A NOTE ON THE COMPARISON PRINCIPLE FOR DEGENERATE SUB-ELLIPTIC EQUATIONS

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Abstract. We show that the comparison principle holds for smooth sub/super-solutions of a class of degenerated sub-elliptic equations that include the sub-elliptic ∞-Laplacian. The equations are defined by a collection of vector fields satisfying Hörmander's rank condition and are left-invariant with respect to a Nilpotent Lie Group.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, simply connected bounded domain and for some integer $m \leq n$, let $\{X_1, \dots, X_m\}$ be a collection of smooth linearly independent vector fields defined on Ω satisfying Hörmander's rank condition

$$\dim\big(\texttt{Lie}(X_1, \dots, X_m)(x)\big) = n, \tag{1.1}$$

at every point $x \in \Omega$ and are left invariant with respect to a nilpotent Lie group $\mathbb{G} = (\mathbb{R}^n, *)$ of step r. In other words, if $\mathfrak{g}_1 = \text{span}\{X_1, \dots, X_m\}$ then we have $\texttt{Lie}(\mathfrak{g}_1) = \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$. The group $\mathbb{G}$ is called a Carnot group of step r. See Section 2 for more details. The more relevant case occurs if $m < n$ as for the case of $m = n$, we are in the Riemannian setting. We consider the equation

$$\mathcal{L}u := -\,\text{Tr}\,\big(a(\mathfrak{X}u)\mathfrak{X}\mathfrak{X}u\big) = -\sum_{i,j=1}^{m} a_{i,j}(\mathfrak{X}u)X_jX_iu = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where $\mathfrak{X}u = (X_1u, \dots, X_mu)$ is the sub-elliptic gradient, $\mathfrak{X}\mathfrak{X}u = (X_jX_iu)_{i,j}$ is the sub-elliptic second derivative matrix, and $a_{i,j} : \mathbb{R}^m \to \mathbb{R}$ are C^1 functions such that the $(m \times m)$ matrix $a(\xi)$ is symmetric and satisfies the ellipticity condition

$$\mathcal{E}(\xi) := \big\langle a(\xi)\,\xi, \xi\big\rangle > 0 \qquad \forall\ \xi \in \mathbb{R}^m \setminus \{0\}. \tag{1.3}$$

Furthermore, we assume that a is homogeneous; i.e. there exist a non-negative constant $\beta \in \mathbb{R}$ such that for any $t \in \mathbb{R}$, we have

$$a(t\,\xi) = |t|^{\beta} a(\xi), \quad \forall\ \xi \in \mathbb{R}^m. \tag{1.4}$$

Note that since a is assumed to be symmetric, the equation (1.2) can also be written as

$$\mathcal{L}u = -\,\text{Tr}\,\big(a(\mathfrak{X}u)\mathfrak{X}\mathfrak{X}^{\star}u\big) = 0,$$

where $\mathfrak{X}\mathfrak{X}^{\star}u$ is the symmetrized sub-elliptic second derivative. More details are presented in Section 2.

The equation (1.2) encompasses a large class of degenerated equations, some examples are in order. The sub-Laplacian $\Delta_{\mathfrak{X}}u = -\,\text{Tr}(\mathfrak{X}\mathfrak{X}u) = 0$ corresponds to (1.2) with $a(\xi) \equiv \mathrm{I}_m$ and $\beta = 0$, and the equation is linear. Among examples of non-linear equations, one of the most important cases is the sub-elliptic ∞-Laplacian equation

$$\Delta_{\mathfrak{X},\infty}u = \big\langle \mathfrak{X}\mathfrak{X}u\,\mathfrak{X}u, \mathfrak{X}u\big\rangle = 0, \tag{1.5}$$

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which corresponds to the equation (1.2) with $a(\xi) = \xi \otimes \xi$, which satisfies (1.4) with $\beta = 2$. We refer to [6, 5, 14] for proofs of a comparison principle in Carnot groups that implies the uniqueness of viscosity solutions of the equation (1.5) with Dirichlet boundary conditions. A generalization of the equation (1.5), given by

$$\big\langle \mathfrak{X}\mathfrak{X}u \, \nabla f(\mathfrak{X}u), \nabla f(\mathfrak{X}u)\big\rangle = \sum_{i,j=1}^{m} \partial_i f(\mathfrak{X}u)\partial_j f(\mathfrak{X}u) X_i X_j u = 0 \tag{1.6}$$

corresponding to $a(\xi) = \nabla f(\xi) \otimes \nabla f(\xi)$, has been studied by C. Wang [14] in the context of minimization of $\|f(\mathfrak{X}u)\|_{L^\infty}$ for a function $f \in C^2(\mathbb{R}^m)$ that is convex, homogeneous of a fixed degree ≥ 1 and $f(\xi) > 0$ for $\xi \neq 0$; the special case $f(\xi) = |\xi|^2$ leads to the equation (1.5). Other notable examples include the normalized sub-elliptic p-Laplacian equation

$$\Delta^N_{\mathfrak{X},p} u = \Delta_{\mathfrak{X}} u + (p-2)\frac{\Delta_{\mathfrak{X},\infty} u}{|\mathfrak{X}u|^2} = 0,$$

corresponding to $a(\xi) = \mathrm{I}_m + (p-2)(\xi \otimes \xi)/|\xi|^2$ for $1 < p < \infty$.

In this paper, we prove the following comparison principle for smooth sub/super-solutions.

Theorem 1.1. *Let $\mathcal{L}$ be as in (1.2) with $a \in C^1(\mathbb{R}^m, \mathbb{R}^{m\times m})$ satisfying (1.3) and (1.4). If there exists $u, v \in C^2(\Omega)$ such that $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ in $\partial\Omega$, then we have $u \leq v$ in Ω.*

We remark that this comparison principle is non-trivial even in the Euclidean case since we need to overcome the case where the gradient vanishes. We follow an argument of Barles and Busca [5], based on the strong maximum principle for sub-solutions. In the case of arbitrary Hörmander vector fields, this comparison principle for linear equations is due to Bony [8]. A comparison principle for viscosity solutions of the equation (1.6) was proved by Wang [14]. Our conditions are more general and our proof does not rely on approximations by solutions of p-Laplace equations. For quasilinear degenerate elliptic equations defined by Hörmander vector fields, the strong maximum principle was proved by Capogna and Zhou [9] and for fully nonlinear equations defined by Hörmander vector fields, the strong maximum principle has been established by Bardi-Goffi [2], see Section 2. We also refer to [4] and [3],for similar results.

Our proof is limited to smooth sub/super solutions. In the Euclidean case and in the case of Carnot groups, arbitrary viscosity sub-solutions and super-solutions can be approximated, respectively, by semi-convex sub-solutions and semi-concave super-solutions (see [11, 1, 14, 6]) that suggests that our proof can potentially be extended to viscosity solutions. This extension will be addressed in a forthcoming paper.

2. Notations and Preliminaries

For a function $u : \Omega \to \mathbb{R}$, let us denote the sub-elliptic gradient and second derivative as

$$\mathfrak{X}u = (X_1 u, \ldots, X_m u) \quad \text{and} \quad \mathfrak{X}\mathfrak{X}u = (X_j X_i u)_{i,j}.$$

The second derivative matrix is not symmetric in general, hence we denote the symmetrized second derivative as

$$\mathfrak{X}\mathfrak{X}^\star u = \frac{1}{2}\left(\mathfrak{X}\mathfrak{X}u + (\mathfrak{X}\mathfrak{X}u)^T\right) = \frac{1}{2}\big(X_i X_j u + X_j X_i u\big)_{i,j},$$

the sub-elliptic divergence of $F = (f_1, \ldots, f_m)$ is defined by

$$\operatorname{div}_H(F) = \sum_{j=1}^{m} X_j^* f_j,$$

where X_j^* are the adjoints with respect to $L^2(\Omega)$. As X_j's are left-invariant, we can choose coordinates so that $X_j^* = -X_j$, without loss of generality. The standard Euclidean dot product on $\mathbb{R}^n$ is denoted by $(\cdot)$, the Euclidean vector fields are denoted as ∂_{x_i} and $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$

is the Euclidean gradient, DF is the Jacobian matrix for a vector function $F:\Omega\to\mathbb{R}^n$ and $D^2u = D(\nabla u) = (\partial_{x_i}\partial_{x_j}u)_{i,j}$ is the Euclidean Hessian.

For symmetric square matrices $A, B\in\mathbb{R}^{k\times k}$, we shall denote $A\le B$ if $\langle A\xi,\xi\rangle \le \langle B\xi,\xi\rangle$ for every $\xi\in\mathbb{R}^k$. From the ellipticity (1.3) and condition (1.4) together, we can conclude

$$\mathcal{E}(\xi) = \langle a(\xi)\,\xi,\xi\rangle \ge a_0|\xi|^{2+\beta} \quad \forall\, \xi\in\mathbb{R}^m, \quad \text{where,} \quad a_0 := \inf_{\zeta\in\mathbb{S}^{m-1}}\langle a(\zeta)\zeta,\zeta\rangle > 0. \tag{2.1}$$

We note that there exists the matrix $\sigma:\Omega\to\mathbb{R}^{n\times m}$ written as

$$\sigma(x) = (\sigma_i^j(x))_{i,j} = [\sigma^1(x),\dots,\sigma^m(x)]$$

for $\sigma^j:\Omega\to\mathbb{R}^n$, such that $X_j = \sigma^j(x)\nabla$. Hence, for any $u:\Omega\to\mathbb{R}$, we have

$$\mathfrak{X}u = \sigma(x)^T\nabla u \quad \text{and} \quad \mathfrak{X}\mathfrak{X}u = \sigma(x)^T D^2u\,\sigma(x) + \{D\sigma(x)\otimes\sigma(x)\}\cdot\nabla u \tag{2.2}$$

where $D\sigma(x)\otimes\sigma(x)$ is a 3-tensor such that $\{D\sigma(x)\otimes\sigma(x)\}\cdot\nabla u$ is a matrix with entries

$$\big(\{D\sigma(x)\otimes\sigma(x)\}\cdot\nabla u\big)_{i,j} = D\sigma^j(x)\sigma^i(x)\cdot\nabla u = \sum_{k,l}\partial_{x_l}\sigma_k^j(x)\sigma_l^i(x)\partial_{x_k}u.$$

Thus, the symmetrized second derivative can be expressed as

$$\mathfrak{X}\mathfrak{X}^\star u = \sigma(x)^T D^2u\,\sigma(x) + g(x,\nabla u), \tag{2.3}$$

where $g(x,\nabla u)\in\mathbb{R}^{m\times m}$ is a matrix with entries

$$g(x,\nabla u)_{i,j} = \frac{1}{2}\big(D\sigma^j(x)\sigma^i(x)\cdot\nabla u + D\sigma^i(x)\sigma^j(x)\cdot\nabla u\big),$$

note that the map $\xi\mapsto g(x,\xi)$ is linear. Assuming that the vector fields are non-zero and smooth, we have $\sigma(x)\neq 0$ and $x\mapsto\sigma(x)$ is smooth for every $x\in\Omega$.

For any function $f:\Omega\to\mathbb{R}$, we shall denote the set of maximum points as

$$\operatorname{argmax}_\Omega(f) = \big\{x\in\Omega : f(x) = \max_\Omega f\big\} \tag{2.4}$$

If the function does not have an interior local maxima in Ω, then $\operatorname{argmax}_\Omega(f)$ is empty. Also, it is clear that if $\Theta:\Omega\to\Omega'$ is invertable, $x\in\operatorname{argmax}_\Omega(f)$ if and only if $\Theta(x)\in\operatorname{argmax}_{\Omega'}(f\circ\Theta^{-1})$. Furthermore, for continuous functions f_n, f, if $f_n\to f$ uniformly as $n\to\infty$ and $x_n\to x\in\Omega$ for $x_n\in\operatorname{argmax}_\Omega(f_n)$, then $x\in\operatorname{argmax}_\Omega(f)$. Also, $\operatorname{argmax}_\Omega(f+c) = \operatorname{argmax}_\Omega(f)$ if c is constant.

Our starting point is the following theorem due to Bardi-Goffi [2], who establish the strong maximum principle for viscosity solutions to fully non-linear sub-elliptic equations determined by Hörmander vector fields.

Theorem 2.1 (Strong Maximum Principle)**.** *Given smooth vector fields $X_1,\dots,X_m$ satisfying Hörmander's condition (1.1), if a function $G:\bar\Omega\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{m\times m}\to\mathbb{R}$ satisfies the following:*

(1) G is lower semicontinuous and for all $r\le s$ and symmetric matrices $Y\le X$,

$$G(x,r,\xi,X)\le G(x,s,\xi,Y);$$

(2) there exists $\phi:(0,1]\to(0,\infty)$ such that for all $\lambda\in(0,1], x\in\Omega, r\in[-1,0], \xi\in\mathbb{R}^m\setminus\{0\}$ and symmetric $X\in\mathbb{R}^{m\times m}$, we have

$$G(x,\lambda r,\lambda\xi,\lambda X)\ge\phi(\lambda)G(x,r,\xi,X);$$

(3) for all $x\in\Omega, \xi\in\mathbb{R}^m\setminus\{0\}, X\in\mathbb{R}^{m\times m}$, the following ellipticity condition holds,

$$\sup_{\gamma>0} G\left(x,0,\xi,X-\gamma\xi\otimes\xi\right) > 0;$$

then, any viscosity sub-solution (resp. super-solution) of the equation $G(x,u,\mathfrak{X}u,\mathfrak{X}\mathfrak{X}^\star u) = 0$ that attains a non-negative (resp. non-positive) maximum (resp. minimum) in Ω, is constant.

Using Theorem 2.1 to our special case leads to the following.

Corollary 2.2. *Given the equation* (1.2) *with* $a : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ *satisfying* (1.3) *and* (1.4)*, any viscosity sub-solution (resp. super-solution) that attains a non-negative (resp. non-positive) maximum (resp. minimum) in* Ω*, is constant.*

Proof. It is easy to see that $G(x,r,\xi,X) = -\operatorname{Tr}\big(a(\xi)X\big)$ satisfies the hypotheses of Theorem 2.1. Indeed, since $a(\xi)$ is symmetric and positive definite, we have $\operatorname{Tr}\big(a(\xi)Z\big) \geq 0$ for all $Z \geq 0$ which implies (1). The homogeneity condition (1.4) leads to (2) and the ellipticity condition (1.3) leads to (3) because we have that

$$G\left(x,0,\xi,X-\gamma\xi\otimes\xi\right) \;=\; \gamma\langle a(\xi)\xi,\xi\rangle - \operatorname{Tr}\big(a(\xi)X\big) > 0,$$

for any $\gamma > \operatorname{Tr}\big(a(\xi)X\big)/\langle a(\xi)\xi,\xi\rangle$. This completes the proof. □

Note that we use the result of Bardi-Goffi for smooth sub/super solutions, so we refer to their article [2] for the definitions and properties of viscosity solutions in the sub-elliptic framework, which we will not use in this manuscript.

Finally, we provide some preliminaries and necessary properties of Carnot Groups. The Lie group $\mathbb{G} = (\mathbb{R}^n, *)$ is connected and simply connected, with the Lie algebra of left-invariant vector fields $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ where $\mathfrak{g}_1 = \operatorname{span}\{X_1,\ldots,X_m\}$ so that from (1.1), $\mathtt{Lie}(\mathfrak{g}_1) = \mathfrak{g}$ and

$$[\,\mathfrak{g}_1,\mathfrak{g}_j\,] = \mathfrak{g}_{j+1} \quad \forall\, j\in\{1,\ldots,r-1\}, \quad \text{and} \quad [\,\mathfrak{g}_1,\mathfrak{g}_r\,] = \{0\}, \tag{2.5}$$

when $\mathbb{G}$ is nilpotent of step r. The group admits a family $\{\delta_\lambda\}_{\lambda>0}$ of automorphisms called dilations that induce a homogeneous structure. The basis $\{X_i\}$'s of $\mathfrak{g}_1$ can be extended to the basis $\{X_{i,j}\}$ of $\mathfrak{g}$ and the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ being a global diffeomorphism, we have the exponential coordinates $x = \exp\big(\sum_{j=1}^r \sum_{i=1}^{m_j} x_{i,j}X_{i,j}\big)$ where $m_j = \dim(\mathfrak{g}_j)$ and in these coordinates, $(x,y) \mapsto x*y$ is a polynomial. Thus, $X_iu(x) = \lim_{t\to 0}\frac{1}{t}\big(u(x*\exp(tX_i)) - u(x)\big)$. The homogeneous norm

$$\|x\| = \left(\sum_{j=1}^{r}\left(\sum_{i=1}^{m_j}|x_{i,j}|^2\right)^{\frac{r!}{j}}\right)^{\frac{1}{2r!}}, \tag{2.6}$$

gives rise to left-invariant metric $d(x,y) = \|y^{-1}*x\|$ satisfying $d(z*x,z*y) = d(x,y)$ and $d(\delta_\lambda x,\delta_\lambda y) = \lambda d(x,y)$ for all $x,y,z\in\mathbb{G}$. We shall denote the distance function dist as

$$\operatorname{dist}(x,E) = \inf\{d(x,y) : y\in E\}$$

for any $x\in\mathbb{G}$ and $E\subset\mathbb{G}$. Note that, up to isomorphisms of such groups, we can regard that in the exponential coordinates, $x^{-1} = -x$, see [7]. Furthermore, if $\|x\|,\|y\| < \nu$ for some $\nu > 0$, there exists a constant $c = c(\mathbb{G},\nu) > 0$ such that the pseudo-triangle inequality

$$\|x*y\| \leq c\left(\|x\| + \|y\|\right) \tag{2.7}$$

and the inequality

$$\|x^{-1}*y*x\| \leq c\,\|y\|^{\frac{1}{r}}, \tag{2.8}$$

hold for all $x,y\in\mathbb{G}$, see [7] and [12]. Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ be the projection corresponding to $\mathfrak{g}\to\mathfrak{g}_j$ and $\mathfrak{X}_j = (X_{1,j},\ldots,X_{m_j,j})$ be the gradient of $\mathfrak{g}_j$. We extend the notation of inner product $\langle\cdot,\cdot\rangle$ to that in any $\mathbb{R}^{m_j}$'s. For any $u\in C^1(\Omega)$, we have the following Taylor's formula

$$u(x) = u(x_0) + \big\langle \mathfrak{X}u(x_0), \pi_1({x_0}^{-1}*x)\big\rangle + o\big(\|{x_0}^{-1}*x\|\big). \tag{2.9}$$

Note that setting $h = {x_0}^{-1}*x$ we have

$$u_h(x_0) = u(x_0*h) = u(x_0) + \big\langle \mathfrak{X}u(x_0), \pi_1(h)\big\rangle + o\big(\|h\|\big). \tag{2.10}$$

Thus, the Taylor series with respect to left invariant vector fields give expansions for right translations.

The (bi-invariant) Haar measure of $\mathbb{G}$ is the Lebesgue measure of $\mathbb{R}^n$, denoted by $|\cdot|$, and for any measurable $E \subseteq \mathbb{R}^n$, we have $|\delta_\lambda(E)| = \lambda^Q |E|$ where $Q = \sum_{j=1}^r j m_j$ is the homogeneous dimension of $\mathbb{G}$, which is also the Hausdorff dimension with respect to the metrics d. We refer to [7] for further details on the structure and properties of Carnot groups.

3. Comparison principle

In this section, we prove the weak comparison principle for smooth super and sub-solutions of the equation, i.e. if $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ in $\partial\Omega$, then $u \leq v$ everywhere in Ω. We extend the approach of Barles and Busca [5], who studied the Euclidean case. This is achieved at increasing levels of generality which directs the method of proving the theorem and also reflects upon the difficulties arising from the degeneracy of the equation (1.2).

We begin with the following preliminary lemma.

Lemma 3.1. *If there exists $u, v \in C^2(\Omega)$ such that $\mathcal{L}u < 0 \leq \mathcal{L}v$ (resp. $\mathcal{L}u \leq 0 < \mathcal{L}v$) in Ω and $u \leq v$ in $\partial\Omega$, then we have $u \leq v$ in Ω.*

Proof. We proceed by contradiction. Assume the contrary, i.e. there exists $x \in \Omega$ such that $u(x) > v(x)$. Hence, there exists at least one $x_0 \in \bar{\Omega}$ such that

$$u(x_0) - v(x_0) = \max_{x \in \Omega} \{u(x) - v(x)\} > 0.$$

Since $u - v \leq 0$ on $\partial\Omega$ from assumption, hence $x_0 \in \Omega$ and the interior maximality at x_0 implies $\nabla u(x_0) = \nabla v(x_0)$ and $D^2 u(x_0) \leq D^2 v(x_0)$. This, together with (2.2) and (2.3), yields

$$\mathfrak{X}u(x_0) = \mathfrak{X}v(x_0) =: \xi_0 \quad \text{and} \quad \mathfrak{X}\mathfrak{X}^\star u(x_0) \leq \mathfrak{X}\mathfrak{X}^\star v(x_0).$$

Using the above together with the equation, we have the following,

$$\begin{aligned} 0 &\leq \operatorname{Tr}\Big(a(\xi_0)(\mathfrak{X}\mathfrak{X}^\star v(x_0) - \mathfrak{X}\mathfrak{X}^\star u(x_0)\Big) \\ &= \operatorname{Tr}\big(a(\mathfrak{X}v(x_0))\mathfrak{X}\mathfrak{X}^\star v(x_0)\big) - \operatorname{Tr}\big(a(\mathfrak{X}u(x_0))\mathfrak{X}\mathfrak{X}^\star u(x_0)\big) \\ &= -\mathcal{L}v(x_0) + \mathcal{L}u(x_0) < 0. \end{aligned}$$

The latter strict inequality of the above leads to a contradiction and the proof is complete. $\square$

The next goal is to relax the assumption $\mathcal{L}u < 0 \leq \mathcal{L}v$ or $\mathcal{L}u \leq 0 < \mathcal{L}v$ of Lemma 3.1 to $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω. This is difficult due to the degeneracy of the equation. Given a sub-solution u, the strategy is to construct small perturbations $u_\lambda = h_\lambda(u)$ for $h_\lambda \in C^2(\mathbb{R})$ and $\lambda > 0$ small enough, so that u_λ are strict sub-solutions and satisfy the assumptions of Lemma 3.1.

The construction of such sub-solutions (resp. super-solutions) can be done using ellipticity (1.3) and the homogeneity condition (1.4). We have the following technical lemma.

Lemma 3.2. *Let $\beta \in \mathbb{R}$ be as in (1.4). Given any $h \in C^2(\mathbb{R})$ and $w \in C^2(\Omega)$, we have*

$$\mathcal{L}(h(w)) = -|h'(w)|^\beta \Big[h''(w)\mathcal{E}(\mathfrak{X}w) - h'(w)\mathcal{L}w\Big]. \tag{3.1}$$

Proof. For any $h \in C^2(\mathbb{R})$ and $w \in C^2(\Omega)$, we note that $\mathfrak{X}(h(w)) = h'(w)\mathfrak{X}w$ and

$$\mathfrak{X}\mathfrak{X}(h(w)) = h'(w)\mathfrak{X}\mathfrak{X}w + h''(w)\mathfrak{X}w \otimes \mathfrak{X}w.$$

Hence, using the above together with (1.4), we have that

$$\begin{aligned} \mathcal{L}(h(w)) &= -\operatorname{Tr}\big(a(\mathfrak{X}(h(w)))\mathfrak{X}\mathfrak{X}(h(w))\big) \\ &= -h'(w)\operatorname{Tr}\big(a(h'(w)\mathfrak{X}w)\mathfrak{X}\mathfrak{X}w\big) - h''(w)\big\langle a(h'(w)\mathfrak{X}w)\mathfrak{X}w, \mathfrak{X}w\big\rangle \\ &= -|h'(w)|^\beta\Big[h''(w)\big\langle a(\mathfrak{X}w)\mathfrak{X}w, \mathfrak{X}w\big\rangle + h'(w)\operatorname{Tr}\big(a(\mathfrak{X}w)\mathfrak{X}\mathfrak{X}^\star w\big)\Big], \end{aligned}$$

for $\beta \in \mathbb{R}$ as in (1.4). The proof is finished. $\square$

Using Lemma 3.2 together with the ellipticity (1.3), we show the following.

Lemma 3.3. *If there exists $u, v \in C^2(\Omega)$ such that $\mathcal{L}u \le 0 \le \mathcal{L}v$ in Ω and $u \le v$ in $\partial\Omega$, and, in addition, if $\mathfrak{X}u$ (resp. $\mathfrak{X}v$) does not vanish at all maximal points of $u - v$, then $u \le v$ in Ω.*

Proof. As before, we shall assume the contrary and establish a contradiction. Without loss of generality, we can regard $u - v \le -\tau < 0$ in $\partial\Omega$ for any $\tau > 0$ arbitrarily small. The arbitrarity of τ would conclude the proof for every $u \le v$ on $\partial\Omega$.

The contrary hypothesis implies $u(x) > v(x)$ for some $x \in \Omega$ and since $u < v$ in $\partial\Omega$, hence maximal points of $u - v$ are in the interior. Thus, $\text{argmax}_\Omega(u - v) \ne \emptyset$ and we have, for any $y \in \text{argmax}_\Omega(u - v)$,

$$u(y) - v(y) = \max_{x\in\Omega} \{u(x) - v(x)\} =: M_0 > 0,$$

and we consider the given condition $\mathfrak{X}u(y) \ne 0$ for all $y \in \text{argmax}_\Omega(u - v)$. Now, let us take $u_\lambda = h_\lambda(u)$ for $\lambda > 0$, defined by

$$h_\lambda(u) = u + \lambda(u - u_0)^2 \quad \text{with} \quad u_0 = \inf_\Omega u,$$

(in fact, any $h_\lambda \in C^2(\mathbb{R})$ with $h'_\lambda, h''_\lambda > 0$ and $h_\lambda \to \text{id}$ as $\lambda \to 0^+$ will do). Thus, we have

$$\|u_\lambda - u\|_{L^\infty} \le 4\lambda\|u\|^2_{L^\infty}, \tag{3.2}$$

for any $\lambda > 0$. For a sequence $x_\lambda \in \text{argmax}_\Omega(u_\lambda - v)$ such that $x_\lambda \to x_0$ up to possible subsequence, as $\lambda \to 0^+$, we have $x_0 \in \text{argmax}_\Omega(u - v)$. Since $\mathfrak{X}u(x_0) \ne 0$, therefore $|\mathfrak{X}u(x_0)| \ge \theta$ for some $\theta > 0$ which implies $\mathfrak{X}u_\lambda(x_0) = h'_\lambda(u)\mathfrak{X}u(x_0) = (1 + 2\lambda(u - u_0))\mathfrak{X}u(x_0) \ne 0$ with $|\mathfrak{X}u_\lambda(x_0)| \ge \theta$. Note that, using (3.2),

$$u_\lambda(x_\lambda) - v(x_\lambda) = \max_{x\in\Omega} \{u_\lambda(x) - v(x)\} > 0,$$

whenever $0 < \lambda < M_0/4\|u\|^2_{L^\infty}$. Furthermore, note that $u - v \le -\tau < 0$ in $\partial\Omega$ along with (3.2) implies $u_\lambda \le v$ in $\partial\Omega$ for any $0 < \lambda \le \tau/4\|u\|^2_{L^\infty}$ leading to $x_\lambda \in \Omega$. The interior maximality at x_λ implies $\nabla u_\lambda(x_\lambda) = \nabla v(x_\lambda)$ and $D^2u_\lambda(x_\lambda) \le D^2v(x_\lambda)$, which together with (2.2) and (2.3), leads to

$$\mathfrak{X}u_\lambda(x_\lambda) = \mathfrak{X}v(x_\lambda) =: \xi_\lambda \quad \text{and} \quad \mathfrak{X}\mathfrak{X}^\star u_\lambda(x_\lambda) \le \mathfrak{X}\mathfrak{X}^\star v(x_\lambda). \tag{3.3}$$

Since $x_\lambda \to x_0$ as $\lambda \to 0^+$, there exists $\lambda_0 = \lambda_0\big(n, \theta, \|u\|_{L^\infty} + \|v\|_{L^\infty}, \text{diam}(\Omega)\big) > 0$, such that whenever $0 < \lambda < \lambda_0$ we have

$$d(x_\lambda, x_0) < \min\big\{\, \text{dist}(x_0, \partial\Omega), \theta/2\|\mathfrak{X}\mathfrak{X}u\|_{L^\infty}\big\},$$

hence $\mathfrak{X}u(x_\lambda) \ne 0$ with $|\mathfrak{X}u(x_\lambda)| \ge \theta/2$. These conditions respectively imply that $x_\lambda \in \Omega$ and $\mathcal{E}(\mathfrak{X}u(x_\lambda)) > 0$ from (1.3). Hence, if $\mathcal{L}u \le 0 \le \mathcal{L}v$ holds in Ω as given, then using (3.1) we obtain

$$\mathcal{L}u_\lambda(x_\lambda) = -|h'_\lambda(u)|^\beta\Big[h''_\lambda(u)\mathcal{E}(\mathfrak{X}u(x_\lambda)) - h'_\lambda(u)\mathcal{L}u(x_\lambda)\Big] < 0; \tag{3.4}$$

therefore, using (3.3) and (3.4) together, we obtain

$$\begin{aligned} 0 &\le \text{Tr}\,\Big(a(\xi_\lambda)(\mathfrak{X}\mathfrak{X}^\star v(x_\lambda) - \mathfrak{X}\mathfrak{X}^\star u_\lambda(x_\lambda)\Big) \\ &= \text{Tr}\,\big(a(\mathfrak{X}v(x_\lambda))\mathfrak{X}\mathfrak{X}^\star v(x_\lambda)\big) - \text{Tr}\,\big(a(\mathfrak{X}u_\lambda(x_\lambda))\mathfrak{X}\mathfrak{X}^\star u_\lambda(x_\lambda)\big) \\ &= -\mathcal{L}v(x_\lambda) + \mathcal{L}u_\lambda(x_\lambda) < 0, \end{aligned}$$

which, as earlier, leads to a contradiction.

In the case of the given condition being non-vanishing of $\mathfrak{X}v$, we can obtain a similar contradiction taking $v_\lambda = h_\lambda(v)$ with $h_\lambda(v) = v - \lambda(v - v_0)^2$ with $v_0 = \inf_\Omega v$ (or any $h_\lambda \in C^2(\mathbb{R})$ with $h'_\lambda > 0 > h''_\lambda$ and $h_\lambda \to \text{id}$ as $\lambda \to 0^+$) and using (3.1). The proof is finished. □

To remove the additional assumption of the non-vanishing gradient at maximal points of Lemma 3.3, we need to investigate how the maxima propagate under point-wise perturbation.

***Proof of Theorem** 1.1.* As before, we shall assume the contrary i.e. $\max_\Omega(u-v)>0$ and since $u\le v$ in $\partial\Omega$, the maxima is attained in the interior. Thus, we have

$$u(x_0)-v(x_0)=\max_{x\in\Omega}\{u(x)-v(x)\}=M_0>0,$$

for an interior point $x_0\in\Omega$. For any $\delta>0$, let $\Omega_\delta:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\delta\}$. Recalling the left-invariance of the metric, note that for any $\delta>0$ and $h\in\mathbb{R}^n$ with $\|h\|<\delta$, we have $d(x*h,x)=\|h\|<\delta<\operatorname{dist}(x,\partial\Omega)$ for any $x\in\Omega_\delta$. Hence, $x*h\in\Omega$ for any $x\in\Omega_\delta$ and $\|h\|<\delta$. Therefore, for any $\delta>0$, let us denote $M_\delta:B_\delta(0)\to\mathbb{R}$ as

$$M_\delta(h)=\max_{x\in\Omega_\delta}\{u(x*h)-v(x)\}\quad\text{and}\quad\mathcal{A}_\delta(h)=\{x\in\Omega_\delta:u(x*h)-v(x)=M_\delta(h)\},\tag{3.5}$$

for any $h\in\mathbb{R}^n$ with $\|h\|<\delta$. Thus, $M_0(0)=M_0$ and $x_0\in\mathcal{A}_0(0)$ and moreover, for every $0<\delta<\operatorname{dist}(x_0,\partial\Omega)$ we have $M_\delta(0)=M_0>0$ and $\mathcal{A}_\delta(0)\neq\emptyset$. This further implies the maxima is in the interior in Ω_δ and therefore, for all $0<\delta'<\delta$ since $\Omega_\delta\subset\Omega_{\delta'}$ we have $M_\delta(0)=M_{\delta'}(0)$.

In the following, we shall denote the right translations $u_h,v_l:\Omega_\delta\to\mathbb{R}$ by $u_h(x):=u(x*h)$ and $v_l(x)=v(x*l)$ for $h,l\in\mathbb{R}^n$ with $\|h\|,\|l\|<\delta$; then, similarly as above, let us denote

$$M_\delta(h,l)=\max_{x\in\Omega_\delta}\{u_h(x)-v_l(x)\}\quad\text{and}\quad\mathcal{A}_\delta(h,l)=\{x\in\Omega_\delta:u_h(x)-v_l(x)=M_\delta(h,l)\},\tag{3.6}$$

and note that (3.5) corresponds to $l=0$. Also note that, if $\mathcal{A}_\delta(0,l)\neq\emptyset$ then the maxima is in the interior in Ω_δ and therefore, $M_\delta(0,l)=M_{\delta'}(0,l)$ for all $0<\delta'<\delta$, similarly as above. Note that $h\mapsto M_\delta(h,l)$ is a Lipschitz function since for $x\in\mathcal{A}_\delta(h,l)$ and $x'\in\mathcal{A}_\delta(h',l)$, we have

$$\begin{aligned}M_\delta(h,l)-M_\delta(h',l)&=u(x*h)-v(x*l)-u(x'*h')+v(x'*l)\\&\le u(x*h)-v(x*l)-u(x*h')+v(x*l)\\&=u(x*h)-u(x*h')\le d(h,h')\|\mathfrak{X}u\|_{L^\infty},\end{aligned}\tag{3.7}$$

where we have use the maximality at x' in the second inequality. A symmetric inequality using the maximality at x gives the bound

$$|M_\delta(h,l)-M_\delta(h',l)|\le d(h,h')\|\mathfrak{X}u\|_{L^\infty}.\tag{3.8}$$

Similarly, $l\mapsto M_\delta(h,l)$ is also a Lipschitz function and by arguing similarly as (3.7) above using maximality in $\mathcal{A}_\delta(h,l)$ and $\mathcal{A}_\delta(h,l')$ we can obtain

$$|M_\delta(h,l)-M_\delta(h,l')|\le d(l,l'))\|\mathfrak{X}v\|_{L^\infty}.\tag{3.9}$$

It is noteworthy that in (3.5) and (3.6), M_δ may be subject to addition by constant, with respect to possible relabelling $u\mapsto\tilde u=u+(\text{const})$ and $v\mapsto\tilde v=v+(\text{const})$. Nevertheless, all properties and arguments remain unchanged as long as $\tilde u\le\tilde v$ on $\partial\Omega$ holds, since the gradient and maximal set are invariant of such relabeling, i.e. $\mathfrak{X}\tilde u=\mathfrak{X}u$ and $\operatorname{argmax}(\tilde u_h-\tilde v_l)=\operatorname{argmax}(u_h-v_l)$.

Now, we divide the rest of the proof into two alternative cases, based on the vanishing behavior of the gradient. This is an adaptation of that of Barles-Busca [5], see also [1]. The goal is to establish contradiction in both cases. We have chosen the right translation in (3.5) and (3.6) of the above because that is what appears in the Taylor expansion of u in (2.10).

Case 1: There exists $0<\delta_0\le\frac14\min\{\operatorname{dist}(x_0,\partial\Omega),M_0/\|\mathfrak{X}v\|_{L^\infty}\}$ and $l_0\in\mathbb{R}^n$, $\|l_0\|\le\delta_0$, such that for all $h\in\mathbb{R}^n$ with $\|h\|<\delta_0$, there exists points $x_h\in\mathcal{A}_{\delta_0}(h,l_0)$ such that we have $\mathfrak{X}u(x_h*h)=0$.

Since $\delta_0<M_0/2\|\mathfrak{X}v\|_{L^\infty}$, from (3.9) we have

$$M_{\delta_0}(0,l_0)\ge M_0-\|l_0\|\|\mathfrak{X}v\|_{L^\infty}\ge M_0/2>0.\tag{3.10}$$

For Case 1, using maximality with $x_h \in \mathcal{A}_{\delta_0}(h, l_0), x_{h'} \in \mathcal{A}_{\delta_0}(h', l_0)$, and (2.9), we note that

$$\begin{aligned} u(x_h * h) - v(x_h * l_0) &\geq u(x_{h'} * h) - v(x_{h'} * l_0) \\ &= u(x_{h'} * h') - v(x_{h'} * l_0) + \big\langle \mathfrak{X}u(x_{h'} * h'), \pi_1(h'^{-1} * h)\big\rangle + o\big(\|h'^{-1} * h\|\big) \\ &= u(x_{h'} * h') - v(x_{h'} * l_0) + o(d(h, h')). \end{aligned}$$

From (3.6) and the above, we have $M_{\delta_0}(h, l_0) \geq M_{\delta_0}(h', l_0) + o(d(h, h'))$. Since this inequality is symmetric with respect to h and h', we conclude that at points of differentiability of the function $h \mapsto M_{\delta_0}(h, l_0)$, we have $\mathfrak{X}M_{\delta_0}(h, l_0) = 0$. Recalling (3.8), since $h \mapsto M_{\delta_0}(h, l_0)$ is Lipschitz, by Rademacher's theorem on Carnot groups [13], it is differentiable at a.e. $\|h\| \leq \delta_0$. Therefore, we have $\mathfrak{X}M_{\delta_0}(h, l_0) = 0$ for a.e. $h \in B_\delta(0)$. It follows then that the Lipschitz constant of $h \mapsto M_{\delta_0}(h, l_0)$ is zero, so that the function $h \mapsto M_{\delta_0}(h, l_0)$ is constant in $B_{\delta_0}(0)$ (see Proposition 4.8 in [10]). Thus, for any $\|h\| < \delta_0$, we have

$$M_{\delta_0}(h, l_0) = M_{\delta_0}(0, l_0).$$

Hence, for any $\tilde{x}_0 \in \mathcal{A}_\delta(0, l_0)$ and $\|h\| < \delta < \delta_0 < \operatorname{dist}(x_0, \partial\Omega)$, using the above with (3.6) and interior maximality at $\tilde{x}_0$, we have

$$\begin{aligned} u(\tilde{x}_0) - v(\tilde{x}_0 * l_0) = M_\delta(0, l_0) = M_{\delta_0}(0, l_0) &= M_{\delta_0}(h, l_0) \\ &= u(x_h * h) - v(x_h * l_0) \geq u(\tilde{x}_0 * h) - v(\tilde{x}_0 * l_0), \end{aligned}$$

leading to $u(\tilde{x}_0) \geq u(\tilde{x}_0 * h)$. Thus, we have a sub-solution u with a local maximum at $\tilde{x}_0 \in \Omega$, which can be converted to a non-negative maximum by adding a large enough positive constant to u. From Corollary 2.2, $u(x) = u(\tilde{x}_0)$ for all $x \in B_\delta(\tilde{x}_0)$. Furthermore, for all $\|h'\| < \delta$, the maximality at $\tilde{x}_0 \in \mathcal{A}_\delta(0, l_0)$ implies

$$u(\tilde{x}_0) - v(\tilde{x}_0 * l_0) \geq u(\tilde{x}_0 * h') - v(\tilde{x}_0 * h' * l_0) = u(\tilde{x}_0) - v(\tilde{x}_0 * h' * l_0),$$

leading to $v(\tilde{x}_0 * h' * l_0) \geq v(\tilde{x}_0 * l_0)$, which also means that the super-solution v has a local minimum at $\tilde{x}_0 * l_0$. By adding a negative constant and converting the super-solution v to have a non-positive minimum at $\tilde{x}_0 * l_0 \in \Omega$, we can use Corollary 2.2 again to conclude $v(x) = v(\tilde{x}_0 * l_0)$ in a neighborhood of $\tilde{x}_0 * l_0 \in \Omega$. Hence, $\{x \in \Omega : u(x) - v(x) = u(\tilde{x}_0) - v(\tilde{x}_0 * l_0)\}$ being both open and closed and Ω being connected, it is the whole of Ω. Thus, for every $x \in \Omega$, we have $u(x) - v(x) = u(\tilde{x}_0) - v(\tilde{x}_0 * l_0) = M_{\delta_0}(0, l_0) > 0$ from (3.10), which contradicts $u \leq v$ in $\partial\Omega$.

Case 2: For any $0 < \delta < \frac{1}{4}\min\{\operatorname{dist}(x_0, \partial\Omega), M_0/\|\mathfrak{X}v\|_{L^\infty}\}$ and any $l \in \mathbb{R}^n$ with $\|l\| \leq \delta$, there exists $h_l \in \mathbb{R}^n$ with $\|h_l\| < \delta$, such that for all $x \in \mathcal{A}_\delta(h_l, l)$ we have $\mathfrak{X}u(x * h_l) \neq 0$.

Here, for $0 < \delta < \frac{1}{4}\min\{\operatorname{dist}(x_0, \partial\Omega), M_0/\|\mathfrak{X}v\|_{L^\infty}\}$, in the following arguments we shall encounter further several upper bounds of δ, all of which are to be considered respectively as the proof proceeds. First, we show the following.

Claim: There exists $h \in \mathbb{R}^n$ with $\|h\| < \delta$, such that for all $x \in \mathcal{A}_\delta(h, h)$ we have $\mathfrak{X}u(x*h) \neq 0$.

Notice that the set $\{x*h_l : x \in \mathcal{A}_\delta(h_l, l)\}$ is contained in a compact set $K_\delta \subset \Omega_{\delta/2}$ independent of l. Therefore, from Case 2, we can regard that

$$|\mathfrak{X}u(x * h_l)| \geq \theta_\delta > 0, \quad \forall\, x \in \mathcal{A}_\delta(h_l, l). \tag{3.11}$$

Hence, let us take any $l_0 \in B_\delta(0)$ and use the hypothesis of Case 2 repeatedly to define the sequence $l_{j+1} = h_{l_j}$ for every $j \in \mathbb{N} \cup \{0\}$. Since $\{l_j\}$ is bounded, up to a sub-sequence we have $l_j \to h$ for some $h \in B_\delta(0)$ and hence $d(l_j, h), d(l_{j+1}, l_j) \to 0^+$ as $j \to \infty$. We show that h satisfies the claim. As $\|h\| < \delta$, it is not hard to see that $M_\delta(h, h) > 0$ from (3.8) and (3.9), when $\delta < M_0/4\|\mathfrak{X}u\|_{L^\infty}$. Furthermore, as in the proof of Lemma 3.3, we can assume without loss of generality, that

$$u(z) - v(z) \le -\tau < 0, \quad \forall\, z \in \partial\Omega, \tag{3.12}$$

for any arbitrarily small $\tau > 0$. Now, for all $x \in \partial\Omega_\delta$, notice that

$$\operatorname{dist}(x * h, \partial\Omega) \le \operatorname{dist}(x, \partial\Omega) + d(x * h, x) = \delta + \|h\| < 2\delta$$

for $\|h\| < \delta$ which leads us to the following for all $x \in \partial\Omega_\delta$,

$$\begin{aligned} u(x * h) - v(x * h) &\le -\tau + \operatorname{dist}(x * h, \partial\Omega)\big(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}\big) \\ &\le -\tau + 2\delta\big(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}\big) \le 0, \end{aligned}$$

if $\delta < \tau/2(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty})$. Thus, we have $u_h \le v_h$ at $\partial\Omega_\delta$, which implies the maxima at $M_\delta(h, h) > 0$ is attained in the interior and $\mathcal{A}_\delta(h, h) \neq \emptyset$. Hence, for any $x \in \mathcal{A}_\delta(h, h)$, we have $u(x * h) - v(x * h) = \max_{\Omega'}(u_h - v_h)$ for any $\Omega' \subset\subset \Omega$ with $x \in \Omega'$. Since $l_j \to h$ as $j \to \infty$, we can also similarly conclude that $\mathcal{A}_\delta(l_j, l_j) \neq \emptyset$ and the maxima is interior for $j \ge j_0$ large enough. Therefore, for any $x \in \mathcal{A}_\delta(h, h)$ and a neighborhood $B \subset\subset \Omega$ with $x \in B$, let us denote $B_j := \{y * l_j * h^{-1} : y \in B\}$; then, note that for $j \ge j_0$ large enough, $d(l_j, h) < \frac{1}{2}\operatorname{dist}(x, \partial B)$ so that $x \in B_j$ and hence we have

$$\begin{aligned} M_\delta(l_j, l_j) &= \max_{y \in B}\{u(y * l_j) - v(y * l_j\} = \max_{z \in B_j}\{u(z * h) - v(z * h\} \\ &= \max_{B_j}(u_h - v_h) = u(x * h) - v(x * h) \\ &= u(x_j * l_j) - v(x_j * l_j) \end{aligned} \tag{3.13}$$

where $x_j := x * h * {l_j}^{-1}$. Recalling (3.6), $x_j \in \mathcal{A}_\delta(l_j, l_j)$ and $d(x_j, x) = d(l_j, h)$, also note that $x \in B_j$ and $x_j = x * h * {l_j}^{-1} \in B$. Now we produce similarly a maximal point in $\mathcal{A}_\delta(h_{l_j}, l_j)$ close to x in order to compare the gradients using (3.11) by suitably relabelling with

$$\tilde{u} = u + c_{1,j}, \qquad \tilde{v} = v + c_{2,j}, \tag{3.14}$$

where $c_{1,j} = M_\delta(l_{j+1}, l_j) - M_\delta(l_{j+1}, l_{j+1})$ and $c_{2,j} = v(x * h) - v(x * h * {l_{j+1}}^{-1} * l_j)$. It is clear from (3.9) and directly, that

$$|c_{1,j}|, |c_{2,j}| \le d(l_{j+1}, l_j)\|\mathfrak{X}v\|_{L^\infty},$$

from which, together with (3.12), we can conclude similarly as above that for $j \ge j_0$ large enough, $\tilde{u} < \tilde{v}$ on $\partial\Omega$. Using (3.13) with $x_{j+1} = x * h * {l_{j+1}}^{-1}$ for $j \ge j_0$ large enough, we obtain

$$\begin{aligned} M_\delta(l_{j+1}, l_j) &= M_\delta(l_{j+1}, l_{j+1}) + c_{1,j} = u(x_{j+1} * l_{j+1}) - v(x_{j+1} * l_{j+1}) + c_{1,j} \\ &= u(x_{j+1} * l_{j+1}) - v(x_{j+1} * l_j) + c_{1,j} - c_{2,j} \\ &= \tilde{u}(x_{j+1} * l_{j+1}) - \tilde{v}(x_{j+1} * l_j). \end{aligned} \tag{3.15}$$

Therefore, relabelling $u \mapsto \tilde{u}$ and $v \mapsto \tilde{v}$ as in (3.14), we can conclude from (3.15) that

$$x'_j := x_{j+1} = x * h * {l_{j+1}}^{-1} \in \mathcal{A}_\delta(l_{j+1}, l_j)$$

and $d(x'_j, x) = d(l_{j+1}, h)$. Therefore, for any $x \in \mathcal{A}_\delta(h, h)$ and $j \ge j_0$ large enough, we regard $d(l_{j+1}, h), d(l_j, h) \le R_\delta^r$ where $0 < R_\delta < \delta$ can be chosen as small as needed; as we established above, there exists $x'_j \in \mathcal{A}_\delta(h_{l_j}, l_j)$ such that $d(x'_j, x) = d(l_{j+1}, h) \le R_\delta^r$. Hence,

$$\begin{aligned} |\mathfrak{X}u(x * h) - \mathfrak{X}u(x'_j * h_{l_j})| &\le \|\mathfrak{X}\mathfrak{X}u\|_{L^\infty}\Big[d(x * h, x * h_{l_j}) + d(x * h_{l_j}, x'_j * h_{l_j})\Big] \\ &\le c\|\mathfrak{X}\mathfrak{X}u\|_{L^\infty}\Big[d(h, h_{l_j}) + d(x, x'_j)^{1/r}\Big] \le 2c\|\mathfrak{X}\mathfrak{X}u\|_{L^\infty} R_\delta, \end{aligned} \tag{3.16}$$

for all $j \geq j_0$ large enough, where we used triangle inequality, (2.8) and $R_\delta^r \leq R_\delta$ for $R_\delta < 1$. Now, since $x_j' \in \mathcal{A}_\delta(h_{l_j}, l_j)$, from (3.11), recall that

$$|\mathfrak{X}u(x_j' * h_{l_j})| \geq \theta_\delta > 0.$$

Therefore, we choose $R_\delta < \theta_\delta/4c\|\mathfrak{X}\mathfrak{X}u\|_{L^\infty}$ so that using the the above together with (3.16), we can conclude $\mathfrak{X}u(x * h) \neq 0$ with $|\mathfrak{X}u(x * h)| \geq \theta_\delta/2 > 0$ for any $x \in \mathcal{A}_\delta(h, h)$. Thus, we have proved the claim.

Note that the interior maximality of $u_h - v_h$ at $x \in \mathcal{A}_\delta(h, h)$ implies $\nabla u_h(x) = \nabla v_h(x)$ and $D^2 u_h(x) \leq D^2 v_h(x)$, which together with (2.2) and (2.3) yields

$$\mathfrak{X}u_h(x) = \mathfrak{X}v_h(x) \quad \text{and} \quad \mathfrak{X}\mathfrak{X}^\star u_h(x) \leq \mathfrak{X}\mathfrak{X}^\star v_h(x). \tag{3.17}$$

However, in order to make use of the left-invariance of the vector fields we need to establish a relation similar to (3.17) but in terms of the left translation. To this end, let us denote

$$\Omega^\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > c\delta^{1/r}\}$$

where $c = c(m, n, r, \operatorname{diam}(\Omega)) > 0$ such that using (2.8) we have for $\|h\| < \delta$ the inequality

$$d(h * x, x) = \|x^{-1} * h * x\| \leq c\|h\|^{1/r} < c\delta^{1/r} < \operatorname{dist}(x, \partial\Omega)$$

for any $x \in \Omega^\delta$. Thus, for any $x \in \Omega^\delta$ and $\|h\| < \delta$, we have $h * x \in \Omega$. Henceforth, let the left-translation $u^h, v^h : \Omega^\delta \to \mathbb{R}$ be defined by $u^h(x) := u(h * x), v^h(x) := v(h * x)$. Furthermore, we denote

$$\Omega(\delta) := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \max\{\delta, c\delta^{1/r}\}\big\} = \Omega_\delta \cap \Omega^\delta$$

so that the conjugation $\mathcal{C}_h : \Omega(\delta) \to \Omega(\delta)$ defined by $\mathcal{C}_h(x) = h^{-1} * x * h$, is well defined. Since $h^{-1} = -h$, we have $(\mathcal{C}_h)^{-1} = \mathcal{C}_{h^{-1}}$. Note also that $u_h = u^h \circ \mathcal{C}_h$ and $v_h = v^h \circ \mathcal{C}_h$.

For any $x \in \mathcal{A}_{\delta_0}(h, l)$ and for every

$$0 < \delta < \min\{\operatorname{dist}(x, \partial\Omega), (\operatorname{dist}(x, \partial\Omega)/c)^r, \delta_0\},$$

we have $x \in \Omega(\delta)$ and the maximum in (3.6) can be taken over $\Omega(\delta)$; in other words for $\|h\|, \|l\| < \delta$ we have $M_\delta(h, l) = \max_{\Omega(\delta)}(u_h - v_l) = u_h(x) - v_l(x)$ for $x \in \mathcal{A}_\delta(h, l)$. Hence, we may write $\mathcal{A}_\delta(h, l) = \operatorname{argmax}_{\Omega(\delta)}(u_h - v_l)$ as denoted in (2.4). Therefore, $x \in \mathcal{A}_\delta(h, h)$ if and only if $\bar{x} = \mathcal{C}_h(x) \in \operatorname{argmax}_{\Omega(\delta)}(u_h \circ \mathcal{C}_{h^{-1}} - v_h \circ \mathcal{C}_{h^{-1}}) = \operatorname{argmax}_{\Omega(\delta)}(u^h - v^h)$. Henceforth, let

$$\bar{\mathcal{A}}_\delta(h, h) = \{x \in \Omega(\delta) : u^h(x) - v^h(x) = \max_{\Omega(\delta)}(u^h - v^h)\} \tag{3.18}$$

with $\|h\| < \delta$ as in the claim, so that $\bar{x} \in \bar{\mathcal{A}}_\delta(h, h)$ iff $x = \mathcal{C}_{h^{-1}}(\bar{x}) \in \mathcal{A}_\delta(h, h)$.

We use (3.12) similarly as above. Notice that for $x \in \partial\Omega(\delta)$ and $\|h\| < \delta$, we have

$$\operatorname{dist}(x * h, \partial\Omega) \leq \operatorname{dist}(x, \partial\Omega) + d(x * h, x) < \delta + \max\{\delta, c\delta^{1/r}\}.$$

Therefore, together with (3.12) we obtain for all $x \in \partial\Omega(\delta)$,

$$\begin{aligned} u(x * h) - v(x * h) &\leq -\tau + \operatorname{dist}(x * h, \partial\Omega)\big(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}\big) \\ &\leq -\tau + (\delta + \max\{\delta, c\delta^{1/r}\})\big(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}\big) \leq 0, \end{aligned}$$

when $\delta < \min\{\tau/2(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}), (\tau/2c(\|\mathfrak{X}u\|_{L^\infty} + \|\mathfrak{X}v\|_{L^\infty}))^r\}$, which further imply

$$u^h(\bar{x}) = u^h(\mathcal{C}_h(x)) = u(x * h) \leq v(x * h) = v^h(\bar{x}), \qquad \text{for all } \bar{x} \in \partial\Omega(\delta). \tag{3.19}$$

Hence, at any $\bar{x} \in \bar{\mathcal{A}}_\delta(h, h)$, the interior maximality of $u^h - v^h$ implies $\nabla u^h(\bar{x}) = \nabla v^h(\bar{x})$ and $D^2 u^h(\bar{x}) \leq D^2 v^h(\bar{x})$ which together with (2.2) and (2.3) yields

$$\mathfrak{X}u^h(\bar{x}) = \mathfrak{X}v^h(\bar{x}) \quad \text{and} \quad \mathfrak{X}\mathfrak{X}^\star u^h(\bar{x}) \leq \mathfrak{X}\mathfrak{X}^\star v^h(\bar{x}). \tag{3.20}$$

Now, on u^h and v^h, we can use left-invariance of the vector fields that imply $X_j u^h(y) = X_j u(h*y)$ and $X_j v^h(y) = X_j v(h*y)$ for all $y \in \Omega(\delta)$. Therefore, from the above claim, we have

$$\mathfrak{X}u^h(\bar{x}) = \mathfrak{X}u(h*\bar{x}) = \mathfrak{X}u(x*h) \neq 0, \qquad \forall\ \bar{x} \in \bar{\mathcal{A}}_\delta(h,h), \tag{3.21}$$

and for any $\bar{y} \in \Omega(\delta)$, letting $y = \mathcal{C}_{h^{-1}}(\bar{y})$, the left-invariance implies

$$\mathcal{L}u^h(\bar{y}) = \mathcal{L}u(h*\bar{y}) = \mathcal{L}u(y*h) \leq 0 \leq \mathcal{L}v(y*h) = \mathcal{L}v(h*\bar{y}) = \mathcal{L}v^h(\bar{y}). \tag{3.22}$$

Thus, (3.22),(3.19) and (3.21) together satisfy the conditions of Lemma 3.3 in $\Omega(\delta)$ for u^h and v^h with $\|h\| < \delta$ as in the above claim for any $\delta > 0$ small enough as shown above. Therefore, we can invoke Lemma 3.3 to conclude $u^h \leq v^h$ everywhere in $\Omega(\delta)$ with $\|h\| < \delta$ as in the claim for any $\delta > 0$ small enough. Since, $u^h \to u, v^h \to v$ and the domain $\Omega(\delta) \to \Omega$ as $\delta \to 0^+$, this contradicts the contrary hypothesis.

Combining both cases, the proof is complete. □

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