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Doctoral Thesis

Shape Optimization of Robin Eigenvalues

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Introduction

The optimization of the eigenvalues of the Laplace operator with respect to the domain is one of the most developed topics in the field of the shape optimization. If shape optimization problems involving Dirichlet and Neumann boundary conditions have been widely studied (and partially solved, see for instance the textbooks [17], [58], [59] and [60]), on the other hand we still have a great amount of open problems involving the third kind of boundary conditions, the so-called Robin conditions. The purpose of this thesis is to investigate some of those open problems and try to solve them.

The starting point of our analysis is the eigenvalue problem for the Robin Laplacian on a Lipschitz domain Ω, i.e.

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega \end{cases}$$

(where $\beta \in \mathbb{R}$ is said boundary parameter and the value of λ clearly depends on Ω). Its weak formulation in $H^1(\Omega)$ leads to the variational representation for the k-th eigenvalue of the Robin-Laplacian on Ω with boundary parameter β via the usual Courant-Fischer min-max formula (see [34])

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx},$$

where $\mathcal{S}_k$ denotes the family of all subspaces of $H^1(\Omega)$ with dimension k. This representation is crucial in our work, since it allows to approach the shape optimization problems via the direct methods of the Calculus of Variations; in addition, it allows to study some remarkable properties of the eigenvalues. The presence of both a boundary and a volume integral at the numerator of the Rayleigh quotient for $\lambda_{k,\beta}(\Omega)$ is the reason why the approach to Robin eigenvalues is usually technically difficult: in general, we have neither a completely controlled behaviour under rescaling of the domain (above all if the boundary parameter is negative, as we will see in Chapter 2), nor monotonicity under set inclusions, in general. These two properties are crucial in solving classical shape optimization problems; see for instance the main result in [29], where a general existence result is given under the hypotheses of decreasing monotonicity under set inclusions, or Chapter 6 (in particular Corollary 6.1.6) in [17], where functionals involving Dirichlet eigenvalues are analysed. In our framework, these properties are lacking; for that reason, the shape optimization of the Robin eigenvalues turns out to be technically challenging. Indeed, in general, it is not possible to prove for the Robin eigenvalues a result of shape optimization just by adapting the proof of its Dirichlet or Neumann counterpart. For instance, the isoperimetric inequality for the first Robin eigenvalue can not be proved via the same symmetrization argument used in the classical Faber-Krahn inequality, since the behaviour of the boundary integral in the Rayleigh quotient is not completely controlled. In view of these difficulties, many authors have been encouraged to find completely new techniques to approach the shape optimization of the Robin eigenvalues.

As already said, throughout our work we made large use of direct methods of Calculus of Variations to obtain our existence results. Our main goal has been to prove the existence of optimal shapes, possibly in a relaxed setting, obtaining in some cases additional information on the structure of optimal shapes. We focused mostly on functionals involving also higher Robin eigenvalues (not only the first one), then we did not look for Faber-Krahn (or reversed Faber-Krahn) inequalities even for technical reasons. Indeed, isoperimetric inequalities for eigenvalues of higher order are generally harder to be obtained and rarer in literature. Indeed, up to our knowledge, only for the second Robin eigenvalue with positive boundary parameter there is a Faber-Krahn-type inequality, due to James Kennedy, see [64]; for higher eigenvalues no similar results are available.

The structure of the thesis is the following.

In Chapter 1, we recall some of the basic tools used in the variational approach to the shape optimization problems; we mainly refer to classic textbooks about the different topics, e.g. [2] for a survey on measure theory and functions of bounded variation, [60] for some properties of the Hausdorff convergences and of some remarkable classes of open domains, [17] for further properties of the Hausdorff convergences and for the convergence in the sense of Mosco of the functional spaces.

In Chapter 2 we refer principally to [23] in order to recall the definition and the variational formulation of the Robin eigenvalues; we point out the most important properties, their analogies with the best-known Dirichlet or Neumann eigenvalues and we report some well known Faber-Krahn (or reversed

Faber-Krahn) inequality for the Robin eigenvalues. We will recall above all the surprising result of Freitas and Kreijcirik in the case of negative boundary parameter (see [50]). As we will see in detail, it is, up to our knowledge, the first result in literature of shape optimization of the eigenvalues of the Laplacian whose solution is not the ball, in general. This surprising fact, together with the almost complete lack of results for higher eigenvalues (see [23], Open Problem 4.33), suggested us to look for some existence result for higher eigenvalues with negative boundary conditions.

This analysis is reported in Chapter 3; here we prove, in different settings, the existence of optimal shapes maximizing Robin eigenvalues with negative boundary parameter. Most of results presented are contained in a recent paper (see [18]), where we also proved some geometric controls of the spectrum that are even crucial to prove the existence of maximizers. Our work is inspired by the approach of Bogosel, Bucur and Giacomini for the maximization of the Steklov eigenvalues [11], where they proved both existence result and geometric control of the eigenvalues.

In Chapter 4 we present a result of existence and regularity for a shape optimization problem on the class convex sets. More precisely, we consider a family of functionals involving the Robin eigenvalues with positive boundary parameter, with a perimeter penalization, and prove that optimal convex shapes exist and have smooth boundary of class C^1. Our main reference is [31], in which we used a rather intuitive cutting technique to obtain the regularity of the boundary of optimal convex shapes.

In Chapter 5 we follow two different approaches to minimize $\lambda_{k,\beta}(\cdot) + P(\cdot)$ among Lipschitz domains. As usual, we tried to relax the problem in a suitable functional setting, in particular, inspired by [24], [25], [26] and [27], we chose the framework of $SBV^{1/2}$-functions. The well posedness itself of the problem has not been trivial and the existence results at the moment are available only for the principal eigenvalue and under the hypothesis of bounded design region, that ensures additional compactness; the elimination of such additional hypothesis and the existence results for higher eigenvalues can be two perspectives of research.

In Chapter 6 we reformulate in the setting of generalized polygons (recalled from [22]) many of the results of the previous chapters. Here, we obtained a nice surprise: the cutting technique developed in [31] to obtain the regularity of the boundary of optimal convex shapes can be transposed on generalized polygons to obtain a sharp estimate on the number of sides of some optimal shapes.

For the sake of completeness, in the first chapter of the appendix we presented some (local) optimality conditions that are valid, in case of simple eigenvalues, for every optimal shape with sufficiently smooth boundary.

We also studied some transposition of the classical properties of the Robin Laplacian (selfadjointness, spectral representation, etc...) and of its eigenvalues in the non-local setting (more precisely, in the framework of the fractional Sobolev spaces). We provide some properties of the general Robin linear problem, we prove that the non-local Robin eigenvalues admit a variational representation via a min-max formula and we show some basic properties of the eigenvalues. We even prove a non-local version (for fractional Sobolev spaces) of Chenais' uniform extension theorem (see [30]) for classical Sobolev spaces.

Contents

Chapter 1

Some basic tools in shape optimization

The study of shape optimization problems can find its historical origin in Dido's Problem. Even though lots of the traditional problems came out by "purely geometric" ideas, this field of study has been developed mostly using techniques of calculus of variation and geometrical analysis. In this introductory chapter, we are going to recall some important definitions and results that will be used throughout the thesis. We consider as known the very basic notions about measure theory, real analysis (L^p and Sobolev spaces, absolute continuity and mutual singularity of measures, etc...), approximate continuity and differentiability and functional analysis (dual spaces, weak convergences, etc...). We will focus only on that classic results and definitions that are directly used in our proofs, mostly if it is worth to emphasize some examples or counterexamples. Our choice is due on one hand to recall some less known results and, on the other hand, to remove ambiguities on some definitions.

Notation. Let us start fixing some notation.

For $x \in \mathbb{R}^d$ and and $r > 0$, $B_r(x)$ will denote the ball of radius r centered in x and, for every $E, F \subset \mathbb{R}^d$, we will denote the Euclidean distance between x and E by $\operatorname{dist}(x, E) := \inf\{|\,x - y| : y \in E\}$ and the Euclidean distance between E and F by $\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$. We will use the symbol $\mathbb{S}^{d-1}$ to denote the unit sphere of $\mathbb{R}^d$.

For every measurable set $E \subseteq \mathbb{R}^d$, we will use the symbols χ_E for the characteristic function of E, E^c for its complement and tE for the rescaled set $\{tx : x \in E\}$ and by $|E|$ its Lebesgue measure. We will denote by $\mathcal{B}(\mathbb{R}^d)$ the σ-algebra of Borel sets of $\mathbb{R}^d$. We will denote by $L^p(\Omega; \mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued functions that are p-summable on Ω with respect to the Lebesgue measure on $\mathbb{R}^d$ and by $W^{k,p}(\Omega;\mathbb{R}^N)$ the Sobolev space of $\mathbb{R}^N$-valued functions that are p-summable on Ω together with their first k weak derivatives. For brevity, we set $H^k(\Omega;\mathbb{R}^N) := W^{k,2}(\Omega;\mathbb{R}^N)$ and we omit $\mathbb{R}^N$ in all the above spaces if $N = 1$. If u is a measurable function, we will denote by S_u the set of approximate discontinuity of u and by J_u its set of jump points, according to the notion of approximate limit (for details, see Section 3.6 in [2]). For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^k$, we will denote by $Lip(f)$ its Lipschitz constant.

In the field of shape optimization, problems are usually set on the class of open Lipschitz domains. To avoid ambiguities about the meaning of Lipschitz domain, we recall their definition (we will use the approach of [35]).

Definition 1.0.1 (Lipschitz domain)**.** Let $E \subset \mathbb{R}^d$ be an open set. We say that E is a Lipschitz domain if, for every $p \in \partial E$, there exist $r > 0$ and a bi-Lipschitz bijection $T_p : B_r(p) \to B_1(0)$ such that

$$T_p(\partial\Omega \cap B_r(p)) = \{x \in B_1(0) : x_n = 0\}$$

and

$$T_p(\Omega \cap B_r(p)) = \{x \in B_1(0) : x_n > 0\}\,.$$

Roughly speaking, a Lipschitz domain E is an open bounded set such that ∂E is locally the graph of a Lipschitz function and E lies locally only at one side of ∂E. Open sets whose topological boundary is even a finite union of Lipschitz curves are not Lipschitz domains. For instance, the set

$$E := \left\{(x,y) \in \mathbb{R}^2 : xy < 0\right\}$$

is not a Lipschitz domain, even if ∂E is a finite union of Lipschitz curves: for the point $p = (0,0) \in \partial E$ there is no bi-Lipschitz map T_p satisfying Definition 1.0.1.

1.1 A short survey on Measure Theory and Geometric Measure Theory

In the following section we summarize some basic notions of Measure Theory and Geometric Measure Theory. For further details, see Chapters 1 and 2 in [2].

A first important tool used throughout the thesis is the s-dimensional Hausdorff measure. We recall its definition and some of the main properties we are going to use in the following (for more details see [2], Chapter 2, Section 8). Throughout the section, we will set

$$\omega_s := \frac{\pi^{s/2}}{\Gamma(1+s/2)}$$

(where $\Gamma(t) = \int_0^{+\infty} s^{t-1}e^{-s}\,ds$ is the Euler Γ function); if $s \in \mathbb{N}$, ω_s is the Lebesgue measure of the unit ball of $\mathbb{R}^s$.

Definition 1.1.1 (Hausdorff pre-measure)**.** Let $s \geq 0$ and $E \subset \mathbb{R}^d$. For every fixed $\delta \in]0,+\infty]$, we define the s-dimensional δ-Hausdorff pre-measure

$$\mathcal{H}^s_\delta(E) := \frac{\omega_s}{2^s} \inf \left\{ \sum_{i\in I} [\mathrm{diam}(E_i)]^s : \mathrm{diam}(E_i) < \delta, \right\},$$

where the infimum is computed among all finite or countable covers $\{E_i\}_{i\in I}$ of E and with the convention that $\mathrm{diam}(\emptyset) = 0$.

Notice that the map $\delta \mapsto \mathcal{H}^s_\delta(E)$ is decreasing in $]0,+\infty]$.

Definition 1.1.2 (Hausdorff measure)**.** Let $s \geq 0$ and $E \subset \mathbb{R}^d$. We define s-dimensional Hausdorff measure of E by

$$\mathcal{H}^s(E) := \lim_{\delta\to 0^+} \mathcal{H}^s(E).$$

Here we summarize some of the main properties of the Hausdorff measures.

Proposition 1.1.3 (main properties of Hausdorff measures)**.** *The following properties hold for Hausdorff measures.*

(i) *$\mathcal{H}^s$ is an outer measure on $\mathbb{R}^d$ for every $s \geq 0$ and, in particular, it is a positive measure on $\mathcal{B}(\mathbb{R}^d)$.*

(ii) *$\mathcal{H}^s$ is invariant under translation and s-homogeneous under homotheties, i.e.*

$$\mathcal{H}^s(E+x) = \mathcal{H}^s(E) \quad \forall x \in \mathbb{R}^d, \quad \mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E) \quad \forall \lambda > 0.$$

(iii) *If $s > d$, $\mathcal{H}^s(E) = 0$ for any $E \subset \mathbb{R}^d$.*

(iv) *If $s > s'$, then*

$$\mathcal{H}^s(E) \geq 0 \ \Rightarrow \ \mathcal{H}^{s'}(E) = +\infty.$$

(v) *If $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a Lipschitz function, then, for any $E \subset \mathbb{R}^d$,*

$$\mathcal{H}^s(f(E)) \leq [Lip(f)]^s \mathcal{H}^s(E).$$

Definition 1.1.4 (Hausdorff dimension)**.** Let $E \subseteq \mathbb{R}^d$. The Hausdorff dimension of E is given by

$$\mathcal{H} - dim(E) := \inf \{ s > 0 : \mathcal{H}^s(E) = 0 \} .$$

Remark 1.1.5. In some remarkable cases, the Hausdorff s-dimensional measure coincides with well known measures and it is explicitly computable.

(i) $\mathcal{H}^0$ is the counting measure on $\mathbb{R}^d$.

(ii) For every piecewise regular set E with $\mathcal{H} - dim(E) = 1$ (e.g. a piecewise regular curve), $\mathcal{H}^1$ is the length of E.

(iii) For every piecewise regular set E with $\mathcal{H} - dim(E) = 2$, $\mathcal{H}^2$ is the area of E.

(iv) For every Lebesgue-measurable set $E \subseteq \mathbb{R}^d$, $\mathcal{H}^d(E) = |E|$.

(iii) For every piecewise regular hypersurface $E \subset \mathbb{R}^d$, it holds

$$\mathcal{H} - dim(E) = d - 1$$

and $\mathcal{H}^{d-1}(E)$ is the $(d-1)$-dimensional area measure of E.

In the following we will use the term finite (real or vector valued) measure to denote a signed measure with finite total variation (see Definition 1.4 in [2]).

Definition 1.1.6 (Borel and Radon measures on $\mathbb{R}^d$)**.** A positive measure μ on $\mathcal{B}(\mathbb{R}^d)$ is called a Borel measure. If μ is finite on the compact sets, it is called a positive Radon measure.

More generally, a (real or vector valued) set function μ defined on the relatively compact sets in $\mathcal{B}(\mathbb{R}^d)$ that is also a (real or vector valued) finite measure on $\mathcal{B}(K)$ for any compact set $K \subset \mathbb{R}^d$ is said a Radon measure on $\mathbb{R}^d$; if, in addition, μ is a (real or vector valued) finite measure on $\mathcal{B}(\mathbb{R}^d)$, we call μ a finite Radon measure on $\mathbb{R}^d$.

We denote by $[\mathcal{M}_{loc}(\mathbb{R}^d)]^m$ (respectively $[\mathcal{M}(\mathbb{R}^d)]^m$) the space of the $\mathbb{R}^m$-valued Radon (respectively $\mathbb{R}^m$-valued Radon) measures on $\mathbb{R}^d$.

If μ is a Radon measure on $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ is a μ-measurable set, we will denote by $\mu \lfloor A$ the restriction of μ to A, i.e. the measure defined by $(\mu \lfloor A)(E) := \mu(E \cap A)$ for every measurable set E.

Remark 1.1.7. The Hausdorff measure $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^d$ for every $s \geq 0$, but it is a positive Radon measure on $\mathbb{R}^d$ only if $s \geq d$.

Hausdorff measures allow to generalize the classical notion of rectifiable set.

Definition 1.1.8 ($\mathcal{H}^s$-rectifiability)**.** Let $s \in [0, d]$ be an integer and let $E \subseteq \mathbb{R}^d$ be $\mathcal{H}^s$-measurable. We say that E is countably $\mathcal{H}^s$-rectifiable if there exist countably many Lipschitz functions $f_n : \mathbb{R}^s \to \mathbb{R}^d$ such that

$$\mathcal{H}^s\left(E \setminus \bigcup_{n\in\mathbb{N}} f_n(\mathbb{R}^s)\right) = 0.$$

If, in addition, $\mathcal{H}^s(E) < +\infty$, we say that E is $\mathcal{H}^s$-rectifiable.

Now we give a useful definition of weak convergence of measures; we denote by $C_c(\mathbb{R}^d)$ the space of continuous functions with compact support and by $C_0(\mathbb{R}^d)$ its completion with respect to the sup-norm $\|\cdot\|_\infty$.

Definition 1.1.9 (weak* convergence of Radon measures)**.** Let $(\mu_n)_n \subset [\mathcal{M}_{loc}(\mathbb{R}^d)]^m$ and $\mu \in [\mathcal{M}_{loc}(\mathbb{R}^d)]^m$; we say that $(\mu_n)_n$ locally weakly* converges to μ if

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} u\, d\mu_n = \int_{\mathbb{R}^d} u\, d\mu$$

for every $u \in C_c(\mathbb{R}^d)$. If $(\mu_n)_n \subset [\mathcal{M}(\mathbb{R}^d)]^m$ and $\mu \in [\mathcal{M}(\mathbb{R}^d)]^m$, we say that $(\mu_n)_n$ weakly* converges to μ if

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} u\, d\mu_n = \int_{\mathbb{R}^d} u\, d\mu$$

for every $u \in C_0(\mathbb{R}^d)$.

We recall some important results about derivation of measures which will be used in some proofs where a measure theoretical approach is required.

Theorem 1.1.10 (Radon-Nikodym)**.** *Let μ be a positive measure and ν a $\mathbb{R}^m$-valued measure on $\mathbb{R}^d$ such that μ is σ-finite. Then, there exists a unique pair ν^a, ν^s of $\mathbb{R}^m$-valued measures such that ν^a is absolutely continuous with respect to μ, ν^s and μ are mutually singular and $\nu = \nu^a + \nu^s$. In addition, there exists a unique function $f \in L^1(\mathbb{R}^d; \mathbb{R}^m)$ such that $\nu^a = f\mu$.*

The function f is called the density of ν with respect to μ and is denoted by ν/μ or by $d\nu/d\mu$. In some situations it is useful to write explicitly the density function; to this aim, we recall the following well known theorem.

Theorem 1.1.11 (Besicovitch derivation Theorem)**.** *Let μ be a positive Radon measure in an open set $\Omega \subseteq \mathbb{R}^d$ and ν a $\mathbb{R}^m$-valued Radon measure. Then, for μ-a.e. x in the support of μ, the limit*

$$f(x) := \lim_{\rho \to 0^+} \frac{\nu(B_\rho(x))}{\mu(B_\rho(x))}$$

exists in $\mathbb{R}^m$ and, moreover, the Radon-Nikodym decomposition $\nu = \nu^a + \nu^s$ is given by

$$\nu^a = f\mu, \qquad \nu^s = \nu \lfloor E,$$

where E is the μ-negligible set

$$E = (\Omega \setminus supp(\mu)) \cup \left\{ x \in supp(\mu) : \lim_{\rho \to 0^+} \frac{|\nu|(B_\rho(x))}{\mu(B_\rho(x))} = +\infty \right\}.$$

1.2 A short survey on functions of bounded variation

In this section, we refer to Chapters 3 (to recall the first definitions and results about functions of bounded variation and sets of finite perimeter) and 4 (for a short survey on special functions of bounded variation) in [2].

To begin, we recall the definition of function of bounded variation (see Definition 3.1 in [2]).

Definition 1.2.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $u \in L^1(\Omega)$. We say that u is a function of bounded variation if $Du = (D_1 u, \dots, D_d u)$, the distributional derivative of u, is a finite R^d-valued Radon measure, i.e. if

$$\int_\Omega u \frac{\partial g}{\partial x_i}\, dx = -\int_\Omega g \, dD_i u \quad \forall g \in C_c^\infty(\Omega)$$

for every $i = 1, \dots, d$. The vector space of all functions of bounded variation on Ω is denoted by $BV(\Omega)$; it is a normed space, endowed with the BV-norm, defined by

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

for any $u \in BV(\Omega)$.

We recall that the Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$ and that, if $u \in BV(\Omega)$ and $Du = 0$, u is constant in any connected component of Ω.

In the following result we give a more precise representation of the Radon-Nikodym decomposition of the gradient measure Du of $u \in BV(\Omega)$. Indeed, according to Radon-Nykodim Theorem, every Radon measure μ on an open set Ω can be splitted, into an absolutely continuous part μ^a and a singular part μ^s with respect to $\mathcal{L}^d$; in our case, the measure Du can be splitted into such two measures $D^a u$ and $D^s u$, but it is interesting to investigate some suitable splitting of the singular part $D^s u$. To this aim, we introduce the following measures

$$D^j u := D^s u \lfloor J_u, \quad D^c u := D^s u \lfloor (\Omega \setminus S_u),$$

called respectively the jump part and the Cantor part of the measure Ju. The following Theorem gives us a complete decomposition of Du (for our purposes).

Theorem 1.2.2 (decomposition of the gradient of a BV function)**.** *Let $u \in BV(\Omega)$. Then the distributional derivative Du is decomposable as follows:*

$$Du = D^a u + D^j u + D^c u,$$

i.e. $D^s u = D^j u + D^a u$; in addition, $D^a u$, $D^j u$ and $D^c u$ are mutually singular, then

$$|Du| = |D^a u| + |D^j u| + |D^c u|.$$

Moreover, $D^a u$ and $D^j u$ can be explicited by

$$D^a u = \nabla u \mathcal{L}^d,$$

where ∇u is the approximate gradient of u and by

$$D^j u = (u^+ - u^-) \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

where ν_u is the direction of the jump and u^+, u^- are the approximate limits on the two sides of J_u. Finally, Du has the following representation

$$Du = \nabla u \mathcal{L}^d + (u^+ - u^-) \nu_u \mathcal{H}^{d-1} \lfloor J_u + D^c u.$$

Notice that the decomposition $D^s u = D^j u + D^a u$ is due to the fact that Du vanishes on the $\mathcal{H}^{d-1}$-negligible set $S_u \setminus J_u$ (see Sections 3.6 and 3.7 in [2]).

Remark 1.2.3. The three parts $D^a u$, $D^s u$, $D^j u$ of the decomposition of Du have different interpretations. Roughly speaking, $D^a u$ is linked with volume integrals, $D^j u$ with $(d-1)$-dimensional surface integrals and $D^c u$ with “fractal objects”. This behaviour is even clearer looking at the respective total variations. Let us consider $u \in C^1_c(\mathbb{R}^d)$. Obviously, $u \in BV(\mathbb{R}^d)$, and its distributional derivative coincides with the classical gradient, then $Du = D^a u = \nabla u \mathcal{L}^d$. Moreover

$$|Du|(\mathbb{R}^d) = |D^a u|(\mathbb{R}^d) = \int_{\mathbb{R}^d} |\nabla u| \, dx.$$

Let us consider now the characteristic function $u := \chi_B$ of a ball $B \subset \mathbb{R}^d$. It is easy to verify that $Du = D^j u = \nu_B \mathcal{H}^{d-1} \lfloor \partial B$ and that

$$|Du|(\mathbb{R}^d) = |D^j u|(\mathbb{R}^d) = \int_{\partial B} d\mathcal{H} = \mathcal{H}^{d-1}(\partial B).$$

Finally, if we consider the well known Cantor-Vitali function u, it belongs to $BV((0,1))$ and its distributional derivative consists only in its Cantor part (concentrated on the Cantor middle third set), since u is continuous ($D^j u = 0$) and piecewise constant ($D^a u = 0$ on any interval where u is constant).

1.2.1 The space SBV

The representation formula introduced at the end of the previous section allows us to define a remarkable subspace of BV. Indeed, the Cantor part is usually hard to handle and not very frequent to be found in shape optimization problems; especially in our thesis, we will deal only with surface and volume energies. According to the ideas roughly presented in Remark 1.2.3, it is convenient to consider functions in BV whose Cantor derivative is null. For the results presented in this section, we refer the reader to Chapter 4 in [2].

Definition 1.2.4 (space $SBV(\Omega)$)**.** Let $u \in BV(\Omega)$. We say that u is a special function of bounded variation if its distributional derivative consists only in the absolutely continuous part $D^a u$ and in the jump part $D^j u$, i.e. if

$$Du = \nabla u \mathcal{L}^d \lfloor (\Omega \setminus J_u) + (u^+ - u^-) \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

namely if $D^c u = 0$. The set of all special functions of bounded variations is a vector space denoted by $SBV(\Omega)$.

We recall that the space $SBV(\Omega)$ is a closed subspace of $BV(\Omega)$ with respect to the BV-norm. An very useful (and used!) result in variational problems involving the SBV-spaces is due to L. Ambrosio and ensures compactness of suitable sequences in SBV.

Theorem 1.2.5 (compactness and lower semicontinuity)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $(u_k)_k$ be a sequence in $SBV(\Omega)$ such that, for some $p \in]1, +\infty[$:*

$$\sup_{k \in \mathbb{N}} \left\{ \|u_k\|_\infty + \int_\Omega |\nabla u_k|^p \, dx + \mathcal{H}^{n-1}(S_{u_k}) \right\} < +\infty.$$

There, there exist a subsequence $(u_{k_h})_h$ and a function $u \in SBV(\Omega)$ such that

$$u_{k_h} \to u \text{ strongly in } L^1_{loc}(\Omega),$$

$$\nabla u_{k_h} \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega, \mathbb{R}^n)$$

and

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{h \to +\infty} \mathcal{H}^{n-1}(J_{u_{k_h}}).$$

Moreover, $Du_{k_h} \overset{*}{\rightharpoonup} Du$ *in* $\mathcal{M}(\Omega; \mathbb{R}^d)$ *and the absolutely continuous part and the jump part converge separately, i.e* $\nabla u_{k_h} \rightharpoonup \nabla u$ *weakly in* $L^1(\Omega; \mathbb{R}^d)$ *and* $D^j u_{k_h} \overset{*}{\rightharpoonup} D^j u$ *in* $\mathcal{M}(\Omega; \mathbb{R}^d)$*.*

Remark 1.2.6. The previous theorem holds even if you replace the L^∞-norm by the BV-norm (A. Braides, Theorem 2.3 in [14]).

The inequality $p > 1$ is strict: if you consider $p = 1$ the theorem does not hold. Indeed, considering the usual sequence $(u_k)_k \subset SBV(0,1)$ of piecewise linear functions converging to the Cantor Vitali function u, they converge in the sense of Ambrosio's Theorem to u, but $u \notin SBV(0,1)$.

1.2.2 Sets of finite perimeter

Among measurable sets, an important role in our work will be played by sets of finite perimeter. We recall the results in Chapter 3, Sections 3 and 4 in [2].

Definition 1.2.7 (sets of finite perimeter)**.** Let $E \subseteq \mathbb{R}^d$ be measurable and let $\Omega \subseteq \mathbb{R}^d$ be open. We define the perimeter of E in Ω as

$$P(E, \Omega) := \left\{ \int_E \operatorname{div}(\varphi)\, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}$$

and we say that E is of finite perimeter in Ω if $P(E, \Omega) < +\infty$. If $\Omega = \mathbb{R}^d$ we simply say that E is of finite perimeter and denote its perimeter by $P(E)$.

Looking at the definition of perimeter, it can be easily checked that, if two measurable sets E and F coincide in Ω up to a $\mathcal{L}^d$-negligible set (i.e. if $|\Omega \cap \setminus (E \Delta F)| = 0$), then $P(E, \Omega) = P(F, \Omega)$ (see Proposition 3.38 in [2]).

An important notion to generalize the topological boundary of a Lipschitz domain and that is strictly linked with the perimeter of a measurable set is the reduced boundary. This allows us to generalize the concept of normal unit vector and, above all, will make us able to relax shape optimization problems involving boundary terms, handling reduced boundaries of finite perimeter sets in the same way as topological boundaries of Lipschitz domains.

Definition 1.2.8 (reduced boundary)**.** Let $E \subset \mathbb{R}^d$ be a measurable set and let Ω the largest open set such that E is locally of finite perimeter in Ω.

The reduced boundary of E, denoted by $\partial^* E$, is the set of the points $x \in \operatorname{supp}|D\chi_E| \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E(B_\rho(x))|}$$

exists in $\mathbb{R}^d$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \partial^* E \to \mathbb{S}^{d-1}$ is called the generalized inner normal vector to E.

It can be proved that $\partial^* E$ is a Borel set and that ν_E is a Borel function. We can also show, via the Besicovitch derivation Theorem, that $D\chi_E = \nu_E |D\chi_E|$ and that the measure $|D\chi_E|$ is concentrated on $\partial^* E$. The link among perimeter, reduced boundary and topological boundary of smooth sets is given in the following proposition.

Proposition 1.2.9. *Let $E \subset \mathbb{R}^d$ be a set of finite perimeter; then, the reduced boundary $\partial^* E$ is countably $\mathcal{H}^{d-1}$-rectifiable and $|D\chi_E| = \mathcal{H}^{d-1} \lfloor \partial^* E$. In addition,*

$$P(E, \Omega) = |D\chi_E|(\Omega) = \mathcal{H}^{d-1}(\partial^* E \cap \Omega)$$

for every open set Ω. If E is a Lipschitz domain, then it is a set of finite perimeter, $\mathcal{H}^{d-1}(\partial E \setminus \partial^* E) = 0$ and $P(E) = \mathcal{H}^{d-1}(\partial E)$.

The choice of sets of finite perimeter is usually the most natural to relax a variational shape optimization problem. Indeed, under suitable hypotheses, sequences of sets of finite perimeters are compact with respect to the convergence in measure and lower semicontinuity of the perimeters is guaranteed. This good behaviour is often useful to prove the existence of optimal shapes, at least in a relaxed setting. The following proposition will be frequently used when dealing with sets of finite perimeter; for its statement, see Proposition 3.39 in [2]; for the proof, we refer to Proposition 3.23 and Proposition 3.38(b) in [2].

Proposition 1.2.10 (Compactness of uniformly bounded sequences of sets of finite perimeter)**.** *Let $A \subset \mathbb{R}^d$ an open bounded set and let $(E_n)_n$ be a sequence of subsets of A with finite perimeter such that*

$$\sup_n P(E_n, A) < +\infty.$$

Then, there exists $E \subseteq A$ with finite perimeter in A such that, up to subsequences,

$$\chi_{E_n} \to \chi_E$$

and

$$P(E, A) \leq \liminf_{n \to \infty} P(E_n, A).$$

Remark 1.2.11. The previous compactness theorem is often used under the hypothesis that $E_n \subset\subset A$, so that we can replace the perimeters in A with the perimeters in the whole of $\mathbb{R}^d$. This will be the case of many existence results in the thesis: in many cases our effort will be to prove that the sets of any optimizing sequence are contained in a fixed bounded open set (e.g. a big ball) and that the perimeters are uniformly bounded. This idea is very recurrent in shape optimization problems; in that cases, the open bounded set A is often said a bounded design region.

1.3 Variational representation of the eigenvalues and other tools of Functional Analysis

The core of this PhD Thesis is the study of some shape problems involving the eigenvalues of a well known elliptic operator: the Laplace operator with Robin boundary conditions. The study of this shape optimization spectral problems has its basis on a good representation of the eigenvalues of such functionals. Fortunately, via some results of Functional Analysis, we are able to represent the eigenvalues of an elliptic operator by a variational formula. The definitions and spectral results presented in this section are taken from [56] and [68].[1] In the following, for every linear operator T, we will denote by $D(T)$ its domain.

Definition 1.3.1 (closed operator)**.** Let X, Y be two complex Banach spaces and let $T : X \to Y$ be a linear operator. We say that T is closed if its graph

$$G_T := \{(u, v) \in X \times Y | u \in D(T),\ v = Tu\}$$

is closed in $X \times Y$.

Definition 1.3.2 (spectrum and resolvent of a Linear operator)**.** Let X be a complex Banach space and let $T : X \to X$ be a linear operator. We define the resolvent set of T by

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}.$$

We define the spectrum of T by

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

[1] Both references are lecture notes of two courses given at the University of Paris-Sud; an extended version of [56] has been published as a book, see [57].

If T is invertible, the operator T^{-1} is often said resolvent operator of T. Let us recall that, in view of the previous definition, all the possible eigenvalues of T belong to $\sigma(T)$. Now, we introduce a property that is strictly linked with the variational representation of eigenvalues that we will use throughout the thesis.

Definition 1.3.3 (operator with compact resolvent)**.** Let X be a complex Banach space and let $T : X \to X$ be a linear operator. We say that T has compact resolvent if there exists $\lambda \in \rho(T)$ such that $(\lambda I - T)^{-1}$ is a compact operator.

In the following we will set lots of our problems in the functional spaces $H^1(\Omega)$, with Ω a bounded Lipschitz domain or $\Omega = \mathbb{R}^d$. The next proposition (see Corollary 4.10 in [68]) gives us a criterion to obtain the compactness of the resolvent for closed operators and will be very useful in the setting of H^1-spaces.

Proposition 1.3.4 (criterion for the compact resolvent of a closed operator on a Banach space)**.** *Let X be a Banach space, $T : D(T) \subseteq X \to X$ be a closed operator and let $\lambda \in \rho(T)$. Then, the operator $(\lambda I - T)^{-1}$ is compact if and only if the embedding $D_T \hookrightarrow X$ is compact.*

In this section we are going to set the results in Hilbert spaces; to avoid ambiguities, we will use the symbol $\langle \cdot, \cdot \rangle_H$ to denote the scalar product in H and the symbol $\| \cdot \|_H$ to denoted the induced norm.

Definition 1.3.5 (selfadjoint operators)**.** Let H be an Hilbert space and let $A : H \to H$ be a linear operator on H. We say that A is selfadjoint if

$$\langle Au, v \rangle_H = \langle u, Av \rangle_H$$

for any $u, v \in H$.

We remark that a selfadjoint operator is also a closed operator. It is worth to emphasize that the selfadjointness is a property depending not only on the form of the operator but also on the particular Hilbert space where the operator is defined.

Now, we recall some definitions concerning the boundedness of linear operators and bilinear forms. We refer again to [68] (Section 2.2) and [56] (Section 4.1).

Definition 1.3.6 (bounded, coercive and semibounded forms)**.** A bilinear form $a : H \times H \to \mathbb{R}$ is said to be

- bounded, if there exists $C > 0$ such that $|q(u,v)| \leq C\|u\|_H\|v\|_H$ for every $u, v \in H$;
- coercive (or elliptic), if there exists $C > 0$ such that $|q(u,u)| \geq C\|u\|_H^2$ for every $u \in H$;
- semibounded from below, if there exists $C \in \mathbb{R}$ such that $q(u,u) \geq C\|u\|_H^2$ for every $u \in H$.

Definition 1.3.7 (semibounded operator)**.** We say that a selfadjoint operator $A : D(A) \subset H \to H$ is semibounded from below if the bilinear form

$$q(u,v) := \langle Au, v \rangle_H$$

is semibounded from below.

The next fundamental result is the so called min-max or max-min principle (also said Poincaré principle or Courant-Fischer formulae) and gives us a suitable representation for our purposes. We will present the same statement as in Section 13.4 in [56], combining Theorem 13.4.1 and Remark 13.4.2. To find more details on the ideas leading to the following formulae we refer the reader to [34].

Theorem 1.3.8 (variational representation formula of the eigenvalues)**.** *Let H an Hilbert space and let $A : H \to H$ be a selfadjoint operator, semibounded from below and with compact resolvent. Then its eigenvalues consist of an increasing sequence*

$$\lambda_1 \leq \lambda_2 \leq \ldots \to +\infty,$$

where every eigenvalue is counted with its multiplicity. Moreover, the eigenvalues can be obtained via the min-max formula

$$\lambda_k = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\langle Au, u \rangle_H}{\|u\|_H}, \tag{1.1}$$

or the max-min formula

$$\lambda_k = \max_{S^\perp \in \mathcal{S}_{k-1}} \min_{u \in S \setminus \{0\}} \frac{\langle Au, u \rangle_H}{\|u\|_H}, \tag{1.2}$$

where $\mathcal{S}_k$ (resp. $\mathcal{S}_{k-1}$) denotes the family of all subspaces of H with dimension k (resp. $k-1$). Finally, the min-max and max-min are attained only at the corresponding eigenfunctions.

The quantity $\frac{\langle Au,u \rangle_H}{\|u\|_H}$ is called Rayleigh quotient.

Remark 1.3.9. In many spectral shape optimization problems the k-dimensional subspaces involved in the min-max formula can be considered as subspaces of a suitable subspace of the maximal domain $D(A)$. Indeed, if A is not selfadjoint on the whole of $D(A)$, we restrict A in such a way that all the hypotheses of Proposition 1.3.8 are satisfied. For instance, as we will see in the next chapter, we will set in $H^1(\Omega)$ the min-max formula for the eigenvalues the Robin Laplacian $-\Delta_\beta$ on a Lipschitz domain Ω: the maximal domain $D(-\Delta_\beta)$ is a proper subspace of $L^2(\Omega)$, but $-\Delta_\beta$ is selfadjoint only if we are allowed to integrate by part the expression $\langle -\Delta_\beta u, v\rangle$, namely if we consider $u, v \in H^1(\Omega)$.

Remark 1.3.10 (weak formulation of spectral functionals)**.** Proposition 1.3.8 has a double utility. On one hand, it represents a very good tool to write explicitly the spectral functional to optimize. On the other hand, it allows us to relax the functional to minimize (resp. maximize), i.e. to find the larger (resp. smaller) lower (resp. upper) semicontinuos functional that is smaller (resp. larger) than the given functional. In other words, one can enlarge the Hilbert space where the operator is defined, possibly obtaining a problem having some optimizers. We will often make use of a sort of "formal" relaxation, i.e. a weak formulation of the given problem where we will consider a formally similar functional. Such a functional will be defined on a wider class of domains or on some suitable functional spaces. This will help us if the problem is not (apparently) solvable directly, possibly concluding that the solution found in this relaxed framework is in fact a solution to the original problem (or it has a one-to-one correspondence). It is worth to emphasize that lots of the "relaxed" functionals we are going to introduce are not proper relaxations of the given functionals; for many details about a rigorous approach to relaxation see, for instance, Chapter 3 in [17], where the authors link relaxation with Optimal Control Problems.

Another important tool in Functional Analysis is the Krein-Rutman Theorem. In spite of its abstract nature, one of its most remarkable applications is to prove the simplicity of the principal eigenvalues of the Laplace operator $-\Delta_\Omega$ with suitable boundary conditions on a bounded connected Lipschitz domain Ω. We report the same statement as in Theorem 1.2.6 in [58].

Theorem 1.3.11 (Krein-Rutman)**.** *Let X be a Banach space and $C \subset X$ a closed cone vertexed in 0 such that $\mathring{C} \neq \emptyset$ and $C \cap (-C) = \{0\}$. Let $T : X \to X$ be a compact operator such that $T(C \setminus \{0\}) \subset \mathring{C}$. Then, the greatest eigenvalue of T is simple and the corresponding eigenvectors are in $\mathring{C} \cup (-\mathring{C})$.*

To prove the above cited simplicity of the first eigenvalue, one applies previous theorem taking as T the resolvent operator of $-\Delta_\Omega$, once it is proved that T is compact on a suitable Banach space and that for some cone of functions C it holds $T(C \setminus \{0\}) \subset \mathring{C}$ (it is often done via some maximum principle). Then, since the eigenvalues of T are the reciprocal of the eigenvalues of $-\Delta_\Omega$, we can conclude that the principal eigenvalue of $-\Delta_\Omega$ is simple. A sketch of that kind of argument, applied to Dirichlet-Laplacian eigenvalues, can be found in [58], Theorem 1.2.5.; in Chapter 2 we will apply an analogous argument to the Robin-Laplacian operator on a connected Lipschitz domain.

1.4 Hausdorff convergences

In this section we introduce a very useful notion of distance among closed sets, the so-called Hausdorff distance, and we will introduce some topologies on open and closed sets induced by that distance; we refer mostly to Chapter 2 in [60] and Section 2.4 and 4.6 in [17]. Indeed, the class of sets of finite perimeter enjoys good properties in terms of compactness and lower semicontinuity under suitable topologies, which are fundamental when one works with direct methods of calculus of variation. The only disadvantage is that no topological properties of converging sequences can be a priori ensured for the limit set in this framework. We can only deduce measure theoretical properties: in general we are not able to say if a convergent sequence of open (or connected, compact, etc...) sets of finite perimeter converges to an open (or connected, compact, etc...) set. This difficulty may be overcome choosing a priori a suitable class of sets satisfying some topological property and a compact topology on this class of sets. The idea is to choose such a compact topology so that also the functional involved in our minimization problem turns out to be semicontinuos.

1.4.1 Hausdorff distance, H-convergence of compact sets, H^c-convergence of bounded open sets

Definition 1.4.1 (Hausdorff topology on closed sets)**.** Let $A, B \subseteq \mathbb{R}^d$ be closed. We define the *Hausdorff distance between A and B* by

$$d_H(A,B) := \max\left\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{x\in B} \operatorname{dist}(x,A)\right\}.$$

The topology induced by this distance is called *Hausdorff topology (or simply H-topology) on closed sets.*

This topology turns out to be good for our purposes since, under not so restrictive hypotheses on the functional, it guarantees compactness and semicontinuity required to apply direct methods of calculus of variation. Moreover, it preserves some topological properties when we consider the limit set of a sequence of sets enjoying that property. In the next proposition are summarized some remarkable properties of the H-convergence presented in Section 2.2.3 of [60].

Proposition 1.4.2 (Properties of the H-convergence)**.**

(i) *A decreasing sequence of non-empty compact sets H-converges to its intersection.*

(ii) *An increasing sequence of non-empty compact sets contained in a compact B H-converges to the closure of its union.*

(iii) *If $(K_n)_n$ is a sequence of compact sets contained in a compact B and $K_n \xrightarrow{H} K$, then*

$$K = \bigcap_{n\in\mathbb{N}} \left(\overline{\bigcup_{p\geq n} K_p} \right) = \left\{ x \in B : \exists x_{n_p} \in K_{n_p}, x_{n_p} \overset{p\to\infty}{\longrightarrow} x \right\}$$
$$= \left\{ x \in B : \exists x_n \in K_n, x_n \overset{n\to\infty}{\longrightarrow} x \right\}$$

(iv) *The inclusion is stable for the Hausdorff convergence: if $K_n \overset{H}{\to} K$, $G_n \overset{H}{\to} G$ and $K_n \subseteq G_n$ for every $n \in \mathbb{N}$, then $K \subseteq G$.*

It is worth emphasizing an important compactness result involving the Hausdorff convergence (see Theorem 2.2.23 in [60]).

Proposition 1.4.3. *Let $B \subset \mathbb{R}^d$ a fixed compact set. Then, the class of the closed sets contained in B is compact in the Hausdorff topology.*

In the following, we will use some corollaries of this result, adding some topological constraints and showing that the obtained class of sets are still compact with respect to Hausdorff convergence.

Remark 1.4.4. Let us consider $A \subset \mathbb{R}^d$ and $A_n \subset \mathbb{R}^d$ for every $n \in \mathbb{N}$. We say that $A_n \to A$ uniformly if, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$A \subseteq A_n + B_\varepsilon(0), \quad A_n \subseteq A + B_\varepsilon(0) \quad \forall n \geq n_\varepsilon.$$

The remarkable fact is that, for closed sets, uniform convergence and Hausdorff convergence coincide (see Remark 2.4.2 in [17]); more precisely

$$d_H(A, B) = \inf \left\{ \varepsilon > 0 : A \subseteq B + B_\varepsilon(0), \quad B \subseteq A + B_\varepsilon(0) \right\}.$$

This is an immediate consequence of the straightforward equivalence

$$A \subseteq B + B_\varepsilon(0) \;\Leftrightarrow\; \sup_{x\in B} \operatorname{dist}(x, A) \leq \varepsilon.$$

This equivalence will be useful when we will try to get compactness results in the Hausdorff convergence in some particular classes of domains with topological constraints.

In shape optimization problems we often have to work with particular classes of open sets and we will would like to endow this classes with a suitable topology recalling the good properties of the Hausdorff topology on closed sets. The counterpart for open sets of the Hausdorff topology is defined below.

Definition 1.4.5 (Hausdorff topology on open sets)**.** Let $D \subset \mathbb{R}^d$ be compact and let $A, B \subset D$ be open. We define the *Hausdorff-complementary distance between A and B* by

$$d_{H^c}(A, B) := d_H(D \setminus A, D \setminus B).$$

The topology induced by this distance is called *Hausdorff-complementary topology (or simply H^c-topology) on open sets.*

The previous definition is independent of the choice of the fixed compact "box" D; in view of this, many authors adopt the notation $d_H(A^c, B^c)$ instead of $d_H(D \setminus A, D \setminus B)$.

Now let us state some remarkable properties of the H^c-convergence (see Section 2.2.3 of [60] for their proofs).

Proposition 1.4.6 (Properties of the H^c-convergence)**.**

(i) *An increasing sequence of open sets contained in a compact B H^c-converges to its union.*

(ii) *A decreasing sequence of open sets H^c-converges to the interior of its intersection.*

(iii) *Let $(A_n)_n$ be a sequence of open sets and A an open set such that $A_n \xrightarrow{H^c} A$. Then, for every $x \in \partial A$, there exists a sequence of points $x_n \in \partial A_n$ such that $x_n \to x$.*

(iv) *The inclusion is stable for the H^c-convergence for open sets: if $A_n \xrightarrow{H} A$, $V_n \xrightarrow{H} V$ and $A_n \subseteq V_n$ for every $n \in \mathbb{N}$, then $A \subseteq V$.*

Similarly to the case of H-topology for compact sets, under reasonable hypotheses we have a compactness result for open sets in the H^c-topology (See Corollary 2.2.24 in [60]).

Proposition 1.4.7. *Let $D \subset \mathbb{R}^d$ a fixed compact set. Then, the class of the open sets contained in D is compact in the Hausdorff-complementary topology.*

Remark 1.4.8. It holds that, if $(\Omega_n)_n$ is a sequence of open sets that H^c-converges to Ω and if K is a compact contained in Ω, then $K \subset \Omega_n$ for n sufficiently large (see Proposition 2.2.15 in [60]). This assertion is straightforwardly verified observing that $\inf_{x \in K} \operatorname{dist}(x, \Omega^c) > 0$ and that $\operatorname{dist}(x, \Omega^c) \leq \operatorname{dist}(x, \Omega_n^c) + d_{H^c}(\Omega_n, \Omega)$, and so also $\inf_{x \in K} \operatorname{dist}(x, \Omega_n^c) > 0$ for n large enough.

This result will be useful to prove the convexity of a H^c-limit set of open convex sets.

The preservation of topological properties sometimes is not enough to guarantee the compactness of an optimizing sequence in a variational problem; in fact, if we have some constraints on the Lebesgue measure or on the perimeter, we would like that the limit sets of an optimizing sequence satisfies the same constraint. In other words, we require some kind of (semi)continuity of the Lebesgue measure or of the perimeters with respect the Hausdorff convergences. We will immediately see that the results are not always positive: this bad behaviour in general can also affect the semicontinuity of the shape functionals in the problem. For the following results and counterexamples, we refer to Section 2.2.3 in [60].

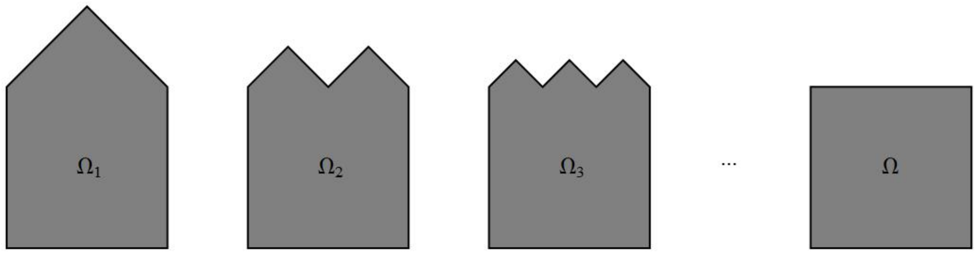
Remark 1.4.9 (Semicontinuity of the Lebesgue measure)**.** The Lebesgue measure is lower semicontinuos with respect to the H^c-convergence (see pag. 34 or Proposition 2.2.21 in [60]), but not continuous, in general. Take, for instance

$$\Omega_n :=]0,1[\setminus \bigcup_{k=1}^{n-1} \left\{ \frac{k}{n} \right\};$$

we have $\Omega_n \xrightarrow{H^c} \emptyset$, $|\Omega_n| = 1$ for every $n \in \mathbb{N}$, but $|\emptyset| = 0 < \liminf_n |\Omega_n| = 1$.

On the other hand, the Lebesgue measure is upper semicontinuos for the H-topology (it is enough to apply the H^c-lower semicontinuity of the Lebesgue measure to the sets K_n^c and K^c), but it is not lower semicontinuos in general. Take for instance an enumeration $\{x_n\}_n$ of the rational points in $[0,1]$ and set $K_n := x_k : k \leq n$. All this compact sets have null Lebesgue measure, but they H-converge to $K := [0,1]$, whose Lebesgue measure is 1.

Remark 1.4.10 (Semicontinuity of the perimeter)**.** In the H^c-complementary topology, the perimeter is neither upper nor lower semicontinuos, in general. As a counterexample for the upper semicontinuity, it is enough to take a sequence of saw-toothed squares Ω_n, with side length 1 and n isosceles right triangular teeth per side. All their perimeters are equal to $4\sqrt{2}$, but they



H^c-converge to the open square Ω with side length 1 and perimeter equal to 4.

To show the lack of lower semicontinuity, we consider in $\mathbb{R}^2$ the annulus

$$\Omega = B_1((0,0)) \setminus \overline{B_{1/2}((0,0))}$$

and the sequence of open sets $(\Omega_n)_n \subset \mathbb{R}^2$ defined as follows. For every fixed $n \in \mathbb{N}$, let us consider all the points of the form

$$x^n_{j,k} := \left(\frac{j}{n}, \frac{k}{n}\right)$$

with $j, k \in \mathbb{Z}$. Let us denote by $x^n_1, \ldots, x^n_{p(n)}$ all the points $x^n_{j,z}$ such that $x^n_{j,k} \in B_{1/2}((0,0))$. Then, we set

$$\Omega_n := B_1((0,0)) \setminus \left(\bigcup_{i=1}^{p(n)} \{x^n_j\}\right).$$

It is easily verifiable that Ω_n H^c-converges to Ω. But

$$P(\Omega_n) = 2\pi < 3\pi = P(\Omega)$$

so we do not have lower semicontinuity of the perimeters, in general.

The previous examples shows that, in general, we only have upper (resp. lower) semicontinuity of the Lebesgue measure in the H-convergence (resp. H^c-convergence), but we do not have continuity of the measures nor semicontinuity of the perimeters in general. One can obtain the required properties only adding some other hypotheses, e.g. on the number of the connected components of the topological boundary of the involved sets, as the following result shows (for a proof, we refer the reader to Theorem 4.4.17 in [3] or Theorem 3.18 in [47]).

Theorem 1.4.11 (Golab). *Let $(K_n)_n$ be a sequence of compact connected sets in $\mathbb{R}^d$ such that $K_n \to K$ in the Hausdorff metric. Then, K is connected and*

$$\mathcal{H}^1(K) \leq \liminf_{n\to+\infty} \mathcal{H}^1(K_n).$$

Notice that the previous result involves only the one dimensional Hausdorff measure, and so it is really useful only when the sets K_n and K are curves in $\mathbb{R}^d$; in particular, we will use Theorem 1.4.11 when dealing with classes of open bounded sets in $\mathbb{R}^2$ whose boundaries are compact curves of $\mathbb{R}^2$.

Once seen that both H and H^c-topology are compact on uniformly bounded sets (respectively compact and open), a natural question arises: which topological properties are preserved to the limit set? A first answer has already been given: if all sets are in a big (compact) box, compact sets H-converge to compact sets, open sets H^c-converge to open sets. We are interested in knowing if other properties are maintained. For the following subsections we refer principally to Chapter 2 in [60].

1.4.2 Convexity and Hausdorff convergence

In this subsection we study how convexity and Hausdorff convergence are linked. We start this with an important remark (see Item 8 in section 2.2.3 of [60]).

Remark 1.4.12. If we consider a H^c-convergent sequence $(\Omega_n)_n$ of convex open sets, then also the limit set Ω is convex. In fact, for every $x, y \in \Omega$, the compact set $\{x, y\}$ is contained in Ω and so, as remarked in 1.4.8, $\{x, y\} \subset \Omega_n$ for n sufficiently large. Then, as Ω_n is convex, $[x, y] \subset \Omega_n$, and thanks to the stability of the H^c-convergence with respect to inclusions, $[x, y] \subset \Omega$ and so Ω is convex.

Using the same argument, it can be proved that, if a sequence $(K_n)_n$ of convex compact sets H-converge to K, then K is a convex compact set.

Notice that in the previous remark we assume a priori that the sequences are convergent, so it is not a compactness theorem. Under the hypotheses of bounded design region (see Remark 1.2.11) we gain the following compactness result (combining Theorem 2.2.23 and Corollary 2.2.24 in [60] with the previous remark).

Theorem 1.4.13. *The class of closed (resp. open) convex sets contained in a bounded design region is compact with respect to the H-convergence (resp H^c-convergence).*

Next proposition shows that, for convex sets, we have continuity for Lebesgue measure and perimeters, which in general is not true, as seen in the previous section. For a proof see Proposition 2.4.3 in [17].

Proposition 1.4.14. *The following results hold for convex sets:*

(i) *If* $A \subseteq B$, *then* $\mathcal{H}^{d-1}(\partial A) \leq \mathcal{H}^{d-1}(\partial B)$;

(ii) *If* A_n, A *are closed (respectively open) and* $A_n \to A$ *with respect to the* H*-topology (respectively* H^c*-topology), then* $\chi_{A_n} \to \chi_A$ *in* L^1*; moreover, if* $\mathcal{H} - dim(A) = \mathcal{H} - dim(A_n)$, *then* $\mathcal{H}^{d-1}(\partial A_n) \to \mathcal{H}^{d-1}(\partial A)$.

(iii) $|A| \leq \rho \mathcal{H}^{d-1}(\partial A)$, *where* ρ *is the radius of the biggest ball contained in* A.

We close this short section recalling an important result due to F. John (see [61]), involving convex sets; this result allows, under some suitable hypotheses, the compactness of a sequence of convex sets with non-empty interior to a convex set with non-empty interior.

Theorem 1.4.15 (John's ellipsoid Theorem)**.** *Let* $K \subset \mathbb{R}^d$ *a compact convex set with non-empty interior. Then, there exists an ellipsoid* $E \subset \mathbb{R}^d$ *centered in* $x_0 \in E$ *such that*

$$E \subseteq K \subseteq x_0 + d(E - x_0)$$

(where the ellipsoid $x_0 + d(E - x_0)$ *is obtained by a dilation of* E *of a factor of* d *and with center* c*).*

The ellipsoid E is often said John's ellipsoid and it is the ellipsoid of maximal volume contained in K.

1.4.3 Connectedness and Hausdorff convergence

If convexity is preserved by Hausdorff convergences for both open and closed sets, we do not have the same result for the connectedness (see Item 9 in section 2.2.3 of [60]).

Remark 1.4.16. The H^c-convergence does not preserve the connectedness of the open sets, unless you are in dimension one (in $\mathbb{R}$ convexity and connection coincide). As counterexamples, we can take in $\mathbb{R}^2$

$$\Omega_n^1 :=]0,2[\times]0,1[\setminus \{1\} \times \left[\frac{1}{n}, 1\right]$$

or

$$\Omega_n^2 := B_2(0) \setminus \left\{ e^{\frac{ik\pi}{n}} : 0 \leq k < n \right\}$$

(see [60], pag. 33, fig. 2.2), which converge respectively to

$$\Omega^1 :=]0,1[\times]0,1[\cup]1,2[\times]0,1[$$

and

$$\Omega^2 := B_2(0) \setminus \partial B_1(0).$$

On the other hand, connection is preserved by the H-convergence of compact sets. In fact, the following proposition holds (see Proposition 2.2.17 in [60]).

Proposition 1.4.17. *Let $(K_n)_n$ a sequence of compact connected sets converging to a compact set K. Then K is connected. More generally, if K_n has at most $p \geq 1$ connected components, then K has at most p connected components.*

Remark 1.4.18 (see Remark 2.2.18 in [60])**.** The previous proposition does not have a counterpart for the H^c-topology (as seen in the counterexample 1.4.16). The only result that we can gain is an application of Proposition 1.4.17 to the complements (in a fixed compact box) of the open sets of a H^c-converging sequence. Precisely, if $\Omega_n \xrightarrow{H^c} \Omega$ into a fixed compact B, then $\#(\Omega^c \cap B) \leq \liminf_n \#(\Omega_n^c \cap B)$.

In dimension $d = 2$ this remark allows us to obtain some important topological information; we remark that if a bounded open set in $\mathbb{R}^2$ is disjoint union of simply connected open sets, then its complement (in the compact B) is a compact connected set. This allow us to prove that the H^c-limit of unions of simply connected set is union of simply connected set. Indeed, let $\Omega_n \subset \mathbb{R}^2$ be a bounded disjoint union of simply connected open sets for every $n \in \mathbb{N}$; if $\Omega_n \xrightarrow{H^c} \Omega$, then

$$1 \leq \#(\Omega^c \cap B) \leq \liminf_n \#(\Omega_n^c \cap B) = 1$$

and so Ω is union of simply connected open sets.

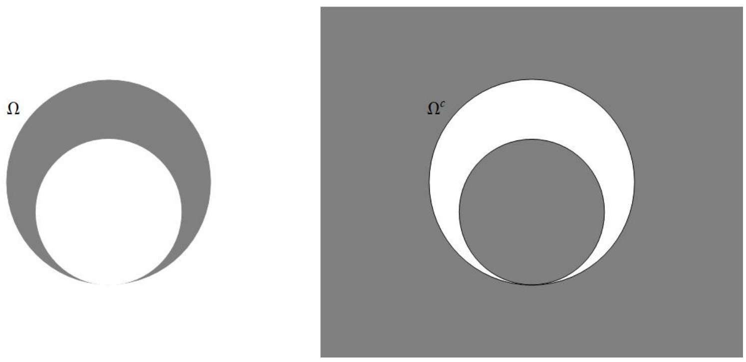


Figure 1.1: The open set Ω is simply connected, its complement Ω^c is connected (the interior small disk and the complement of the bigger disk are connected by the point of tangency).

1.4.4 Uniform cone properties and Hausdorff convergence

In this section we focus on a class of opens sets satisfying some uniform regularity conditions. For the proofs of this section and for some other references, see Section 2.4 in [60].

In particular, we require that the cones have uniform direction. We introduce the following notation. For every $y \in \mathbb{R}^d$, ξ unit vector and $\varepsilon > 0$, we define open cone with vertex y, direction ξ and size ε the open set:

$$C(y, \xi, \varepsilon) := \left\{ z \in \mathbb{R}^d : (z - y) \cdot \xi > \cos(\varepsilon)|z - y|, \quad 0 < |z - y| < \varepsilon \right\}.$$

Notice that such cones have height and opening depending on the same parameter ε.

Definition 1.4.19 (ε-cone property)**.** Let $\Omega \subset \mathbb{R}^d$ be open. We say that Ω has the ε-cone property if, for every $x \in \partial\Omega$, there exists a unit vector ξ_x such that, for every $y \in \overline{\Omega} \cap B_\varepsilon(x)$, one has $C(y, \xi_x, \varepsilon) \subset \Omega$. We denote by $\mathcal{C}_\varepsilon$ the family of all sets in $\mathbb{R}^d$ satisfying the ε-property.

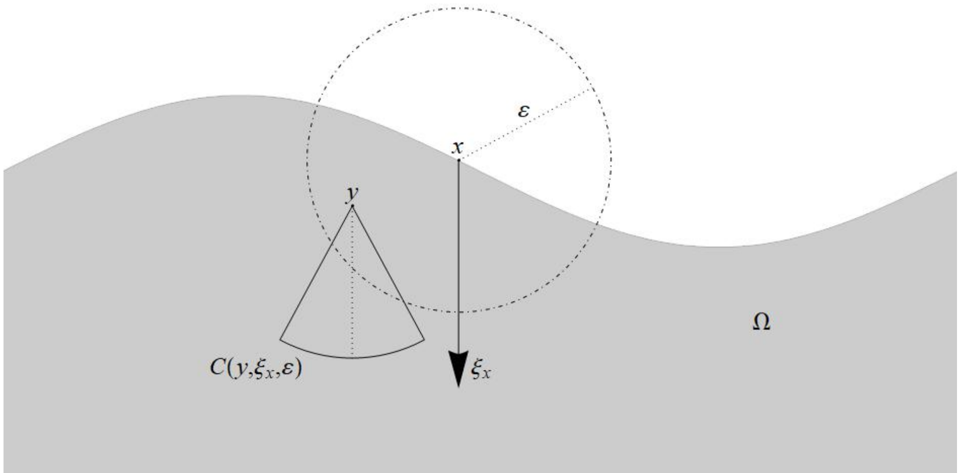


Figure 1.2: ε-cone property in $\mathbb{R}^2$.

It is worth to emphasize that in the previous definition the direction of the cone is uniform for all the cone vertexed in $B_\varepsilon(x) \cap \Omega$.

Remark 1.4.20. If $\Omega \in \mathcal{C}_\varepsilon$, then $\overline{\Omega}^c \in \mathcal{C}_\varepsilon$.

Indeed, let us consider $x \in \partial\Omega$ and let us prove that $C(y, -\xi_x, \varepsilon) \subset \Omega^c$ for every $y \in \overline{\Omega}^c \cap B_\varepsilon(x)$. Let us fix $y \in \overline{\Omega}^c \cap B_\varepsilon(x)$, $z \in C(y, -\xi_x, \varepsilon)$ and let us suppose that $z \in \Omega$. Since $\Omega \in \mathcal{C}_\varepsilon$, then $C(z, \xi_x, \varepsilon) \subset \Omega$; moreover, by construction, $y \in C(z, \xi_x, \varepsilon) \subset \Omega$ and this is in contradiction with $y \in \Omega^c$. Then, even $\overline{\Omega}^c$ satisfies the ε-cone property.

The following proposition links the ε-cone property with the Lipschitz domains (see Proposition 2.4.7 in [60]).

Proposition 1.4.21. *Let $\Omega \subset \mathbb{R}^d$ be an open set such that $\partial\Omega$ is bounded. Then, Ω is a Lipschitz domain if and only if there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{C}_\varepsilon$.*

In other words, Lipschitz domains satisfy some ε-cone property and vice versa.

Remark 1.4.22 (see Remark 2.4.8 in [60])**.** The sets in $\mathcal{C}_\varepsilon$ are equilipschitz, i.e. the Lipschitz constants in Definition 1.0.1 are the same for all sets in $\mathcal{C}_\varepsilon$ and depend only on ε.

Remark 1.4.23. The fact that the direction of the axes of the cones are locally uniform is necessary in order to have the equivalence stated in 1.4.21. Indeed, the open set

$$E = \big\{(x, y) \in \mathbb{R}^2 : xy > 0\big\}$$

satisfy a uniform interior and exterior $\pi/2$-cone property if we allow the cones to rotate even locally, but it is not a Lipschitz domain. Then, $E \notin \mathcal{C}_\varepsilon$ for any ε: the point $(0, 0)$ does not satisfy the property in Definition 1.4.19 for any choice of $\varepsilon > 0$.

Remark 1.4.24. All the previous properties remain valid even if the uniform height of the cone or the uniform size of the neighbourhood of the boundary point in Definition 1.4.19 are chosen not equal to ε, but still uniformly, see [30]. In other words, if we choose cones of opening ε in Definition 1.4.19, we can choose a suitable uniform height h for the cones and a suitable uniform radius r for the neighbourhoods instead of ε itself. Since this does not modify the validity of the results above, in the following, once chosen the uniform opening ε of the cones, we will only speak of "uniform cone property" or "ε-cone property", without specifying the uniform height and radius chosen.

Since the natural setting for many shape optimization problems is the family of Lipschitz domains of $\mathbb{R}^d$, possibly satisfying some constraints (e.g., the family of Lipschitz domain with fixed measure or perimeter), we ask ourselves if the class $\mathcal{C}_\varepsilon$ is compact under some suitable topology. In the next result (see Proposition 2.4.10 in [60]) we give a positive answer to that question.

Proposition 1.4.25 (compactness and semicontinuity under Hausdorff convergences)**.** *For any $\varepsilon > 0$, the class $\mathcal{C}_\varepsilon$ is H^c-compact, i.e., for every equibounded sequence $(\Omega_n)_n \subset \mathcal{C}_\varepsilon$, there exists $\Omega \in \mathcal{C}_\varepsilon$ such that, up to subsequences,*

Ω_n H^c-converges to Ω. Moreover, $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\mathbb{R}^d)$ (i.e. the convergence is also in measure) and, up to the same subsequence above, the compact sets $\overline{\Omega}_n$ and $\partial\Omega_n$ H-converge respectively to $\overline{\Omega}$ and $\partial\Omega$.

Remark 1.4.26. Let $(A_n)_n$ be a uniformly bounded sequence of open sets of $\mathbb{R}^d$ satisfying a ε-cone property with $|A_n| \geq m > 0$ for every $n \in \mathbb{N}$; it is easy to show that

$$\overline{A_n} \xrightarrow{H} \overline{A} \qquad \Leftrightarrow \qquad \mathring{A}_n \xrightarrow{H^c} \mathring{A}.$$

The implication "$\Leftarrow$" follows by the previous proposition. The converse implication follows by the definition of uniform convergence of sets and its equivalence with the H-topology on compact sets. Let us fix $\delta > 0$. Since $\overline{A_n}, \overline{A}$ are contained in a compact set $B \subset \mathbb{R}^d$, we have that there exists $n_\delta \in \mathbb{N}$ such that, for every $n \geq n_\delta$

$$\overline{A_n} \subset \overline{A} + B_\delta, \quad \overline{A} \subset \overline{A_n} + B_\delta.$$

Since A_n, A have regular boundaries, we deduce that

$$D \setminus \mathring{A}_n \subset (D \setminus \mathring{A}) + B_\delta, \quad D \setminus \mathring{A} \subset (D \setminus \mathring{A}_n) + B_\delta.$$

Then $D \setminus \mathring{A}_n$ H-converges to $D \setminus \mathring{A}$ and so $\mathring{A}_n$ H^c-converges to $\mathring{A}$.

In view of the previous equivalence, in lots of problems involving uniformly regular sets, we will speak only of *Hausdorff convergence*, specifying if the involved sets are open or closed only where necessary.

A very important result is the following uniform extension theorem for H^k-spaces on $\mathcal{C}_\varepsilon$. It has been proved by D. Chenais in 1975, see [30].

Theorem 1.4.27 (uniform extension theorem for $H^k(\Omega)$)**.** *Let $\varepsilon > 0$ and $D \subset \mathbb{R}^d$ be a bounded design region. Then, there exist a positive constant C depending on ε, D and k such that, for every $\Omega \subset D$, $\Omega \in \mathcal{C}_\varepsilon$, there exists an extension operator*

$$E_\Omega : H^k(\Omega) \to H^k(\mathbb{R}^d)$$

such that

$$\|E_\Omega\| \leq C.$$

Roughly speaking, the previous theorem ensures that H^k-functions of uniformly regular sets can be extended to the whole of $\mathbb{R}^d$ with the same constant. This theorem turns out to be very useful in the case of H^c-converging sequences of uniformly regular open sets. For instance, we will use this result in Theorem 4.1.3, where the uniformly regular sets are bounded convex sets. It is worth to emphasize that the result is still valid for fractional Sobolev spaces H^s: we will give a proof of the result in Theorem B.3.4 in Appendix[2].

Even though lots of shape optimization problems are set in the class of Lipschitz domains (possibly satisfying some constraint), it is not possible to use the previous compactness result in such a general class. When we study that problems using direct methods of Calculus of Variations, one of the main difficulty is to prove that any optimizing sequence of Lipschitz domains converge to a Lipschitz set; one way could be to show that all the set in the sequence enjoy some ε-cone property, but it is usually hard, a priori. For that reason, two different approaches are used: either the ε-cone regularity is a priori inferred to obtain existence in some subclasses of $\mathcal{C}_\varepsilon$, or the problem is relaxed in a more general setting (open sets, finite perimeter sets,...) to obtain compactness with respect to other topologies.

1.5 Continuity under deformations: Mosco convergence

In this section we will summarize some necessary tools to study the so-called shape continuity, i.e. the continuity (or, at least, the semicontinuity) of a shape functional when the admissible domains vary in a class of sets endowed with a suitable topology; we refer principally to Sections 4.5 and 7.2 in [17]. Indeed, in many shape optimization problems solved via direct methods of the Calculus of Variation, the most challenging part in proving existence of optimal shapes is to prove that there is (semi)continuity of the functional with respect to the chosen topology, at least for an optimizing sequence $(\Omega_n)_n$, and that such sequence is compact with respect to the same topology. This problem, as we will see in the following chapters, is often linked with the behaviour of the functional spaces $X(\cdot)$ (e.g. $H^1(\cdot)$ or $H^1_0(\cdot)$) involved in the variational formulation of the problem. In other words, we need to study how the functional spaces $X(\Omega_n)$ vary and whether they converge to $X(\Omega)$, whenever Ω_n converge to Ω in some sense. The idea is to deduce that information directly working on some convergence (in the sense of Hausdorff, in measure, etc.) on the class of admissible domains.

The key point in this setting is to look for some topology that is weak enough to guarantee the compactness of optimizing sequences but strong enough to ensure at least semicontinuity of the shape functional. Moreover, such a topology has to entail the convergence of the functional spaces described above.

[2]The result is proved with a slightly different statement.

A good idea is to look at an important notion of convergence of functionals introduced by E. De Giorgi and T. Franzoni in [45].

Definition 1.5.1 (Γ-convergence)**.** Let X be a topological space and let

$$F_n : X \to [0, +\infty[$$

be a functional on X for every $n \in \mathbb{N}$. We say that the sequence $(F_n)_n$ Γ-converges to a functional $F : X \to [0, +\infty[$ if the following two conditions are satisfied:

(i) for every sequence $(x_n)_n \subset X$ converging to some $x \in X$, it holds

$$F(x) \leq \liminf_{n \to +\infty} F_n(x_n);$$

(ii) for every $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to x such that

$$F(x) \geq \limsup_{n \to +\infty} F_n(x_n).$$

In this case, F is said the Γ-limit of F_n.

In terms of minimization, Γ-convergence is very useful. Indeed, it has been proved that the Γ-limit functional F is lower semicontinuos; moreover, if x_n is a minimizer for F_n, every cluster point of the sequence $(x_n)_n$ is a minimizer for F (roughly speaking, as some authors say, “minimizers converge to minimizers”).

Now, our aim is to find a convergence of functional spaces that recalls the good variational properties of the Γ-convergence and that is strongly linked to some convergence of domains. The following notion of convergence of Banach spaces is very useful in shape optimization problems. In particular, in our research work, it will be used to obtain convergence of H^1 spaces of H^c-converging domains. For the following definition and further details we refer the reader to Section 4.5 in [17], where the general case and some applications to the H_0^1 spaces are treated.

Definition 1.5.2 (convergence in the sense of Mosco)**.** Let X be a Banach space and $(G_n)n$ a sequence of closed subsets of X. We define weak upper and strong lower limits in the sense of Kuratowski the spaces

$$w - \limsup_{n \to +\infty} G_n := \left\{ u \in X : \exists\, (n_k)_k,\ \exists\, u_{n_k} \in G_{n_k} \text{ s.t. } u_{n_k} \rightharpoonup u \text{ weakly in } X \right\},$$

$$s - \liminf_{n \to +\infty} G_n := \left\{ u \in X : \exists\, u_{n_k} \in G_{n_k} \text{ s.t. } u_n \to u \text{ strongly in } X \right\}.$$

We say that G_n converges to the closed subspace G in the sense of Mosco (or briefly: G_n Mosco-converges to G) if

$$w - \limsup_{n\to+\infty} G_n \subseteq G$$

and

$$G \subseteq s - \liminf_{n\to+\infty} G_n,$$

i.e. if

$$w - \limsup_{n\to+\infty} G_n = G = s - \liminf_{n\to+\infty} G_n.$$

The convergence in the sense of Mosco is really useful to handle the H^1 spaces on moving domains: under suitable topological constraints, Mosco convergence is equivalent to convergence in measure and Hausdorff convergence. An important result is the following theorem holding in $\mathbb{R}^2$ (see [17], Theorem 7.2.1).

Proposition 1.5.3. *Let $(\Omega_n)_n$ be a sequence of open domains in $\mathbb{R}^2$ such that $\#\Omega_n^c \leq l$ such that Ω_n H^c-converges to some Ω. Then $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$ if and only if $|\Omega_n|$ converges to $|\Omega|$. In particular, the Mosco convergence holds if $\sup_n(\#\Omega_n) < +\infty$ and $\sup_n \mathcal{H}^{d-1}(\partial\Omega_n) < +\infty$.*

Recalling the results in previous sections linking Hausdorff convergences to convergence in measure, the previous proposition turns out to be very useful when dealing with some classes of planar domains where some extra topological constraint is a priori inferred and guarantees the convergence in measure. For instance, taking into account a sequence $(\Omega_n)_n$ of disjoint unions of simply connected open sets of $\mathbb{R}^2$ that H^c-converge to an open set Ω, if the convergence is also in measure, then we also have $H^1(\Omega_n) \to H^1(\Omega)$. If we require some regularity of the boundary, we have a Mosco convergence result holding in any dimension d (see Proposition 7.2.7 in [17]).

Proposition 1.5.4. *Let Ω_n, Ω be open domains in a bounded design region $B \subset \mathbb{R}^d$ satisfying a uniform cone condition. If Ω_n H^c-converges to Ω, then $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$.*

Remark 1.5.5. Let us take Ω_n, Ω as in Proposition 1.5.4 and $u_n \in H^1(\Omega_n)$ such that $\|u_n\|_{H^1(\Omega_n)} < C$, with $C > 0$ independent on n. Using Definition 1.5.2, it is possible to prove that there exists $u \in H^1(\Omega)$ such that, up to subsequences, $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\mathbb{R}^d)$ and $\tilde{\nabla} u_n \to \tilde{\nabla} u$ weakly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$, where we denoted by $\tilde{f}$ the zero extension of the function f outside its domain (Ω_n and Ω for $u_n, \nabla u_n$ and $u, \nabla u$, respectively).

So far, we recalled some important tools used in many shape optimization problems solved via direct methods of Calculus of Variations. In the next chapter, we start focusing on the Robin eigenvalues and their properties, the main topic of the thesis.

Chapter 2

The Robin Laplacian and its eigenvalues

In this chapter we summarize the main properties of the eigenvalues of the Robin Laplacian and we recall some remarkable results in shape optimization problems involving such eigenvalues. We will principally refer to [23], were one can find most of the results stated about Robin eigenvalues, their proofs (here omitted or only sketched) and further references to get acquainted about the topic. In the first section, we recall the properties of the operator that allow us to use the variational formula (1.1) and we highlight some properties of the eigenfunctions. In Section 2, we focus on some remarkable properties of the eigenvalues, mostly that properties that are related to the variation of the domains (monotonicity under inclusions, behaviour under dilatations, etc.). Finally, in Section 3, we recall some well known results in shape optimization, introducing the problems studied in the next chapters.

2.1 Definition of the eigenvalues, variational formula and some properties of the eigenfunctions

Definition 2.1.1. Let $\beta \in \mathbb{R}$ be a fixed real number and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. A number $\lambda \in \mathbb{R}$ is an eigenvalue of the Robin problem for the Laplace operator (or, briefly, a Robin eigenvalue) with boundary parameter β if there exists a non-zero function $u \in H^1(\Omega)$ solving the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

(here n is the outer normal on $\partial\Omega$), i.e., in the weak sense:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} uv \, d\mathcal{H}^{d-1} = \lambda \int_\Omega uv \, dx \quad \forall v \in H^1(\Omega).$$

To handle the Robin eigenvalues in shape optimization problems via direct methods of Calculus of Variations, it is very useful to use a variational representation based on the bilinear form associated to weak formulation of problem (2.1) above. To this aim, let us consider the symmetric bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} uv \, d\mathcal{H}^{d-1}.$$

It is the bilinear form associated to problem (2.1) in $H^1(\Omega)$. Due to the trace inequality in $H^1(\Omega)$, the corresponding quadratic form $a(u) := a(u,u)$ is bounded from above and semibounded from below: there exist two positive constants c_1, c_2, depending only on Ω and β, such that, for all $u \in H^1(\Omega)$,

$$a(u) + c_1 \|u\|^2_{L^2(\Omega)} \geq c_2 \|u\|^2_{H^1(\Omega)},$$

i.e. a is $L^2(\Omega)$-elliptic. If we consider the operator on $L^2(\Omega)$ associated with a, given by

$$\begin{aligned} D(-\Delta_\beta) = \Big\{ & u \in L^2(\Omega) : \Delta u \in L^2(\Omega), \\ & \exists \, \frac{\partial u}{\partial n} \text{ (in the sense of distributions) and } = -\beta u \text{ in } L^2(\partial\Omega) \Big\}, \\ -\Delta_\beta u = & -\Delta u, \end{aligned}$$

it is selfadjoint and bounded from below in $H^1(\Omega)$ and its resolvent is compact, as the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By the spectral theorem for selfadjoint operators with compact resolvent (Proposition 1.3.8), its eigenvalues form an increasing sequence

$$\lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \ldots \to +\infty$$

(where each eigenvalue is repeated according to its multiplicity, which is finite). Moreover, since $-\Delta_\beta$ is self-adjoint, for every $k \in \mathbb{N}$ the k-th eigenvalue, that we will denote by the symbol $\lambda_{k,\beta}(\Omega)$, is given by the usual min-max formula (1.1)

$$\begin{aligned} \lambda_{k,\beta}(\Omega) &= \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\langle -\Delta_\beta u, u \rangle}{\|u\|^2_{L^2(\Omega)}} = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{a(u)}{\|u\|^2_{L^2(\Omega)}} \\ &= \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx}, \end{aligned} \tag{2.2}$$

or by the max-min formula (1.2)

$$\begin{aligned}\lambda_{k,\beta}(\Omega) &= \max_{S^\perp\in\mathcal{S}_{k-1}}\min_{u\in S\setminus\{0\}}\frac{\langle -\Delta_\beta u, u\rangle}{\|u\|^2_{L^2(\Omega)}} = \max_{S^\perp\in\mathcal{S}_{k-1}}\min_{u\in S\setminus\{0\}}\frac{a(u)}{\|u\|^2_{L^2(\Omega)}}\\ &= \max_{S^\perp\in\mathcal{S}_{k}}\min_{u\in S\setminus\{0\}}\frac{\displaystyle\int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial\Omega} u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\,dx},\end{aligned} \tag{2.3}$$

where $\mathcal{S}_k$ (resp.$\mathcal{S}_{k-1}$) denotes the family of all subspaces of $H^1(\Omega)$ with dimension k (resp. $k-1$). Notice that $\lambda_{k,\beta}(\Omega)$ is achieved only at the corresponding eigenfunctions. For technical simplicity, throughout the thesis we will always use the min-max formula (2.2) to handle $\lambda_{k,\beta}(\Omega)$. It is worth to emphasize that Problem (2.1) (and then the eigenvalues) are invariant under rotations and translations of the set Ω.

Remark 2.1.2 (regularity of eigenfunctions, see Proposition 4.1 in [23])**.** Let us observe that, as the eigenfunctions solve the Helmholtz equation $-\Delta u = \lambda u$ in Ω, then they are analytic in Ω; moreover, the eigenfunctions are continuous on the whole of $\overline{\Omega}$. To prove that assertion in the case of positive boundary parameter $\beta \geq 0$ we refer to Corollary 5.5 in [36], where it has been proved that every eigenfunction ψ is in $L^\infty(\Omega)$, and to Corollary 2.9 in [71], where it is proved that weak solutions of the inhomogeneous Robin problem are continuous on $\overline{\Omega}$. An alternative proof of this fact is given in Lemma 2.1 in [19], where it is proved that every eigenfunction ψ of a more general problem belongs to $C(\overline{\Omega})\cap C^1(\Omega)$. The case of negative boundary parameter is a particular case of Corollary 4.2 in [38], where it has been proved that eigenfunctions are in $C(\overline{\Omega})\cap C^\infty(\Omega)$.

As the eigenfunctions are continuous on Ω, we ask ourselves if it is possible to know some a priori estimates on the maximal and minimal value, or at least some information on the sign of the eigenfunctions. We will see in the following chapters that above all the second request is technically important in some problems involving particular classes of Lipschitz domains. Even if the previous two request are not solvable, in general, it is worth to emphasize a very useful result in that direction. It provides a strictly positive lower bound for an eigenfunction for the first Robin eigenvalue of a connected Lipschitz domain when the boundary parameter β is positive. A proof of this result is contained in [6], Theorem 6.11(j), where the authors use a technique based on C_0-semigroups.

Proposition 2.1.3 (Strictly positive first eigenfunctions)**.** *Let Ω be a connected Lipschitz domain and let $\beta > 0$. Then there exist $\alpha > 0$ and a first Robin eigenfunction $u \in C(\overline{\Omega})$ such that $u \geq \alpha$.*

If we consider a Lipschitz domain consisting in more than one connected component, the previous result holds for the restriction of u on each of the connected components where u is not identically zero.

2.2 Properties of the eigenvalues

In this section we will summarize some remarkable properties of $\lambda_{k,\beta}(\Omega)$, referring mainly to Sections 4.2, 4.3 and 4.4 in [23]. We will emphasize the analogies and the differences between the Robin-Laplacian eigenvalues and the eigenvalues of the Laplacian with other well known boundary conditions (e.g. Dirichlet or Neumann). It will be highlighted that some good properties, for instance non-negativity or monotonicity, cannot be a priori inferred, unlike it happens for other problems.

The technical interest in studying the Robin problems is that, in lots of cases, the presence of the boundary term in the variational representation (2.2) does not allow to simplify some arguments in the proofs or to remove the dependence of some estimates on the boundary parameter β (or other parameters of the problem). In view of these peculiarities, authors looked (and still look!) for approaches that are different from the standard ideas (used mostly in Dirichlet or Neumann problems).

In the following we will use the notation $R(u, \Omega, \beta)$ to denote the Rayleigh quotient in (2.2) and (2.3). If one or two variables are omitted, it means that they are a priori fixed. Moreover, we will denote by $\lambda_k(\Omega)$, $\mu_k(\Omega)$ and $\sigma_k(\Omega)$ respectively the k-th Dirichlet eigenvalue, the k-th Neumann eigenvalue and the k-th Steklov eigenvalue for the Laplacian.

To begin, we summarize some properties of the map $\beta \mapsto \lambda_{k,\beta}(\Omega)$.

Remark 2.2.1 (dependence the boundary parameter)**.** Let us fix a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ and a function $u \in H^1(\Omega), u \neq 0$; then, the function

$$\beta \mapsto R(u, \Omega, \beta)$$

is increasing in $\mathbb{R}$. Consequently, passing to the min-max formula (2.2), even the function

$$\beta \mapsto \lambda_{k,\beta}(\Omega)$$

is increasing in $\mathbb{R}$. More precisely, it is a piecewise analytic function of β (the points of non analyticity given by the intersections of the eigencurves when $\lambda_{\beta,k}(\Omega)$ is not simple); if $\lambda_{\beta,k}(\Omega)$ is simple, its derivative is given by the formula

$$\frac{d}{d\beta}\lambda_{k,\beta}(\Omega) = \frac{\int_{\partial\Omega}\psi_k^2\,d\sigma}{\int_\Omega \psi_k^2\,dx},$$

where ψ_k is an eigenfunction for $\lambda_{\beta,k}$ (a proof of this fact is given in [4] for $\beta > 0$, but the poof holds also for $\beta \leq 0$). In particular, in $\beta = 0$, for the first eigenvalue it holds

$$\frac{d}{d\beta}\lambda_{1,\beta}(\Omega)\bigg|_{\beta=0} = \frac{\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} \tag{2.4}$$

(see, for instance, [52],[54] and [66]). The asymptotic behaviour of the map is described by the following equalities (see [23], resp. Prop. 4.5 and Prop. 4.8):

$$\lim_{\beta\to+\infty}\lambda_{\beta,k}(\Omega) = \sup\{\lambda_{\beta,k}(\Omega) : \beta\in\mathbb{R}\} = \lambda_k(\Omega),$$

$$\lim_{\beta\to-\infty}\lambda_{\beta,k}(\Omega) = -\infty.$$

Moreover, the map $\beta \mapsto R(u,\Omega,\beta)$ is linear in β, then the function $\beta \mapsto \lambda_{1,\beta}(\Omega)$, as infimum of linear functions, is concave.

Remark 2.2.2 (link with Dirichlet, Neumann and Steklov eigenvalues)**.** Let us observe that $\mu_k(\Omega) = \lambda_{k,0}(\Omega)$, i.e., if $\beta = 0$, the Robin eigenvalue coincide with the k-th Neumann eigenvalue. Moreover, if we replace $H^1(\Omega)$ by the smaller space $H_0^1(\Omega)$ in (2.2), we obtain $\lambda_{k,\beta}(\Omega) \leq \lambda_k(\Omega)$ for every $\beta\in\mathbb{R}$. Then, for every $\beta > 0$, it holds

$$\mu_k(\Omega) \leq \lambda_{k,\beta}(\Omega) \leq \lambda_k(\Omega),$$

where the lower estimate is a consequence of the increasing monotonicity of the map $\beta \mapsto \lambda_{k,\beta}(\Omega)$. Notice that the upper estimate could be even found heuristically letting β go to $+\infty$ in (2.1) and using again the fact that $\beta \mapsto \lambda_{k,\beta}(\Omega)$ is increasing (see also the asymptotic behaviour of $\lambda_{k,\beta}(\Omega)$ at the bottom of Remark 2.2.1).

Finally, if $\beta = -\sigma_k(\Omega) < 0$, then $\lambda_{k,\beta}(\Omega) = 0$.

Before studying the positivity of the Robin eigenvalues, we recall that Dirichlet eigenvalues $\lambda_k(\Omega)$ and Neumann eigenvalues $\mu_k(\Omega)$ are non-negative for every possible choice of the Lipschitz domain Ω. In particular, Dirichlet eigenvalues are all strictly positive, Neumann eigenvalues are strictly positive if $k \geq N+1$, where $\#\Omega$ (the number of connected components of Ω) equals N. We ask ourselves if some information on the sign of the Robin eigenvalues can be found.

Remark 2.2.3 (sign of the Robin eigenvalues)**.** If $\beta \geq 0$, it is clear that every eigenvalue of every bounded, Lipschitz, connected, domain is non negative. Let us focus on the case of $\beta < 0$. By (2.2), for the first eigenvalue we have that

$$\lambda_{1,\beta}(\Omega) = \min_{u \in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx + \beta \int_{\partial\Omega} u^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\, dx}.$$

If we consider a constant non-zero function, for instance $\chi_\Omega \in H^1(\Omega)$, we have

$$\lambda_{1,\beta}(\Omega) \leq \beta \frac{\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} < 0. \tag{2.5}$$

In the same way, if we take a Lipschitz domain Ω consisting at least in $k \geq 1$ connected components $\Omega_1, \dots, \Omega_k$, we can consider as a test space for the computation of $\lambda_{k,\beta}(\Omega)$ the k-dimensional subspace $V := \operatorname{span}\{\chi_{\Omega_1}, \dots, \chi_{\Omega_k}\}$ and obtain again

$$\lambda_{k,\beta}(\Omega) \leq \beta \frac{\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} < 0.$$[1]

By the previous estimates we deduce that we can not say if $\lambda_{k,\beta}(\Omega)$ is non-negative, in general, when the boundary parameter is negative. It is obvious that, for every fixed Lipschitz domain Ω and boundary parameter, there exists a maximal order $k \in \mathbb{N}$ such that we have $\lambda_{j,\beta}(\Omega) \leq 0$ for every $j \leq k$ and $\lambda_{j,\beta}(\Omega) \geq 0$ for every $j \geq k$. In the same way, fixed $k \in \mathbb{N}$ and the domain Ω, letting β vary in $\mathbb{R}_-$, in view of the increasing monotonicity of the map $\beta \mapsto \lambda_{k,\beta}(\Omega)$, we obtain that $\lambda_{k,\beta}(\Omega) \geq 0$ if and only if $-\sigma_k(\Omega) \leq \beta \leq 0$ and the equality holds if and only if $\beta = -\sigma_k(\Omega)$.

Once we studied some information about the sign of the eigenvalues, we are interested in understanding some information about simplicity of eigenvalues. An important tool in that direction is the Krein-Rutman Theorem 1.3.11.

Remark 2.2.4 (simplicity of the first eigenvalue)**.** If Ω is connected, then the first eigenvalue is simple: this is a consequence of the Krein-Rutman Theorem 1.3.11, applied taking as T the resolvent operator of $-\Delta_\beta$ on Ω, C the closed cone $C := \left\{u \in C(\overline{\Omega}) : u \geq 0\right\}$ and the condition $T(C \setminus \{0\}) \subset \mathring{C}$ satisfied in view of the strong maximum principle. Passing to the reciprocal of the eigenvalues of T, i.e. to the eigenvalues of $-\Delta_\beta$, we conclude that $\lambda_{1,\beta}(\Omega)$ is simple. We can highlight that simplicity of the first eigenvalue occurs only if Ω is connected. If Ω has $k > 1$ connected components, say $\Omega_1, \dots, \Omega_k$, one

[1] A sharper estimate will be presented in Chapter 3.

can consider an eigenfunction $u \in H^1(\Omega)$ for $\lambda_{1,\beta}(\Omega)$ and, for every $j \leq k$, the j-dimensional subspace

$$V_j := \text{span}\left\{u\chi_{\Omega_1}, u\chi_{\Omega_2}, \ldots, u\chi_{\Omega_{j-1}}, u\chi_{\Omega_j \cup \ldots \cup \Omega_k}\right\} \subset H^1(\Omega)$$

as a test space to compute $\lambda_{j,\beta}(\Omega)$ via the min-max formula (2.2). Then, we easily obtain

$$\lambda_{1,\beta}(\Omega) = \ldots = \lambda_{k,\beta}(\Omega).$$

Let us emphasize another difference of Robin eigenvalues with Dirichlet and Neumann eigenvalues. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain; let us fix $t \geq 0$ and consider the set $t\Omega := \{tx : x \in \Omega\}$. It is well known that, for every $k \geq 1$, Dirichlet eigenvalue λ_k and Neumann eigenvalue μ_k rescale as follows:

$$\lambda_k(t\Omega) = \frac{1}{t^2}\lambda_k(\Omega), \quad \mu_k(t\Omega) = \frac{1}{t^2}\mu_k(\Omega).$$

How do Robin eigenvalues rescale?

Remark 2.2.5 (scaling property and behaviour under dilations)**.** To perform the rescaling of the Robin eigenvalues, we consider the standard change of variables

$$t\Omega \ni x \mapsto x' := x/t \in \Omega.$$

Hence, for every $u \in H^1(t\Omega)$, we obtain

$$\begin{aligned} R\left(u(\cdot), t\Omega, \beta\right) &= \frac{\displaystyle\int_{t\Omega} |\nabla u(x)|^2 \, dx + \beta \int_{\partial(t\Omega)} u^2(x) \, d\mathcal{H}^{d-1}(x)}{\displaystyle\int_{t\Omega} u^2(x) \, dx} \\ &= \frac{\displaystyle t^{d-2} \int_{\Omega} |\nabla u(tx')|^2 \, dx' + t^{d-1}\beta \int_{\partial\Omega} u^2(tx') \, d\mathcal{H}^{d-1}(x')}{\displaystyle t^d \int_{\Omega} u^2(tx') \, dx'} \\ &= \frac{1}{t^2} R\left(u\left(t\ \cdot\right), \Omega, t\beta\right). \end{aligned}$$

Then, apply the min-max formula (2.2) and observe that all the functions of $H^1(\Omega)$ can be expressed as $u\,(tx)$, where $u \in H^1(t\Omega)$ and $x \in \Omega$; even the vice versa holds. Then, the previous equality leads, for every $k \in \mathbb{N}$ and for every $t > 0$, to the following scaling formula

$$\lambda_{k,\beta}(t\Omega) = \frac{1}{t^2}\lambda_{k,t\beta}(\Omega), \tag{2.6}$$

holding for every real value of the boundary parameter β. In particular, if $\beta \geq 0$ and $t > 1$, we obtain

$$\begin{aligned}
R\left(u(\cdot), t\Omega, \beta\right) &= \frac{t^{d-2} \displaystyle\int_{\Omega} |\nabla u(tx')|^2 \, dx' + t^{d-1}\beta \int_{\partial\Omega} u^2(tx') \, d\mathcal{H}^{d-1}(x')}{t^d \displaystyle\int_{\Omega} u^2(tx') \, dx'} \\
&\leq \frac{t^{-2} \displaystyle\int_{\Omega} |\nabla u(tx')|^2 \, dx' + t^{-1}\beta \int_{\partial\Omega} u^2(tx') \, d\mathcal{H}^{d-1}(x')}{\displaystyle\int_{\Omega} u^2(tx') \, dx'} \\
&\leq \frac{\displaystyle\int_{\Omega} |\nabla u(tx')|^2 \, dx' + \beta \int_{\partial\Omega} u^2(tx') \, d\mathcal{H}^{d-1}(x')}{\displaystyle\int_{\Omega} u^2(tx') \, dx'} \\
&= R\left(u\left(t\ \cdot\right), \Omega, \beta\right).
\end{aligned}$$

Passing to the min-max formula (2.2), we obtain the inequality

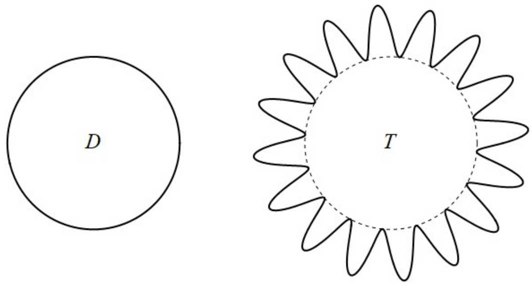
$$\lambda_{k,\beta}(t\Omega) \leq \lambda_{k,\beta}(\Omega), \tag{2.7}$$

i.e. the Robin eigenvalues with positive boundary parameter are monotonically decreasing under dilatations.

To conclude this remark, we emphasize that we do not have any scale invariance property in general (differently from Dirichlet or Neumann eigenvalues) nor any rescaling of the eigenvalues if we let t vary, since also the boundary parameter is rescaled (see (2.6)). Moreover, the monotonicity under dilatations (2.7) holds only for eigenvalues with positive boundary parameter; no general results are known in the case of negative boundary parameter.

Let us recall that, for Dirichlet eigenvalues, it holds $\lambda_k(\Omega_1) \leq \lambda_k(\Omega_2)$ if $\Omega_2 \subseteq \Omega_1$, but for Neumann eigenvalues there is no monotonicity under inclusions, in general (for some example see Section 1.3.2 in [58]). In Remark 2.2.5 there is a result of decreasing monotonicity under inclusions; is there any general result in that direction? The answer is negative; some counterexamples for the first eigenvalue are presented in [69], for both positive and negative β. If one takes a disk $D \subset \mathbb{R}^2$ and a set $T \subset \mathbb{R}^2$ obtained by the disk D and the union of "tentacles" (i.e. rapidly oscillating smooth boundary, see figure below), it holds $D \subset T$ and, for a suitable choice of the parameters describing the boundary, one has $\lambda_{1,\beta}(D) \leq \lambda_{1,\beta}(T)$ for $\beta > 0$ (see [39]).

On the other hand, if D_1 and D_2 are concentric disks with $D_1 \subset D_2$, it holds $\lambda_{1,\beta}(D_1) \geq \lambda_{1,\beta}(D_2)$ in view of the monotonicity under dilatations. Then, no general results of monotonicity under inclusions can be stated for $\beta > 0$.



In an analogous way, for $\beta < 0$, taking the same sets $D \subset T$ above, one can choose the parameters in such a way that one has $\lambda_{1,\beta}(D) \geq \lambda_{1,\beta}(T)$. On the contrary, if D_1 and D_2 are concentric disk with $D_1 \subset D_2$, it holds $\lambda_{1,\beta}(D_1) \leq \lambda_{1,\beta}(D_2)$ (see, e.g., Theorem 1 or Corollary 4 in [52]). Then, also for $\beta < 0$, there is no monotonicity under inclusions, in general.

Both previous examples are based on the fact that a rapid oscillation of the boundary, in general, causes the eigenvalues to increase if $\beta > 0$ and to decrease if $\beta < 0$ (in view of the large increase of the term $\int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}$, see [39]).

We ask ourselves if there exist some result of monotonicity under inclusion, holding at least for some classes of sets or for some particular value of the boundary parameter. Next result is one of the most general ones involving domain monotonicity for the principal Robin eigenvalue (see [52], Theorem 1). From now on, in this section, we will always consider the dimension $d \geq 2$.

Theorem 2.2.6. *Let $B \subset \mathbb{R}^d$ a ball and let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain contained in B. Then, for every $\beta > 0$, we have*

$$\lambda_{1,\beta}(B) \leq \lambda_{1,\beta}(\Omega)$$

and, for every $\beta < 0$,

$$\lambda_{1,\beta}(\Omega) \leq \lambda_{1,\beta}(B).$$

We ask ourselves if the previous result holds also for higher eigenvalues. The answer, in general, is negative, as we can see by the following proposition (see [23], Proposition 4.2).

Theorem 2.2.7. *Let $k \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with at most k connected components. Then, for every ball $B \subset \mathbb{R}^d$, there exists $\beta > 0$ such that*

$$\lambda_{k,\beta}(B) \geq \lambda_{k,\beta}(\Omega).$$

In particular, for any ball B containing Ω, there exists $\beta > 0$ such that the previous inequality is satisfied.

2.3 Isoperimetric inequalities

In this section we summarize some of the most important isoperimetric results concerning the Robin eigenvalues. We first recall two of the best-known isoperimetric inequalities for spectral problems of the Laplace operator (see [17] for a complete discussion):

- *Faber-Krahn* inequality for the first Dirichlet eigenvalue: $\lambda_1(B) \leq \lambda_1(\Omega)$, i.e. among smooth sets of prescribed measure, the ball is the only minimizer of λ_1;

- *Szëgo-Weinberger* inequality for the second Neumann eigenvalue: $\mu_2(B) \geq \mu_2(\Omega)$, i.e. among smooth sets of prescribed measure, the ball is the only maximizer of μ_2.

Notice that for the Dirichlet eigenvalues one deals with a minimization problem and for the Neumann eigenvalues, on the contrary, one deals with a maximization problem. Also for the Robin problem we have two possible behaviours, depending on the sign of β; for that reason we split the discussion in two parts.

2.3.1 The case of positive boundary parameter

When $\beta > 0$, Robin eigenvalues are bounded from below (by zero), then it is reasonable to loon for minimizers of $\lambda_{1,\beta}$ among sets of prescribed measure. The first important result in that direction is the Faber-Krahn inequality for $\lambda_{1,\beta}$. It is due to Bossel and Daners; a prove was given in [13] for smooth domains in two dimensions and, in higher dimension, in [37].

Theorem 2.3.1 (Bossel-Daners)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B \subset \mathbb{R}^d$ be a ball such that $|B| = |\Omega|$. Then, for any $\beta > 0$,*

$$\lambda_{1,\beta}(B) \leq \lambda_{1,\beta}(\Omega) \tag{2.8}$$

and the equality holds if and only if Ω is a ball.

In 2010 it has been proved by Bucur and Giacomini that the previous result remains valid in the class of measurable sets of prescribed measure m (see [24]); more precisely, defining the eigenvalues in a weaker sense involving SBV-functions, they proved that the principal eigenvalue is still minimized by the ball of measure m (more precisely, by the zero extension of the first eigenfunction of a ball of measure m).

Theorem 2.3.2 (Bucur-Giacomini)**.** *Let $m > 0$ be given and let $u : \mathbb{R}^d \to \mathbb{R}$, $u \in SBV(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Assume that $|\{u \neq 0\}| = m$. Then, if B is the ball of measure m, it holds*

$$\int_{\mathbb{R}^d} |\nabla u|^2\, dx + \beta \int_{J_u} \left[(u^+)^2 + (u^-)^2\right]\, d\mathcal{H}^{d-1} \geq \lambda_{1,\beta}(B) \int_{\mathbb{R}^d} u^2\, dx.$$

The equality holds if and only if u is the first Robin eigenfunction of a ball of measure m, extended by zero outside the ball.

This variational formulation suggested to the same authors a free discontinuity approach to prove the existence of minimizers for higher eigenvalues with measure constraint (see [27]). The techniques and the tools used to obtain that result are reprise in Chapter 5, where we replace the measure constraint with the penalization of the perimeter. For that reason, we refer to Chapter 5 for the weak formulation of the functional and for a short survey on the existence results in the setting.

In the following, when referring to Theorem 2.3.1 or Theorem 2.3.2, we will often write "Faber-Krahn inequality", in view of the analogous result in the Dirichlet case.

Remark 2.3.3 (global estimates on $\lambda_{1,\beta}(B_r)$)**.** It is worth to highlight the following estimates on the first eigenvalue on a ball of radius r (see e.g. [65]); for every $\beta > 0$ it holds

$$\frac{\beta}{4r(1+\beta r)} \leq \lambda_{1,\beta}(B_r) \leq \frac{C_d \beta}{r(1+\beta r)}, \tag{2.9}$$

where $C_d > 0$ is a dimensional constant. This implies that $\lambda_{1,\beta}(B_r)$ is infinitesimal as the radius r explodes and explodes as the radius r tends to zero. The same estimates hold replacing B_r by a bounded convex domain Ω and r by the inradius r_Ω of Ω.

For higher eigenvalues, some results go in the same direction of the Dirichlet case. In the following theorem is proved that $\lambda_{2,\beta}$ is minimized by the disjoint union of two equal balls (for a proof we refer to [64] and [63]).

Theorem 2.3.4 (Kennedy)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B_2 \subset \mathbb{R}^d$ be any disjoint union of two equal balls each of volume $|\Omega|/2$. Then, for any $\beta > 0$,*

$$\lambda_{2,\beta}(B_2) < \lambda_{2,\beta}(\Omega) \tag{2.10}$$

and the equality holds if and only if Ω is a disjoint union of two equal balls.

For higher eigenvalues, the situation is more involved, due to the different behavior of the eigenvalues as $\beta > 0$ becomes larger ($\lambda_{k,\beta}(\Omega)$ is close to its Dirichlet counterpart $\lambda_k(\Omega)$) and as $\beta > 0$ becomes smaller ($\lambda_{k,\beta}(\Omega)$ is close to its Neumann counterpart $\mu_k(\Omega)$), see Remark 2.2.2. Some results depending on the value of β are expected, and the following theorem, proved in [64], goes in that direction.

Theorem 2.3.5 (Kennedy)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B_k \subset \mathbb{R}^d$ be any disjoint union of k equal balls each of volume $|\Omega|/k$. Then, there exists $\beta_0(\Omega) > 0$ such that*

$$\lambda_{k,\beta}(B_k) < \lambda_{k,\beta}(\Omega) \tag{2.11}$$

for every $\beta \in [0, \beta_0(\Omega))$ and the equality holds if and only if Ω is a disjoint union of k equal balls.

Notice that the threshold $\beta_0(\Omega)$ depends on the set Ω; it would be interesting to remove that dependence, proving that the threshold β_0 in Theorem 2.3.5 depends only on k, on the fixed measure $m > 0$ of the admissible domains and on the dimension d.

2.3.2 The case of negative boundary parameter: the conjecture of Bareket and the result of Freitas and Kreijcirik

In the following section we summarize some well known facts about optimal shapes in the case of negative boundary parameter. We start noticing that for $\beta < 0$, the principal Robin eigenvalues are bounded from above but unbounded from below. Indeed, it is possible to consider a sequence $(\Omega_n)_n$ of Lipschitz domains having the same measure m and such that $\mathcal{H}^{d-1}(\partial\Omega_n)$ positively diverges (e.g., Lipschitz domains having rapidly oscillating boundaries). Then, by (2.5), we have $\lambda_{1,\beta}(\Omega_n) \to -\infty$ and we conclude that $\lambda_{1,\beta}$ cannot have minimizers among sets of given measure. This behaviour suggests to look for maximizers of $\lambda_{1,\beta}$ in suitable classes of admissible sets.

Let us recall the longstanding *Conjecture of Bareket* (1977, see [9]):

Let $\Omega \subset \mathbb{R}^d$ be a bounded and sufficiently smooth domain and denote by B a ball of $\mathbb{R}^d$ with $|B| = |\Omega|$. Then, for every $\beta < 0$, $\lambda_{1,\beta}(\Omega) \le \lambda_{1,\beta}(B)$.[2]

[2]This version of the statement is due to F. Brock and D. Daners, see [15]. The original conjecture of M. Bareket was stated in dimension 2.

In other words, M. Bareket conjectured that a reversed Faber-Krahn inequalities holds for any negative value of the boundary parameter. For many years the conjecture has been supported. Indeed, recalling the derivation formula of the principal eigenvalue with respect to β in zero, see (2.4), one can notice that the derivative attains its smallest value on the ball B, among Lipschitz domains of measure m. Moreover, since

$$\lambda_{1,0}(\Omega) = \mu_1(\Omega) = 0 = \mu_1(B) = \lambda_{1,0}(B),$$

in view of the smoothness of the eigencurves, this yields that the inequality conjectured by Bareket holds for every β in a small neighbourhood of 0, a priori depending on Ω:

$$\lambda_{1,\beta}(\Omega) \leq \lambda_{1,\beta}(B) \quad \forall \beta \in [\beta_0(\Omega), 0]. \tag{2.12}$$

Further evidences of the conjecture are provided by Bareket herself in [9]. More recently, in 2015, V. Ferone, C. Nitsch and C. Trombetti proved in[48] the local maximality of the ball in any space dimension, a result that seemed to support Bareket's conjecture.

It was thus surprising when, again in 2015, P. Freitas and D. Kreijcirik disproved the general validity of the conjecture in [50]. More precisely they proved that the ball is not a maximizer among sets of given measure, in general, even in $\mathbb{R}^2$. They showed that there exists a spherical shell of the same measure of the ball whose principal eigenvalue is strictly larger than $\lambda_{1,\beta}(B)$ for large values of β (depending on the shell). The result is the following (for a proof see the Theorem 1 in [50] or Theorem 4.31 in [23]).

Theorem 2.3.6. *Let $B_r \subset \mathbb{R}^d$ be a ball of radius $r > 0$. Then, there exist a spherical shell*

$$A_{r_1,r_2} := \left\{ x \in \mathbb{R}^d : r_1 < |x| < r_2 \right\},$$

with the same volume as B_r, such that

$$\lambda_{1,\beta}(B_r) < \lambda_{1,\beta}(A_{r_1,r_2})$$

for every sufficiently large negative value of β.

The previous theorem is very important in literature, since it is the first well known result of optimization of the principal eigenvalue of the Laplace operator where the optimizer is not a ball, in general. The proof is based on an asymptotic expansion of $\lambda_{1,\beta}(B_r)$ and $\lambda_{1,\beta}(A_{r_1,r_2})$ as $\beta \to -\infty$ and on a comparison of the expansions. More precisely, one has (see Theorem 3 in [50])

$$\lambda_{1,\beta}(B_r) = -\beta^2 + \frac{d-1}{r}\beta + o(\beta),$$

$$\lambda_{1,\beta}(A_{r_1,r_2}) = -\beta^2 + \frac{d-1}{r_2}\beta + o(\beta),$$

as $\beta \to -\infty$; subtracting side by side we have that, for large negative values of β, the difference $\lambda_{1,\beta}(B_r) - \lambda_{1,\beta}(A_{r_1,r_2})$ has to be negative.

On the other hand, for smooth domains in the plane, a reverse Faber-Krahn inequality holds for small negative values of β (see Theorem 2 in [50]).

Theorem 2.3.7. *Let $m > 0$ and let $B_m \subset \mathbb{R}^2$ the disk of measure m. Then, there exist a negative constant β_0 depending only on m such that*

$$\lambda_{1,\beta}(\Omega) \le \lambda_{1,\beta}(B_m)$$

for every $\beta \in [\beta_0, 0]$ and every Ω of measure m with C^2 boundary.

In other words, the estimate in the previous theorem improves (2.12) in the plane, removing the dependence of b_0 on Ω and then making the estimate uniform.

In addition to the previous result, several aspects of the problem to maximize $\lambda_{1,\beta}$ have been investigated by Antunes, Freitas and Krejcirik in an interesting work of 2017 (see [5]). One of the most remarkable ones is that the ball maximizes $\lambda_{1,\beta}$ among smooth sets of fixed perimeter (see Theorem 2 in [5]).

Theorem 2.3.8. *For every $\beta \le 0$ and for every bounded domain $\Omega \subset \mathbb{R}^2$ with C^2 boundary, we have*

$$\lambda_{1,\beta}(\Omega) \le \lambda_{1,\beta}(B),$$

where B is a disk having the same perimeter as Ω.

For higher eigenvalues, there are very few results; some of those are based again on the asymptotic expansion of the eigenvalues of balls and spherical shells (see Section 4.5.2, Theorem 4.21 and the following Proposition 4.42 in [23]).

Proposition 2.3.9 ([23], Proposition 4.42)**.** *Let $k \ge 1$ and let $B \subset \mathbb{R}^d$ a ball. There exist a spherical shell*

$$A_{r_1,r_2} := \left\{x \in \mathbb{R}^d : r_1 < |x| < r_2\right\},$$

with the same volume as B and a constant $\beta_0(k) < 0$ such that

$$\lambda_{k,\beta}(B) < \lambda_{k,\beta}(A_{r_1,r_2})$$

for $\beta < \beta_0(k)$.

We can notice that a general existence result for maximizers of $\lambda_{k,\beta}$ for any $k \in \mathbb{N}$ and $\beta < 0$ has been an open problem for some years (see [23], Open Problem 4.33). In the next chapter we give some answers to that question, at least in a relaxed setting or inferring some extra topological information. More precisely, we will prove existence of maximizers for $\lambda_{k,\beta}$ among measurable sets with either fixed measure or perimeter and we will study also particular cases, when we restrict ourselves to simply connected sets of $\mathbb{R}^2$ or to domains satisfying some geometric constraint.

Chapter 3

Shape optimization problems for the k-th eigenvalue of the Robin Laplacian with negative boundary parameter

3.1 Introduction

In this chapter it will be given a first answer to the question set at the end of Chapter 2: we will prove that, for every $\beta < 0$, maximizers for $\lambda_{k,\beta}$ exist among suitable classes of measurable sets of given volume (or given perimeter, or satisfying some topological constraints). An approach via direct methods of Calculus of Variations to this problem is new, up to our knowledge, even if some important results have been found, above all for the principal eigenvalue. As seen in Chapter 2, M. Bareket conjectured in 1977 that for $k = 1$ the ball maximizes $\lambda_{1,\beta}$. In 2015, Freitas and Krejcirik proved in [50] that even in $\mathbb{R}^2$ the solution is, in general, not the disc. Precisely, if the boundary parameter β is larger than a fixed threshold β_1 (depending on the area m), then the first eigenvalue of a suitably chosen annulus is strictly greater than the same eigenvalue computed for the disk of the same measure. On the opposite sense, if β is smaller than another threshold β_2, then the ball is the only maximizer.

We are interested in studying the following problem

$$\max\left\{F(\lambda_{1,\beta}(\Omega),\dots,\lambda_{k,\beta}(\Omega)) : \Omega \subset \mathbb{R}^d \text{ bounded and Lipschitz}, |\Omega| = m\right\}, \tag{3.1}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is non-decreasing in each variable and upper semicontinuos and $|\Omega|$ the Lebesgue measure of Ω. Notice that the motivating problem

$$\max\left\{\lambda_{k,\beta}(\Omega) : \Omega \subset \mathbb{R}^d \text{ bounded and Lipschitz, } |\Omega| = m\right\},$$

is a particular case of (3.1).

The core of this Chapter are the results in [18], where we give the proof of the existence of an optimal solution to problem (3.1) (independent of the precise knowledge of its shape) and to the two dimensional problem in the class of simply connected sets. Precisely, we will prove an existence result for (3.1) in a relaxed sense, obtaining also some information about the structure of the optimal sets. To this aim, we extend the variational definition of $\lambda_{k,\beta}(\Omega)$ to measurable sets with finite perimeter and to arbitrary simply connected sets in $\mathbb{R}^2$ having a topological boundary of finite length in the spirit of the relaxation of the Steklov eigenvalues in [11]. At the moment, we are not able to prove any regularity of those optimal sets, but we prove some properties of the optimal sets: they are bounded with a controlled diameter, they have a controlled perimeter and not more than a number of "connected" components depending on m, k, d and β. Surprisingly, one could expect that this number is not larger than k, but we are not able to prove it, since a strange phenomenon due to the uncontrolled behaviour of the eigenvalues to rescaling occurs. In two dimensions of the space, we prove also existence of a solution in the class of unions of pairwise disjoint open, simply connected sets. As expected for Robin boundary conditions, the geometry of optimal sets will depend on the mass m.

We will gain the existence of maximizers as a consequence of some geometric control of the spectrum. For the Steklov spectrum, such results have been proved by Colbois, Girouard and El Soufi [32] where they get upper bounds for the eigenvalues by a quantity involving the isoperimetric ratio of the set and by Bogosel, Bucur and Giacomini [11] where such bounds are obtained in terms of diameter. The case of Robin boundary conditions is more tricky. Contrary to the Steklov problem, we have simultaneously both negative and positive eigenvalues and they do not obey any homogeneity law. Consequently, the control of the spectrum by homogeneous geometric quantities is more involved and less explicit. Nevertheless, roughly speaking, both results state that larger is either the isoperimetric ratio or the diameter of a connected set, then lower is its k-th Robin eigenvalue. We point out that, for positive boundary parameter, the isoperimetric ratio and the diameter do not play any role on the control of the spectrum.

Throughout the chapter, to emphasize that the boundary parameter is negative, we will denote by $-\beta < 0$ the boundary parameter and we still write

$\lambda_{k,\beta}$ instead of $\lambda_{k,-\beta}$. Then, for every $k \in \mathbb{N}$, the k-th eigenvalue is given by the min-max formula

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx}, \tag{3.2}$$

where $\mathcal{S}_k$ denotes the family of all k-dimensional subspaces of $H^1(\Omega)$.

3.2 Some preliminary tools and properties

In the following, an important role will be played by the spherical shells of $\mathbb{R}^d$; as in the two dimensional case, we will use often the term *annulus* to denote a spherical shell of $\mathbb{R}^d$ for any dimension d. To simplify the notation, we will denote the annulus centred in x and of radii $r < R$ with the symbol $A_{r,R}(x)$.

A first important result concerning spherical shells is the following relative isoperimetric inequality, proved in [11], Lemma 2.2. In this inequality, the isoperimetric constant depends neither on the measure of the set, nor on the annulus involved, but depends only on the dimension of the space.

Lemma 3.2.1 (Uniform relative isoperimetric inequality in annuli)**.** *Let $m > 0$ be given. Then there exist two positive constants $c = c(d)$ and $w = w(m, d)$ such that, for every $r \geq 0$, $l \geq w$ and every measurable set $E \subseteq A_{r,r+l}(0)$ with $|E| \leq m$, we have*

$$|E|^{\frac{d-1}{d}} \leq cP(E, A_{r,r+l}(0)).$$

Remark 3.2.2. Looking at the proof of the previous lemma (see the proof of Lemma 2.2 in [11], in particular the Steps 2 and 3), we notice a consequence of the choice of w: this constant is chosen in such a way that, if we consider a measurable set E containing the spherical shell $A_{r,r+l}(0)$ for some $r \geq 0$, then we necessarily have $|E| > m$. This remark will allow us to understand the structure of possible optimal sets in the crucial Lemma 3.4.1.

In order to use the direct methods of the Calculus of Variation to maximize (3.1), we need some upper semicontinuity properties. The following proposition, proved in [11], Proposition 2.3, gives us a lower semicontinuity result which will be useful in Section 3.6 to gain an existence result in dimension d.

Proposition 3.2.3. *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets of finite perimeter of $\mathbb{R}^d$ and let $E \subset \mathbb{R}^d$ of finite perimeter such that*

$$\limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial^* E_n) < +\infty \qquad \textit{and} \qquad \chi_{E_n} \overset{L^1(\mathbb{R}^d)}{\longrightarrow} \chi_E.$$

Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$. Then

$$\int_{\partial^* E} u^2 \, d\mathcal{H}^{d-1} \leq \liminf_{n\to\infty} \int_{\partial^* E_n} u_n^2 \, d\mathcal{H}^{d-1}.$$

The fact that we can use neither non-negativity nor monotonicity or rescaling properties to get the existence of maximizers is the one of the greatest difficulties in solving the problem; indeed, in many well known proofs of existence for spectral shape optimization problems, we can rescale optimizing sequences or make topological assumptions thanks to some properties here lacking. As we will see in the further sections, we will overcome this obstacle, at least in a relaxed setting.

Remark 3.2.4. Using the strict negativity of the first eigenvalue, we can emphasize another difference between the Robin-negative and the Dirichlet case. We focus on the optimal shapes for the second eigenvalue in both cases and we remark that optimal shapes are different, in general. Indeed, we know that the union of two and equal disjoint balls of measure $m/2$ minimizes the second eigenvalue of the Dirichlet Laplacian among all shapes of prescribed Lebesgue measure m. Moreover, if we compute the second eigenvalue of the Robin Laplacian with any negative boundary parameter $-\beta$ for the union of two disjoint equal balls of measure $m/2$, this eigenvalue is necessarily strictly negative, as it coincides with the first Robin eigenvalue of each ball. On the other hand, the second eigenvalue of a ball of measure m can be strictly positive, if β is below a certain threshold. More precisely, for any admissible Ω, when β equals the second Steklov eigenvalue $\sigma_2(\Omega)$, it holds $\lambda_{2,\beta}(\Omega) = 0$ (see [11] for details). Then, as the Robin eigenvalues are increasing functions of the boundary parameter, $\lambda_{2,\beta}(\Omega) > 0$ if and only $\beta < \sigma_2(\Omega)$ (i.e. the negative boundary parameter $-\beta$ takes values in $]-\sigma_2(\Omega), 0[$). Then, we can conclude that it is not possible, in general, that the union of two equals balls maximizes $\lambda_{2,\beta}$.

3.3 The relaxed Robin eigenvalues

In order to get an existence result, we need to extend the notion of Robin eigenvalues. In the classical setting, in facts, it turns out to be hard to find directly an existence result in the class of Lipschitz domain, as neither compactness nor upper semicontinuity of the Rayleigh quotient are preserved (unless you do not add some topological constraints, see Section 3.8). A natural way to proceed is to consider a larger class of sets which includes Lipschitz domains and which is endowed of a topology which guarantees upper semicontinuity of the relaxed eigenvalues and compactness of the maximizing sequences. To this aim, it is useful the choice of sets of finite perimeter.

Definition 3.3.1 (Relaxed Robin eigenvalues)**.** Let $\Omega \subset \mathbb{R}^d$ a set of finite perimeter and let $k \in \mathbb{N}$. We define the k-th relaxed eigenvalue for the Robin problem with parameter $-\beta$ the quantity

$$\tilde{\lambda}_{k,\beta}(\Omega) := \inf_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx - \beta \int_{\partial^*\Omega} u^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\, dx},$$

where $\mathcal{S}_k$ denotes the class of all k-dimensional subspaces of $H^1(\mathbb{R}^d)$ which are also k-dimensional subspaces of $L^2(\Omega)$.

We compute $\tilde{\lambda}_{k,\beta}(\Omega)$ considering the above defined class $\mathcal{S}_k$ since we want that, if Ω is bounded and Lipschitz, then $\tilde{\lambda}_{k,\beta}(\Omega) = \lambda_{k,\beta}(\Omega)$. Indeed, if we do not require that the admissible k-dimensional subspaces of $H^1(\mathbb{R}^d)$ are also k-dimensional subspaces of $L^2(\Omega)$, we could find a k-dimensional subspace generated by k functions $u_1, \dots, u_k$ which are linearly independent in $H^1(\Omega)$ and such that there exists $j \in \{1, \dots, k\}$ for which u_j restricted on Ω is null. In this case we would find a subspace of $H^1(\Omega)$ with dimension less than or equal to $k-1$, and it is not admissible in the min-max formula to compute $\lambda_{k,\beta}(\Omega)$.

It is useful to recall the following definition by Section 2.1 in [11].

Definition 3.3.2 (well separated sets)**.** Let $A, B \subseteq \mathbb{R}^d$. We say that A and B are well separated if there exist two open sets E_A, E_B such that, up to negligible sets, $A \subseteq E_A$, $B \subseteq E_B$ and $\operatorname{dist}(E_A, E_B) > 0$.

In other words, if you have two sufficiently regular finite perimeter sets A and B which are well separated, you can extend any $u_A \in H^1(A)$ and $u_B \in H^1(B)$ in such a way that their supports lie at positive distance.

Remark 3.3.3. If a $V_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$ is an admissible k-dimensional subspace of $H^1(\mathbb{R}^d)$ for the computation of $\tilde{\lambda}_{k,\beta}(\Omega)$ and $\varphi_1, \dots, \varphi_k$ have disjoint support, then we can assume that there exists an index $1 \le j \le k$ such that the Rayleigh quotient attains its maximum in V_k on φ_j.

This follows from the inequalities

$$\min_{i=1,\dots,k} \frac{a_i}{b_i} \le \frac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} \le \max_{i=1,\dots,k} \frac{a_i}{b_i}, \tag{3.3}$$

where $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{R}$ with $b_1, \dots, b_k > 0$. To prove (3.3) let us suppose by contradiction that either

$$\frac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} < \min_{i=1,\ldots,k} \frac{a_i}{b_i} \tag{3.4}$$

or

$$\frac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} > \max_{i=1,\ldots,k} \frac{a_i}{b_i}. \tag{3.5}$$

Let us see the first inequality; it can be written equivalently as

$$\begin{cases} \dfrac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} < \dfrac{a_1}{b_1} \\ \qquad\qquad\qquad \ldots \\ \dfrac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} < \dfrac{a_k}{b_k} \end{cases}$$

Then, multiplying both sides of each inequality by the product of the denominators we have

$$\begin{cases} (a_1 + \ldots + a_k)b_1 < a_1(b_1 + \ldots + b_k) \\ \qquad\qquad\qquad \ldots \\ (a_1 + \ldots + a_k)b_k < a_k(b_1 + \ldots + b_k) \end{cases}$$

Summing the inequalities side by side we obtain

$$(a_1 + \ldots + a_k)(b_1 + \ldots + b_k) < (a_1 + \ldots + a_k)(b_1 + \ldots + b_k),$$

which is false, then (3.4) is not true. Reasoning in the same way, we prove that (3.5) is not true, then we can conclude that (3.3) holds.

Now, let V_k and $\varphi_1, \dots, \varphi_k$ be as above and let

$$\varphi := \sum_{i=1}^{k} a_i \varphi_i$$

attaining the maximum for the Rayleigh quotient in V_k. Since the functions φ_i have disjoint supports, we have

$$\left(\sum_{i=1}^{k} a_i \varphi_i \right)^2 = \sum_{i=1}^{k} a_i^2 \varphi_i^2$$

and

$$\left| \sum_{i=1}^{k} a_i \nabla \varphi_i \right|^2 = \sum_{i=1}^{k} a_i^2 |\nabla \varphi_i|^2.$$

Then,

$$\max_{u\in V_k\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx - \beta\int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\,dx} = \frac{\displaystyle\int_\Omega |\nabla \varphi|^2\,dx - \beta\int_{\partial^*\Omega} \varphi^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega \varphi^2\,dx}$$

$$= \frac{\displaystyle\sum_{i=1}^k a_i^2\left(\int_\Omega |\nabla\varphi_i|^2\,dx - \beta\int_{\partial^*\Omega}\varphi_i^2\,d\mathcal{H}^{d-1}\right)}{\displaystyle\sum_{i=1}^k\int_\Omega a_i^2\varphi_i^2\,dx}$$

$$\geq \min_{i=1,\dots,k} \frac{\displaystyle\int_\Omega |\nabla\varphi_i|^2\,dx - \beta\int_{\partial^*\Omega}\varphi_i^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega a_i^2\varphi_i^2\,dx}.$$

We can straightforwardly verify that relaxed eigenvalues enjoy some properties which recall those of classical eigenvalues.

Proposition 3.3.4 (properties of the relaxed eigenvalues)**.** *Let $\Omega \subset \mathbb{R}^d$ be a set of finite perimeter and $\beta > 0$ be fixed.*

(a) *If Ω is open, bounded and Lipschitz, then for every $k \in \mathbb{N}$ it holds*

$$\tilde{\lambda}_{k,\beta}(\Omega) = \lambda_{k,\beta}(\Omega)$$

(classical setting).

(b) *For every $k \in \mathbb{N}$ and for every $t > 0$ one has*

$$\tilde{\lambda}_{k,\beta}(t\Omega) = \frac{1}{t^2}\tilde{\lambda}_{k,t\beta}(\Omega)$$

(scaling property).

(c) *For every Ω of finite perimeter and for every $\beta > 0$ one has*

$$\tilde{\lambda}_{1,\beta}(\Omega) \leq -\beta\frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|} < 0$$

(strict negativity of the first relaxed eigenvalue).

(d) *For every Ω of finite perimeter given by a disjoint union of $N \geq k$ well separated sets with positive Lebesgue measure and for every $\beta > 0$ one has*

$$\tilde{\lambda}_{k,\beta}(\Omega) \leq -\frac{\beta}{\omega_d}\left(\frac{N-k}{|\Omega|}\right)^{1/d} < 0.$$

Proof. Item (a) is a consequence of the choice of the admissible subspaces for the computation of $\tilde{\lambda}_{k,\beta}(\Omega)$. Item (b) and item (c) follow by the same computations made in the classical setting in (2.5) and (2.6), replacing the topological boundary with the reduced one. To prove item (d), we start remarking that there exist k of the N well separated parts of Ω, say $\Omega_1, \ldots, \Omega_k$, with measure less than $|\Omega|/(N-k)$. Then, we consider the k-dimensional test space V for $\lambda_{k,\beta}$ spanned by the k characteristic functions $\chi_{\Omega_1}, \ldots, \chi_{\Omega_k} \in H^1(\mathbb{R}^d)$. Since we can suppose that the maximum is attained on one of these functions, say χ_{Ω_j} (see 3.3.3), we have

$$\begin{aligned}
\lambda_{k,\beta}(\Omega) &\leq \max_{u \in V} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx - \beta \int_{\partial^*\Omega} u^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\, dx} \\
&= \max_{(c_1,\ldots,c_k)\in\mathbb{R}^k} -\beta \frac{\displaystyle\sum_{i=1}^k c_i^2 \mathcal{H}^{d-1}(\partial^*\Omega_i)}{\displaystyle\sum_{i=1}^k c_i^2 |\Omega_i|} = -\beta \frac{\mathcal{H}^{d-1}(\partial^*\Omega_j)}{|\Omega_j|} \\
&\leq -\frac{\beta}{\omega_d} \frac{1}{|\Omega_j|^{1/d}} \leq -\frac{\beta}{\omega_d} \left(\frac{N-k}{|\Omega|}\right)^{1/d}.
\end{aligned}$$

□

Observe that we are not able to say what is the lowest non negative relaxed eigenvalue; it is neither possible to deduce any monotonicity with respect to the domain or scale invariance property in general for $\tilde{\lambda}_{k,\beta}(\cdot)$, as remarked for $\lambda_{k,\beta}(\cdot)$.

3.4 A fundamental lemma and some properties of good candidates

In this section we will prove Lemma 3.4.1, which involves well separated sets and will be crucial in our analysis. This lemma will give us an important alternative to distinguish "good" and "bad" sets in terms of maximization of $\tilde{\lambda}_{k,\beta}(\Omega)$. We refer to Lemma 4.1 in [11], where a similar kind of result has been proved for the Steklov eigenvalues. The main difference (and difficulty) with respect to the Steklov case is that, in our situation, we do not have monotonicity under homotheties; then our result is valid for sets of finite perimeter whose measure is smaller than a fixed value depending on the parameters of the problem.

Lemma 3.4.1. *Let $\beta > 0$, $A > 0$, $c = c(d) > 0$ the isoperimetric constant for the uniform relative isoperimetric inequality in annuli, $m \in \left]0, \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}\right]$ and $w = w(m, d) > 0$ the width constant in the uniform relative isoperimetric inequality in annuli. Then, there exists $L = L(m, d, \beta, A) > w$ such that, for every $r \geq 0$, $l \geq L$ and for every measurable set $E \subseteq A_{r,r+l}(0)$ with finite perimeter and with $|E| \leq m$, we have at least one of the following possibilities.*

(a) *There exists $\varphi \in H_0^1\left(A_{r,r+l}(0)\right)$ such that*

$$\int_{\partial^* E} \varphi^2 \, d\mathcal{H}^{d-1} > 0, \quad \int_E \varphi^2 \, dx > 0$$

and

$$\frac{\displaystyle\int_E |\nabla\varphi|^2 \, dx - \beta \int_{\partial^* E} \varphi^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_E \varphi^2 \, dx} \leq -A.$$

(b) *We have*

$$\left| E \cap A_{r+\frac{l-w}{2}, r+\frac{l+w}{2}}(0) \right| = 0,$$

i.e., up to negligible sets, E lies outside an annulus of width w.

Proof. Let $L > 0$ such that

$$\frac{L - w}{2} > \sqrt{\frac{2c}{\beta}} m^{\frac{1}{2d}} \sum_{j=1}^{\infty} \frac{1}{\left(2^{\frac{1}{2d}}\right)^k}. \tag{3.6}$$

Let $r \geq 0$, $l \geq L$ and $E \subseteq A_{r,r+l}(0)$ measurable, with finite perimeter and such that $|E| \leq m$. Moreover, let us assume that $|E| > 0$, otherwise situation (b) would occur trivially.

Let us suppose that assertion (a) does not hold and let us show that situation (b) occurs. Let us consider the functions m_1 and p_1 defined, for every $t \in \left[0, \frac{l-w}{2}\right]$, by

$$m_1(t) := \left| E \cap \left(A_{r,r+t}(0) \cup A_{r+l-t,r+l}(0)\right)\right|$$

and

$$p_1(t) := P\left(E, A_{r+t,r+l-t}(0)\right).$$

If $p_1(t) = 0$ for some t, situation (b) takes place trivially. Then, we can assume $p_1(t) > 0$ and consider, for every $t > 0$, the function $\varphi_{1,t} \in H_0^1\left(A_{r,r+l}(0)\right)$ defined by

$$\varphi_{1,t}(x) := \left[\frac{1}{t} \operatorname{dist}\left(x, A^c_{r,r+l}(0)\right)\right] \wedge 1.$$

Notice that $\int_E \varphi_{1,t}^2\, dx > 0$ ($|E| > 0$ and $\varphi_{1,t}$ is not the zero function) and that

$$\int_{\partial^* E} \varphi_{1,t}^2\, d\mathcal{H}^{d-1} \geq p_1(t) > 0$$

for any t. Then, since situation (a) cannot occur, we have that

$$-A < \frac{\displaystyle\int_E |\nabla\varphi_{1,t}|^2\, dx - \beta \int_{\partial^* E} \varphi_{1,t}^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_E \varphi_{1,t}^2\, dx} \leq \frac{\dfrac{1}{t^2} m_1(t) - \beta p_1(t)}{\displaystyle\int_E \varphi_{1,t}^2\, dx}.$$

Observe that there exists $t_1 \in \left]0, \frac{l-w}{2}\right]$ such that $0 < m(t_1) = \frac{|E|}{2}$. We claim that $t_1 < \frac{l-w}{2}$. To prove this fact, consider $\varphi_1 := \varphi_{1,t_1}$; we obtain

$$-A < \frac{\dfrac{1}{t_1^2}\dfrac{|E|}{2} - \beta p_1(t_1)}{\displaystyle\int_E \varphi_1^2\, dx}.$$

If the numerator is non negative we have

$$\frac{1}{t_1^2}\frac{|E|}{2} \geq \beta p_1(t_1) \geq \frac{\beta}{c}\left(\frac{|E|}{2}\right)^{\frac{d-1}{d}},$$

where we used the uniform relative isoperimetric inequality in annuli and the fact that both $E \cap (A_{r,r+t_1}(0) \cup A_{r+l-t_1,r+l}(0))$ and $E \cap A_{r+t_1,r+l-t_1}(0)$ have measure $\frac{|E|}{2}$. Last inequality yields the estimate

$$t_1 \leq \sqrt{\frac{c}{\beta}} |E|^{\frac{1}{2d}} \frac{1}{2^{\frac{1}{2d}}} < \sqrt{\frac{2c}{\beta}} m^{\frac{1}{2d}} \frac{1}{2^{\frac{1}{2d}}} < \frac{l-w}{2}.$$

On the other hand, if $\frac{1}{t_1^2}\frac{|E|}{2} - \beta p_1(t_1) < 0$, it holds

$$-A < \frac{\frac{1}{t_1^2}\frac{|E|}{2} - \beta p_1(t_1)}{m} \leq \frac{\frac{1}{t_1^2}\frac{m}{2} - \frac{\beta}{c}\left(\frac{m}{2}\right)^{\frac{d-1}{d}}}{m}.$$

By an easy computation we have

$$\frac{1}{t_1^2}\frac{m}{2} - \frac{\beta}{2c}\left(\frac{m}{2}\right)^{\frac{d-1}{d}} > \frac{\beta}{2c}\left(\frac{m}{2}\right)^{\frac{d-1}{d}} - Am \geq 0,$$

since $m \leq \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$. Thus we obtain the estimate

$$t_1 < \sqrt{\frac{2c}{\beta}} m^{\frac{1}{2d}} \frac{1}{2^{\frac{1}{2d}}} < \frac{l-w}{2}$$

and this completely proves the claim on t_1.

We now proceed as above, reasoning on the annulus $A_{r+t_1,r+l-t_1}(0)$ and the set $E \cap A_{r+t_1,r+l-t_1}(0)$, whose measure is $\frac{|E|}{2}$. For every $t \in \left[0, \frac{l-w}{2} - t_1\right]$ we define the quantities

$$m_2(t) := |E \cap (A_{r+t_1,r+t_1+t}(0) \cup A_{r+l-t_1-t,r+l-t_1}(0))|$$

and

$$p_2(t) := P\left(E, A_{r+t_1+t,r+l-t_1-t}(0)\right).$$

As a test function, we consider

$$\varphi_{2,t}(x) := \left[\frac{1}{t}\mathrm{dist}\left(x, A^c_{r+t_1,r+l-t_1}(0)\right)\right] \wedge 1.$$

Using the same arguments as above, we can find $t_2 \in \left]0, \frac{l-w}{2} - t_1\right[$ such that

$$|E \cap (A_{r+t_1,r+t_1+t_2}(0) \cup A_{r+l-t_1-t_2,r+l-t_1}(0))| = |E \cap A_{r+t_1,r+l-t_1}(0)| = \frac{|E|}{4}$$

and such that the following estimate is satisfied:

$$t_2 < \sqrt{\frac{2c}{\beta}} m^{\frac{1}{2d}} \frac{1}{\left(2^{\frac{1}{2d}}\right)^2} < \frac{l-w}{2} - t_1.$$

Thanks to the choice of L, we can carry out the argument infinitely many times, obtaining a sequence $(t_n)_n \subset \left]0, \frac{l-w}{2}\right[$ such that

$$|E \cap A_{r+t_1+\cdots+t_n,r+l-t_1-\ldots-t_n}| = \frac{|E|}{2^n}$$

and

$$E \cap A_{r+\frac{l-w}{2},r+\frac{l+w}{2}} \subseteq E \cap A_{r+t_1+\cdots+t_n,r+l-t_1-\ldots-t_n}$$

for every $n \in \mathbb{N}$. Letting n go to infinity, we obtain that situation (b) occurs, completing the proof. □

Remark 3.4.2. Let us notice that the bound a priori on the measure of E is not restrictive in the general application of the previous lemma (we will use a trick to overcome this restriction on the admissible volumes in Proposition 3.4.3). In particular, the "critical measure" depends decreasingly on the modulus A of the threshold for the Rayleigh quotients. We do not know if it is possible to prove an analogous result removing this bound (as in Lemma 4.1 in [11]), since, as mentioned before, the problem is not scale invariant and it is not known if there is any monotonicity property, in general. Notice that, as

A decreases to 0, the bound on the volumes goes to $+\infty$ and then the lemma is applicable with a wider range of volumes.

Nevertheless, the meaning of Lemma 3.4.1 is that, within a certain range of measures, if the Rayleigh quotient for a set E is greater than a fixed bound (and reasonably better for the maximization), then, up to negligible sets, E must lie in a small annulus or in two well separated small annuli at distance greater than a constant depending only on the measure and on the dimension of the space.

The following proposition allows us to understand some properties that good candidates to be maximizers for $\tilde{\lambda}_{k,\beta}(\Omega)$ have to satisfy. We will follow the ideas of Proposition 5.4 in [11]; in that proof, authors apply Lemma 4.1 of [11] (whose counterpart for Robin eigenvalues is the previous Lemma 3.4.1) relatively to the measure m of the given set. In our context, there is a crucial difference: we can not apply Lemma 3.4.1 directly with the measure m of the given set Ω, since such measure does not satisfy the bound in the statement of Lemma 3.4.1, in general. The idea to overcome the difficulty is rather natural: we thought to cut Ω by intersection with a suitable family of annuli, in such a way that at least k "slices" of Ω have volume less than the fixed threshold.

Proposition 3.4.3 (a priori bound on the diameter)**.** *Let $\Omega \subset \mathbb{R}^d$ of finite perimeter, $|\Omega| = m$ and let $A > 0$ such that $\tilde{\lambda}_{k,\beta}(\Omega) > -A$. Then, up to negligible sets, Ω is union of N well separated and bounded sets of finite perimeter*

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_N,$$

with $N < \frac{mA^d\omega^d}{\beta^d} + k$, with $\omega = \omega(d) > 0$ isoperimetric constant in $\mathbb{R}^d$ and $diam(\Omega_j) \leq D(m, \beta, d, k, A)$, i.e. the diameters of the well separated sets are uniformly bounded.

Proof. Without loss of generality, we can consider the origin as a point of density one for Ω. Let $c = c(d) > 0$ be the dimensional constant in the relative isoperimetric inequality for annuli and set $m^* := \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$. Let us consider the constants $w = w(m^*, d) = w(A, \beta, d)$ and $L = L(m^*, d, \beta, A) = L(d, \beta, A)$ of Lemma 3.4.1 applied to m^* and define the annuli

$$A_j := A_{jL,(j+1)L}(0) := \left\{x \in \mathbb{R}^d : jL < |x| < (j+1)L\right\}$$

for $j = 0, \ldots, \left\lfloor \frac{m}{m^*} \right\rfloor + k$. By construction, there exist k of this annuli, say $A_{n_1}, \ldots, A_{n_k}$, such that $|\Omega \cap A_{n_h}| \leq m^*$. For each $h = 1, \ldots, k$, let us apply Lemma 3.4.1 to each set $\Omega \cap A_{n_h}$: either there exist $\varphi_h \in H^1_0(A_{n_h})$ such that

$$\int_{\partial^*(\Omega \cap A_{n_h})} \varphi_h^2 \, d\mathcal{H}^{d-1} > 0, \quad \int_{\Omega \cap A_{n_h}} \varphi_h^2 \, dx > 0$$

and

$$\frac{\displaystyle\int_{\Omega\cap A_{n_h}} |\nabla\varphi_h|^2 \, dx - \beta \int_{\partial^*(\Omega\cap A_{n_h})} \varphi_h^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega\cap A_{n_h}} \varphi_h^2 \, dx} \leq -A$$

or

$$\left|\Omega \cap A_{n_h} \cap A_{n_h L + \frac{n_h L - w}{2}, n_h L + \frac{n_h L + w}{2}}\right| = 0.$$

Observe that there exists one of the sets $\Omega \cap A_{n_h}$ in which the first case is not satisfied. In fact, if the first situation occurred in all $\Omega \cap A_{n_h}$, there would exist $\varphi_1, \ldots, \varphi_k$ as above, with trivial extension to the whole space belonging to $H^1(\mathbb{R}^d)$ and disjoint supports. Moreover, such φ_h would be supported in A_{n_h} and would have null trace on ∂A_{n_h}, then

$$\int_{\Omega\cap A_{n_h}} |\nabla\varphi_h|^2 \, dx = \int_{\Omega} |\nabla\varphi_h|^2 \, dx,$$

$$\int_{\partial^*(\Omega\cap A_{n_h})} \varphi_h^2 \, d\mathcal{H}^{d-1} = \int_{\partial^*\Omega} \varphi_h^2 \, d\mathcal{H}^{d-1}$$

and

$$\int_{\Omega\cap A_{n_h}} \varphi_h^2 \, dx = \int_{\Omega} \varphi_h^2 \, dx.$$

Hence, for each $h = 1, \ldots, k$, we would have

$$\frac{\displaystyle\int_{\Omega} |\nabla\varphi_h|^2 \, dx - \beta \int_{\partial^*\Omega} \varphi_h^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega} \varphi_h^2 \, dx} \leq -A.$$

In view of the Remark 3.3.3, we could assume that the maximum of the Rayleigh in span $\{\varphi_1, \ldots, \varphi_k\}$ is attained in some φ_j. Then, by the usual minimax formula, we would obtain that $\tilde{\lambda}_{k,\beta}(\Omega) \leq -A$, in contradiction with the hypotheses.

Let now be $p \in \{1, \ldots, k\}$ such that A_{n_p} does not satisfy the first alternative and let us set

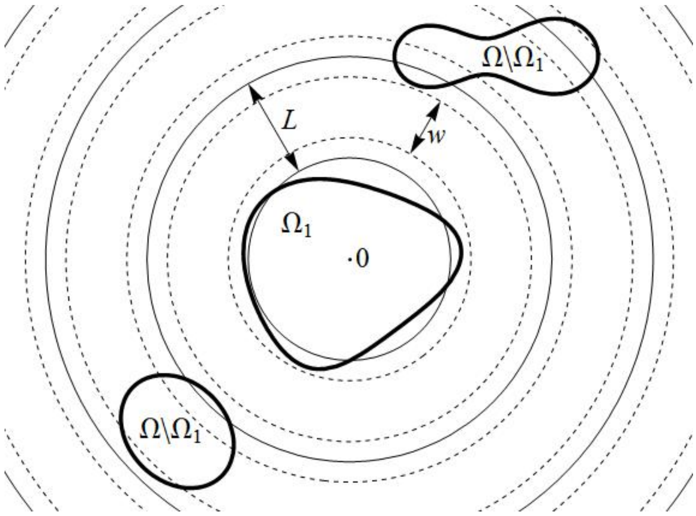
$$\Omega_1 := \Omega \cap B_{n_p L + \frac{L}{2}}(0).$$

Observe that Ω_1 is bounded, with finite perimeter and well separated from $\Omega \setminus \Omega_1$, with

$$\mathrm{dist}(\Omega_1, \Omega \setminus \Omega_1) \geq w.$$

Moreover

$$\mathrm{diam}(\Omega_1) \leq 2\left(\left\lfloor \frac{m}{m^*} \right\rfloor + k + 1\right) L,$$



with the right hand side depending on A, m, β, d, k.

Let us now consider the set $\Omega \setminus \Omega_1$, whose measure is less than m, translate the set in such a way that the origin is a point of density one for $\Omega \setminus \Omega_1$ and repeat the same arguments as above to build Ω_2; with the same bound on the diameter.

The argument could be carried on to build the following well separated parts of Ω, whose diameters are still uniformly bounded by $2\left(\lfloor \frac{m}{m^*} \rfloor + k + 1\right) L$. Notice that we cannot repeat the argument infinitely many times. In fact, if this was the case, up to negligible sets we would have

$$\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n, \quad m = |\Omega| = \sum_{n\in\mathbb{N}} |\Omega_n|$$

and, for each well separated set Ω_n we consider its characteristic function whose Rayleigh quotient is less than or equals $-\frac{\beta}{\omega_d}|\Omega_n|^{-\frac{1}{d}}$. Notice that $|\Omega_n| \to 0$; so, if we take k of these well separated sets, say $\Omega_n^1, \dots, \Omega_n^k$, with measure smaller than a threshold, we obtain $\tilde{\lambda}_{k,\beta}(\Omega) < -A$ (taking span $\left\{\chi_{\Omega_n^1}, \dots, \chi_{\Omega_n^k}\right\}$ as a test space for $\tilde{\lambda}_{k,\beta}(\Omega)$).

Moreover, the number N of the well separated parts is controlled by a constant depending on m, β, d, k, A. Indeed, we can assume that $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ with $N \geq k$ (possibly counting also Ω_j which are negligible). Then, we can order the sets Ω_j in such a way that $|\Omega_j| \leq \frac{m}{N-k}$ for every $j \leq k$ and consider k test functions $\varphi_1, \dots, \varphi_k \in H^1(\mathbb{R}^d)$ with disjoint supports and such that $0 \leq \varphi_j \leq 1$, $\varphi_j = 1$ on Ω_j and $\varphi_j = 0$ on $\Omega \setminus \Omega_j$. Let us consider $V :=$ span $\{\varphi_1, \dots, \varphi_k\}$ and, without loss of generality, assume that the Rayleigh quotient attains in φ_1 its maximum for the space V. Then we have

$$
\begin{aligned}
-A < \tilde{\lambda}_{k,\beta}(\Omega) &\leq \max_{u \in V \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \beta \int_{\partial^*\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx} \\
&\leq \frac{\displaystyle\int_\Omega |\nabla \varphi_1|^2 \, dx - \beta \int_{\partial^*\Omega} \varphi_1^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega \varphi_1^2 \, dx} \\
&\leq -\beta \frac{\mathcal{H}^{d-1}(\partial^*\Omega_1)}{|\Omega_1|} \leq -\frac{\beta}{\omega} |\Omega_1|^{-\frac{1}{d}} \leq -\frac{\beta}{\omega} \left(\frac{m}{N-k} \right)^{-\frac{1}{d}}
\end{aligned}
$$

and this yields

$$
N < \frac{mA^d \omega^d}{\beta^d} + k,
$$

completing the proof. □

Remark 3.4.4. Notice that, as A decreases to 0, the bound on the number of well separated components goes to k: this fact suggests that, if $\tilde{\lambda}_{k,\beta}(\Omega)$ is non negative, it is union of at most k well separated parts (up to negligible sets).

Observe that we have again a result depending on the measure of the set Ω: both diameters and number of connected components are bounded by a quantity depending on $|\Omega|$. We would have expected this behaviour, since, as already said, the problem is not scale invariant.

3.5 Isoperimetric and isodiametric control of the spectrum

The next step for our purposes is to have some uniform control on the perimeters to ensure compactness of maximizing sequences. For our approach, we have been inspired by [32], where authors proved several results for the isoperimetric control of the Steklov spectrum of Riemannian manifolds satisfying some additional hypotheses. A crucial tool in this setting is the following Lemma 3.5.1, proved in [55], Corollary 3.12, in a more general setting.

We will use the following notation: if A is the annulus $A_{r,R}(x)$, we will denote with $2{\cdot}A$ the annulus $A_{\frac{r}{2},2R}(x)$.

Lemma 3.5.1. *Let ν be a finite, non negative, non atomic Radon measure on $\mathbb{R}^d$. Then, for every $k \in \mathbb{N}$, there exist a family $\mathcal{A}$ of k annuli in $\mathbb{R}^d$ such that*

(a) *there exists a positive constant γ_d depending only on the dimension d such that*
$$\nu(A) \geq \gamma_d \frac{\nu(\mathbb{R}^d)}{k};$$

(b) *the annuli $\{2{\cdot}A\}_{A\in\mathcal{A}}$ are disjoint.*

Lemma 3.5.2 (Isoperimetric control of the relaxed spectrum)**.** *Let $\Omega \subset \mathbb{R}^d$ of finite perimeter and let $|\Omega| = m$. Then, there exist two a positive constants $C_1 = C_1(d)$, $C_2 = C_2(d)$ such that*

$$\begin{aligned}\tilde{\lambda}_{k,\beta}(\Omega) \leq -C_2 &\left(\frac{k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \dfrac{\beta}{C_1}\mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|} \right)^{-} \\ &+ \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}\chi_{]0,+\infty[}\left(k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1}\mathcal{H}^{d-1}(\partial^*\Omega) \right)\end{aligned}$$

Remark 3.5.3. Lemma 3.5.2 could be equivalently stated in the following way: under the hypotheses above, there exists two positive constants C_1 and C_2, depending only on d, such that if
$$\mathcal{H}^{d-1}(\partial^*\Omega) > \frac{C_1}{\beta}k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}},$$
then
$$\tilde{\lambda}_{k,\beta}(\Omega) \leq \frac{C_2 k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta\mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|},$$
otherwise, if
$$\mathcal{H}^{d-1}(\partial^*\Omega) \leq \frac{C_1}{\beta}k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}},$$
then
$$\tilde{\lambda}_{k,\beta}(\Omega) \leq \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}.$$

Proof of Lemma 3.5.2. Let us consider Ω as in the hypotheses and apply Lemma 3.5.1 to the finite non atomic measure $\nu := \mathcal{H}^{d-1}\lfloor\partial^*\Omega$: there exist $2k$ annuli $A_1, \dots, A_{2k}$ in $\mathbb{R}^d$ and a positive constant γ_d depending only on the d such that, for every $i = 1, \dots, 2k$
$$\mathcal{H}^{d-1}(\partial^*\Omega \cap A_i) \geq \gamma_d \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k} \tag{3.7}$$
and, if $i \neq j$,
$$2{\cdot}A_i \cap 2{\cdot}A_j = \emptyset. \tag{3.8}$$

In particular, we can order the $2k$ annuli in such a way that

$$|\Omega \cap 2{\cdot}A_i| \le \frac{|\Omega|}{k}. \tag{3.9}$$

Now let us write each annulus as

$$A_i = \{r_{1,i} < |x - x_i| < r_{2,i}\}$$

and consider the functions h_i defined as follows

$$h_i(x) := \begin{cases} \dfrac{1}{r_{2,i}}\mathrm{dist}(x, \mathbb{R}^d \setminus 2{\cdot}A_i) & \text{if } x \notin B_{r_{2,i}}(x_i), \\ 1 & \text{if } x \in A_i, \\ \dfrac{1}{r_{1,i}}\mathrm{dist}(x, \mathbb{R}^d \setminus 2{\cdot}A_i) & \text{if } x \in B_{r_{1,i}}(x_i). \end{cases} \tag{3.10}$$

Observe that $h_i \in H^1(\mathbb{R}^d)$, $h_i = 0$ on $\mathbb{R}^d \setminus 2{\cdot}A_i$ and

$$|\nabla h_i| = \begin{cases} \dfrac{1}{r_{2,i}} & \text{in } B_{2r_{2,i}}(x_i) \setminus B_{r_{2,i}}(x_i), \\ 0 & \text{in } A_i \cup (2{\cdot}A_i)^c, \\ \dfrac{1}{r_{1,i}} & \text{in } B_{r_{1,i}}(x_i) \setminus B_{\frac{r_{1,i}}{2}}(x_i). \end{cases}$$

Denoting by $R(u)$ the Rayleigh quotient for any admissible function u, we estimate for every $i = 1, \dots, k$ the quantity $R(h_i)$ as follows:

$$\begin{aligned} R(h_i) &= \frac{\displaystyle\int_\Omega |\nabla h_i|^2\,dx - \beta \int_{\partial^*\Omega} h_i^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega h_i^2\,dx} \\ &\le \frac{\left(\displaystyle\int_{2\cdot A_i} |\nabla h_i|^d\,dx\right)^{\frac{2}{d}} |\Omega \cap 2{\cdot}A_i|^{\frac{d-2}{d}} - \beta \mathcal{H}^{d-1}(\partial^*\Omega \cap A_i)}{\displaystyle\int_\Omega h_i^2\,dx}, \end{aligned} \tag{3.11}$$

where we used the Hölder inequality. Observe that the quantity

$$\left(\int_{2\cdot A_i} |\nabla h_i|^d\,dx\right)^{\frac{2}{d}}$$

is a positive constant depending only on d. So, using estimates (3.7) and (3.8), we obtain by (3.11)

$$R(h_i) \le \frac{\left(\displaystyle\int_{2\cdot A_i} |\nabla h_i|^d\,dx\right)^{\frac{2}{d}} \left(\dfrac{|\Omega|}{k}\right)^{\frac{d-2}{d}} - \beta\gamma_d \dfrac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k}}{\displaystyle\int_\Omega h_i^2\,dx}. \tag{3.12}$$

Let us consider now the positive constants

$$C_2(d) := \max_{i=1,\dots,2k} \left(\int_{2\cdot A_i} |\nabla h_i|^d \, dx \right)^{\frac{2}{d}}, C_1(d) := \frac{2C_2}{\gamma_d},$$

which depend only on the dimension d. Observe that, if

$$\mathcal{H}^{d-1}(\partial^*\Omega) > \frac{C_1}{\beta} k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}},$$

then the numerator of the right hand side of (3.12) is negative, then we can continue the estimate using again (3.9):

$$\begin{aligned} R(h_i) &\leq \frac{C_2 \left(\dfrac{|\Omega|}{k} \right)^{\frac{d-2}{d}} - \dfrac{C_2}{C_1} \beta \dfrac{\mathcal{H}^{d-1}(\partial^*\Omega)}{k}}{|\Omega \cap 2\cdot A_i|} \\ &\leq \frac{C_2 \left(\dfrac{|\Omega|}{k} \right)^{\frac{d-2}{d}} - \dfrac{C_2}{C_1} \beta \dfrac{\mathcal{H}^{d-1}(\partial^*\Omega)}{k}}{\dfrac{|\Omega|}{k}} \qquad (3.13) \\ &\leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1} \beta \mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|}. \end{aligned}$$

Since $h_1, \dots, h_k$ have disjoint supports in view of (3.8), we achieve the thesis in the first case by considering the space $S_k := \operatorname{span}\{h_1, \dots, h_k\}$.

Now, let $\mathcal{H}^{d-1}(\partial^*\Omega) \leq \frac{C_1}{\beta} k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}}$. In view of Lemma 3.5.1 applied to the finite non atomic measure $\mathcal{L}^d \lfloor \Omega$, there exists a family of k annuli $B_1, \dots, B_k$ such that

$$|\Omega \cap B_i| \geq \gamma_d \frac{|\Omega|}{k} \qquad (3.14)$$

and, if $i \neq j$,

$$2\cdot B_i \cap 2\cdot B_j = \emptyset. \qquad (3.15)$$

Consider the test functions $l_1, \dots, l_k$ for the annuli $B_1, \dots, B_k$, constructed in the same way as the h_i in (3.10), and observe that their supports are disjoint, thanks to (3.15). Now, let us estimate $R(l_i)$:

$$
\begin{aligned}
R(l_i) &= \frac{\displaystyle\int_\Omega |\nabla l_i|^2\, dx - \beta \int_{\partial^*\Omega} l_i^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega l_i^2\, dx} \\
&\leq \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap B_i)}{\displaystyle\int_\Omega l_i^2\, dx}.
\end{aligned}
\tag{3.16}
$$

If the numerator of the last fraction is negative, then $R(l_i) < 0$. Otherwise we can estimate $R(l_i)$ using (3.14):

$$
\begin{aligned}
R(l_i) &\leq \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap B_i)}{|\Omega \cap B_i|} \\
&\leq \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap B_i)}{\gamma_d \dfrac{|\Omega|}{k}} \\
&\leq \frac{C_2 k}{\gamma_d |\Omega|^{\frac{2}{d}}} = \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}.
\end{aligned}
$$

In both cases $R(l_i) \leq \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}$; reasoning as above on span $\{l_1, \ldots, l_k\}$, we complete the proof. □

Remark 3.5.4. Let us observe that we have two different situations (then two different estimates) depending again on the measure of Ω. Fortunately, as we will see in the existence theorem 3.6.1, this will not prevent us to prove that maximizing sequences of admissible sets have uniformly bounded perimeters. In other words, Lemma 3.5.2 will give us the needed isoperimetric control of the spectrum that ensures the compactness of maximizing sequences.

Furthermore, we point out that Lemma 3.4.1, Proposition 3.4.3 and Lemma 3.5.2 hold also for Lipschitz domains, replacing the reduced boundary with the topological boundary: this allows us to have the same results also in the classical setting.

Now, recalling the ideas in Proposition 4.3 in [11], we would like to obtain also an isodiametric control of the Robin spectrum. This will be done in the following Proposition 3.5.5, where we give an upper estimate for $\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega)$ for every open, bounded, Lipschitz and connected set Ω of prescribed measure m. This estimate depends only on the dimension d of the space, the measure m of Ω, the index k and the boundary parameter β and it is valid both for $\lambda_{k,\beta}(\Omega) \geq 0$ and for $\lambda_{k,\beta}(\Omega) < 0$.

Proposition 3.5.5 (isodiametric control of the spectrum)**.** *Let $m > 0$ and $A > 0$, $\Omega \subset \mathbb{R}^d$ be an open, bounded, Lipschitz and connected set of measure m, let $C_1 = C_1(d), C_2 = C_2(d) > 0$ the dimensional constants in Lemma 3.5.2, $\omega = \omega(d)$ the Lebesgue measure of the unit ball in $\mathbb{R}^d$, $c = c(d)$ the dimensional constant in the isoperimetric inequality in annuli 3.2.1, m^* and L as in Lemma 3.4.1. Then, if*

$$\mathrm{diam}(\Omega) > 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L,$$

it holds

$$\lambda_{k,\beta}(\Omega) < -A \tag{3.17}$$

otherwise,

$$\begin{aligned}
&\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega)\\
&\le \omega\left(\frac{2C_2}{(\omega m)^{1/d}}k^{2/d} - \frac{2C_2\beta d}{C_1}\right)\chi_{]0,+\infty[}\left(\mathcal{H}^{d-1}(\partial\Omega) - \frac{C_1}{\beta}m^{\frac{d-2}{d}}k^{2/d}\right)\\
&+ \frac{C_1 L}{m^{2/d}}\left[2k^2 + \left(\frac{2^{2d}c^d m + 2\beta^d}{\beta^d}\right)k\right]\left[1 - \chi_{]0,+\infty[}\left(\mathcal{H}^{d-1}(\partial\Omega) - \frac{C_1}{\beta}m^{\frac{d-2}{d}}k^{2/d}\right)\right].
\end{aligned} \tag{3.18}$$

Proof. The proof is obtained following the same arguments as in Proposition 4.3 in [11], recalling the "slicing trick by annuli" in Proposition 3.4.3 and treating separately the two different situations of Lemma 3.5.2 (see Remark 3.5.3).

Without loss of generality, let us consider the origin as a point of density 1 for Ω. Let us suppose that

$$\mathrm{diam}(\Omega) > 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L$$

and define the $k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1$ concentric annuli

$$A_j := A_{jL,(j+1)L}(0) := \left\{x \in \mathbb{R}^d : jL < |x| < (j+1)L\right\}, \quad j = 0, \dots, \left\lfloor \frac{m}{m^*} \right\rfloor + k.$$

By construction, there exist k of this annuli, say $A_{n_1}, \dots, A_{n_k}$, such that $|\Omega \cap A_{n_h}| \le m^*$. For each $h = 1, \dots, k$, let us apply Lemma 3.4.1 to each set $\Omega \cap A_{n_h}$ and observe that, as Ω is connected, in each annulus the first alternative takes place. Then, there exist k functions $\varphi_1, \dots, \varphi_k \in H^1(\Omega)$ such that $\mathrm{supp}(\varphi_j) \subset A_{n_j}$ and

$$\frac{\displaystyle\int_\Omega |\nabla \varphi_j|^2\, dx - \beta \int_{\partial\Omega} \varphi_j^2\, d\sigma}{\displaystyle\int_\Omega \varphi_j^2\, dx} \le -A$$

for every $j = 1, \dots, k$. Then, the space $S := \mathrm{span}\left\{\varphi_1, \dots, \varphi_k\right\}$ is admissible for the computation of $\lambda_{k,\beta}(\Omega)$ and, in view of the Remark 3.3.3, we could assume that the maximum of the Rayleigh quotient in S is attained in some φ_j. Then we have

$$\lambda_{k,\beta}(\Omega) \leq \max_{u \in S \setminus \{0\}} = R(\varphi_j) \leq -A$$

i.e., passing to the minimum among admissible subspaces, (3.17).

Let now be

$$\mathrm{diam}(\Omega) \leq 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L$$

and suppose that the first case of Lemma 3.5.2 (see Remark 3.5.3) holds true, so

$$\mathcal{H}^{d-1}(\partial\Omega) < \frac{C_1}{\beta} m^{\frac{d-2}{d}} k^{2/d}.$$

Then, again by Lemma 3.5.2,

$$\lambda_{k,\beta}(\Omega) \leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta \mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} < 0,$$

i.e. $\lambda_{k,\beta}(\Omega)$ is necessarily strictly negative. Using the isoperimetric inequality and the isodiametric inequality

$$|\Omega| \leq \omega_d \left(\frac{\mathrm{diam}(\Omega)}{2}\right)^d$$

we obtain

$$\begin{aligned}
\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) &\leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta \mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} \frac{2|\Omega|^{1/d}}{\omega^{1/d}} \\
&\leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta d \omega^{1/d} |\Omega|^{\frac{d-1}{d}}}{|\Omega|^{\frac{d-1}{d}}} \frac{2}{\omega^{1/d}} \\
&= \frac{2C_2}{(\omega m)^{1/d}} k^{2/d} - \frac{2C_2 \beta d}{C_1}.
\end{aligned}$$

Let us suppose now that the second case of Lemma 3.5.2 holds true, i.e.

$$\mathcal{H}^{d-1}(\partial\Omega) \geq \frac{C_1}{\beta} m^{\frac{d-2}{d}} k^{2/d}.$$

In that case, if $\lambda_{k,\beta}(\Omega) < 0$, we trivially have $\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) < 0$; otherwise, if $\lambda_{k,\beta}(\Omega) \geq 0$, using the upper bound on $\mathrm{diam}(\Omega)$ we have

$$\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) \leq \frac{C_1 L}{m^{2/d}} \left[2k^2 + \left(\frac{2^{2d} c^d m + 2\beta^d}{\beta^d}\right) k\right],$$

getting (3.18) and completing the proof of the proposition. $\square$

Remark 3.5.6. In Proposition 3.4.3 we proved that, if a set Ω is a good candidate to maximize $\tilde{\lambda}_{k,\beta}$, it must have controlled diameter (notice that 3.4.3 still holds in the classical setting). Another evidence of this behaviour of good candidates is given by Proposition 3.5.5: we proved that if the diameter of a Lipschitz connected domain Ω is not controlled by the constant $2\left(k + \lfloor\frac{m}{m^*}\rfloor + 1\right)L$, then Ω is not a good candidate to maximize $\lambda_{k,\beta}(\Omega)$.

3.6 Existence of optimal shapes in the class of measurable sets

In this section, following Section 5 of [11], we will get an existence result for a relaxed version of the problem (3.1), involving the relaxed Robin eigenvalues.

We focus ourselves on the maximization problem

$$\max\left\{F(\tilde{\lambda}_1^\beta(\Omega), \dots, \tilde{\lambda}_{k,\beta}(\Omega)) : \Omega \subset \mathbb{R}^d \text{ has finite perimeter and } |\Omega| = m\right\}, \tag{3.19}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ non decreasing in each variable and upper semicontinuos (as in (3.1)). To avoid trivial situations, we will infer another reasonable hypothesis on F.[1] Our existence result is based on an adaptation of Theorem 5.6 in [11] to our setting, with an important difference: in our case, the uniform bound on the perimeters of a maximizing sequence have to be deduced treating separately the two situation in Lemma 3.5.2. Nevertheless, in both situation, we are able to get such a uniform bound and then to gain the compactness of a maximizing sequence.

In this context, we will denote with B_m the ball of measure $m > 0$.

Theorem 3.6.1 (Existence of optimal domains)**.** *Let $F : \mathbb{R}^k \to \mathbb{R}$ non decreasing in each variable and upper semicontinuos. Moreover, assume that there exist Ω_0 admissible, $A > 0$ and $A_1, \dots, A_k \in]0, A[$ such that*

$$F(\tilde{\lambda}_{1,\beta}(\Omega_0), \dots, \tilde{\lambda}_{k,\beta}(\Omega_0)) > F(-A_1, \dots, -A_k). \tag{3.20}$$

Then, Problem (3.19) has at least a solution. Moreover, up to negligible sets, each optimal set is bounded and can be written as the union of at most $\frac{m|\lambda_{k,\beta}(B_m)|^d\omega^d}{\beta^d} + k$ equibounded well separated sets of finite perimeter.

Proof. Let $(\Omega_n)_n$ be a maximizing sequence for $F(\tilde{\lambda}_{1,\beta}(\cdot), \dots, \tilde{\lambda}_{k,\beta}(\cdot))$. In view of the assumption, we observe that any admissible domain E such that

[1] In [18] the additional hypothesis is given in a smarter form, involving the coercivity of the function F; here, the further hypothesis is given in a slightly weaker form in the statement of Theorem 3.6.1.

$\tilde{\lambda}_{h,\beta}(E) \leq -A_h$ for every $j = 1, \ldots, k$ can not be optimal. Then, it is not restrictive to assume that $\tilde{\lambda}_{h,\beta}(\Omega_n) > -A_h$ for every $h = 1, \ldots, k$. Moreover, without loss of generality, we can assume that $\tilde{\lambda}_{h,\beta}(\Omega_n) \geq \tilde{\lambda}_{h,\beta}(B_m) = \lambda_{h,\beta}(B_m)$ for every $n \in \mathbb{N}$.[2] By Lemma 3.5.2 we have that either the perimeter of Ω_n is less than a constant depending on m, d, β, k or

$$\lambda_{h,\beta}(B_m) \leq \tilde{\lambda}_{h,\beta}(\Omega_n) \leq \frac{C_2 h^{\frac{2}{d}} m^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta \mathcal{H}^{d-1}(\partial^*\Omega_n)}{m}.$$

Via a straightforward computation, we obtain that also in this second case it holds $\mathcal{H}^{d-1}(\partial^*\Omega_n) \leq C = C(m, d, \beta, k)$. Hence, we deduce that

$$\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\partial^*\Omega_n) < +\infty. \tag{3.21}$$

Thanks to proposition 3.4.3, we can write

$$\Omega_n = \Omega_n^1 \cup \ldots \cup \Omega_n^{N_n}, \quad N_n \leq \frac{m|\lambda_{k,\beta}(B_m)|^d \omega^d}{\beta^d} + k,$$

with $\Omega_n^1, \ldots, \Omega_n^{N_n}$ equibounded and well separated. Up to translations, we can assume that the Ω_n are contained in a fixed ball of $\mathbb{R}^d$; so, by (3.21) and Proposition 1.2.10, we deduce that there exists $\Omega \subset \mathbb{R}^d$ of finite perimeter such that, up to subsequences, that $\chi_{\Omega_n} \to \chi_\Omega$ strongly in L^1, $|\Omega| = m$ and $P(\Omega) \leq \liminf_n P(\Omega_n)$.

Using the same arguments as in Theorem 5.6 in [11], let us show that such admissible set Ω is a solution of the problem (3.19). We claim that

$$\limsup_{n\to\infty} \tilde{\lambda}_{h,\beta}(\Omega_n) \leq \tilde{\lambda}_{h,\beta}(\Omega). \tag{3.22}$$

To prove it, let us fix $\varepsilon > 0$ and let $V_h := \mathrm{span}\,\{u_1, \ldots, u_h\} \subset H^1(\mathbb{R}^d)$ be an admissible subspace for the computation of $\tilde{\lambda}_{h,\beta}(\Omega)$ such that

$$\tilde{\lambda}_{h,\beta}(\Omega) \leq \max_{u\in V_h\setminus\{0\}} R(u) - \varepsilon.$$

For each $n \in \mathbb{N}$, let

$$u_n := \sum_{j=1}^{h} \alpha_j^n u_j$$

[2]If $\lambda_{h,\beta}(B_m) < -A_h$ the assumption comes trivially from the previous remark on optimal sets; if $\lambda_{h,\beta}(B_m) \geq -A_h$, we can assume again $\tilde{\lambda}_{h,\beta}(\Omega_n) \geq \lambda_{h,\beta}(B_m)$ since F is non decreasing and $(\Omega_n)_n$ is a maximizing sequence.

attaining the maximum

$$\max_{v\in V_h\setminus\{0\}} \frac{\displaystyle\int_{\Omega_n} |\nabla v|^2\,dx - \beta\int_{\partial^*\Omega_n} v^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega_n} v^2\,dx}.$$

We can also assume that

$$\sum_{j=1}^{h}(\alpha_j^n)^2 = 1$$

(it is not restrictive). So, up to subsequences, there exist $\alpha_1,\ldots,\alpha_h$ such that α_j^n converges to α_j as n goes to infinity. Hence, set

$$u := \sum_{j=1}^{h}\alpha_j u_j,$$

we have that $u_n \to u$ strongly in $H^1(\mathbb{R}^d)$. By the convergence of χ_{Ω_n} to χ_Ω and Proposition 3.2.3 we obtain that

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega_n}|\nabla u_n|^2\,dx &= \int_\Omega |\nabla u|^2\,dx\\ \lim_{n\to\infty}\int_{\Omega_n} u_n^2\,dx &= \int_\Omega u^2\,dx\\ \liminf_{n\to\infty}\int_{\partial^*\Omega_n} u_n^2\,d\mathcal{H}^{d-1} &\geq \int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}.\end{aligned} \tag{3.23}$$

We finally have

$$\begin{aligned}\limsup_{n\to\infty}\tilde\lambda_{h,\beta}(\Omega_n) &\leq \limsup_{n\to\infty}\frac{\displaystyle\int_{\Omega_n}|\nabla u_n|^2\,dx - \beta\int_{\partial^*\Omega_n}u_n^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega_n}u_n^2\,dx}\\ &\leq \frac{\displaystyle\int_\Omega|\nabla u|^2\,dx - \beta\int_{\partial^*\Omega}u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\,dx} \leq \tilde\lambda_{h,\beta}(\Omega)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ we get the required upper semicontinuity (3.22).

Thanks to the assumptions on F we deduce that

$$\begin{aligned}F(\tilde\lambda_{1,\beta}(\Omega),\ldots,\tilde\lambda_{k,\beta}(\Omega)) &\geq \limsup_{n\to\infty}F(\tilde\lambda_{1,\beta}(\Omega_n),\ldots,\tilde\lambda_{k,\beta}(\Omega_n))\\ &= \sup_{\substack{|E|=m\\ \mathcal{H}^{d-1}(\partial^*E)<+\infty}} F(\tilde\lambda_{1,\beta}(E),\ldots,\tilde\lambda_{k,\beta}(E)).\end{aligned}$$

Then Ω is optimal. Finally, the bound on the number of well separated parts $\Omega_1, \dots, \Omega_N$ and the uniform bound on the diameters of the sets Ω_j, $j = 1, \dots, N$, can be achieved reasoning as in Proposition 3.4.3, replacing A_h with $|\lambda_{h,\beta}(B_m)|$ for each $h = 1, \dots, k$. □

As said above, the assumption (3.20) has been made to avoid trivial situation such as F constant on $\mathbb{R}^k$, for which every admissible Ω would be a solution.

Remark 3.6.2 (Isoperimetric control for the first eigenvalue)**.** Observe that, if $k = 1$, the uniform bound on the perimeters of a maximizing sequence would be straightforwardly achieved. In fact, considering a maximizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ such that, without loss of generality, $\tilde{\lambda}_{1,\beta}(\Omega_n) \geq \lambda_{1,\beta}(B_m)$, by computing the Rayleigh quotient on a test function constant on Ω we have

$$-|\lambda_{1,\beta}(B_m)| \leq \tilde{\lambda}_{1,\beta}(\Omega_n) \leq -\beta\frac{\mathcal{H}^{d-1}(\partial^*\Omega_n)}{|\Omega_n|} = -\beta\frac{\mathcal{H}^{d-1}(\partial^*\Omega_n)}{m}.$$

It implies that

$$\mathcal{H}^{d-1}(\partial^*\Omega_n) \leq \frac{m}{\beta}|\lambda_{1,\beta}(B_m)|,$$

which gives us the required equiboundedness of the perimeters.

To conclude this section, we state a straightforward corollary to Theorem 3.6.1.

Corollary 3.6.3. *The problem*

$$\max\left\{\lambda_{k,\beta}(\Omega) : \Omega \subset \mathbb{R}^d \textit{ has finite perimeter and } |\Omega| = m\right\}$$

has at least a solution Ω for every $k \in \mathbb{N}$. Moreover, up to negligible sets, each optimal set is bounded and can be written as the union of at most $\frac{m|\lambda_{k,\beta}(B_m)|^d\omega^d}{\beta^d}+$ k equibounded and well separated sets of finite perimeter.

Proof. It is straightforward consequence of Theorem 3.6.1 applied with

$$F(x_1, \dots, x_k) := x_k.$$

□

3.7 Existence among (union of) simply connected open sets in $\mathbb{R}^2$

A natural idea in shape optimization problems is to look for optimal domains belonging to classes of open sets of $\mathbb{R}^2$, possibly relaxing the functional involved. In the case of the Robin eigenvalues with negative boundary parameter, it is interesting to consider the class of open, bounded sets $\Omega \subset \mathbb{R}^2$ that are and union of simply connected sets, with $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$. In the following, for the sake of brevity, as habitual in the field of shape optimization, we will say that an open set Ω is "simply connected" if it is union of simply connected sets, even if Ω is not connected by arcs. For instance, a disjoint union of two balls will be considered simply connected.

In the framework of simply connected sets, it is convenient to define the relaxed eigenvalues by using the topological boundary and not the reduced one. We will work in $H^1(\Omega)$, extending the function by zero in Ω^c and defining the traces on the boundary as in the SBV setting.

The idea to restrict ourselves on the class of unions of simply connected open sets has been inspired by the existence results in Section 6 of [11]. In that case, authors proved that in the class of open sets with fixed measure and whose topological boundaries have finite length, maximizers for the (relaxed) Steklov eigenvalues exist and are union of at most k simply connected sets (Theorem 6.4). Their argument is based on the fact that, for Steklov eigenvalues, starting from an admissible domain Ω of measure m, with possible holes, one can always build a simply connected competitor $\tilde{\Omega}$ whose Steklov eigenvalues are larger than the respective eigenvalues of Ω (see Lemma 6.5 and Lemma 6.6 in [11]). More precisely, $\tilde{\Omega}$ is built by filling in the possible holes of Ω, obtaining a set $\hat{\Omega}$ whose relaxed Steklov eigenvalues are larger than the Steklov eigenvalues of Ω; then, rescaling $\hat{\Omega}$ (by a scaling factor less than 1) in order to satisfy the measure constraint, we obtain the simply connected competitor $\tilde{\Omega}$ whose eigenvalues are still larger (since the scaling factor is less than 1, see Item (b) in Lemma 6.2 of [11]). In our framework, it is not possible to proceed in the same way, since we are not able to compare the Robin eigenvalues of Ω and $\hat{\Omega}$ (built as above) and we do not have any good scaling property. So, the simply connectedness of admissible domains will be a priori required.

We first give a suitable definition of relaxed eigenvalues.

Definition 3.7.1 (relaxed eigenvalues for simply connected open sets in $\mathbb{R}^2$)**.** Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and simply connected such that $\mathcal{H}^1(\partial\Omega) < +\infty$.

For every $k \in \mathbb{N}$ we set

$$\bar{\lambda}_{k,\beta}(\Omega) := \inf_{S \in \mathcal{S}_k} \sup_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx - \beta \int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2 \right]\, d\mathcal{H}^1}{\displaystyle\int_\Omega u^2\, dx}, \tag{3.24}$$

where $\mathcal{S}_k$ denotes the space of all k-dimensional subspaces of $H^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3.7.2. Let us observe that the definition above is correct. Indeed, as $\partial\Omega$ is $\mathcal{H}^1$-rectifiable and $u \in H^1(\Omega) \cap L^\infty(\Omega)$, if we consider the extension by zero on Ω, we obtain a function in $SBV(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ (still denoted by u) which has the jumps on $\partial\Omega$. In other words, we can speak about u^+ and u^- on $\partial\Omega$, the upper and lower approximate limits of u pointwisely $\mathcal{H}^1$-a.e. on $\partial\Omega$. In general, $u^+(x)$ and $u^-(x)$ are not different, since $\partial\Omega$ can be larger than the jump set J_u of u. For smooth domains, $\{u^+(x), u^-(x)\} = \{u(x), 0\}$, where $u(x)$ is the value of the boundary trace of u from the inside and 0 is th trace of the zero function outside the domain. We remark that it is necessary to work with the whole topological boundary, not only with the reduced one, to consider also simply connected sets with fractures where some functions in $H^1(\Omega)$ have different traces on the two sides of the fracture: this behavior is possible when dealing with the H^c-topology on this class of sets.

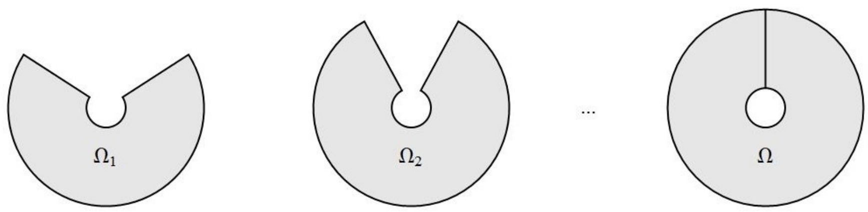


Figure 3.1: The sequence of open annular sectors $(\Omega_n)_n$ H^c-converges to the set Ω, an annulus cut on a radial segment. The function given by the polar coordinate ϑ (counted counter-clockwise starting from the cut) is $H^1(\Omega)$, but it has traces 0 and 2π respectively on the left side and the right side of the segment.

Moreover, in terms of the application of the min-max formula, the H^1 spaces of simply connected open sets having the same connected components with a different placement in the space, are equivalent, even if the connected components lie at null distance.

In view of this, we can assume that the connected components of the admissible sets are well separated.

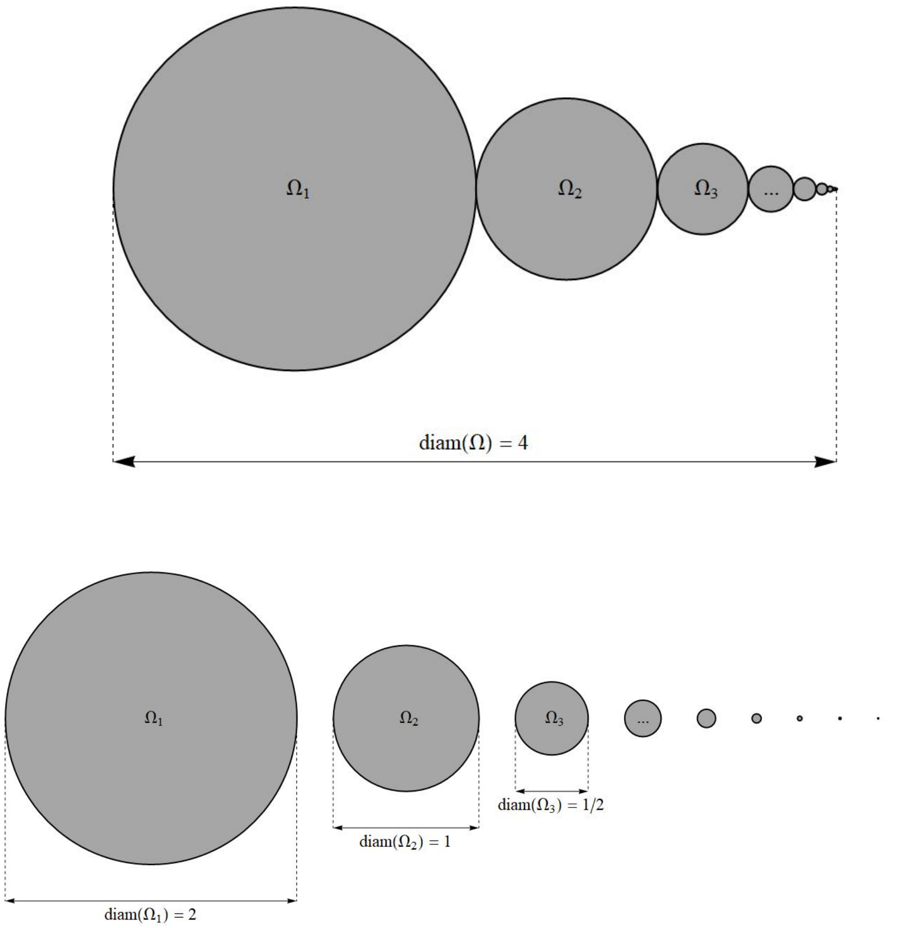


Figure 3.2: Two sets Ω and Ω' given by different disjoint unions of the open sets $\Omega_n := B_{1/2^n}$ $(n \in \mathbb{N})$. Both unions are simply connected according to our definition and the two spaces $H^1(\Omega) \cap L^\infty(\Omega)$ and $H^1(\Omega') \cap L^\infty(\Omega')$ are equivalent for the computation of $\lambda_{k,\beta}$.

The problem we are going to study is the following:

$$\max\Big\{ F(\bar{\lambda}_{1,\beta}(\Omega), \dots, \bar{\lambda}_{k,\beta}(\Omega)) : \\ \Omega \subset \mathbb{R}^2 \text{ open, bounded, simply connected, } \mathcal{H}^1(\partial\Omega) < +\infty,\ |\Omega| = m \Big\}, \tag{3.25}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ as in the hypotheses of Theorem 3.6.1.

It is a well known fact (see 1.4.18) that the class of sets of $\mathbb{R}^2$ which are contained in a box and are union simply connected open sets is compact with respect to the H^c-topology. Then, we need some conditions that ensure us that the limit set is still admissible for our problem. Moreover, we need some upper semicontinuity result for $\bar{\lambda}_{k,\beta}$; to this aim it is necessary to gain the lower semicontinuity of the curvilinear integral $\int_{\partial\Omega}[(u^+)^2+(u^-)^2]\;d\mathcal{H}^1$. The next theorem provides us the required semicontinuity of the boundary integral and is based on the ideas in Proposition 2.6 in [11], but there are some technical differences that it is worth to emphasize. The main difference is that in next theorem we consider functions with possible jumps, while in the above cited proposition authors proved the lower semicontinuity result for functions in $H^1(\mathbb{R}^2)$, then without jumps.

Theorem 3.7.3. *Let $\Omega \subseteq \mathbb{R}^2$ be open, $k \in \mathbb{N}$, and $(K_n)_n, K \subset \Omega$ be compact sets with at most k connected components, such that $\limsup_n \mathcal{H}^1(K_n) < +\infty$ and $K_n \to K$ in the Hausdorff metric. Let $u_n \in H^1(\Omega \setminus K_n)$ be such that*

$$\limsup_{n\to+\infty} \|u_n\|_{H^1(\Omega\setminus K_n)} + \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right]\;d\mathcal{H}^1 < +\infty. \tag{3.26}$$

Then, there exists $u \in H^1(\Omega \setminus K)$ such that, up to subsequences, we have

$$u_n \to u \quad \textit{strongly in } L^2_{loc}(\Omega),$$

$$\nabla u_n \rightharpoonup \nabla u \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^2),$$

and

$$\int_K \left[(u^+)^2 + (u^-)^2\right]\;d\mathcal{H}^1 \leq \liminf_{n\to+\infty} \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right]\;d\mathcal{H}^1. \tag{3.27}$$

Proof. Our proof is based on the same ideas as in Proposition 2.6 in [11], with some technical differences that will be highlighted step by step.

Without loss of generality we can assume $u_n \geq 0$ and $k = 1$ (i.e. K_n, K are connected). In view of the weak compactness in $L^2(\Omega)$ ($\|u_n\|_{L^2(\Omega)}$ is uniformly bounded), of the convergence in the sense of Hausdorff and in measure of the open set $\Omega \setminus K_n$ to $\Omega \setminus K$ and then of the convergence in the sense of Mosco of $H^1(\Omega \setminus K_n)$ to $H^1(\Omega \setminus K)$, there exists $u \in H^1(\Omega \setminus K)$ such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2),$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2).$$

Moreover, by (3.26), since $\limsup_n \mathcal{H}^1(K_n) < +\infty$, each function u_n^2 belongs to $SBV(\Omega)$ and then, by Proposition 5.1.4 (see Theorem 3.3 in [24]), we get that $u_n \to u$ strongly in $L^2_{loc}(\Omega)$. Notice that Proposition 5.1.4 is not sufficient to obtain (3.27), as in general K_n and K are larger than J_{u_n} and J_u, respectively.

Notice that we can not use directly Proposition 2.6 in [11] as there authors consider functions $u_n \in H^1(\mathbb{R}^2)$, then without jumps on K_n.

Without loss of generality, we can assume, by truncation, that there exists $M > 0$ such that, for every $n \in \mathbb{N}$, $u_n \le M$ a.e.; moreover, we can assume that Ω is bounded and smooth in order to work in a smooth neighbourhood of K.

We divide the proof of (3.27) in several steps.

Step 1. In view of the hypotheses on u_n, u, we deduce that $u_n, u, u_n^2, u^2 \in SBV(\Omega)$,

$$J_{u_n^2} = J_{u_n} \subseteq K_n, J_{u^2} = J_u \subseteq K$$

and that the SBV-traces $u_n^\pm, u^\pm$ are well defined, as the functions are in L^∞ and K_n, K are $\mathcal{H}^1$-countably rectifiable. Following the measure theoretic approach in Step 1 of the proof of Proposition 2.6 in[11], let us define the sequence of positive Radon measures $(\mu_n)_n \subset \mathcal{M}_b(\Omega)$ by setting, for any Borel set $A \subseteq \Omega$

$$\mu_n(A) := \int_{A \cap K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1$$

(in the above cited result the measure μ_n is defined considering the boundary integral of a function u_n that has no jumps on K_n; here, we consider both SBV-traces $u_n^\pm$). In view of the previous assumptions, the sequence $(|\mu_n|(\Omega))_n$ is equibounded, then we can assume that there exists $\mu \in \mathcal{M}_b(\mathbb{R}^2)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$. Let now be $\nu := \mathcal{H}^1 \lfloor K$. Since $u \in SBV(\Omega)$ and K is $\mathcal{H}^1$-countably rectifiable, for $\mathcal{H}^1$-a.e. point $x \in K$ the following facts hold true:

(a) K admits an approximate tangent line l_x at x, since K is $\mathcal{H}^1$-countably rectifiable;

(b) x is either an approximate jump point or a an approximate continuity point for u;

(c) the Radon-Nikodym derivative $d\mu/d\nu$ is given by

$$\frac{d\mu}{d\nu}(x) = \lim_{\rho \to 0^+} \frac{\mu\left(\overline{Q_{2\rho}(x)}\right)}{\nu\left(\overline{Q_{2\rho}(x)}\right)}.$$

Let us prove that

$$\frac{d\mu}{d\nu}(x) \ge u^+(x)^2 + u^-(x)^2. \tag{3.28}$$

for $\mathcal{H}^1$-a.e. $x \in K$. Indeed, if (3.28) holds, denoting by μ^a and μ^s respectively the absolutely continuous and the singular part of μ with respect to ν we have

$$\int_K \left[(u^+)^2 + (u^-)^2\right] \, d\mathcal{H}^1 \leq \int_K \frac{d\mu}{d\nu} \, d\mathcal{H}^1 = \mu^a(\Omega) \leq \mu^a(\Omega) + \mu^s(\Omega)$$
$$= \mu(\Omega) \leq \liminf_{n\to+\infty} \mu_n(\Omega) = \liminf_{n\to+\infty} \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1$$

i.e. the required lower semicontinuity in (3.27).

Step 2. Let $x \in K$ satisfying the previous properties (a), (b), (c) and show some geometric properties of K near x, using the same blow up argument as in Step 2 of Proposition 2.6 in [11] (the only difference is that, since we could have two different traces of the function u across K, a further rotation could be needed to associate u^+ and u^- respectively to the upper and lower halfplanes determined by l_x). Without loss of generality, we can suppose that $x = 0$ (we will then prove (3.28) in the form $\frac{d\mu}{d\nu}(0) \geq u^+(0)^2 + u^-(0)^2$) and that the approximate tangent line to K at the point 0, say $l := l_0$, is horizontal (i.e. $l \subset \{x_2 = 0\}$). Moreover, possibly rotating by π, we can assume that the approximate limit associated to the upper halfplane $\{x_2 > 0\}$ is u^+. For every $\varepsilon > 0$, let us set

$$K_\varepsilon := \frac{1}{\varepsilon} K;$$

by the definition of approximate tangent line, we obtain that

$$\mathcal{H}^1 \lfloor K_\varepsilon \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l \tag{3.29}$$

weakly* in $\mathcal{M}_b(\mathbb{R}^2)$ as ε goes to 0. Let us prove that, for every $r > 0$

$$K_\varepsilon \cap \overline{Q_{2r}(0)} \to l \cap \overline{Q_{2r}(0)} \tag{3.30}$$

in the Hausdorff topology as ε goes to 0.

Let $(\varepsilon_n)_n \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$. Since the Hausdorff topology is compact on the class of compact connected sets, for every fixed $m \in \mathbb{N}$, the sequence of compact connected sets

$$\left(K_{\varepsilon_n} \cap \overline{Q_{2m}(0)}\right)_n$$

converge to a compact connected set K_0^m up to a subsequence (possibly depending on m). In particular, we can choose, via a diagonal argument, a subsequence $(\varepsilon_{n_h})_h \subseteq (\varepsilon_n)_n$ that realizes the above H-convergence for every $m \in \mathbb{N}$, i.e.

$$K_{\varepsilon_{n_h}} \cap \overline{Q_{2m}(0)} \overset{h\to\infty}{\longrightarrow} K_m^0 \quad \forall\, m \in \mathbb{N}$$

in the Hausdorff topology. Let us remark that, for every $m \in \mathbb{N}$, it holds

$$K_0^m \subseteq K_0^{m+1}$$

and

$$K_0^{m+1} \cap \overline{Q_{2m}(0)} = K_0^m \cap \overline{Q_{2m}(0)}.$$

Let us set

$$K_0 := \bigcup_{m \in \mathbb{N}} K_0^m$$

and prove that $K_0 = l$.

- $K_0 \subseteq l$. Let us assume by contradiction that there exists $\xi \in K_0 \setminus l$. Since l is closed, there exists $\eta > 0$ such that $\overline{B_\eta(\xi)} \cap l = \emptyset$. Since $\mathcal{H}^1 \lfloor K_{\varepsilon_{n_h}} \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l$ weakly* in $\mathcal{M}_b(\mathbb{R}^2)$, then

$$0 = (\mathcal{H}^1 \lfloor l)(\overline{B_\eta(\xi)}) \geq \limsup_{h \to +\infty} (\mathcal{H}^1 \lfloor K_{\varepsilon_{n_h}})(\overline{B_\eta(\xi)}) = \lim_{h \to +\infty} \mathcal{H}^1(K_{\varepsilon_{n_h}} \cap \overline{B_\eta(\xi)}). \tag{3.31}$$

  $K_{\varepsilon_{n_h}}$ is connected and $\mathcal{H}^1$-rectifiable; hence, it is also connected by arcs. Let us consider a sequence $(\xi_{\varepsilon_{n_h}})_h$, converging to ξ, with $\xi_{\varepsilon_{n_h}} \in K_{\varepsilon_{n_h}}$. For sufficiently large $h \in \mathbb{N}$, every point $\xi_{\varepsilon_{n_h}}$ is connected to $0 \in K_{\varepsilon_{n_h}}$ by an arc of positive length contained in $K_{\varepsilon_{n_h}} \cap \overline{B_\eta(\xi)}$, against (3.31).

- $l \subseteq K_0$. If we assume by contradiction that there exists $\xi \in l \setminus K_0$, then, for some $\eta > 0$, we have $\overline{B_\eta(\xi)} \cap K_{\varepsilon_{n_h}} = \emptyset$ for sufficiently large $h \in \mathbb{N}$, against the weak* convergence $\mathcal{H}^1 \lfloor K_\varepsilon \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l$.

Then $K_0 = l$ and this implies (3.30).

Step 3. Now, we follow the same ideas as in Step 3 of Proposition 2.6 in [11] and look for a sufficient estimate to get 3.28 and conclude the proof of the theorem via a slicing argument. Such a sufficient estimate (see (3.34)) will be obtained with the same blow up argument in [11]. The main difference with our proof is that the limit function of our blow up is a piecewise constant function, with a possible jump on the line $\{x_2 = 0\}$, while the limit function in [11] is a constant function.

Let us observe that

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap K)}{2\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap K)}{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap l)} \cdot \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap l)}{2\varepsilon} = 1.$$

Then, for every positive, decreasing, infinitesimal sequence $(\varepsilon_m)_m$, it holds

$$\begin{aligned}\frac{d\mu}{d\nu}(0) &= \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{\nu\left(\overline{Q_{2\varepsilon_m}(0)}\right)} = \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{\mathcal{H}^1\left(\overline{Q_{2\varepsilon_m}(0)} \cap K\right)} \\ &= \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m}.\end{aligned}$$

Moreover, by the weak* convergence $\mu_n \overset{*}{\rightharpoonup} \mu$ we have

$$\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right) \geq \limsup_{m\to+\infty} \mu_n\left(\overline{Q_{2\varepsilon_m}(0)}\right).$$

Hence, there exists a sequence of index $(n_m)_m$ such that

$$\varepsilon_m^2 + \mu\left(\overline{Q_{2\varepsilon_m}(0)}\right) \geq \mu_{n_m}\left(\overline{Q_{2\varepsilon_m}(0)}\right)$$

and that, setting

$$\hat{K}_m := \frac{1}{\varepsilon_m} K_{n_m} \cap \overline{Q_2(0)},$$

we have the convergence

$$\hat{K}_m \to l \cap \overline{Q_2(0)}$$

in the Hausdorff metric.

Let us define the function v_m by

$$v_m(y) := u_{n_m}(\varepsilon_m y) \quad (y \in Q_2(0)).$$

Observe that, in view of the strong L^2-convergence $u_n \to u$, we can choose the sequence $(n_m)_m$ in such a way that

$$v_m \to u^\pm \quad \text{strongly in } L^2(Q_2(0)) \tag{3.32}$$

where $u^\pm$ is the piecewise constant function given by

$$u^\pm(x_1, x_2) := \begin{cases} u^+(0) & \text{if } x_2 > 0 \\ u^-(0) & \text{if } x_2 < 0 \end{cases}$$

(instead, in Step 3 of Proposition [11], the limit function of the blow up is the constant function $u(0)$). Moreover, the absolutely continuous part of the gradients of the functions v_m satisfy the convergence

$$\nabla v_m \to 0 \quad \text{strongly in } L^2(Q_2(0)) \tag{3.33}$$

Indeed, (3.32) is a consequence of the strong L^2-convergence $u_n \to u$ and of the convergence of $\hat{K}_m$ to $l \cap \overline{Q_2(0)}$. Convergence (3.33) is due to the strong L^2-convergence $\nabla u_n \to \nabla u$ and to the equality

$$\int_{Q_2(0)} |\nabla v_m(y)|^2 \, dy = \int_{Q_2(0)} |\nabla(u_{n_m}(\varepsilon_m y))|^2 \, dy = \int_{Q_{2\varepsilon_m}(0)} |\nabla u_{n_m}(x)|^2 \, dx;$$

in addition, (3.33) implies that $(\nabla u_m)_m$ is uniformly bounded in $L^2(Q_2(0); \mathbb{R}^2)$.

In view of the choice of $(n_m)_m$, we obtain

$$\begin{aligned}\frac{d\mu}{d\nu}(0) &= \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m} \geq \liminf_{m\to+\infty} \frac{\mu_{n_m}\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m} \\ &= \liminf_{m\to+\infty} \frac{1}{2\varepsilon_m} \int_{K_{n_m}\cap\overline{Q_{2\varepsilon_m}(0)}} \left[(u_{n_m}^+)^2 + (u_{n_m}^-)^2\right] \, d\mathcal{H}^1 \\ &= \frac{1}{2} \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1.\end{aligned}$$

Hence, if we prove that

$$\frac{1}{2} \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1 \geq u^+(0)^2 + u^-(0)^2, \tag{3.34}$$

we will get estimate (3.28), concluding the proof of the theorem. We point out that all the arguments to perform the previous extractions of subsequences are the same as in Step 3 of Proposition 2.6 in [11].

Step 4. To prove (3.34), we consider the one-dimensional sections of the functions v_m and u. We follow again the approach of Proposition 2.6 in [11], more precisely of the Step 4; the main difference is that now we have to consider "piecewise Sobolev" functions with a finite number of jumps. So, an important difficulty is that we have to gain some convergence of the traces of that functions across their jump sets. The key point of that approach is to prove that, up to subsequences, the piecewise Sobolev functions weakly converge in their intervals of approximate continuity. In such a way, since the weak H^1 convergence entails the uniform convergence, we gain the required continuity of the traces.

Now, in view of the countably $\mathcal{H}^1$-rectifiability of $\hat{K}_m$, we can apply the area formula (Theorem 2.71 in [2]) to the left hand side below, obtaining the inequality

$$\begin{aligned}&\liminf_{m\to+\infty} \int_{-1}^{1} \int_{(\hat{K}_m)_{x_1}} \left[v_m^+(x_1,s)^2 + v_m^-(x_1,s)^2\right] \, d\mathcal{H}^0(s) \, dx_1 \\ &\leq \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1,\end{aligned}$$

where

$$(\hat{K}_m)_{x_1} := \left\{ s \in [-1,1] : (x_1, s) \in \hat{K}_m \right\}.$$

Let us consider a sequence $(m_k)_k$ realizing the liminf in the left hand side above, i.e.

$$\begin{aligned} &\lim_{k\to+\infty} \int_{-1}^{1} \int_{(\hat{K}_{m_k})_{x_1}} \left[v_{m_k}^{+}(x_1,s)^2 + v_{m_k}^{-}(x_1,s)^2 \right] \, d\mathcal{H}^0(s)\, dx_1 \\ &\leq \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^{+})^2 + (v_m^{-})^2 \right] \, d\mathcal{H}^1. \end{aligned} \tag{3.35}$$

Notice that we can assume

$$\limsup_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^{+})^2 + (v_m^{-})^2 \right] \, d\mathcal{H}^1 < +\infty \tag{3.36}$$

and that, for a.e. $x_1 \in [-1,1]$, $\mathcal{H}^0((\hat{K}_m)_{x_1}) < +\infty$. We deduce that, for a.e. $x_1 \in [-1,1]$, $v_{m_k}^2(x_1,\cdot)$ is a $SBV(-1,1)$ function, with a finite number of jumps (in $(\hat{K}_m)_{x_1}$) (so that it is piecewise H^1 in $(-1,1)$) and it holds

$$v_{m_k}(x_1,\cdot) \to u^{\pm} \quad \text{strongly in } L^2(-1,1) \tag{3.37}$$

where we denoted for brevity again by $u^{\pm}$ the one dimensional section of the piecewise constant function $u^{\pm}$. Now, let us assume that, for every $x_1 \in]-1,1[$, the slices $(\hat{K}_{m_k})_{x_1}$ are definitely non-empty, i.e. there exists $N(x_1) \in \mathbb{N}$ such that

$$(\hat{K}_{m_k})_{x_1} \neq \emptyset \tag{3.38}$$

for every $k \geq N(x_1)$. Then, it makes sense to consider the convergence

$$(\hat{K}_{m_k})_{x_1} \to (x_1, 0) \tag{3.39}$$

in the Hausdorff metric (it is a consequence of the fact that $\hat{K}_{m_k} \to l \cap \overline{Q_2(0)}$).

Now, let us define

$$i(x_1,k) := \inf \left\{ y : y \in (\hat{K}_{m_k})_{x_1} \right\},$$

$$s(x_1,k) := \sup \left\{ y : y \in (\hat{K}_{m_k})_{x_1} \right\}.$$

Notice that $v_{m_k}(x_1,\cdot) \in H^1\left( ]-1,1[\setminus (\hat{K}_{m_k})_{x_1} \right)$, then, in particular, the function is in both in $H^1(]-1, i(x_1,k)[)$ and $H^1(]s(x_1,k),1[)$. Moreover, since both points $i(x_1,k)$ and $s(x_1,k)$ converge to 0, the open interval $]i(x_1,k), s(x_1,k)[$ converges to the empty set and the intervals $]-1, i(x_1,k)[$ and $]s(x_1,k),1[$ converge respectively to $]-1,0[$ and $]0,1[$. Combining all this facts, we deduce that it is sufficient to prove that $v_{m_k}(x_1, \cdot)$ converges weakly in $H^1(]-1, i(x_1,k)[)$ and $H^1(]s(x_1,k), 1[)$ respectively to $u^-(0)$ and $u^-(0)$, then uniformly. More precisely, we apply the same arguments as in Step 4 of Proposition 2.6 in [11] on each one of the intervals $]-1, i(x_1,k)[$ and $]s(x_1,k), 1[$, possibly rescaling all the moving domains in such a way that all the intervals have length 1.

Let us fix $\varepsilon > 0$. In view of and of the equiboundedness of $(\|\nabla v_m\|_{L^2(Q_2(0))})_m$, we observe that the absolutely continuous parts of the derivatives of v_{m_k} are equibounded in $L^2(-1,1)$; then, applying estimates (3.35), (3.36) Fatou's Lemma we have

$$\begin{aligned}
&\int_{-1}^{1} \liminf_{k\to+\infty} \Bigg[\varepsilon\Big(\int_{-1}^{1} |\partial_2 v_{m_k}(x_1,s)|^2 ds \Big) \\
&\qquad\qquad + \int_{(\hat{K}_{m_k})_{x_1}} \left[v^+_{m_k}(x_1,s)^2 + v^-_{m_k}(x_1,s)^2 \right] d\mathcal{H}^0(s) \Bigg] \, dx_1 \\
&\leq \liminf_{k\to+\infty} \Bigg[\varepsilon\Big(\int_{Q_2(0)} |\partial_2 v_{m_k}|^2 dx \Big) \\
&\qquad\qquad + \int_{-1}^{1} \int_{(\hat{K}_{m_k})_{x_1}} \left[v^+_{m_k}(x_1,s)^2 + v^-_{m_k}(x_1,s)^2 \right] d\mathcal{H}^0(s) \Bigg] \, dx_1 < +\infty.
\end{aligned}$$

The integrand function on the left hand side is finite for a.e. $x_1 \in]-1,1[$; then, there exists a subsequence $(v_{m_{k_h}})_h$ (possibly depending on x_1) that realizes the above liminf and so

$$\begin{aligned}
&\lim_{h\to+\infty} \Bigg[\varepsilon\Big(\int_{-1}^{1} |\partial_2 v_{m_{k_h}}(x_1,s)|^2 ds \Big) \\
&\qquad + \int_{(\hat{K}_{m_{k_h}})_{x_1}} \left[v^+_{m_{k_h}}(x_1,s)^2 + v^-_{m_{k_h}}(x_1,s)^2 \right] d\mathcal{H}^0(s) \Bigg] < +\infty.
\end{aligned} \tag{3.40}$$

We deduce that $(v_{m_{k_h}}(x_1,\cdot))_h$ is uniformly bounded in $H^1\Big((-1,1) \setminus (\hat{K}_{m_k})_{x_1}\Big)$ and then that the sequences

$$\Big(\|v_{m_{k_h}}(x_1,\cdot)\|_{H^1(]-1,i(x_1,k)[)}\Big)_h \quad , \qquad \Big(\|v_{m_{k_h}}(x_1,\cdot)\|_{H^1(]s(x_1,k),1[)}\Big)_h$$

are uniformly bounded. In view of the previous remarks, we deduce that we have the required convergence of the traces.

Now, we can conclude the proof recalling again the same arguments as in

Proposition 2.6 in [11]. Applying Fatou's Lemma we get

$$\begin{aligned}
u^+(0)^2 + u^-(0)^2 &= \int_{(x_1,0)} u^+(0)^2 + u^-(0)^2 \, d\mathcal{H}^0(s) \\
&\le \liminf_{h\to+\infty} \int_{(\hat{K}_{m_{k_h}})_{x_1}} v^+_{m_{k_h}}(x_1,s)^2 + v^-_{m_{k_h}}(x_1,s)^2 \, d\mathcal{H}^0(s) \\
&\le \liminf_{k\to+\infty} \Bigg[\varepsilon \Big(\int_{-1}^{1} |\partial_2 v_{m_k}(x_1,s)|^2 ds \Big) \\
&\quad + \int_{(\hat{K}_{m_k})_{x_1}} \left[v^+_{m_k}(x_1,s)^2 + v^-_{m_k}(x_1,s)^2 \right] d\mathcal{H}^0(s) \Bigg].
\end{aligned}$$

Integrating in the variable x_1 and using Fatou's Lemma and (3.35), we conclude that

$$\begin{aligned}
2\left(u^+(0)^2 + u^-(0)^2\right) &= \int_{-1}^{1} \left[u^+(0)^2 + u^-(0)^2 \right] \, dx_1 \\
&\le \liminf_{k\to+\infty} \Bigg[\varepsilon \Big(\int_{Q_2(0)} |\partial_2 v_{m_k}|^2 dx \Big) \\
&\quad + \int_{-1}^{1} \int_{(\hat{K}_{m_k})_{x_1}} \left[(v^+_{m_k})^2 + (v^-_{m_k})^2 \right] d\mathcal{H}^0(s) \, dx_1 \Bigg] \\
&\le \varepsilon C + \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v^+_m)^2 + (v^-_m)^2 \right] \, d\mathcal{H}^1.
\end{aligned}$$

Let us assume that (3.38) does not hold, i.e. that there exist $a \in]-1,1[$ and a subsequence $(m_{k_h})_h$ (generally depending on a) such that $(\hat{K}_{m_{k_h}})_a = \emptyset$. Then, for every $x_1 \neq a$ (take for instance $x_1 > a$) and $h \in \mathbb{N}$ sufficiently large, it necessarily holds $(\hat{K}_{m_{k_h}})_{x_1} \neq \emptyset$; otherwise, in view of the Hausdorff convergence $\hat{K}_{m_{k_h}} \to l \cap \overline{Q_2(0)}$, we would have

$$\hat{K}_{m_{k_h}} \cap (]a, x_1[\times \mathbb{R}) \neq \emptyset$$

and then the set $K_{m_{k_h}}$ would not be connected, obtaining a contradiction. Hence, we apply the previous argument to $x_1 \neq a$ to get (3.34) also in this case, concluding the proof. $\square$

To ensure compactness of maximizing sequences, we need some results analogous to Proposition 3.4.3 and Lemma 3.5.2. Fortunately, these results are still valid in our context, with analogous proofs. The only difference is that, in the integrals over the curves $\partial\Omega$, we have to estimate the traces of an admissible function possibly from both sides of $\partial\Omega$.

Proposition 3.7.4. *Let $\Omega \subset \mathbb{R}^2$ open and simply connected with $|\Omega| = m$; let $A > 0$ such that $\bar{\lambda}_{k,\beta}(\Omega) > -A$. Then, Ω has at most N connected components $\Omega_1, \dots, \Omega_N$, with $N = N(m, \beta, k, A)$, and diam$(\Omega_j) \leq D(m, \beta, k, A)$, i.e. the diameters of connected components are uniformly bounded.*

Proof. The proof is based on the same argument as in the proof of Proposition 3.4.3 (see also Proposition 6.3 in [11]): we use the counterpart of Lemma 3.4.1 for two-dimensional simply connected sets with $\mathcal{H}^1$-finite boundary; to be precise, in this context, we can consider as critical measure $m^* := \frac{1}{2}\left(\frac{\beta}{Ac}\right)^2$. Then, we prove the uniform bound on the number of well separated parts of Ω (again as in Proposition 3.4.3) and, finally, we deduce by Remark 3.7.2 that the well separated sets are exactly the connected components of Ω. □

Lemma 3.7.5 (Isoperimetric control of the relaxed spectrum)**.** *Let $\Omega \subset \mathbb{R}^2$ an open simply connected set with $\mathcal{H}^1(\partial\Omega) < +\infty$ and let $|\Omega| = m$. Then, there exist two a positive constants C_1, C_2 such that*

$$\bar{\lambda}_{k,\beta}(\Omega) \leq -C_2 \left(\frac{k - \frac{\beta}{C_1}\mathcal{H}^1(\partial\Omega)}{|\Omega|} \right)^{-} + \frac{C_1 k}{|\Omega|}\chi_{]0,+\infty[}\left(k - \frac{\beta}{C_1}\mathcal{H}^1(\partial\Omega)\right)$$

Proof. The proof of this lemma is based on the same arguments as in the proof of Lemma 3.5.2 in the case $d = 2$. We apply Lemma 3.5.1 to the finite non-atomic measures $\mathcal{H}^1\lfloor\partial\Omega$ and $\mathcal{L}^2\lfloor\Omega$ and, after defining the annuli A_i and the test functions h_i, we obtain the constants

$$C_2 := \max_{i=1,\dots,2k} \int_{2\cdot A_i} |\nabla h_i|^2, \qquad C_1 := \frac{C_2}{\gamma_2}$$

(notice that the constant C_1 here is half the constant C_1 found in Lemma 3.5.2, because we consider both traces of h_i on $\partial\Omega$). □

Remark 3.7.6. Another interesting way to prove the bound on the diameter of good candidates is to use Lemma 3.7.5 and the fact that we are in dimension two. Indeed, Lemma 3.7.5, applied to the sets of a maximizing sequence $(\Omega_n)_n$, implies that the sequence $(\mathcal{H}^1(\partial\Omega_n))_n$ is uniformly bounded, with at most $N(A, m, \beta, k)$ connected components. In view of this fact, the sum of the diameters of the connected components of Ω_n is bounded by a constant independent of $n \in \mathbb{N}$. Moreover, for every admissible set Ω, the variational expression of $\bar{\lambda}_{k,\beta}(\Omega)$ involves the functional space $H^1(\Omega) \cap L^\infty(\Omega)$, that is invariant under translations of the connected components of Ω. Notice that N is independent on n; then we can conclude that $(\text{diam}(\Omega_n))_n$ is uniformly bounded.

Now we are able to prove the main result in this section.

Theorem 3.7.7. *Problem* (3.25) *admits at least a solution* $\Omega \subset \mathbb{R}^2$ *in the class of open, bounded, simply connected sets with* $|\Omega| = m$ *and* $\mathcal{H}^1(\partial\Omega) < +\infty$. *Moreover,* Ω *has at most* $N(k, \beta, m)$ *connected components, the uniform bound given as in Proposition 3.7.4.*

Proof. Let $(\Omega_n)_n$ be a maximizing sequence of open, simply connected sets in $\mathbb{R}^2$ such that $\mathcal{H}^1(\partial\Omega_n) < +\infty$ and $|\Omega_n| = m$ for every $n \in \mathbb{N}$. Let us prove that there exists an open, simply connected set $\Omega \subset \mathbb{R}^2$ such that, up to subsequences, $\Omega_n \to \Omega$ in the H^c-topology, $\mathcal{H}^1(\Omega) < +\infty$, $|\Omega| = m$ and, for every $h \in \mathbb{N}$ it holds

$$\bar{\lambda}_{h,\beta}(\Omega) \geq \limsup_{n\in\mathbb{N}} \bar{\lambda}_{h,\beta}(\Omega_n). \tag{3.41}$$

Let us observe that, as in Theorem 3.6.1, we can suppose that $\bar{\lambda}_{h,\beta}(\Omega_n) \geq \lambda_{\beta,k}(B_m)$ and then, in view of Lemma 3.7.5, we have

$$\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial\Omega_n) < +\infty.$$

Moreover, as shown in Proposition 3.7.4, Ω_n has at most $N(h, \beta, m)$ uniformly bounded (and possibly empty) connected components, say $\Omega_n^1, \dots, \Omega_n^N$. Then, the sequence $(\Omega_n)_n$ is uniformly bounded in a fixed open bounded set $B \subset \mathbb{R}^2$, since $\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial\Omega_n) < +\infty$ implies that $\sup_{n\in\mathbb{N}} \operatorname{diam}(\Omega_n) < +\infty$ in $\mathbb{R}^2$. Moreover, as $\#(\Omega_n^c) = 1$, there exists an open set $\Omega \subset B$ such that $\Omega_n \to \Omega$ in the H^c-topology and $\#(\Omega^c) = 1$, i.e. Ω is simply connected (see Remark 1.4.18). In view of the structure of the well separated parts of Ω_n, the boundary of each of them is connected ($\#(\partial\Omega_n^j) = 1$) and then $\#(\partial\Omega_n) \leq N(h, \beta, d, m)$. In view of Golab Theorem 1.4.11, we deduce that $\mathcal{H}^1(\partial\Omega) < +\infty$. These uniform bounds on $\#(\partial\Omega_n)$ and $\mathcal{H}^1(\partial\Omega_n)$ ensure us that $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\mathbb{R}^2)$ (see, for instance, Theorem 7.4.7 in [17]), then

$$|\Omega| = \lim_{n\to+\infty} |\Omega_n| = m,$$

i.e Ω is an admissible set for (3.25). Moreover, the H^c-convergence of Ω_n to Ω implies that $H^1(\Omega_n) \to H^1(\Omega)$ in the sense of Mosco (see Proposition 1.5.3).

Let now $\varepsilon > 0$ and let S an admissible vector space in the min-max formula for $\bar{\lambda}_{h,\beta}(\Omega)$ such that

$$\bar{\lambda}_{h,\beta}(\Omega) \geq \max_{u\in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \beta \int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2\right] \, d\mathcal{H}^1}{\displaystyle\int_\Omega u^2 \, dx} - \varepsilon. \tag{3.42}$$

Let $\{u_j : j = 1, \dots, h\}$ a $L^2(\Omega)$-orthonormal basis for S. Then, for every $j = 1, \dots, h$, there exists $v_j^n \in H^1(\Omega_n)$ such that, denoting by the same symbol the extension by zero of a function outside its support, $v_j^n \to u_j$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v_j^n \to \nabla u_j$ strongly in $L^2(\mathbb{R}^2; \mathbb{R}^2)$. Now, since $\{u_1, \dots, u_h\}$ is $L^2(\Omega)$-orthonormal and $\Omega_n \to \Omega$ in $L^1(\mathbb{R}^2)$, we deduce that, for $n \in \mathbb{N}$ sufficiently large, $\{v_1^n, \dots, v_h^n\}$ can be chosen linearly independent in $L^2(\Omega)$. Let $S_n := \operatorname{span}\{v_1^n, \dots, v_h^n\}$; it is an admissible subspace for the computation of $\bar{\lambda}_{h,\beta}(\Omega)$. Let

$$v^n = \sum_{j=1}^{h} \alpha_j^n v_j^n \in S_n$$

realizing the maximum for the relaxed Rayleigh quotient $\bar{R}$ on S_n:

$$\max_{w \in S_n} \bar{R}(w) = \bar{R}(v^n).$$

Without loss of generality, we can assume

$$\sum_{j=1}^{h} (\alpha_j^n)^2 = 1.$$

Then, up to subsequences, $\alpha_j^n \to \alpha_j$ in $\mathbb{R}$, with

$$\sum_{j=1}^{h} (\alpha_j)^2 = 1.$$

Setting

$$v = \sum_{j=1}^{h} \alpha_j u_j,$$

we have that $v \in S \setminus \{0\}$, $v^n \to v$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v^n \to \nabla v$ in $L^2(\mathbb{R}^2; \mathbb{R}^2)$. Using (3.42), the continuity of the volume integrals and the lower semicontinuity of the boundary integral (see Theorem 3.7.3), we have

$$\begin{aligned}
\limsup_{n \to +\infty} \bar{\lambda}_{h,\beta}(\Omega_n) &\le \limsup_{n \to +\infty} \sup_{w \in S_n} \bar{R}(w) \le \limsup_{n \to +\infty} \bar{R}(v^n) + \varepsilon \\
&\le \bar{R}(v) + \varepsilon \le \max_{u \in S \setminus \{0\}} \bar{R}(u) \le \bar{\lambda}_{h,\beta}(\Omega) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ we obtain that $\bar{\lambda}_{h,\beta}(\cdot)$ is upper semicontinuos for maximizing sequences, which are also compact; then reasoning in 3.6.1, we conclude that (3.25) admits a solution. $\square$

3.8 Existence of maximizers under geometric constraints

So far we took into account weak formulations of Problem (3.1) in different setting by the class of Lipschitz domains. In this sections we will restrict ourselves to families of Lipschitz domains satisfying some geometrical constraints. In particular, we will present some results holding in any dimension d for uniformly regular sets and for convex sets.

3.8.1 Existence among uniformly regular sets

An existence result for Problem (3.1) is obtained when we focus on the class of Lipschitz domains of fixed volume satisfying some ε-cone property. Recalling the notation of Chapter 1, we denote by $C_{\varepsilon,\varepsilon}$ the cone of height ε and opening ε.

To begin, let us fix $m > 0$ and $\varepsilon > 0$ such that the family

$$\left\{\Omega \subset \mathbb{R}^d : |\Omega| = m, \Omega \in \mathcal{C}_\varepsilon\right\}$$

is non-empty.[3] That assumption is usual when the admissible sets satisfy some ε-cone property and, at the same time, a measure constraint (see, for instance the statement of Theorem 4.3.1 in [60]).

We consider the maximization problem

$$\max\left\{F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) : \Omega \subset \mathbb{R}^d, |\Omega| = m, \Omega \in \mathcal{C}_\varepsilon, \right\}, \tag{3.43}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is as in the hypotheses of Theorem 3.6.1.

Observe that, instead of considering the relaxed version of the eigenvalues $\tilde{\lambda}_{k,\beta}$, we can work with the classical eigenvalues $\lambda_{k,\beta}$, since for Lipschitz domains the two definitions coincide. To prove the following theorem we follow again the ideas in Theorem 5.6 in [11] and in Section 3.6.

Theorem 3.8.1 (existence of maximizers in $\mathcal{C}_\varepsilon$)**.** *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfying the hypotheses of $m > 0$ and $\varepsilon > 0$ such that the family $\left\{\Omega \subset \mathbb{R}^d : |\Omega| = m, \Omega \in \mathcal{C}_\varepsilon\right\}$ is non-empty. Then, Problem (3.43) has at least a solution. Moreover, every solution is a bounded Lipschitz domain with at most N connected components, with the bounds on the diameter and on the number of connected components depending only on $m, \beta, d, k, \varepsilon$.*

[3] The family of sets above is empty, e.g., if m is smaller than the volume of a cone $C(x, \xi, \varepsilon)$, see Definition 1.4.19.

Proof. The proof uses the same arguments as in Theorem 3.6.1. First of all, let us observe that, in view of the assumptions on m and ε, there exists at least an admissible set $E \in \mathcal{C}_\varepsilon$ such that $|E| = m$. Let $(\Omega_n)_n$ be a maximizing sequence for (3.43); we can assume without loss of generality that $\lambda_{k,\beta}(\Omega_n) \geq \lambda_{k,\beta}(E)$ for every $n \in \mathbb{N}$. As in the case of measurable sets, by Proposition 3.5.2 we have that

$$\sup_n \mathcal{H}^{d-1}(\partial\Omega_n) < +\infty. \tag{3.44}$$

Moreover, by 3.4.3, Ω_n is disjoint union of at most N equibounded open Lipschitz sets Ω_n^j satisfying the ε-cone property. In view of the uniform regularity, the uniform bound on $\mathrm{diam}(\Omega_n^j)$ depends also on ε and even the upper bound N on the number of the connected components depends on ε (N cannot exceed the value $m/|C(x,\xi,\varepsilon)|$ for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{S}^{d-1}$, since every connected component has to contain at least a cone of the same size).

So, by Proposition 1.4.25, we deduce that there exists an open set $\Omega \in \mathcal{C}_\varepsilon$ such that, up to subsequences,

$$\Omega_n \xrightarrow{H^c} \Omega,\ \chi_{\Omega_n} \xrightarrow{L^1(\mathbb{R}^d)} \chi_\Omega \text{ and } \mathcal{H}^{d-1}(\partial\Omega_n) \to \mathcal{H}^{d-1}(\partial\Omega)$$

(so, in particular, $|\Omega| = m$).

To show that such admissible set Ω is a solution of the problem (3.43), we argue in the same way as in Theorem 3.6.1. $\square$

Notice that the bound $m/|C(x,\xi,\varepsilon)|$ for $\#\Omega$ is not sharp: a deeper analysis could give us a strictly smaller value.

3.8.2 Existence among convex sets

If we restrict ourselves on convex sets of given volume, Problem (3.1) still admits a solution. The interest in studying the problem is that the convexity hypothesis provides extra compactness that make some arguments very immediate. For instance, isoperimetric control of the spectrum and uniform bounds on the diameter can be obtained in a different and easier way than in the case of measurable sets (and than also differently than in [11]). In the following of this paragraph, we will mainly point out such simplifications.

Let us fix $m > 0$ and consider the maximization problem

$$\max\left\{F(\lambda_{1,\beta}(\Omega),\dots,\lambda_{k,\beta}(\Omega)) : \Omega \subset \mathbb{R}^d \text{ convex}, |\Omega| = m\right\}, \tag{3.45}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is as in the hypotheses of Theorem 3.6.1.

First, we state Lemma 3.8.2 and Lemma **??**, that are the counterpart we already proved in the context of sets of finite perimeter.

Lemma 3.8.2. *Let $\Omega \subset \mathbb{R}^d$ be an open convex set such that $|\Omega| = m$ and let $A > 0$ such that $\lambda_{k,\beta}(\Omega) > -A$. Then the diameter of Ω is less than a positive constant $D = D(m, \beta, d, k, A)$.*

Proof. The proof follows by Lemma 3.5.2, but in this framework it can be obtained even arguing by contradiction. Indeed, if we suppose that there exists a sequence $(\Omega_n)_n$ of admissible convex sets of measure m such that $\lambda_{k,\beta}(\Omega_n) > -A$ and $\mathrm{diam}(\Omega_n)$ diverges, this implies, in view of John's Ellipsoid Theorem 1.4.15, that the sequence $(\rho_n)_n$ of the inradii of the inscribed ellipsoids $(I_n)_n$ vanishes as n goes to $+\infty$ (and then, the sequence $(d\rho_n)_n$ of inradii of the circumscribed ellipsoids $(C_n)_n$ vanishes, too). By 1.4.14(iii) we have that both $\mathcal{H}^{d-1}(\partial I_n)$ and $\mathcal{H}^{d-1}(\partial C_n)$ diverge, and then $\mathcal{H}^{d-1}(\partial \Omega_n)$ diverges too (notice that $\mathcal{H}^{d-1}(\partial \Omega_n)$ has the same asymptotic behaviour of $\mathcal{H}^{d-1}(\partial I_n)$ and $\mathcal{H}^{d-1}(\partial C_n)$). Then, $\lambda_{k,\beta}(\Omega_n)$ negatively diverges, in contradiction with the lower bound $-A$. $\square$

Remark 3.8.3. A shorter way to prove the previous lemma by contradiction is the following. If the diameter for good candidates to maximize $\lambda_{k,\beta}(\Omega)$ was not unifromly bounded, then $\mathcal{H}^{d-1}(\partial \Omega_n) \to +\infty$ as in the previous Lemma, obtaining a contradiction with the following Lemma 3.8.4.

Lemma 3.8.4 (Isoperimetric control for convex sets)**.** *Let $\Omega \subset \mathbb{R}^d$ a closed convex set such that $|\Omega| = m$. Then, there exist two positive constants $C_1 = C_1(d)$, $C_2 = C_2(d)$ such that*

$$\lambda_{k,\beta}(\Omega) \le -C_2 \left(\frac{k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{\beta}{C_1} \mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} \right)^{-} + \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}} \chi_{]0,+\infty[} \left(k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1} \mathcal{H}^{d-1}(\partial\Omega) \right).$$

Remark 3.8.5 (uniform bound on the perimeters of good candidates)**.** As in Theorem 3.6.1, the previous lemma provides the estimate

$$\sup_n \mathcal{H}^{d-1}(\partial\Omega_n) < +\infty.$$

This last uniform bound could also be deduced by proposition 1.4.14(i): as all the convex sets Ω_n are all contained in a ball whose diameter is given in 3.8.2, all their perimeters are uniformly bounded by the surface area of the boundary of that ball.

The main result of this section is the following.

Theorem 3.8.6 (existence of a convex maximizer)**.** *Problem* (3.45) *has at least a solution.*

Proof. The thesis is achieved once seen that every maximizing sequence is compact and using the upper semicontinuity of the functional. □

3.9 Existence results with perimeter constraints

If we infer a priori a uniform bound on the perimeter, a result of optimality can be found in all situations we has been found replacing the volume constraint with a perimeter constraint; in such cases the isoperimetric control of the spectrum is already achieved, as we bound a priori the perimeters of admissible sets. A result of such type can be found in Proposition 4.1 in [20], where authors give the proof of the existence of a finite perimeter set maximizing

$$\max\left\{\tilde{\lambda}_{1,\beta}(\Omega) : |\Omega| < +\infty,\ P(\Omega) \le c\right\},$$

basing their arguments on the analysis of maximizing sequences presented in [18]. Moreover, in [20], authors point out that is not clear whether optimal shapes saturate the constraint on the perimeter, i.e. we do not know whether an optimal shape Ω satisfy the equality $P(\Omega) = c$. This fact is due to the uncontrollable behaviour of $\lambda_{1,\beta}(\Omega)$ under rescaling. The following proposition generalizes Proposition 4.1 in [20] (having, essentially, the same proof) to the same family of functionals in Theorem 3.6.1; we only sketch the proof, for which we refer to Proposition 4.1 in [20] and Theorem 3.6.1.

Proposition 3.9.1. *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfy the same hypotheses as in Theorem 3.6.1. Then,*

$$\max\left\{F(\tilde{\lambda}_{1,\beta}(\Omega), \dots, \tilde{\lambda}_{k,\beta}(\Omega)) : |\Omega| < +\infty,\ P(\Omega) \le c\right\} \tag{3.46}$$

admits a solution of finite perimeter with a uniformly bounded number of well separated sets of finite perimeter lying at positive distance.

Proof. The compactness of a maximizing sequence $(\Omega_n)_n$ with respect to the convergence in measure can be achieved in the same way as in Proposition 4.1 of [20], since the sequence $(P(\Omega_n))_n$ is uniformly bounded from above (by c) and also the diameters are uniformly bounded in view of Proposition 3.4.3. Moreover, using the same arguments as in Proposition 4.1 of [20], we can observe that there exist two positive constants $0 < m_1 < m_2$ such that $m_1 < |\Omega_n| < m_2$. Naming Ω a limit set of the compact sequence $(\Omega_n)_n$, we deduce that Ω is a set with finite and, above all, strictly positive measure.

Moreover, since the perimeters are lower semicontinuos (see again the proofs of Theorem 3.6.1 or Proposition 4.1 in [20]), we deduce that $P(\Omega) \leq c$; then, Ω is an admissible set and, in view of the upper semicontinuity of every relaxed eigenvalue $\tilde{\lambda}_{1,\beta}(\cdot)$, we can conclude that Ω is a maximizer for Problem (3.46).

□

Remark 3.9.2. In the same way, every other existence theorem throughout the chapter (see Theorem 3.7.7 for planar simply connected sets, Theorem 3.8.1 for uniformly regular sets and Theorem 3.8.6 for convex sets in any dimension) can be rephrased replacing the measure constraint with an upper bound on the perimeters. We omit the proofs of this new results since they are a straightforward adaptation of the proofs of the original theorems (where the admissible sets satisfy a measure constraint), keeping into account the same remarks as in of Proposition 3.9.1 and in Proposition 4.1 in [20] (in order to consider admissible sets satisfying an upper bound on the perimeter).

3.10 Further remarks

The smoothness of the boundary of optimal sets is an unsolved problem, which probably is very difficult. For now, progress in this direction was done only for the spectral problems involving Dirichlet boundary conditions or Robin boundary conditions with positive parameter (but in this last case only for local minimizers of energy type functionals). Provided the optimal set was smooth and $\lambda_{k,\beta}$ was simple, the optimality condition reads (see [5])

$$\int_{\partial\Omega} \Big(|\nabla_{\partial\Omega} u|^2 - (\lambda_{k,\beta}(\Omega) + \beta^2 + \beta\mathcal{H})u^2 \Big) V \cdot n \, d\mathcal{H}^{d-1} = 0, \tag{3.47}$$

for every smooth vector field satisfying $\int_{\partial\Omega} V \cdot n \, d\mathcal{H}^{d-1} = 0$. Above, $\mathcal{H}$ stands for the mean curvature of $\partial\Omega$.

The following inequality holds true

$$\lambda_{1,\beta}(\Omega) \leq -\beta^2.$$

We refer the reader to Giorgi and Smits [53, Theorem 2.3] and to Daners and Kennedy [40, Lemma 2.1] in the context of Lipschitz sets, but it can naturally be extended to the first relaxed eigenvalue in both frameworks of measurable or simply connected sets. For smooth planar domains the following inequality was proved in [5]

$$\lambda_{1,\beta}(\Omega) < -\beta^2 - \frac{2\pi}{\mathcal{H}^1(\partial\Omega)}\beta.$$

It remains unclear how many connected components (or well separated sets) should have the solution of problem (3.1). Even for $k = 1$ the connectedness of the optimal shape is not straightforward. Assume that the optimal set consists on two well separated sets. One of them will give the first eigenvalue, and the second one could possibly be cancelled. Nevertheless, erasing some connected component would make that the measure of the set is not anymore satisfying the constraint. Contrary to the case of positive boundary parameter, the behavior of the eigenvalues to dilations is not controlled, then we cannot dilate the remaining components.

Assuming the smoothness of the boundary of the optimal set for $\lambda_{1,\beta}$ in two dimensions of the space, one can prove its connectedness [4]. Indeed, if the connected component Ω_1 giving the first eigenvalue uses less measure than allowed, then the constraint $\int_{\partial\Omega_1} V \cdot n \, d\mathcal{H}^{d-1} = 0$ on the admissible vector fields can be removed, leading to $|\nabla_{\partial\Omega_1} u|^2 - (\lambda_{k,\beta}(\Omega_1) + \beta^2 + \beta\mathcal{H})u^2 = 0$ on $\partial\Omega_1$. Consequently,

$$\lambda_{1,\beta}(\Omega_1) + \beta^2 + \beta\mathcal{H} \geq 0$$

at almost every point of the boundary. This would contradict $\lambda_{1,\beta}(\Omega_1) < -\beta^2 - \frac{2\pi}{\mathcal{H}^1(\partial\Omega_1)}\beta$, after summation over $\partial\Omega_1$.

[4]This remark, due to James Kennedy, can be found in Section 6 of [18].

Chapter 4

Existence and regularity of optimal convex shapes for functionals involving the Robin eigenvalues

We closed the previous chapter emphasizing that sometimes it looks very difficult to prove the regularity of optimal shapes when the boundary parameter is negative. In this chapter we consider a different situation in which it is possible to prove the regularity of the optimal shapes. In particular, we present the results in [31], where we prove the existence of convex solutions for a problem involving Robin eigenvalues with positive boundary parameter and we show the C^1 regularity of their boundaries.

We are interested in solving in $\mathbb{R}^d$ the following problem

$$\min\left\{F(\lambda_{1,\beta}(\Omega),\dots,\lambda_{k,\beta}(\Omega))+\Lambda P(\Omega):\Omega\subset\mathbb{R}^d \text{ bounded and convex}\right\},\tag{4.1}$$

where $\Lambda>0$, $F:\mathbb{R}^k\to\mathbb{R}$ is non decreasing and upper semicontinuos in each variable and $P(\Omega)$ is the perimeter in the sense of De Giorgi (for Lipschitz set it holds $P(\Omega)=\mathcal{H}^{d-1}(\partial\Omega)$). The prototypic problem of this family is

$$\min\left\{\lambda_{1,\beta}(\Omega)+\ldots+\lambda_{k,\beta}(\Omega)+\Lambda P(\Omega):\Omega\subset\mathbb{R}^d \text{ bounded and convex}\right\}.$$

We will prove an existence result for the problem (4.1) and, under slightly stronger hypotheses on F, we will be able to gain also regularity of the optimal sets. The main result of the chapter is the following theorem.

Theorem 4.0.1. *Let $F:\mathbb{R}^k\to\mathbb{R}$ be non decreasing and lower semicontinuos in each variable. Then, Problem* (4.1) *admits at least a solution.*

Moreover, if F is differentiable and its partial derivative with respect to the first variable is strictly positive, then every optimal solution has C^1 boundary.

The result presented in the chapter, up to our knowledge, is new in literature, although similar problems have already been studied in different settings (e.g. with Dirichlet boundary conditions) and using different techniques. The proof of existence of optimal convex shapes (Theorem 4.1.3) follows a standard approach used in shape optimization problems with geometrical constraints: the idea is to work with the Hausdorff topologies and show that minimizing sequences of open convex sets enjoy some properties that assure compactness. The proof of the regularity of the convex solutions (Theorem 4.3.3) is based on a cutting argument that allow us to show that, in order to minimize (4.1), it is more convenient to remove corners.

The structure of the chapter is the following. In Section 4.1 we obtain the existence of optimal convex shapes minimizing (4.1) via direct methods of calculus of variations, proving that $\lambda_{k,\beta}$ is lower semicontinuos and that minimizing sequences are compact and do not degenerate or stretch indefinitely in any direction. Then, in Section 4.2, we estimate the gap between $\lambda_{k,\beta}(\Omega)$ and $\lambda_{k,\beta}(\Omega_\varepsilon)$, where Ω is an admissible set with a singularity point on the boundary and Ω_ε is another convex competitor obtained by a suitable cut of Ω around the singularity point. Finally, in 4.3, we introduce the family of convex energy subsolutions for (4.1), that generalizes the definition of solutions, then we complete the proof of Theorem 4.0.1 showing the regularity of the boundaries of the energy subsolutions. We close the chapter showing some related results and making some remarks about the use of the technique presented.

4.1 Existence of convex minimizers

In this section we will prove the existence of bounded convex minimizers for the problem (4.1) using the direct methods of calculus of variations. In view of this strategy, we need the lower semicontinuity of the functional in (4.1) with respect the Hausdorff topology, that ensures compactness of minimizing sequences of convex sets. We even prove a continuity result for the eigenvalues (Proposition 4.1.1, as a consequence of a deep analysis made in [28]) and we show that a minimizing sequence satisfy some uniform cone property; then, in view of Remark 1.4.26, the results presented the following sections are valid even for convex closed sets.

The first step to prove the existence of minimizers is the following proposition, whose prove is obtained in [28] as a consequence of more general stability results for elliptic problems with some Robin-type boundary condition (we refer the reader in particular to Theorem 3.2 and Corollary 3.4 in [28], where the convergence of the spectrum of the Robin Laplacian is proved).

Proposition 4.1.1 (Continuity of $\lambda_{k,\beta}$)**.** *Let $(\Omega_n)_n$ be a sequence of open convex sets converging to an open, non empty, convex set Ω in the Hausdorff topology and let Ω_n, Ω be contained in a compact set $B \subset \mathbb{R}^d$. Then, for every $k \in \mathbb{N}$,*

$$\lambda_{k,\beta}(\Omega) = \lim_{n\to+\infty} \lambda_{k,\beta}(\Omega_n).$$

Proof. The proof of this result is a direct consequence of Theorem 3.2 and Corollary 3.4 in [28], once proved that $\mathcal{H}^{d-1}(\partial\Omega_n)$ converges to $\mathcal{H}^{d-1}(\partial\Omega_n)$. In view of the hypotheses, the sets Ω_n and Ω satisfy a uniform cone property; then, by Proposition 1.4.25, we deduce the convergence of the perimeters. □

Remark 4.1.2 (Lower semicontinuity of $\lambda_{k,\beta}$)**.** Even if the previous proposition provides the continuity of the eigenvalues, in order to prove the existence of minimizers it is sufficient to have the lower semicontinuity of the eigenvalues. Indeed, in [31] we proved (independently by the results in [28]) the following lower semicontinuity result.

Let $(\Omega_n)_n$ be a sequence of open convex sets converging to an open, non empty, convex set Ω in the Hausdorff topology and let Ω_n, Ω be contained in a compact set $B \subset \mathbb{R}^d$. Then, for every $k \in \mathbb{N}$,

$$\lambda_{k,\beta}(\Omega) \le \liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n).$$

Proof. Without loss of generality, we can suppose $\liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n) < +\infty$.

Let V_n be an admissible space for the computation of $\lambda_{k,\beta}(\Omega_n)$ such that

$$\lambda_{k,\beta}(\Omega_n) = \max_{V_n} R_{\Omega_n}.$$

Let $\{u_1^n, \dots, u_k^n\} \subset H^1(\Omega_n)$ an $L^2(\Omega_n)$-orthonormal basis for V_n. Without loss of generality we can suppose the sequence $(\lambda_{k,\beta}(\Omega_n))_n$ bounded. So, for every $i = 1, \dots, k$,

$$\int_{\Omega_n} |\nabla u_i^n|^2\, dx + \beta \int_{\partial\Omega_n} (u_i^n)^2\, d\sigma < C.$$

Then, $\sup_n \|u_n^i\|_{H^1(\Omega_n)} < +\infty$ for every $i = 1, \dots, k$. Moreover, by Proposition 1.5.4, $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco; then, by Remark 1.5.5, there exist $u_1, \dots, u_k \in H^1(\Omega)$ such that, up to subsequences, $\tilde{u}_n^i \to u^i$ strongly in $L^2(\mathbb{R}^d)$ and $\tilde{\nabla u_i^n} \rightharpoonup \tilde{\nabla} u_i$ weakly in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Notice that $u_1, \dots, u_k$ are linearly independent in $H^1(\Omega)$, since Ω_n converges to Ω also in measure; hence, the linear space $V := \operatorname{span}\{u_1, \dots, u_k\}$ is a competitor for the computation of $\lambda_{k,\beta}(\Omega)$. Let $w = \sum \alpha_i u_i$ realizing the maximum of the Rayleigh quotient $R_\Omega(\cdot)$

on V and let $w_n := \sum \alpha_i u_i^n \in V_n$. Observe that, up to subsequences, $w_n \to w$ strongly in $L^2(\mathbb{R}^d)$ and $\chi_{\Omega_n}\nabla w^n \rightharpoonup \chi_\Omega \nabla w$ weakly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$. Since Ω, Ω_n are convex, uniformly bounded and converge to a convex set, they satisfy a uniform cone property. Then, in view of Theorem 1.4.27, there exists a family of uniformly bounded operators that extends w_n and w to the whole of $\mathbb{R}^d$ in such a way that $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^d)$. We can thus apply Proposition 3.2.3 to w_n, w and $\partial\Omega_n, \partial\Omega$ to have the lower semicontinuity of the boundary integrals. Finally, we have the convergence of the volume integrals at the denominator and the lower semicontinuity of the L^2-norms of the gradients in the Rayleigh quotient. Using the fact that $w_n \in V_n$, we conclude that

$$\begin{aligned}\lambda_{k,\beta}(\Omega) &\leq \max_V R_\Omega = R_\Omega(w) \leq \liminf_{n\to+\infty} R_{\Omega_n}(w_n) \leq \liminf_{n\to+\infty} \max_{V_n} R_{\Omega_n} \\ &= \liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n),\end{aligned}$$

obtaining the required lower semicontinuity of the eigenvalues. □

We are now in a position to prove the existence of solutions of (4.1). The key point of the following theorem is to proof that the diameters of the sets of a minimizing sequence are uniformly bounded and that the limit set does not degenerate in any direction.

Theorem 4.1.3. *Problem* (4.1) *admits at least a bounded convex minimizer.*

Proof. Let $(\Omega_n)_n$ be a minimizing sequence of admissible sets. From the optimality of $(\Omega_n)_n$, we have that $\sup_n \mathcal{H}^{d-1}(\partial\Omega_n) < +\infty$. Then, via isoperimetric inequality, we also have $\sup_n |\Omega_n| < +\infty$. Without loss of generality, up to translations and rotations we can suppose that

$$\operatorname{diam}(\Omega_n) = \mathcal{H}^1(\Omega_n \cap \{x_2 = \ldots = x_d = 0\})$$

and that

$$\min_{i=2,\ldots,d}\left(\max_{\Omega_n} x_i - \min_{\Omega_n} x_i\right) = \max_{\Omega_n} x_d - \min_{\Omega_n} x_d$$

i.e. the width of Ω_n is minimal on the direction of the axe x_d. We claim that $\sup_n \operatorname{diam}(\Omega_n) < +\infty$ and that, up to subsequences,

$$\lim_n \left(\max_{\Omega_n} x_d - \min_{\Omega_n} x_d\right) > 0. \tag{4.2}$$

We start proving (4.2) arguing by contradiction. Let us suppose that the limit in (4.2) is zero; define

$$\Omega_n^0 := \Omega_n \cap \{x_d = 0\}$$

and, for every $x' \in \Omega_n^0$, the segment

$$l_n(x') := \{(x', x_d) \in \Omega_n\}\,.$$

Let us consider an admissible function $u \in H^1(\Omega_n)$ for the computation of the Robin eigenvalues of Ω_n and observe that, for every $x' \in \Omega_n^0$, the function $x_d \mapsto u(x', x_d)$ is admissible for the computation of the Robin eigenvalues of $l_n(x')$. Then we have

$$\begin{aligned}
R(u) &= \frac{\displaystyle\int_{\Omega_n} |\nabla u|^2\, dx + \beta \int_{\partial\Omega_n} u^2\, d\sigma}{\displaystyle\int_{\Omega_n} u^2\, dx} \\
&= \frac{\displaystyle\int\limits_{\Omega_{n_0}} dx' \int\limits_{l_n(x')} \left[|\nabla_{x'} u|^2 + \left(\frac{\partial u}{\partial x_d}\right)^2 \right] dx_d + \beta \int\limits_{\Omega_{n_0}} dx' \int\limits_{\partial l_n(x')} u^2(x', x_d) d\mathcal{H}^0(x_d)}{\displaystyle\int_{\Omega_{n_0}} dx' \int_{l_n(x')} u^2\, dx_d} \\
&\geq \frac{\displaystyle\int_{\Omega_{n_0}} \left(\int_{l_n(x')} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n(x')} u^2(x', x_d)\, d\mathcal{H}^0(x_d) \right) dx'}{\displaystyle\int_{\Omega_{n_0}} \left(\int_{l_n(x')} u^2\, dx_d \right) dx'} \\
&\geq \min_{x' \in \Omega_n^0} \frac{\displaystyle\int_{l_n(x')} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n(x')} u^2(x', x_d)\, d\mathcal{H}^0(x_d)}{\displaystyle\int_{l_n(x')} u^2\, dx_d}.
\end{aligned} \tag{4.3}$$

Now, the term on the last side is a minimum computed among one dimensional Rayleigh quotients on segments. Thanks to the monotonicity under homotheties (2.7) and to the fact that all the $l_n(x')$ are homothetical, we can conclude that the required minimum is achieved on the longest segment $l_n^{max} := l_n(x_{max})$, so

$$R(u) \geq \frac{\displaystyle\int_{l_n^{max}} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n^{max}} u^2(x_{max}, x_d)\, d\mathcal{H}^0(x_d)}{\displaystyle\int_{l_n(x')} u^2\, dx_d} \geq \lambda_{1,\beta}(l_n^{max})$$

as $u(x_{max}, \cdot)$ is a competitor in the computation of $\lambda_{1,\beta}(l_n^{max})$. Let us observe that, as we are contradicting (4.2), the length of l_n^{max} tends to zero as n goes to infinity; then, by estimates (2.9), $\lambda_{1,\beta}(l_n^{max})$ tends to $+\infty$, so $R(u) = +\infty$ for every admissible function $u \in H^1(\Omega_n)$, which is impossible.

To prove that the diameters of the Ω_n sets are uniformly bounded, we argue straightforwardly by contradiction. If the sequence of the diameters was unbounded, as the Ω_n are convex and uniformly bounded in measure, the product

$$\prod_{j=1}^{d}\left(\max_{\Omega_n} x_j - \min_{\Omega_n} x_j\right)$$

has to be uniformly bounded in measure. In view of our assumptions, as the diameter of Ω_n tends to infinity, necessarily the first term of the product diverges and so at least the smallest among the remaining $d-1$ terms has to vanish; in other words, we would have

$$\lim_n \left(\max_{\Omega_n} x_d - \min_{\Omega_n} x_d\right) = 0,$$

in contradiction with (4.2).

Then $(\Omega_n)_n$ is an equibounded sequence of convex sets which converge (up to subsequences) to a bounded convex set Ω in the Hausdorff topology; moreover, by Proposition 1.4.14, the convergence is also in measure. In addition, thanks to (4.2), the limit set Ω is not degenerate (i.e. it has positive measure) and

$$P(\Omega) \leq \liminf_{n\to+\infty} P(\Omega_n).$$

Finally, thanks to the continuity of the Robin eigenvalues (Proposition 4.1.1, but the lower semicontinuity is sufficient, see Remark 4.1.2) and to the monotonicity and lower semicontinuity of the function F in each variable, we obtain

$$F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \leq \liminf_{n\to+\infty} F(\lambda_{1,\beta}(\Omega_n), \dots, \lambda_{k,\beta}(\Omega_n)),$$

so we can conclude that Ω is a minimizer of (3.1). □

Remark 4.1.4. The previous existence theorem is still valid if, instead of penalizing the perimeter as in Problem (4.1), we imposee a uniform constraint on the measures, on the perimeters or on the diameters of the admissible convex sets and minimize only the functional $F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega))$.

4.2 Estimates on the cut set

The aim of the results presented in this section is to show that, under some additional hypotheses, the optimal convex shapes have C^1 boundary. We will prove this regularity result for a larger class of sets, the so-called *energy subsolutions* for Problem (4.1) (see Definition 4.3.1); we will see that optimal sets for Problem (4.1) are also energy subsolutions. The technique to prove the regularity of the boundary is rather intuitive: supposing, by contradiction, that an energy subsolution Ω has a singularity point x_0 for its boundary, we cut a suitable "ε-neighbourhood" of x_0, obtaining a convex competitor Ω_ε. Then, comparing the values of (3.1) for Ω and Ω_ε, we observe that there exists a cut set strictly better than the optimal set Ω, obtaining a contradiction. The key point of this approach is to estimate the gap between $\lambda_{h,\beta}(\Omega)$ and $\lambda_{h,\beta}(\Omega_\varepsilon)$, for every $h \in \mathbb{N}$.

We will distinguish between the case $d = 2$ and $d > 2$, since the arguments used are based on 2-dimensional sections and on a lower bound on the ratio between two surface areas; in dimension larger than two this is not immediate as in $\mathbb{R}^2$.

4.2.1 The case $d = 2$

In the planar setting, many assumptions can be done. First of all, as the boundary of a convex 2-dimensional set is a Lipschitz curve, the only singularity points for the outer normal are sharp corners, in which we can distinguish two different tangent lines. In the following, without loss of generality, we will assume that Ω has only a singularity point coinciding with the origin, that Ω lies in the halfplane $\{x_1 > 0\}$ and that the bisector of the corner between the distinguished tangent lines is $\{x_2 = 0\}$.

In any dimension d the situation is more involved. In this case, the singularity set for the outer normal is at most a $(d-2)$-dimensional locally Lipschitz variety and it can happen that we have in every singularity point more than one couple of distinguished tangent hyperplanes (e.g., in $\mathbb{R}^3$ the vertex of a circular cone has infinite couples of tangent planes). In every case, without loss of generality, we can assume that, chosen a singularity point for the boundary, it coincides with the origin and that Ω lies in the halfspace $\{x_1 > 0\}$; moreover, we can rotate Ω and chose one of the couples of distinguished tangent hyperplanes in such a way that their bisector is the hyperplane $\{x_d = 0\}$.

Under these assumptions, for every dimension d and every $\varepsilon > 0$, we define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{x_1 > \varepsilon\}, \; m_\varepsilon := \Omega \setminus \Omega_\varepsilon, \; \sigma_\varepsilon := \Omega \cap \{x_1 = \varepsilon\}, \; s_\varepsilon := \partial\Omega \setminus \partial\Omega_\varepsilon. \quad (4.4)$$

Let us observe that, in view of our choice, the origin turns out to realize the maximum

$$\max_{x \in \overline{m_\varepsilon}} \operatorname{dist}(x, \sigma_\varepsilon) = \varepsilon.$$

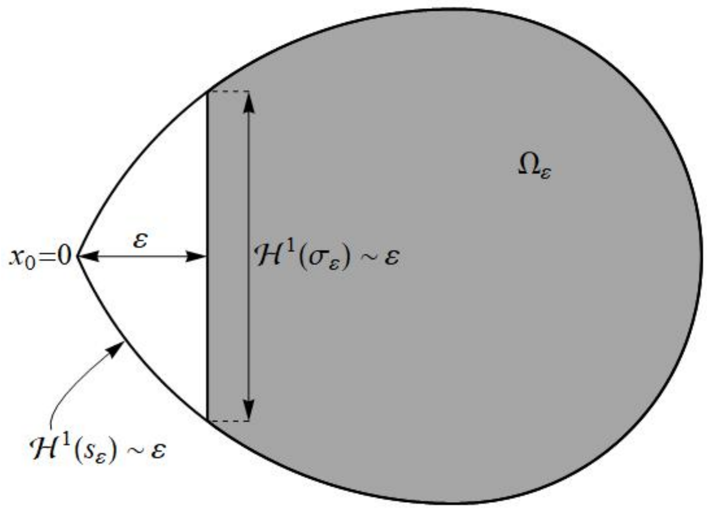


Figure 4.1: Cutting procedure in dimension $d = 2$.

Lemma 4.2.1. *Let $\Omega \subset \mathbb{R}^2$ be an admissible set for* (3.1) *with a singularity point for the outer normal. Then, there exist $\varepsilon_0 > 0$ and $C = C(\Omega) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\lambda_{1,\beta}(\Omega_\varepsilon) \leq \lambda_{1,\beta}(\Omega) - C\varepsilon. \tag{4.5}$$

Proof. First of all let us remark that both $\mathcal{H}^1(\sigma_\varepsilon)$ and $\mathcal{H}^1(s_\varepsilon)$ are infinitesimal of the same order of ε and that $|m_\varepsilon|$ is infinitesimal of the same order of ε^2 as ε goes to 0; moreover, there exists a constant $C_1 > 1$, depending only on the set Ω, such that $\mathcal{H}^1(s_\varepsilon) \geq C_1 \mathcal{H}^1(\sigma_\varepsilon)$.

Under the same assumptions of the beginning of the section and using the same notation as in (4.4), let us compare $\lambda_{1,\beta}(\Omega_\varepsilon)$ with $\lambda_{1,\beta}(\Omega)$. Let us consider $u \in H^1(\Omega)$ a $L^2(\Omega)$-normalized eigenfunction for $\lambda_{1,\beta}(\Omega)$ positively bounded from below (it exists in view of Proposition 2.1.3); its restriction on Ω_ε is a test function for $\lambda_{1,\beta}(\Omega_\varepsilon)$ and it holds

$$\begin{aligned}
\lambda_{1,\beta}(\Omega_\varepsilon) &\leq \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} u^2\, d\sigma}{\displaystyle\int_{\Omega_\varepsilon} u^2\, dx} \\
&\leq \frac{\displaystyle\int_{\Omega} |\nabla u|^2\, dx + \beta \int_{\partial\Omega} u^2\, d\sigma + \beta \int_{\sigma_\varepsilon} u^2\, d\sigma - \beta \int_{s_\varepsilon} u^2\, d\sigma}{\displaystyle 1 - \int_{m_\varepsilon} u^2\, dx} \qquad (4.6) \\
&\leq \left[\lambda_{1,\beta}(\Omega) + \beta \int_{\sigma_\varepsilon} u^2\, d\sigma - \beta \int_{s_\varepsilon} u^2\, d\sigma\right] \left(1 + 2\int_{m_\varepsilon} u^2\, dx\right) \\
&\leq \lambda_{1,\beta}(\Omega) + \beta \left(\int_{\sigma_\varepsilon} u^2\, d\sigma - \int_{s_\varepsilon} u^2\, d\sigma\right) + C_2\varepsilon^2
\end{aligned}$$

for ε small enough. Fixed $\delta > 0$, with $(u(0)+\delta)^2 \leq C_1(u(0)-\delta)^2$, there exists $\varepsilon_0 > 0$ such that

$$0 < u(0) - \delta < u(x) < u(0) + \delta$$

for every $x \in \overline{m_{\varepsilon_0}}$, since $u \in C(\overline{\Omega})$ and $u(0) > 0$. In particular, as m_ε is decreasing in ε with respect to inclusions, we can choose ε_0 small enough to satisfy (4.6). Then we obtain

$$\begin{aligned}
\lambda_{1,\beta}(\Omega_\varepsilon) &\leq \lambda_{1,\beta}(\Omega) + \beta\left(\mathcal{H}^1(\sigma_\varepsilon)(u(0)+\delta)^2 - \mathcal{H}^1(s_\varepsilon)(u(0)-\delta)^2\right) + C_2\varepsilon^2 \\
&\leq \lambda_{1,\beta}(\Omega) + \beta\mathcal{H}^1(\sigma_\varepsilon)\left((u(0)+\delta)^2 - C_1(u(0)-\delta)^2\right) + C_2\varepsilon^2 \\
&\leq \lambda_{1,\beta}(\Omega) - \beta C_3\varepsilon + C_2\varepsilon^2 \leq \lambda_{1,\beta}(\Omega) - C\varepsilon,
\end{aligned}$$

where the last constant C takes into account all the previous constants and depends only on the domain Ω. □

The behaviour of higher order eigenvalues is studied in the following lemma.

Lemma 4.2.2. *Let $\Omega \subset \mathbb{R}^2$ be an admissible set for* (3.1) *with a singularity point for the outer normal. Then, for every $h \in \mathbf{N}, h \geq 2$,*

$$\lambda_{h,\beta}(\Omega_\varepsilon) \leq \lambda_{h,\beta}(\Omega) + o(\varepsilon). \qquad (4.7)$$

Proof. Let us consider h eigenfunctions for the Laplacian Robin, say $u_1, \dots, u_h$, associated to $\lambda_{1,\beta}(\Omega), \dots, \lambda_{h,\beta}(\Omega)$ such that they form an L^2-orthonormal basis of $S := \operatorname{span}\{u_1, \dots, u_h\}$. Let us consider S as a test space for the computation of $\lambda_{h,\beta}(\Omega_\varepsilon)$; precisely, we can restrict ourselves to the subset of $\operatorname{span}\{u_1|_{\Omega_\varepsilon}, \dots, u_h|_{\Omega_\varepsilon}\}$ of functions of the form $\sum_{i=1}^h \alpha_i^\varepsilon u_i$ with $\sum_{i=1}^h (\alpha_i^\varepsilon)^2 = 1$.

This compactness hypothesis on the coefficients ensures us that, up to subsequences, $\alpha_i^\varepsilon \to \alpha_i \in [-1,1]$ and

$$\sum_{i=1}^{h} \alpha_i^\varepsilon u_i \longrightarrow \sum_{i=1}^{h} \alpha_i u_i$$

strongly in $H^1(\Omega)$. In the following we will denote by $(\overline{\alpha}_1^\varepsilon, \dots, \overline{\alpha}_h^\varepsilon)$ and $(\overline{\alpha}_1, \dots, \overline{\alpha}_h)$ two h-ple of coefficients that maximizes $R^\beta_{\Omega_\varepsilon}$ and R^β_Ω in S. We claim that $\overline{\alpha}_i^\varepsilon \to 0$ if $\lambda_{i,\beta}(\Omega) < \lambda_{h,\beta}(\Omega)$. To prove this claim, observe first that $\lambda_{h,\beta}(\Omega) = \max_{u\in S} R(u)$, since at least $u_h \in S$ is associated to $\lambda_{h,\beta}(\Omega)$. Then we have

$$\begin{aligned}\lambda_{h,\beta}(\Omega) &= \max_{\substack{\alpha_1,\dots,\alpha_h\in\mathbf{R}\\ \sum_i \alpha_i^2=1}} \frac{\displaystyle\int_\Omega \Big|\sum_i \alpha_i \nabla u_i\Big|^2 dx + \beta \int_{\partial\Omega} \Big(\sum_i \alpha_i u_i\Big)^2 d\sigma}{\displaystyle\int_\Omega \Big(\sum_i \alpha_i u_i\Big)^2 dx} \\ &= \sum_{i,j} \overline{\alpha}_i \overline{\alpha}_j \left(\int_\Omega \nabla u_i \cdot \nabla u_j \, dx + \beta \int_{\partial\Omega} u_i u_j \, d\sigma\right)\end{aligned} \tag{4.8}$$

Let us compute the quantity between brackets using the Robin boundary conditions, integrating by parts and recalling that the u_i and u_j belong to an orthonormal basis of eigenfunctions:

$$\begin{aligned}\int_\Omega \nabla u_i \cdot \nabla u_j \, dx + \beta \int_{\partial\Omega} u_i u_j \, d\sigma &= \int_\Omega \nabla u_i \cdot \nabla u_j \, dx - \int_{\partial\Omega} u_i \frac{\partial u_j}{\partial n} \, d\sigma \\ &= -\int_\Omega u_i \Delta u_j \, dx = \lambda_{j,\beta}(\Omega) \int_\Omega u_i u_j \, dx = \lambda_{j,\beta}(\Omega)\delta_{ij}.\end{aligned}$$

So, by (4.8), we obtain

$$\lambda_{h,\beta}(\Omega) = \sum_{i,j} \overline{\alpha}_i \overline{\alpha}_j \lambda_{j,\beta}(\Omega)\delta_{ij} = \sum_i \overline{\alpha}_i^2 \lambda_{i,\beta}(\Omega),$$

that implies that all the coefficients related to $\lambda_{i,\beta}(\Omega) < \lambda_{h,\beta}(\Omega)$ have to be 0.

In view of this remark, for any $\varepsilon > 0$ sufficiently small, we estimate $\lambda_{h,\beta}(\Omega_\varepsilon)$

using S as a test space:

$$
\begin{aligned}
\lambda_{h,\beta}(\Omega_\varepsilon) &= \max_{\substack{\alpha_1^\varepsilon,\dots,\alpha_h^\varepsilon\in\mathbf{R}\\ \sum_i(\alpha_i^\varepsilon)^2=1}} \frac{\displaystyle\int_{\Omega_\varepsilon}\Big|\sum_i \alpha_i^\varepsilon \nabla u_i\Big|^2 dx + \beta\int_{\partial\Omega_\varepsilon}\Big(\sum_i \alpha_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle\int_{\Omega_\varepsilon}\Big(\sum_i \alpha_i^\varepsilon u_i\Big)^2 dx}\\
&\le \frac{\displaystyle\int_{\Omega}\Big|\sum_i \overline{\alpha}_i^\varepsilon \nabla u_i\Big|^2 dx + \beta\int_{\partial\Omega}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle 1-\int_{m_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx}\\
&+ \frac{\displaystyle\beta\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma - \beta\int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle 1-\int_{m_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx}\\
&\le \lambda_{h,\beta}(\Omega) + \beta\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma - \beta\int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma + C|m_\varepsilon|.
\end{aligned}
\tag{4.9}
$$

Observe that, for every $i = 1,\dots,h$, $\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i \to 0$; moreover, following the remark at the beginning of Lemma 4.2.1 about the infinitesimal order of $\mathcal{H}^1(\sigma_\varepsilon)$, $\mathcal{H}^1(s_\varepsilon)$ and $|m_\varepsilon|$, by (4.9) we have

$$
\begin{aligned}
\lambda_{h,\beta}(\Omega_\varepsilon) &\le \lambda_{h,\beta}(\Omega) + \beta\Bigg[\int_{\sigma_\varepsilon}\Big(\sum_i (\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i + \overline{\alpha}_i u_i\Big)^2 d\sigma\\
&- \int_{s_\varepsilon}\Big(\sum_i (\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i + \overline{\alpha}_i u_i\Big)^2 d\sigma\Bigg] + C\varepsilon^2\\
&\le \lambda_{h,\beta}(\Omega) + \beta\left(\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma\right) + o(\varepsilon).
\end{aligned}
\tag{4.10}
$$

To estimate the boundary integrals in the last term, we have to distinguish two cases. If

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 \neq 0,$$

then, for any sufficiently small ε, we can proceed as in Lemma 4.2.1 and conclude that

$$\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma \le 0.$$

If

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 = 0,$$

the uniform continuity of the eigenfunctions u_i on m_ε implies that both integrands go to zero as ε goes to zero and so the boundary integrals are infinitesimal of higher order than ε. In both cases, by (4.10) we obtain

$$\lambda_{h,\beta}(\Omega_\varepsilon) \leq \lambda_{h,\beta}(\Omega) + o(\varepsilon).$$

□

Remark 4.2.3. Let us compare the results of the previous lemmas. In Lemma 4.2.1 we proved that, after a small cut, the first eigenvalue decreases by a term of the same order as the perimeter. On the other hand, in Lemma 4.2.2, we proved that a small cut could at most increase $\lambda_{h,\beta}$ ($h \geq 2$) by a term infinitesimal of higher order than the perimeter. In other words, the possible increase of $\lambda_{h,\beta}$ ($h \geq 2$) is infinitesimal of higher order than the decrease of $\lambda_{1,\beta}$.

4.2.2 The case $d > 2$

The case of higher dimension is quite different. Recalling the notation introduced in (4.4), the key point is to prove that the ratio $\mathcal{H}^{d-1}(s_\varepsilon)/\mathcal{H}^{d-1}(\sigma_\varepsilon)$ has a lower bound strictly greater than 1, as in the planar case. It is not trivial at a first sight, but fortunately this obstacle can be overcome taking into account suitable 2-dimensional sections of Ω around the singularity point of the boundary. We will get the required lower estimate in the following lemmas, the first holding in dimension 3, the second holding in any dimension. We chose to expose separately the cases of dimension $d = 3$ and of higher dimension for a better clarity for the reader, although the proofs are quite similar.

Lemma 4.2.4. *Let $\Omega \subset \mathbb{R}^3$ be an admissible set for* (3.1) *with a singularity point at the origin for the outer normal and let us consider s_ε and σ_ε as in* (4.4). *Then there exists $C > 1$ such that $\mathcal{H}^2(s_\varepsilon)/\mathcal{H}^2(\sigma_\varepsilon) \geq C$ for every $\varepsilon > 0$.*

Proof. Let us use the same convention as in (4.4), so that the origin is a singularity point for $\partial\Omega$, and let us assume that the outer normal to $\partial\Omega$ is discontinuous in the origin with respect to the direction of the x_2 axe. Let us consider two distinguished tangent hyperplanes at the singularity point; without loss of generality we can assume that the bisector of the two planes is the plane $\{x_2 = 0\}$ their intersection V is the line $\{x_1 = x_2 = 0, \}$. Under these assumptions, the orthogonal projection V_ε of V onto σ_ε is a segment on the line $\{x_1 = \varepsilon, x_2 = 0\}$, and it can be expressed by

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ a_\varepsilon \leq x_3 \leq b_\varepsilon \end{cases}$$

with $a_\varepsilon \leq 0 \leq b_\varepsilon$. Moreover, the orthogonal space $V^\perp$ is a 2-dimensional plane and the section $c_\varepsilon := s_\varepsilon \cap V^\perp$ is given by a Lipschitz curve with a corner point in the origin. Notice that

$$V^\perp = \{x_3 = 0\}$$

and that, denoting by l_ε the segment $\sigma_\varepsilon \cap V^\perp$, the curve c_ε is the graph of a concave function defined on l_ε. So, as in the planar setting (see 4.2.1), there exists $\alpha > 0$ such that

$$\frac{\mathcal{H}^1(c_\varepsilon)}{\mathcal{H}^1(l_\varepsilon)} \geq 1 + \alpha \tag{4.11}$$

for every $\varepsilon > 0$ sufficiently small.

The idea to estimate $\mathcal{H}^2(\sigma_\varepsilon)$ in terms of $\mathcal{H}^2(s_\varepsilon)$ is based on the Fubini's theorem: we will split the 2-dimensional surface integral in two 1-dimensional integrals in the variables x_2, x_3 and we will estimate uniformly from above the 1-dimensional sections of s_ε with the 1-dimensional sections of σ_ε. Let us define

$$l_\varepsilon(x_3) := \sigma_\varepsilon \cap (V^\perp + x_3)$$

the 1-dimensional slice of σ_ε passing through $(0, 0, x_3)$ and parallel to $l_\varepsilon = l_\varepsilon(0)$. If we denote by

$$c_\varepsilon(x_3) := \sigma_\varepsilon \cap (V^\perp + x_3),$$

then $c_\varepsilon(0) = c_\varepsilon$. Moreover, by continuity and (4.11), the above constant $\alpha > 0$ can be chosen in such a way that

$$\frac{\mathcal{H}^1(c_\varepsilon(x_3))}{\mathcal{H}^1(l_\varepsilon(x_3))} \geq 1 + \alpha \tag{4.12}$$

for every $a_\varepsilon/2 \leq x_3 \leq b_\varepsilon/2$. Let us remark that, in every case, the ratio above is greater or equal than 1 for every $x_3 \in]a_\varepsilon, b_\varepsilon[$.

Recalling that s_ε is the graph of a concave function $\phi = \phi(x_2, x_3)$ on σ_ε, let us estimate from below the area of $s_\varepsilon \cap \{x_3 \geq 0\}$:

$$
\begin{aligned}
&\mathcal{H}^2(s_\varepsilon \cap \{x_3 \geq 0\}) \\
&\quad = \int_{\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}} \sqrt{1+|\nabla\phi|^2}\, dx_2\, dx_3 + \mathcal{H}^2(s_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} dx_3 \int_{l_\varepsilon(x_3)} \sqrt{1+|\nabla\phi|^2}\, dx_2 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} dx_3 \int_{l_\varepsilon(x_3)} \sqrt{1+(\partial_{x_2}\phi)^2}\, dx_2 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} \mathcal{H}^1(c_\varepsilon(x_3))\, dx_3 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq (1+\alpha) \int_0^{b_\varepsilon/2} \mathcal{H}^1(l_\varepsilon(x_3))\, dx_3 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq (1+\alpha)\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}) + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad = \mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}) + \alpha\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}).
\end{aligned}
\tag{4.13}
$$

Let us notice that, as $\sigma_\varepsilon \cap \{x_3 \geq 0\}$ is convex, there exists a positive constant $\gamma < 1$ depending only on Ω such that

$$\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}) \geq \gamma\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}).$$

Then, replacing the estimate in (4.13), we obtain

$$\mathcal{H}^2(s_\varepsilon \cap \{x_3 \geq 0\}) \geq (1+\alpha\gamma)\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}). \tag{4.14}$$

Reasoning for $x_3 \leq 0$ as above we obtain

$$\mathcal{H}^2(s_\varepsilon \cap \{x_3 \leq 0\}) \geq (1+\alpha\gamma)\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \leq 0\}),$$

that combined with (4.14) gives us the thesis (with $C = 1+\alpha\gamma$). □

In higher dimension we obtain the same result, after noticing that we can reason similarly to the previous Lemma on each dimension that is orthogonal to a suitable 2-dimensional section.

Lemma 4.2.5. *Let $\Omega \subset \mathbb{R}^d$ ($d > 3$) be an admissible set for* (3.1) *with a singularity point at the origin and let us consider s_ε and σ_ε as in* (4.4)*. Then, there exists $C > 1$ such that $\mathcal{H}^{d-1}(s_\varepsilon)/\mathcal{H}^{d-1}(\sigma_\varepsilon) \geq C$ for every $\varepsilon > 0$.*

Proof. Let us start remarking that, in view of the assumptions below (4.4), the intersection V between two distinguished tangent $(d-1)$-dimensional hyperplanes at the singularity point is a $(d-2)$-dimensional hyperplane whose orthogonal projection onto σ_ε, say V_ε, is a convex, $(d-2)$-dimensional set. Let us observe also that $c_\varepsilon := s_\varepsilon \cap V^\perp$ is given by a Lipschitz curve with a corner point in the origin and that, told $l_\varepsilon := \sigma_\varepsilon \cap V^\perp$, there exists a constant $\alpha > 0$ such that the same estimate as in (4.11). To achieve the thesis, it is enough to repeat the same argument in the second part of 4.2.4 on each of $d-2$ (orthogonal) segments passing by the orthogonal projection of the origin onto σ_ε and parallel to the first $d-2$ Cartesian axes. $\square$

Now we are in a position to state the analogous of 4.2.1 and 4.2.2; we omit the proof as it is straightforward, replacing ε by the surface area $\mathcal{H}^{d-1}(\sigma_\varepsilon)$.

Lemma 4.2.6. *Let $\Omega \subset \mathbb{R}^d$ be an admissible set for* (3.1) *with a singularity point for the outer normal. Then, there exists $\varepsilon_0 > 0$ ad $C = C(\Omega) > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\lambda_{1,\beta}(\Omega_\varepsilon) \le \lambda_1^\beta(\Omega) - C\mathcal{H}^{d-1}(\sigma_\varepsilon).$$

Moreover, for every $h \in \mathbb{N}, h \ge 2$

$$\lambda_{h,\beta}(\Omega_\varepsilon) \le \lambda_h^\beta(\Omega) + o(\mathcal{H}^{d-1}(\sigma_\varepsilon)).$$

4.3 Regularity of optimal convex shapes

The aim of this section is to prove a regularity result for the optimal shape whose existence has been proved in 4.1, to complete the proof of Theorem 4.0.1. We will prove a more general result, i.e. the C^1-regularity of the boundary for a larger class of sets.

Definition 4.3.1 (Energy subsolutions)**.** Let $\Omega \subset \mathbb{R}^d$ be a convex bounded set. Ω is said an energy subsolution for problem (3.1) if, for every convex set $\tilde{\Omega} \subseteq \Omega$, it holds

$$F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \le F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})).$$

Remark 4.3.2. Intuitively, a convex set Ω is an energy subsolution when its "energy" $F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega))$ is minimal compared to his convex subsets. Roughly speaking, thanks to the monotonicity of F and of $\lambda_{h,\beta}$ under dilatations, if $\tilde{\Omega} \subset \Omega$ is an admissible set and we focus only on the energy term, it is convenient to rescale $\tilde{\Omega}$ to obtain a wider convex set with lower energy; on the other hand, this increases the perimeter term in (3.1), as the two terms seem to be in competition. This behaviour suggests us that a convex solution should balance the two competing terms with the lowest energy possible. In view of this, let us remark that every minimizer Ω of (3.1) is also an energy subsolution; in fact, for every $\tilde{\Omega} \subset \Omega$, using the monotonicity of the perimeter of convex sets under inclusions, we have

$$\begin{aligned}F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) + \Lambda P(\Omega) &\leq F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})) + \Lambda P(\tilde{\Omega}) \\ &\leq F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})) + \Lambda P(\Omega),\end{aligned}$$

that implies $F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \leq F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega}))$.

The following theorem will give us the required regularity for energy subsolutions. To prove it, we will argue by contradiction, supposing that an energy subsolution Ω has at least a singularity point for the outer normal and cutting a piece of Ω around this point; the obtained cut subset will give a strictly smaller energy than Ω, in contradiction with the definition of energy subsolution.

Theorem 4.3.3 (regularity of the energy subsolutions)**.** *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfy the same hypotheses as in* (3.1) *and, in addition, let it be differentiable in each variable each variable with strictly positive derivative with respect to the first variable. Then, every energy subsolution for problem* (3.1) *has C^1 boundary.*

Proof. Let Ω be an energy subsolution for (3.1) and consider Ω_ε and σ_ε as in (4.4). Considering a Taylor expansion of F and the results in Lemma 4.2.6, we obtain for sufficiently small ε

$$\begin{aligned}&F(\lambda_{1,\beta}(\Omega_\varepsilon), \dots, \lambda_{k,\beta}(\Omega_\varepsilon)) \\ &\qquad = F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \\ &\qquad + \sum_{h=1}^{k} \frac{\partial F}{\partial x_h}(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \cdot (\lambda_{h,\beta}(\Omega_\varepsilon) - \lambda_{h,\beta}(\Omega)) + o(\mathcal{H}^{d-1}(\sigma_\varepsilon)) \\ &\qquad \leq F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) - \frac{\partial F}{\partial x_1}(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \cdot (C\mathcal{H}^{d-1}(\sigma_\varepsilon)) \\ &\qquad + o(\mathcal{H}^{d-1}(\sigma_\varepsilon)) \\ &\qquad < F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega))\end{aligned}$$

in contradiction with the fact that Ω is an energy subsolution. □

Proof of Theorem 4.0.1. Problem (3.1) admits a convex bounded solution Ω thanks to 4.1.3; by Remark 4.3.2, this solution is also an energy subsolution, then, under the additional hypotheses of F, by Theorem 4.3.3, Ω has C^1 boundary. □

4.3.1 Further remarks

Using analogous techniques, it is possible to prove an existence and regularity result for the problem

$$\min \{\lambda_{1,\beta}(\Omega) : \Omega \subseteq D, \Omega \text{ open and convex}, |\Omega| \leq m\}, \tag{4.15}$$

where D is a bounded design region. In this case, if the ball of measure m is not contained in D, the problem is not trivially solved.

Proposition 4.3.4. *Problem* (4.15) *admits a convex solution with C^1 boundary.*

Proof. The existence of a convex solution is due to a standard compactness argument for uniformly bounded, open, convex sets and to the lower semicontinuity of the Rayleigh quotient; the regularity arises from the estimate

$$\lambda_{1,\beta}(\Omega_\varepsilon) \leq \lambda_{1,\beta}(\Omega) - C\mathcal{H}^{d-1}(\sigma_\varepsilon)$$

in Lemma 4.2.6. □

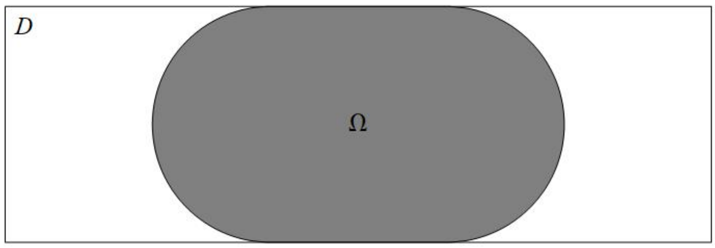


Figure 4.2: An admissible solution to problem (4.15) in a rectangular domain D; even the possible junctions with the boundary of the bounded design region must be C^1.

The interesting fact of the previous result is that the regularity of the optimal shapes does not depend on the regularity of the boundary of the bounded design region. Indeed, even if we consider a bounded design region with rough domain, the optimal shapes still have C^1 boundary.

In a similar way, the problem

$$\min \{\lambda_{k,\beta}(\Omega) + \Lambda P(\Omega) : \Omega \text{ bounded and convex}\}, \tag{4.16}$$

with $\Lambda > 0$, admits a regular solution.

Proposition 4.3.5. *Problem* (4.16) *admits a convex solution with* C^1 *boundary.*

Proof. The existence is gained with the same arguments in 4.1.3. The regularity is obtained by contradiction as a consequence of 4.2.6: the gap between $P(\Omega)$ and $P(\Omega_\varepsilon)$ decreases to zero more slowly than the difference $\lambda_{k,\beta}(\Omega) - \lambda_{k,\beta}(\Omega_\varepsilon)$. □

Existence and regularity of convex solutions for Problem (4.16) are almost a byproduct of the deeper analysis made throughout the chapter for more general problems. In next chapter we will set (4.16) in a more general framework and we will see that it is neither of easy solution nor of immediate formulation.

Chapter 5

Minimization of the k-th eigenvalue of the Robin-Laplacian with perimeter penalization

In [27] the authors minimize $\lambda_{k,\beta}$ with a measure constraint in a relaxed setting. Our aim is to replace the measure constraint with some constraint on the perimeter. More precisely, in continuity with the problem studied in Chapter 4, we chose to use a penalization term and to study the problem

$$\min\left\{\lambda_{k,\beta}(\Omega)+\mathcal{H}^{d-1}(\partial\Omega)|\Omega\subset\mathbb{R}^d \text{ is an open Lipschitz domain}\right\}. \tag{5.1}$$

In Chapter 4 we studied existence and regularity of minimizers in the class of convex domains, but here we would like to have a more general result, possibly relaxing the problem.

A first idea could be to relax the eigenvalues in any dimension d in the same sense of the relaxed eigenvalues $\tilde{\lambda}_{k,\beta}$ in Chapter 3, taking into account sets of finite perimeter and boundary integrals defined on the reduced boundary. Unfortunately, this strategy leads to a ill posed problem. Indeed, even for the first eigenvalue, a minimizing sequence could converge in some sense to an admissible set but the eigenfunctions could not converge to a function in $H^1(\mathbb{R}^d)$ (see the example presented in figure 3.1, page 73). On the other hand, also the idea to use the relaxed eigenvalues in the sense of $\bar{\lambda}_{k,\beta}$ fails in general, as we should have additional topological hypotheses to obtain some existence result, even in $\mathbb{R}^2$.

To avoid those situations, we have to move into suitable spaces of functions of bounded variation and relax Problem (5.1) to obtain a well posed free discontinuity problem.

5.1 The $SBV^{\frac{1}{2}}$-spaces: a good setting for the weak formulation

In this section, we recall some facts and some definitions about $SBV^{\frac{1}{2}}$-spaces, that are very useful to handle free discontinuity problems involving boundary integrals. Our main references are two papers by D. Bucur and A. Giacomini: [24], where the $SBV^{\frac{1}{2}}$-framework is introduced to approach in a variational way the Faber-Krahn inequality and [27], where many results are generalized to vector valued functions whose components are (in some sense) in a $SBV^{\frac{1}{2}}$-space.

The definition of the space $SBV^{1/2}(\mathbb{R}^d)$ is recalled below (see Definition 3.1 in [24]).

Definition 5.1.1 (the space $SBV^{1/2}(\mathbb{R}^d)$)**.** Let $u : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. We say that $u \in SBV^{\frac{1}{2}}(\mathbb{R}^d)$ if $u \geq 0$ a.e. in $\mathbb{R}^d$ and $u^2 \in SBV(\mathbb{R}^d)$.

Now, we recall some fine properties of the functions in $SBV^{\frac{1}{2}}(\mathbb{R}^d)$; we refer the reader to Lemma 1 in [24], where such properties are stated and proved. In order to avoid ambiguities between positive (resp. negative) part and interior (resp. exterior) trace, throughout this chapter, for any measurable function u, we will denote by $\gamma_1(u)$ and $\gamma_2(u)$ the traces on the two sides of the jump set J_u from the directions of $-\nu_u$ and ν_u (whenever the traces and ν_u are well defined).

Proposition 5.1.2. *Let $u \in SBV^{\frac{1}{2}}(\mathbb{R}^d)$. The following items hold true*

(i) u is a.e. approximately differentiable with approximate gradient ∇u such that
$$\nabla(u^2) = 2u\nabla u \quad a.e. \ in\ \mathbb{R}^d.$$

(ii) J_u is $\mathcal{H}^{d-1}$-countably rectifiable with normal vector ν_u such that $D^j(u^2)$ is given by
$$D^j(u^2) = [\gamma_2(u)^2 - \gamma_1(u)^2]\nu_u \mathcal{H}^{d-1} \lfloor J_u.$$

(iii) For every $\varepsilon > 0$ we have $(u-\varepsilon)^+ \in SBV(\mathbb{R}^d)$.

Now, to cover also the case of higher eigenvalues, we recall the $SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d;\mathbb{R}^k)$ spaces, introduced in [27], Definition 3.4, in order to consider also vector valued functions whose components change sign.

Definition 5.1.3 (the space $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$)**.** Let $u = (u_1, \dots, u_k) : \mathbb{R}^d \to \mathbb{R}^k$ be a measurable function. We say that $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ if $(u_i)^+, (u_i)^- \in SBV^{\frac{1}{2}}(\mathbb{R}^d)$ for every $i = 1, \dots, k$ and

$$\int_{\mathbb{R}^d} |\nabla u|^2\, dx + \int_{J_u} [|\gamma_1(u)|^2 + |\gamma_2(u)|^2]\, d\mathcal{H}^{d-1} < +\infty.$$

If $k = 1$, we denote the space simply by $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d)$.

Now, we recall an important compactness and lower semicontinuity result in the functional space $SBV^{\frac{1}{2}}(\mathbb{R}^d)$. For the detailed proof see Theorem 3.3 in [24].

Proposition 5.1.4 (compactness and lower semicontinuity in $SBV^{\frac{1}{2}}(\mathbb{R}^d)$)**.** *Let $(u_n)_n \subset SBV^{\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} |\nabla u_n|^2\, dx + \int_{J_{u_n}} \left[\gamma_1(u_n)^2 + \gamma_2(u_n)^2\right]\, d\mathcal{H}^{d-1} + \int_{\mathbb{R}^d} u_n^2\, dx \leq C$$

for some $C > 0$. Then, there exists $u \in SBV^{\frac{1}{2}}(\mathbb{R}^d)$, $\Phi \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ and a subsequence $(u_{n_k})_k$ such that the following items hold true.

(i) Compactness: $u_{n_k} \to u$ strongly in $L^2_{loc}(\mathbb{R}^d)$ and

$$\nabla u_{n_k} \rightharpoonup \Phi \quad \textit{weakly in } L^2(\mathbb{R}^d;\mathbb{R}^d)$$

with

$$\Phi \chi_{supp(u)} = \nabla u$$

(ii) Lower semicontinuity: for every open set $A \subseteq \mathbb{R}^d$ we have

$$\int_A |\nabla u|^2\, dx \leq \liminf_{k\to+\infty} \int_A |\nabla u_{n_k}|^2\, dx$$

and

$$\int_{J_u \cap A} \left[\gamma_1(u)^2 + \gamma_2(u)^2\right]\, d\mathcal{H}^{d-1} \leq \liminf_{k\to+\infty} \int_{J_{u_{n_k}} \cap A} \left[\gamma_1(u_{n_k})^2 + \gamma_2(u_{n_k})^2\right]\, d\mathcal{H}^{d-1}.$$

The counterpart of the previous result for the functional space $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ is the following proposition (see Proposition 3.6 in [27]).

Proposition 5.1.5 (compactness and lower semicontinuity in $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$)**.** *Let $(u_n)_n \subset SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ such that*

$$\int_{\mathbb{R}^d} |\nabla u_n|^2\, dx + \int_{J_{u_n}} \left[|\gamma_1(u_n)|^2 + |\gamma_2(u_n)|^2\right]\, d\mathcal{H}^{d-1} + \int_{\mathbb{R}^d} |u_n|^2\, dx \leq C$$

for some $C > 0$. Then, there exists $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$, $\Phi \in L^2(\mathbb{R}^d;\mathbb{R}^{kd})$ and a subsequence $(u_{n_h})_h$ such that the following items hold true.

(i) Compactness: $u_{n_h} \to u$ strongly in $L^2_{loc}(\mathbb{R}^d;\mathbb{R}^k)$ and

$$\nabla u_{n_h} \rightharpoonup \Phi \quad \textit{weakly in } L^2(\mathbb{R}^d;\mathbb{R}^{kd})$$

with

$$\Phi \chi_{supp(u)} = \nabla u$$

(ii) Lower semicontinuity: for every open set $A \subseteq \mathbb{R}^d$ we have

$$\int_A |\nabla u|^2\, dx \leq \liminf_{h\to+\infty} \int_A |\nabla u_{n_h}|^2\, dx$$

and

$$\int_{J_u \cap A} \left[|\gamma_1(u)|^2 + |\gamma_2(u)|^2\right]\, d\mathcal{H}^{d-1} \leq \liminf_{h\to+\infty} \int_{J_{u_{n_h}} \cap A} \left[|\gamma_1(u_{n_h})|^2 + |\gamma_2(u_{n_h})|^2\right]\, d\mathcal{H}^{d-1}.$$

In the following, for every vector valued function $u = (u_1, \dots, u_k)$, we will denote by

$$V(u) := \{u_1, \dots, u_k\}.$$

In [27] the authors introduced the functional space $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ in order to give a weak formulation of the k-th Robin eigenvalue. A key point in their free discontinuity approach is to ensure that the k components of a function in $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ are $L^2(\mathbb{R}^d)$-linearly independent, in order to recall a k-ple of $L^2(\Omega)$-linearly independent functions in $H^1(\Omega)$ that generate an admissible space for $\lambda_{k,\beta}(\Omega)$. To this aim, we recall a useful functional space (see Definition 3.9 in [27]).

Definition 5.1.6 (the space $\mathcal{F}_k(\mathbb{R}^d)$)**.** Let $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$. We say that $u \in \mathcal{F}_k(\mathbb{R}^d)$ if $\{u_1, \dots, u_k\}$ are linearly independent in $L^2(\mathbb{R}^d)$.

In [24], [25] and [27] authors relaxed Robin eigenvalues using $SBV^{\frac{1}{2}}$-functions in order to prove the existence of minimal "shapes" among sets of prescribed measure respectively for the first Robin eigenvalue (see [24] for an approach similar to the proof of the Bossel-Daners inequality and [25] for a completely variational approach) and for the higher eigenvalues (see [27]). The idea is classical when using a free discontinuity approach: one replaces the dependence on a domain Ω with the dependence on a SBV-function u, whose support will be "identified" with Ω and whose jump set u will be identified with $\partial^*\Omega$ (or $\partial\Omega$, if Ω is a sufficiently smooth domain). In particular, in [24] authors introduced a weak formulation for the first eigenvalue involving functions in $SBV^{\frac{1}{2}}(\mathbb{R}^d)$; the idea is that, since the first eigenfunction of a Lipschitz domain can be taken of constant sign (in particular non negative), then the admissible functions in the weak formulation can be taken a priori non negative. For the higher eigenvalues the situation is a bit more complicated, since the eigenfunctions can change their sign in general; moreover, every k-dimensional subspace V_k is admissible for $\lambda_{k,\beta}(\Omega)$ and can be spanned by a basis $\{u_1,\dots,u_k\}$ of functions that are linearly independent $L^2(\Omega)$, so it is reasonable to choose for the weak formulation of the k-th eigenvalue a family vector valued functions whose components are L^2-linearly independent. Reasoning in that way, in [27], Definition 3.9, the authors relaxed the Robin eigenvalues as follows.

Definition 5.1.7 (relaxed eigenvalues)**.** Let $u\in\mathcal{F}_k(\mathbb{R}^d)$. We define the k-th relaxed Robin eigenvalue for the function u by

$$R_{k,\beta}(u):=\max_{v\in V(u)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}|\nabla v|^2\,dx+\beta\int_{J_u}[(v^+)^2+(v^-)^2]\,d\mathcal{H}^{d-1}}{\int_{\mathbb{R}^d}v^2\,dx}.$$

Remark 5.1.8. The functional $R_{k,\beta}(u)$ is finite and well defined for every $u\in\mathcal{F}_k(\mathbb{R}^d)$, since it holds $V(u)\subset SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d)$ (see Lemma 3.8 in [27]) and then term $\int_{\mathbb{R}^d}|\nabla v|^2\,dx+\beta\int_{J_u}[(v^+)^2+(v^-)^2]\,d\mathcal{H}^{d-1}$ is finite for every $v\in V(u)$.

Notice that, if $\Omega\subset\mathbb{R}^d$ is a Lipschitz domain and $u_1,\dots,u_k\in H^1(\Omega)$ are the first k Robin eigenfunction on Ω with boundary parameter $\beta>0$, then $u=(u_1,\dots,u_k)$, extended by zero outside Ω, belongs to $\mathcal{F}_k(\mathbb{R}^d)$ and it holds

$$J_u=\partial\Omega$$

and

$$R_{k,\beta}(u)=\lambda_{k,\beta}(\Omega).$$

(see Remark 3.11 in [27]).

It is worth to highlight that, if $v=\sum_{j=1}^k a_ju_j$ realizes $R_{k,\beta}(u)$ in $V(u)$, then also λv realizes $R_{k,\beta}(u)$ for every $\lambda\neq 0$; in particular, v can be chosen such that $\|v\|_{L^2(\mathbb{R}^d)}=1$.

A key result in our proofs is the lower semicontinuity of the term $R_{k,\beta}$. We omit the proof as it is contained in Proposition 3.12 in [27].

Proposition 5.1.9 (lower semicontinuity of $R_{k,\beta}$). *Let $(u_n)_n \subset \mathcal{F}_k$ and $u \in \mathcal{F}_k$ such that $u_n \to u$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$. Then*

$$R_{k,\beta}(u) \le \liminf_{n\to+\infty} R_{k,\beta}(u_n).$$

Once we replace $\lambda_{k,\beta}(\Omega)$ by $R_{k,\beta}(u)$, we need to understand how to relax the boundary term. We found two possible ways to replace the boundary term:

- penalizing the perimeter of a set where an admissible function is non-null ("free Robin" problem);
- penalizing all the jumps of the admissible functions ("jump Robin" problem).

In both cases we have to pay attention to the way we relax the problem, as the starting ideas above are not sufficient for the well-posedness of the problems. The following sections are aimed to this; we will prove some starting existence results (only for the first eigenvalue and in abounded design region), highlighting what are the technical difficulties to overcome and the research perspectives to study the problem.

Remark 5.1.10. Even if we are able to prove the lower semicontinuity of the weak functionals for a general order $k \in \mathbb{N}$ (see Propositions 5.2.1 and 5.3.1), the existence results in the following sections involve only the first weak eigenvalue. The problem is that, at the moment, we do not know if a (minimizing) sequence in $\mathcal{F}_k$ converge to an admissible function in $\mathcal{F}_k$. If, $k = 1$ and we prescribe $\|u_n\|_{L^2(\mathbb{R}^d)} = C$ for some positive constant C, then we are sure that the possible limit function u belongs to $\mathcal{F}_1(\mathbb{R}^d)$. For $k > 1$, even if we assume that each component satisfy $\|u_n^j\|_{L^2(\mathbb{R}^d)} = C$, this is not true. Indeed, as we will see, for minimizing sequences we can assume that

$$\int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2] d\mathcal{H}^{d-1} + \int_{\mathbb{R}^d} u_n^2\,dx \le C$$

for some positive constant C. So (see Proposition 3.8 in [27]), there exists $u \in SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d;\mathbb{R}^k)$ such that, up to subsequence, $u_n \to u$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$. But it is not said, in general, that $u \in \mathcal{F}_k(\mathbb{R}^d)$ if $k > 1$. An example in $\mathbb{R}^2$ can be given by the sequence of functions

$$u_n(x,y) = \begin{cases} (x, x^{1+\frac{1}{n}}) & \text{in } [0,1]^2, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $(u_n)_n \subset \mathcal{F}_2(\mathbb{R}^2)$ and u_n converges strongly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$ to

$$u(x,y) = \begin{cases} (x,x) & \text{in } [0,1]^2, \\ 0 & \text{otherwise,} \end{cases}$$

that is not a function in $\mathcal{F}_2(\mathbb{R}^2)$.

Notice that the equality $SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d) \setminus \{0\} = \mathcal{F}_1(\mathbb{R}^d)$ holds. That means that, if we are able to prove that the limit sequence of functions in $\mathcal{F}_1(\mathbb{R}^d)$ is non null (e.g., if it has strictly positive $L^2(\mathbb{R}^d)$-norm), this is sufficient to conclude that such function is also in $\mathcal{F}_1(\mathbb{R}^d)$ (this is not possible for higher k, as seen in the previous example).

5.2 The "free Robin" problem

We would like to relax in SBV Problem (5.1) taking into account the reduced boundary of the set where an admissible function is non-null. Intuitively, we start taking into account the minimization problem

$$\min\left\{R_{k,\beta}(u) + \mathcal{H}^{d-1}(\partial^* \{u \neq 0\}) | u = (u_1,\ldots,u_k) \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)\right\}. \quad (5.2)$$

Even though it seems a very natural relaxation of Problem (5.1), Problem (5.2) is not well posed. Indeed, it is possible to consider a concave Lipschitz set Ω with strictly positive first Robin eigenfunction and then to consider an admissible SBV function v_ε which holds v on Ω and $\varepsilon > 0$ on $\operatorname{conv}(\Omega) \setminus \Omega$. In this way the value of $R_{k,\beta}(v_\varepsilon)$ is very close to $R_{k,\beta}(v_\varepsilon)$ but the term of perimeter can sensibly decrease.

This example shows that Problem (5.2) is not stable under small perturbations of the variable u. To avoid this phenomenon, our idea is to link the boundary term with the perimeter of a set of finite measure Ω where u is supported, i.e. such that $u = 0$ outside Ω. In such a way, we allow the support of the function u to "invade" the set Ω (even if it is not clear whether this phenomenon actually occurs, in general). More precisely, we study the problem

$$\begin{aligned}\min\big\{R_{k,\beta}(u) + P(\Omega) \mid \Omega \subset \mathbb{R}^d,\ |\Omega| + P(\Omega) < +\infty,& \\ u \in \mathcal{F}_k(\mathbb{R}^d),\ u = 0 \in \Omega^c\big\}.&\end{aligned} \quad (5.3)$$

where the bound on the measure is to avoid trivial situations (e.g., taking $\Omega = \mathbb{R}^d$). In other words, we box a priori the support of an admissible function u in an unknown set Ω and then perform the minimization among all admissible couples (u, Ω). Notice that, if Ω is a bounded Lipschitz domain and u is an eigenfunction for $\lambda_{k,\beta}(\Omega)$ we have

$$R_{k,\beta}(u) + P(\Omega) = \lambda_{k,\beta}(\Omega) + \mathcal{H}^{d-1}(\partial\Omega),$$

i.e. we are in the classical setting.

The weak formulation (5.3) is not the most natural we could imagine, as one of the main ideas in the field of free discontinuity problems is to replace n-ples of variables of different nature (open sets, curves, functions) with only one variable function (think to the weak formulation of the Mumford-Shah functional, see [41]). On the other hand, as we will see, some very useful information about optimal sets (if they exist) can be found

- if Ω is a solution of the problem, then it turns out to be a perimeter supersolution (in the sense of De Philippis and Velichkov, see [42]);
- if a set is a perimeter supersolution, then it satisfies a density estimate, so we gain some regularity ([42]).

The plan to study the problem is the following. We start looking for some lower semicontinuity theorem involving both $R_{k,\beta}(u)$ and $P(\Omega)$; then, we focus on proving the existence in a bounded design region in the case $k = 1$. Finally, recalling the results in [42], we prove that, for every minimizing pair (u, Ω), the set Ω can be chosen open.

5.2.1 A lower semicontinuity result

In the next proposition we obtain lower semicontinuity of the functional $R_{k,\beta}(u)+P(\Omega)$ with respect to both variables, in some product topology. In particular, $R_{k,\beta}(u) + P(\Omega)$ is lower semicontinuos u with respect to the $L^2(\mathbb{R}^d;\mathbb{R}^k)$ strong topology and in Ω with respect to the convergence in measure.

Proposition 5.2.1. *Let $(\Omega_n)_n$ a sequence of uniformly bounded sets of finite perimeter such that*

$$\sup_{n\in\mathbb{N}} P(\Omega_n) < +\infty.$$

Let $u_n, u \in \mathcal{F}_k(\mathbb{R}^d)$ such that $u_n \to u$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$ and $u_n = 0$ in Ω_n^c. Then, there exists $\Omega \subset \mathbb{R}^d$ bounded set of finite perimeter such that, up to subsequences, $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^d)$, $u = 0$ in Ω^c and

$$R_{k,\beta}(u) + P(\Omega) \le \liminf_{n\to+\infty} R_{k,\beta}(u_n) + P(\Omega_n). \tag{5.4}$$

Proof. By Proposition 2.3.6 and Proposition 2.3.10 in [58], we have that, up to subsequences, $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^d)$ and

$$P(\Omega) \leq \liminf_{n\to+\infty} P(\Omega_n). \tag{5.5}$$

Moreover, by Proposition 3.12 in [27], we have that

$$R_{k,\beta}(u) \leq \liminf_{n\to+\infty} R_{k,\beta}(u_n). \tag{5.6}$$

Summing (5.5) and (5.5) and using the superadditivity of the liminf, we obtain (5.4). Finally, the L^2-convergence of u_n to u implies that $u = 0$ in Ω^c. $\square$

5.2.2 Existence of optimal shapes in a bounded design region

In this short section we provide a first existence result for (5.3), under the hypotheses of bounded design region and in the case $k = 1$.

Theorem 5.2.2. *Let $D \subset \mathbb{R}^d$ open and bounded. Then, the problem*

$$\min\left\{R_{1,\beta}(u) + P(\Omega) | \Omega \subset D,\ P(\Omega) < +\infty,\ u \in \mathcal{F}_1(\mathbb{R}^d), u = 0 \textit{ in } \Omega^c\right\} \tag{5.7}$$

admits a solution.

Proof. Let $(\Omega_n, u_n)_n$ be a minimizing sequence for Problem (5.7). Without loss of generality, we can suppose that

$$R_{1,\beta}(u_n) + P(\Omega_n) \leq C,$$

for some positive constant C independent of n, and that $\|u_n\|_{L^2(\mathbb{R}^d)} = 1$ for every $n \in \mathbb{N}$. Then, both $(P(\Omega_n))_n$ and $(u_n)_n$ are uniformly bounded, respectively in $\mathbb{R}$ and in $SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d)$ (i.e.

$$\int_{\mathbb{R}^d} |\nabla u_n|^2\, dx + \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2] d\mathcal{H}^{d-1} + \int_{\mathbb{R}^d} u_n^2\, dx \leq C$$

for some positive constant C). So, by Proposition 5.1.5, there exists $u \in SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d)$ such that, up to subsequences, $u_n \to u$ strongly in $L^2(B)$. Moreover, by Proposition 1.2.10, possibly passing to a subsequence, there exists $\Omega \subset B$ of finite perimeter such that $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(B)$. Let us remark that, in view of both convergences, we have that $u = 0$ in Ω^c (see the last part of the proof of Proposition 5.2.1). Then, the couple (u, Ω) turns out to be admissible for (5.7) and, in view of the lower semicontinuity of the functional (Proposition 5.2.1), we obtain

$$R_{1,\beta}(u) + P(\Omega) \leq \liminf_{n\to+\infty} R_{1,\beta}(u_n) + P(\Omega_n) = \inf_{(v,A)} \{R_{1,\beta}(v) + P(A)\},$$

so (u, Ω) is a solution for (5.7). □

Remark 5.2.3. We are not able, in general to prove that optimal shapes have to be bounded, except if we are in $\mathbb{R}^2$. In that case, we can remove the hypotheses of bounded design region: the bound on the diameter will be a consequence of the uniform bound on the perimeters of the sets on a minimizing sequence, assumed without loss of generality in the proof of Theorem 5.2.2.

5.2.3 A regularity issue

In this section we prove some topological regularity of the solutions to Problem (5.3), if they exist (for $k = 1$, in a bounded design region it is true, as seen in the previous section). More precisely, we will see that an optimal finite perimeter set Ω can always be replaced by an open set strictly linked with Ω. We start the section defining an important class of sets.

Definition 5.2.4 (perimeter supersolution)**.** Let $\Omega \subset \mathbb{R}^d$ a set of finite perimeter. We say that Ω is a *perimeter supersolution* if $|\Omega| < +\infty$ and, for every $\tilde{\Omega} \supset \Omega$ of finite perimeter, then $P(\tilde{\Omega}) \geq P(\Omega)$.

It is immediate to prove the following result.

Proposition 5.2.5. *If* (u, Ω) *is a minimizing couple for the problem* (5.3)*, then* Ω *is a perimeter supersolution.*

Proof. Let $\tilde{\Omega} \subset \mathbb{R}^d$ be a set of finite perimeter such that $\tilde{\Omega} \supset \Omega$. Since $u = 0$ in $\Omega^c \supset \tilde{\Omega}^c$, the couple $(u, \tilde{\Omega})$ is admissible for (5.3). Using the optimality of (u, Ω) we obtain

$$R_{k,\beta}(u) + P(\Omega) \leq R_{k,\beta}(u) + P(\tilde{\Omega}),$$

i.e. $P(\Omega) \leq P(\tilde{\Omega})$. □

Remark 5.2.6. As highlighted in [42] by the authors, perimeter supersolutions are sets having positive mean curvature (see Definition A.1.3) at least in a weak sense. For instance, a bounded convex set Ω is an example of perimeter supersolution. Notice that in that case, if (u, Ω) is a minimizing couple for (5.3), then also $(u, \overline{\Omega})$ and $(u, \mathring{\Omega})$ are optimal.

Now, we introduce the following density estimate.

Definition 5.2.7 (exterior density estimate)**.** Let $\Omega \subset \mathbb{R}^d$ a set of finite perimeter. We say that Ω *satisfies an exterior density estimate* if there exists a positive dimensional constant $c = c(d)$ such that, for every $x \in \mathbb{R}^d$, one of the following situations occurs:

(i) there exists $r > 0$ such that $B_r(x) \subset \Omega$ a.e.;

(ii) for every $r > 0$, it holds $|B_r(x) \setminus \Omega| > c|B_r(x)|$.

The next results link the previous density estimate with the perimeter supersolutions, ensuring that there exist open optimal shapes for (5.3). The proofs of the following two propositions (holding for any perimeter supersolution, not only for the solutions of (5.3)) are omitted, as they can be found in [42].

Proposition 5.2.8. *Let $\Omega \subset \mathbb{R}^d$ be a perimeter supersolution. Then, Ω satisfies an exterior density estimate. In particular, if (u,Ω) is a solution of* (5.3)*, then Ω satisfies an exterior density estimate.*

Proposition 5.2.9. *Let $\Omega \subset \mathbb{R}^d$ a set of finite perimeter satisfying an exterior density estimate. Then, the set of the points of density 1 for Ω*

$$\Omega_1 = \left\{ x \in \mathbb{R}^d : \exists \lim_{r\to 0^+} \frac{|\Omega \cap B_r(x)|}{|B_r(x)|} \right\}$$

is open. In particular, for every perimeter supersolution Ω, Ω_1 is open.

Now, we are ready for the main theorem of this short section.

Theorem 5.2.10 (existence of an open solution)**.** *Let (u,Ω) a solution for* (5.3)*. Then, (u,Ω_1) is also a solution for* (5.3) *and Ω_1 is open.*

Proof. It is sufficient to remark that the couple (u,Ω_1) is admissible for (5.3), $P(\Omega_1) = P(\Omega)$ since $|\Omega \Delta \Omega_1| = 0$ and Ω_1 is open in view of Proposition 5.2.9. □

5.2.4 Further remarks and perspectives

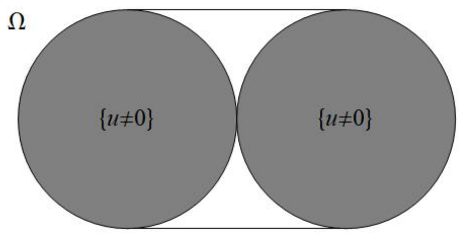
We have been able to prove only an existence result for the principal "eigenvalue", because, as seen in Remark 5.1.10, it could happen that a sequence in $\mathcal{F}_k(\mathbb{R}^d)$ converge to a function $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k) \setminus \mathcal{F}_k(\mathbb{R}^d)$. It would also be interesting to remove the bounded design region hypotheses, but it seems technically difficult in view of the presence of a couple of variables. Nevertheless, we think that both questions could be linked with some condition ensuring the

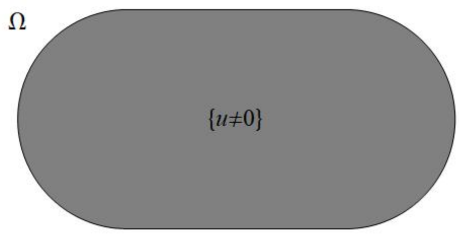
L^2-orthonormality (or, at least, linear independence) of the components of the function u (see the proof of Theorem 4.6 in [27]).

Once one has the existence of minimizers, even in a bounded design region, one of the most challenging problems is to understand if the couple $(u, supp(u))$ can be considered optimal for (5.3), i.e. if u is defined a.e. in the optimal domain Ω. For $k = 1$, we expect the minimizer should be the couple $(u, supp(u))$, where u is the first eigenfunction of a ball B; the fact that B is a perimeter supersolution supports our conjecture. For higher eigenvalues, to conjecture a possible result about the optimality of the couple $(u, supp(u))$, we made some numerical simulations on the second eigenvalue using the software FreeFem++. We focused on a two dimensional case: we compared the numerical values of (5.3) on the couples (u_1, Ω) and (u_2, Ω), where u_1 and u_2 are respectively second eigenfunction of the disjoint union of two tangent disks of radium 1 and the second eigenfunction of the respective stadium Ω. The results are summarized in the following table, where we omitted the term $P(\Omega)$ (that is the same for both couples):

β	$R_{2,\beta}(u_1)$	$R_{2,\beta}(u_2)$
0	0.21046	0.76095
0.5	0.88508	1.57319
1	1.73291	2.18306
1.5	2.25629	2.64573
2	2.67311	3.00244
5	4.01315	4.07245
10	4.77408	4.62919
20	5.24496	4.95788
100	5.67036	5.24609
1000	5.77229	5.31400

It turns out that, for large values of β, it is convenient to define the function u on the whole of the perimeter supersolution Ω, otherwise, for small values of β, u can be considered supported on a subset of Ω that could not be a perimeter supersolution.





That situation suggests us to follow a research direction to find a critical value $\bar{\beta} > 0$ for the boundary parameter such that, if $0 < \beta < \bar{\beta}$, then $|\Omega \setminus supp(u)| > 0$, while, if $\beta > \bar{\beta}$, then $supp(u) = \Omega$.

Notice that the "free Robin" approach on one hand gives a certain regularity of the optimal shapes, but, on the other hand, does not model a relaxed version of the original problem (5.1), in general. Indeed, as seen above, it can happen that in some optimal couples (u, Ω), even if Ω is a Lipschitz domain, u is not an eigenfunction for $\lambda_{\beta,k}(\Omega)$. To overcome the problem and link the variable function u with the boundary term, we found another weak formulation of (5.1) that we present in the following section.

5.3 The "jump Robin" problem

In this section we relax again Problem (5.1) in the SBV setting, linking also the boundary term with the support of the relaxed "eigenfunction". Since we are intuitively led to identify the support of a SBV function with an "admissible set" and the jump set of the function with some (topological or reduced) boundary of this set, we start taking into account the minimization problem

$$\min\left\{R_{k,\beta}(u) + \mathcal{H}^{d-1}(J_u) | u = (u_1, \ldots, u_k) \in \mathcal{F}_k\right\}. \tag{5.8}$$

Problem (5.8) is not well posed. Take, for instance, the sequence of concentric balls $(B_n(0))_n$ and consider, for each ball, $v_n = (v_1^n, \ldots, v_k^n)$, where $v_1^n, \ldots, v_k^n$ are the first k linearly independent Dirichlet eigenfunction for $B_n(0)$ (whose trivial extension by zero is continuous on the whole of $\mathbb{R}^d$). Then we have

$$R_{k,\beta}(v_n) + \mathcal{H}^{d-1}(J_{v_n}) = \lambda_k(B_n(0)) \longrightarrow 0.$$

This example shows us that the right way to replace the perimeter is to consider not only the jump set for an admissible function v, but also the reduced boundary of the set where the components of v are non null. In particular, we will study the problem

$$\min\left\{R_{k,\beta}(u) + \mathcal{H}^{d-1}\left(J_u \cup \bigcup_{i=1}^{k} \partial^*(\{v_i \neq 0\})\right) | u \in \mathcal{F}_k(\mathbb{R}^d)\right\}. \tag{5.9}$$

5.3.1 A compactness and lower semicontinuity result

We start proving a compactness and lower semicontinuity result for the functional.

Proposition 5.3.1. *Let $\Omega \subset \mathbb{R}^d$ open and bounded and let $u_n \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ such that*

$$\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega |u_n|^2\,dx + \int_{J_{u_n}} [(u_n)^+]^2 + [(u_n)^-]^2\,d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^k \partial^*(\{u_i^n \neq 0\})\right) < C$$

for some $C > 0$ independent on $n \in \mathbb{N}$. Then, there exists $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ such that u_n weakly converges to u in the sense of Proposition 5.1.5 and, in addition,

$$R_{k,\beta}(u) + \mathcal{H}^{d-1}\left(J_u \cup \bigcup_{i=1}^k \partial^*(\{u_i \neq 0\})\right) \le \liminf_{n\to+\infty} R_{k,\beta}(u_n) + \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^k \partial^*(\{u_i^n \neq 0\})\right).$$

Proof. Let us observe that, since $(u_n)_n$ satisfies the hypotheses of Proposition 5.1.5, there exists a function $u \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$ realizing the required weak convergence. In particular, $u_n \to u$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$; then, in view of Proposition 5.1.9, one has

$$R_{k,\beta}(u) \le \liminf_{n\to+\infty} R_{k,\beta}(u_n).$$

We need to prove the lower semicontinuity of the boundary term. For every $\varepsilon > 0$ and every $v \in SBV_{\pm}^{\frac{1}{2}}(\mathbb{R}^d;\mathbb{R}^k)$, we denote by v^ε the function whose components satisfy

$$v_i^\varepsilon := (v_i^+ \vee \varepsilon)\cdot \chi_{\{v_i>0\}} + (v_i^- \vee \varepsilon)\cdot \chi_{\{v_i<0\}}.$$

Let us observe that $v_i^\varepsilon \in SBV(\Omega)$ (Proposition 5.1.2) and that, for every sufficiently small $\varepsilon > 0$, $u_n^\varepsilon \to u^\varepsilon$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$. In addition,

$$J_{u^\varepsilon} \subseteq J_u \cup \bigcup_{i=1}^k \partial^*(\{u_i \neq 0\})$$

and, for every $n \in \mathbb{N}$,

$$J_{u_n^\varepsilon} \subseteq J_{u_n} \cup \bigcup_{i=1}^k \partial^*(\{u_i^n \neq 0\}), \tag{5.10}$$

the above set inclusions holding up to a $\mathcal{H}^{d-1}$-negligible set. Let us observe that

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^{d-1}(J_{u^\varepsilon}) = \mathcal{H}^{d-1}\left(J_u \cup \bigcup_{i=1}^k \partial^*(\{u_i \neq 0\})\right) \tag{5.11}$$

and, for every $n \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^{d-1}(J_{u_n^\varepsilon}) = \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^k \partial^*(\{u_i^n \neq 0\})\right),$$

Let us use the Ambrosio-Braides compactness and lower semicontinuity theorem (Theorem 1.2.5 and Remark 1.2.6) to conclude the proof. We need to show that

$$\int_\Omega |\nabla u_n^\varepsilon|^2\,dx + \mathcal{H}^{d-1}(J_{u_n^\varepsilon}) + \|u_n^\varepsilon\|_{BV(\Omega)} < C$$

for some $C > 0$ independent on n. It holds

$$\int_\Omega |\nabla u_n^\varepsilon|^2\,dx \leq \int_\Omega |\nabla u_n|^2\,dx < C,$$

$$\mathcal{H}^{d-1}\left(J_{u_n^\varepsilon}\right) \leq \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^k \partial^*(\{u_i^n \neq 0\})\right) < C$$

and

$$\begin{aligned}
&\|u_n^\varepsilon\|_{BV(\Omega)} = \|u_n^\varepsilon\|_{L^1(\Omega)} + |Du_n^\varepsilon|(\Omega)\\
&= \int_\Omega |u_n^\varepsilon|\,dx + \int_\Omega |\nabla u_n^\varepsilon|\,dx + \int_{J_{u_n^\varepsilon}} \left|(u_n^\varepsilon)^+ - (u_n^\varepsilon)^-\right|\,d\mathcal{H}^{d-1}\\
&\leq |\Omega|^{1/2}\left(\int_\Omega |u_n^\varepsilon|^2\,dx\right)^{1/2} + C_2|\Omega|^{1/2}\left(\int_\Omega |\nabla u_n|^2\,dx\right)^{1/2}\\
&+ C_3\left(\varepsilon\mathcal{H}^{d-1}(J_{u_n^\varepsilon}) + \int_{J_{u_n}} \left|(u_n)^+ - (u_n)^-\right|\,d\mathcal{H}^{d-1}\right)\\
&\leq |\Omega|^{1/2}\left(\int_\Omega |u_n^\varepsilon|^2\,dx\right)^{1/2} + C_2|\Omega|^{1/2}\left(\int_\Omega |\nabla u_n|^2\,dx\right)^{1/2}\\
&+ C_3\left(\varepsilon\mathcal{H}^{d-1}(J_{u_n^\varepsilon}) + \mathcal{H}^{d-1}(J_{u_n})^{1/2}\left(\int_{J_{u_n}} |(u_n)^+|^2 + |(u_n)^-|^2\,d\mathcal{H}^{d-1}\right)^{1/2}\right)\\
&\leq C_4 + C_5\left(\int_\Omega |\nabla u_n|^2\,dx + \int_{J_{u_n}} |(u_n)^+|^2 + |(u_n)^-|^2\,d\mathcal{H}^{d-1}\right)^{1/2}\\
&\leq C_6,
\end{aligned}$$

were the constants are independent on $n \in \mathbb{N}$. In view of the Ambrosio-Braides compactness and lower semicontinuity theorem and in view of the inclusion (5.10), we have

$$\mathcal{H}^{d-1}(J_{u_\varepsilon}) \leq \liminf_{n\to+\infty} \mathcal{H}^{d-1}(J_{u_n^\varepsilon}) \leq \liminf_{n\to+\infty} \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^{k} \partial^*(\{u_i^n \neq 0\})\right).$$

Then, by equality (5.11), since $\varepsilon > 0$ is arbitrary small it holds

$$\mathcal{H}^{d-1}\left(J_u \cup \bigcup_{i=1}^{k} \partial^*(\{u_i \neq 0\})\right) \leq \liminf_{n\to+\infty} \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^{k} \partial^*(\{u_i^n \neq 0\})\right),$$

concluding the proof. □

Notice that, to gain the previous lower semicontinuity result, we use the hypotheses that all the functions are supported in the same bounded design region Ω, in order to apply the result of Ambrosio and Braides.

5.3.2 Existence of minimizers in a bounded design region

If we look at the proof of Proposition 5.3.1 and we try to repeat the same arguments for a sequence of functions in $\mathcal{F}_k(\mathbb{R}^d)$, it not said that their limit is still a function in $\mathcal{F}_k(\mathbb{R}^d)$, as seen in Remark 5.1.10. Since at the moment we are not able to prove that linear independence of the components is somehow preserved, we prove an existence result $k = 1$ under the hypothesis of bounded design region. More precisely, we fix an open bounded set $D \subset \mathbb{R}^d$ and study the problem

$$\min\left\{R_{k,\beta}(u) + \mathcal{H}^{d-1}\left(J_u \cup \partial^*(\{u \neq 0\})\right) | u \in \mathcal{F}_1(\mathbb{R}^d), u = 0 \text{ a.e. in } B^c\right\}. \tag{5.12}$$

Proposition 5.3.2. *Problem* (5.12) *admits a solution.*

Proof. Let $B \subset \mathbb{R}^d$ be open and bounded and let $(u_n)_n \subset \mathcal{F}_1(\mathbb{R}^d)$ be a minimizing sequence for (5.12) in the bounded design region B. Without loss of generality, we can assume that $\|u_n\|_{L^2(B)} = 1$ and that the sequence

$$\left(R_{k,\beta}(u_n) + \mathcal{H}^{d-1}\left(J_{u_n} \cup \partial^*(\{u^n \neq 0\})\right)\right)_n$$

is uniformly bounded; then

$$\int_B |\nabla u_n|^2\,dx + \int_B |u_n|^2\,dx + \int_{J_{u_n}} |(u_n)^+|^2 + |(u_n)^-|^2\,d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}\left(J_{u_n} \cup \partial^*(\{u^n \neq 0\})\right) \leq C$$

for some positive constant $C > 0$. In view of Proposition 5.3.1, there exists $u \in SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^d)$ such that $u_n \to u$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^k)$ realizing the lower semicontinuity of the functional. Then, since $\|u\|_{L^2(\Omega)} = 1$, we deduce that $u \in \mathcal{F}_1(\mathbb{R}^d)$ and then u is a minimizer for (5.12). $\square$

5.4 Further remarks and perspectives

As clearly repeated throughout the chapter, even for Problem (5.9) at the moment we are able to prove only an existence result for $k = 1$ in a bounded design region. It reasonable to expect that such a minimizer could be a suitable ball B, or better, a function supported in B (we guess the first eigenfunction of B itself). The perspective is to approach to this Faber-Krahn type inequality following the ideas in [24] and [25], where a similar functional framework has been treated.

To remove the hypotheses of bounded design region, we can prove some boundedness result; for instance, as done in Theorem 4.6 in [27], we can show that some family of minimizers have to be necessarily with bounded support. It would imply the choice of minimizing sequences with uniformity bounded support.

In both sections, we had the problem to ensure that sequences in $\mathcal{F}_k(\mathbb{R}^d)$ converge to functions in $\mathcal{F}_k$. The fact that it is not possible, at a first sight, suggest us to investigate on that direction, finding some arguments ensuring linear independence of the components of the limit function. For instance, it could be an idea to build minimizing sequences in $\mathcal{F}_k(\mathbb{R}^d)$ that converge to a function still in $\mathcal{F}_k(\mathbb{R}^d)$ starting from a suitable sequence of orthonormal eigenfunctions. This idea has been used in [27], to obtain the existence of minimizers (in a weak sense) for $\lambda_{k,\beta}$ among sets satisfying a measure constraint. In this paper, authors used a density argument based on a result of G. Cortesani and R. Toader (see Theorem 3.1 in [33]) to show that the infimum of the original problem coincide with the infimum of the free discontinuity problem. Then, they consider a sequence $(\Omega_n)_n$ minimizing $\lambda_{k,\beta}$ and build a minimizing sequences $(u_n)_n$ for $R_{k,\beta}$ setting $u_n = (u_n^1, \dots, u_n^k)$, where $u_n^1, \dots, u_n^k$ are the k first eigenfunction of Ω_n (assumed orthonormal and extended by zero outside Ω_n). In such a way it is assured that the possible limit function is in $\mathcal{F}_k(\mathbb{R}^d)$. Such a strategy provides a good tool to prove boundedness of the supports of minimizers built via this procedure. We would like to obtain a similar result in our framework, at least to find an argument assuring that the infima of Problems (5.1) and (5.9) coincide. To this aim, we conjecture that, for every $u \in \mathcal{F}_k(\mathbb{R}^d)$, there exists a sequence $(\Omega_n)_n$ of bounded Lipschitz domains in $\mathbb{R}^d$

such that

$$\liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n) + P(\Omega_n) \leq R_{k,\beta}(u) + \mathcal{H}^{d-1}\left(J_{u_n} \cup \bigcup_{i=1}^{k} \partial^*(\{u_i^n \neq 0\})\right).$$

If we prove this result, we have the required equality of the infima.

Chapter 6

The case of polygons

In this chapter we focus our interest on some problems studied in the previous chapters, now in the framework of suitable families of polygons. For the choice of the class of admissible polygons and for some preliminary result we refer to [22], where an appropriate family of admissible polygons have been used. We start defining the family of simple polygons.

Definition 6.0.1 (simple polygon, see [21])**.** A simple polygon is the open bounded planar region P delimited by a finite number of not self-intersecting line segments (called sides) which are pairwise joined (at their endpoints called vertices) to form a closed path.

Let us denote by $\mathcal{P}_N$ the family of simple polygons with at most N sides. Notice that simple polygons are connected and simply connected.

In the following, we will use as a key tool the H^c-convergence, as it preserves many topological properties of polygonal domains. The only disadvantage of this approach is that, in general, a sequence of simple polygons in $\mathcal{P}_N$ does not H^c-converge to a simple polygon in $\mathcal{P}_N$, as shown in the figures below.

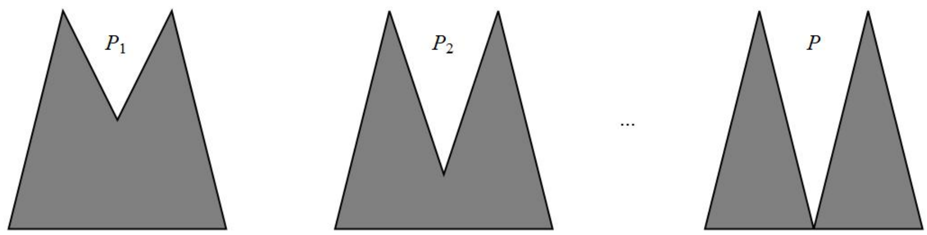


Figure 6.1: The sequence $(P_n)_n \subset \mathcal{P}_5$ H^c-converges to P, that is not a simple polygon.

To overcome this problem and for the sake of well posedness, we choose to follow the approach in [21] and set our shape optimization problems in a wider class of sets in order to admit polygons in a more general sense.

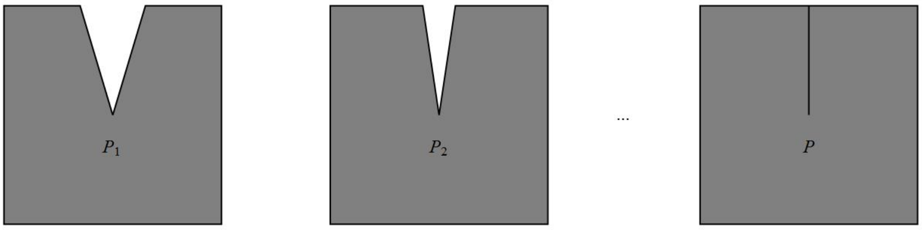


Figure 6.2: The sequence $(P_n)_n \subset \mathcal{P}_7$ H^c-converges to the "degenerate polygon" P, which has a boundary given by line segments, but is not a simple polygon.

Definition 6.0.2 (generalized polygon, see [21])**.** We say that an open set $P \subset \mathbb{R}^2$ is a generalized polygon with at most N sides if there exists a sequence $(P_n)_n$ of simple polygons in $\mathcal{P}_N$ such that P_n locally H^c-converges to P, i.e. if the sequence $(P_n \cap B)_n$ H^c-converges to $P \cap B$ for every ball $B \subset \mathbb{R}^2$.

We denote by $\overline{\mathcal{P}_N}$ the class of generalized polygons with at most N sides.

Remark 6.0.3. The following facts hold true for the family $\overline{\mathcal{P}_N}$ (see [21] for details).

(i) $\overline{\mathcal{P}_N}$ is closed with respect to the local H^c-convergence (in the sense of Definition 6.0.2).

(ii) Every $\Omega \in \overline{\mathcal{P}_N}$ is simply connected, since Ω^c is connected (see Proposition 1.4.17 and Remark 1.4.18).

(iii) $\Omega \in \overline{\mathcal{P}_N}$ may be disconnected; each connected component of Ω is delimited by a finite number of line segments (still called the sides of Ω), which are pairwise joined at their endpoints (still called vertices of) to form a closed path, possibly containing self-intersections; in particular, Ω has at most N sides, counted with their multiplicity.

(iv) Every $\Omega \in \overline{\mathcal{P}_N}$ has has finite Lebesgue measure (see Proposition 2.2.21 in [60]) and is bounded (otherwise, in view of the bound on the number of sides, necessarily Ω would have two parallel sides with infinite length, contradicting the fact that $|\Omega| < +\infty$.

Remark 6.0.4. Let us observe that the number of sides is lower semicontinuos for locally H^c-converging sequences $(P_n)_n \subset \overline{\mathcal{P}_N}$.

Notice that this fact does not hold if the number of sides is not bounded a priori (the sequence $(R_n)_n$ of regular n-gons of measure m centered at a point $x_0 \in \mathbb{R}^2$ H^c-converges to the disk of measure m centered at x_0).

Now we need to set the variational problem, possibly relaxing the definition of $\lambda_{k,\beta}$ in order to consider the case of generalized polygons and to preserve the semicontinuity of the spectral functionals. Following the approach of Section 3.7 in Chapter 3, we set, for every $\beta \in \mathbb{R}$

$$\bar{\lambda}_{k,\beta}(P) := \inf_{S\in\mathcal{S}_k} \sup_{u\in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2\right]\,d\mathcal{H}^1}{\displaystyle\int_\Omega u^2\,dx},$$

where $\mathcal{S}_k$ denotes the set of all k-dimensional subspaces of $H^1(P) \cap L^\infty(P)$, u^+ and u^- are the approximate limsup and liminf of u and the boundary integral on ∂P is considered in the sense of the SBV traces. As remarked in the previous chapters, this definition is well posed. Moreover, if P is a simple polygon, then $\bar{\lambda}_{k,\beta}(P) = \lambda_{k,\beta}(P)$. Since the kind of optimization depends on the sign of the boundary parameter β, we split the discussion into two parts.

6.1 Positive boundary parameter: existence results and open problems

Let us fix $\beta > 0$ and let us study the problem

$$\min\left\{F(\bar{\lambda}_{1,\beta}(P),\dots,\bar{\lambda}_{k,\beta}(P)) : P \in \overline{\mathcal{P}_N}, |P| \le m\right\}, \tag{6.1}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is lower semicontinuos and non decreasing in each variable. We start proving a preliminary lemma concerning the number of sides of possible optimal generalized polygons. The idea is to compare the value of (6.1) on a given generalized polygon $P \in \overline{\mathcal{P}_n}$ and on another polygon obtained cutting P around a convex corner. That idea recalls the technique used in Chapter 4 to prove regularity of convex minimizers.

Remark 6.1.1. In next lemma we will make use of a recent weaker version of Theorem 2.1.3, due to D. Bucur (see [16]) and different by other results giving lower bounds for the first Robin eigenfunction (e.g., in [6] and [51] is given the original version of Theorem 2.1.3, with argument based on semigroup acting on the Lipschitz boundary). We state such more general result in a suitable way for our purposes: if P is a connected generalized polygon and a function $u \in \mathcal{H}^1(P)$ realizes $\bar{\lambda}_{1,\beta}(P)$ for some $\beta > 0$, then there exists $\alpha > 0$ such that $u \ge \alpha$.

Lemma 6.1.2. *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfy the same hypotheses as in* (6.1) *and, in addition, let it be differentiable in each variable with strictly positive derivative* *with respect to the first variable. For every polygon* $P \in \overline{\mathcal{P}_N}$*, there exists a generalized polygon* $P' \in \overline{\mathcal{P}_{N+1}}$ *such that* $|P'| \leq m$ *and*

$$F(\bar{\lambda}_{1,\beta}(P'), \ldots, \bar{\lambda}_{k,\beta}(P')) < F(\bar{\lambda}_{1,\beta}(P), \ldots, \bar{\lambda}_{k,\beta}(P)).$$

In particular, $P \in \overline{\mathcal{P}_N}$ *can not be a minimizer for* (6.1) *in* $\overline{\mathcal{P}_{N+1}}$.

Proof. The proof is based on a similar argument as in Theorem 4.3.3. Let us consider a function u realizing $\lambda_{1,\beta}$ and choose a connected component of P where u is non null (in order to apply the result in Remark 6.1.1). Let us consider a convex corner with vertex x_0 and suppose that, up to a rotation and a translation, $x_0 = 0$ and the bisector of the corner in x_0 is the line $x_1 = 0$. Without loss of generality, in view of Remark 6.1.1, we can assume that there are no sides vertexed in x_0 that have self-intersections (otherwise, one applies the following arguments only at one side of the self-intersection, see figure below).

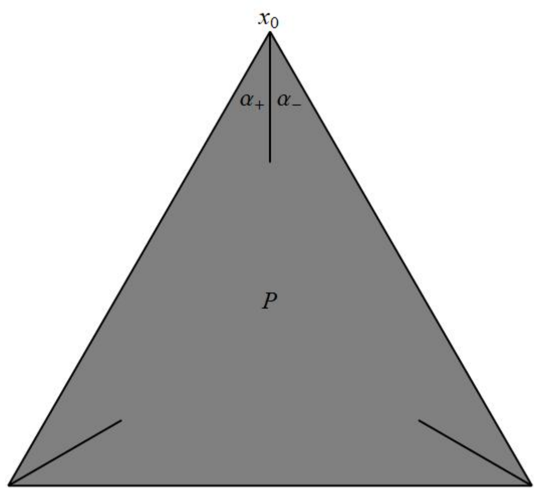


Figure 6.3: In the polygon P all convex corners are determined by self-intersected sides; choosing x_0 as above, without loss of generality, we can apply the arguments of the proof only on one of the corners α_+ and α_-.

For every $\varepsilon > 0$ let us define the sets

$$P_\varepsilon := \Omega \cap \{x_2 < -\varepsilon\}\,,\; m_\varepsilon := P \backslash P_\varepsilon,\; b_\varepsilon := P \cap \{x_2 = -\varepsilon\}\,,\; l_\varepsilon := \partial P \backslash \partial P_\varepsilon. \quad (6.2)$$

Notice that, for ε sufficiently small, P_ε has $N+1$ sides.

In view of Remark 6.1.1, we can choose a strictly positive "eigenfunction" for $\bar{\lambda}_{1,\beta}(P)$ to obtain that

$$\bar{\lambda}_{1,\beta}(P_\varepsilon) \leq \bar{\lambda}_{1,\beta}(P) - C\varepsilon$$

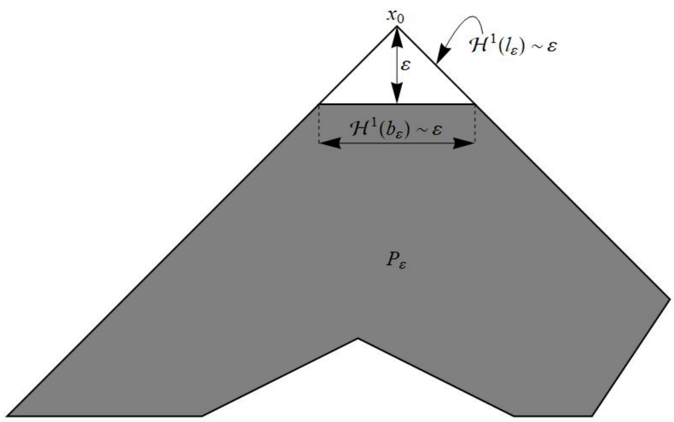


Figure 6.4: The cutting procedure of $P \in \overline{\mathcal{P}_7}$.

for sufficiently small ε, in an analogous way to Lemma 4.2.1. To estimate the higher eigenvalues, in view of our assumptions we can proceed as in Lemma 4.2.2 obtaining

$$\bar{\lambda}_{k,\beta}(P_\varepsilon) \leq \bar{\lambda}_{k,\beta}(P) + o(\varepsilon).$$

The hypotheses on F lead to the first assertion, once we set $P' := P_\varepsilon \in \overline{\mathcal{P}_{N+1}}$ for a suitable value of $\varepsilon > 0$.

In particular, if we consider a generalized polygon $P \in \overline{\mathcal{P}_N} \subset \overline{\mathcal{P}_{N+1}}$, the corresponding generalized polygon P' (built as above) gives us a strictly lower value for Problem (6.1) in $\overline{\mathcal{P}_{N+1}}$, then P cannot be a minimizer in $\overline{\mathcal{P}_{N+1}}$. $\square$

The following theorem gives us an existence result in $\overline{\mathcal{P}_N}$.

Theorem 6.1.3. *Problem* (6.1) *admits a solution $P \in \overline{\mathcal{P}_N}$ with exactly N sides. Moreover, the sequence of the minima $(m_N)_N$ for* (6.1) *(labelled on the number of sides), is strictly decreasing in N.*

Proof. Let us consider a minimizing sequence $(P_n)_n$ for (6.1). Let us suppose that the diameters of the polygons P_n are not uniformly bounded. Since the number of sides is uniformly bounded, the polygons P_n can have at most $N/3$ well separated components (in the case that P_n is union of $N/3$ open triangles).

Then, under the assumption $\sup_n \operatorname{diam}(P_n) = +\infty$, the only possible behaviour is that P_n becomes longer (since the diameters diverge) and thinner (since the measure is bounded a priori) along some directions, creating sharp spikes. Take one of such directions, say the line $x_2 = 0$. Hence, any projection l_n^C of the set P_n on the line $x_1 = C$, has to be union of segments of length tending to 0, if C is sufficiently large. Proceeding as in Theorem 4.1.3, we obtain that a lower bound for every admissible relaxed Rayleigh quotient $\bar{R}_{P_n}(u)$ is given by $\bar{\lambda}_{1,\beta}(l_n^{max})$, where l_n^{max} is the longest among the l_C^n sets. In particular, by the generalized Faber-Krahn inequality (Theorem 2.3.2),

$$\bar{R}_{P_n}(u) \geq \lambda_{1,\beta}(B_{l_n^{max}}),$$

where $B_{l_n^{max}}$ is the segment (1-dimensional ball) with the same length as l_n^{max}. Since $\mathcal{H}^1(B_{l_n^{max}})$ vanishes as n tends to infinity, we obtain that $\lambda_{1,\beta}(B_{l_n^{max}})$ positively diverges, obtaining that any sequence $\bar{R}_{P_n}(u)$ of admissible Rayleigh quotients on P_n diverges, against the hypotheses of minimality on $(P_n)_n$.

Hence, we can suppose that the sequence $(P_n)_n$ is uniformly bounded, say $P_n \subset B_R(0)$ for some $R > 0$ and every $n \in \mathbb{N}$. In view of Remark 6.0.3, there exists a generalized polygon $P \in \overline{\mathcal{P}_N}$, $P \subset B_R(0)$, such that $P_n \to P$ in the H^c-topology and $|P| \leq m$. Moreover, the H^c-convergence of P_n to P implies that $H^1(P_n) \to H^1(P)$ in the sense of Mosco (see Proposition 1.5.3).

To prove that P is a minimizer for (6.1), let us fix $\varepsilon > 0$ and consider an admissible h-dimensional test space V_n for $\bar{\lambda}_{h,\beta}(P_n)$ such that

$$\max_{w\in V_n} \bar{R}_{P_n}(w) \leq \bar{\lambda}_{h,\beta}(P_n) + \varepsilon.$$

Let us consider a $L^2(P_n)$-orthonormal basis of V_n, say $\{u_1^n, \dots, u_h^n\}$. In view of Mosco convergence, for every $j = 1, \dots, h$ there exist $u_j \in H^1(P)$ such that $u_n^j \to u_j$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla u_j^p \rightharpoonup \nabla u_j$ weakly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$ (here we denote with the same symbol the extension of the functions outside their domains). Let V be the h-dimensional vector space spanned by $\{u_1, \dots, u_h\}$ (in view of the L^2-convergence we can suppose the u_j functions linearly independent) and let us consider $v := \sum_{j=1}^h \alpha_j u_j$ such that

$$\bar{R}_P(v) = \max_{w\in V} \bar{R}_P(w).$$

Let us consider $v_n := \sum_{j=1}^h \alpha_j u_j^n \in V_n$ and observe that $v_n \to v$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Thanks to continuity of the volume integrals at the denominator and to the lower semicontinuity of the gradient integral and of the boundary integral (see Theorem 3.7.3), we obtain

$$\begin{aligned}\lambda_{k,\beta}(P) &\leq \max_{w\in V} \bar{R}_P(w) = \bar{R}_P(v) \leq \liminf_{n\to+\infty} \bar{R}_{P_n}(v_n)\\ &\leq \liminf_{n\to+\infty} \max_{w\in V_n} \bar{R}_{P_n}(w) \leq \liminf_{n\to+\infty} \lambda_{h,\beta}(P_n) + \varepsilon.\end{aligned}$$

Letting ε go to 0, we obtain that P is a minimizer for Problem (6.1).

Moreover, P has exactly N sides. Indeed, if it had less than N sides, say $N - K$ sides, we can apply K times Lemma 6.1.2 to obtain a polygon P' with exactly N sides, $|P'| \leq m$ and

$$F(\bar{\lambda}_{1,\beta}(P'), \dots, \bar{\lambda}_{k,\beta}(P')) < F(\bar{\lambda}_{1,\beta}(P), \dots, \bar{\lambda}_{k,\beta}(P)),$$

contradicting the minimality of P.

We also deduce that every minimizer in $\overline{\mathcal{P}_N}$ cannot be a minimizer in $\overline{\mathcal{P}_{N+1}}$ and this implies that the sequence of minima $(m_N)_N$ is strictly decreasing. $\square$

Remark 6.1.4. If we do not require that the number of sides of the admissible polygons is bounded, than Problem (6.1) does not have a solution in general. An easy counterexample is given by

$$\min\left\{\bar{\lambda}_{1,\beta}(P) : P \text{ generalized polygon}, |P| \leq m\right\},$$

whose infimum is given by $\lambda_{1,\beta}(B_m)$, where B_m is the ball of measure m: this value is not attained on any finite perimeter set of measure m, except on the ball itself.

If we require a little more regularity on the admissible polygons, it is natural to study the following problem:

$$\min\left\{F(\lambda_{1,\beta}(P), \dots, \lambda_{k,\beta}(P)) : P \in \mathcal{P}_N, |P| \leq m, P \text{ convex}\right\}. \tag{6.3}$$

Notice that, in view of the convexity hypotheses, it is not necessary to consider also degenerate polygons. An existence result is obtainable as a corollary to Theorem 6.1.3.

Corollary 6.1.5. *Problem* (6.3) *admits a solution with exactly N sides. Moreover, the sequence of the minima $(m_N)_N$ for* (6.3), *is strictly decreasing in N.*

Proof. The proof is a consequence of Theorem 6.1.3 applied with the regularity hypotheses in (6.3). $\square$

6.1.1 Further remarks and open problems

Our choice to set problem (6.1) in a relaxed setting ensures us a more general existence result; whether the problem is or not a relaxation of

$$\min\left\{F(\lambda_{1,\beta}(P), \dots, \lambda_{k,\beta}(P)) : P \in \mathcal{P}_N, |P| \leq m\right\}$$

is not clear. In other words, we would like to understand if optimal shapes for (6.1) are simple polygons. An idea could be to assume that a minimizer P is in $\overline{\mathcal{P}_N} \setminus \mathcal{P}_N$ and build admissible simple polygons that are better than P in term of optimization (e.g. deleting some fractures). If we are able to prove that, we can conclude that solutions to (6.1) are, in fact, simple polygons.

It could be interesting to find some isoperimetric inequalities on polygons, at least for a low number of sides. The main difficulty in a first approach to the problem is that it is not possible to transpose the same argument used to prove the Faber-Krahn inequality for the first Robin eigenvalue in [13] and [37]. Indeed, their results are based on the radiality of the first eigenfunction of the disk and on a comparison between the level sets of such function (that are smooth) and the level sets of an eigenfunction of any domain.

It seems that very few results on isoperimetric inequalities on polygons are already available. An interesting work in that direction is the recent paper [49] by P. Freitas and J. Kennedy, where the authors proved that, for every $\beta > 0$, the square minimizes $\lambda_{1,\beta}$ and the union of two equal square minimizes $\lambda_{2,\beta}$ in the class of disjoint unions of rectangles with prescribed area $A > 0$ ([49], Theorem 4.1 and Corollary 4.2). In addition, for higher eigenvalues they proved that the union of k disjoint equal squares minimizes $\lambda_{k,\beta}$ only for small values of the β, the bound of beta being directly proportional to the ratio $(k/A)^{1/2}$ ([49], Theorem B). The proof of such results are based on the explicit expression of the eigenvalues for such particular domains (the choice to work on rectangles is due to the possibility to represent explicitly the eigenfunctions via separation of variables). A smart technique could be to find an explicit expression of the first eigenfunction for the regular N-gone R_N and then to compute via smooth transformations any first eigenfunction $u \in H^1(P_N)$ for the $\lambda_{1,\beta}(P_N)$, where P_N is a general, admissible N-gone P_N, hoping to obtain some estimates like $\lambda_{1,\beta}(R_N) \leq R_{P_N}(u)$. In that sense, for $N = 3$, a book by McCartin [67] seems to be useful; in that reference are given explicit representations of the Robin eigenfunctions for the equilateral triangle. One could try to manipulate them in order to obtain eigenfunctions of general triangles, even if it seems very technical.

6.2 Negative boundary parameter: existence results and open problems

If the amount of references for the Robin problems with positive boundary parameter on polygons is very limited, for $\beta < 0$ nothing seems to be done. In this short section we give some existence result and some properties of the optimal sets. In the following we use the notation of Chapter 3: we consider a negative boundary parameter $-\beta < 0$ and write $\lambda_{k,\beta}$ instead of $\lambda_{k,-\beta}$. We focus on the problem

$$\max\left\{F(\bar{\lambda}_{1,\beta}(P), \dots, \bar{\lambda}_{k,\beta}(P)) : P \in \overline{\mathcal{P}_N}, |P| = m\right\} \tag{6.4}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is non increasing and upper semicontinuos in each variable. The following theorem gives us an existence results for polygons with at most

N sides.

Theorem 6.2.1. *Problem* (6.4) *admits a solution* $P \in \overline{\mathcal{P}_N}$ *with at most*

$$\min\left\{\frac{N}{3}, \frac{m|\lambda_{k,\beta}(R_m^N)|^d\omega^d}{\beta^d} + k\right\}$$

well separated components, where R_m^N *is the regular* N*-gon of measure* m.

Proof. Let us consider a maximizing sequence $(P_n)_n$ for (6.4). Using the same tools as in Chapter 3, we can assume that $\sup_n \mathcal{H}^1(\partial P_n) < +\infty$ and so, since we are in $\mathbb{R}^2$, also $\sup_n \operatorname{diam}(P_n) < +\infty$. Then, up to subsequences, the sequence $(P_n)_n$ H^c-converge to a polygon $P \in \mathcal{P}_{N,\varepsilon}$. The convergence is also in measure and then, by Proposition 1.5.3, $H^1(P_n)$ converges to $H^1(P)$ in the sense of Mosco. Then, repeating the same arguments in Theorem 3.7.7, we can conclude that problem (6.4) admits P as a solution. Moreover, the number of well separated components of every solution is less than both $N/3$ and $\frac{m|\lambda_{k,\beta}(R_m^N)|^d\omega^d}{\beta^d} + k$: the first bound depends on the fact that P has at most N sides, the second is a consequence of the same arguments in 3.7.7, replacing the admissible set B_m (the ball of measure m) by the regular N-gon of measure m R_m^N. □

Notice that, in view of the definition of the class $\overline{\mathcal{P}_N}$, we speak only of well separated components, not of connected components, as we are not allowed to separate connected components having some shared sides.

If we require a little more regularity on the admissible polygons, it is natural to study the problem

$$\min\left\{F(\lambda_{1,\beta}(P), \ldots, \lambda_{k,\beta}(P)) : P \in \mathcal{P}_N, |P| = m, P \text{ convex}\right\}, \tag{6.5}$$

where we can remove the uniform cone hypothesis. A trivial corollary to Theorem 6.2.1 is the following.

Corollary 6.2.2. *Problem* (6.5) *admits a solution.*

Proof. The proof is given combining Theorems 6.2.1 and 3.8.6. □

6.2.1 Further remarks and open problems

A first interesting problem to study could be to prove that optimizers for (6.4) have exactly N sides, we are not able to apply a similar version Lemma 6.1.2: the argument used in the case of positive boundary parameter fails in this setting.

It could be interesting to find some isoperimetric inequalities on polygons, at least for a low number of sides. At the moment no results are known (up to our knowledge), but we can imagine that they are based on the explicit representation of the eigenfunctions. For instance, an idea to study isoperimetry of triangles can start from such a representation. Starting from any triangle of fixed area, one takes the isosceles triangle with same basis and height and, via an affine transformation, modify the eigenfunction of such isosceles triangle into an admissible function for the generic triangle. A good starting point in that direction could be the study of Chapter 7 in [67].

We end this chapter reporting a recent field of research involving polygons with Robin conditions on the boundary: the study of the honeycomb conjecture (roughly speaking, “the optimal shape for the cells of an honeycomb is the regular hexagon”) for the Robin Laplacian eigenvalues. Indeed, trying to prove that conjecture, some authors prove isoperimetric inequalities on convex polygons for some functionals linked with Robin eigenvalues. To get acquainted on the topic see [22], where some isoperimetric results are shown and where authors highlight the lack of more general Faber-Krahn inequalities for Robin eigenvalues on polygons.

Appendix A

Some necessary conditions of optimality

In this brief chapter we are going to recall (mostly from Chapter 5 in [60], in particular Sections 5.4, 5.6 and 5.7) a procedure to obtain necessary conditions on optimal sets for some spectral shape optimization problem. In particular, we will focus on the Robin eigenvalues, recalling some necessary conditions of optimality when the involved sets are regular. We will focus also on the case of multiple eigenvalues as, up to our knowledge, this situation has not been treated in detail yet, although it does not seem unreasonable to generalize some similar results obtained for other spectral problems involving the Dirichlet Laplacian eigenvalues (see, for instance [10]). We started from the question: if a problem admits optimal shapes (possibly satisfying additional topological conditions), do the optimal shapes are also "local optima"? In other words, do the shape functional can be derived in some sense to have optimality condition in an analogous way to real valued functions defined on $\mathbb{R}^k$?

A.1 Optimality condition for a simple eigenvalue

A.1.1 Shape derivative: definitions and key results

An important tool used to obtain optimality conditions is the so called shape derivative, that is a sort of first variation of a shape functional under deformations by smooth vector fields. In this short survey we refer mostly to Section 5.4 in [60]; we will recall the definitions of the main tools used to derive a boundary value problem; in particular, we will focus on the derivation of eigenvalue of the Laplace operator with Robin conditions.

We first recall the definition of tangential gradient (see Definition 5.4.5 in

[60]).

Definition A.1.1 (tangential gradient)**.** Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^1 boundary and let $g \in C^1(\partial\Omega)$. The tangential gradient of g on $\partial\Omega$ is defined by

$$\nabla_{\partial\Omega} g := \nabla\tilde{g} - (\nabla\tilde{g} \cdot n)n,$$

where n is the outer normal unit vector on $\partial\Omega$ and $\tilde{g} : \mathbb{R}^d \to \mathbb{R}$ is a C^1 extension of g.

It can be proved that the definition above is independent on the extension $\tilde{g}$ (we refer again to [60], more precisely yo the remark just below Definition 5.4.5). It is convenient to recall also the definition of tangential divergence (see Definition 5.4.6 in [60]).

Definition A.1.2 (tangential divergence)**.** Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^1 boundary and let $V \in C^1(\partial\Omega; \mathbb{R}^d)$. We define the tangential divergence of V by

$$\mathrm{div}_{\partial\Omega} V := \mathrm{div}\tilde{V} - \tilde{V}' n \cdot n,$$

where n is the outer normal unit vector on $\partial\Omega$ and $\tilde{V} : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 extension of V.

As in the case of the tangential gradient, the definition is independent on the extension $\tilde{V}$.

Another important notion in this framework is the mean curvature of a surface (here we recall Definition 5.4.7 in [60]).

Definition A.1.3 (mean curvature)**.** Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^2 boundary. We define the mean curvature of $\partial\Omega$ by

$$\mathcal{H} := \mathrm{div}_{\partial\Omega} n,$$

where n is the outer normal unit vector on $\partial\Omega$.

To extend the tangential gradient and the tangential divergence to a more general functional setting, we recall the definition of Sobolev spaces on a smooth topological boundary.

Definition A.1.4 (Sobolev Spaces on $\partial\Omega$)**.** Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^2 boundary. We define the Sobolev Space $W^{1,1}(\partial\Omega; \mathbb{R}^k)$ as the closure of $C^1(\Omega; \mathbb{R}^k)$ with respect to the norm

$$\|u\|_{W^{1,1}(\partial\Omega;\mathbb{R}^d)} := \int_{\partial\Omega} |u| \, d\mathcal{H}^{d-1} + \int_{\partial\Omega} |\nabla_{\partial\Omega} u| \, d\mathcal{H}^{d-1},$$

omitting $\mathbb{R}^k$ if $k = 1$. We define the Sobolev Space $W^{2,1}(\partial\Omega)$ by

$$W^{2,1}(\partial\Omega) := \left\{ u \in W^{1,1}(\partial\Omega) : \nabla_{\partial\Omega} u \in W^{1,1}(\partial\Omega; \mathbb{R}^d) \right\}$$

As remarked in Section 5.4 of [60], the definitions of tangential gradient and tangential divergence can be extended respectively to real valued functions in $W^{1,1}(\partial\Omega)$ and to vector valued functions in $W^{1,1}(\partial\Omega;\mathbb{R}^d)$.

Now, we recall the definition of Laplace-Beltrami operator, that generalizes to a smooth hypersurface the notion of Laplace operator (see Definition 5.4.11 in [60])

Definition A.1.5 (Laplace-Beltrami operator)**.** Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^2 boundary. We define the Laplace-Beltrami operator on $\partial\Omega$ by

$$\Delta_{\partial\Omega} u := \operatorname{div}_{\partial\Omega}\left(\nabla_{\partial\Omega} u\right),$$

for every $u \in W^{2,1}(\partial\Omega)$.

Remark A.1.6 (see Proposition 5.4.12 and Theorem 5.4.13 in [60])**.** For every domain of class C^2 and every $u \in C^2(\overline{\Omega})$, the next decomposition of the Laplace operator on $\partial\Omega$ holds (see Formula (5.54) in [60]):

$$\Delta u = \Delta_{\partial\Omega} u + \mathcal{H}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2}.$$

By density, we can extend that formula to functions in $H^3(\Omega)$. Moreover, for every $f \in H^2(\Omega)$ and every $g \in H^3(\Omega)$, the following integration by parts formula holds (see Formula (5.59) in [60])

$$\int_{\partial\Omega} \nabla_{\partial\Omega} f \cdot \nabla_{\partial\Omega} g \, d\mathcal{H}^{d-1} = -\int_{\partial\Omega} f \Delta_{\partial\Omega} g \, d\mathcal{H}^{d-1}. \tag{A.1}$$

A.1.2 How to derive a boundary value problem

In the following section we refer to Section 5.6 and Paragraph 5.4.4 of [60], in order to give survey on how to derive (heuristically!) a boundary value problem when a domain varies under smooth perturbations. Our aim is to give to the reader a short user's guide to derive every boundary value problem, even if it is not linear.

We start defining the normed space (see Paragraph 5.4.4 in [60])

$$C^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d) := C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$$

endowed with the $W^{1,\infty}$-norm. Let us fix a smooth vector field $V \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and let us consider a function Φ defined by

$$\begin{aligned} [0,T[&\to C^{1,\infty}, \\ t &\mapsto \Phi(t), \end{aligned}$$

where $\Phi(t)$ is a function in $C^{1,\infty}$, derivable in 0, such that

$$\Phi(0) = I, \quad \Phi'(0) = V.$$

For instance, an admissible Φ is given, for every $t \in [0, T[$, by

$$\begin{aligned} &\mathbb{R}^d \to \mathbb{R}^d, \\ &x \mapsto [\Phi(t)](x) := x + tV(x). \end{aligned}$$

Let us denote by

$$\Omega_t := [\Phi(t)](\Omega) \tag{A.2}$$

the family of smooth domains obtained as images of Ω by the function $\Phi(t)$ (for all the previous notation we refer to the whole of Chapter 5 in [60]).

The first step is to define the outer normal unit vector on every boundary $\partial\Omega_t$ in such a way that the family of normal unit vectors $(n_t)_t$ varies smoothly in t and that, for $t = 0$, we obtain the outer normal unit vector to $\partial\Omega$. To this aim, we recall the following result (see Proposition 5.4.14 in [60]).

Proposition A.1.7 (extension of a normal vector field on a varying domain)**.** *Let Ω be a domain of class C^2 and let $(\Omega_t)_t$ the family of domains defined in* (A.2)*. Let $n \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ an extension of the outer normal unit vector to $\partial\Omega$ (still denoted by n). Let us define*

$$w(t) := \left[{}^t(D\Phi(t)^{-1})n\right] \circ \Phi(t)^{-1};$$

then, the function

$$n_t := \frac{w(t)}{\|w(t)\|}$$

is an extension of n to $\partial\Omega_t$, $n_t \in C^0(\mathbb{R}^d; \mathbb{R}^d)$ and the map $t \mapsto n_t$ is continuous in $[0, T[$ and admits right first derivative in 0. Moreover, for every continuous extension $\tilde{n}_t$ of n to $\partial\Omega_t$, the first derivative in $t = 0$ is given by

$$\left.\frac{\partial \tilde{n}_t}{\partial t}\right|_{t=0} = -\nabla_{\partial\Omega}(V \cdot n) - (D\tilde{n}_0 \cdot n)(V \cdot n)$$

on $\mathbb{R}^d$. In particular, considering the extension n_t above, it holds

$$\left.\frac{\partial n_t}{\partial t}\right|_{t=0} = -\nabla_{\partial\Omega}(V \cdot n) - (Dn \cdot n)(V \cdot n)$$

Once we define the above extension of the normal unit vector, we are able to pass to a derivation of a large family of boundary value problems; for all the following arguments we refer to Section 5.6 in [60]. Let us consider

$$\begin{cases} A(t, u_t) = f & \text{in } \Omega_t, \\ B(t, u_t) = g & \text{in } \partial\Omega_t, \end{cases} \tag{A.3}$$

on Ω_t, where u_t is a solution of Problem (A.3) and the operators $A(t,\cdot)$ and $B(t,\cdot)$ act on suitable functional spaces respectively on Ω_t and $\partial\Omega_t$. The idea is to derive the boundary value problem with respect to t in $t=0$. Before starting with the formal derivation of the problem, let us remark that the map $t \to u_t$ is differentiable (by an application of implicit function theorem, as explained in Section 5.6 of [60]). Once shown such derivability, we can consider the function

$$u' := \left.\frac{\partial u_t}{\partial t}\right|_{t=0}.$$

Under suitable hypotheses of regularity on A, B, f, g (that allow the derivation of the problem below, see the end of Section 5.6 in [60], also for further references), it can be proved that u' is a solution to the problem

$$\begin{cases} \partial_t A(0,u) + \partial_u A(0,u)\cdot u' = 0 & \text{in } \Omega, \\ \partial_t B(0,u) + \partial_u B(0,u)\cdot u' = \dfrac{\partial}{\partial n}\left(g - B(0,u)\right)(V\cdot n) & \text{in } \partial\Omega. \end{cases} \tag{A.4}$$

(the computations are quite standard; we refer again to Section 5.6 in [60]). In next paragraph, we will apply these ideas to the derivation of the Robin eigenvalues problem.

A.1.3 Necessary condition of optimality for a simple Robin eigenvalue

Now, we are going to recall an application of the results in the previous paragraph to the derivation of simple eigenvalues of the Laplacian operator with Robin boundary conditions. For many details we refer to Section 5.7 in [60], where the computation is made in the case of Dirichlet and Neumann eigenvalues (that can be easily transposed to the Robin case); we also refer to some works where the computation is made (or recalled), e.g. [8], where other references to the original computation can be found.

The reason why we deal with the case of simple eigenvalues is easily understandable looking at the following example in finite dimension (see the example at the beginning of Section 5.7 in [60]). Let $A_t \in \mathbb{R}^{2,2}$ be the square matrix defined by

$$A_t := \begin{pmatrix} 1-t & 0 \\ 0 & 1+t \end{pmatrix}.$$

If we order its two eigenvalues $\lambda_1(A_t), \lambda_2(A_t)$ in non decreasing order, we have that

$$\lambda_1(A_t) = 1+t, \quad \lambda_2(A_t) = 1-t, \quad \forall t < 0,$$

$$\lambda_1(A_0) = \lambda_2(A_0) = 1$$

and

$$\lambda_1(A_t) = 1 - t, \quad \lambda_2(A_t) = 1 + t, \quad \forall t > 0.$$

In other words

$$\lambda_1(A_t) = 1 - |t|, \quad \lambda_2(A_t) = 1 + |t|, \quad \forall t \in \mathbb{R},$$

then neither $t \mapsto \lambda_1(A_t)$ nor $t \mapsto \lambda_2(A_t)$ are differentiable in $t = 0$. The reason is that, since the eigenvalue λ_1 is multiple in $t = 0$, the two (analytic) curves

$$t \mapsto 1 - t, \quad t \mapsto 1 + t,$$

describing the roots of the characteristic polynomial, meet in 0, creating a corner. Before the corner $(t < 0)$ $1 + t < 1 - t$, so $\lambda_1(A_t) = 1 + t$ and $\lambda_2(A_t) = 1 - t$; after the corner (t>0), the two analytic functions exchange their role, i.e. $\lambda_1(A_t) = 1 - t < 1 + t = \lambda_2(A_t)$. Then, we could not differentiate the eigenvalues of the matrix A_0, because the two functions are not differentiable in $t = 0$.

If we let a domain Ω vary under a smooth vector field $\Phi(t)$ in the same way as in (A.2), the same situation may occur: we could have that a Robin eigenvalue of $\Omega = \Omega_0$ is multiple and then the curves, given by $t \mapsto \lambda_{h,\beta}(\Omega_t)$, intersect in $t = 0$. Such curves are not differentiable, in general (as in the example above); then it is not possible to find a first order necessary condition based on the derivative of $t \mapsto \lambda_{k,\beta}(\Omega_t)$, to say that $\lambda_{k,\beta}(\Omega)$ is a local optimum. Whether the curves can be reordered across the intersection to have all analytic curves, is in interesting question that we treat in the next section.

To overcome the problem, in this section we consider only the case of simple eigenvalues: under this hypothesis, it is possible to work only on one differentiable curve $t \mapsto \lambda_k(\Omega_t)$, then we are able to write a first order condition based on the derivative $\lambda'_{k,\beta}(\Omega_t)$.

We denote by $\mathcal{V}_0$ the family of the smooth volume preserving vector fields and by $E_{\lambda_{k,\beta}(\Omega)}$ the eigenspace relative to the eigenvalue $\lambda_{k,\beta}(\Omega)$.

Proposition A.1.8 (derivation of a simple Robin eigenvalue and necessary condition of optimality)**.** *Let Ω be a C^3 domain and let $V \in \mathcal{V}_0$. If $\lambda_{\beta,k}(\Omega)$ is simple, then*

$$\lambda'_{k,\beta}(\Omega) = \int_{\partial\Omega} \left[|\nabla_{\partial\Omega} u|^2 - (\lambda_{k,\beta}(\Omega) + \beta^2 + \beta\mathcal{H})u^2 \right] V \cdot n \, d\mathcal{H}^{d-1}$$

for some eigenfunction $u \in E_{\lambda_{k,\beta}(\Omega)}$. In particular, if Ω is (locally) optimal, it holds

$$\int_{\partial\Omega} \left[|\nabla_{\partial\Omega} u|^2 - (\lambda_{k,\beta}(\Omega) + \beta^2 + \beta\mathcal{H})u^2 \right] V \cdot n \, d\mathcal{H}^{d-1} = 0$$

for some eigenfunction $u \in E_{\lambda_{k,\beta}(\Omega)}$.

Sketch of the proof. We just sketch the proof as the result is already known in literature and its proof follows the ideas in Theorems 5.7.1 and 5.7.2 of [60], where there are the derivatives of the Dirichlet and Neumann eigenvalues, respectively. We then consider the boundary value problem

$$\begin{cases} -\Delta u_t = \lambda_{k,\beta}(\Omega_t) u_t & \text{in } \Omega_t, \\ \frac{\partial u_t}{\partial n_t} + \beta u_t = 0 & \text{in } \partial\Omega_t, \end{cases} \tag{A.5}$$

with the condition of normalization

$$\int_{\Omega_t} u_t^2 \, dx = 1. \tag{A.6}$$

Provided that the qualitative hypotheses of the previous section are satisfied, we derive in $t = 0$ the boundary value problem (A.5)as (A.3), with

$$A(t, u_t) = -\nabla u_t - \lambda_{k,\beta}(\Omega_t), \ B(t, u_t) = \frac{\partial u_t}{\partial n_t} + \beta u_t, \ f = g = 0;$$

in addition, we derive also the condition of normalization (A.6), obtaining $\int_\Omega u \, u' \, dx = 0$. We obtain the correspondent of the boundary value problem (A.4) (with our choice of A, B, f, g). Taking the first equation of this new boundary value problem on Ω, the thesis is obtained multiplying such equation by u, integrating on Ω, applying the Robin boundary condition and the condition $\int_\Omega u \, u' \, dx = 0$. □

A.2 Remarks and perspectives

Let us observe that all the previous considerations about necessary conditions of optimality hold considering both positive and negative boundary parameter in the Robin problem, so the approach is the same for minimality conditions when dealing with positive boundary parameter and maximality conditions when dealing with negative boundary parameter.

A.2.1 Some remarks about derivation of multiple eigenvalues

To perform a derivation of multiple eigenvalues, we have to pay attention to the matricial example at the beginning of the chapter. Under some suitable hypotheses, it is possible to reorder the eigenvalues in such a way that every eigencurve $t \mapsto \lambda_k(t)$ turns out to be analytic. In such a way, we can imagine that an optimality condition can be given in two different ways: either $t = 0$ is a stationary point (of minimum or maximum) for the map $t \mapsto \lambda_{k,\beta}(\Omega_t)$ (in the case of simple eigenvalues) or in $t = 0$ there exist right and left derivative of $t \mapsto \lambda_{k,\beta}(\Omega_t)$ and have opposite sign (obtaining a minimum/maximum point with a cusp, in case of multiple eigenvalues).

The key point in this framework is to understand if a map of the type $t \mapsto \Omega_t$ (see (A.2)) perturb analytically the spectrum of a linear operator in the sense of Kato (see [62]) or, in other words, if the map

$$t \mapsto \lambda_{k,\beta}(\Omega_t)$$

is analytic near $t = 0$. An important result is the following.

Proposition A.2.1. *The map $t \mapsto \lambda_{k,\beta}(\Omega_t)$ is analytic in a neighbourhood of $t = 0$.*

The proof of the previous proposition can be found combining Theorem 4.4, Chapter VII, Paragraph 6 in [62] (holding for Dirichlet eigenvalues) with the remark at the bottom of page 425, where the author explain the technical substitutions to be done to obtain the result for Neumann or Robin eigenvalues.

To our knowledge, even this analyticity is not enough to ensure the possibility to gain optimality conditions for multiple eigenvalues. Indeed, to our purposes, we would need a result such as: "if $\lambda_{k,\beta}(\Omega)$ is an eigenvalue for Ω with multiplicity m, then there exist m distinguished analytic maps $t \mapsto \lambda^{(1)}_{k,\beta}(t) \in \mathbb{R}, \dots, t \mapsto \lambda^{(m)}_{k,\beta}(t) \in \mathbb{R}$ and $u_{1,t}, \dots, u_{m,t} \in H^1(\Omega_t)$ such that, for every $i = 1, \dots, m$ and t sufficiently small,

$$\begin{cases} -\Delta u_{i,t} = \lambda^{(i)}_{k,\beta}(t) u_{i,t} & \text{in } \Omega_t, \\ \frac{\partial u_{i,t}}{\partial n} + \beta u_{i,t} = 0 & \text{in } \partial\Omega_t \end{cases}$$

and $\lambda^{(i)}_{k,\beta}(0) = \lambda_{\beta,k}(\Omega)$". This request comes from some works about the same problem on Dirichlet eigenvalues, were, also for the perturbed problem, the boundary condition remains of the Dirichlet type. In our case, it is clear that the perturbed problem is still a Robin problem, but is not clear whether the boundary condition remains the same (i.e. if we have to replace β by some $\beta(t)$).

It is worth to investigate how such multiple Robin eigenvalues can be treated; as remarked above, some references about the Dirichlet problem suggest a way to follow; we cite again the papers [10], [12] and [46] to get acquainted on the problem.

Appendix B

Some properties and results about non-local Robin-Laplacian and fractional Sobolev Spaces

In this chapter we will present some transposition in the non-local setting of the classical properties of the Robin Laplacian (selfadjointness, spectral representation, etc...) and its eigenvalues.[1] In Section 1, we recall (by [1], [43] and [44]) the necessary tools to understand the following of the chapter and we provide some results about the linear problem associated to the non-local Robin Laplacian. In the Section 2, we prove that the operator satisfies the hypotheses of Theorem 1.3.8 to have a min-max formula for the eigenvalues and we prove some basic properties of the eigenvalues, highlighting analogies and differences with the local case. In last section, as a byproduct of our analysis, we prove a non-local version of Chenais' uniform extension theorem 1.4.27 (see [30] for the original reference), as we think it could be useful to develop some techniques to approach non-local shape optimization problems involving uniformly regular sets. This last section is inspired by Section 5 in [43], where an extension theorem for fractional Sobolev spaces on a fixed Lipschitz domain is given.

[1]The results of the first two sections are obtained in a joint work with A. Carbotti, Università del Salento.

B.1 Some preliminary tools and the linear problem

The natural non-local counterpart of the classical Laplace operator is the fractional Laplace operator $(-\Delta)^s$. For the setting of the problem and for the preliminary results we refer to [1], [43] and [44].

Definition B.1.1 (fractional Laplace operator)**.** Let $s \in (0,1)$. We define the fractional Laplacian operator $(-\Delta)^s$ setting, for every u in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$,

$$(-\Delta)^s u(x) := c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}}\, dy,$$

where $P.V.$ stands for "in the sense of the principal value" and $c_{d,s}$ is a dimensional constant depending on d and s given by

$$c_{d,s} := \left(\int_{\mathbb{R}^d} \frac{1-\cos(\zeta_1)}{|\zeta|^{d+2s}}\, d\zeta \right)^{-1}.$$

The previous definition can be intended also in a weak sense, for functions that are not smooth (see [1] for some references on the topic). To our purposes, it is sufficient to consider functions in a suitable Hilbert space.

Definition B.1.2. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. We define the space $H^s(\Omega)$ by

$$H^s(\Omega) := \left\{ u \in L^2(\Omega);\ \frac{u(x) - u(y)}{|x-y|^{\frac{d}{2}+s}} \in L^2(\Omega \times \Omega) \right\}. \tag{B.1}$$

These space $H^s(\Omega)$ is the non-local counterpart of the classical Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$; it is very useful to extend the fractional Laplacian to an Hilbert space of non-smooth functions, as we will see in the following. Now, to give the non-local counterpart of the normal derivative, we define the operator below.

Definition B.1.3 (non-local normal derivative)**.** Let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz set. For every $u \in L^2(\mathbb{R}^d)$, we define the non-local normal derivative outside $\overline{\Omega}$ by

$$\mathcal{N}_s u(x) := c_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy \text{ for } x \in \mathbb{R}^d \setminus \overline{\Omega} \tag{B.2}$$

where $c_{d,s}$ is the same normalization constant in the definition of fractional Laplacian.

Now, we introduce the linear Robin problem for the non-local Laplacian; as in the local setting, we deal with solutions in a weak sense (in the sense of Sobolev spaces). To this aim, we define the following Hilbert space:

$$H^s_{\Omega,g} := \left\{ u : \mathbb{R}^d \to \mathbb{R} \quad \text{measurable} \, ; \|u\|_{H^s_{\Omega,g}} < \infty \right\} \tag{B.3}$$

where

$$\begin{aligned} \|u\|^2_{H^s_{\Omega,g}} :&= \|u\|^2_{L^2(\Omega)} + \beta \|u\|^2_{L^2\left(\mathbb{R}^d \setminus \Omega\right)} + \left\| |g|^{1/2} u \right\|^2_{L^2\left(\mathbb{R}^d \setminus \Omega\right)} \\ &+ \int_{\mathbb{R}^{2d} \setminus (\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy \end{aligned} \tag{B.4}$$

Definition B.1.4 (non-local Robin linear problem)**.** Let $f \in L^2(\Omega)$ and $g \in L^1\left(\mathbb{R}^d \setminus \Omega\right)$. We say that $u \in H^s_{\Omega,g}$ is a solution of the Robin-Laplacian linear problem on Ω with source f and boundary (or external) condition g if u solves

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega \\ \mathcal{N}_s u + \beta u = g \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \end{cases} \tag{B.5}$$

in a weak sense, i.e. if u solves

$$\begin{aligned} \int\int_{\mathbb{R}^{2d} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+2s}} dxdy + \beta \int_{\mathbb{R}^d \setminus \Omega} uvdx \\ = \int_{\mathbb{R}^d \setminus \Omega} gvdx + \int_{\Omega} fvdx \end{aligned} \tag{B.6}$$

for every $v \in H^s_{\Omega,g}$.

To handle the weak formulation (B.6) above, it is very useful the following non-local version of the integration by parts formulae, see Lemma 3.2 in [44].

Lemma B.1.5 (non-local divergence theorem and integration by parts formula)**.** *Let $u, v \in C^2_b\left(\mathbb{R}^d\right)$. Then*

$$\int_{\Omega} (-\Delta)^s u dx = -\int_{\mathbb{R}^d \setminus \Omega} \mathcal{N}_s u dx \tag{B.7}$$

and

$$\int_{\mathbb{R}^{2d} \setminus (\Omega^c)^2} \frac{(u(x) - (u(y))(v(x) - (v(y))}{|x-y|^{d+2s}} \, dx \, dy = \int_{\Omega} v (-\Delta)^s u dx \int_{\mathbb{R}^d \setminus \Omega} v \mathcal{N}_s u \, dx. \tag{B.8}$$

Using the previous lemma we are able to prove the uniqueness for solutions of (B.5) when the boundary parameter β is non-negative.

Theorem B.1.6 (Uniqueness for solutions of (B.5))**.** *If $\beta \geq 0$, problem* (B.5) *admits a unique solution.*

Proof. Let u_1, u_2 be two solutions (B.5); then $w = u_1 - u_2$ solves

$$\begin{cases} (-\Delta)^s w = 0 \text{ in } \Omega \\ \beta w + \mathcal{N}_s w = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

in a weak sense, i.e.

$$\int\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{|w(x) - w(y)|^2}{|x-y|^{d+2s}} dxdy + \beta \int_{\mathbb{R}^d\setminus\Omega} w^2 dx = 0$$

where we chose $w \in H^s_{\Omega,0}$ itself as a test function. It follows that w vanishes a.e. in $\mathbb{R}^d \setminus \Omega$ and that $W(x,y) := w(x) - w(y)$ is null a.e. $\mathbb{R}^{2d} \setminus (\Omega^c)^2$. In particular, W is null on the cylinder $\Omega \times \mathbb{R}^d$, hence we deduce that $w = 0$ a.e. in $\mathbb{R}^d$, proving the unicity of the (weak) solution. □

The following proposition characterizes the weak solutions as critical points[2] of an associated energy functional.

Proposition B.1.7. *Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^d \setminus \Omega)$. Let $I : H^s_{\Omega,g} \to \mathbb{R}$ be the functional defined as:*

$$\begin{aligned} I[u] := {} & \frac{c_{ds}}{4} \int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy + \frac{\beta}{2} \int_{\mathbb{R}^d\setminus\Omega} u^2 dx \\ & - \int_\Omega fudx - \int_{\mathbb{R}^d\setminus\Omega} gudx. \end{aligned} \tag{B.9}$$

Then, u is a critical point of I if and only if u is a weak solution of (B.5)*.*

Proof. Firstly we observe that the functional is well defined on $H^s_{\Omega,g}$; indeed we have, fixed $u \in H^s_{\Omega,g}$

$$\left| \int_\Omega fudx \right| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \|u\|_{H^s_{\Omega,g}} \tag{B.10}$$

and

$$\left| \int_{\mathbb{R}^d\setminus\Omega} gudx \right| \leq \|g\|^{1/2}_{L^1(\mathbb{R}^d\setminus\Omega)} \left\| |g|^{1/2} u \right\|_{L^2(\mathbb{R}^d\setminus\Omega)} \leq C \|u\|_{H^s_{\Omega,g}} \tag{B.11}$$

Therefore, if $u \in H^s_{\Omega,g}$ we have that

$$|I[u]| \leq C \|u\|_{H^s_{\Omega,g}} < +\infty.$$

[2] In the sense of the Gâteau derivative; for more details see [70]

Now we compute the first variation of I.
Fixed ε such that $|\varepsilon| < 1$ and $v \in H^s_{\Omega,g}$, then $u + \varepsilon v \in H^s_{\Omega,g}$, and so we have

$$\begin{aligned}
I[u+\varepsilon v] &= \frac{c_{n,s}}{4}\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{|(u+\varepsilon v)(x) - (u+\varepsilon v)(y)|^2}{|x-y|^{d+2s}} dxdy \\
&+ \frac{\beta}{2}\int_{\mathbb{R}^d\setminus\Omega} (u+\varepsilon v)^2 dx - \int_\Omega f(u+\varepsilon v)\, dx - \int_{\mathbb{R}^d\setminus\Omega} g(u+\varepsilon v)\, dx \\
&= I[u] + \varepsilon\Bigg(\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} dxdy \\
&+ \beta\int_{\mathbb{R}^d\setminus\Omega} uvdx - \int_\Omega fvdx - \int_{\mathbb{R}^d\setminus\Omega} gvdx\Bigg) \\
&+ \varepsilon^2\left(\frac{c_{n,s}}{4}\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy + \frac{\beta}{2}\int_{\mathbb{R}^d\setminus\Omega} v^2 dx\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{I[u+\varepsilon v] - I[u]}{\varepsilon} \\
&\quad= \frac{c_{n,s}}{2}\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} dxdy + \beta\int_{\mathbb{R}^d\setminus\Omega} uvdx \\
&\quad- \int_\Omega fvdx - \int_{\mathbb{R}^d\setminus\Omega} gvdx,
\end{aligned}$$

which means that

$$\begin{aligned}
I'[u](v) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} dxdy + \beta\int_{\mathbb{R}^d\setminus\Omega} uvdx \\
&- \int_\Omega fvdx - \int_{\mathbb{R}^d\setminus\Omega} gvdx.
\end{aligned}$$

Therefore, u is a critical point of I if and only if u is a weak solution of (B.5). □

B.2 The non-local Robin Laplacian operator and its eigenvalues

In this section we focus on the non-Local Robin Laplacian eigenvalues problem, i.e. on the problem

$$\begin{cases} (-\Delta)^s u = \lambda_{k,\beta}(\Omega)\, u \text{ in } \Omega \\ \mathcal{N}_s u + \beta u = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \end{cases} \tag{B.12}$$

or, in a weak form,

$$\int\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}dxdy+\beta\int_{\mathbb{R}^d\setminus\Omega} uvdx$$
$$=\lambda_{k,\beta}(\Omega)\int_\Omega uvdx$$

for every $v\in H^s_{\Omega,0}$.

Before introducing the Robin-Laplacian operator, observe thst $H^s_{\Omega,0}\subset L^2(\mathbb{R}^d)\cap H^s(\Omega)$.

Definition B.2.1. We define the Robin Laplacian operator on $L^2(\Omega)$ as the linear extension of the operator $(-\Delta)^s$ to the space

$$D\left((-\Delta)^s\right):=\{u\in L^2(\mathbb{R}^d)\cap H^s(\Omega)\ ;\ (-\Delta)^s u\in L^2(\Omega),$$
$$\beta u+\mathcal{N}_s u=0 \text{ in } \mathbb{R}^d\setminus\overline{\Omega}\}.$$

A first important property, based on the integration by parts formula, is the following.

Proposition B.2.2 (Selfadjointness of Robin fractional Laplacian)**.** *The Robin fractional Laplacian* $((-\Delta)^s, D((-\Delta)^s))$ *is selfadjoint in* $L^2(\Omega)$.

Proof. Let $\langle\cdot,\cdot\rangle$ be the scalar product in $L^2(\Omega)$ and $u,v\in D\left((-\Delta)^s\right)$. then, the integration by parts formula (B.8) allows us to write

$$\begin{aligned}\langle(-\Delta)^s u,v\rangle &=\int_\Omega(-\Delta)^s uvdx\\ &=\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}dxdy+\int_{\mathbb{R}^d\setminus\Omega}\mathcal{N}_s uvdx\\ &=\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}dxdy-\beta\int_{\mathbb{R}^d\setminus\Omega} uvdx\\ &=\int_{\mathbb{R}^{2d}\setminus(\Omega^c)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}dxdy+\int_{\mathbb{R}^d\setminus\Omega} u\mathcal{N}_s vdx\\ &=\int_\Omega u(-\Delta)^s vdx=\langle u,(-\Delta)^s v\rangle,\end{aligned}$$

proving the selfadjointness of $(-\Delta)^s$ on its domain. □

Now, we want to show that the eigenvalues of the Robin fractional Laplacian admit a variational representation via the Courant-Fischer min-max and max-min formulae (1.1) and (1.2). To do this, we use the spectral Theorem 1.3.8, once we prove that the associated quadratic form is semibounded from below.

Proposition B.2.3. *Let us consider the quadratic form associated to the Robin fractional Laplacian*

$$Q_\beta(u) := \iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,dxdy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx.$$

Then, $Q_\beta(u)$ is semibounded from below in $H^s(\Omega)$, i.e. there exists $\gamma\in\mathbb{R}$ such that

$$Q_\beta(u) \ge \gamma\|u\|^2_{H^s(\Omega)} \tag{B.13}$$

Proof. To prove (B.13), we notice that if $\beta \le 0$, we have

$$\beta\|u\|^2_{\mathbb{R}^d\setminus\Omega} \ge \beta\|u\|^2_{\mathbb{R}^d}$$

and then

$$Q_\beta(u) \ge \beta\|u\|^2_{H^s(\mathbb{R}^d)} \ge \beta C_{ext}\|u\|^2_{H^s(\Omega)},$$

where $C_{ext} > 0$ is the norm of the extension operator of $H^s(\Omega)$ to the whole of $\mathbb{R}^d$.

On the other hand, if $\beta > 0$, we can choose $\gamma = 0$ to get trivially (B.13). A more accurate analysis leads us to show that

$$Q_\beta(u) + \beta\|u\|^2_{L^2(\Omega)} \ge \gamma\|u\|^2_{H^s(\Omega)},$$

with $\gamma = \min\{1,\beta\}$. □

In view of the previous two propositions and of the compactness of the embedding $H^s(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we can apply Theorem 1.3.8 for semibounded selfadjoint operators with compact resolvent. Then, the eigenvalues of the fractional Robin Laplacian form an increasing and positive diverging sequence

$$\lambda_{1,\beta}(\Omega) \le \lambda_{2,\beta}(\Omega) \le \ldots \to +\infty.$$

Moreover, $\lambda_{k,\beta}(\Omega)$ can be represented by

$$\begin{aligned}\lambda_{k,\beta}(\Omega) &= \min_{S\in\mathcal{S}_k}\max_{u\in S\setminus\{0\}} \frac{Q_\beta(u)}{\|u\|^2_{L^2(\Omega)}}\\ &= \min_{S\in\mathcal{S}_k}\max_{u\in S\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,dx\,dy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx}{\displaystyle\int_\Omega u^2\,dx},\end{aligned} \tag{B.14}$$

or by

$$
\begin{aligned}
\lambda_{k,\beta}(\Omega) &= \max_{S^\perp \in \mathcal{S}_{k-1}} \min_{u\in S\setminus\{0\}} \frac{Q_\beta(u)}{\|u\|^2_{L^2(\Omega)}} \\
&= \max_{S^\perp \in \mathcal{S}_{k-1}} \min_{u\in S\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u\,(x)-u\,(y))^2}{|x-y|^{d+2s}}\,dx\,dy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx}{\displaystyle\int_\Omega u^2\,dx},
\end{aligned}
\tag{B.15}
$$

where we denote by $\mathcal{S}_k$ (resp. $\mathcal{S}_{k-1}$) the family of all k-dimensional (resp. $(k-1)$-dimensional) subspaces of $L^2(\mathbb{R}^d)\cap H^s(\Omega)$. We recall that the equality

$$
\lambda_{k,\beta}(\Omega) = \frac{\displaystyle\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{(u\,(x)-u\,(y))^2}{|x-y|^{d+2s}}\,dx\,dy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx}{\displaystyle\int_\Omega u^2\,dx}
$$

holds whenever $u\in L^2(\mathbb{R}^d)\cap H^s(\Omega)$ is an eigenfunction relative to the eigenvalue $\lambda_{k,\beta}(\Omega)$.

Some interesting properties are summarized in the following proposition.

Proposition B.2.4 (monotonicity w.r.t. the boundary parameter and under dilations)**.** *For every bounded Lipschitz domain Ω and every $k\in\mathbb{N}$, the map $\beta\mapsto\lambda_{k,\beta}(\Omega)$ is monotonically increasing in $\mathbb{R}$. Moreover, the non-local Robin eigenvalues are monotonically decreasing under dilation, i.e., for every open bounded Lipschitz set $\Omega\subset\mathbb{R}^d$, $\beta>0$, $t\geq 1$ and $k\in\mathbb{N}$, it holds*

$$
\lambda_{k,\beta}(t\Omega)\leq\lambda_{k,\beta}(\Omega).
$$

Proof. The first statement is straightforward to prove.

To show the monotonicity under dilations, let $t\geq 1$ and let us observe that there is a one-to-one correspondence (given by the homothety $x\in\Omega\mapsto x' := tx\in t\Omega$) between functions $u\in L^2(\mathbb{R}^d)\cap H^s(\Omega)$ and functions $v\in L^2(\mathbb{R}^d)\cap H^s(t\Omega)$ and between k-dimensional subspaces of $L^2(\mathbb{R}^d)\cap H^s(\Omega)$ and k-dimensional subspaces of $L^2(\mathbb{R}^d)\cap H^s(t\Omega)$. Let us consider $u\in L^2(\mathbb{R}^d)\cap H^s(\Omega)$ and $v\in L^2(\mathbb{R}^d)\cap H^s(t\Omega)$ such that

$$
v(tx)=u(x)\quad \text{a.e. } x\in\mathbb{R}^d.
$$

We have

$$
\int_{t\Omega} v^2\,dx' = t^d\int_\Omega u^2\,dx,
$$

$$\int_{\mathbb{R}^d\setminus t\Omega} v^2\,dx' = t^n \int_{\mathbb{R}^d\setminus\Omega} u^2\,dx$$

and

$$\iint_{\mathbb{R}^{2d}\setminus(t\Omega^c)^2} \frac{\left(v\left(x'\right)-v\left(y'\right)\right)^2}{\left|x'-y'\right|^{d+2s}}\,dx'\,dy' = t^{d-2s}\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left|x-y\right|^{d+2s}}\,dx\,dy.$$

Hence, computing the Rayleigh quotients we obtain

$$\begin{aligned}
R_\beta^{t\Omega}(v) &= \frac{\displaystyle\iint_{\mathbb{R}^{2d}\setminus(t\Omega^c)^2} \frac{\left(v\left(x'\right)-v\left(y'\right)\right)^2}{\left|x'-y'\right|^{d+2s}}\,dx'\,dy' + \beta\int_{\mathbb{R}^d\setminus t\Omega} v^2\,dx'}{\displaystyle\int_{t\Omega} v^2\,dx'} \\
&= \frac{t^{-2s}\displaystyle\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left|x-y\right|^{d+2s}}\,dx\,dy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx}{\displaystyle\int_{\Omega} u^2\,dx} \\
&\le \frac{\displaystyle\iint_{\mathbb{R}^{2d}\setminus(\Omega^c)^2} \frac{\left(u\left(x\right)-u\left(y\right)\right)^2}{\left|x-y\right|^{d+2s}}\,dx\,dy + \beta\int_{\mathbb{R}^d\setminus\Omega} u^2\,dx}{\displaystyle\int_{\Omega} u^2\,dx} = R_\beta^{\Omega}(u).
\end{aligned}$$

Passing to the min-max formula we conclude the proof. □

Remark B.2.5 (non-local scaling property)**.** Let us observe that, as a byproduct of the previous proposition, the Robin eigenvalues satisfy the following scaling property

$$\lambda_{k,\beta}(t\Omega) = t^{-2s}\lambda_{k,t^{2s}\beta}(\Omega)$$

for every $t > 0$. Notice that letting $s \to 1^-$, we do not obtain the "local scaling property" (2.6) $\lambda_{k,\beta}(t\Omega) = t^{-2}\lambda_{k,t\beta}(\Omega)$. This probably depends on the different nature of the second addendum in the Rayleigh quotients: in the local case, it is a surface integral, in the non-local case it is a volume integral.

B.3 Uniform extension Theorem for uniformly regular open sets

As remarked in Chapter 1, Theorem 1.4.27, when dealing with a sequence $(\Omega_n)_n$ of extension domains which are uniformly regular (i.e. satisfying the same ε-cone property), it could be necessary to ensure that the extension operators $E_n : H^1(\Omega_n) \to H^s(\mathbb{R}^d)$ are uniformly bounded. Throughout this section we will prove this result in a non-local setting, with a slightly different statement to Theorem 1.4.27. More precisely, we will prove that the extensions operators E_n above are equibounded. This result, up to our knowledge, is not still present in literature, even if it is natural to wonder whether the extension operators E_n can be considered equibounded and even if the result we are inspired by has been published some years ago in [43]. For many of the arguments of our proofs we refer to [43], in which we can find the extension theorem for a Lipschitz domain $\Omega \subset \mathbb{R}^d$.

We start with a lemma involving compactly supported functions. We prove that, if $u_n \in H^s(\Omega_n)$ is identically zero in a neighbourhood of $\partial\Omega_n$, then the sequence $(u_n)_n$ can be extended uniformly to the whole of $\mathbb{R}^d$. We base our argument on an adaptation of the proof of Lemma 5.1 in [43] and on the fact that, if Ω_n H^c-converges to a non-empty open set Ω, then there exists a non-empty compact set with positive measure contained in all the Ω_n sets.

Lemma B.3.1. *Let $(\Omega_n)_n$ be a sequence of uniformly regular bounded open sets H^c-converging to $\Omega \subset \mathbb{R}^d$. Let $K \subset \mathbb{R}^d$ be compact such that $K \subset \Omega_n$ for every $n \in \mathbb{N}$. For every $u_n \in H^s(\Omega_n)$ such that $u_n = 0$ in $\Omega_n \setminus K$, we set*

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in \Omega_n, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega_n. \end{cases}$$

Then, $\tilde{u}_n \in H^s(\mathbb{R}^d)$ and there exists a positive constant $C = C(\Omega)$ such that

$$\|\tilde{u}_n\|_{H^s(\mathbb{R}^d)} \leq C\|u_n\|_{H^s(\Omega_n)}. \tag{B.16}$$

Proof. Let us remark that, in view of the Hausdorff convergence of Ω_n to Ω, we have that $K \subset \Omega$. Since $\|\tilde{u}_n\|_{L^2(\mathbb{R}^d)} = \|u_n\|_{L^2(\Omega_n)}$, to prove (B.16) it is sufficient to show that the Gagliardo seminorm of $\tilde{u}_n$ in $\mathbb{R}^d$ is bounded by $\|u_n\|_{H^s(\Omega_n)}$. It holds

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))^2}{|x-y|^{d+2s}}\,dx\,dy \\ &= \int_{\Omega_n}\int_{\Omega_n} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))^2}{|x-y|^{d+2s}}\,dx\,dy + 2\int_{\Omega_n}\int_{\mathbb{R}^d\setminus\Omega_n} \frac{u_n^2(x)}{|x-y|^{d+2s}}\,dy\,dx. \end{aligned} \tag{B.17}$$

Moreover, for a.e. $y \in \mathbb{R}^d \setminus K$ and a.e $x \in \Omega_n$

$$\frac{u_n^2(x)}{|x-y|^{d+2s}} \leq \chi_K(x)u_n^2(x) \sup_{x\in K} \frac{1}{|x-y|^{d+2s}} = \chi_K(x)u_n^2(x)\frac{1}{dist(y,\partial K)^{d+2s}}. \tag{B.18}$$

Let $D := dist(\partial K, \partial\Omega) > 0$. Since Ω_n H^c-converges to Ω, we have that the uniform convergence holds for the boundaries; in particular,

$$\partial\Omega \subset \partial\Omega_n + B_{D/2} \qquad \text{and} \qquad \partial\Omega_n \subset \partial\Omega + B_{D/2}.$$

We deduce that, for every $y \in \mathbb{R}^d \setminus \Omega_n$, it holds

$$dist(y, \partial K) \geq \frac{D}{2}.$$

Then, by (B.18), we have

$$\begin{aligned}\int_{\Omega_n}\int_{\mathbb{R}^d\setminus\Omega_n} \frac{u_n^2(x)}{|x-y|^{d+2s}}\,dy\,dx &\leq \int_{\Omega_n}\int_{\mathbb{R}^d\setminus\Omega_n} \chi_K(x)u_n^2(x)\frac{1}{dist(y,\partial K)^{d+2s}}\,dy\,dx\\ &\leq C\|u_n\|^2_{L^2(\Omega_n)},\end{aligned}$$

with $C > 0$ depending only on Ω. Combining with (B.17) we obtain estimate (B.16). □

Now, we state a result of extension by reflection of a function defined on the halfspace $\mathbb{R}^d_+$. For a proof of this result, see Lemma 5.2 in [43].

Lemma B.3.2 (Reflection lemma)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and symmetric with respect to the plane $x_1 = \ldots = x_{d-1} = 0$ and consider the set*

$$\Omega_+ := \{x \in \Omega : x_d > 0\}.$$

Let $u \in H^s(\Omega_+)$; for every $x \in \Omega$, we define

$$\bar{u}(x) := \begin{cases} u(x_1, \ldots, x_{d-1}, x_d) & \text{if } x_d > 0,\\ -u(x_1, \ldots, x_{d-1}, -x_d) & \text{if } x_d < 0.\end{cases}$$

Then, the function $\bar{u} : \Omega \to \mathbb{R}$ belongs to $H^s(\Omega)$ and

$$\|\bar{u}\|_{H^s(\Omega)} \leq 4\|u\|_{H^s(\Omega_+)}.$$

Now we analyse the behaviour of the functions u_n under truncations by cut-off functions ψ_n. If we require that the cut-off functions are equilipchitz, then also the truncation operators are uniformly bounded. The proof is based on the same arguments as in Lemma 5.3 in [43], considering a uniform Lipschitz constant for all the functions ψ_n of the sequence. We remark that in the uniform extension Theorem B.3.4 we will use a less general case of Lemma B.3.3.

Lemma B.3.3 (Truncation lemma)**.** *Let $(\Omega_n)_n$ be a sequence of uniformly regular bounded open sets H^c-converging to $\Omega \subset \mathbb{R}^d$. For every $n \in \mathbb{N}$, let $u_n \in H^s(\Omega_n)$ and let $\psi_n \in Lip(\Omega_n)$ with $(\psi_n)_n$ equilipschitz and $0 \leq \psi_n \leq 1$. Then, $\psi_n u_n \in H^s(\Omega_n)$ and there exists a positive constant $C = C(\Omega) > 0$ such that*

$$\|\psi_n u_n\|_{H^s(\Omega_n)} \leq C\|u_n\|_{H^s(\Omega_n)}. \tag{B.19}$$

Proof. Since $\|\psi_n u_n\|_{L^2(\Omega_n)} \le \|u_n\|_{L^2(\Omega_n)}$, to conclude the proof it is sufficient to show that the Gagliardo seminorm of $\psi_n u_n$ is bounded by $\|u_n\|_{H^s(\Omega_n)}$. By the definition of Gagliardo seminorm, summing and subtracting $\psi_n(x)u_n(y)$ we obtain

$$\begin{aligned}
&\int_{\Omega_n}\int_{\Omega_n} \frac{\big(\psi_n(x)u_n(x)-\psi_n(y)u_n(y)\big)^2}{|x-y|^{d+2s}}\,dx\,dy \\
&\le 2\Bigg(\int_{\Omega_n}\int_{\Omega_n} \frac{\big(u_n(x)-u_n(y)\big)^2}{|x-y|^{d+2s}}\,dx\,dy \\
&+\int_{\Omega_n}\int_{\Omega_n} \frac{u_n^2(x)\,\big(\psi_n(x)-\psi_n(y)\big)^2}{|x-y|^{d+2s}}\,dy\,dx\Bigg).
\end{aligned} \tag{B.20}$$

Let us estimate the second addendum in the right hand side of (B.20). Denoting by L the uniform Lipschitz constant for $(\psi_n)_n$, it holds

$$\begin{aligned}
&\int_{\Omega_n}\int_{\Omega_n} \frac{u_n^2(x)\,\big(\psi_n(x)-\psi_n(y)\big)^2}{|x-y|^{d+2s}}\,dy\,dx \\
&=\int_{\Omega_n}\int_{\Omega_n\cap|x-y|\le 1} \frac{u_n^2(x)\,\big(\psi_n(x)-\psi_n(y)\big)^2}{|x-y|^{d+2s}}\,dy\,dx \\
&+\int_{\Omega_n}\int_{\Omega_n\cap|x-y|\ge 1} \frac{u_n^2(x)\,\big(\psi_n(x)-\psi_n(y)\big)^2}{|x-y|^{d+2s}}\,dy\,dx \\
&\le L^2\int_{\Omega_n}\int_{\Omega_n\cap|x-y|\le 1} \frac{u_n^2(x)}{|x-y|^{d+2(s-1)}}\,dy\,dx \\
&+\int_{\Omega_n}\int_{\Omega_n\cap|x-y|\ge 1} \frac{u_n^2(x)}{|x-y|^{d+2s}}\,dy\,dx \\
&\le L^2\int_{\Omega_n}\underbrace{\int_{|x-y|\le 1} \frac{u_n^2(x)}{|x-y|^{d+2(s-1)}}\,dy}_{C(d,s)}\,dx \\
&+\int_{\Omega_n}\underbrace{\int_{\cap|x-y|\ge 1} \frac{u_n^2(x)}{|x-y|^{d+2s}}\,dy}_{C(d,s)}\,dx \\
&\le C(d,s,L)\|u_n\|^2_{L^2(\Omega_n)}.
\end{aligned} \tag{B.21}$$

Combining (B.21) with (B.20), we obtain (B.19). □

Now we are ready to proof the main result of this section. We will follow the same approach as in Theorem 5.4 in [43], applied to a sequence $(\Omega_n)_n$ and an open set Ω as above, combining the fact that we can assume that a finite open covering of $\partial\Omega$ is also a finite open covering of $\partial\Omega_n$ (up to subsequences) and that the boundaries of all the sets are equilipschitz (i.e., the bi-Lipschitz maps in Definition 1.0.1 are equilipschitz for all the sets Ω_n and Ω).

Theorem B.3.4 (non-local uniform extension)**.** *Let $(\Omega_n)_n$ be a sequence of uniformly regular open bounded sets H^c-converging to an open set Ω and let $u_n \in H^s(\Omega_n)$. Then, there exists $\tilde{u}_n \in H^s(\mathbb{R}^d)$ such that $\tilde{u}_n$ is an extension of u_n to the whole of $\mathbb{R}^d$ and*

$$\|u_n\|_{H^s(\mathbb{R}^d)} \leq C\|u_n\|_{H^s(\Omega_n)},$$

with $C > 0$ depending only on s, d and Ω.

Proof. Let us consider a finite covering $\bigcup_{j=1}^{l} B_j$ by balls of the compact set $\partial\Omega$ and let $\delta := \operatorname{dist}\left(\partial\left(\bigcup_{j=1}^{l} B_j\right), \partial\Omega\right) > 0$. In view of the equivalence between Hausdorff convergence and uniform convergence of uniformly regular compact sets, there exist a subsequence (a tail of the sequence), named again $(\Omega_n)_n$, such that $\partial\Omega_n \subset \partial\Omega + B_\delta$. Let us consider the covering of the whole space

$$\mathbb{R}^d = \left(\mathbb{R}^d \setminus (\partial\Omega + B_\delta) \bigcup_{j=1}^{l} B_j\right).$$

There exists a partition of the unity related to this covering, namely $l+1$ smooth functions $\psi_0, \psi_1, \ldots, \psi_l$ such that $\operatorname{spt}\psi_0 \subset \mathbb{R}^d \setminus (\partial\Omega + B_\delta)$, $\operatorname{spt}\psi_j \subset B_j$ for every $j = 1, \ldots, l$, $0 \leq \psi_j \leq 1$ for every $j = 0, \ldots, l$ and $\sum_{j=0}^{l} \psi_j = 1$. By Lemma B.3.3, we have that $\psi_0 u_n \in H^s(\Omega_n)$; moreover, we can extend $\psi_0 u_n$ to the whole of $\mathbb{R}^d$, since $\psi_0 u_n \equiv 0$ in a neighbourhood of $\partial\Omega_n$. More precisely, the extension $\tilde{\psi_0 u_n} \in H^s(\mathbb{R}^d)$ is given by

$$\tilde{\psi_0 u_n}(x) := \begin{cases} \psi_0 u_n(x) & \text{if } x \in \Omega_n \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega_n \end{cases}$$

and it holds

$$\|\tilde{\psi_0 u_n}\|_{H^s(\mathbb{R}^d)} \leq C_1 \|\psi_0 u_n\|_{H^s(\Omega_n)} \leq C_2 \|u_n\|_{H^s(\Omega_n)}, \tag{B.22}$$

where the first inequality follows by Lemma B.3.1, the second inequality follows by Lemma B.3.3 and the constants $C_1, C_2 > 0$ depend only on Ω, s, d.

Now, for every $n \in \mathbb{N}$ (for every index of the relabelled subsequence) and $j = 1, \ldots, l$, it holds that $B_j \cap \Omega_n$ is a non-empty open set. Let us denote by Q the cube centered at the origin with side equal to 2, by Q_+ the half cube

$$Q_+ := \{x \in Q : x_d > 0\}$$

and by Q_0 the $(d-1)$-dimensional cube

$$Q_0 := \{x \in Q : x_d = 0\}.$$

In view of the regularity assumptions satisfied by the sets Ω_n, for any $n \in \mathbb{N}$ and $j = 1, \ldots, k$, there exists a bi-Lipschitz isomorphism $T_{n,j} : Q \to B_j$ such that

$$T_{n,j}(Q) = B_j, \quad T_{n,j}(Q_+) = B_j \cap \Omega, \quad T_{n,j}(Q_0) = B_j \cap \partial\Omega$$

and

$$C_1 \leq Lip(T_{n,j}) + Lip(T_{n,j}^{-1}) \leq C_2,$$

with $C_1, C_2 > 0$ independent on n and j, since all the Ω_n sets satisfy the same ε-cone property (see Remark 1.4.22).

For any $\hat{x} \in Q_+$, let us define

$$v_{n,j}(\hat{x}) := u_n(T_{n,j}(\hat{x})).$$

Let us show that the function $v_{n,j}$ belongs to $H^s(Q_+)$. Using the change of variable $x = T_{n,j}(\hat{x})$, we have

$$\begin{aligned}
\int_{Q_+}\int_{Q_+} & \frac{(v_{n,j}(\hat{x}) - v_{n,j}(\hat{y}))^2}{|\hat{x}-\hat{y}|^{d+2s}}\, d\hat{x}\, d\hat{y} \\
&= \int_{Q_+}\int_{Q_+} \frac{(u_n(T_{n,j}(\hat{x})) - u_n(T_{n,j}(\hat{y})))^2}{|\hat{x}-\hat{y}|^{d+2s}}\, d\hat{x}\, d\hat{y} \\
&= \int_{\Omega_n\cap B_j}\int_{\Omega_n\cap B_j} \frac{(u_n(x) - u_n(y))^2}{|T_{n,j}^{-1}(x) - T_{n,j}^{-1}(y)|^{d+2s}} \det(DT_{n,j}^{-1})\, dx\, dy \\
&\leq C \int_{\Omega_n\cap B_j}\int_{\Omega_n\cap B_j} \frac{(u_n(x) - u_n(y))^2}{|x-y|^{d+2s}}\, dx\, dy,
\end{aligned} \tag{B.23}$$

with $C > 0$ independent on n and j. This proves that $v_{n,j} \in H^s(Q_+)$.

By Lemma B.3.2, for each function $v_{n,j}$ there exist an extension to Q, say $\bar{v}_{n,j}$, such that $\bar{v}_{n,j} \in H^s(Q)$ and

$$\|\bar{v}_{n,j}\|_{H^s(Q)} \leq 4\|v_{n,j}\|_{H^s(Q_+)}.$$

For any $x \in B_j$, we define

$$w_{n,j}(x) := \bar{v}_{n,j}(T_{n,j}^{-1}(x)).$$

The function $w_{n,j}$ belongs to $H^s(B_j)$. Indeed, proceeding as in (B.23), we have

$$\begin{aligned}
&\int_{B_j}\int_{B_j}\frac{(w_{n,j}(x)-w_{n,j}(y))^2}{|x-y|^{d+2s}}\,dx\,dy\\
&\qquad=\int_{B_j}\int_{B_j}\frac{\left(\bar v_{n,j}(T_{n,j}^{-1}(x))-\bar v_{n,j}(T_{n,j}^{-1}(y))\right)^2}{|x-y|^{d+2s}}\,dx\,dy\\
&\qquad=\int_Q\int_Q\frac{(\bar v_{n,j}(\hat x)-\bar v_{n,j}(\hat y))^2}{|T_{n,j}(\hat x)-T_{n,j}(\hat y)|^{d+2s}}\det(DT_{n,j})\,d\hat x\,d\hat y\\
&\qquad\le C\int_Q\int_Q\frac{(\bar v_{n,j}(\hat x)-\bar v_{n,j}(\hat y))^2}{|\hat x-\hat y|^{d+2s}}\,d\hat x\,d\hat y,
\end{aligned}\tag{B.24}$$

with $C>0$ independent on n and j.

Notice that, for every $x\in B_j\cap\Omega_n$, it holds

$$w_{n,j}(x):=\bar v_{n,j}(T_{n,j}^{-1}(x))=v_{n,j}(T_{n,j}^{-1}(x))=u_n(x),$$

then $\psi_j u_n=\psi_j w_{n,j}$ on $B_j\cap\Omega_n$. Moreover, $\psi_j w_{n,j}$ has compact support in B_j, then, by Lemma B.3.1, there exists an extension $\widetilde{\psi_j w_{n,j}}\in H^s(\mathbb{R}^d)$ satisfying

$$\|\widetilde{\psi_j w_{n,j}}\|_{H^s(\mathbb{R}^d)}\le C\|\psi_j w_{n,j}\|_{H^s(B_j)}$$

for some $C>0$ depending only on d,s,Ω. Using Lemma B.3.1, Lemma B.3.2, Lemma B.3.3 and estimates (B.23) and (B.24), we have

$$\begin{aligned}
\|\widetilde{\psi_j w_{n,j}}\|_{H^s(\mathbb{R}^d)}&\le C\|\psi_j w_{n,j}\|_{H^s(B_j)}\le C\|w_{n,j}\|_{H^s(B_j)}\\
&\le C\|\bar v_{n,j}\|_{H^s(Q)}\le C\|v_{n,j}\|_{H^s(Q_+)}\le C\|u_n\|_{H^s(\Omega_n\cap B_j)}
\end{aligned}\tag{B.25}$$

(to simplify the notation, in the previous estimate we used the symbol C to denote all the positive constants depending only on d,s,Ω, which are possibly different).

Now, let us define

$$\tilde u_n:=\widetilde{\psi_0 u_n}+\sum_{j=1}^{l}\widetilde{\psi_j w_{n,j}}.$$

The function $\tilde u_n$ is defined on the whole of $\mathbb{R}^d$ and $\tilde u_n|_{\Omega_n}=u_n$. Moreover, by (B.22) and (B.25), we conclude that

$$\|u_n\|_{H^s(\mathbb{R}^d)}\le C\|u_n\|_{H^s(\Omega_n)},$$

with $C>0$ depending only on s, d and Ω, so $\tilde u_n$ is the required extension of u_n. □

As already remarked at the beginning of the section, we did not find any reference containing a similar result, even if it is reasonable that some authors already studied the problem.

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