabla_\tau u(y)|\, d\mathcal{H}^{n-1} \\
& + \frac{r}{n-2} \int_{\partial B^n_r} |M_2(\nabla_\tau u(y))|\, d\mathcal{H}^{n-1} \,.
\end{aligned}
$$

Now, setting

$$
F_1(r) := \int_{\partial B^n_r} |\nabla_\tau u|\, d\mathcal{H}^{n-1}, \quad F_2(r) := \int_{\partial B^n_r} |M_2(\nabla_\tau u)|\, d\mathcal{H}^{n-1}
$$

we have

$$
\int_0^1 F_1(r)\, dr \le \int_{B^n} |\nabla u| dx < \infty\,, \qquad \int_0^1 F_2(r)\, dr \le \int_{B^n} |M_2(\nabla u)| dx < \infty\,.
$$

Thus we get necessarily that $\liminf_{r\to 0} r\,\big(F_1(r) + F_2(r)\big) = 0$ and hence, definitely,

$$
\liminf_{r\to 0} \mathbf{M}(G_w \llcorner (B^n_r \setminus B^n_\delta) \times \mathbb{R}^2) = 0\,.
$$

It remains to estimate the energy of w on B^n_δ. We have

$$
\nabla w(x) = \frac{r}{\delta} \nabla v \left(\frac{r}{\delta} x\right), \qquad M_2(\nabla w) = \frac{r^2}{\delta^2} M_2 \left(\nabla v \left(\frac{r}{\delta} x\right)\right)\,, \quad x \in B^n_\delta\,.
$$

Then

$$
\begin{aligned}
& \mathbf{M}(G_w \llcorner B^n_\delta \times \mathbb{R}^2) \le \\
& |B^n_\delta| + \frac{r}{\delta} \int_{B^n_\delta} \left|\nabla v \left(\frac{r}{\delta} x\right)\right| dx + \frac{r^2}{\delta^2} \int_{B^n_\delta} \left| M_2 \left(\nabla v \left(\frac{r}{\delta} x\right)\right)\right| dx = \\
& |B^n_\delta| + \left(\frac{\delta}{r}\right)^{n-1} \int_{B^n_r} |\nabla v(y)| dy + \left(\frac{\delta}{r}\right)^{n-2} \int_{B^n_r} |M_2(\nabla v(y))| dy < \infty\,.
\end{aligned}
$$

Therefore, recalling that $n \ge 3$, letting $r_j \to 0$ along a suitable sequence, and choosing $\delta = \delta(r_j)$ small w.r.t. r_j we find $w_j : B^n_{r_j} \to \mathbb{R}^2$ smooth with $w_j = u$ on $\partial B^n_{r_j}$ such that:

$$
\lim_{j\to\infty} \mathbf{M}(G_{w_j} \llcorner B^n_{r_j} \times \mathbb{R}^2) = 0
$$

and the proof is complete. □

4.4 Removal of high codimension singularities

Proof of Theorem 3.7. Without loss of generality, we can assume that $\Sigma = \Delta$, with Δ an $(n-k)$-simplex, and that $\Delta = \{0_{\mathbb{R}^k}\} \times \widehat{\Delta}$, where we use the notation $x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Consider the neighborhood $\Delta_\varepsilon = \{(\widetilde{x}, \widehat{x}) \in \mathbb{R}^n : \widehat{x} \in \widehat{\Delta}\,, |\widetilde{x}| \le r_\varepsilon(\widehat{x})\}$, where r_ε is given by (4.10), and for $r > 0$ denote $\widetilde{B}_r := \{\widetilde{x} \in \mathbb{R}^k : |\widetilde{x}| \le r\}$.

Let $\delta = \delta(\varepsilon) < \varepsilon$ and $v : \Delta_\varepsilon \to \mathbb{R}^2$ be smooth such that $v_{|\partial\Delta_\delta} = u_{|\partial\Delta_\delta}$, and define the map $w_\varepsilon : B^n \to \mathbb{R}^2$ as

$$w_\varepsilon(x) = \begin{cases} u(x) & \text{in } B^n \setminus \Delta_\varepsilon, \\ u\left(r_\varepsilon(\widehat{x})\widetilde{x}/|\widetilde{x}|, \widehat{x}\right) & \text{in } \Delta_\varepsilon \setminus \Delta_\delta, \\ v(\varepsilon\widetilde{x}/\delta, \widehat{x}) & \text{in } \Delta_\delta. \end{cases}$$

In order to estimate the energy on $\Delta_\varepsilon \setminus \Delta_\delta$, let us start proving that

$$\liminf_{\varepsilon\to 0} \int_{\Delta_\varepsilon\setminus\Delta_\delta} |\nabla w_\varepsilon(x)| dx = 0. \tag{4.11}$$

For this purpose, we introduce an adapted orthonormal basis $(\nu, \tau_1, \ldots, \tau_{k-1})$ in $\mathbb{R}^k$ so that ν is the outward unit normal to $\partial\widetilde{B}_r$ at a point $\widetilde{y} \in \partial B_r$. Then, for any $x \in \Delta_\varepsilon \setminus \Delta_\delta$, with

$$\widetilde{y} = \widetilde{y}(x) = r_\varepsilon(\widehat{x})\,\frac{\widetilde{x}}{\rho} \in \partial B_{r_\varepsilon(\widehat{x})}\,, \quad \rho := |\widetilde{x}| \tag{4.12}$$

we obtain $\partial_\nu w_\varepsilon(\widetilde{x}, \widehat{x}) = 0$,

$$\partial_{\tau_a} w_\varepsilon(\widetilde{x}, \widehat{x}) = \frac{r_\varepsilon(\widehat{x})}{\rho}\,\partial_{\tau_a} u(\widetilde{y}(x), \widehat{x})\,, \quad \alpha = 1, \ldots, k-2 \tag{4.13}$$

and denoting $\widehat{x} = (x_{k+1}, \ldots, x_n)$

$$\partial_{x_\beta} w_\varepsilon(\widetilde{x}, \widehat{x}) = \partial_{x_\beta} u(\widetilde{y}(x), \widehat{x}) + \partial_\nu u(\widetilde{y}(x), \widehat{x})\,\partial_{x_\beta} r_\varepsilon(\widehat{x})\,, \quad \beta = k+1, \ldots, n\,. \tag{4.14}$$

Therefore, since the distance function is 1-Lipschitz and $r_\varepsilon(\widehat{x}) \le \varepsilon$, there exists a positive constant c, only depending on k and n, such that

$$|\nabla w_\varepsilon(\widetilde{x}, \widehat{x})| \le c\,\frac{\varepsilon}{\rho}\,|\nabla u(\widetilde{y}(x), \widehat{x})|\,.$$

Using the change of variable in (4.12) and Fubini's theorem, we estimate:

$$\begin{aligned} \int\limits_{\Delta_\varepsilon\setminus\Delta_\delta} |\nabla w_\varepsilon(x)|\,dx \le{}& c\,\varepsilon \int\limits_{\Delta_\varepsilon} \rho^{-1}\,|\nabla u(\widetilde{y}(x), \widehat{x})|\,dx \\ ={}& c\,\varepsilon \int\limits_{\widehat{\Delta}} \Big(\int\limits_0^{r_\varepsilon(\widehat{x})} \rho^{-1} \Big(\int_{\partial\widetilde{B}_\rho} |\nabla u(\widetilde{y}(x), \widehat{x})|\,d\mathcal{H}^{k-1} \Big)\,d\rho \Big)\,d\widehat{x} \\ ={}& c\,\varepsilon \int\limits_{\widehat{\Delta}} \Big(\int\limits_0^{r_\varepsilon(\widehat{x})} \frac{\rho^{k-2}}{r_\varepsilon(\widehat{x})^{k-1}} \Big(\int\limits_{\partial\widetilde{B}_{r_\varepsilon(\widehat{x})}} |\nabla u(\widetilde{y}, \widehat{x})|\,d\mathcal{H}^{k-1} \Big)\,d\rho \Big)\,d\widehat{x} \\ ={}& \frac{c}{k-1}\,\varepsilon \int\limits_{\widehat{\Delta}} \Big(\int\limits_{\partial\widetilde{B}_{r_\varepsilon(\widehat{x})}} |\nabla u(\widetilde{y}, \widehat{x})|\,d\mathcal{H}^{k-1} \Big)\,d\widehat{x}\,, \end{aligned} \tag{4.15}$$

where we used that $k > 2$. Setting

$$F(\varepsilon) := \int_{\widehat{\Delta}} \Big(\int_{\partial\widetilde{B}_{r_\varepsilon(\widehat{x})}} |\nabla u(\widetilde{y}, \widehat{x})|\,d\mathcal{H}^{k-1} \Big)\,d\widehat{x}$$

since for each $\varepsilon_0 > 0$ small

$$\int_0^{\varepsilon_0} F(\varepsilon)\,d\varepsilon = \int_{\Delta_{\varepsilon_0}} |\nabla u(x)|\,dx < \infty\,,$$

then, necessarily $\liminf_{\varepsilon\to 0} \varepsilon\,F(\varepsilon) = 0$ and we obtain (4.11).

We now show that

$$\liminf_{\varepsilon\to 0}\int_{\Delta_\varepsilon\setminus\Delta_\delta}|M_2(\nabla w_\varepsilon(x))|dx=0. \tag{4.16}$$

We use again the adapted frame. This time, with an obvious notation, recalling that $\partial_\nu w_\varepsilon(\widetilde{x},\widehat{x})=0$, by (4.13) we get for $1\le\alpha_1<\alpha_2\le k-1$

$$|M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))_{\tau_{\alpha_1}\tau_{\alpha_2}}|=\Big(\frac{r_\varepsilon(\widehat{x})}{\rho}\Big)^2|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{\tau_{\alpha_1}\tau_{\alpha_2}}|\,,$$

whereas by (4.14), for each $k+1\le\beta_1<\beta_2\le n$ we can estimate

$$\begin{aligned}|M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))_{x_{\beta_1}x_{\beta_2}}|\le\ &|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{x_{\beta_1}x_{\beta_2}}|\\ &+|\partial_{x_{\beta_1}}r_\varepsilon(\widehat{x})|\,|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{\nu x_{\beta_2}}|\\ &+|\partial_{x_{\beta_2}}r_\varepsilon(\widehat{x})|\,|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{\nu x_{\beta_1}}|\,.\end{aligned}$$

Finally, for each $\alpha=1,\dots,k-1$ and $\beta=k+1,\dots,n$ we have:

$$\begin{aligned}&|M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))_{\tau_\alpha x_\beta}|\le\\ &\quad\frac{r_\varepsilon(\widehat{x})}{\rho}\Big(|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{\tau_\alpha x_\beta}|+|\partial_{x_\beta}r_\varepsilon(\widehat{x})|\,|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))_{\tau_\alpha\nu}|\Big)\,.\end{aligned}$$

We thus definitely obtain the upper bound:

$$|M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))|\le c\Big(1+\varepsilon+\frac{r_\varepsilon(\widehat{x})}{\rho}+\Big(\frac{r_\varepsilon(\widehat{x})}{\rho}\Big)^2\Big)\,|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))|$$

for some absolute constant c, possibly depending on Δ. Therefore, as in (4.15), this time we estimate:

$$\begin{aligned}&\int\limits_{\Delta_\varepsilon\setminus\Delta_\delta}|M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))|\,dx\le c\int\limits_{\Delta_\varepsilon}\Big(1+\varepsilon+\frac{r_\varepsilon(\widehat{x})}{\rho}+\Big(\frac{r_\varepsilon(\widehat{x})}{\rho}\Big)^2\Big)|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))|\,dx\\ &=c\int\limits_{\widehat{\Delta}}\int\limits_0^{r_\varepsilon(\widehat{x})}\Big(1+\varepsilon+\frac{r_\varepsilon(\widehat{x})}{\rho}+\Big(\frac{r_\varepsilon(\widehat{x})}{\rho}\Big)^2\Big)\int\limits_{\partial\widetilde{B}_\rho}|M_2(\nabla u(\widetilde{y}(x),\widehat{x}))|\,d\mathcal{H}^{k-1}d\rho\,d\widehat{x}\\ &=c\int\limits_{\widehat{\Delta}}\int\limits_{\partial\widetilde{B}_{r_\varepsilon(\widehat{x})}}|M_2(\nabla u(\widetilde{y},\widehat{x}))|\,d\mathcal{H}^{k-1}d\widehat{x}\cdot\int\limits_0^{r_\varepsilon(\widehat{x})}\Big(1+\varepsilon+\frac{r_\varepsilon(\widehat{x})}{\rho}+\Big(\frac{r_\varepsilon(\widehat{x})}{\rho}\Big)^2\Big)\Big(\frac{\rho}{r_\varepsilon(\widehat{x})}\Big)^{k-1}d\rho\\ &\le c\,C(k)\,\varepsilon\,G(\varepsilon)\end{aligned}$$

for some dimensional constant $C(k)$, where we used again that $k>2$, $r_\varepsilon(\widehat{x})\le\varepsilon$, and we have denoted

$$G(\varepsilon):=\int_{\widehat{\Delta}}\Big(\int_{\partial\widetilde{B}_{r_\varepsilon(\widehat{x})}}|M_2(\nabla u(\widetilde{y},\widehat{x}))|\,d\mathcal{H}^{k-1}\Big)\,.$$

Since again for $\varepsilon_0>0$ small

$$\int_0^{\varepsilon_0}G(\varepsilon)\,d\varepsilon=\int_{\Delta_{\varepsilon_0}}|M_2(\nabla u(x))|dx<\infty\,,$$

then, necessarily $\liminf_{\varepsilon\to 0}\varepsilon\,G(\varepsilon)=0$ and we obtain (4.16).

On the other hand, for almost every $x\in\Delta_\delta$

$$\nabla_{\widetilde{x}}w_\varepsilon(x)=\frac{\varepsilon}{\delta}\nabla_{\widetilde{x}}v\left(\frac{\varepsilon}{\delta}\widetilde{x},\widehat{x}\right),\qquad\nabla_{\widehat{x}}w_\varepsilon(x)=\nabla_{\widehat{x}}v\left(\frac{\varepsilon}{\delta}\widetilde{x},\widehat{x}\right).$$

whereas for $1\le i<j\le n$, with an obvious notation,

$$\begin{aligned}M_2(\nabla w_\varepsilon)_{ij}&=\frac{\varepsilon^2}{\delta^2}M_2\left(\nabla v\left(\frac{\varepsilon}{\delta}\widetilde{x},\widehat{x}\right)\right)_{ij} && i,j\le k,\\ M_2(\nabla w_\varepsilon)_{ij}&=\frac{\varepsilon}{\delta}M_2\left(\nabla v\left(\frac{\varepsilon}{\delta}\widetilde{x},\widehat{x}\right)\right)_{ij} && i\le k,\quad j>k,\\ M_2(\nabla w_\varepsilon)_{ij}&=M_2\left(\nabla v\left(\frac{\varepsilon}{\delta}\widetilde{x},\widehat{x}\right)\right)_{ij} && i,j>k,\end{aligned}$$

so that (recalling that $0 < \delta < \varepsilon$) definitely

$$|\nabla w_\varepsilon(\widetilde{x},\widehat{x})| \le \frac{\varepsilon}{\delta}\,|\nabla v(\widetilde{y}(x),\widehat{x})|\,, \quad |M_2(\nabla w_\varepsilon(\widetilde{x},\widehat{x}))| \le \Big(\frac{\varepsilon}{\delta}\Big)^2 |M_2(\nabla v(\widetilde{y}(x),\widehat{x}))|\,,$$

where we have denoted

$$\widetilde{y} = \widetilde{y}(x) = \frac{\varepsilon}{\delta}\widetilde{x}\,. \tag{4.17}$$

Therefore, changing variable by (4.17), we get

$$\begin{aligned}
&\mathbf{M}(G_{w_\varepsilon} \llcorner \Delta_\delta \times \mathbb{R}^2) \le \\
&\quad \le |\Delta_\delta| + \frac{\varepsilon}{\delta}\int_{\Delta_\delta} |\nabla v(\widetilde{y}(x),\widehat{x})|\,d\widetilde{x}\,d\widehat{x} + \Big(\frac{\varepsilon}{\delta}\Big)^2 \int_{\Delta_\delta} |M_2(\nabla v(\widetilde{y}(x),\widehat{x})|\,d\widetilde{x}\,d\widehat{x} \\
&\quad = |\Delta_\delta| + \Big(\frac{\delta}{\varepsilon}\Big)^{k-1} \int_{\Delta_\varepsilon} |\nabla v(\widetilde{y},\widehat{x})|\,d\widetilde{x}\,d\widehat{x} + \Big(\frac{\delta}{\varepsilon}\Big)^{k-2} \int_{\Delta_\varepsilon} |M_2(\nabla v(\widetilde{y},\widehat{x})|\,d\widetilde{x}\,d\widehat{x}\,.
\end{aligned}$$

In conclusion, since $k \ge 3$, letting $\varepsilon_j \to 0$ along a suitable sequence, and choosing $\delta = \delta(\varepsilon_j)$ small w.r.t. ε_j, on account of (4.11) and (4.16) we find

$$\lim_{j\to\infty} \mathbf{M}(G_{w_{\varepsilon_j}} \llcorner \Delta_{\varepsilon_j} \times \mathbb{R}^2) = 0$$

and the proof is complete. □

5 Final remarks and open questions

In this final section, we briefly discuss whether our approach in terms of currents extends (with the expected modifications) to the wider class of maps $u \in BV(B^n, \mathbb{S}^1)$ with finite relaxed energy (0.2). We then show that in case of both dimension and codimension at least equal to three, the corresponding relaxed area functional fails to be subadditive as a set function, even in the Sobolev case.

5.1 The BV case.

The optimal lifting Cartesian current satisfies again (1.13) (see [30]), where the distribution $\mathrm{Div}_{\bar\alpha}\,\mathbf{m}_u$ is defined as in (1.5), with an obvious extension of the adjoint notation to the $\mathbb{R}^{2\times n}$-valued measure Du. In case $D^J u = 0$, recalling that $D_i u^j = \nabla_i u^j\,\mathcal{L}^n + (D^C u)_i^j$, we define the graph current G_u in such a way that for every $(n-1)$-form $\omega = \omega^{(1)}$ as in (1.3), this time we have

$$G_u(\omega^{(1)}) = \sum_{j=1}^2 \sum_{i=1}^n \int_{B^n} \phi_i^j(x,u(x))\,dD_i u^j\,.$$

It turns out that properties (1.4) and (1.14) continue to hold. Therefore, the lower bound given by Proposition 3.2 readily extends. On the other hand, the upper bound inequality holds true provided that one is able to find a sequence $\{u_k\} \subset W^{1,1}(B^n, \mathbb{S}^1)$ converging to u strictly in the BV-sense and satisfying $\lim_k \mathbf{M}(\mathbb{P}(u_k)) = \mathbf{M}(\mathbb{P}(u))$, compare [22]. In conclusion, we expect that our Main Result, Theorem 0.2, extends to the wider class of maps $u \in BV(B^n, \mathbb{S}^1)$ such that $D^J u = 0$.

When $D^J u \neq 0$, instead, even in low dimension $n = 2$, we are very far from having an explicit formula, and even a characterization of the class of maps in $BV(B^2, \mathbb{S}^1)$ with finite relaxed energy (0.2). The situation is much more complicate, since homological tools similar to the ones exploited in this paper fail to detect the energy gap.

For example, let $u \in BV(B^2, \mathbb{S}^1)$ be the symmetric triple junction map, so that u is constant in each third of the unit disk, with the three constant values α, β, γ equal to the vertices of an equilateral triangle $T_{\alpha\beta\gamma}$ inscribed in the unit circle $\mathbb{S}^1$. According to (1.5) and using the decomposition formula [30, (4.6)], we get $|\,\mathrm{Div}\,\mathbf{m}_u|(B^2) = |T_{\alpha\beta\gamma}|$, i.e. the area of the triangle $T_{\alpha\beta\gamma}$, recovering the exact energy gap (compare [4]).

On the other hand, by slightly modifying Example 4.5 from [5], so that the vertices of the two triangles in the target space belong to $\mathbb{S}^1$, one obtains a piecewise constant map in $BV(B^2, \mathbb{S}^1)$, with jump set equal to the union of twelve radii, in such a way that $|\,\mathrm{Div}\,\mathbf{m}_u|(B^2) = 0$, but the energy gap is positive. By enforcing this modification also in Example 4.6 of [5], one obtains even a piecewise constant map in $BV(B^2, \mathbb{S}^1)$ with infinite energy gap. Therefore, the relevant (topological) singularity at the origin is not seen by any reasonable definition of the current of the singularities $\mathbb{P}(u)$.

5.2 A counterexample to the measure property

We finally come back to the main conjecture in this framework: for each function $u \in BV(B^n, \mathbb{R}^2)$ with finite relaxed energy (0.2), the localized functional $B \mapsto \overline{\mathcal{A}}_{BV}(u, B)$ is subadditive on open sets, and hence it can be extended to a Borel measure on B^n. That property should follow since the topology induced by strict convergence in BV is stronger enough, compared to the L^1-topology, see (0.1).

On the other hand, we now see that locality fails to hold when both dimension and codimension are strictly larger than two. This drawback is due to the fact that energy concentration in the relaxation process may occur on one-dimensional sets. Therefore, strict convergence fails to be strong enough in order to guarantee uniqueness of the Cartesian current enclosing the graph of Sobolev functions u in $W^{1,1}(B^3, \mathbb{R}^3)$. As in [1], we build up our counterexample by means of the vortex map.

Denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, and let $u : B^3 \to \mathbb{S}^2$ be given by $u(x) = x/|x|$. Then, $u \in W^{1,p}(B^3, \mathbb{S}^2)$ for each exponent $1 \le p < 3$. Moreover, the cofactor function $\operatorname{cof} \nabla u$ belongs to $W^{1,q}(B^3, \mathbb{R}^{3\times 3})$ for $1 \le q < 3/2$, and $\det \nabla u = 0$ $\mathcal{L}^3$-a.e. on B^3, by the area formula. Then, for each open set $B \subset B^3$, the 3-dimensional area of the graph of the restriction $u_{|B}$ satisfies:

$$\mathcal{A}(u, B) = \int_B \sqrt{1 + |\nabla u|^2 + |\operatorname{cof} \nabla u|^2}\, dx < \infty\,.$$

The graph 3-current G_u is well-defined as before, in terms of the pull back of the graph map w.r.t. the approximate gradient, and actually G_u is i.m. rectifiable in $\mathcal{R}_3(B^3 \times \mathbb{R}^3)$, with finite mass $\mathbf{M}(G_u) = \mathcal{A}(u, B^3)$, and a non zero boundary,

$$(\partial G_u) \llcorner B^3 \times \mathbb{R}^3 = -\delta_{0_{\mathbb{R}^3}} \times [\![\mathbb{S}^2]\!]\,,$$

see [20, Sec. 3.2.2]. Roughly speaking, there are two qualitatively different ways to fill the hole in the graph of u: inserting a ball $\delta_{0_{\mathbb{R}^3}} \times [\![D^3]\!]$, where D^3 is the (naturally oriented) unit ball in the target space, or a cylinder $[\![L]\!] \times [\![\mathbb{S}^2]\!]$, where $[\![\mathbb{S}^2]\!] := \partial [\![B^3]\!]$ and L is any oriented line segment connecting a point in the boundary ∂B^3 of the domain to the origin $0_{\mathbb{R}^3}$. Therefore both the 3-currents T_1 and T_2,

$$T_1 := G_u + \delta_{0_{\mathbb{R}^3}} \times [\![D^3]\!]\,, \qquad T_2 := G_u + [\![L]\!] \times [\![\mathbb{S}^2]\!]\,,$$

are Cartesian currents in $B^3 \times \mathbb{R}^3$. Furthermore, it is not difficult to find two sequences $\{u_k^{(i)}\} \subset C^\infty(B^3, \mathbb{R}^3)$, where $i = 1, 2$, satisfying the following properties:

- $u_k^{(i)} \to u$ strongly in $W^{1,1}(B^3, \mathbb{R}^3)$, and hence strictly in BV;
- $G_{u_k^{(i)}} \rightharpoonup T_i$ weakly in $\mathcal{D}_3(B^3 \times \mathbb{R}^3)$;
- $\mathcal{A}(u_k^{(i)}) \to \mathbf{M}(T_i)$, as $k \to \infty$, where

$$\mathbf{M}(T_1) = \mathcal{A}(u) + \frac{4\pi}{3}\,, \quad \mathbf{M}(T_2) = \mathcal{A}(u) + 4\pi\,.$$

The smooth functions $u_k^{(1)}$ are equal to $x/|x|$ outside the ball $B^3_{1/k}$, where they cover once and with the appropriate orientation the ball D^3 in the target space.

The smooth functions $u_k^{(2)}$, instead, take values in the unit sphere $\mathbb{S}^2$. They are equal to $x/|x|$ outside the ball $B^3_{1/k}$ and a small conical neighborhood U_k of the segment L with opening angle $1/k$, so that $\mathcal{L}^3(L_k) \to 0$. Moreover, they are smoothly extended on $U_k \setminus B^3_{1/k}$ in such a way that on each radius $r \in (1/k, 1)$ they cover almost all the unit sphere $\mathbb{S}^2$. This can be done in such a way that the Brouwer degree of the smooth map $u_k^{(2)}{}_{|\partial B^3_r} : \partial B^3_r \to \mathbb{S}^2$ is equal to zero, for any $r \in (1/k, 1)$. Therefore, each $u_k^{(2)}$ can be smoothly extended to the smaller ball $B^3_{1/k}$ by taking values in $\mathbb{S}^2$ and hence by paying a small amount of extra energy.

Remark 5.1. In dimension $n = 2$, instead of $n = 3$, the analogous to sequence $\{u_k^{(2)}\}$ does not converge to the vortex map $u(x) = x/|x|$ in the strict BV sense, and hence in $W^{1,1}$, too. In fact, the gradient of $u_k^{(2)}$ in U_k is of the order of $c_n k^{2-n}$ for some absolute constant $c_n > 0$, and hence in case $n = 2$ we get

$$|Du_k^{(2)}|(B^2) = \int_{B^2} |\nabla u_k^{(2)}|\, dx \to \int_{B^2} |\nabla u|\, dx + c_2 > |Du|(B^2)\,,$$

as $k \to \infty$. Whence, only L^1-convergence (or weak*-BV convergence) of $u_k^{(2)}$ to u holds, see also [30].

By the previous construction, denoting for each open set $B \subset B^3$

$$\overline{\mathcal{A}}_{BV}(u, B) := \inf\Big\{\liminf_{k\to\infty} \mathcal{A}(u_k, B) \mid \{u_k\} \subset C^1(B, \mathbb{R}^3),\ u_k \stackrel{BV}{\to} u\Big\},$$

clearly $\overline{\mathcal{A}}_{BV}(u, B^3) < \infty$. Let now B^3_r be the open ball centered at the origin and with radius r. Using the sequence $\{u^{(1)}_k\}$ and a slicing argument, as e.g. in [1, Lemma 5.2], we can find a radius $r_3 \in (0, 1)$ such that if $r > r_3$, then

$$\overline{\mathcal{A}}_{BV}(u, B^3_r) = \mathcal{A}(u, B^3_r) + \frac{4\pi}{3}.$$

On the other hand, using the sequence $\{u^{(2)}_k\}$ we obtain for any $0 < r \le 1$ the inequality

$$\overline{\mathcal{A}}_{BV}(u, B^3_r) \le \mathcal{A}(u, B^3_r) + 4\pi\, r.$$

In conclusion, arguing exactly as in [1, Thm. 5.1], it turns out that the localized functional $B \mapsto \overline{\mathcal{A}}_{BV}(u, B)$ fails to be subadditive.

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