HÖLDER REGULARITY OF CONTINUOUS SOLUTIONS TO BALANCE LAWS AND APPLICATIONS IN THE HEISENBERG GROUP

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Abstract. We prove Hölder regularity of any continuous solution u to a $1d$ scalar balance law, when the source term is bounded and the flux is nonlinear of order $\ell \in \mathbb{N}$ with $\ell \geq 2$. Moreover, we prove that at almost every point (t, x) it holds $u(t, x + h) - u(t, x) = o(|h|^{\frac{1}{\ell}})$ as $h \to 0$. We apply the results to provide a new proof the Rademacher theorem for intrinsic Lipschitz functions in the first Heisenber group.

1. Introduction

The effect of non-linearity in conservation laws is two-folds: on the one hand it prevents the existence of smooth solutions due to shocks formation. On the other hand it has some regularizing effects. The most striking example is the estimate by Oleinik [**Ole**], which proves that L^∞ initial data are immediately regularized to BV_{loc} at positive times, for entropy solutions of the classical Burgers' equation

$$u_t + [u^2/2]_x = 0.$$

Furthermore, [**ADL**] proved that BV_{loc} regularity improves to SBV_{loc}, then generalized to more general $1d$ scalar balance laws in [**Robyr**] and to systems of conservation and balance laws in [**BC**]. Similar results have been proven for Hamilton-Jacobi equations in several space dimensions [**BDLR**].

A possible approach to prove the BV_{loc} estimate in the scalar case relies on the existence of a family non-crossing (backward) characteristics [**Daf_book**] along which the solution is constant. The non-crossing condition is a geometric constraint which forces BV_{loc} regularity. This approach has been extended to the more general conservation law of the form:

$$u_t + [f(u)]_x = 0, \tag{1.1}$$

see [**BGJ**; **AGV**] for the case of convex fluxes f and [**Daf**; **Cheng**; **Mar**], for the case of general smooth fluxes. The final result is that a fractional regularity of entropy solutions can be obtained quantifying the non-linearity of the flux f.

Motivated by some applications to the theory of rectifiable sets in the Heisenberg group in the present paper we take into account the presence of the source $g \in L^\infty$ in (1.1), i.e.

$$u_t + [f(u)]_x = g, \tag{1.2}$$

We prove a regularity result for the continuous solutions of (1.2) where $f \in C^\ell(\mathbb{R})$ is nonlinear of order $\ell \geq 2$ (see Definition A.1). More precisely, we first prove that every continuous solution to (1.2) is indeed locally $1/\ell-$Hölder continuous and, even better,

$$u(t, x + h) - u(t, x) = o\left(\sqrt[\ell]{|h|}\right) \quad \text{as} \quad h \to 0 \tag{1.3}$$

at Lebesgue almost every point (t, x). It is known that, even with analytic fluxes f and continuous sources terms g, continuous solutions to (1.2) improve to Hölder continuous regularity but it does not go beyond. Indeed, even with quadratic flux continuous solutions to (2.1) might exhibit nasty fractal behaviour and in general they are neither Sobolev nor BV [**KSC**; **ABC3**]. Moreover, if the flux f is strictly convex but not analytic the result generally fails [**ABC3**], which motivates the finite order nonlinearity Assumption A.1 on the flux. Similar results have been obtained also for conservation laws in several space dimensions by means of the kinetic formulation (see for example [**LPT**; **GL**]), nevertheless the conjectured optimal regularity has not been proven yet. We mention that also in the case of Hamilton-Jacobi equations, optimal Hölder regularity has been recently proved (see [**CV**] and references therein).

In order to describe the application of our result to the theory of rectifiable sets in the Heisenberg group we recall that the notion of Lipschitz submanifolds in sub-Riemannian geometry was introduced, at least in the setting of Carnot groups, by B. Franchi, R. Serapioni and F. Serra Cassano in a series of papers [**FSSC**; **FSSC2**; **FS**] through the theory of intrinsic Lipschitz graphs (see also [**CMPSC1**; **CMPSC2**]). Roughly speaking, a subset $S \subset \mathbb{G}$ of a Carnot group $\mathbb{G}$ is intrinsic Lipschitz if at each point $P \in S$ there is an intrinsic cone with vertex P and fixed opening, intersecting S only in P. Remarkably, this notion turned out to be the right one in the setting of the intrinsic rectifiability in the simplest Carnot group, namely the Heisenberg group $\mathbb{H}^n$. Indeed, it was proved in [**FS**] that the notion of rectifiable set in terms of an intrinsic regular hypersurfaces is equivalent to the one in terms of intrinsic Lipschitz graphs. We also remark that intrinsic Lipschitz functions played a crucial role in the recent paper [**NY**], where the longstanding question of determining the approximation ratio of the Goemans-Linial algorithm for the Sparsest Cut Problem was settled, see also [**NY2**] for different applications. We address the interested reader to [**FSSC2**; **Vit**] for a complete introduction to the theory of intrinsic Lipschitz functions. One of the main open questions in this area of research is whether a Rademacher type theorem holds. Namely, assume that a splitting $\mathbb{G} = \mathbb{W}\mathbb{V}$ of a Carnot group $\mathbb{G}$ is fixed, is it true that every intrinsically Lipschitz function $u : \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable almost everywhere? In [**FMS**; **FSSC2**], it is proved that the answer is *yes* if $\mathbb{G}$ is of step two or a group of type $\star$ and $\mathbb{V} = \mathbb{R}$. More recently, D. Vittone in [**Vit**] proved that the answer is also *yes* in the case of the Heisenberg group without any a priori assumption on the splitting. We also address the interested reader to [**AM**; **AM2**; **LDM**] for further partial results. Remarkably, in [**JNGV**] the authors constructed intrinsic Lipschitz graphs of codimension 2 in Carnot groups which are not intrinsically differentiable almost everywhere. Insipired by the results contained in [**ASCV**], in [**BCSC**], the first named author together with Bigolin and Serra Cassano provided a characterization of intrinsic Lipschitz graphs in the Heisenberg group in terms of a system of non linear first order PDEs. Moreover, they proved the equivalence of different notions of continuous weak solutions to the equation

$$u_t + [u^2/2]_x = g, \tag{1.4}$$

where g is a bounded function. It turns out that the question whether an intrinsically Lipschitz function u is intrinsically differentiable or is equivalent to (1.3). This observation allows us to provide a completely PDEs based proof of the Rademacher theorem in $\mathbb{H}^1$. We point out that the proof provided in [**FSSC2**] requires some deep and nontrivial results in geometric measure theory, namely the fact that the subgraph of an intrinsic Lipschitz function u is a set with locally finite $\mathbb{H}-$perimeter and that at almost every point of the graph of u there is an approximate tangent plane. Another interesting feature of our approach is its potential applicability to those Carnot groups where the aforementioned geometric tools are not yet available e.g. the Engel group.

Structure of the paper: In § 2 we revise an elementary estimate on continuous solutions to balance laws when the flux is convex. When the source is bounded, such estimate is the key to Hölder continuity and Lipschitz continuity along characteristics. We prove in § 3 Hölder regularity of any continuous solution u to a $1d$ scalar balance law, when the source term is bounded and the flux is nonlinear of order ℓ in the sense of § A. The Hölder regularity is then refined in § 4, where we show that at Lebesgue almost every point (t,x) it holds $u(t,x+h) - u(t,x) = o(|h|^{\frac{1}{\ell}})$ as $h \to 0$. In § 5 we finally prove the Rademacher theorem for intrinsic Lipschitz functions in $\mathbb{H}^1$.

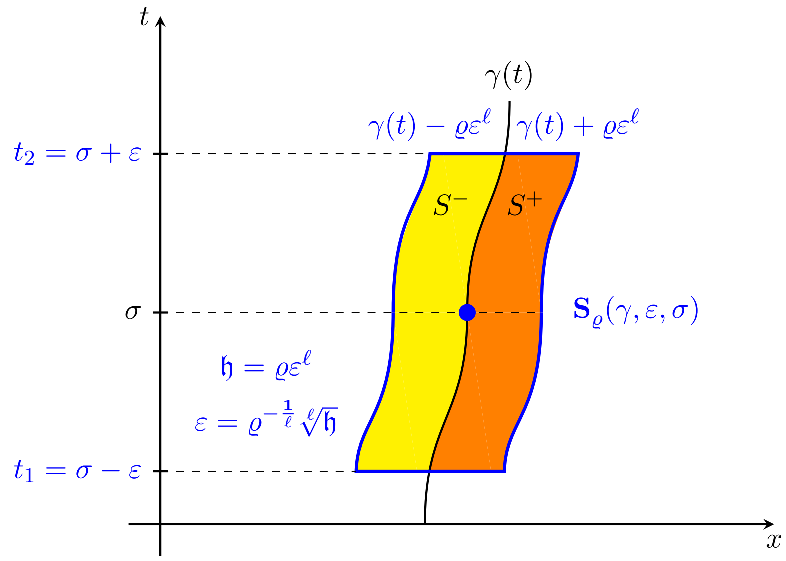

FIGURE 1. A region S_ϱ of the Vitali covering (4.2) and $S^\pm(\gamma|_{[t_1,t_2]},\mathfrak{h})$ in Proposition 2.2.

2. Preliminary results

We recall the definition of characteristics of the equation

$$\partial_t u + [f(u)]_x = g\,, \qquad f\in C^\ell(\mathbb{R}), \ell\in\mathbb{N},\ g \text{ bounded.} \tag{2.1}$$

Definition 2.1. A characteristic associated to a continuous solution u to (2.1) is any function $\gamma\in C^1(I)$, defined on a real interval $I\subset\mathbb{R}$ and satisfying the ordinary differential equation

$$\dot\gamma(t) = f'(u(s,\gamma(t))) \qquad \text{for } t\in I.$$

The following well-known result will be helpful later on (see [**Daf2**; **ABC1**]):

Proposition 2.2. *Let $g\in L^1_{\rm loc}(\Omega)$, $\Omega\subset\mathbb{R}^2$ be open and assume that $u\in C^0(\Omega)$ is a distributional solution of (2.1). Given a characteristic $\gamma:[t_1,t_2]\to\mathbb{R}$ and $\mathfrak{h}>0$, assume that the following sets*

$$S^+(\gamma,\mathfrak{h}) \doteq \{(t,x)\in[t_1,t_2]\times\mathbb{R} : x\in[\gamma(t),\gamma(t)+\mathfrak{h}]\},$$
$$S^-(\gamma,\mathfrak{h}) \doteq \{(t,x)\in[t_1,t_2]\times\mathbb{R} : x\in[\gamma(t)-\mathfrak{h},\gamma(t)]\}$$

are contained in Ω. If f is convex on the interval $u(S^+(\gamma,\mathfrak{h})\cup S^-(\gamma,\mathfrak{h}))$, then

$$\int_{\gamma(t_2)}^{\gamma(t_2)+\mathfrak{h}} u(t_2,x)dx - \int_{\gamma(t_1)}^{\gamma(t_1)+\mathfrak{h}} u(t_1,x)dx \le \int_{S^+(\gamma,\mathfrak{h})} g\,dxdt \tag{2.2a}$$

$$\int_{\gamma(t_2)-\mathfrak{h}}^{\gamma(t_2)} u(t_2,x)dx - \int_{\gamma(t_1)-\mathfrak{h}}^{\gamma(t_1)} u(t_1,x)dx \ge \int_{S^-(\gamma,\mathfrak{h})} g\,dxdt. \tag{2.2b}$$

The analogous statement holds for concave fluxes f reversing the inequalities (2.2a) and (2.2b).

Remark 2.3. Dividing estimates (2.2a) and (2.2b) by $\mathfrak{h}$ and letting $\mathfrak{h}\to 0$ we get that $t\mapsto u(t,\gamma(t))$ is Lipschitz continuous with constant bounded by $\|g\|_{L^\infty}$. The same result was proven in [**ABC1**] under the assumption that $\mathscr{L}^1(\overline{\mathrm{Infl}(f)})=0$, where $\mathrm{Infl}(f)$ denotes the set of inflections points of f.

3. Hölder regularity

The Hölder regularity of continuous solutions to (1.4) in [**BSC**; **BCSC**] has been generalized in [**Car**] to the case of ℓ-nonlinear convex fluxes. We describe in § A what we mean by ℓ-nonlinear function: here we only mention that, in particular, are ℓ-nonlinear the functions for which at any point there is a derivative of order between 2 and ℓ which does not vanish. In this section we extend the local $\frac{1}{\ell}$-Hölder regularity of continuous solutions without convexity hypothesis, so that it applies, for example, to $f(u) = u^3$.

Proposition 3.1. *Let $f \in C^2(\mathbb{R})$ be nonlinear of order ℓ according to Definition A.1. Consider an open connected set $\Omega \subset \mathbb{R}^2$ and let $u \in C^0(\Omega)$ and $g \in L^\infty(\Omega)$ be such that*

$$\partial_t u + [f(u)]_x = g \quad \text{in } \mathcal{D}'(\Omega).$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = \frac{1}{\ell}$.

We first prove the following claim.

Lemma 3.2. *In the hypothesis of Proposition 3.1, consider $(t, x_1), (t, x_2) \in \Omega$ such that f'' does not vanish in the closed interval I with endpoints $u(t, x_1) = u_1$ and $u(t, x_2) = u_2$: set*

$$q = \frac{\min_I f''}{\max_J f''}\,, \quad G = \|g\|_{L^\infty}\,, \quad \tau = \frac{q}{2G}|u_2 - u_1|\,, \quad C = \|f''\|_{L^\infty(B_{G\tau}(I))}\,, \quad L = \|f'\|_{L^\infty(B_{G\tau}(I))}\,,$$

given any $J \supset B_{\frac{1}{2}|u_2-u_1|}(I)$. If $\|g\|_{L^\infty} = 0$ rather fix above any $G > 0$. Assume that

$$B_{|x_1-x_2|+L\tau+CG\tau^2}(t, x_1) \subset \Omega\,. \tag{3.1}$$

Then, denoting by c_ℓ the constant of nonlinearity in Definition A.1, we get

$$|u_2 - u_1| \le \mathsf{c}|x_2 - x_1|^{\frac{1}{\ell}}, \qquad \text{with } \mathsf{c} = \sqrt[\ell]{\frac{4G}{qc_\ell}}.$$

Proof of Lemma 3.2. We consider two characteristics γ_1, γ_2 passing through $(t, x_1), (t, x_2)$. Without loss of generality we assume $x_1 < x_2$. We can also assume $f'(u_1) > f'(u_2)$, the argument in the opposite case is similar considering evolution in the past.

Since γ_1 is a characteristic, then

$$\gamma_1(t+\tau) = x_1 + \int_t^{t+\tau} f'(u(s, \gamma_1(s)))ds. \tag{3.2}$$

By Remark 2.3 the Lipschitz function $u(\cdot, \gamma_1(\cdot))$ restricted to $[t, t+\tau]$ has image in $[u_1 - G\tau, u_1 + G\tau]$. By the chain rule on $\sigma \mapsto f'(u(\sigma, \gamma_1(\sigma)))$ we estimate

$$\left|f'(u(s, \gamma_1(s))) - f'(u_1)\right| = \left|\int_t^s \partial_\sigma f'(u(\sigma, \gamma_1(\sigma)))d\sigma\right| \le CG(s-t) \qquad \forall s \in [t, t+\tau]. \tag{3.3}$$

Estimating the integral in (3.2) by (3.3), and doing similarly with γ_2, we see that

$$\gamma_1(t+\tau) \ge x_1 + f'(u_1)\tau - \frac{CG}{2}\tau^2\,. \tag{3.4a}$$

$$\gamma_2(t+\tau) \le x_2 + f'(u_2)\tau + \frac{CG}{2}\tau^2. \tag{3.4b}$$

Notice that by the choice of τ and (3.1) the set $\{(s, \gamma_1(s)), (s, \gamma_2(s)) : s \in [t, t+\tau]\} \subset \Omega$ and since $2G\tau \le |u_2 - u_1|$ we have

$$(u_2 - G\tau, u_2 + G\tau) \cap (u_1 - G\tau, u_1 + G\tau) = \emptyset\,.$$

In particular there is no $s \in [t, t+\tau)$ for which $\gamma_1(s) = \gamma_2(s)$, and therefore $\gamma_1(s) < \gamma_2(s)$ for every $s \in [t, t+\tau)$. Concatenating the previous inequalities (3.4), as $\gamma_1(\tau) - \gamma_2(\tau) \le 0$ we have

$$f'(u_1) - f'(u_2) \le \frac{x_2 - x_1}{\tau} + CG\tau.$$

Lemma A.5 allows to absorb $CG\tau = \frac{q}{2}\|f''\|_{L^\infty(B_{\tau G}(I))}|u_2 - u_1|$ in the l.h.s., so that

$$\left(f'(u_1) - f'(u_2)\right)|u_1 - u_2|q \le 4G(x_2 - x_1).$$

By Lemma A.4 we have $f'(u_1) - f'(u_2) \ge c_\ell|u_1 - u_2|^{\ell-1}$, thus we arrive to the claim

$$|u_1 - u_2| \le \left(\frac{4G}{qc_\ell}\right)^{\frac{1}{\ell}} \sqrt[\ell]{x_2 - x_1}\,. \qquad \square$$

Example 3.3. The solution $u(t,x) = \operatorname{sign} x \cdot \sqrt{|x|}$ to $\partial_t u + \partial_x u^2 = \operatorname{sign} x$ has Hölder constant $\sqrt{2} = \left(\frac{2\|g\|_{L^\infty}}{qc_\ell}\right)^{\frac{1}{\ell}}$.

Proof of Proposition 3.1. Given any compact set $K \subset \Omega$ whose t-sections are intervals, set $J = u(K)$. By Proposition A.5 the set $Z = \{v \in J \;:\; f''(v) = 0\}$ is finite: set

$$\bar{\mathsf{c}} = \frac{\max\{|w-v| \;:\; w, v \in J\}}{\max\{|w-v| \;:\; B_\delta(Z)\cap[v,w]=\emptyset,\; w,v\in J\}} \qquad \delta = \frac{1}{3}\min\{|v-w| \;:\; v, w \in Z\}\,.$$

Observe that for every $x < y$ it is possible to find $x_1, x_2 \in [x,y]$ such that $|u(t,x) - u(t,y)| \le \bar{c}|u(t,x_1) - u(t,x_2)|$ and $B_\delta(Z) \cap [x_1,x_2] = \emptyset$, so that for some $c > 0$[1] by Lemma 3.2 we have

$$|u(t,x)-u(t,y)| \le \bar{\mathsf{c}}|u(t,x_1)-u(t,x_2)| \le \bar{\mathsf{c}}\mathsf{c}|x_1-x_2|^{\frac{1}{\ell}} \le \bar{\mathsf{c}}\mathsf{c}|y-x|^{\frac{1}{\ell}}\,.$$

We thus proved that u is $\frac{1}{\ell}-$Hölder in space on compact sets K whose t-sections are intervals. In order to recover the regularity in time, recall that u is $\|g\|_{L^\infty}-$Lipschitz along characteristics by Remark 2.3: as in at Step 2 in the proof of [**Car**] we get $\frac{1}{\ell}-$Hölder continuity on subsets of Ω of the form $\{(t,x) \in [t_1,t_2]\times\mathbb{R} \;:\; \gamma_1(t) \le x \le \gamma_2(t)\}$, with γ_1, γ_2 characteristics. Since the interior of such characteristic regions cover any compact subset of Ω we get the thesis. □

Remark 3.4. By looking at the proof of Proposition 3.1 we have that the ℓ-Hölder constant of u on a rectangle K depends only on f restricted to $u(K)$. More precisely it depends on c_ℓ and on the minimum distance between zeros of f''.

4. Finer Hölder regularity

The main result of this section is the following:

Proposition 4.1. *Let $f \in C^2$ be nonlinear of order ℓ as in Definition A.1. Let $\Omega \subset \mathbb{R}^2$ and $u \in C^0(\Omega)$, $g \in L^\infty(\Omega)$ be such that*

$$\partial_t u + [f(u)]_x = g \qquad \text{in } \mathcal{D}'(\Omega).$$

Then for $\mathscr{L}^2-$a.e. $(t,x) \in \Omega$ it holds

$$A(t,x) \doteq \limsup_{y\to x} \frac{|u(t,y)-u(t,x)|}{\sqrt[\ell]{|y-x|}} = 0. \tag{4.1}$$

By Remark 3.4, the function A is locally essentially bounded.

We will prove (4.1) for points (t,x) which are Lebesgue points of g with respect to the following suitable coverings of Ω. Such covering was first introduced in [**Car**] for proving that for $\mathscr{L}^2-$a.e. (t,x) the derivative of $\sigma \mapsto u(\sigma,\gamma(\sigma))$ along any characteristic γ through (t,x) is the Lebesgue value of g: for being self-contained, we prove this again in § 5 (Case 2) for the quadratic flux; the same argument works for ℓ-nonlinear fluxes by the estimates in this section.

Given $\varrho > 0$, define regions translating a characteristic γ of $\mathfrak{h} = \varrho\varepsilon^\ell$, between times $\sigma \pm \varepsilon$:

$$\begin{aligned}\mathcal{V}_\varrho &\doteq \left\{S_\varrho(\gamma,\varepsilon,\sigma) \;\middle|\; \sigma\in\mathbb{R},\,\varepsilon>0,\,\gamma\in C^1((\sigma-\varepsilon,\sigma+\varepsilon) \;:\; \dot\gamma(t) = f'(u(t,\gamma(t)))\right\}\,,\\ S_\varrho(\gamma,\varepsilon,\sigma) &\doteq \left\{(t,x)\in[\sigma-\varepsilon,\sigma+\varepsilon]\times\mathbb{R} \;:\; \gamma(t)-\varrho\varepsilon^\ell \le x \le \gamma(t)+\varrho\varepsilon^\ell\right\}.\end{aligned} \tag{4.2}$$

Proposition 4.2. *Consider any sequence $\{\varrho_j\}_{j\in\mathbb{N}}$ and $\mathcal{V}_{\varrho_j}$ as in (4.2). For every $q \in L^\infty(\Omega)$ a.e. $(t,x) \in \Omega$ is a Lebesgue point of q with respect to the covering $\mathcal{V}_{\varrho_j}$, namely there is $O \subset \Omega$ with $\mathscr{L}^2(\Omega\setminus O) = 0$ such that for every $(t,x) \in O$ and every $j \in \mathbb{N}$ it holds*

$$\lim_{\substack{(t,x)\in S\in\mathcal{V}_{\varrho_j}\\ \operatorname{diam}(S)\downarrow 0}} \frac{1}{\mathscr{L}^2(S)}\int_S |q-q(t,x)|\,dxdt = 0.$$

[1]Doing a rough estimate, there is not point in looking for constants better than $\mathsf{c} = \sqrt[\ell]{\frac{\min_{J\setminus B_\delta(Z)}|f''|}{\max_J|f''|}\frac{4G}{c_\ell}}$.

The proof of the above proposition can be found in [**Car**]: the key point is that one can define through the sets $S_\varrho(\gamma,\varepsilon,\sigma)$ a quasi-distance for which the measure $\mathscr{L}^2$ is doubling so that Lebesgue differentiation theorem applies.

We are now in position to prove Proposition 4.1.

Proof of Proposition 4.1. Since $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ by Proposition 3.1, for any $c\in\mathbb{R}$ it holds $A(t,x)=0$ at every (t,x) which is a point of density one of the set $u^{-1}(\{c\})$. Since $Z=\{v\ :\ f''(v)=0\}$ is discrete by Lemma A.5, the statement thus holds for $\mathscr{L}^2$-a.e. $(t,x)\in u^{-1}(Z)$.

We now prove the claim for $\mathscr{L}^2$-a.e. $(t,x)\in\Omega\setminus u^{-1}(Z)$, namely such that $f''(u(t,x))\neq 0$. Given two sequences $\varrho_j,\delta_j\downarrow 0$, $\varrho_j+\delta_j<1$, we consider the coverings $\mathcal{V}_{\varrho_j}$ in (4.2) and we set

$$r_j=\frac{\varrho_j^\ell}{1+\delta_j}\quad\text{and denote } \varepsilon=\varepsilon_j(\mathfrak{h})=\frac{\sqrt[\ell]{(1+\delta_j)\mathfrak{h}}}{\varrho_j}\ \text{ for } \mathfrak{h}>0\,,$$

so that $r_j\varepsilon^\ell=\mathfrak{h}$ and $\varrho_j\varepsilon^\ell=\varrho_j^{-(\ell-1)}(1+\delta_j)\mathfrak{h}$. We prove that $A(t,x)=0$ for every $(t,x)\in\Omega\setminus u^{-1}(Z)$ which is a Lebesgue point of g with respect to each covering $\mathcal{V}_{\varrho_j}$ for $j\in\mathbb{N}$. Proposition 4.2 grants that such points have full Lebesgue measure in $\Omega\setminus u^{-1}(Z)$.

Up to a translation we may assume that $(t,x)=(0,0)$: let $x_k\to 0$ be a sequence such that $A(0,0)=\lim_{k\to\infty}\frac{|u(0,x_k)-u(0,0)|}{\sqrt[\ell]{|x_k|}}$. Up to a symmetry with respect to the axis $x=0$, and up to repeating the following argument for negative times, is not restrictive to consider the case for which $x_k\downarrow 0$, $u(0,x_k)<u(0,0)$ and $f''(u(0,0))>0$. Denote by γ_0 and γ_k two characteristics such that $\gamma_0(0)=0$ and $\gamma_k(0)=x_k$. Set $u_0(t)=u(t,\gamma_0(t))$ and $u_k(t)=u(t,\gamma_k(t))$. Let us fix $j\in\mathbb{N}$ and denote by S_k the set $S_{\varrho_j}(\gamma_0,\varepsilon_j(x_k),0)$. With the standard convention $\inf\emptyset=+\infty$, set

$$t_k^*=\varepsilon_j(x_k)\wedge\inf\{t>0\ \ :\ u_0(t)\le u_k(t)\}\ .$$

Notice that $t_k^*>0$, being u continuous and $u_k(0)<u_0(0)$. Since $x_k,t_k^*\to 0$ as $k\to\infty$ and u is continuous, there is $\bar k$ such that for every $k>\bar k$ the function u restricted to S_k takes values in $u^{-1}(\{f''>0\})$. We will assume $k\ge\bar k$ in the following of the proof: thus

$$\dot\gamma_0(t)=f'(u_0(t))\ge f'(u_k(t))=\dot\gamma_k(t)\qquad\text{in } [0,t_k^*]$$

and $\gamma_k\le\gamma_0+x_k$ by the initial condition, so that by the choice $\varrho_j\varepsilon^\ell>(1+\delta_j)\mathfrak{h}$ we have

$$\{(t,x)\in[0,t_k^*]\times\mathbb{R}:\gamma_0(t)-\delta_jx_k\le x\le\gamma_k(t)+\delta_jx_k\}\subset S_k\,. \tag{4.3}$$

If $t\in[0,t_k^*]$, Proposition 2.2 gives the one sided estimates

$$\int_{\gamma_k(t)}^{\gamma_k(t)+\delta_jx_k}u(t,x)dx-\int_{x_k}^{x_k+\delta_jx_k}u(0,x)dx\le\int_0^t\int_{\gamma_k(t)}^{\gamma_k(t)+\delta_jx_k}g(s,x)dxds\,, \tag{4.4a}$$

$$\int_{\gamma_0(t)-\delta_jx_k}^{\gamma_0(t)}u(t,x)dx-\int_{-\delta_jx_k}^{0}u(0,x)dx\ge\int_0^t\int_{\gamma_0(t)-\delta_jx_k}^{\gamma_0(t)}g(s,x)dxds\,. \tag{4.4b}$$

Denote by c the $\frac{1}{\ell}$-Hölder constant of u restricted to $S_{\bar k}$. Then when $\{t\}\times[x-r,x+r]\subset S_k$

$$\int_x^{x+r}|u(t,q)-u(t,x)|\,dq\le\mathsf{c}r^{\frac{\ell+1}{\ell}},\qquad\int_{x-r}^{x}|u(t,q)-u(t,x)|\,dq\le\mathsf{c}r^{\frac{\ell+1}{\ell}}. \tag{4.5}$$

By (4.3) and by Proposition 4.2 in particular for every $t\in[0,t_k^*]$ there is $\eta_k\downarrow 0$ such that

$$\left|\int_0^t\int_{\gamma_k(t)}^{\gamma_k(t)+\delta_jx_k}g(s,x)dxds-t\delta_jx_kg(0,0)\right|\le\eta_k\mathscr{L}^2(S_k)\,. \tag{4.6}$$

Subtracting equations (4.4) and dividing by δ_jx_k, from (4.3), (4.5) and (4.6) we deduce that

$$u_0(t)-u_k(t)\ge u_0(0)-u_k(0)-2\frac{\eta_k\mathscr{L}^2(S_k)}{\delta_jx_k}-2\mathsf{c}\sqrt[\ell]{\delta_jx_k}\qquad\text{for every } t\in[0,t_k^*]. \tag{4.7}$$

Case 1: there is $t \in [0, t_k^*]$ such that $u_0(t) - u_k(t) \le \frac{1}{2}(u_0(0) - u_k(0))$. From (4.7) it holds

$$\begin{aligned} \frac{u(0,0) - u(0,x_k)}{\sqrt[\ell]{x_k}} &\le 4\frac{\mathscr{L}^2(S_k)}{\delta_j x_k \sqrt[\ell]{x_k}}\eta_k + 4\mathsf{c}\frac{\sqrt[\ell]{\delta_j x_k}}{\sqrt[\ell]{x_k}} \\ &= \frac{16(1+\delta_j)^{\frac{\ell+1}{\ell}}}{\delta_j \varrho_j^\ell}\eta_k + 4\mathsf{c}\sqrt[\ell]{\delta_j}. \end{aligned} \tag{4.8}$$

Case 2: for every $t \in [0, t_k^*]$ it holds $u_0(t) - u_k(t) > \frac{1}{2}(u_0(0) - u_k(0))$. In particular we have $t_k^* = \frac{\sqrt[\ell]{(1+\delta_j)x_k}}{\varrho_j}$ and γ_k, γ_0 do not intersect in $[0, t_k^*]$. By Lemma A.4, for every $t \in [0, t_k^*]$ it holds

$$\begin{aligned} \gamma_0'(t) - \gamma_k'(t) &= f'(u_0(t) - f'(u_k(t)) \\ &\ge 2c_\ell \left(u_0(t) - u_k(t)\right)^{\ell-1} \ge \frac{c_\ell}{2^{\ell-2}}\left(u_0(0) - u_k(0)\right)^{\ell-1} \end{aligned}$$

Integrating with respect to time in $[0, t_k^*]$ and imposing that $\gamma_0(t_k^*) \le \gamma_k(t_k^*)$ we obtain

$$0 \ge \gamma_0(t_k^*) - \gamma_k(t_k^*) \ge \gamma_0(0) - \gamma_k(0) + \frac{c_\ell}{2^{\ell-1}}\left(u_0(0) - u_k(0)\right)^{\ell-1} t_k^*.$$

Recalling that $\gamma_0(0) = 0$ and $\gamma_k(0) = x_k$ this implies

$$\frac{u(0,0) - u(0,x_k)}{\sqrt[\ell]{x_k}} \le 2\left(\frac{x_k}{t_k^* c_\ell}\right)^{\frac{1}{\ell-1}} \frac{1}{\sqrt[\ell]{x_k}} = 2\left(\frac{\varrho_j}{c_\ell \sqrt[\ell]{1+\delta_j}}\right)^{\frac{1}{\ell-1}} \tag{4.9}$$

Taking first the limit as $k \to \infty$, where η_k vanishes, (4.8) and (4.9) show that $A(0,0) = 0$ because δ_j and ϱ_j can be taken arbitrarily small. □

5. A new proof of the Rademacher theorem for intrinsic Lipschitz functions

Aim of this section is to provide a new proof of the Rademacher theorem for intrinsic Lipschitz functions in the first Heisenberg group $\mathbb{H}^1$. We start by recalling the main definitions, we refer the interested reader to [**SC**] for a complete introduction to the subject.

We denote the points of $\mathbb{H}^1 \equiv \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ by

$$P = [z,t] = [x+iy, t] = (x,y,t), \qquad z \in \mathbb{C},\ x,y \in \mathbb{R},\ t \in \mathbb{R}.$$

If $P = [z,t]$, $Q = [z',t'] \in \mathbb{H}^1$ and $r > 0$, the group operation reads as

$$P \cdot Q := \left[z + z', t + t' - \frac{1}{2}\operatorname{Im}(\langle z, \bar{z'}\rangle)\right]. \tag{5.1}$$

The group identity is the origin 0 and one has $[z,t]^{-1} = [-z,-t]$. In $\mathbb{H}^1$ there is a natural one parameter group of non isotropic dilations $\delta_r(P) := [rz, r^2 t]$, $r > 0$. The group $\mathbb{H}^1$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$$

and with the left-invariant and homogeneous distance

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty.$$

The metric d_∞ is equivalent to the standard Carnot-Carathéodory distance. It follows that the Hausdorff dimension of $(\mathbb{H}^1, d_\infty)$ is 4, whereas its topological dimension is clearly 3. The Lie algebra $\mathfrak{h}_1$ of left invariant vector fields is (linearly) generated by

$$X = \partial_x - \frac{1}{2}y\partial_t, \quad Y = \partial_y + \frac{1}{2}x\partial_t, \quad T = \partial_t$$

and the only nonvanishing commutators are

$$[X,Y] = T.$$

In the spirit of [**ASCV**] we set $\mathbb{W} := \{(x,y,t) \in \mathbb{H}^1 :\ x = 0\} \equiv \mathbb{R}^2$. Therefore, if $A \in \mathbb{W}$, we write $A = (y,t)$.

Following [**ASCV**] we define the graph quasidistance as:

Definition 5.1. For $A=(y,t)$, $B=(y',t')\in\omega$ we define

$$d_\varphi(A,B)=|y-y'|+\left|t'-t-\frac{1}{2}(\varphi(A)+\varphi(B))(y'-y)\right|^{1/2}.$$

An intrinsic differentiable structure can be induced on $\mathbb{W}$ by means of d_φ, see [**ASCV**]. We remind that a map $L:\mathbb{W}\to\mathbb{R}$ is $\mathbb{W}$-linear if it is a group homeomorphism and $L(ry,r^2t)=rL(y,t)$ for all $r>0$ and $(y,t)\in\mathbb{W}$. We recall then the notion of φ-differentiablility.

Definition 5.2. Let $\varphi:\omega\subset\mathbb{W}\to\mathbb{R}$ be a fixed continuous function, and let $A_0\in\omega$ and $\psi:\omega\to\mathbb{R}$ be given. We say that ψ is φ-differentiable at A_0 if there is an $\mathbb{W}$-linear functional $L:\mathbb{W}\to\mathbb{R}$ such that

$$\lim_{A\to A_0}\frac{\psi(A)-\psi(A_0)-L(A_0^{-1}\cdot A)}{\mathrm{d}_\varphi(A_0,A)}=0. \tag{5.2}$$

It is well known [**ASCV**] that given a $\mathbb{W}$-linear functional $L:\mathbb{W}\to\mathbb{R}$ there exists a unique $\hat{w}\in\mathbb{R}$ such that $L(A)=L((y,t))=\hat{w}y$. Let us now introduce the concept of intrinsic Lipschitz function.

Definition 5.3. Let $\varphi:\omega\subset\mathbb{W}\to\mathbb{R}$. We say that φ is an intrinsic Lipschitz continuous function and we write $\varphi\in Lip_{\mathbb{W}}(\omega)$ if there exists a constant $L>0$ such that

$$|\varphi(A)-\varphi(B)|\le L d_\varphi(A,B)\quad \forall A,B\in\omega.$$

We say that φ is locally intrinsic Lipschitz and we write $\varphi\in Lip_{\mathbb{W},loc}(\omega)$ if $\varphi\in Lip_{\mathbb{W},loc}(\omega')$ for any $\omega'\Subset\omega$.

The following characterization is proved in [**BCSC**].

Theorem 5.4. *Let $\omega\subset\mathbb{R}^2$ be an open set and let $\varphi:\omega\to\mathbb{R}$ be a continuous. Then $\varphi\in Lip_{\mathbb{W},loc}(\omega)$ if and only if there exists $g\in L^\infty(\omega)$ such that*

$$\varphi_y+[\varphi^2/2]_t=g\quad in\ \mathcal{D}'(\omega).$$

Let us finally recall the following Rademacher type Theorem, proved in [**FSSC2**] (see also [**Vit**]). As recalled in the Introduction the original proof requires deep results in geometric measure theory, namely the fact that the subgraph of an intrinsic Lipschitz function φ is a set with locally finite $\mathbb{H}$-perimeter and that at almost every point of the graph of φ there is an approximate. Here instead we propose a completely PDEs based proof which uses the results of Section 4.

Theorem 5.5. *If $\varphi\in Lip_{\mathbb{W}}(\omega)$ then φ is φ-differentiable $\mathcal{L}^2-$a.e. in ω.*

More precisely, we prove φ-differentiability at those points which are Lebesgue points of g for all the coverings (4.2), for any fixed sequence ρ_i decreasing to 0.

Proof. Let $A_0=(t,x)$ be a Lebesgue point of g with respect to the coverings $\mathcal{V}_{\varrho_j}$ of Lemma 4.2 and for simplicity we assume $A_0=(0,0)$. We show that any sequence $A_n=(t_n,x_n)\to A_0$ has a (not relabeled) subsequence for which

$$\lim_{n\to\infty}\frac{|\varphi(A_n)-\varphi(A_0)-g(0,0)t_n|}{d_\varphi(A_n,A_0)}=0.$$

We can assume $t_n,x_n'\in(0,\frac{1}{1+\|g\|_{L^\infty}})$. We recall the explicit expression

$$d_\varphi(A_n,A_0)=|t_n-t|+\left|x_n-x-\frac{\varphi(A_n)+\varphi(A_0)}{2}\cdot(t_n-t)\right|^{1/2}.$$

Let $\gamma_n\in C^1([0,t_n])$ be a characteristic of φ through (t_n,x_n) and set $A_n'=(0,\gamma_n(0))=(0,x_n')$: by Remark (2.3)

$$\left|x_n-\frac{\varphi(A_n)+\varphi(A_0)}{2}\cdot t_n-x_n'\right|=\left|\int_0^{t_n}\varphi(s,\gamma(s))\,ds-\frac{\varphi(A_n)+\varphi(A_0)}{2}\cdot t_n\right|\le\frac{\|g\|_{L^\infty}}{4}t_n^2. \tag{5.3}$$

Up to a (not relabeled) subsequence one of the following two cases is satisfied.

Case 1: it holds $\lim_{n\to\infty} \frac{|t_n|}{\sqrt{|x'_n|}} = 0$. In this case it follows from Proposition 4.1 and Remark 2.3 that:

$$\begin{aligned}|\varphi(A_n) - \varphi(A_0) - g(0,0)t_n| &\leq \|g\|_{L^\infty}|t_n| + |\varphi(A_n) - \varphi(A'_n)| + |\varphi(A'_n) - \varphi(A_0)| \\ &\leq 2\|g\|_{L^\infty} t_n + o\left(\sqrt{x'_n}\right) = o\left(\sqrt{x'_n}\right) .\end{aligned}$$

We conclude since (5.3) and $t_n = o(\sqrt{x'_n})$ imply $d_\varphi(A_n, A_0) = \sqrt{x'_n} \cdot (1 + o(\sqrt{x'_n}))$.

Case 2: there is $\delta \in (0,1)$ such that $\frac{|t_n|}{\sqrt{|x'_n|}} \geq \delta > 0$. Let $\varepsilon \in (0,1)$ and consider $S^\varepsilon_n = \{(s,x) \in [0,t_n] \times \mathbb{R} : \gamma(s) - \varepsilon t_n^2 \leq x \leq \gamma(s) - \varepsilon t_n^2\}$. Denote by c the $\frac{1}{2}$−Hölder constant of φ in a ball B centered at A_0. For n sufficiently large $A_n, A'_n \in B$: by estimates like (4.4)-(4.5) we get

$$|\varphi(A_n) - \varphi(A_0) - g(0,0)t_n| \leq 4\mathsf{c}\sqrt{\varepsilon} t_n + \frac{1}{2\varepsilon t_n^2} \int_{S^\varepsilon_n} |g - g(0,0)|. \tag{5.4}$$

Just for notational convenience, suppose $\varrho_0 = 1$. Since $S^\varepsilon_n \subset S_1\left(\gamma_n, t_n\sqrt{\frac{1}{\delta^2} + \varepsilon}, 0\right) \in \mathcal{V}_1$ and $(0,0)$ is a Lebesgue point of g with respect to the covering $\mathcal{V}_1$, then

$$\eta_n \doteq \frac{1}{\mathscr{L}^2\left(S_1\left(\gamma_n, t_n\sqrt{\frac{1}{\delta^2} + \varepsilon}, 0\right)\right)} \int_{S^\varepsilon_n} |g - g(0,0)|$$

vanishes as $n \to \infty$. The measure of the set is $4\left(\frac{1}{\delta^2} + \varepsilon\right)^{\frac{3}{2}} t_n^3$. It follows from (5.4) that

$$\begin{aligned}\frac{|\varphi(A_n) - \varphi(A_0) - g(0,0)t_n|}{t_n} &\leq 4\mathsf{c}\sqrt{\varepsilon} + \frac{\mathscr{L}^2\left(S_1\left(\gamma_n, t_n\sqrt{\frac{1}{\delta^2} + \varepsilon}, 0\right)\right)}{2\varepsilon t_n^3}\eta_n \\ &= 4\mathsf{c}\sqrt{\varepsilon} + \frac{2\left(\frac{1}{\delta^2} + \varepsilon\right)^{\frac{3}{2}}}{\varepsilon}\eta_n.\end{aligned}$$

Recalling that $|t_n| \leq d_\varphi(A_n, A_0)$, the claim follows by letting $n \to \infty$ since ε is arbitrarily small. □

Appendix A. Nonlinear functions of a given order

We specify the assumption of finite order nonlinearity, for brevity also called ℓ-nonlinearity.

Definition A.1. Let $I \subset \mathbb{R}$ be an interval. A function f differentiable at v is "nonlinear of order $\ell > 1$ with constant $c > 0$ at v" if there exists $\delta > 0$ such that for every $v + h \in I \cap B_\delta(v)$ one has

$$|f(v+h) - f(v) - f'(v)h| \geq c\,|h|^\ell \,.$$

If the inequality holds for all $v, v+h \in I$ we call f "nonlinear in I of order ℓ with constant c".

Of course, if f is nonlinear of order $\ell > 1$ with constant $c > 0$ at v then it is also nonlinear at v of every order $\ell' \geq \ell$ with any constant $0 < c' \leq c$ just because $|h|^{\ell'-\ell} \leq 1$ when $0 < |h| \leq \delta < 1$.

Remark A.2. The best order of nonlinearity at a point v is the order ℓ of the first non-vanishing derivative at the point, higher than the first. It can be proved just applying the definition of differentiability ℓ-times recursively, and the constant is arbitrarily close to $\frac{1}{\ell!}|f^{(\ell)}(v)|$.

Lemma A.3. *Suppose $f \in C^\ell(I)$. If $\sum_{k=2}^{\ell} \frac{1}{k!}\left|f^{(k)}\right| \geq c$ in the interval I and I has length d then f is nonlinear in I of order ℓ with constant $c \cdot \min\{d^{2-\ell}, 1\}$. Moreover, if $k \in \mathbb{N}$ is the minimum exponent of nonlinearity of f at v with c_0 the infimum of the relative constants, then $f^{(k)}(v) = k!c_0$ and $f^{(j)}(v) = 0$ for $j = 2, \dots, k-1$.*

Proof. Let $v, v+h \in I$. Just by Taylor's expansion, if $f^{(k)}(v) = 0$ for $2 \leq k < j \leq \ell$ one has

$$|f(v+h) - f(v) - f'(v)h| = \left|\sum_{k=2}^{j-1} \frac{1}{k!} f^{(k)}(v)h^k + \frac{1}{j!} f^{(j)}(\xi)h^j\right| \geq c|h|^j \geq \frac{c}{d^{\ell-j}}|h|^\ell \,,$$

thanks to the assumtion $\frac{1}{j!}\left|f^{(j)}\right| \geq c$ and that K has diameter d, so that $|h|^{\ell-j} \leq d^{\ell-j}$.

Suppose now that k is the minimum exponent of nonlinearity of f at v: by the first part of the statement in particular $f^{(j)}(v) = 0$ for $j = 2, \dots, k-1$ and that $|f^{(k)}(v)| \leq c_0 k!$. If

$$|f(v+h) - f(v) - f'(v)h| \geq (c_0 - \varepsilon)|h|^\ell \qquad \forall \varepsilon > 0\,,\ |h| \leq \delta$$

then is ξ between v and $v+h$ such that $f(v+h) - f(v) - f'(v)h = \frac{f^{(k)}}{k!}(\xi)$, thus $|f^{(k)}(v)| \geq c_0 k!$. ☐

Nonlinearity provides a lower bound on the increments also of the first derivative of f:

Lemma A.4. *Suppose f is nonlinear in an interval I of order ℓ with constant c. Then*

$$|f'(v) - f'(w)| \geq 2c|v-w|^{\ell-1} \qquad \forall v, w \in I\,. \tag{A.1}$$

Proof. Just by Definition (A.1) of nonlinearity when $h = w - v$ we have

$$|f'(w)h - f'(v)h| = |f(v+h) - f(v) - f'(v)h + f(w-h) - f(w) + f'(w)h| \geq 2c|h|^\ell \,.$$ ☐

Lemma A.5. *If $f \in C^2(J)$ is nonlinear of order ℓ and constant c in a compact interval J, then:*

(1) When $\ell = 2$ the second derivative of f has lower bound $2c$ in J and it holds

$$|f'(v) - f'(w)| \geq q_{w,v} \cdot \|f''\|_{L^\infty(J)} \cdot |v-w|\,, \tag{A.2}$$

for all $v, w \in J$, with $q_{w,v} \doteq \frac{2c}{\|f''\|_{L^\infty(J)}}$ belonging in $(0,1]$ and independent on w, v.

(2) When $\ell > 2$ the set $Z = \{v \in J : f''(v) = 0\}$ is finite and inequality (A.2) holds whenever $[w,v] \cap Z = \emptyset$ with $q_{w,v} \doteq \frac{\min_{[w,v]}|f''|}{\max_J |f''|}$ belonging in $(0,1]$.

Proof. (1) When $\ell = 2$ the statement follows immediately by (A.1).

(2) At any accumulation point $\bar v$ of D necessarily $f^{(j)}(\bar v) = 0$ for every $j \geq 2$, thus by Taylor's expansion we would have $f(\bar v + h) - f(\bar v) - f'(\bar v)h = o(h^\ell)$: this would contradict the finite order of nonlinearity of f. In particular, also when $\ell > 2$ the set D must be discrete.

If $[u,v] \cap K = \emptyset$ then f is nonlinear of order 2 in $[u,v]$ and $|f'(v) - f'(w)| = |f''(\xi)||v - w|$ with $\xi \in (v,w)$. Writing $|f''(\xi)| \geq \min_{[u,v]}|f''| \cdot \frac{\max_J |f''|}{\max_J |f''|}$ one has the claim. ☐

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