A RELAXATION APPROACH TO THE MINIMISATION OF THE NEO-HOOKEAN ENERGY IN 3D

MARCO BARCHIESI, DUVAN HENAO, CARLOS MORA-CORRAL, AND RÉMY RODIAC

Abstract. Despite its high significance in nonlinear elasticity, the neo-Hookean energy is still not known to admit minimisers in some appropriate admissible class. Using ideas from relaxation theory, we propose a larger minimisation space and a modified functional that coincides with the neo-Hookean energy on the original space. This modified energy is the sum of the neo-Hookean energy and a term penalising the singularities of the inverse deformation. The new functional attains its minimum in the larger space, so the initial question of existence of minimisers of the neo-Hookean energy is thus transformed into a question of regularity of minimisers of this new energy.

1. Introduction

1.1. Overview of the problem. The neo-Hookean model, given its widespread use, is highly significant in nonlinear elasticity. In this model, minimisers of the neo-Hookean energy

$$E(\boldsymbol{u}) = \int_\Omega \left[|D\boldsymbol{u}|^2 + H(\det D\boldsymbol{u}) \right] \mathrm{d}\boldsymbol{x} \tag{1.1}$$

are sought in a space of orientation-preserving maps (i.e., with $\det D\boldsymbol{u} > 0$ a.e.) satisfying some injectivity conditions (e.g., $\boldsymbol{u}$ one-to-one a.e.) in order to avoid interpenetration of matter. Here $H : (0,\infty) \to [0,\infty)$ is a convex function such that

$$\lim_{t\to\infty} \frac{H(t)}{t} = \lim_{s\to 0} H(s) = \infty, \tag{1.2}$$

$\Omega \subset \mathbb{R}^3$ represents the reference configuration of an elastic body and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is the deformation map. Unfortunately, the coercivity of the neo-Hookean energy is not sufficient to apply the current theories in calculus of variations to deduce existence of a minimiser in an appropriate space. Indeed, the neo-Hookean energy is a borderline case of energies that admit minimisers, like

$$\int_\Omega [|D\boldsymbol{u}|^p + H(\det D\boldsymbol{u})] \,\mathrm{d}\boldsymbol{x}$$

with $p > 2$ or

$$\int_\Omega \left[|D\boldsymbol{u}|^2 + H(\det D\boldsymbol{u}) + \widetilde{H}(|\operatorname{cof} D\boldsymbol{u}|) \right] \mathrm{d}\boldsymbol{x},$$

where $\widetilde{H}$ is superlinear at infinity; cf., e.g., [3, 4, 29, 32, 31, 21, 22, 23, 24] and references therein. The difficulty one has to face in minimising the neo-Hookean energy is due to the lack of compactness of the minimisation space with respect to the H^1 convergence, as shown by an example of Conti & De Lellis [12]; see also [6, 13]. For some results on existence of minimisers of the energy (1.1) in the axisymmetric setting we refer to [25, 7], but we emphasise that the goal of this article is to consider the general 3D case.

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When one cannot prove the existence of minimisers via the direct method of calculus of variations, a common strategy consists in splitting the difficulty into two steps. The first step, called relaxation, aims at obtaining existence of minimisers of a modified energy in a bigger and less regular space, with the requirement that the modified energy coincides with the original one in the original space. The purpose of the second step is to prove regularity of one of the minimisers obtained in the previous step and to show that it actually belongs to the original smaller space. Our goal in this paper is to implement the first step for the minimisation of the neo-Hookean energy and to transform the existence problem into a regularity problem for a modified energy. The new energy we propose is motivated by our previous work [7] where the same problem was considered in the particular case of axisymmetric deformations.

Before entering into details, we note that the process of relaxation gives rise to natural spaces in calculus of variations. For instance, minimising sequences of $\|Du\|_{L^1(\Omega)}$ among $W^{1,1}$ functions with prescribed Dirichlet data are not compact, and a larger space more suitable to the problem is the space of functions of bounded variation (BV). Another example is the minimisation of the Dirichlet energy on the space $H^1_{\boldsymbol{b}}(\Omega;\mathbb{S}^2)\cap C^0(\overline{\Omega},\mathbb{S}^2)$ of continuous unit-valued H^1 maps with prescribed Dirichlet data $\boldsymbol{b}$ on $\partial\Omega$, $\Omega\subset\mathbb{R}^3$, a problem extensively studied beginning with the pioneering works [10, 20, 8]. Since $H^1_{\boldsymbol{b}}\cap C^0$ is not weakly compact, the relaxation leads to the minimisation of a modified energy functional in the larger space of unit-valued maps in $H^1_{\boldsymbol{b}}$ that satisfy the boundary condition but are not necessarily continuous.

1.2. **Setting and statement of the main result.** We now describe more precisely our minimisation setting. We work with a strong form of the Dirichlet boundary condition, namely, we choose $\widetilde{\Omega}$ a smooth bounded domain of $\mathbb{R}^3$ such that $\widetilde{\Omega}\subset\Omega$, and we require that deformations coincide with a bounded C^1 orientation-preserving diffeomorphism $\boldsymbol{b}:\Omega\to\mathbb{R}^3$ not only on $\partial\Omega$ but on the whole of $\Omega\setminus\widetilde{\Omega}$. We define

$$\Omega_{\boldsymbol{b}}:=\boldsymbol{b}(\Omega)\quad\text{ and }\quad\widetilde{\Omega}_b:=\boldsymbol{b}(\widetilde{\Omega}).$$

Since interpenetration of matter is physically unrealistic, we require the deformations to be one-to-one a.e. We recall that $\boldsymbol{u}:\Omega\to\mathbb{R}^3$ being one-to-one a.e. means that there exists a set N of zero Lebesgue measure such that $\boldsymbol{u}|_{\Omega\setminus N}$ is one-to-one. We also ask these maps to be orientation preserving, i.e., to satisfy $\det D\boldsymbol{u}>0$ a.e. Moreover, we will impose that maps in $\mathcal{A}$ satisfy the *divergence identities:*

$$\operatorname{Div}\big((\operatorname{adj}D\boldsymbol{u})\boldsymbol{g}\circ\boldsymbol{u}\big)=(\operatorname{div}\boldsymbol{g})\circ\boldsymbol{u}\,\det D\boldsymbol{u}\quad\forall\boldsymbol{g}\in C^1_c(\mathbb{R}^3,\mathbb{R}^3),\tag{1.3}$$

where Div is the distributional divergence in Ω. Maps satisfying the divergence identities (see [28, 33, 30, 32]) enjoy extra regularity, as shown in [5], and in fact do not present cavitation or create new surface (see [22]). All these requirements lead us to try to work with the minimisation space

$$\mathcal{A}:=\{\boldsymbol{u}\in H^1(\Omega,\mathbb{R}^3):\boldsymbol{u}=\boldsymbol{b}\text{ in }\Omega\setminus\widetilde{\Omega},\ \boldsymbol{u}\text{ is one-to-one a.e.},\ \det D\boldsymbol{u}>0\text{ a.e.},$$
$$\text{identity (1.3) holds and }E(\boldsymbol{u})<\infty\}.$$

Unfortunately, this space is not closed with respect to the H^1 weak convergence, and one has to face a problem of lack of compactness, as shown by Conti & De Lellis in their example [12, Theorem 6.1]. This non-compactness impedes the application of the direct method of calculus of variations. As mentioned in the introduction, our strategy is the following: we seek a larger space $\mathcal{B}$ that is *compact* for sequences with equibounded energy. On that space, we want a *lower semicontinuous* energy F coinciding with E on $\mathcal{A}$. By using the direct method of calculus of variations, one can then obtain that the energy F admits a minimiser $\boldsymbol{u}$ on $\mathcal{B}$. Then, the existence problem of a minimiser for E is reduced to showing that $\boldsymbol{u}$ belongs (hopefully) to $\mathcal{A}$.

Our choice for the family $\mathcal{B}$ and the energy F is driven by the following two facts. First, the geometric image of a map $\boldsymbol{u}$ in $\mathcal{A}$, defined in Definition 2.3 and which can be thought of as $\boldsymbol{u}(\Omega)$, can be shown to be equal to $\Omega_{\boldsymbol{b}}$ in a measure theoretic sense. We refer to [7, Proposition 4.11] for a proof in the axisymmetric setting. In the general case the proof is exactly as in [5, Theorem 4.1] but using the Brezis-Nirenberg degree, cf. [11, 12], instead of the Brouwer degree; the conclusion is that there exists an open set U with $\widetilde{\Omega} \Subset U \Subset \Omega$ such that

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U) = \boldsymbol{b}(U) \text{ a.e.},$$

hence $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) = \Omega_{\boldsymbol{b}}$. Second, the inverse of a map in $\mathcal{A}$ belongs to $W^{1,1}(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$, cf. [7, Proposition 4.12] in the axisymmetric case. In the general case, one can apply [5, Lemma 5.1] to show that condition INV holds; as before one uses the Brezis-Nirenberg degree. Then, with [24, Theorem 3.4] one concludes that $\boldsymbol{u}^{-1}$ is Sobolev, first in some open set V with $\widetilde{\Omega}_{\boldsymbol{b}} \Subset V \Subset \Omega_{\boldsymbol{b}}$ and then in $\Omega_{\boldsymbol{b}}$.

However, this last condition is not stable: as shown by Conti & De Lellis in their example, the weak H^1 limit of a sequence in $\mathcal{A}$ can have a limit with an inverse that is not in $W^{1,1}$ but only in BV. This motivates us to define

$$\begin{aligned}\mathcal{B} := \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3) : \boldsymbol{u} = \boldsymbol{b} \text{ in } \Omega \setminus \widetilde{\Omega},\ \boldsymbol{u} \text{ is one-to-one a.e.}, \det D\boldsymbol{u} > 0 \text{ a.e.},\\ \Omega_{\boldsymbol{b}} = \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \text{ a.e.}, \boldsymbol{u}^{-1} \in BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3), \text{ and } E(\boldsymbol{u}) < \infty\},\end{aligned}$$

where $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega)$ is the geometric image defined in Definition 2.3. As explained before, the inclusion $\mathcal{A} \subset \mathcal{B}$ holds. Then, we extend E on $\mathcal{B}$ by defining

$$F(\boldsymbol{u}) := E(\boldsymbol{u}) + 2\|D^s\boldsymbol{u}^{-1}\|, \tag{1.4}$$

for $\boldsymbol{u} \in \mathcal{B}$. Here $D^s\boldsymbol{u}^{-1}$ is the singular part of the distributional gradient of the inverse, $|D^s\boldsymbol{u}^{-1}|$ is the total variation of $D^s\boldsymbol{u}^{-1}$ (which is itself a positive Radon measure), and $\|D^s\boldsymbol{u}^{-1}\|$ is the norm of the measure $D^s\boldsymbol{u}^{-1}$, so that $\|D^s\boldsymbol{u}^{-1}\| = |D^s\boldsymbol{u}^{-1}|(\Omega_{\boldsymbol{b}})$.

The definition of F is inspired by our previous works [7] where we have proved that F admits a minimiser among axially symmetric maps belonging to $\mathcal{B}$. In the present paper we extend this result to maps without any symmetry. Another feature of the energy F is that $\|D^s\boldsymbol{u}^{-1}\|$ has an expression resembling the notion of minimal connections that was introduced by Brezis-Coron-Lieb in [10] and which also appears in the relaxed energy for harmonic maps; cf. [8]. We refer to [6, Theorem 1.3] for more on this expression.

Our main theorem is the compactness of the class $\mathcal{B}$ and the lower semicontinuity of the functional F with respect to the weak convergence in H^1. This provides the existence of minimisers for the energy F in $\mathcal{B}$.

Theorem 1.1. *Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\mathcal{B}$ such that $\{F(\boldsymbol{u}_j)\}_j$ is equibounded. Then there exists $\boldsymbol{u} \in \mathcal{B}$ such that, up to a subsequence, $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$ and*

$$\liminf_{j\to\infty} F(\boldsymbol{u}_j) \geq F(\boldsymbol{u}).$$

In particular, the energy F has a minimiser in $\mathcal{B}$.

We remark that, by definition of the relaxed energy, we have $F(\boldsymbol{u}) \leq E_{\mathrm{rel}}(\boldsymbol{u})$ for every $\boldsymbol{u}$ in the weak H^1 closure of maps in $\mathcal{A}$. Here the relaxed energy is defined abstractly by

$$E_{\mathrm{rel}}(\boldsymbol{u}) := \inf\{\liminf_{j\to\infty} E(\boldsymbol{u}_j) : \{\boldsymbol{u}_j\}_j \subset \mathcal{A} \text{ and } \boldsymbol{u}_j \rightharpoonup \boldsymbol{u} \text{ in } H^1(\Omega, \mathbb{R}^3)\}.$$

It is desirable that F coincides with the relaxation of E, in order to get, possibly, a negative result: if none of the minimisers of the relaxed energy belong to $\mathcal{A}$, then E has no minimisers in $\mathcal{A}$. It is important to mention that the factor 2 in formula (1.4) appearing in front of $\|D^s\boldsymbol{u}^{-1}\|$ is sharp, as shown in [6]: there exists a map $\boldsymbol{u}$ in $\mathcal{B} \setminus \mathcal{A}$ (the nasty one provided by Conti & De

Lellis) and a sequence $\{\boldsymbol{u}_j\}_j$ in $\mathcal{A}$ such that $\lim_{j\to\infty} E(\boldsymbol{u}_j) = F(\boldsymbol{u})$. However, we are not able to prove that F coincides with the relaxed energy $E_{\rm rel}$ at the moment.

A final remark is that we focus mainly on the Dirichlet part of the neo-Hookean energy, i.e., on $|D\boldsymbol{u}|^2$. But some recent results in [13] seem to indicate that if the convex function H satisfies stronger coercivity properties then the compactness of the minimisation space could be restored.

In the last part of the paper, we develop further the connection with harmonic map theory. Set $H^1_{\boldsymbol{b}}(\Omega,\mathbb{S}^2) := \{\boldsymbol{u} \in H^1(\Omega,\mathbb{S}^2) : \boldsymbol{u} = \boldsymbol{b} \text{ on } \partial\Omega\}$, for suitable Dirichlet data $\boldsymbol{b}$. In [20] Hardt & Lin showed that the minimum of $\int_\Omega |D\boldsymbol{u}|^2 \mathrm{d}\boldsymbol{x}$ on $H^1_{\boldsymbol{b}}$ can be strictly less than the infimum on $H^1_{\boldsymbol{b}} \cap C^0(\overline{\Omega};\mathbb{S}^2)$, and in [8, Corollaries 2 and 3] Bethuel, Brezis & Coron established that

$$\inf_{\boldsymbol{u}\in H^1_{\boldsymbol{b}}\cap C^0} \int_\Omega |D\boldsymbol{u}|^2 \mathrm{d}\boldsymbol{x} = \min_{\boldsymbol{u}\in H^1_{\boldsymbol{b}}} \left[\int_\Omega |D\boldsymbol{u}|^2 \mathrm{d}\boldsymbol{x} + 2\Big(\sup_{\substack{\phi\in C^1(\Omega)\\ \|\nabla\phi\|_\infty\leq 1}} \langle \mathrm{Div}\ \big((\mathrm{adj}\, D\boldsymbol{u})\boldsymbol{u}\big), \phi\rangle \Big) \right].$$

In the case of unit-valued maps $\boldsymbol{u}$ that are smooth except for a finite number of singularities, the expression involving Div $\big((\mathrm{adj}\, D\boldsymbol{u})\boldsymbol{u}\big)$ is the minimal connection length, introduced in [10], multiplied by $4\pi = \mathcal{H}^2(\mathbb{S}^2)$. Intuitively, the natural analogue of the expression by Bethuel, Brezis & Coron is Div $\big((\mathrm{adj}\, D\boldsymbol{u})\boldsymbol{g}\circ\boldsymbol{u}\big)$, where $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ is some unit-valued extension of the normal to the singular set in the deformed configuration. However, since in the neo-Hookean case that singular surface is not necessarily smooth and its shape and location are unknown, by duality one is lead to consider the functional

$$L(\boldsymbol{u}) = \sup_{\substack{\phi\in C^1(\Omega),\ \|\nabla\phi\|_\infty\leq 1\\ \boldsymbol{g}\in C^1_c(\mathbb{R}^3;\mathbb{R}^3),\ \|\boldsymbol{g}\|_\infty\leq 1}} \langle \mathrm{Div}\ \big((\mathrm{adj}\, D\boldsymbol{u})\boldsymbol{g}\circ\boldsymbol{u}\big), \phi\rangle.$$

In [6, Proposition 5.9] this expression was shown to coincide, for axisymmetric maps, with $\|D^s\boldsymbol{u}^{-1}\|$ and with the mass of the defect current associated to $\boldsymbol{u}$, as introduced by Giaquinta, Modica & Souček [18, 19]. In this article we show, in Proposition 5.3, that the proof of lower semicontinuity by Bethuel, Brezis & Coron can be extended to $E(\boldsymbol{u}) + 2L(\boldsymbol{u})$. This functional, which could play the same role as F in the relaxation strategy, does not involve the inverse map $\boldsymbol{u}^{-1}$, and this might provide some advantage in future studies of the problem.

The paper is organised as follows. We start in Section 2 by recalling some definitions and preliminary results. In Section 3 we prove the compactness of sequences of maps in $\mathcal{B}$ with a uniform bound on the neo-Hookean energy and on the BV norm of their inverses. Section 4 is devoted to the proofs of the lower semicontinuity of F in $\mathcal{B}$ and of Theorem 1.1. We prolong our study of the relaxation of the neo-Hookean energy in Section 5 by establishing the lower semicontinuity of $E(\boldsymbol{u}) + 2L(\boldsymbol{u})$.

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2. Notation and preliminaries

Let U be an open set of $\mathbb{R}^3$. For a vectorial map $\boldsymbol{u} : U \to \mathbb{R}^3$ we denote by $D\boldsymbol{u}$ its distributional Jacobian matrix. When $\boldsymbol{u}$ is in $BV(U, \mathbb{R}^3)$ we let $D\boldsymbol{u} = D^a\boldsymbol{u} + D^s\boldsymbol{u} = D^a\boldsymbol{u} + D^j\boldsymbol{u} + D^c\boldsymbol{u}$ be the standard decomposition of $D\boldsymbol{u}$, where $D^a\boldsymbol{u}$ denotes the absolutely continuous part of $D\boldsymbol{u}$ with respect to the Lebesgue measure, and $D^s\boldsymbol{u}$ denotes its orthogonal part. It is itself divided into the jump part $D^j\boldsymbol{u}$ and the Cantor part $D^c\boldsymbol{u}$. We will also use the notion of approximate differentiability (see, e.g., [16, Section 3.1.2], [31, Definition 2.3] or [23, Section 2.3]); if $\boldsymbol{u} : U \to \mathbb{R}^3$ is approximately differentiable we denote by $\nabla\boldsymbol{u}$ its approximate differential. Due to the Calderón-Zygmund theorem, every $\boldsymbol{u} \in BV(\Omega, \mathbb{R}^3)$ is approximately differentiable a.e. and $D^a\boldsymbol{u} = \nabla\boldsymbol{u}\, \mathcal{L}^3$. In particular, with a small abuse of notation, for Sobolev maps $D\boldsymbol{u} = \nabla\boldsymbol{u}$ a.e. The same notation applies to a scalar function $\phi : U \to \mathbb{R}$.

The Lebesgue measure of a measurable set $A \subset \mathbb{R}^3$ is denoted by $|A|$. We say that two sets A, B are equal a.e. and we write $A = B$ a.e. if $|A \setminus B| = |B \setminus A| = 0$. Given a measurable set $A \subset \mathbb{R}^3$ and a point $\boldsymbol{x} \in \mathbb{R}^3$, we define the density of A at $\boldsymbol{x}$ by

$$D(A, \boldsymbol{x}) := \lim_{r \to 0} \frac{|B(\boldsymbol{x}, r) \cap A|}{|B(\boldsymbol{x}, r)|} \tag{2.1}$$

when the limit exists. Here $B(\boldsymbol{x}, r)$ is the open ball centred at $\boldsymbol{x}$ of radius r.

The set of 3×3 matrices with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}^{3\times 3}$, while $\mathbb{R}^{3\times 3}_+$ is its subset of matrices with positive determinant. The adjoint and cofactor of $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ are denoted by $\operatorname{adj} \boldsymbol{A}$ and $\operatorname{cof} \boldsymbol{A}$, respectively, so that $\boldsymbol{A}(\operatorname{adj} \boldsymbol{A}) = (\operatorname{adj} \boldsymbol{A})\boldsymbol{A} = (\det \boldsymbol{A}) \operatorname{Id}$ and $\operatorname{cof} \boldsymbol{A} = (\operatorname{adj} \boldsymbol{A})^T$.

We recall the area formula of Federer ([31, Proposition 2.6] and [16, Theorem 3.2.5 and Theorem 3.2.3]). We will use the notation $\mathcal{N}(\boldsymbol{u}, A, \boldsymbol{y})$ for the number of preimages of a point $\boldsymbol{y}$ in the set A under $\boldsymbol{u}$. In this section, Ω is any bounded domain of $\mathbb{R}^3$.

Proposition 2.1. *Let $\boldsymbol{u}$ be approximately differentiable a.e. in Ω, and denote the set of approximate differentiability points of $\boldsymbol{u}$ by Ω_d. Then, for any measurable set $A \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^3 \to \mathbb{R}$,*

$$\int_A (\varphi \circ \boldsymbol{u})\, |\det \nabla\boldsymbol{u}|\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \varphi(\boldsymbol{y})\, \mathcal{N}(\boldsymbol{u}, \Omega_d \cap A, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

whenever either integral exists. Moreover, if a map $\psi : A \to \mathbb{R}$ is measurable and $\bar{\psi} : \boldsymbol{u}(\Omega_d \cap A) \to \mathbb{R}$ is given by

$$\bar{\psi}(\boldsymbol{y}) := \sum_{\boldsymbol{x} \in \Omega_d \cap A,\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}} \psi(\boldsymbol{x})$$

then $\bar{\psi}$ is measurable and

$$\int_A \psi(\varphi \circ \boldsymbol{u})\, |\det \nabla\boldsymbol{u}|\, \mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{u}(\Omega_d \cap A)} \bar{\psi}\, \varphi\, \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{y} \in \boldsymbol{u}(\Omega_d \cap A), \tag{2.2}$$

whenever the integral on the left-hand side of (2.2) *exists.*

We observe that the previous proposition implies that, for $\boldsymbol{u}$ approximately differentiable a.e.,

$$|\boldsymbol{u}(N \cap \Omega_d)| = 0 \quad \text{whenever } |N| = 0. \tag{2.3}$$

We will need to work with a set of points satisfying more properties than just approximate differentiability.

Definition 2.2. *Let $\boldsymbol{u}$ be approximately differentiable a.e. and such that $\det D\boldsymbol{u} \neq 0$ a.e. We define Ω_0 as the set of $\boldsymbol{x} \in \Omega$ for which the following are satisfied:*

1. *the approximate differential of $\boldsymbol{u}$ at $\boldsymbol{x}$ exists and equals $D\boldsymbol{u}(\boldsymbol{x})$.*
2. *there exist $\boldsymbol{w} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ and a compact set $K \subset \Omega$ of density 1 at $\boldsymbol{x}$ such that $\boldsymbol{u}|_K = \boldsymbol{w}|_K$ and $\nabla\boldsymbol{u}|_K = D\boldsymbol{w}|_K$,*

(3) $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$.

It can be seen from [16, Theorem 3.1.8], Rademacher's Theorem and Whitney's Theorem that Ω_0 is a set of full Lebesgue measure in Ω, i.e., $|\Omega \setminus \Omega_0| = 0$.

Definition 2.3. *For any measurable set A of Ω, the geometric image of A under an a.e. approximately differentiable map $\boldsymbol{u}$ is defined by*

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A) := \boldsymbol{u}(A \cap \Omega_0),$$

with Ω_0 as in Definition 2.2.

We will need the following result.

Lemma 2.4. *(*[31, Lemma 2.5]*) Let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be approximately differentiable in almost all Ω and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 2.2. Then for every $\boldsymbol{x} \in \Omega_0$ and every measurable set $A \subset \Omega$,*

$$D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A), \boldsymbol{u}(\boldsymbol{x})) = 1 \text{ whenever } D(A, \boldsymbol{x}) = 1,$$

where the density is defined in (2.1)*.*

In order to define the inverse of maps which are approximately differentiable, one-to-one a.e. and such that $\det \nabla \boldsymbol{u} \neq 0$ a.e., we first give the following lemma.

Lemma 2.5. *(*[21, Lemma 3]*) Let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be approximately differentiable in almost all Ω, one-to-one a.e., and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for a.e. $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 2.2. Then $u|_{\Omega_0}$ is one-to-one.*

Definition 2.6. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be approximately differentiable in almost all Ω, one-to-one a.e., and suppose that* $\det D\boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for a.e. $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 2.2. Then we define the inverse $\boldsymbol{u}^{-1}$ as the map $\boldsymbol{u}^{-1} : \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \to \mathbb{R}^3$ that sends every $\boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega)$ to the only $\boldsymbol{x} \in \Omega_0$ such that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}$.*

Proposition 2.7. *Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$ be such that* $\det D\boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for a.e. $\boldsymbol{x} \in \Omega$ and $\boldsymbol{u}$ is one-to-one a.e. Let Ω_0 be as in Definition 2.2. Then $\boldsymbol{u}^{-1}$ is approximately differentiable at every $\boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) = \boldsymbol{u}(\Omega_0)$ and its approximate differential satisfies*

$$\nabla \boldsymbol{u}^{-1}\big(\boldsymbol{u}(\boldsymbol{x})\big) = D\boldsymbol{u}(\boldsymbol{x})^{-1} = \frac{\operatorname{adj} D\boldsymbol{u}(\boldsymbol{x})}{\det D\boldsymbol{u}(\boldsymbol{x})} \text{ for every } \boldsymbol{x} \in \Omega_0. \tag{2.4}$$

In particular, if we assume that $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) = \Omega_{\boldsymbol{b}}$ *a.e. then $\nabla \boldsymbol{u}^{-1} \in L^1(\Omega_{\boldsymbol{b}})$ with*

$$\|\nabla \boldsymbol{u}^{-1}\|_{L^1(\Omega_{\boldsymbol{b}})} \leq \frac{1}{\sqrt{3}} \|D\boldsymbol{u}\|^2_{L^2(\Omega)}. \tag{2.5}$$

If we assume furthermore that $\boldsymbol{u}^{-1} \in BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$ then $D^a \boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x})) = D\boldsymbol{u}(\boldsymbol{x})^{-1}$ for a.e. $\boldsymbol{x} \in \Omega$.

Proof. The proof is adapted from [22, Theorem 2 iii)]. Let $\boldsymbol{x}_0 \in \Omega_0$ and define $\boldsymbol{y}_0 := \boldsymbol{u}(\boldsymbol{x}_0)$ and $\boldsymbol{F} := D\boldsymbol{u}(\boldsymbol{x}_0)$. Thanks to Definition 2.2, $\boldsymbol{F}$ is invertible and thanks to Lemma 2.4 we have $D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega), \boldsymbol{y}_0) = 1$. Define, for each $\delta > 0$,

$$E_\delta := \left\{ \boldsymbol{x} \in \Omega_0 \setminus \{\boldsymbol{x}_0\} : \frac{|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}_0) - \boldsymbol{F}(\boldsymbol{x} - \boldsymbol{x}_0)|}{|\boldsymbol{x} - \boldsymbol{x}_0|} < \delta \right\}.$$

Since $\boldsymbol{u}$ is approximately differentiable at $\boldsymbol{x}_0$ and the set Ω_0 is of full measure in Ω, we deduce that $D(E_\delta, \boldsymbol{x}_0) = 1$ for all $\delta > 0$. Now for each $\varepsilon > 0$ we set

$$A_\varepsilon := \left\{ \boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \setminus \{\boldsymbol{u}(\boldsymbol{x}_0)\} : \frac{|\boldsymbol{u}^{-1}(\boldsymbol{y}) - \boldsymbol{x}_0 - D\boldsymbol{u}(\boldsymbol{x}_0)^{-1}(\boldsymbol{y} - \boldsymbol{u}(\boldsymbol{x}_0))|}{|\boldsymbol{y} - \boldsymbol{u}(\boldsymbol{x}_0)|} > \varepsilon \right\}.$$

Let $\boldsymbol{x} \in \Omega_0 \setminus \{\boldsymbol{x}_0\}$ and $\boldsymbol{y} := \boldsymbol{u}(\boldsymbol{x})$. Thanks to Lemma 2.5, we have $\boldsymbol{y} \neq \boldsymbol{y}_0$. Set $\boldsymbol{r} := \boldsymbol{y} - \boldsymbol{y}_0 - \boldsymbol{F}(\boldsymbol{x} - \boldsymbol{x}_0)$. Then

$$\frac{|\boldsymbol{x} - \boldsymbol{x}_0 - \boldsymbol{F}^{-1}(\boldsymbol{y} - \boldsymbol{y}_0)|}{|\boldsymbol{y} - \boldsymbol{y}_0|} \leq |\boldsymbol{F}^{-1}| \frac{|\boldsymbol{r}|}{|\boldsymbol{x} - \boldsymbol{x}_0|} \frac{|\boldsymbol{x} - \boldsymbol{x}_0|}{|\boldsymbol{y} - \boldsymbol{y}_0|} \leq |\boldsymbol{F}^{-1}| \frac{|\boldsymbol{r}|}{|\boldsymbol{x} - \boldsymbol{x}_0|} \frac{1}{\left|\boldsymbol{F} \frac{\boldsymbol{x}-\boldsymbol{x}_0}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right| - \frac{|\boldsymbol{r}|}{|\boldsymbol{x}-\boldsymbol{x}_0|}}.$$

This shows that if $\boldsymbol{u}(E_\delta) \cap A_\varepsilon \neq \varnothing$ for some $\delta, \varepsilon > 0$, then

$$\varepsilon > |\boldsymbol{F}^{-1}| \frac{\delta}{\inf\{|\boldsymbol{F}\boldsymbol{v}| : |\boldsymbol{v}| = 1\} - \delta}. \tag{2.6}$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that (2.6) does not hold and, hence, $\boldsymbol{u}(E_\delta) \cap A_\varepsilon = \varnothing$. As $D(E_\delta, \boldsymbol{x}_0) = 1$, then by Lemma 2.4, $D(\mathbb{R}^3 \setminus \boldsymbol{u}(E_\delta), \boldsymbol{y}_0) = 0$, and, hence, $D(A_\varepsilon, \boldsymbol{y}_0) = 0$. This proves that $\boldsymbol{u}^{-1}$ is approximately differentiable at $\boldsymbol{u}(\boldsymbol{x}_0)$ and its approximate differential is equal to $D\boldsymbol{u}(\boldsymbol{x}_0)^{-1}$ and, thus, (2.4) holds.

We now assume that $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) = \Omega_{\boldsymbol{b}}$ a.e. Propositions 2.1 and 2.7 as well as the matrix inequality

$$|\boldsymbol{A}|^2 \geq \sqrt{3}\, |\operatorname{cof} \boldsymbol{A}|, \qquad \boldsymbol{A} \in \mathbb{R}^{3\times 3} \tag{2.7}$$

(see [7, Lemma 2.6]) show that

$$\int_{\Omega_{\boldsymbol{b}}} |\nabla \boldsymbol{u}^{-1}(\boldsymbol{y})| \, \mathrm{d}\boldsymbol{y} = \int_\Omega |\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{3}} \int_\Omega |D\boldsymbol{u}(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}.$$

Finally, if in addition $\boldsymbol{u}^{-1} \in BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$, then $D^a \boldsymbol{u}^{-1} = \nabla \boldsymbol{u}^{-1}$ a.e., which implies the conclusion. □

We now make explicit the divergence identities (1.3) by means of the functional $\mathcal{E}$ introduced in [21].

Definition 2.8. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be measurable and approximately differentiable a.e. Suppose that* $\det \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega)$ *and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{3\times 3})$. *For every $\boldsymbol{f} \in C^1_c(\Omega \times \mathbb{R}^3, \mathbb{R}^3)$, define*

$$\mathcal{E}(\boldsymbol{u}, \boldsymbol{f}) := \int_\Omega [\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) + \det \nabla \boldsymbol{u}(\boldsymbol{x}) \operatorname{div}_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))] \, \mathrm{d}\boldsymbol{x},$$

where $D\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ denotes the derivative of $\boldsymbol{f}(\cdot, \boldsymbol{y})$ evaluated at $\boldsymbol{x}$, while $\operatorname{div} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ *is the divergence of $\boldsymbol{f}(\boldsymbol{x}, \cdot)$ evaluated at $\boldsymbol{y}$.*

By definition of distributional divergence, the divergence identities hold if and only if $\mathcal{E}(\boldsymbol{u}, \phi\, \boldsymbol{g}) = 0$ for all $\phi \in C^1_c(\Omega)$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$. In fact, by density of sums of functions of separate variables (see, e.g., [27, Corollary 1.6.5]), this holds if and only if $\mathcal{E}(\boldsymbol{u}, \boldsymbol{f}) = 0$ for all $\boldsymbol{f} \in C^1_c(\Omega \times \mathbb{R}^3, \mathbb{R}^3)$. Of course, $\phi\, \boldsymbol{g}$ denotes the function $\phi(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{y})$ for $(\boldsymbol{x}, \boldsymbol{y}) \in \Omega \times \mathbb{R}^3$.

3. Compactness of maps with equibounded F energy in $\mathcal{B}$

In this section we show that the set of deformation maps such that their geometric image is $\Omega_{\boldsymbol{b}}$ and whose inverses are in $BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$ is compact for the weak convergence in H^1 if we assume a uniform bound on the neo-Hookean energy and on the BV norm of the inverses. Furthermore, those bounds also provide that the weak H^1 limit is one-to-one a.e. and satisfies that $\det D\boldsymbol{u} > 0$ a.e. In this respect, a uniform bound on the BV norm of the inverses of a sequence of deformation maps plays the same role as a uniform bound on the surface energy defined in [21]. An intermediate step is to show the validity of the divergence identities for $\boldsymbol{u}^{-1}$.

We start with the following variant of [21, Theorem 2].

Proposition 3.1. *Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\mathcal{B}$. Assume that $\{F(\boldsymbol{u}_j)\}_j$ is equibounded and that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$. Then*

i) $\det D\boldsymbol{u} \neq 0$ *a.e.*

ii) $\boldsymbol{u}$ *is one-to-one a.e.*
iii) $\operatorname{im_G}(\boldsymbol{u},\Omega)=\Omega_{\boldsymbol{b}}$ *a.e.*
iv) $\boldsymbol{u}^{-1}\in BV(\Omega_{\boldsymbol{b}},\mathbb{R}^3)$
v) up to a subsequence, $\boldsymbol{u}_j^{-1}\overset{*}{\rightharpoonup}\boldsymbol{u}^{-1}$ *in* $BV(\Omega_{\boldsymbol{b}},\mathbb{R}^3)$ *and* $\boldsymbol{u}_j^{-1}\to\boldsymbol{u}^{-1}$ *a.e.*
vi) $\det D\boldsymbol{u}_j\rightharpoonup|\det D\boldsymbol{u}|$ *in* $L^1(\Omega_{\boldsymbol{b}})$.

Proof. Since $\sup_j\int_\Omega H(\det D\boldsymbol{u}_j)<\infty$, by using the De la Vallée-Poussin criterion we can find $\theta\in L^1(\Omega)$ such that
$$\det D\boldsymbol{u}_j\rightharpoonup\theta \text{ in } L^1(\Omega).$$
Besides, an application of Fatou's lemma along with the properties of H in (1.2) gives $\theta>0$ a.e. in Ω. Passing to a subsequence we can also assume that $\boldsymbol{u}_j\to\boldsymbol{u}$ a.e.

We first want to show that the Jacobian determinant of $\boldsymbol{u}$ is different from zero. Let $\psi:\Omega_{\boldsymbol{b}}\to\mathbb{R}$ be continuous and bounded. As $\boldsymbol{u}_j$ is one-to-one a.e., an application of the change of variables formula (Proposition 2.1) shows that
$$\int_\Omega\psi(\boldsymbol{u}_j(\boldsymbol{x}))\det D\boldsymbol{u}_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\boldsymbol{b}}}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$
Since $\{\psi(\boldsymbol{u}_j)\}_j$ is equibounded in L^∞, a standard convergence result (see, e.g., [17, Proposition 2.61]) shows that
$$\int_\Omega\psi(\boldsymbol{u}(\boldsymbol{x}))\,\theta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\boldsymbol{b}}}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{3.1}$$
By approximation, the above formula remains valid for any ψ bounded Borel. Let $V:=\{\boldsymbol{x}\in\Omega_d:\det D\boldsymbol{u}(\boldsymbol{x})=0\}$. Formula (2.3) shows that $|\boldsymbol{u}(V)|=0$. Let U be a Borel set such that $\boldsymbol{u}(V)\subset U$ and $|U|=0$. By taking $\psi=\chi_U$, we obtain
$$\int_V\theta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0.$$
Since θ is positive a.e., necessarily $|V|=0$, proving i).

Set $\boldsymbol{v}_j:=\boldsymbol{u}_j^{-1}$. Thanks to (2.5), from the assumption that $\{F(\boldsymbol{u}_j)\}_j$ is equibounded, we find that both the absolutely continuous and the singular parts of $D\boldsymbol{v}_j$ are equibounded, so $\|D\boldsymbol{v}_j\|$ is equibounded. By compactness, up to a subsequence, $\boldsymbol{v}_j$ converges weakly* and a.e. in $BV(\Omega_{\boldsymbol{b}},\mathbb{R}^3)$ to some $\boldsymbol{v}$.

Now we apply a similar argument leading to (3.1). For $\varphi:\Omega\to\mathbb{R}$ and $\psi:\Omega_{\boldsymbol{b}}\to\mathbb{R}$ both continuous and bounded, we obtain first
$$\int_\Omega\varphi(\boldsymbol{x})\psi(\boldsymbol{u}_j(\boldsymbol{x}))\det D\boldsymbol{u}_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\boldsymbol{b}}}\varphi(\boldsymbol{v}_j(\boldsymbol{y}))\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$
then
$$\int_\Omega\varphi(\boldsymbol{x})\psi(\boldsymbol{u}(\boldsymbol{x}))\theta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\boldsymbol{b}}}\varphi(\boldsymbol{v}(\boldsymbol{y}))\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$
and, finally, that the above formula is valid for all φ and ψ bounded Borel. As a consequence, with Ω_d the set of approximate differentiability of $\boldsymbol{u}$, Proposition 2.1 shows that
$$\int_{\mathbb{R}^3}\psi(\boldsymbol{y})\sum_{\substack{\boldsymbol{x}\in\Omega_d\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}}}\varphi(\boldsymbol{x})\frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|}\mathrm{d}\boldsymbol{y}=\int_{\Omega_{\boldsymbol{b}}}\varphi(\boldsymbol{v}(\boldsymbol{y}))\psi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$
for any φ and ψ bounded Borel. Since the formula holds for all ψ, then
$$\sum_{\substack{\boldsymbol{x}\in\Omega_d\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}}}\varphi(\boldsymbol{x})\frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|}=0\quad\text{for a.e. }\boldsymbol{y}\in\mathbb{R}^3\setminus\Omega_{\boldsymbol{b}}$$

and

$$\sum_{\substack{\boldsymbol{x}\in\Omega_d\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}}} \varphi(\boldsymbol{x})\frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|} = \varphi(\boldsymbol{v}(\boldsymbol{y})) \quad \text{for a.e. } \boldsymbol{y}\in\Omega_{\boldsymbol{b}}. \tag{3.2}$$

From the first identity we obtain $\boldsymbol{u}(\Omega_d)\subset\Omega_{\boldsymbol{b}}$ a.e., because

$$\int_{\{\boldsymbol{x}\in\Omega_d:\boldsymbol{u}(\boldsymbol{x})\notin\Omega_{\boldsymbol{b}}\}} \varphi(\boldsymbol{x})\theta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{u}(\Omega_d)\setminus\Omega_{\boldsymbol{b}}} \sum_{\substack{\boldsymbol{x}\in\Omega_d\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}}} \varphi(\boldsymbol{x})\frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|}\mathrm{d}\boldsymbol{y} = 0,$$

so taking $\varphi=\chi_\Omega$ and using the positivity of θ we obtain $|\{\boldsymbol{x}\in\Omega_d:\boldsymbol{u}(\boldsymbol{x})\notin\Omega_{\boldsymbol{b}}\}|=0$, and, by (2.3), $|\boldsymbol{u}(\Omega_d)\setminus\Omega_{\boldsymbol{b}}|=0$.

Now consider the second identity. Any point in $\Omega_{\boldsymbol{b}}$ for which formula (3.2) is valid is necessarily in $\boldsymbol{u}(\Omega_d)$, since otherwise the left-hand side would always be zero, whereas one may find φ such that the right-hand side is not zero. This shows that $|\Omega_{\boldsymbol{b}}\setminus\boldsymbol{u}(\Omega_d)|=0$ and, hence, $\boldsymbol{u}(\Omega_d)=\Omega_{\boldsymbol{b}}$ a.e. By property (2.3), $\boldsymbol{u}(\Omega_d)=\boldsymbol{u}(\Omega_0)$ a.e., so iii) is proved.

Let $\Omega'_{\boldsymbol{b}}$ be the set of $\boldsymbol{y}\in\Omega_{\boldsymbol{b}}$ for which formula (3.2) is valid. Fix $\boldsymbol{y}_0\in\Omega'_{\boldsymbol{b}}$ and set $\boldsymbol{x}_0:=\boldsymbol{v}(\boldsymbol{y}_0)$. Taking any bounded Borel φ such that $\varphi(\boldsymbol{x}_0)=0$ we obtain from (3.2) that

$$\sum_{\substack{\boldsymbol{x}\in\Omega_d\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}_0}} \varphi(\boldsymbol{x})\frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|} = 0.$$

As there is no composition with $\boldsymbol{u}$ or $\boldsymbol{v}$ in the formula above, by approximation it remains valid for any bounded measurable φ with $\varphi(\boldsymbol{x}_0)=0$. Taking φ to be the characteristic function of $\{\boldsymbol{x}\in\Omega_d\setminus\{\boldsymbol{x}_0\}:\theta(\boldsymbol{x})>0\}$ we obtain

$$\sum_{\substack{\boldsymbol{x}\in\Omega_d\setminus\{\boldsymbol{x}_0\}\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}_0,\ \theta(\boldsymbol{x})>0}} \frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|} = 0,$$

so there is no $\boldsymbol{x}\in\Omega_d\setminus\{\boldsymbol{x}_0\}$ such that $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}_0$ and $\theta(\boldsymbol{x})>0$. This shows that $\boldsymbol{u}$ is one-to-one in $\{\boldsymbol{x}\in\Omega_d:\boldsymbol{u}(\boldsymbol{x})\in\Omega'_{\boldsymbol{b}},\ \theta(\boldsymbol{x})>0\}$. Thanks to i), this set is of full measure in Ω (see, e.g., [5, Lemma 2.8 (c)] for a proof, which in fact is an easy consequence of Proposition 2.1). This proves ii).

Now we take $\varphi=\chi_{\{\boldsymbol{x}_0\}}$ in (3.2) and obtain that

$$\sum_{\substack{\boldsymbol{x}\in\Omega_d\cap\{\boldsymbol{x}_0\}\\ \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{y}}} \frac{\theta(\boldsymbol{x})}{|\det D\boldsymbol{u}(\boldsymbol{x})|} = 1,$$

which implies that $\boldsymbol{x}_0\in\Omega_d$, $\boldsymbol{u}(\boldsymbol{x}_0)=\boldsymbol{y}_0$ and $\theta(\boldsymbol{x}_0)=|\det D\boldsymbol{u}(\boldsymbol{x}_0)|$. Equality $\boldsymbol{u}(\boldsymbol{x}_0)=\boldsymbol{y}_0$ says that $\boldsymbol{v}=\boldsymbol{u}^{-1}$ in $\Omega'_{\boldsymbol{b}}$ and, hence, $\boldsymbol{v}=\boldsymbol{u}^{-1}$ a.e. This proves iv) and v). Finally, equality $\theta(\boldsymbol{x}_0)=|\det D\boldsymbol{u}(\boldsymbol{x}_0)|$ says that $\theta=|\det D\boldsymbol{u}|$ in the set $\{\boldsymbol{x}\in\Omega_d:\boldsymbol{u}(\boldsymbol{x})\in\Omega'_{\boldsymbol{b}}\}$, which, as before, has full measure in Ω. Thus, $\theta=|\det D\boldsymbol{u}|$ a.e. and, hence, vi) holds. □

It will be instrumental to consider the inverses of maps in $\mathcal{B}$ in the proof of lower semicontinuity of F. We first study the continuity of the cofactor gradient of these inverses.

Lemma 3.2. *Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\mathcal{B}$. Assume that $\{F(\boldsymbol{u}_j)\}_j$ is equibounded and that $\boldsymbol{u}_j\rightharpoonup\boldsymbol{u}$ in $H^1(\Omega,\mathbb{R}^3)$. Then, up to a subsequence, $\operatorname{cof}\nabla\boldsymbol{u}_j^{-1}\rightharpoonup\operatorname{cof}\nabla\boldsymbol{u}^{-1}$ in $L^1(\Omega_{\boldsymbol{b}},\mathbb{R}^{3\times3})$.*

Proof. The proof is divided into two steps.

Step 1: Weak convergence in the sense of measures.* Let $\boldsymbol{\psi} \in C(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ be bounded. By Propositions 2.1 and 2.7, and using formula $\operatorname{adj} \boldsymbol{A}^{-1} \det \boldsymbol{A} = \boldsymbol{A}$ for any invertible matrix $\boldsymbol{A}$, we find

$$\int_{\Omega_{\boldsymbol{b}}} \operatorname{adj} \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = \int_{\Omega} D\boldsymbol{u}_j(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{u}_j(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x}. \tag{3.3}$$

Up to a subsequence, we know that $\boldsymbol{u}_j \to \boldsymbol{u}$ a.e. Hence the dominated convergence theorem shows that $\psi \circ \boldsymbol{u}_j \to \psi \circ \boldsymbol{u}$ in L^q for all $q < \infty$. This implies that

$$\int_{\Omega} D\boldsymbol{u}_j(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{u}_j(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \to \int_{\Omega} D\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{u}(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x}.$$

Thanks to Proposition 3.1, formula (3.3) also holds for $\boldsymbol{u}$ in place of $\boldsymbol{u}_j$, so we conclude

$$\int_{\Omega_{\boldsymbol{b}}} \operatorname{adj} \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \to \int_{\Omega_{\boldsymbol{b}}} \operatorname{adj} \nabla \boldsymbol{u}^{-1}(\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

This shows that $\operatorname{adj} \nabla \boldsymbol{u}_j^{-1} \overset{*}{\rightharpoonup} \operatorname{adj} \nabla \boldsymbol{u}^{-1}$ in the sense of measures.

Step 2: Equiintegrability. We first show that $\{\det \nabla \boldsymbol{u}_j^{-1}\}_j$ is equiintegrable. By Propositions 2.1 and 2.7,

$$\int_{\Omega} H(\det D\boldsymbol{u}_j(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} = \int_{\Omega} H\left(\frac{1}{\det \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y})}\right) \det \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Since $H_1(t) := H(1/t)t$ is convex and satisfies (1.2), the De la Vallée-Poussin criterion shows that $\{\det \nabla \boldsymbol{u}_j^{-1}\}_j$ is equiintegrable, as $\{F(\boldsymbol{u}_j)\}_j$ is bounded.

Let $V \subset \Omega_{\boldsymbol{b}}$ be a Borel set. On the one hand, as in (3.3),

$$\int_V |\operatorname{adj} \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y})| \, \mathrm{d}\boldsymbol{y} = \int_{\boldsymbol{u}_j^{-1}(V)} |D\boldsymbol{u}_j(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x}.$$

On the other hand,

$$|\boldsymbol{u}_j^{-1}(V)| = \int_V \det \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Take a fixed $\varepsilon > 0$. Since the sequence $\{D\boldsymbol{u}_j\}_j$ is equiintegrable, there exists $\delta > 0$ such that for any measurable $A \subset \Omega$ with $|A| < \delta$ we have that for all $j \in \mathbb{N}$

$$\int_A |D\boldsymbol{u}_j(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} < \varepsilon.$$

We apply this to $A = \boldsymbol{u}_j^{-1}(V)$, complementing it with the equiintegrability of $\det \nabla \boldsymbol{u}_j^{-1}$, which gives that there exists $\eta > 0$ such that

$$\text{if } |V| < \eta \text{ then } \int_V \det \nabla \boldsymbol{u}_j^{-1}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} < \delta.$$

This shows that the sequence $\{\operatorname{adj} \nabla \boldsymbol{u}_j^{-1}\}_j$ is equiintegrable, which finishes the proof thanks to Step 1. □

The next proposition shows that inverses of maps in $\mathcal{B}$ satisfy the divergence identities.

Proposition 3.3. *Let $\boldsymbol{u}$ be a map in $\mathcal{B}$. Then $\boldsymbol{u}^{-1}$ satisfies the divergence identities in $\Omega_{\boldsymbol{b}}$.*

Proof. Let $\phi \in C^1_c(\Omega_{\boldsymbol{b}})$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Then

$$\begin{aligned}\mathcal{E}(\boldsymbol{u}^{-1}, \phi \boldsymbol{g}) &= \int_{\Omega_{\boldsymbol{b}}} \left[\boldsymbol{g}(\boldsymbol{u}^{-1}(\boldsymbol{y})) \cdot \left(\operatorname{cof} \nabla \boldsymbol{u}^{-1}(\boldsymbol{y})\, D\phi(\boldsymbol{y})\right) + \phi(\boldsymbol{y}) \operatorname{div} \boldsymbol{g}(\boldsymbol{u}^{-1}(\boldsymbol{y})) \det \nabla \boldsymbol{u}^{-1}(\boldsymbol{y})\right] \mathrm{d}\boldsymbol{y} \\ &= \int_{\Omega} \left[\boldsymbol{g}(\boldsymbol{x}) \cdot \left(\operatorname{cof} \nabla \boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x}))\, D\phi(\boldsymbol{u}(\boldsymbol{x}))\right) + \phi(\boldsymbol{u}(\boldsymbol{x})) \operatorname{div} \boldsymbol{g}(\boldsymbol{x}) \det \nabla \boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x}))\right] \det D\boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega} \left[\boldsymbol{g}(\boldsymbol{x}) \cdot \left(D\boldsymbol{u}(\boldsymbol{x})^T D\phi(\boldsymbol{u}(\boldsymbol{x}))\right) + \phi(\boldsymbol{u}(\boldsymbol{x})) \operatorname{div} \boldsymbol{g}(\boldsymbol{x})\right] \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.4}$$

By the chain rule (e.g., [15, Theorem 4.2.2.4]), $\phi \circ \boldsymbol{u} \in H^1(\Omega)$ and

$$D(\phi \circ \boldsymbol{u})(\boldsymbol{x}) = D\boldsymbol{u}(\boldsymbol{x})^T D\phi(\boldsymbol{u}(\boldsymbol{x})), \qquad \text{a.e. } \boldsymbol{x} \in \Omega. \tag{3.5}$$

Let us see that $\operatorname{tr}(\phi \circ \boldsymbol{u}) = 0$, where tr is the trace operator on $\partial\Omega$. Since $|\boldsymbol{u} - \boldsymbol{b}| \in H^1(\Omega)$ and $\boldsymbol{u} = \boldsymbol{b}$ in $\Omega \setminus \widetilde{\Omega}$, we have that $|\boldsymbol{u} - \boldsymbol{b}| \in H^1_0(\Omega)$ (see, e.g., [9, Proposition 9.18]), so $\operatorname{tr} |\boldsymbol{u} - \boldsymbol{b}| = 0$ (see, e.g., [14, Theorem 5.5.2]). Now, by [15, Theorem 5.3.2], for $\mathcal{H}^2$-a.e. $\boldsymbol{x}_0 \in \partial\Omega$,

$$0 = \operatorname{tr}(|\boldsymbol{u} - \boldsymbol{b}|)(\boldsymbol{x}_0) = \lim_{r \to 0} \frac{1}{r^n} \int_{B(\boldsymbol{x}_0, r) \cap \Omega} |\boldsymbol{u} - \boldsymbol{b}|$$

and, hence, using that ϕ is Lipschitz,

$$\operatorname{tr}(|\phi \circ \boldsymbol{u} - \phi \circ \boldsymbol{b}|)(\boldsymbol{x}_0) = \lim_{r \to 0} \frac{1}{r^n} \int_{B(\boldsymbol{x}_0, r) \cap \Omega} |\phi \circ \boldsymbol{u} - \phi \circ \boldsymbol{b}| = 0,$$

so $\operatorname{tr}(\phi \circ \boldsymbol{u}) = \operatorname{tr}(\phi \circ \boldsymbol{b}) = 0$, since $\phi \circ \boldsymbol{b}$ is continuous, $\operatorname{spt} \phi \subset \Omega_{\boldsymbol{b}}$ and $\Omega_{\boldsymbol{b}} \cap \boldsymbol{b}(\partial\Omega) = \varnothing$. Therefore, by integration by parts (e.g., [15, Theorem 4.3.1]),

$$\int_{\Omega} (\phi \circ \boldsymbol{u})(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = -\int_{\Omega} D(\phi \circ \boldsymbol{u})(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

which shows, thanks to (3.4) and (3.5), that $\mathcal{E}(\boldsymbol{u}^{-1}, \phi \boldsymbol{g}) = 0$, i.e., $\boldsymbol{u}^{-1}$ satisfies the divergence identities. □

Proposition 3.4. *Let* $\{\boldsymbol{u}_j\}_j$ *be a sequence in* $\mathcal{B}$. *Assume that* $\{F(\boldsymbol{u}_j)\}_j$ *is equibounded and that* $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ *in* $H^1(\Omega, \mathbb{R}^3)$. *Then* $\det \nabla \boldsymbol{u}_j^{-1} \rightharpoonup \det \nabla \boldsymbol{u}^{-1}$ *in* $L^1(\Omega_{\boldsymbol{b}})$ *and* $\det D\boldsymbol{u} > 0$ *a.e. in* Ω.

Proof. By Proposition 3.1 v) and Lemma 3.2 we have $\boldsymbol{u}_j^{-1} \rightharpoonup \boldsymbol{u}^{-1}$ in $BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$ and $\operatorname{cof} \nabla \boldsymbol{u}_j^{-1} \rightharpoonup \operatorname{cof} \nabla \boldsymbol{u}^{-1}$ in $L^1(\Omega_{\boldsymbol{b}}, \mathbb{R}^{3\times 3})$, up to a subsequence.

As seen in Step 2 of Lemma 3.2, the sequence $\{\det \nabla \boldsymbol{u}_j^{-1}\}_j$ is equiintegrable, so by Fatou's lemma we get a $\widehat{\theta} \in L^1(\Omega_{\boldsymbol{b}})$ positive a.e. such that $\det \nabla \boldsymbol{u}_j^{-1} \rightharpoonup \widehat{\theta}$ in $L^1(\Omega_{\boldsymbol{b}})$. Then, Lemma 3.2, Propositions 3.3 and 3.1, and the remark after Definition 2.8 allow us to apply [21, Theorem 1] on the inverse and conclude that $\widehat{\theta} = \det \nabla \boldsymbol{u}^{-1}$ a.e. Thus $\det \nabla \boldsymbol{u}^{-1} > 0$ a.e. and, by Proposition 2.7, $\det D\boldsymbol{u} > 0$ a.e. □

As an immediate consequence of Propositions 3.1 and 3.4, we obtain the main conclusion of this section, namely, the compactness of maps in $\mathcal{B}$ with equibounded energy F.

Corollary 3.5. *Let* $\{\boldsymbol{u}_j\}_j \subset \mathcal{B}$ *be such that* $\{F(\boldsymbol{u}_j)\}_j$ *is bounded. Then there exists* $\boldsymbol{u} \in \mathcal{B}$ *such that, up to a subsequence,* $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ *in* $H^1(\Omega, \mathbb{R}^3)$.

4. Lower semicontinuity of F in $\mathcal{B}$ and proof of the theorem

The key argument to prove the lower semicontinuity of the energy F is to use a change of variables to express the neo-Hookean energy in terms of the inverse deformations. This allows to control from below the Dirichlet part of the neo-Hookean energy by the L^1 norm of the absolutely continuous part of the gradient of the inverses with the optimal constant 2. We then use the theory of BV functions to obtain the desired lower semicontinuity.

The following lemma presents two matrix inequalities. We first recall that the constant in inequality (2.7) is optimal. However, we would need a constant 2 instead of $\sqrt{3}$, since this would provide the right constant in front of the total variation in (1.4), as explained in the introduction and as shown by the construction in [6].

Lemma 4.1. *a)* $|\boldsymbol{A}|^2 \geq 2|\operatorname{cof}\boldsymbol{A}| - 2\max\{1, \det\boldsymbol{A}\}$ *for all* $\boldsymbol{A} \in \mathbb{R}^{3\times 3}_+$.

b) $\frac{|\operatorname{cof}\boldsymbol{B}|^2}{\det\boldsymbol{B}} \geq 2|\boldsymbol{B}| - 2\max\{1, \det\boldsymbol{B}\}$ *for all* $\boldsymbol{B} \in \mathbb{R}^{3\times 3}_+$.

Proof. We start by proving a). The three terms $|\boldsymbol{A}|$, $|\operatorname{cof}\boldsymbol{A}|$ and $\det\boldsymbol{A}$ are invariant under multiplication by rotations. Therefore, by singular value decomposition, we can assume that $\boldsymbol{A}$ is diagonal with positive diagonal elements $v_1 \leq v_2 \leq v_3$. In this case, we have

$$|\boldsymbol{A}|^4 - 4|\operatorname{cof}\boldsymbol{A}|^2 = \left(v_1^2 + v_2^2 + v_3^2\right)^2 - 4\left(v_2^2 v_3^2 + v_1^2 v_3^2 + v_1^2 v_2^2\right) = \left(v_1^2 + v_2^2 - v_3^2\right)^2 - 4v_1^2 v_2^2,$$

so

$$|\boldsymbol{A}|^4 \geq 4|\operatorname{cof}\boldsymbol{A}|^2 - 4v_1^2 v_2^2.$$

Taking into account that if $a \geq b \geq 0$ then $\sqrt{a^2 - b^2} \geq a - b$, from the inequality above we get

$$|\boldsymbol{A}|^2 \geq 2|\operatorname{cof}\boldsymbol{A}| - 2v_1 v_2.$$

If $v_1 v_2 \leq 1$ then we are done. If $v_1 v_2 > 1$, then $v_3 \geq v_2 > 1$ and therefore

$$1 < v_1 v_2 < v_1 v_2 v_3 = \det\boldsymbol{A} = \max\{1, \det\boldsymbol{A}\}.$$

This shows a). By taking $\boldsymbol{A} = \boldsymbol{B}^{-1} = \frac{\operatorname{adj}\boldsymbol{B}}{\det\boldsymbol{B}}$, since $\operatorname{cof}\boldsymbol{A} = \frac{\boldsymbol{B}^T}{\det\boldsymbol{B}}$, we get from a) that

$$\frac{|\operatorname{cof}\boldsymbol{B}|^2}{(\det\boldsymbol{B})^2} \geq 2\frac{|\boldsymbol{B}|}{\det\boldsymbol{B}} - 2\max\{1, \frac{1}{\det\boldsymbol{B}}\},$$

and therefore b). □

The key idea in providing the lower semicontinuity of the functional F is to work with the inverse maps, by moving from the reference configuration to the deformed one. In order to do so, we need to check some ingredients: the behaviour of polyconvexity with respect to the passage to the inverse, as well as the limit behaviour of the cofactor and the determinant of the inverse of a sequence of deformations in $\mathcal{B}$.

For the first ingredient we use a result due to Ball [2, Theorem 2.6] (see also [26, Proposition 1.1]).

Lemma 4.2. *If* $W : \mathbb{R}^{3\times 3}_+ \to \mathbb{R}$ *is polyconvex, then so is* $\boldsymbol{A} \mapsto W(\boldsymbol{A}^{-1})\det\boldsymbol{A}$. *In particular, the functional* $\boldsymbol{A} \mapsto \frac{|\operatorname{cof}\boldsymbol{A}|^2}{\det\boldsymbol{A}}$ *is polyconvex.*

Naturally, the function $\widetilde{W}(\boldsymbol{A}) = \frac{|\operatorname{cof}\boldsymbol{A}|^2}{\det\boldsymbol{A}}$ corresponds to the choice $W(\boldsymbol{A}) = |\boldsymbol{A}|^2$. In fact, one can check directly that $\widetilde{W}$ is polyconvex by expressing

$$\widetilde{W}(\boldsymbol{A}) = h(|\operatorname{cof}\boldsymbol{A}|, \det\boldsymbol{A})$$

with $h : (0,\infty)^2 \to \mathbb{R}$ defined as $h(a,b) = \frac{a^2}{b}$. The function $h(\cdot, b)$ is monotone increasing for each $b > 0$, while h is convex. Therefore, $\widetilde{W}$ is polyconvex.

We are now ready to prove the lower semicontinuity.

Proposition 4.3. *The functional F is lower semicontinuous in $\mathcal{B}$ in the weak topology of $H^1(\Omega,\mathbb{R}^3)$.*

Proof. Consider the following functional defined for maps $\boldsymbol{w}$ that are inverses of maps in $\mathcal{B}$:

$$\hat{E}(\boldsymbol{w}) := \int_{\Omega_{\boldsymbol{b}}} \frac{|\operatorname{cof}\nabla\boldsymbol{w}|^2}{\det\nabla\boldsymbol{w}}\,\mathrm{d}\boldsymbol{y}.$$

By Propositions 2.1 and 2.7, $\int_\Omega |D\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} = \hat{E}(\boldsymbol{u}^{-1})$. Let V be the singular set of $D\boldsymbol{u}^{-1}$, i.e., $V\subset\Omega_{\boldsymbol{b}}$ is any Borel set with $|V|=0$ such that $|D^s\boldsymbol{u}^{-1}|(\Omega_{\boldsymbol{b}}\setminus V)=0$.

Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\mathcal{B}$ converging weakly in $H^1(\Omega,\mathbb{R}^3)$ to some $\boldsymbol{u}\in\mathcal{B}$. For the proof of lower semicontinuity of F we can assume, without loss of generality, that $\lim_j F(\boldsymbol{u}_j)$ exists and is finite. As shown in step 2 of Lemma 3.2, $\{\det\nabla\boldsymbol{u}_j^{-1}\}_j$ is equiintegrable. For $\varepsilon>0$, let V_ε an open neighbourhood of V such that $|V_\varepsilon|<\varepsilon$ and $\int_{V_\varepsilon}\det\nabla\boldsymbol{u}_j^{-1}<\varepsilon$ for any $j\in\mathbb{N}$.

Next, by Lemma 4.1 b) we have

$$\hat{E}(\boldsymbol{u}_j^{-1}) \geq \int_{\Omega_{\boldsymbol{b}}\setminus V_\varepsilon} \frac{|\operatorname{cof}\nabla\boldsymbol{u}_j^{-1}|^2}{\det\nabla\boldsymbol{u}_j^{-1}}\,\mathrm{d}\boldsymbol{y} + 2\int_{V_\varepsilon}|\nabla\boldsymbol{u}_j^{-1}|\,\mathrm{d}\boldsymbol{y} - 4\varepsilon$$

and, hence,

$$\hat{E}(\boldsymbol{u}_j^{-1}) + 2\|D^s\boldsymbol{u}_j^{-1}\| \geq \int_{\Omega_{\boldsymbol{b}}\setminus V_\varepsilon} \frac{|\operatorname{cof}\nabla\boldsymbol{u}_j^{-1}|^2}{\det\nabla\boldsymbol{u}_j^{-1}}\,\mathrm{d}\boldsymbol{y} + 2|D\boldsymbol{u}_j^{-1}|(V_\varepsilon) - 4\varepsilon.$$

Now we use the convergences $\operatorname{cof}\nabla\boldsymbol{u}_j^{-1}\rightharpoonup\operatorname{cof}\nabla\boldsymbol{u}^{-1}$ in $L^1(\Omega_{\boldsymbol{b}},\mathbb{R}^{3\times3})$ (Lemma 3.2), $\det\nabla\boldsymbol{u}_j^{-1}\rightharpoonup\det\nabla\boldsymbol{u}^{-1}$ in $L^1(\Omega_{\boldsymbol{b}})$ (Proposition 3.4), $\det D\boldsymbol{u}_j\rightharpoonup\det D\boldsymbol{u}$ in $L^1(\Omega)$ (Proposition 3.1 vi) and Proposition 3.4) and $\boldsymbol{u}_j^{-1}\overset{*}{\rightharpoonup}\boldsymbol{u}^{-1}$ in $BV(\Omega_{\boldsymbol{b}},\mathbb{R}^3)$ (Proposition 3.1 v)), as well as Lemma 4.2 and the convexity of H, to obtain

$$\lim_{j\to\infty}F(\boldsymbol{u}_j) \geq \int_{\Omega_{\boldsymbol{b}}\setminus V_\varepsilon}\frac{|\operatorname{cof}\nabla\boldsymbol{u}^{-1}|^2}{\det\nabla\boldsymbol{u}^{-1}}\,\mathrm{d}\boldsymbol{y} + \int_\Omega H(\det D\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + 2|D\boldsymbol{u}^{-1}|(V_\varepsilon) - 4\varepsilon.$$

By using that $|D\boldsymbol{u}^{-1}|(V_\varepsilon)\geq|D\boldsymbol{u}^{-1}|(V)=\|D^s\boldsymbol{u}^{-1}\|$, sending ε to zero and going back to the reference configuration in the integral in $\Omega_{\boldsymbol{b}}$, we conclude. □

With Corollary 3.5 and Proposition 4.3, the proof of Theorem 1.1 is immediate.

5. Connection with harmonic map theory

In this section we propose another minimisation space $\widetilde{\mathcal{B}}$ and another functional $\widetilde{F}$ that admits a minimiser in $\widetilde{\mathcal{B}}$ and such that if such a minimiser is regular enough then it is a minimiser of the neo-Hookean energy in the original space $\mathcal{A}$. One feature of this new energy is that it does not involve the inverse of the deformation map. It bears some similarity with the relaxed energy of Bethuel-Brezis-Coron in the context of harmonic maps [8]. We note that in [6, Proposition 5.9], in the axisymmetric setting, it was shown that $\mathcal{B}=\widetilde{\mathcal{B}}$ and $F=\widetilde{F}$ and that this latter energy can be expressed in terms of Cartesian currents, cf. [18, 19].

We define

$$\widetilde{\mathcal{B}} := \{\boldsymbol{u}\in H^1(\Omega,\mathbb{R}^3) : \boldsymbol{u}=\boldsymbol{b} \text{ in } \Omega\setminus\widetilde{\Omega},\ \det D\boldsymbol{u}>0 \text{ a.e.}, \boldsymbol{u} \text{ one-to-one a.e.},\ \operatorname{im_G}(\boldsymbol{u},\Omega)=\Omega_{\boldsymbol{b}} \text{ a.e. and } E(\boldsymbol{u})<+\infty\}$$

and

$$L(\boldsymbol{u}) = \sup\{\mathcal{E}(\boldsymbol{u},\phi\boldsymbol{g}) : \phi\in C_c^1(\Omega),\ \|D\phi\|_\infty\leq1,\ \boldsymbol{g}\in C_c^1(\mathbb{R}^3,\mathbb{R}^3),\ \|\boldsymbol{g}\|_\infty\leq1\},$$

where $\mathcal{E}$ is defined in Definition 2.8. The new energy we propose is, for $\boldsymbol{u}\in\widetilde{\mathcal{B}}$,

$$\widetilde{F}(\boldsymbol{u}) = E(\boldsymbol{u}) + 2L(\boldsymbol{u}).$$

We will use the following elementary observation. Any $\phi \in C_c^1(\Omega)$ is Lipschitz with constant $\|D\phi\|_\infty$. Indeed, it is well known that the Lipschitz constant M of ϕ satisfies $M \geq \|D\phi\|_\infty$ and, if Ω is convex, then $M = \|D\phi\|_\infty$. Now consider the extension $\widetilde{\phi}$ of ϕ by zero. Then $\widetilde{\phi} \in C_c^1(\mathbb{R}^3)$ and $\|D\widetilde{\phi}\|_\infty = \|D\phi\|_\infty$. Thus, the Lipschitz constant $\widetilde{M}$ of $\widetilde{\phi}$ is $\|D\phi\|_\infty$, whereas $M \leq \widetilde{M}$. This proves the claim.

Of course, $\mathcal{B} \subset \widetilde{\mathcal{B}}$. The following result shows that, under the assumption $L(\boldsymbol{u}) < \infty$, we have $\mathcal{B} = \widetilde{\mathcal{B}}$ and the quantities $\|D^s\boldsymbol{u}^{-1}\|$ and $L(\boldsymbol{u})$ are comparable.

Lemma 5.1. *a)* $L(\boldsymbol{u}) \leq \|D^s\boldsymbol{u}^{-1}\|$ *for all* $\boldsymbol{u} \in \mathcal{B}$.
b) Any $\boldsymbol{u} \in \widetilde{\mathcal{B}}$ *with* $L(\boldsymbol{u}) < \infty$ *satisfies* $\boldsymbol{u} \in \mathcal{B}$ *and* $\|D^s\boldsymbol{u}^{-1}\| \leq 3L(\boldsymbol{u})$.

Proof. The first part of the proof is common for a) and b). Let $\boldsymbol{u} \in \widetilde{\mathcal{B}}$. By Definition 2.8, Propositions 2.1 and 2.7 and the relation $\boldsymbol{A}^{-1} = \frac{\operatorname{adj}\boldsymbol{A}}{\det\boldsymbol{A}}$ valid for $\boldsymbol{A} \in \mathbb{R}^{3\times3}_+$, we have, for $\phi \in C_c^\infty(\Omega)$ and $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$, that

$$\begin{aligned}\mathcal{E}(\boldsymbol{u}, \phi\boldsymbol{g}) &= \int_\Omega [(\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x})D\phi(\boldsymbol{x})) \cdot \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) + \det D\boldsymbol{u}(\boldsymbol{x})\phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))] \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega_{\boldsymbol{b}}} \left[\left(\nabla\boldsymbol{u}^{-1}(\boldsymbol{y})^T D\phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\right) \cdot \boldsymbol{g}(\boldsymbol{y}) + \phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \right] \mathrm{d}\boldsymbol{y}.\end{aligned}$$

As $\phi \in C_c^1(\Omega)$ we have $\phi \circ \boldsymbol{u}^{-1} \in L^\infty(\Omega_{\boldsymbol{b}})$ and, by [1, Proposition 3.71] and Proposition 2.7, that

$$\nabla(\phi \circ \boldsymbol{u}^{-1}) = (\nabla\boldsymbol{u}^{-1})^T D\phi(\boldsymbol{u}^{-1}), \tag{5.1}$$

which is in $L^1(\Omega_{\boldsymbol{b}})$ thanks to (2.5). We can then write

$$\mathcal{E}(\boldsymbol{u}, \phi\boldsymbol{g}) = \langle \phi \circ \boldsymbol{u}^{-1}, \operatorname{div} \boldsymbol{g} \rangle_{\mathcal{D}'(\Omega_{\boldsymbol{b}})} + \langle \nabla(\phi \circ \boldsymbol{u}^{-1}), \boldsymbol{g} \rangle_{\mathcal{D}'(\Omega_{\boldsymbol{b}},\mathbb{R}^3)}. \tag{5.2}$$

Now we prove a). If we assume that $\boldsymbol{u} \in \mathcal{B}$, formula (5.2) can be expressed as

$$\mathcal{E}(\boldsymbol{u}, \phi\boldsymbol{g}) = \langle -D(\phi \circ \boldsymbol{u}^{-1}) + \nabla(\phi \circ \boldsymbol{u}^{-1}), \boldsymbol{g} \rangle_{\mathcal{D}'(\Omega_{\boldsymbol{b}},\mathbb{R}^3)} = -\langle D^s(\phi \circ \boldsymbol{u}^{-1}), \boldsymbol{g} \rangle_{\mathcal{D}'(\Omega_{\boldsymbol{b}},\mathbb{R}^3)}. \tag{5.3}$$

Assume $\|D\phi\|_\infty \leq 1$ and $\|\boldsymbol{g}\|_\infty \leq 1$. From (5.3) and the chain rule in BV (see, e.g., [1, Theorem 3.96]), using that ϕ is 1-Lipschitz we find that

$$\begin{aligned}|\mathcal{E}(\boldsymbol{u}, \phi\boldsymbol{g})| \leq \|D^s(\phi \circ \boldsymbol{u}^{-1})\| &\leq \int_{J_{\boldsymbol{u}^{-1}}} |[\phi \circ \boldsymbol{u}^{-1}]| \, \mathrm{d}\mathcal{H}^2 + \|D^c\boldsymbol{u}^{-1}\| \\ &\leq \int_{J_{\boldsymbol{u}^{-1}}} |[\boldsymbol{u}^{-1}]| \, \mathrm{d}\mathcal{H}^2 + \|D^c\boldsymbol{u}^{-1}\| = \|D^s\boldsymbol{u}^{-1}\|,\end{aligned}$$

where $[\phi \circ \boldsymbol{u}^{-1}]$ denotes the jump of $\phi \circ \boldsymbol{u}^{-1}$, and analogously for $[\phi \circ \boldsymbol{u}^{-1}]$, and $J_{\boldsymbol{u}^{-1}}$ is the jump set of $\boldsymbol{u}^{-1}$. Taking suprema in ϕ and $\boldsymbol{g}$, part a) is proved.

Now we show b). If we assume that $\boldsymbol{u} \in \widetilde{\mathcal{B}}$ and $L(\boldsymbol{u}) < \infty$ then, by using (5.2), (5.1) and (2.5) we find that, for $\|D\phi\|_\infty \leq 1$ and $\|\boldsymbol{g}\|_\infty \leq 1$,

$$|\langle \phi \circ \boldsymbol{u}^{-1}, \operatorname{div} \boldsymbol{g} \rangle_{\mathcal{D}'(\Omega_{\boldsymbol{b}})}| \leq L(\boldsymbol{u}) + \|\nabla\boldsymbol{u}^{-1}\|_{L^1(\Omega_{\boldsymbol{b}})} \leq L(\boldsymbol{u}) + \frac{1}{\sqrt{3}}\|D\boldsymbol{u}\|^2_{L^2(\Omega)}.$$

Hence we find that $\phi \circ \boldsymbol{u}^{-1} \in BV(\Omega_{\boldsymbol{b}})$ for every $\phi \in C_c^1(\Omega)$ such that $\|D\phi\|_{L^\infty} \leq 1$. By choosing $\phi(\boldsymbol{x}) = x_i$ for $i = 1, 2, 3$ in an open set U such that $\widetilde{\Omega}_{\boldsymbol{b}} \Subset U \Subset \Omega_{\boldsymbol{b}}$, with x_i the i-th coordinate function, we see that $\boldsymbol{u}^{-1} \in BV(U, \mathbb{R}^3)$. As $\boldsymbol{u}^{-1} = \boldsymbol{b}^{-1}$ in $\Omega_{\boldsymbol{b}} \setminus \widetilde{\Omega}_{\boldsymbol{b}}$ we conclude that $\boldsymbol{u}^{-1} \in BV(\Omega_{\boldsymbol{b}}, \mathbb{R}^3)$. Thus, $\boldsymbol{u} \in \mathcal{B}$. We can then use (5.3) to say that $\|D^s(\phi \circ \boldsymbol{u}^{-1})\| \leq L(\boldsymbol{u})$ and thus, by using once more $\phi(\boldsymbol{x}) = x_i$ for $i = 1, 2, 3$, we find that

$$\|D^s\boldsymbol{u}^{-1}\| \leq \sum_{i=1}^3 \|D^s u_i^{-1}\| = \sum_{i=1}^3 |D^s u_i^{-1}|(U) \leq 3L(\boldsymbol{u}).$$

This conclues the proof of b). $\square$

Corollary 5.2. *Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\widetilde{\mathcal{B}}$. Assume that $\{\widetilde{F}(\boldsymbol{u}_j)\}_j$ is equibounded and that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$. Then $\boldsymbol{u}$ is in $\widetilde{\mathcal{B}}$ and $\det D\boldsymbol{u}_j \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\Omega)$.*

Proof. From Lemma 5.1 we obtain that $\{\boldsymbol{u}_j\}_j$ is in $\mathcal{B}$ and $\{F(\boldsymbol{u}_j)\}_j$ is equibounded. From Corollary 3.5 we obtain that the limit $\boldsymbol{u}$ is in $\mathcal{B} \subset \widetilde{\mathcal{B}}$. Finally, Proposition 3.1 vi) and Proposition 3.4 show that $\det D\boldsymbol{u}_j \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\Omega)$. $\square$

We now prove the lower semicontinuity of $\widetilde{F}$ in $\widetilde{\mathcal{B}}$.

Proposition 5.3. *Let $\{\boldsymbol{u}_j\}_j$ be a sequence in $\widetilde{\mathcal{B}}$ such that $\{\widetilde{F}(\boldsymbol{u}_j)\}_j$ is equibounded and such that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$ then*

$$\widetilde{F}(\boldsymbol{u}) \leq \liminf_{j \to +\infty} \widetilde{F}(\boldsymbol{u}_j).$$

Proof. By Corollary 5.2, the term $\int_\Omega H(\det D\boldsymbol{u}) \mathrm{d}\boldsymbol{x}$ is semicontinuous. We shall check the semicontinuity of $\|D\boldsymbol{u}\|^2_{L^2(\Omega)} + 2L(\boldsymbol{u})$. As the supremum of lower semicontinuous functions is lower semicontinuous, it suffices to check the semicontinuity of

$$\boldsymbol{u} \mapsto \|D\boldsymbol{u}\|^2_{L^2(\Omega)} + 2\mathcal{E}(\boldsymbol{u}, \phi\, \boldsymbol{g})$$

for each $\phi \in C^1_c(\Omega)$, $\boldsymbol{g} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$ such that $\|D\phi\|_\infty \leq 1$ and $\|\boldsymbol{g}\|_\infty \leq 1$. So take any such ϕ and $\boldsymbol{g}$, and let $\{\boldsymbol{u}_j\}_j$ be a sequence of maps in $\widetilde{\mathcal{B}}$ with $\{\widetilde{F}(\boldsymbol{u}_j)\}_j$ equibounded such that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in H^1 and, without loss of generality, $\boldsymbol{u}_j \to \boldsymbol{u}$ a.e. We have seen in Corollary 5.2 that $\det D\boldsymbol{u}_j \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\Omega)$. Hence thanks to [17, Proposition 2.61] we have that

$$D_j := \int_\Omega \det D\boldsymbol{u}_j \operatorname{div} \boldsymbol{g}(\boldsymbol{u}_j)\phi \, \mathrm{d}\boldsymbol{x} \to \int_\Omega \det D\boldsymbol{u} \operatorname{div} \boldsymbol{g}(\boldsymbol{u})\phi \, \mathrm{d}\boldsymbol{x}.$$

We now deal with the other term in $\mathcal{E}(\boldsymbol{u}, \phi\, \boldsymbol{g})$, which is

$$\int_\Omega (\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x}) D\phi(\boldsymbol{x})) \cdot \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} = \int_\Omega (\operatorname{adj} D\boldsymbol{u}(\boldsymbol{x}) \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))) \cdot D\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

We can see that

$$\operatorname{adj} D\boldsymbol{u}\, \boldsymbol{g}(\boldsymbol{u}) = (\partial_2 \boldsymbol{u} \wedge \partial_3 \boldsymbol{u} \cdot \boldsymbol{g}(\boldsymbol{u}), \partial_3 \boldsymbol{u} \wedge \partial_1 \boldsymbol{u} \cdot \boldsymbol{g}(\boldsymbol{u}), \partial_1 \boldsymbol{u} \wedge \partial_2 \boldsymbol{u} \cdot \boldsymbol{g}(\boldsymbol{u}))^T.$$

Set $\boldsymbol{v}_j := \boldsymbol{u}_j - \boldsymbol{u}$. Thanks to the bilinearity of the vectorial product, by writing $\boldsymbol{u}_j = \boldsymbol{u} + \boldsymbol{v}_j$ we can check that

$$\partial_2 \boldsymbol{u}_j \wedge \partial_3 \boldsymbol{u}_j = \partial_2 \boldsymbol{u} \wedge \partial_3 \boldsymbol{u} + \partial_2 \boldsymbol{u} \wedge \partial_3 \boldsymbol{v}_j + \partial_2 \boldsymbol{v}_j \wedge \partial_3 \boldsymbol{u} + \partial_2 \boldsymbol{v}_j \wedge \partial_3 \boldsymbol{v}_j.$$

Hence we can write

$$\int_\Omega (\operatorname{adj} D\boldsymbol{u}_j\, \boldsymbol{g}(\boldsymbol{u}_j)) \cdot D\phi \, \mathrm{d}\boldsymbol{x} = A_j + B_j + C_j$$

with

$$A_j = \int_\Omega \boldsymbol{g}(\boldsymbol{u}_j) \cdot (\partial_2 \boldsymbol{u} \wedge \partial_3 \boldsymbol{u} \partial_1 \phi + \partial_3 \boldsymbol{u} \wedge \partial_1 \boldsymbol{u} \partial_2 \phi + \partial_1 \boldsymbol{u} \wedge \partial_2 \boldsymbol{u} \partial_3 \phi) \, \mathrm{d}\boldsymbol{x},$$

$$B_j = \int_\Omega \boldsymbol{g}(\boldsymbol{u}_j) \cdot (\partial_2 \boldsymbol{v}_j \wedge \partial_3 \boldsymbol{u} + \partial_2 \boldsymbol{u} \wedge \partial_3 \boldsymbol{v}_j) \partial_1 \phi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{g}(\boldsymbol{u}_j) \cdot (\partial_3 \boldsymbol{v}_j \wedge \partial_1 \boldsymbol{u} + \partial_3 \boldsymbol{u} \wedge \partial_1 \boldsymbol{v}_j) \partial_2 \phi \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \boldsymbol{g}(\boldsymbol{u}_j) \cdot (\partial_1 \boldsymbol{v}_j \wedge \partial_2 \boldsymbol{u} + \partial_1 \boldsymbol{u} \wedge \partial_2 \boldsymbol{v}_j) \partial_3 \phi \, \mathrm{d}\boldsymbol{x},$$

$$C_j = \int_\Omega \boldsymbol{g}(\boldsymbol{u}_j) \cdot (\partial_2 \boldsymbol{v}_j \wedge \partial_3 \boldsymbol{v}_j \partial_1 \phi + \partial_3 \boldsymbol{v}_j \wedge \partial_1 \boldsymbol{v}_j \partial_2 \phi + \partial_1 \boldsymbol{v}_j \wedge \partial_2 \boldsymbol{v}_j \partial_3 \phi) \, \mathrm{d}\boldsymbol{x}.$$

Now we can see that $A_j \to \int_\Omega (\operatorname{adj} D\boldsymbol{u}\, \boldsymbol{g}(\boldsymbol{u})) \cdot D\phi \, \mathrm{d}\boldsymbol{x}$, since we have that $\boldsymbol{g}(\boldsymbol{u}_j) \to \boldsymbol{g}(\boldsymbol{u})$ a.e. and we can apply the dominated convergence theorem. Besides, we have that $B_j \to 0$ since $D\boldsymbol{v}_j \rightharpoonup \boldsymbol{0}$ in $L^2(\Omega)$ and $\boldsymbol{g}(\boldsymbol{u}_j) \to \boldsymbol{g}(\boldsymbol{u})$ in $L^2(\Omega)$.

Now we treat the term C_j. We express $C_j = \int_\Omega V_j \cdot D\phi \, \mathrm{d}\boldsymbol{x}$ with

$$V_j = (\boldsymbol{g}(\boldsymbol{u}_j) \cdot \partial_2 \boldsymbol{v}_j \wedge \partial_3 \boldsymbol{v}_j, \boldsymbol{g}(\boldsymbol{u}_j) \cdot \partial_3 \boldsymbol{v}_j \wedge \partial_1 \boldsymbol{v}_j, \boldsymbol{g}(\boldsymbol{u}_j) \cdot \partial_1 \boldsymbol{v}_j \wedge \partial_2 \boldsymbol{v}_j)^T.$$

By [8, Lemma 1], we have $|V_j| \leq \frac{1}{2}|D\boldsymbol{v}_j|^2$, and, hence, since $\|D\phi\|_\infty \leq 1$,

$$|C_j| = \left| \int_\Omega V_j \cdot D\phi \, \mathrm{d}\boldsymbol{x} \right| \leq \int_\Omega |V_j| \, \mathrm{d}\boldsymbol{x} \leq \frac{1}{2} \int_\Omega |D\boldsymbol{v}_j|^2 \, \mathrm{d}\boldsymbol{x}.$$

Thus we have seen that

$$\begin{aligned} \|D\boldsymbol{u}_j\|^2_{L^2(\Omega)} + 2\mathcal{E}(\boldsymbol{u}_j, \phi \boldsymbol{g}) &= \int_\Omega \left[|D\boldsymbol{u}|^2 + |D\boldsymbol{v}_j|^2 + 2D\boldsymbol{u} \cdot D\boldsymbol{v}_j \right] \mathrm{d}\boldsymbol{x} + 2\left(D_j + A_j + B_j + C_j\right) \\ &\geq \int_\Omega \left[|D\boldsymbol{u}|^2 + 2D\boldsymbol{u} \cdot D\boldsymbol{v}_j \right] \mathrm{d}\boldsymbol{x} + 2\left(D_j + A_j + B_j\right), \end{aligned}$$

so

$$\liminf_{j\to\infty} \left[\|D\boldsymbol{u}_j\|^2_{L^2(\Omega)} + 2\mathcal{E}(\boldsymbol{u}_j, \phi \boldsymbol{g}) \right] \geq \|D\boldsymbol{u}\|^2_{L^2(\Omega)} + 2\mathcal{E}(\boldsymbol{u}, \phi \boldsymbol{g}),$$

which gives the desired semicontinuity. □

As a consequence of Corollary 5.2 and Proposition 5.3, we obtain the existence of minimisers of $\widetilde{F}$ in $\widetilde{\mathcal{B}}$.

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(Marco Barchiesi) Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, Via Weiss 2 - 34128 Trieste, Italy.
Email address: barchies@gmail.com

(Duvan Henao) Instituto de Ciencias de la Ingeniería, Universidad de O'Higgins. Rancagua, Chile & Center for Mathematical Modeling.
Email address: duvan.henao@uoh.cl

(Carlos Mora-Corral) Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain.
Email address: carlos.mora@uam.es

(R. Rodiac) Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France & Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
Email address: rrodiac@mimuw.edu.pl