PLATEAU BORDERS IN SOAP FILMS AND GAUSS' CAPILLARITY THEORY

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Abstract. We provide, in the setting of Gauss' capillarity theory, a rigorous derivation of the equilibrium law for the three dimensional structures known as *Plateau borders* which arise in "wet" soap films and foams. A key step in our analysis is a complete measure-theoretic overhaul of the homotopic spanning condition introduced by Harrison and Pugh in the study of Plateau's laws for two-dimensional area minimizing surfaces ("dry" soap films). This new point of view allows us to obtain effective compactness theorems and energy representation formulae for the homotopic spanning relaxation of Gauss' capillarity theory which, in turn, lead to prove sharp regularity properties of energy minimizers. The equilibrium law for Plateau borders in wet foams is also addressed as a (simpler) variant of the theory for wet soap films.

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1. Introduction

1.1. **Overview.** Equilibrium configurations of soap films and foams are governed, at leading order, by the balance between surface tension forces and atmospheric pressure. This balance is expressed by the *Laplace–Young law of pressures*, according to which such systems can be decomposed into smooth interfaces with constant mean curvature equal to the pressure difference across them, and by the *Plateau laws*, which precisely postulate which arrangements of smooth interfaces joined together along lines of "singular" points are stable, and thus observable.

The physics literature identifies two (closely related) classes of soap films and foams, respectively labeled as "dry" and "wet". This difference is either marked in terms of the amount of liquid contained in the soap film/foam [WH99, Section 1.3], or in terms of the scale at which the soap film/foam is described [CCAE$^+$13, Chapter 2, Section 3 and 4].

In the dry case, Plateau laws postulates that (i) interfaces can only meet in three at a time forming 120-degree angles along lines of "Y-points"; and (ii) lines of Y-points can only

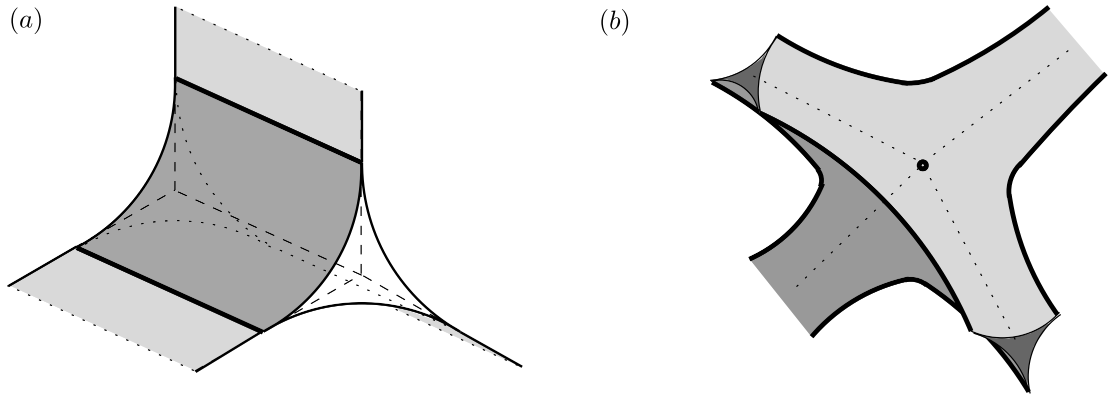

FIGURE 1.1. (a) A Plateau border develops around a "wet" line of Y-points. The wet region is bounded by interfaces of *negative* constant mean curvature. The equilibrium condition which needs to hold across the transition lines (here depicted in bold) between the negatively curved interfaces of a Plateau border and the incoming dry interfaces is that these interfaces meet tangentially. In the case of soap films, where the dry interfaces have zero mean curvature, the jump in the mean curvature across the transition lines implies a discontinuity in the gradient of the unit normal. (b) An arrangement of Plateau borders near a tetrahedral singularity. The transition lines are again depicted in bold. The incoming dry interfaces are omitted for clarity.

meet in fours at isolated "T-points", where six interfaces asymptotically form a perfectly symmetric tetrahedral angle; see, e.g. [WH99, Equilibrium rules A1, A2, page 24].

In the wet case, small but positive amounts of liquid are bounded by negatively curved interfaces, known as *Plateau borders*, and arranged near ideal lines of Y-points or isolated T-points; see Figure 1.1 and [WH99, Fig. 1.8 and Fig. 1.9]. A "third Plateau law" is then postulated to hold across the transition lines between wet and dry parts of soap films/foams, and can be formulated as follows:

$$\text{\textit{the unit normal to a soap film/foam changes continuously}} \atop \text{\textit{across the transition lines between wet and dry interfaces}}\,; \tag{1.1}$$

see, e.g., [WH99, Equilibrium rule B, page 25] and [CCAE$^+$13, Section 4.1.4]. It is important to recall that Plateau borders play a crucial role in determining the mechanical properties of the many physical and biological systems in which they are observed. As a sample of older and newer papers discussing Plateau borders, we mention here [LL65, JP92, BR97, KHS99, LC99, KHS00, GKJ05, SM15]. Postulate (1.1) is assumed in all these works.

The goal of this paper is answering the natural problem of rigorously deriving the equilibrium condition for Plateau borders (1.1) in the context of Gauss' capillarity theory. Since the case of soap films is much harder and interesting from the mathematical viewpoint, we will postpone the discussion of foams until the very last section of this introduction. The main highlight is that, in addressing Plateau borders of soap films, we will develop a new "theory of spanning" for surfaces of geometric measure theory (GMT) which will find further applications in the two companion papers [MNR23a, MNR23b]; see the closing of this overview for more details about these additional applications.

We now give an informal description of our approach. The starting point is [MSS19], where the idea is introduced of modeling soap films as regions E of positive volume $|E| = v$ contained in the complement $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ of a "wire frame" $\mathbf{W}$ ($n = 2$ is the physical case, although the planar case $n = 1$ is also quite interesting in applications). We

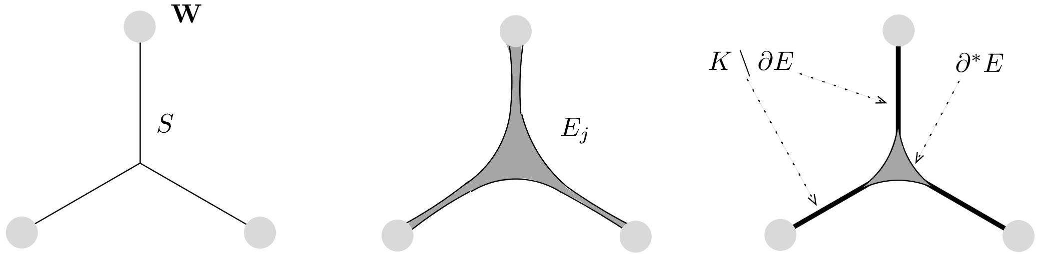

FIGURE 1.2. Emergence of collapsing along a minimizing sequence $\{E_j\}_j$ for the minimization of $\mathcal{H}^n(\Omega \cap \partial E)$ among sets $E \subset \Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ with $|E| = v$ and $\Omega \cap \partial E$ spanning $\mathbf{W}$, when $n = 1$ and $\mathbf{W}$ is the union of three disks in the plane. Notice that for this choice of $\mathbf{W}$ the minimization of $\mathcal{H}^n(S)$ among $S \subset \Omega$ such that S is spanning $\mathbf{W}$ is solved by three segments meeting at Y-point. Collapsing is intuitively related to the presence of Y-type and T-type singularities.

associate to E the surface tension energy $\mathcal{H}^n(\Omega \cap \partial E)$ (where $\mathcal{H}^n$ stands for n-dimensional (Hausdorff) measure, i.e., area when $n = 2$ and length when $n = 1$), and minimize $\mathcal{H}^n(\Omega \cap \partial E)$ under the constraints that $|E| = v$ (for some given $v > 0$) and

$$\Omega \cap \partial E \text{ is spanning } \mathbf{W} \,. \tag{1.2}$$

From the mathematical viewpoint the meaning assigned to (1.2) is, of course, the crux of the matter. In the informal spirit of this overview, we momentarily leave the concept of "spanning" only intuitively defined.

As proved in [KMS22a], this minimization process leads to the identification of *generalized minimizers* in the form of pairs (K, E) with $E \subset \Omega$, $|E| = v$, and such that

$$\Omega \cap \partial E \subset K \text{ and } K \text{ is spanning } \mathbf{W} \,. \tag{1.3}$$

These pairs are minimizing in the sense that

$$\mathcal{H}^n(\Omega \cap \partial E) + 2\,\mathcal{H}^n(K \setminus \partial E) \le \mathcal{H}^n(\Omega \cap \partial E') \,, \tag{1.4}$$

whenever $E' \subset \Omega$, $|E'| = v$ and $\Omega \cap \partial E'$ is spanning $\mathbf{W}$.

If $K = \Omega \cap \partial E$, then generalized minimizers are of course minimizers in the proper sense. If not, the *collapsed interface* $K \setminus \partial E$ is a surface whose positive area has to be counted with a multiplicity factor 2 (which arises from the asymptotic collapsing along $K \setminus \partial E$ of oppositely oriented boundaries in minimizing sequences $\{E_j\}_j$, see Figure 1.2). We expect collapsing to occur whenever the Plateau problem for $\mathbf{W}$ admits one minimizer S with Plateau-type singularities. Whenever this happens, a *wetting conjecture* is made: sequences $\{(K_{v_j}, E_{v_j})\}_j$ of generalized minimizers with $|E_{v_j}| = v_j \to 0^+$ as $j \to \infty$ will be such that the set of Plateau's singularities $\Sigma(S)$ of S is such that $\sup\{\mathrm{dist}(x, E_{v_j}) : x \in \Sigma(S)\} \to 0$. Thus we expect that Plateau's singularities are never "left dry" in the small volume capillarity approximation of the Plateau problem.

A lot of information about generalized minimizers can be extracted from (1.4), and this is the content of [KMS22a, KMS21, KMS22b]. With reference to the cases when $n = 1$ or $n = 2$, one can deduce from (1.4) that if $\mathcal{H}^n(K \setminus \partial E) > 0$, then $K \setminus \partial E$ is a smooth minimal surface (a union of segments if $n = 1$) and that ∂E contains a regular part $\partial^* E$ that is a smooth constant mean curvature surface (a union of circular arcs if $n = 1$) with *negative* curvature. This is of course strongly reminiscent of the behavior of Plateau borders, and invites to analyze the validity of (1.1) in this context. A main obstacle is that, due to serious technical issues (described in more detail later on) related to how minimality is expressed in (1.4), it turns out to be very difficult to say much about the "transition line"

$$\partial E \setminus \partial^* E$$

between the zero and the negative constant mean curvature interfaces in K, across which one should check the validity of (1.1). More precisely, all that descends from (1.4) and a direct application of Allard's regularity theorem [All72] is that *$\partial E\setminus\partial^* E$ has empty interior in K*. Far from being a line in dimension $n = 2$, or a discrete set of points when $n = 1$, the transition line $\partial E \setminus \partial^* E$ could very well have positive $\mathcal{H}^n$-measure and be everywhere dense in K! With such poor understanding of $\partial E \setminus \partial^* E$, proving the validity of (1.1) – that is, the continuity of the unit normals to $K \setminus \partial E$ and $\partial^* E$ in passing across $\partial E \setminus \partial^* E$ – is of course out of question.

We overcome these difficulties by performing a major measure-theoretic overhaul of the Harrison–Pugh homotopic spanning condition [HP16, HP17] used in [MSS19, KMS22a, KMS21, KMS22b] to give a rigorous meaning to (1.2), and thus to formulate the homotopic spanning relaxation of Gauss' capillarity discussed above.

The transformation of this purely topological concept into a measure-theoretic one is particularly powerful. Its most important consequence for the problem discussed in this paper is that it allows us to upgrade the partial minimality property (1.4) of (K, E) into the full minimality property

$$\mathcal{H}^n(\Omega \cap \partial E) + 2\,\mathcal{H}^n(K \setminus \partial E) \le \mathcal{H}^n(\Omega \cap \partial E') + 2\,\mathcal{H}^n(K' \setminus \partial E') \tag{1.5}$$

whenever $E' \subset \Omega$, $|E'| = v$, $\Omega \cap \partial E' \subset K'$ and K' is spanning $\mathbf{W}$. The crucial difference between (1.4) and (1.5) is that the latter is much more efficient than the former when it comes to study the regularity of generalized minimizers (K, E), something that is evidently done by energy comparison with competitors (K', E'). Such comparisons are immediate when working with (1.5), but they are actually quite delicate to set up when we only have (1.4). In the latter case, given a competitor (K', E'), to set up the energy comparison with (K, E) we first need to find a sequence of non-collapsed competitors $\{E'_j\}_j$ (with $E'_j \subset \Omega$, $|E'_j| = v$, and $\Omega\cap\partial E'_j$ spanning $\mathbf{W}$) such that $\mathcal{H}^n(\Omega\cap\partial E'_j) \to \mathcal{H}^n(\Omega\cap\partial E')+2\,\mathcal{H}^n(K'\setminus\partial E')$. Intuitively, E'_j needs to be a δ_j-neighborhood of $K' \cup E'$ for some $\delta_j \to 0^+$ and the energy approximation property has to be deduced from the theory of Minkowski content. But applying the theory of Minkowski content to (K', E') (which is the approach followed, e.g., in [KMS22b]) requires (K', E') to satisfy rectifiability and uniform density properties that substantially restrict the class of available competitors (K', E').

In contrast, once the validity of (1.5) is established, a suitable generalization (Theorem 1.2) of the partition theorem of sets of finite perimeter into indecomposable components [ACMM01, Theorem 1] combined with a subtle variational argument (see Figure 1.7) allows us to show that, in any ball $B \subset\subset \Omega$ with sufficiently small radius and for some sufficiently large constant Λ (both depending just on (K, E)), the connected components $\{U_i\}_i$ of $B \setminus (K \cup E)$ satisfy a perturbed area minimizing property of the form

$$\mathcal{H}^n(B \cap \partial U_i) \le \mathcal{H}^n(B \cap \partial V) + \Lambda\,|U_i \Delta V|\,, \tag{1.6}$$

with respect to *completely arbitrary perturbations* $V \subset B$, $V\Delta U_i \subset\subset B$. By a classical theorem of De Giorgi [DG60, Tam84], (1.6) implies (away from a closed singular set of codimension at least 8, which is thus empty if $n \le 6$) the $C^{1,\alpha}$-regularity of $B\cap\partial U_i$ for each i, and thus establishes *the continuity of the normal stated in* (1.1). In fact, locally at each x on the transition line, K is the union of the graphs of two $C^{1,\alpha}$-functions $u_1 \le u_2$ defined on an n-dimensional disk, having zero mean curvature above the interior of $\{u_1 = u_2\}$, and opposite constant mean curvature above $\{u_1 < u_2\}$. We can thus exploit the regularity theory for double-membrane free boundary problems devised in [Sil05, FGS15] to deduce that the transition line $\partial E \setminus \partial^* E$ is indeed $(n-1)$-dimensional, and to improve the $C^{1,\alpha}$-regularity of $B \cap \partial U_i$ to $C^{1,1}$-regularity. Given the mean curvature jump across $\partial E \setminus \partial^* E$ we have thus established the *sharp* degree of regularity for minimizers of the homotopic spanning relaxation of Gauss' capillarity theory.

The measure-theoretic framework for homotopic spanning conditions laid down in this paper provides the starting point for additional investigations that would otherwise seem unaccessible. In two forthcoming companion papers we indeed establish (i) the convergence towards Plateau-type singularities of energy-minimizing diffused interface solutions of the Allen–Cahn equation [MNR23a], and (ii) some sharp convergence theorems for generalized minimizers in the homotopic spanning relaxation of Gauss' capillarity theory in the vanishing volume limit, including a proof of the above mentioned wetting conjecture [MNR23b].

The rest of this introduction is devoted to a rigorous formulation of the results presented in this overview. We begin in Section 1.2 with a review of the Harrison and Pugh homotopic spanning condition in relation to the classical Plateau problem and to the foundational work of Almgren and Taylor [Alm76, Tay76]. In Section 1.3 we introduce the new measure-theoretic formulation of homotopic spanning and discuss its relation to the measure-theoretic notion of *essential connectedness* introduced by Cagnetti, Colombo, De Philippis and the first-named author in the study of symmetrization inequalities [CCDPM17, CCDPM14]. In Section 1.4 we introduce the *bulk* and *boundary* spanning relaxations of Gauss' capillarity theory, state a general closure theorem for "generalized soap films" that applies to both relaxed problems (Theorem 1.4). In Section 1.5 we prove the existence of generalized soap film minimizers (Theorem 1.5) and their convergence in energy to solutions to the Plateau problem. A sharp regularity theorem (Theorem 1.6) for these minimizers, which validates (1.1), is stated in Section 1.6. Finally, in Section 1.7 we reformulate the above results in the case of foams, see in particular Theorem 1.7.

1.2. **Homotopic spanning: from Plateau's problem to Gauss' capillarity.** The theories of currents and of sets of finite perimeter, i.e. the basic distributional theories of surface area at the center of GMT, fall short in the task of modeling Plateau's laws. Indeed, two-dimensional area minimizing currents in $\mathbb{R}^3$ are carried by smooth minimal surfaces, and thus cannot model Y-type[1] and T-type singularities. This basic issue motivated the introduction of **Almgren minimal sets** as models for soap films in [Alm76]: these are sets $S \subset \mathbb{R}^{n+1}$ that are relatively closed in a given open set $\Omega \subset \mathbb{R}^{n+1}$, and satisfy $\mathcal{H}^n(S) \le \mathcal{H}^n(f(S))$ whenever $f : \Omega \to \Omega$ is a *Lipschitz* (not necessarily injective) map with $\{f \ne \mathrm{id}\} \subset\subset \Omega$. Taylor's historical result [Tay76] validates the Plateau laws in this context, by showing that, when[2] $n = 2$, Almgren minimal sets are locally $C^{1,\alpha}$-diffeomorphic either to planes, to Y-cones, or to T-cones.

The issue of proposing and solving a formulation of Plateau's problem whose minimizers are Almgren minimal sets, and indeed admit Plateau-type singularities, is quite elusive, as carefully explained in [Dav14]. In this direction, a major breakthrough has been obtained by Harrison and Pugh in [HP16] with the introduction of a new spanning condition, which, following the presentation in [DLGM17a], can be defined as follows:

Definition A (Homotopic spanning (on closed sets)). Given a closed set $\mathbf{W} \subset \mathbb{R}^{n+1}$ (the "wire frame"), a **spanning class for W** is a family $\mathcal{C}$ of smooth embeddings of $\mathbb{S}^1$ into

$$\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$$

[1] Currents modulo 3 are compatible with Y-type singularities, but not with T-type singularities.

[2] Similar regularity assertions hold when $n = 1$ (by elementary methods) and, in much more recent developments, when $n \ge 3$ [CES22].

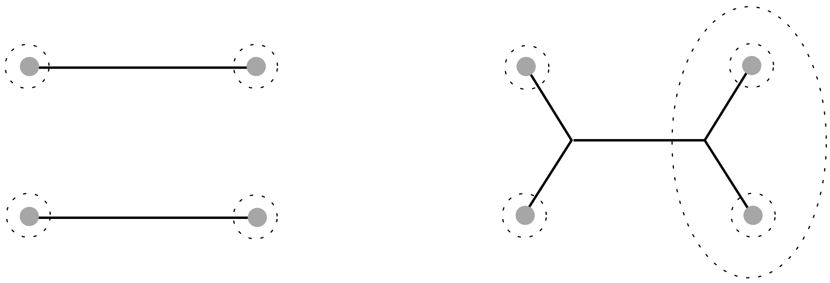

FIGURE 1.3. The dashed lines denote the embeddings of $\mathbb{S}^1$ whose homotopy classes relative to Ω generate different spanning classes $\mathcal{C}$, to which there correspond different minimizers of ℓ.

that is *closed under homotopies in* Ω, that is, if $\Phi : [0,1] \times \mathbb{S}^1 \to \Omega$ is smooth family of embeddings $\Phi_t = \Phi(t, \cdot) : \mathbb{S}^1 \to \Omega$ with $\Phi_0 \in \mathcal{C}$, then $\Phi_t \in \mathcal{C}$ for every $t \in (0,1]$. A set S, contained and relatively closed in Ω, is said to be **$\mathcal{C}$-spanning W** if

$$S \cap \gamma \neq \varnothing\,, \qquad \forall \gamma \in \mathcal{C}\,.$$

Denoting by $\mathcal{S}(\mathcal{C})$ the class of sets S $\mathcal{C}$-spanning $\mathbf{W}$, one can correspondingly formulate the **Plateau problem** (with homotopic spanning)

$$\ell = \ell(\mathcal{C}) := \inf \big\{ \mathcal{H}^n(S) : S \in \mathcal{S}(\mathcal{C}) \big\}\,. \tag{1.7}$$

Existence of minimizers of ℓ holds as soon as $\ell < \infty$, and minimizers S of ℓ are Almgren minimal sets in Ω [HP16, DLGM17a] that are indeed going to exhibit Plateau-type singularities (this is easily seen in the plane, but see also [BM21] for a higher dimensional example). Moreover, given a same $\mathbf{W}$, different choices of $\mathcal{C}$ are possible and can lead to different minimizers, see Figure 1.3. Finally, the approach is robust enough to provide the starting point for several important extensions [DPDRG16, DR18, HP17, FK18, DLDRG19, DPDRG20], including higher codimension, anisotropic energies, etc.

The study of soap films as minimizers of Gauss's capillarity energy with small volume and under homotopic spanning conditions has been initiated in [MSS19, KMS22a], with the introduction of the model

$$\psi(v) := \inf \Big\{ \mathcal{H}^n(\Omega \cap \partial E) : |E| = v\,,\ \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \Big\}\,, \tag{1.8}$$

where $E \subset \Omega$ is an open set with smooth boundary. Without the spanning condition, at small volumes, minimizers of $\mathcal{H}^n(\Omega \cap \partial E)$ would be small diffeomorphic images of half-balls [MM16]. However, the introduction of the $\mathcal{C}$-spanning constraint rules out small droplets, and forces the exploration of a different part of the energy landscape of $\mathcal{H}^n(\Omega \cap \partial E)$. As informally discussed in Section 1.1, this leads to the emergence of generalized minimizers (K, E). More precisely, in [KMS22a] the existence is proved of (K, E) in the class

$$\begin{aligned} \mathcal{K} = \Big\{ (K, E) : & \ K \text{ is relatively closed and } \mathcal{H}^n\text{-rectifiable in } \Omega,\ E \text{ is open,} \\ & E \text{ has finite perimeter in } \Omega, \text{ and } \Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K \Big\}\,, \end{aligned} \tag{1.9}$$

(where $\partial^* E$ denotes the reduced boundary of E) such that, for every competitor E' in $\psi(v)$, it holds

$$\mathcal{H}^n(\Omega \cap \partial^* E) + 2\,\mathcal{H}^n(\Omega \cap (K \setminus \partial^* E)) \le \mathcal{H}^n(\Omega \cap \partial E')\,. \tag{1.10}$$

Starting from (1.10) one can apply Allard's regularity theorem [All72] and various *ad hoc* comparison arguments [KMS21, KMS22b] to prove that $\Omega \cap \partial^* E$ is a smooth hypersurface with constant mean curvature (negative if $\mathcal{H}^n(K \setminus \partial^* E) > 0$), $\Omega \cap (\partial E \setminus \partial^* E)$ has empty

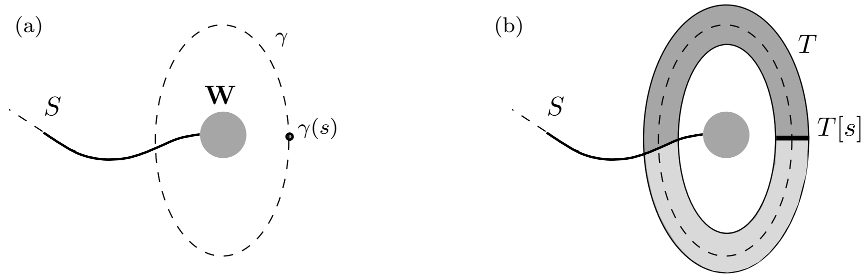

FIGURE 1.4. (a) Homotopic spanning according to Harrison–Pugh: S must intersect every curve $\gamma \in \mathcal{C}$, in particular, the $\mathcal{C}$-spanning property may be lost by removing a single point from S; (b) Homotopic spanning based on essential connectedness: for a.e. section $T[s]$ of the tube T around a curve $\gamma \in \mathcal{C}$, the union $T[s] \cup S$ (essentially) disconnects T (i.e., divides T into two non-trivial parts, depicted here with two different shades of gray).

interior in K, and that $K \setminus (\Sigma \cup \partial E)$ is a smooth minimal hypersurface, where Σ is a closed set with codimension at least 8.

1.3. **Measure theoretic homotopic spanning.** In a nutshell, the idea behind our measure theoretic revision the Harrison–Pugh homotopic spanning condition is the following. Rather than asking that $S \cap \gamma(\mathbb{S}^1) \neq \varnothing$ for every $\gamma \in \mathcal{C}$, as done in Definition A, we shall replace γ with an open "tube" T containing $\gamma(\mathbb{S}^1)$, and ask that S, with the help of a generic "slice" $T[s]$ of T, "disconnects" T itself into two nontrivial regions T_1 and T_2; see Figure 1.4. The key to make this idea work is, of course, giving a proper meaning to the word "disconnects".

To this end, we recall the notion of **essential connectedness** introduced in [CCDPM17, CCDPM14] in the study of the rigidity of equality cases in Gaussian and Euclidean perimeter symmetrization inequalities. Essential connectedness is the "right" notion to deal with such problems since it leads to the formulation of sharp rigidity theorems, and can indeed be used to address other rigidity problems (see [CPS20, Per22, Dom23]). This said, it seems remarkable that the very same notion of what it means for "one Borel set to disconnect another Borel set" proves to be extremely effective also in the context of the present paper, which is of course very far from the context of symmetrization theory.

Denoting by $T^{(t)}$ ($0 \le t \le 1$) the **points of density** t of a Borel set $T \subset \mathbb{R}^{n+1}$ (i.e., $x \in T^{(t)}$ if and only if $|T \cap B_r(x)|/\omega_{n+1}\, r^{n+1} \to t$ as $r \to 0^+$, where ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$), and by $\partial^e T = \mathbb{R}^{n+1} \setminus (T^{(0)} \cup T^{(1)})$ the **essential boundary** of T, given Borel sets S, T, T_1 and T_2 in $\mathbb{R}^{n+1}$, and given $n \ge 0$, we say that S **essentially disconnects** T **into** $\{T_1, T_2\}$, if

$$\begin{aligned} &\{T_1, T_2\} \text{ is a non-trivial Borel partition of } T\,, \\ &\text{and } T^{(1)} \cap \partial^e T_1 \cap \partial^e T_2 \text{ is } \mathcal{H}^n\text{-contained in } S\,. \end{aligned} \tag{1.11}$$

(For example, if K is a set of full $\mathcal{L}^1$-measure in $[-1, 1]$, then $S = K \times \{0\}$ essentially disconnects the unit disk in $\mathbb{R}^2$.) Moreover, we say that T is **essentially connected**[3] if $\varnothing$ does not essentially disconnect T. The requirement that $\{T_1, T_2\}$ is a non-trivial Borel partition of T means that $|T \Delta (T_1 \cup T_2)| = 0$ and $|T_1|\,|T_2| > 0$. By saying that "E is $\mathcal{H}^n$-contained in F" we mean that $\mathcal{H}^n(E \setminus F) = 0$. We also notice that, in (1.11), we have $T^{(1)} \cap \partial^e T_1 \cap \partial^e T_2 = T^{(1)} \cap \partial^e T_i$ ($i = 1, 2$), a fact that is tacitly and repeatedly considered in the use of (1.11) in order to shorten formulas.

[3]Whenever T is of locally finite perimeter, being essentially connected is equivalent to being indecomposable.

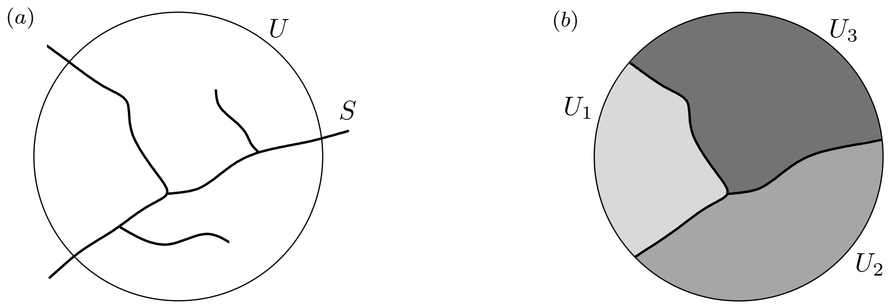

FIGURE 1.5. An example of induced essential partition. The union of the boundaries of the U_i's (inside of U) is contained in S, and the containment may be strict. However, the part of S not contained in $U \cap \bigcup_i \partial U_i$ is not such to disconnect any of the U_i's. In particular, each U_i is essentially connected.

With this terminology in mind, we introduce the following definition:

Definition B (Measure theoretic homotopic spanning)**.** Given a closed set $\mathbf{W}$ and a spanning class $\mathcal{C}$ for $\mathbf{W}$, the **tubular spanning class** $\mathcal{T}(\mathcal{C})$ associated to $\mathcal{C}$ is the family of triples (γ, Φ, T) such that $\gamma \in \mathcal{C}$, $T = \Phi(\mathbb{S}^1 \times B_1^n)$, and[4]

$$\Phi : \mathbb{S}^1 \times \mathrm{cl}\, B_1^n \to \Omega \text{ is a diffeomorphism with } \Phi|_{\mathbb{S}^1 \times \{0\}} = \gamma\,.$$

When $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, the **slice of** T defined by $s \in \mathbb{S}^1$ is

$$T[s] = \Phi(\{s\} \times B_1^n)\,.$$

Finally, we say that a Borel set $S \subset \Omega$ is **$\mathcal{C}$-spanning W** if for each $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$ has the following property:

$$\begin{aligned} &\text{for } \mathcal{H}^n\text{-a.e. } x \in T[s] \\ &\exists \text{ a partition } \{T_1, T_2\} \text{ of } T \text{ s.t. } x \in \partial^e T_1 \cap \partial^e T_2 \\ &\text{and s.t. } S \cup T[s] \text{ essentially disconnects } T \text{ into } \{T_1, T_2\}\,. \end{aligned} \tag{1.12}$$

Before commenting on (1.12), we notice that the terminology of Definition B is coherent with that of Definition A thanks to the following theorem.

Theorem 1.1. *Given a closed set $\mathbf{W} \subset \mathbb{R}^{n+1}$, a spanning class $\mathcal{C}$ for $\mathbf{W}$, and a set S relatively closed in Ω, then S is $\mathcal{C}$-spanning $\mathbf{W}$ in the sense of Definition A if and only if S is $\mathcal{C}$-spanning $\mathbf{W}$ in the sense of Definition B.*

Theorem 1.1 is proved in Appendix A. There we also comment on the delicate reason why, in formulating (1.12), the partition $\{T_1, T_2\}$ must be allowed to depend on specific points $x \in T[s]$. This would not seem necessary by looking at the simple situation depicted in Figure 1.4, but it is actually so when dealing with more complex situations; see Figure A.1.

Homotopic spanning according to Definition B is clearly stable under modifications of S by $\mathcal{H}^n$-negligible sets, but there is more to it. Indeed, even a notion like "$\mathcal{H}^n(S \cap T) > 0$ for every $T \in \mathcal{T}(\mathcal{C})$" would be stable under modifications by $\mathcal{H}^n$-negligible sets, and would probably look more appealing in its simplicity. The catch, of course, is finding an extension of Definition A for which compactness theorems, like Theorem 1.4 below, hold true. This is evidently not the case, for example, if one tries to work with a notion like "$\mathcal{H}^n(S \cap T) > 0$ for every $T \in \mathcal{T}(\mathcal{C})$".

The first key insight on Definition B is that, if restricted to Borel sets S that are locally $\mathcal{H}^n$-finite in Ω, then it can be reformulated in terms of partitions into indecomposable

[4]Here $B_1^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\mathbb{S}^1 = \{s \in \mathbb{R}^2 : |s| = 1\}$.

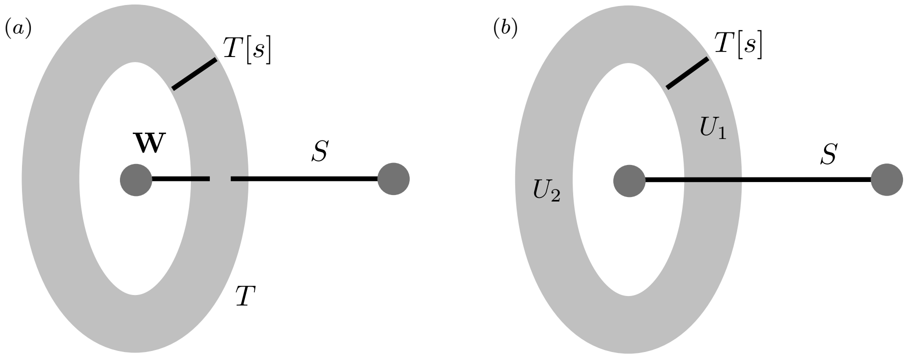

FIGURE 1.6. With $\mathbf{W}$ consisting of two disks in the plane, and T a test tube for the $\mathcal{C}$-spanning condition: (a) S consists of a segment with a gap: since the gap is inside of T, the essential partition of T induced by $S \cup T[s]$ consists of only one set, $U_1 = T$, so that $T \cap \partial^* U_1 = \varnothing$ and (1.14) cannot hold; (b) S consists of a full segment and in this case (with the possible exception of a choice of s such that $T[s]$ is contained in S), the essential partition of T induced by $S \cup T[s]$ consists of two sets $\{U_1, U_2\}$, such that $T[s] \subset T \cap \partial^* U_1 \cap \partial^* U_2$; in this case (1.14) holds.

sets of finite perimeter. This is the content of the following theorem, whose case $S = \varnothing$ corresponds to the standard decomposition theorem for sets of finite perimeter [ACMM01, Theorem 1]. For an illustration of this result, see Figure 1.5.

Theorem 1.2 (Induced essential partitions (Section 2))**.** *If $U \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter and $S \subset \mathbb{R}^{n+1}$ is a Borel set with $\mathcal{H}^n(S \cap U^{(1)}) < \infty$, then there exists a unique*[5] **essential partition** *$\{U_i\}_i$* **of** *U* **induced by** *S, that is to say, $\{U_i\}_i$ is a countable partition of U modulo Lebesgue negligible sets such that, for each i, S does not essentially disconnect U_i.*

Given U and S as in the statement of Theorem 1.2 we can define[6] the **union of the** (reduced) **boundaries** (relative to U) **of the essential partition** induced by S on U by setting[7]

$$\mathrm{UBEP}(S;U) = U^{(1)} \cap \bigcup_i \partial^* U_i \,. \tag{1.13}$$

Two properties of UBEP's which well illustrate the concept are: first, if $\mathcal{R}(S)$ denotes the rectifiable part of S, then $\mathrm{UBEP}(S;U)$ is $\mathcal{H}^n$-equivalent to $\mathrm{UBEP}(\mathcal{R}(S);U)$; second, if S^* is $\mathcal{H}^n$-contained in S, then $\mathrm{UBEP}(S;U)$ is $\mathcal{H}^n$-contained in $\mathrm{UBEP}(S;U)$; both properties are proved in Theorem 2.1 (an expanded restatement of Theorem 1.2).

We can use the concepts just introduced to provide an alternative and technically more workable characterization of homotopic spanning in the measure theoretic setting. This is the content of our first main result, which is illustrated in Figure 1.6.

Theorem 1.3 (Homotopic spanning for locally $\mathcal{H}^n$-finite sets (Section 3))**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is a closed set in $\mathbb{R}^{n+1}$, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and $S \subset \Omega$ is locally $\mathcal{H}^n$-finite in Ω, then S is $\mathcal{C}$-spanning $\mathbf{W}$ if and only if for every $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ we have that, for $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$,*

$$T[s] \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(S \cup T[s]; T) \,. \tag{1.14}$$

[5] Uniqueness is meant modulo relabeling and modulo Lebesgue negligible modifications of the U_i's.

[6] Uniquely modulo $\mathcal{H}^n$-null sets thanks to Federer's theorem recalled in (1.37) below.

[7] Given a Borel set E, we denote by $\partial^* E$ its reduced boundary relative to the maximal open set A wherein E has locally finite perimeter.

1.4. **Direct Method on generalized soap films and Gauss' capillarity.** The most convenient setting for addressing the existence of minimizers in Gauss' capillarity theory is of course that of sets of finite perimeter [Fin86, Mag12]. However, if the notion of homotopic spanning is limited to closed sets, as it is the case when working with Definition A, then one cannot directly use homotopic spanning on sets of finite perimeter, and this is the reason behind the specific formulation (1.8) of $\psi(v)$ used in [MSS19, KMS22a]. Equipped with Definition B we can now formulate Gauss' capillarity theory with homotopic spanning conditions directly on sets of finite perimeter. We shall actually consider *two* different possible formulations

$$\psi_{\rm bk}(v) = \inf\Big\{\mathcal{H}^n(\Omega\cap\partial^*E) : |E| = v \text{ and } \Omega\cap(\partial^*E\cup E^{(1)}) \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,,$$

$$\psi_{\rm bd}(v) = \inf\Big\{\mathcal{H}^n(\Omega\cap\partial^*E) : |E| = v \text{ and } \Omega\cap\partial^*E \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,,$$

where the subscripts "bk" and "bd" stand to indicate that the spanning is prescribed via the *bulk* of E (that is, in measure theoretic terms, via the set $\Omega\cap(\partial^*E\cup E^{(1)})$ or via the (reduced) boundary of E. Inspired by the definition of the class $\mathcal{K}$ introduced in (1.9), we also introduce the class $\mathcal{K}_{\rm B}$ of **generalized soap films** defined by

$$\mathcal{K}_{\rm B} = \Big\{(K,E) : K \text{ and } E \text{ are Borel subsets of } \Omega,\qquad E \text{ has locally finite perimeter in } \Omega \text{ and } \partial^*E\cap\Omega \overset{\mathcal{H}^n}{\subset} K\Big\}\,. \tag{1.15}$$

Here the subscript "B" stands for "Borel", and $\mathcal{K}_{\rm B}$ stands as a sort of measure-theoretic version of $\mathcal{K}$.

In the companion paper [Nov23] the following relaxation formulas for problems $\psi_{\rm bk}$ and $\psi_{\rm bd}$ are proved,

$$\psi_{\rm bk}(v) = \Psi_{\rm bk}(v)\,,\qquad \psi_{\rm bd}(v) = \Psi_{\rm bd}(v)\,,\qquad \forall v>0\,, \tag{1.16}$$

where the following minimization problems on $\mathcal{K}_{\rm B}$ are introduced

$$\Psi_{\rm bk}(v) = \inf\Big\{\mathcal{F}_{\rm bk}(K,E) : (K,E)\in\mathcal{K}_{\rm B}\,, |E| = v\,, K\cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,, \tag{1.17}$$

$$\Psi_{\rm bd}(v) = \inf\Big\{\mathcal{F}_{\rm bd}(K,E) : (K,E)\in\mathcal{K}_{\rm B}\,, |E| = v\,, K \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,. \tag{1.18}$$

Here $\mathcal{F}_{\rm bk}$ and $\mathcal{F}_{\rm bd}$ are the relaxed energies defined for $(K,E)\in\mathcal{K}_{\rm B}$ and $A\subset\Omega$ as

$$\mathcal{F}_{\rm bk}(K,E;A) = 2\,\mathcal{H}^n(A\cap K\cap E^{(0)}) + \mathcal{H}^n(A\cap\partial^*E)\,, \tag{1.19}$$

$$\mathcal{F}_{\rm bd}(K,E;A) = 2\,\mathcal{H}^n(A\cap K\setminus\partial^*E) + \mathcal{H}^n(A\cap\partial^*E)\,, \tag{1.20}$$

(We also set, for brevity, $\mathcal{F}_{\rm bk}(K,E) := \mathcal{F}_{\rm bk}(K,E;\Omega)$ and $\mathcal{F}_{\rm bd}(K,E) := \mathcal{F}_{\rm bd}(K,E;\Omega)$.) We refer to these problems, respectively, as the "bulk-spanning" or "boundary-spanning" Gauss' capillarity models. In this paper we shall directly work with these relaxed models. In particular, the validity of (1.16), although of definite conceptual importance, is not playing any formal role in our deductions.

A first remark concerning the advantage of working with the relaxed problems $\Psi_{\rm bk}$ and $\Psi_{\rm bd}$ rather than with their "classical" counterparts $\psi_{\rm bk}$ and $\psi_{\rm bd}$ is that while the latter two with $v=0$ are trivial (sets with zero volume have zero distributional perimeter), the problems $\Psi_{\rm bk}(0)$ and $\Psi_{\rm bd}(0)$ are actually non-trivial, equal to each other, and amount to a measure-theoretic version of the Harrison–Pugh formulation of Plateau's problem ℓ introduced in (1.7): more precisely, if we set

$$\ell_{\rm B} := \frac{\Psi_{\rm bk}(0)}{2} = \frac{\Psi_{\rm bd}(0)}{2} = \inf\Big\{\mathcal{H}^n(S) : S \text{ is a Borel set } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,, \tag{1.21}$$

then, by Theorem 1.1, we evidently have $\ell_{\rm B}\le\ell$; and, as we shall prove in the course of our analysis, we actually have that $\ell=\ell_{\rm B}$ as soon as $\ell<\infty$.

Our second main result concerns the applicability of the Direct Method on the competition classes of $\Psi_{\rm bk}(v)$ and $\Psi_{\rm bd}(v)$.

Theorem 1.4 (Direct Method for generalized soap films (Sections 4 and 5))**.** *Let* $\mathbf{W}$ *be a closed set in* $\mathbb{R}^{n+1}$*,* $\mathcal{C}$ *a spanning class for* $\mathbf{W}$*,* $\{(K_j,E_j)\}_j$ *be a sequence in* $\mathcal{K}_{\rm B}$ *such that* $\sup_j \mathcal{H}^n(K_j)<\infty$*, and let a Borel set* E *and Radon measures* $\mu_{\rm bk}$ *and* $\mu_{\rm bd}$ *in* Ω *be such that* $E_j \stackrel{\rm loc}{\to} E$ *and*

$$\mathcal{H}^n \llcorner (\Omega\cap\partial^*E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j)\cap E_j^{(0)}) \stackrel{*}{\rightharpoonup} \mu_{\rm bk}\,,$$
$$\mathcal{H}^n \llcorner (\Omega\cap\partial^*E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j)\setminus\partial^*E_j) \stackrel{*}{\rightharpoonup} \mu_{\rm bd}\,,$$

as $j\to\infty$*. Then:*

(i) Lower semicontinuity: *the sets*

$$K_{\rm bk} := \Big(\Omega\cap\partial^*E\Big)\cup\Big\{x\in\Omega\cap E^{(0)}:\theta^n_*(\mu_{\rm bk})(x)\ge 2\Big\}\,,$$
$$K_{\rm bd} := \Big(\Omega\cap\partial^*E\Big)\cup\Big\{x\in\Omega\setminus\partial^*E:\theta^n_*(\mu_{\rm bd})(x)\ge 2\Big\}\,,$$

are such that $(K_{\rm bk},E),(K_{\rm bd},E)\in\mathcal{K}_{\rm B}$ *and*

$$\mu_{\rm bk} \ge \mathcal{H}^n \llcorner (\Omega\cap\partial^*E) + 2\,\mathcal{H}^n \llcorner (K_{\rm bk}\cap E^{(0)})\,,$$
$$\mu_{\rm bd} \ge \mathcal{H}^n \llcorner (\Omega\cap\partial^*E) + 2\,\mathcal{H}^n \llcorner (K_{\rm bd}\setminus \partial^*E)\,,$$

with

$$\liminf_{j\to\infty}\mathcal{F}_{\rm bk}(K_j,E_j)\ge\mathcal{F}_{\rm bk}(K_{\rm bk},E)\,,\qquad \liminf_{j\to\infty}\mathcal{F}_{\rm bd}(K_j,E_j)\ge\mathcal{F}_{\rm bd}(K_{\rm bd},E)\,.$$

(ii) Closure: *we have that*

if $K_j\cup E_j^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for every* j*,*
then $K_{\rm bk}\cup E^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*,*

and that

if K_j *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for every* j*,*
then $K_{\rm bd}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*.*

The delicate part of Theorem 1.4 is proving the closure statements. This will require first to extend the characterization of homotopic spanning from locally $\mathcal{H}^n$-finite sets to generalized soap films (Theorem 3.1), and then to discuss the behavior under weak-star convergence of the associated Radon measures of the objects appearing in conditions like (1.14) (Theorem 4.1).

1.5. **Existence of minimizers in $\Psi_{\rm bk}(v)$ and convergence to ℓ.** From this point onward, we focus our analysis on the bulk-spanning relaxation $\Psi_{\rm bk}(v)$ of Gauss' capillarity. There are a few important reasons for this choice: (i) from the point of view of physical modeling, working with the boundary or with the bulk spanning conditions seem comparable; (ii) the fact that $\Psi_{\rm bk}(0)=\Psi_{\rm bd}(0)$ suggest that, at small values of v, the two problems should actually be equivalent (have the same infima and the same minimizers); (iii) the bulk spanning variant is the one which is relevant for the approximation of Plateau-type singularities with solutions of the Allen–Cahn equations discussed in [MNR23a]; (iv) despite their similarities, carrying over the following theorems for both problems would require the repeated introduction of two versions of many arguments, with a significant increase in length, and possibly with at the expense of clarity.

The following theorem provides the starting point in the study of $\Psi_{\rm bk}(v)$.

Theorem 1.5 (Existence of minimizers and vanishing volume limit for $\Psi_{\rm bk}$ (Section 6))**.** *If* $\mathbf{W}$ *is a compact set in* $\mathbb{R}^{n+1}$ *and* $\mathcal{C}$ *is a spanning class for* $\mathbf{W}$ *such that* $\ell < \infty$*, then*

$$\ell_{\rm B} = \ell\,, \tag{1.22}$$

and, moreover:

(i) Existence of minimizers and Euler–Lagrange equation: *for every* $v > 0$ *there exist minimizers* (K,E) *of* $\Psi_{\rm bk}(v)$ *such that* $(K,E) \in \mathcal{K}$ *and both* E *and* K *are bounded; moreover, there is* $\lambda \in \mathbb{R}$ *such that*

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} {\rm div}^K X \, d\mathcal{H}^n + 2 \int_{K \cap E^{(0)}} {\rm div}^K X \, d\mathcal{H}^n\,, \tag{1.23}$$

for every $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ *with* $X \cdot \nu_\Omega = 0$ *on* $\partial\Omega$*;*

(ii) Regularity from the Euler–Lagrange equations: *if* $(K,E) \in \mathcal{K}$ *is a minimizer of either* $\Psi_{\rm bk}(v)$*, then there is a closed set* $\Sigma \subset K$*, with empty interior in* K*, such that* $K \setminus \Sigma$ *is a smooth hypersurface; moreover,* $K \setminus (\Sigma \cup \partial E)$ *is a smooth minimal hypersurface,* $\Omega \cap \partial^* E$ *is a smooth hypersurface with mean curvature constantly equal to* λ*, and* $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$*; in particular,* $\Omega \cap (\partial E \setminus \partial^* E)$ *has empty interior in* K*;*

(iii) Convergence to the Plateau problem: *if* (K_j, E_j) *is a sequence of minimizers for* $\Psi_{\rm bk}(v_j)$ *with* $v_j \to 0^+$*, then there exists a minimizer* S *of* ℓ *such that, up to extracting subsequences, as Radon measures in* Ω*,*

$$\mathcal{H}^n \llcorner (\partial^* E_j \cap \Omega) + 2\mathcal{H}^n \llcorner (K_j \cap E_j^{(0)}) \stackrel{*}{\rightharpoonup} 2\mathcal{H}^n \llcorner S\,, \tag{1.24}$$

as $j \to \infty$*; In particular,* $\Psi_{\rm bk}(v) \to 2\,\ell = \Psi_{\rm bk}(0)$ *as* $v \to 0^+$*.*

The conclusions of Theorem 1.5 about $\Psi_{\rm bk}(v)$ can be read in parallel to the conclusions about $\psi(v)$ obtained in [KMS22a]. The crucial difference is that, in place of the "weak" minimality inequality (1.10), which in this context would be equivalent to $\mathcal{F}_{\rm bk}(K,E) \le \mathcal{H}^n(\Omega \cap \partial^* E')$ for every competitor E' in $\psi_{\rm bk}(v)$, we now have the proper minimality inequality

$$\mathcal{F}_{\rm bk}(K,E) \le \mathcal{F}_{\rm bk}(K',E') \tag{1.25}$$

for every competitor (K',E') in $\Psi_{\rm bk}(v)$. Not only the final conclusion is stronger, but the proof is also entirely different: whereas [KMS22a] required the combination of a whole bestiary of specific competitors (like the cup, cone, and slab competitors described therein) with the full force of Preiss' theorem, the approach presented here seems more robust as it does not exploit any specific geometry, and it is squarely rooted in the basic theory of sets of finite perimeter.

1.6. **Equilibrium across transition lines in wet soap films.** We now formalize the validation of (1.1) for soap films in the form of a sharp regularity theorem for minimizers (K,E) of $\Psi_{\rm bk}(v)$.

The starting point to obtain this result is the connection between homotopic spanning and partitions into indecomposable sets of finite perimeter established in Theorem 1.3/Theorem 3.1. This connection hints at the possibility of showing that if (K,E) is a minimizer of $\Psi_{\rm bk}(v)$, then the elements $\{U_i\}_i$ of the essential partition of Ω induced by $K \cup E^{(1)}$ are actually (Λ, r_0)-minimizers of the perimeter in Ω, i.e., there exist Λ and r_0 positive constants such that

$$P(U_i; B_r(x)) \le P(V; B_r(x)) + \Lambda\,|V \Delta U_i|\,,$$

whenever $V \Delta U_i \subset\subset \Omega$ and ${\rm diam}\,(V \Delta U_i) < r_0$. The reason why this property is not obvious is that proving the (Λ, r_0)-minimality of U_i requires working with *arbitrary local competitors* V_i of U_i. However, when working with homotopic spanning conditions, checking the admissibility of competitors is the notoriously delicate heart of the matter – as

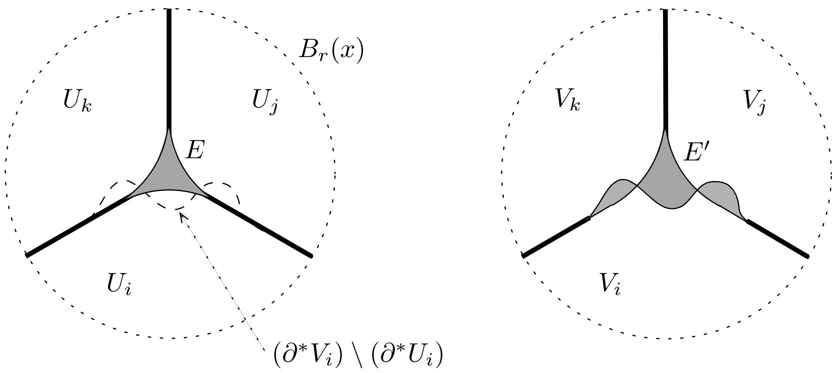

FIGURE 1.7. On the left, a minimizer (K,E) of $\Psi_{\rm bk}(v)$, and the essential partition induced by (K,E) in a ball $B_r(x)$; the multiplicity 2 part of $K \cap B_r(x)$ are depicted with bold lines, to distinguish them from the multiplicity one parts in $B_r(x) \cap \partial^* E$. On the right, a choice of (K',E') that guarantees both the energy gap identity (1.26) and the $\mathcal{H}^n$-containment (1.27) needed to preserve homotopic spanning. The volume constraint can of course be restored as a lower order perimeter perturbation by taking a diffeomorphic image of (K',E'), an operation that trivially preserves homotopic spanning.

reflected in the fact that only very special classes of competitors have been considered in the literature (see, e.g., the cup and cone competitors and the Lipschitz deformations considered in [DLGM17a], the slab competitors and exterior cup competitors of [KMS22a], etc.).

The idea used to overcome this difficulty, which is illustrated in Figure 1.7, is the following. By Theorem 1.2, we can locally represent $\mathcal{F}_{\rm bk}(K,E;B_r(x))$ as the sum of perimeters $P(U_i;B_r(x))+P(U_j;B_r(x))+P(U_k;B_r(x))$. Given a local competitor V_i for U_i we can carefully define a competitor (K',E') so that the elements of the essential partition induced by $K' \cup (E')^{(1)}$ in Ω, that can be used to represent $\mathcal{F}_{\rm bk}(K',E';B_r(x))$ as the sum $P(V_i;B_r(x))+P(V_j;B_r(x))+P(V_k;B_r(x))$, are such that

$$\mathcal{F}_{\rm bk}(K',E';B_r(x)) - \mathcal{F}_{\rm bk}(K,E;B_r(x)) = P(V;B_r(x)) - P(U_i;B_r(x))\,. \tag{1.26}$$

The trick is that by suitably defining K' and E' we can recover the entirety of $B_r(x) \cap \partial^* U_j$ and $B_r(x) \cap \partial^* U_k$ by attributing different parts of these boundaries to different terms in the representation of $\mathcal{F}_{\rm bk}(K',E';B_r(x))$. In other words we are claiming that things can be arranged so that we still have

$$B_r(x) \cap \big(\partial^* U_j \cap \partial^* U_k\big) \overset{\mathcal{H}^n}{\subset} K' \cup (E')^{(1)}\,. \tag{1.27}$$

The fact that we have been able to preserve all but one reduced boundary among those of the elements of the essential partition of $B_r(x)$ induced by (K,E) is enough to shows that $K' \cup (E')^{(1)}$ is still $\mathcal{C}$-spanning $\mathbf{W}$ by means of Theorem 1.3/Theorem 3.1.

By the regularity theory of (Λ, r_0)-perimeter minimizers (see, e.g. [Mag12, Part III]) we can deduce the $C^{1,\alpha}$-regularity of the elements of the partition (away from a closed singular set with area minimizing dimensional bounds). This is already sufficient to prove the continuity of the normal across $\Omega \cap (\partial E \setminus \partial^* E)$, but it also allows us to invoke the regularity theory for free boundaries in the double membrane problem, and to obtain the following sharp regularity result, with which we conclude our introduction.

Theorem 1.6 (Equilibrium along transition lines for soap films (Section 7))**.** *If* $\mathbf{W}$ *is a compact set in* $\mathbb{R}^{n+1}$, $\mathcal{C}$ *is a spanning class for* $\mathbf{W}$ *such that* $\ell < \infty$, $v > 0$, *and* (K_*,E_*) *is a minimizer of* $\Psi_{\rm bk}(v)$, *then there is* $(K,E) \in \mathcal{K}$ *such that* K *is* $\mathcal{H}^n$*-equivalent to* K_*, E

is Lebesgue equivalent to E_*, (K,E) *is a minimizer of* $\Psi_{\rm bk}(v)$, *both* E *and* K *are bounded,* $K\cup E$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$, *and*

$$K\cap E^{(1)}=\varnothing\,; \tag{1.28}$$

in particular, K *is the disjoint union of* $\Omega\cap\partial^*E$, $\Omega\cap(\partial E\setminus\partial^*E)$, *and* $K\setminus\partial E$.

Moreover, there is a closed set $\Sigma\subset K$ *with the following properties:*

(i): $\Sigma=\varnothing$ *if* $1\le n\le 6$, Σ *is locally finite in* Ω *if* $n=7$, *and* $\mathcal{H}^s(\Sigma)=0$ *for every* $s>n-7$ *if* $n\ge 8$;

(ii): $(\Omega\cap\partial^*E)\setminus\Sigma$ *is a smooth hypersurface with constant mean curvature (denoted by* λ *if computed with respect to* ν_E*);*

(iii): $(K\setminus\partial E)\setminus\Sigma$ *is a smooth minimal hypersurface;*

(iv): *if* $\Omega\cap(\partial E\setminus\partial^*E)\setminus\Sigma\neq\varnothing$, *then* $\lambda<0$; *moreover, for every* $x\in\Omega\cap(\partial E\setminus\partial^*E)\setminus\Sigma$, K *is the union of two* $C^{1,1}$*-hypersurfaces that detach tangentially at* x; *more precisely, there are* $r>0$, $\nu\in\mathbb{S}^n$, $u_1,u_2\in C^{1,1}(\mathbf{D}_r^\nu(x))$ *such that*

$$u_1(x)=u_2(x)=0\,,\qquad u_1\le u_2\ \text{ on }\mathbf{D}_r^\nu(x)\,,$$

with $\{u_1<u_2\}$ *and* ${\rm int}\{u_1=u_2\}$ *both non-empty, and*

$$\mathbf{C}_r^\nu(x)\cap K = \cup_{i=1,2}\big\{y+u_i(y)\,\nu: y\in\mathbf{D}_r^\nu(x)\big\}\,, \tag{1.29}$$

$$\mathbf{C}_r^\nu(x)\cap\partial^*E = \cup_{i=1,2}\big\{y+u_i(y)\nu: y\in\{u_1<u_2\}\big\}\,, \tag{1.30}$$

$$\mathbf{C}_r^\nu(x)\cap E = \big\{y+t\,\nu: t\in\big(u_1(y),u_2(y)\big)\big\}\,. \tag{1.31}$$

Here,

$$\mathbf{D}_\nu^r(x)=x+\{y\in\nu^\perp:|y|<r\}\,,$$
$$\mathbf{C}_\nu^r(x)=x+\{y+t\,\nu:y\in\nu^\perp\,,|y|<r\,,|t|<r\}\,.$$

(v): *we have*

$$\Gamma:=\Omega\cap(\partial E\setminus\partial^*E)=\Gamma_{\rm reg}\cup\Gamma_{\rm sing}\,,\qquad \Gamma_{\rm reg}\cap\Gamma_{\rm sing}=\varnothing\,,$$

where: $\Gamma_{\rm reg}$ *is relatively open in* Γ *and for every* $x\in\Gamma_{\rm reg}$ *there are* $r>0$ *and* $\beta\in(0,1)$ *such that* $\Gamma_{\rm reg}\cap B_r(x)$ *is a* $C^{1,\beta}$*-embedded* $(n-1)$*-dimensional manifold;* $\Gamma_{\rm sing}$ *is relatively closed in* Γ *and can be partitioned into a family* $\{\Gamma^k_{\rm sing}\}_{k=0}^{n-1}$ *where, for each* k, $\Gamma^k_{\rm sing}$ *is locally* $\mathcal{H}^k$*-rectifiable in* Ω.

1.7. **Equilibrium across transition lines in wet foams.** Based on the descriptions provided in [WH99, CCAE+13], an effective mathematical model for dry foams at equilibrium in a container is that of locally perimeter minimizing clusters, originating with different terminology in [Alm76], and presented in [Mag12, Part IV] as follows. Given an open set $\Omega\subset\mathbb{R}^{n+1}$, a locally perimeter minimizing clusters is a finite Lebesgue partition $\{U_i\}_i$ of Ω into sets of finite perimeter such that, for some $r_0>0$,

$$\sum_i P(U_i;B)\le\sum_i P(V_i;B) \tag{1.32}$$

whenever $B\subset\subset\Omega$ is a ball with radius less than r_0, and $\{V_i\}_i$ is a Lebesgue partition of Ω with $V_i\Delta U_i\subset\subset B$ and $|V_i|=|U_i|$ for every i. The previously cited results of Almgren and Taylor [Alm76, Tay76] imply that, up to modification of the U_i's by sets of zero Lebesgue measure, when $n=2$, $K=\Omega\cap\bigcup_i\partial U_i$ is a closed subset of Ω that is locally $C^{1,\alpha}$-diffeomorphic to a plane, a Y-cone, or a T-cone; moreover, the part of K that is a surface is actually smooth and each of its connected component has constant mean curvature. Similar results holds when $n=1$ (by elementary methods) and when $n\ge 3$ (by exploiting [CES22]).

The theory for the relaxed capillarity energy $\mathcal{F}_{\rm bk}$ developed in this paper provides an option for modeling wet foams. Again based on the descriptions provided in [WH99, CCAE+13], the following seems to be a reasonable model for wet foams at equilibrium in a container. Given an open set $\Omega \subset \mathbb{R}^{n+1}$ we model wet foams by introducing the class

$$\mathcal{K}_{\rm foam}$$

of those $(K,E) \in \mathcal{K}_{\rm B}$ such that, for some positive constants Λ_0 and r_0,

$$\mathcal{F}_{\rm bk}(K,E;B) \le \mathcal{F}_{\rm bk}(K',E';B) + \Lambda_0\,|E\Delta E'| \tag{1.33}$$

whenever B is a ball compactly contained in Ω and with radius less than r_0, and $(K',E') \in \mathcal{K}_{\rm B}$ is such that $(K\Delta K') \cup (E\Delta E') \subset\subset B$ and there are finite Lebesgue partitions $\{U_i\}_i$ and $\{U_i'\}_i$ of B induced, respectively, by $K \cup E^{(1)}$ and by $K' \cup (E')^{(1)}$, such that $|U_i| = |U_i'|$ for every i. Notice that inclusion of the term $\Lambda_0\,|E\Delta E'|$ in (1.33) allows for the inclusion of energy perturbations due to gravity or other forces. Lemma 7.1 will clarify that by taking $(K,E) \in \mathcal{K}_{\rm foam}$ with $|E| = 0$ we obtain a slightly more general notion of dry foam than the one proposed in (1.32).

Theorem 1.7 (Equilibrium along transition lines for soap films (Section 8))**.** *If $\Omega \subset \mathbb{R}^{n+1}$ is open and $(K_*,E_*) \in \mathcal{K}_{\rm foam}$, then there is $(K,E) \in \mathcal{K} \cap \mathcal{K}_{\rm foam}$ such that K is $\mathcal{H}^n$-equivalent to K_*, E Lebesgue equivalent to E_*, $K \cap E^{(1)} = \varnothing$, and such that, for every ball $B \subset\subset \Omega$, the open connected components $\{U_i\}_i$ of $B \setminus (K \cup E)$ are such that each U_i is (Lebesgue equivalent to an) open set with $C^{1,\alpha}$-boundary in $B \setminus \Sigma$. Here Σ is a closed subset of Ω with $\Sigma = \varnothing$ if $1 \le n \le 6$, Σ locally finite in Ω if $n = 7$, and $\mathcal{H}^s(\Sigma) = 0$ for every $s > n-7$ if $n \ge 8$.*

Organization of the paper. The sections of the paper contain the proofs of the main theorems listed above, as already specified in the statements. To these section we add three appendices. In Appendix A, as already noted, we prove the equivalence of Definition A and Definition B. In Appendix B we prove that, with some regularity of $\partial\Omega$, *every* minimizing sequence of $\Psi_{\rm bk}(v)$ is converging to a minimizers, without need for modifications at infinity: this is, strictly speaking, not needed to prove Theorem 1.5, but it is a result of its own conceptual interest, it will be crucial for the analysis presented in [MNR23a], and it is easily discussed here in light of the proof of Theorem 1.5. Finally, Appendix C contains an elementary lemma concerning the use of homotopic spanning in the plane that, to our knowledge, has not been proved in two dimensions.

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Notation. Sets and measures: We denote by $B_r(x)$ (resp., $B_r^k(x)$) the open ball of center x and radius r in $\mathbb{R}^{n+1}$ (resp., $\mathbb{R}^k$), and omit (x) when $x = 0$. We denote by ${\rm cl}\,(X)$, ${\rm int}(X)$, and $I_r(X)$ the closure, interior and open ε-neighborhood of $X \subset \mathbb{R}^k$. We denote by $\mathcal{L}^{n+1}$ and $\mathcal{H}^s$ the Lebesgue measure and the s-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$, $s \in [0,n+1]$. If $E \subset \mathbb{R}^k$, we set $|E| = \mathcal{L}^k(E)$ and $\omega_k = |B_1^k|$. We denote by $E^{(t)}$, $t \in [0,1]$, the **points of density** t of a Borel set $E \subset \mathbb{R}^{n+1}$, so that E is $\mathcal{L}^{n+1}$-equivalent to $E^{(1)}$, and, for every pair of Borel sets $E, F \subset \mathbb{R}^{n+1}$,

$$(E \cup F)^{(0)} = E^{(0)} \cap F^{(0)}\,. \tag{1.34}$$

We define by $\partial^e E = \mathbb{R}^{n+1} \setminus (E^{(0)} \cup E^{(1)})$ the **essential boundary** of E. Given Borel sets $E_j, E \subset \Omega$ we write

$$E_j \to E\,, \qquad E_j \stackrel{\rm loc}{\to} E\,,$$

when, respectively, $|E_j \Delta E| \to 0$ or $|(E_j \Delta E) \cap \Omega'| \to 0$ for every $\Omega' \subset\subset \Omega$, as $j \to \infty$. Given a Radon measure μ on $\mathbb{R}^{n+1}$, the k-dimensional lower density of μ is the Borel function $\theta^k_*(\mu) : \mathbb{R}^{n+1} \to [0, \infty]$ defined by

$$\theta^k_*(\mu)(x) = \liminf_{r \to 0^+} \frac{\mu(\mathrm{cl}\,(B_r(x)))}{\omega_k r^k}\,.$$

We repeatedly use the fact that, if $\theta^k_*(\mu) \geq \lambda$ on some Borel set K and for some $\lambda \geq 0$, then $\mu \geq \lambda\, \mathcal{H}^k \llcorner K$; see, e.g. [Mag12, Theorem 6.4].

Rectifiable sets: Given an integer $0 \leq k \leq n+1$, a Borel set $S \subset \mathbb{R}^{n+1}$ is **locally $\mathcal{H}^k$-rectifiable** in an open set Ω if S is locally $\mathcal{H}^k$-finite in Ω and S can be covered, modulo $\mathcal{H}^k$-null sets, by a countable union of Lipschitz images of $\mathbb{R}^k$ in $\mathbb{R}^{n+1}$. We say that S is **purely $\mathcal{H}^k$-unrectifiable** if $\mathcal{H}^k(S \cap M) = 0$ whenever M is a Lipschitz image of $\mathbb{R}^k$ into $\mathbb{R}^{n+1}$. Finally, we recall that if S is a locally $\mathcal{H}^k$-finite set in Ω, then there is a pair $(\mathcal{R}(S), \mathcal{P}(S))$ of Borel sets, uniquely determined modulo $\mathcal{H}^k$-null sets, and that are thus called, with a slight abuse of language, *the* **rectifiable part** and *the* **unrectifiable part** of S, so that $\mathcal{R}(S)$ is locally $\mathcal{H}^k$-rectifiable in Ω, $\mathcal{P}(S)$ is purely $\mathcal{H}^k$-unrectifiable, and $S = \mathcal{R}(S) \cup \mathcal{P}(S)$; see, e.g. [Sim83, 13.1].

Sets of finite perimeter: If E is a Borel set in $\mathbb{R}^{n+1}$ and $D1_E$ is the distributional derivative of the characteristic function of E, then we set $\mu_E = -D1_E$. If A is the *largest open set* of $\mathbb{R}^{n+1}$ such that μ_E is a Radon measure in A (of course it could be $A = \varnothing$), then E is of locally finite perimeter in A and the reduced boundary $\partial^* E$ of E is defined as the set of those $x \in A \cap \mathrm{spt}\mu_E$ such that $\mu_E(B_r(x))/|\mu_E|(B_r(x))$ has a limit $\nu_E(x) \in \mathbb{S}^n$ as $r \to 0^+$. Moreover, we have the general identity (see [Mag12, (12.12) & pag. 168])

$$A \cap \mathrm{cl}\,(\partial^* E) = A \cap \mathrm{spt}\mu_E = \Big\{ x \in A : 0 < |E \cap B_r(x)| < |B_r(x)|\ \forall r > 0 \Big\} \subset A \cap \partial E\,. \quad (1.35)$$

By De Giorgi's rectifiability theorem, $\partial^* E$ is locally $\mathcal{H}^n$-rectifiable in A, $\mu_E = \nu_E\, \mathcal{H}^n \llcorner (A \cap \partial^* E)$ on A, and $\partial^* E \subset A \cap E^{(1/2)} \subset A \cap \partial^e E$, and

$$(E - x)/r \stackrel{\mathrm{loc}}{\to} H_{E,x} := \big\{ y \in \mathbb{R}^{n+1} : y \cdot \nu_E(x) < 0 \big\}\,, \qquad \text{as } r \to 0^+\,. \quad (1.36)$$

By a result of Federer,

$$A \text{ is } \mathcal{H}^n\text{-contained in } E^{(0)} \cup E^{(1)} \cup \partial^* E\,; \quad (1.37)$$

in particular, $\partial^* E$ is $\mathcal{H}^n$-equivalent to $A \cap \partial^e E$, a fact frequently used in the following. By *Federer's criterion for finite perimeter*, if Ω is open and E is a Borel set, then

$$\mathcal{H}^n(\Omega \cap \partial^e E) < \infty \qquad \Rightarrow \qquad E \text{ is of finite perimeter in } \Omega\,, \quad (1.38)$$

see [Fed69, 4.5.11]. If E and F are of locally finite perimeter in Ω open, then so are $E \cup F$, $E \cap F$, and $E \setminus F$, and by [Mag12, Theorem 16.3], we have

$$\Omega \cap \partial^*(E \cup F) \stackrel{\mathcal{H}^n}{=} \Omega \cap \Big\{ \big(E^{(0)} \cap \partial^* F\big) \cup \big(F^{(0)} \cap \partial^* E\big) \cup \{\nu_E = \nu_F\} \Big\}\,, \quad (1.39)$$

$$\Omega \cap \partial^*(E \cap F) \stackrel{\mathcal{H}^n}{=} \Omega \cap \Big\{ \big(E^{(1)} \cap \partial^* F\big) \cup \big(F^{(1)} \cap \partial^* E\big) \cup \{\nu_E = \nu_F\} \Big\}\,, \quad (1.40)$$

$$\Omega \cap \partial^*(E \setminus F) \stackrel{\mathcal{H}^n}{=} \Omega \cap \Big\{ \big(E^{(1)} \cap \partial^* F\big) \cup \big(F^{(0)} \cap \partial^* E\big) \cup \{\nu_E = -\nu_F\} \Big\}\,, \quad (1.41)$$

where $\{\nu_E = \pm\nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \pm\nu_F(x)\}$. By exploiting Federer's theorem (1.37), (1.39), (1.40), and (1.41) we can also deduce (the details are left to the reader)

$$(E \cap F)^{(0)} \stackrel{\mathcal{H}^n}{=} E^{(0)} \cup F^{(0)} \cup \{\nu_E = -\nu_F\}\,, \quad (1.42)$$

$$(E \setminus F)^{(0)} \stackrel{\mathcal{H}^n}{=} E^{(0)} \cup F^{(1)} \cup \{\nu_E = \nu_F\}\,. \quad (1.43)$$

Finally, combining (1.39), (1.41), and (1.43), we find

$$\partial^*(E\Delta F) \overset{\mathcal{H}^n}{=} (\partial^* E)\Delta(\partial^* F)\,. \tag{1.44}$$

Partitions: Given a Radon measure μ on $\mathbb{R}^{n+1}$ and Borel set $U \subset \mathbb{R}^{n+1}$ we say that $\{U_i\}_i$ is a **μ-partition of U** if $\{U_i\}_i$ is an at most countable family of Borel subsets of U such that

$$\mu\Big(U \setminus \bigcup_i U_i\Big) = 0\,, \qquad \mu(U_i \cap U_j) = 0 \quad \forall i, j\,; \tag{1.45}$$

and we say that $\{U_i\}_i$ is a **monotone μ-partition** if, in addition to (1.45), we also have $\mu(U_i) \ge \mu(U_{i+1})$ for every i. When $\mu = \mathcal{L}^{n+1}$ we replace "μ-partition" with "Lebesgue partition". When U is a set of finite perimeter in $\mathbb{R}^{n+1}$, we say that $\{U_i\}_i$ is a **Caccioppoli partition** of U if $\{U_i\}_i$ is a Lebesgue partition of U and each U_i is a set of finite perimeter in $\mathbb{R}^{n+1}$: in this case we have

$$\partial^* U \overset{\mathcal{H}^n}{\subset} \bigcup_i \partial^* U_i\,, \qquad 2\,\mathcal{H}^n\Big(U^{(1)} \cap \bigcup_i \partial^* U_i\Big) = \sum_i \mathcal{H}^n(U^{(1)} \cap \partial^* U_i)\,, \tag{1.46}$$

see, e.g., [AFP00, Section 4.4]; moreover,

$$1 \le \#\Big\{i : x \in \partial^* U_i\Big\} \le 2\,, \qquad \forall x \in \bigcup_i \partial^* U_i\,, \tag{1.47}$$

thanks to (1.36) and to the fact that there cannot be three disjoint half-spaces in $\mathbb{R}^{n+1}$.

2. Induced essential partitions (Theorem 1.2)

Given a Borel set S, we say that a Lebesgue partition $\{U_i\}_i$ of U is **induced by** S if, for each i,

$$U^{(1)} \cap \partial^e U_i \text{ is } \mathcal{H}^n\text{-contained in } S\,. \tag{2.1}$$

We say that $\{U_i\}_i$ is *an* **essential partition of** U **induced by** S if it is a Lebesgue partition of U induced by S such that, for each i,

$$S \text{ does not essentially disconnect } U_i\,. \tag{2.2}$$

The next theorem, which expands the statement of Theorem 1.2, shows that when $\mathcal{H}^n$-finite sets uniquely determine induced essential partitions on sets of finite perimeter.

Theorem 2.1 (Induced essential partitions)**.** *If $U \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter and $S \subset \mathbb{R}^{n+1}$ is a Borel set with $\mathcal{H}^n(S \cap U^{(1)}) < \infty$, then there exists an essential partition $\{U_i\}_i$ of U induced by S such that each U_i is a set of finite perimeter and*

$$\sum_i P(U_i; U^{(1)}) \le 2\,\mathcal{H}^n(S \cap U^{(1)})\,. \tag{2.3}$$

Moreover: **(a):** *if S^* is a Borel set with $\mathcal{H}^n(S^* \cap U^{(1)}) < \infty$, S^* is $\mathcal{H}^n$-contained in S, $\{V_j\}_j$ is a Lebesgue partition*[8] *of U induced by S^*, and $\{U_i\}_i$ is the essential partition of U induced by S, then*

$$\textstyle\bigcup_j \partial^* V_j \text{ is } \mathcal{H}^n\text{-contained in } \bigcup_i \partial^* U_i\,; \tag{2.4}$$

(b): *if S and S^* are $\mathcal{H}^n$-finite sets in $U^{(1)}$, and either*[9] *$S^* = \mathcal{R}(S)$ or S^* is $\mathcal{H}^n$-equivalent to S, then S and S^* induce $\mathcal{L}^{n+1}$-equivalent essential partitions of U.*

Proof of Theorem 1.2. Immediate consequence of Theorem 2.1. □

[8] Notice that here we are not requiring that S^* does not essentially disconnect each V_j, i.e., we are not requiring that $\{V_j\}_j$ is an essential partition induced by S^*. This detail will be useful in the applications of this theorem.

[9] Here $\mathcal{R}(S)$ denotes the $\mathcal{H}^n$-rectifiable part of S.

The proof of Theorem 2.1 follows the main lines of the proof of [ACMM01, Theorem 1], which is indeed the case $S = \varnothing$ of Theorem 2.1. We premise to this proof two lemmas that will find repeated applications in later sections too. To introduce the first lemma, we notice that, while it is evident that if S is $\mathcal{C}$-spanning $\mathbf{W}$ and S is $\mathcal{H}^n$-contained into some Borel set S^*, then S^* is also $\mathcal{C}$-spanning $\mathbf{W}$, however, it is not immediately clear if the rectifiable part $\mathcal{R}(S)$ of S (which may not be $\mathcal{H}^n$-equivalent to S) retains the $\mathcal{C}$-spanning property.

Lemma 2.2. *If $\mathbf{W}$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, S is $\mathcal{C}$-spanning $\mathbf{W}$, and $\mathcal{H}^n \llcorner S$ is a Radon measure in Ω, then $\mathcal{R}(S)$ is $\mathcal{C}$-spanning $\mathbf{W}$. Moreover, the sets T_1 and T_2 appearing in* (1.12) *are sets of finite perimeter.*

Proof. We make the following *claim*: if T is open, $T^{(1)} \overset{\mathcal{H}^n}{\subset} T$, $\mathcal{H}^n \llcorner Z$ is a Radon measure in an open neighborhood of T, and Z essentially disconnects T into $\{T_1, T_2\}$, then

$$T_1 \text{ and } T_2 \text{ are of locally finite perimeter in } T\,, \tag{2.5}$$
$$\mathcal{R}(Z) \text{ essentially disconnects } T \text{ into } \{T_1, T_2\}\,. \tag{2.6}$$

Indeed: Since T is open, we trivially have $T \subset T^{(1)}$, and hence T is $\mathcal{H}^n$-equivalent to $T^{(1)}$. Taking also into account that Z essentially disconnects T into $\{T_1, T_2\}$, we thus find

$$T \cap \partial^e T_1 \cap \partial^e T_2 \overset{\mathcal{H}^n}{=} T^{(1)} \cap \partial^e T_1 \cap \partial^e T_2 \overset{\mathcal{H}^n}{\subset} Z \cap T^{(1)} \overset{\mathcal{H}^n}{\subset} Z \cap T\,.$$

By Federer's criterion (1.38) and the $\mathcal{H}^n$-finiteness of Z in an open neighborhood of T we deduce (2.5). By Federer's theorem (1.37), $\partial^e T_i$ is $(\mathcal{H}^n \llcorner T)$-equivalent to $\partial^* T_i$, which combined with the $\mathcal{H}^n$-equivalence of $T^{(1)}$ and T gives

$$\partial^e T_1 \cap \partial^e T_2 \cap T^{(1)} \overset{\mathcal{H}^n}{=} \partial^* T_1 \cap \partial^* T_2 \cap T\,.$$

Since $\partial^* T_1 \cap \partial^* T_2 \cap T$ is $\mathcal{H}^n$-rectifiable and $\partial^e T_1 \cap \partial^e T_2 \cap T^{(1)} \overset{\mathcal{H}^n}{\subset} Z$, we conclude that $\mathcal{H}^n(\partial^e T_1 \cap \partial^e T_2 \cap T^{(1)} \cap \mathcal{P}(Z)) = 0$. Hence,

$$\partial^e T_1 \cap \partial^e T_2 \cap T^{(1)} \overset{\mathcal{H}^n}{\subset} \mathcal{R}(Z)\,,$$

and (2.6) follows.

To prove the lemma: Let $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, let J be of full measure such that (A.1) holds for every $s \in J$, so that, for every $s \in J$ one finds that for $\mathcal{H}^n$-a.e. $x \in T[s]$ there is a partition $\{T_1, T_2\}$ of T with $x \in \partial^e T_1 \cap \partial^e T_2$ and such that $S \cup T[s]$ essentially disconnects T into $\{T_1, T_2\}$. By applying the claim with $Z = S \cup T[s]$, we see that $\mathcal{R}(S \cup T[s])$ essentially disconnects T into $\{T_1, T_2\}$, and that T_1 and T_2 have locally finite perimeter in T. On noticing that $\mathcal{R}(S \cup T[s])$ is $\mathcal{H}^n$-equivalent to $\mathcal{R}(S) \cup T[s]$, we conclude the proof. □

The second lemma is just a simple compactness statement for finite perimeter partitions.

Lemma 2.3 (Compactness for partitions by sets of finite perimeter)**.** *If U is a bounded open set and $\{\{U_i^j\}_{i=1}^\infty\}_{j=1}^\infty$ is a sequence of Lebesgue partitions of U into sets of finite perimeter such that*

$$\sup_j \sum_{i=1}^\infty P(U_i^j) < \infty\,, \tag{2.7}$$

then, up to extracting a subsequence, there exists a Lebesgue partition $\{U_i\}_{i\in\mathbb{N}}$ of U such that for every i and every $A \subset U$ open,

$$\lim_{j\to\infty} |U_i^j \Delta U_i| = 0\,, \qquad P(U_i; A) \le \liminf_{j\to\infty} P(U_i^j; A)\,. \tag{2.8}$$

Moreover,

$$\lim_{i\to\infty}\limsup_{j\to\infty}\sum_{k=i+1}^\infty |U_k^j|^s=0\,,\qquad \forall s\in\Big(\frac{n}{n+1},1\Big)\,. \tag{2.9}$$

Proof. Up to a relabeling we can assume each $\{U_i^j\}_i$ is monotone. By (2.7) and the boundedness of U, a diagonal argument combined with standard lower semicontinuity and compactness properties of sets of finite perimeter implies that we can find a not relabeled subsequence in j and a family $\{U_i\}_i$ of Borel subsets of U with $|U_i|\ge|U_{i+1}|$ and $|U_i\cap U_j|=0$ for every $i\ne j$, such that (2.8) holds. We are thus left to prove (2.9) and

$$\Big|U\setminus\bigcup_{i=1}^\infty U_i\Big|=0\,. \tag{2.10}$$

We start by noticing that for each i there is $J(i)\in\mathbb{N}$ such that $|U_k^j|\le 2\,|U_k|$ for every $j\ge J(i)$ and $1\le k\le i$. Therefore if $k\ge i+1$ and $j\ge J(i)$ we find $|U_k^j|\le|U_i^j|\le 2\,|U_i|$, so that, if $j\ge J(i)$,

$$\sum_{k=i+1}^\infty |U_k^j|^s\le C(n)\sum_{k=i+1}^\infty P(U_k^j)|U_k^j|^{s-(n/(n+1))}\le C\,|U_i|^{s-(n/(n+1))}\,, \tag{2.11}$$

where we have also used the isoperimetric inequality and (2.7). Since $|U_i|\to0$ as $i\to\infty$ (indeed, $\sum_i|U_i|\le|U|<\infty$), (2.11) implies (2.9). To prove (2.10), we notice that if we set $M=|U\setminus\cup_iU_i|$, and we assume that M is positive, then up to further increasing the value of $J(i)$ we can require that

$$|U_k^j|\le|U_k|+\frac{M}{2^{k+2}}\,,\qquad \forall 1\le k\le i\,,\forall j\ge J(i)\,, \tag{2.12}$$

(in addition to $|U_k^j|\le 2\,|U_k|$). By (2.12) we obtain that, if $j\ge J(i)$, then

$$|U|-\sum_{k=i+1}^\infty|U_k^j|=\sum_{k=1}^i|U_k^j|\le\sum_{k=1}^i|U_k|+\frac{M}{2^{k+2}}\le|U|-M+\sum_{k=1}^i\frac{M}{2^{k+2}}\le|U|-\frac{M}{4}\,. \tag{2.13}$$

Rearranging (2.13) and using the sub-additivity of $z\mapsto z^s$ we conclude that

$$(M/4)^s\le\sum_{k=i+1}^\infty|U_k^j|^s\,.$$

We obtain a contradiction with $M>0$ by letting $i\to\infty$ and by using (2.9). □

Proof of Theorem 2.1. Let $\mathcal{U}(S)$ be the set of all the monotone Lebesgue partitions of U induced by S. We notice that $\mathcal{U}(S)\ne\varnothing$, since $\mathcal{U}(S)$ contains the trivial partition with $U_1=U$ and $U_i=\varnothing$ if $i\ge2$. If $U_i\in\{U_i\}_i$ for $\{U_i\}_i\in\mathcal{U}(S)$, then ∂^eU_i is $\mathcal{H}^n$-contained in $\partial^eU\cup(U^{(1)}\cap S)$, which, by Federer's criterion applied to U and $\mathcal{H}^n(S\cap U^{(1)})<\infty$, has finite $\mathcal{H}^n$-measure; it follows then that U_i is a set of finite perimeter due to Federer's criterion. We now fix $s\in(n/(n+1),1)$, and consider a maximizing sequence $\{\{U_i^j\}_i\}_j$ for

$$m=\max\Big\{\sum_{i=1}^\infty|U_i|^s:\{U_i\}_i\in\mathcal{U}(S)\Big\}\,.$$

By standard arguments concerning reduced boundaries of disjoint sets of finite perimeter (see, e.g. [Mag12, Chapter 16]), we deduce from (2.1) that for every j,

$$\begin{aligned}\sum_{i=1}^\infty\mathcal{H}^n\llcorner\partial^*U_i^j&=\sum_{i=1}^\infty\mathcal{H}^n\llcorner(\partial^*U_i^j\cap U^{(1)})+\sum_{i=1}^\infty\mathcal{H}^n\llcorner(\partial^*U_i^j\cap\partial^*U)\\&\le 2\,\mathcal{H}^n\llcorner(S\cap U^{(1)})+\mathcal{H}^n\llcorner\partial^*U\,.\end{aligned} \tag{2.14}$$

Also, due to the sub-additivity of $z \mapsto z^s$ and the general fact that $\partial^e(A\cap B) \subset \partial^e A \cup \partial^e B$, we can refine $\{U_i^j\}_i$ by replacing each U_i^j with the disjoint family

$$\big\{U_i^j \cap U_k^\ell : k \ge 1\,, 1 \le \ell < j\big\}\,,$$

thus obtaining a new sequence in $\mathcal{U}(S)$ which is still maximizing for m. As a consequence of this remark, we can assume without loss of generality that the considered maximizing sequence $\{\{U_i^j\}_i\}_j$ for m has the additional property that

$$U \cap \bigcup_i \partial^* U_i^j \subset U \cap \bigcup_i \partial^* U_i^{j+1}\,, \qquad \forall j\,. \tag{2.15}$$

Thanks to (2.14) we can apply Lemma 2.3 and, up to extracting a subsequence in j, we can find a Lebesgue partition $\{U_i\}_{i\in\mathbb{N}}$ of U by sets of finite perimeter which satisfies (2.8) and (2.9). Moreover, after taking a subsequence, we may assume that $\mathcal{H}^n \llcorner \partial^* U_i^j \overset{*}{\rightharpoonup} \mu_i$ for some Radon measures μ_i such that $\mathcal{H}^n \llcorner \partial^* U_i \le \mu_i$ [Mag12, Prop. 12.15]. Therefore, by (2.8), Federer's theorem for reduced boundaries, and by (2.1) for $\{U_i^j\}_i$, we see that

$$\begin{aligned}\mathcal{H}^n \llcorner (\partial^* U) + \sum_{i=1}^\infty \mathcal{H}^n \llcorner (\partial^* U_i \cap U^{(1)}) &= \sum_{i=1}^\infty \mathcal{H}^n \llcorner (\partial^* U_i) \le \mathrm{w}^* \lim_{j\to\infty} \sum_{i=1}^\infty \mathcal{H}^n \llcorner (\partial^* U_i^j)\\ = \mathrm{w}^* \lim_{j\to\infty} \mathcal{H}^n \llcorner (\partial^* U) + \sum_{i=1}^\infty \mathcal{H}^n \llcorner (\partial^e U_i^j \cap U^{(1)}) &\le \mathcal{H}^n \llcorner (\partial^* U) + 2\mathcal{H}^n \llcorner (S \cap U^{(1)})\,.\end{aligned}$$

By subtracting $\mathcal{H}^n \llcorner (\partial^* U)$ from both sides, we deduce (2.3).

We now show, first, that $\{U_i\}_i \in \mathcal{U}(S)$ (i.e., we check the validity of (2.1) on $\{U_i\}_i$), and then that S does not essentially disconnect any of the U_i. This will complete the proof of the first part of the statement.

To prove that $U^{(1)} \cap \partial^e U_i \overset{\mathcal{H}^n}{\subset} S$, let us introduce the $\mathcal{H}^n$-rectifiable set S_0 defined by

$$S_0 = U^{(1)} \cap \bigcup_{i,j} \partial^* U_i^j\,. \tag{2.16}$$

By $\{U_i^j\}_i \in \mathcal{U}(S)$, S_0 is contained into S modulo $\mathcal{H}^n$-null sets. Therefore, in order to prove (2.1) it will be enough to show that

$$U^{(1)} \cap \partial^* U_i \overset{\mathcal{H}^n}{\subset} S_0\,, \qquad \forall i\,. \tag{2.17}$$

Should this not be the case, it would be $\mathcal{H}^n(U^{(1)} \cap \partial^* U_i \setminus S_0) > 0$ for some i. We could thus pick $x \in U^{(1)} \cap \partial^* U_i$ such that

$$\theta^n(\mathcal{H}^n \llcorner (U^{(1)} \cap \partial^* U_i \setminus S_0))(x) = 1\,. \tag{2.18}$$

Since $\theta^n(\mathcal{H}^n \llcorner \partial^* U_i)(x) = 1$ and $S_0 \subset U^{(1)}$ this implies $\mathcal{H}^n(S_0 \cap B_r(x)) = \mathrm{o}(r^n)$, while $\partial^* U_i \subset U_i^{(1/2)}$ gives $|U_i \cap B_r(x)| = (\omega_{n+1}/2)\, r^{n+1} + \mathrm{o}(r^{n+1})$. Therefore, given $\delta > 0$ we can find $r > 0$ such that

$$\mathcal{H}^n(S_0 \cap B_r(x)) < \delta\, r^n\,, \qquad \min\big\{|U_i \cap B_r(x)|, |U_i \setminus B_r(x)|\big\} \ge \Big(\frac{\omega_{n+1}}{2} - \delta\Big)\, r^{n+1}\,,$$

and then exploit the relative isoperimetric inequality and (2.8) to conclude that

$$\begin{aligned} c(n)\,\Big[\Big(\frac{\omega_{n+1}}{2} - \delta\Big)\, r^{n+1}\Big]^{n/(n+1)} &\le P(U_i; B_r(x)) \le \liminf_{j\to\infty} P(U_i^j; B_r(x))\\ &\le \mathcal{H}^n(S_0 \cap B_r(x)) \le \delta\, r^n\,,\end{aligned}$$

where in the next to last inequality we have used the definition (2.16) of S_0. Choosing $\delta > 0$ small enough we reach a contradiction, thus deducing that $\{U_i\}_i \in \mathcal{U}(S)$.

Taking into account the subadditivity of $z \mapsto z^s$, in order to prove that S does not essentially disconnect any U_i it is sufficient to show that $\{U_i\}_i$ is a maximizer of m. To see this, we notice that $|U_i^j \Delta U_i| \to 0$ as $j \to \infty$ implies

$$m = \lim_{j\to\infty} \sum_{i=1}^{k} |U_i^j|^s + \sum_{i=k+1}^{\infty} |U_i^j|^s = \sum_{i=1}^{k} |U_i|^s + \lim_{j\to\infty} \sum_{i=k+1}^{\infty} |U_i^j|^s \,,$$

so that, letting $k \to \infty$ and exploiting (2.9), we conclude that

$$m = \sum_{i=1}^{\infty} |U_i|^s \,. \tag{2.19}$$

This completes the proof of the first part of the statement (existence of essential partitions).

Let now S, S^*, $\{U_i\}_i$, and $\{U_j^*\}_j$ be as in statement (a) – that is, S^* is $\mathcal{H}^n$-contained in S, $\{U_i\}_i$ is an essential partition of U induced by S, and, for every j, $\{U_j^*\}_j$ is a Lebesgue partition of U induced by S^* – and set $Z = \cup_i \partial^* U_i$ and $Z^* = \cup_j \partial^* U_j^*$. Arguing by contradiction with (2.4), let us assume $\mathcal{H}^n(Z^* \setminus Z) > 0$. By the definition of Lebesgue partition we have that $Z \setminus U^{(1)}$ and $Z^* \setminus U^{(1)}$ are both $\mathcal{H}^n$-equivalent to $\partial^* U$. Therefore we have $\mathcal{H}^n((Z^* \setminus Z) \cap U^{(1)}) > 0$. Since $U^{(1)}$ is $\mathcal{H}^n$-equivalent to the union of the $\{U_i^{(1)} \cup \partial^* U_i\}_{i\in I}$ we can find $i \in I$ and $j \in J$ such that $\mathcal{H}^n(U_i^{(1)} \cap \partial^* U_j^*) > 0$. This implies that both $(U_i \cap U_j^*)^{(1/2)}$ and $(U_i \setminus U_j^*)^{(1/2)}$ are non-empty, and thus that $\{U_j^* \cap U_i, U_i \setminus U_j^*\}$ is a non-trivial Borel partition of U_i. Since

$$U_i^{(1)} \cap \partial^e (U_j^* \cap U_i) \overset{\mathcal{H}^n}{\subset} U^{(1)} \cap \partial^* U_j^* \overset{\mathcal{H}^n}{\subset} S^* \,,$$

we conclude that S^* is essentially disconnecting U_i, against the fact that S is not essentially disconnecting U_i and the fact that S^* is $\mathcal{H}^n$-contained in S.

We finally prove statement (b). Let $\{U_i\}_{i\in I}$, and $\{U_j^*\}_{j\in J}$ be essential partitions of U induced by S and S^* respectively. Given $i \in I$ such that $|U_i| > 0$, there is at least one $j \in J$ such that $|U_i \cap U_j^*| > 0$. We *claim* that it must be $|U_i \setminus U_j^*| = 0$. Should this not be the case, $\partial^* U_j^*$ would be essentially disconnecting U_i, thus implying that S^* (which contains $\partial^* U_j^*$) is essentially disconnecting U_i. Now, either because we are assuming that S^* is $\mathcal{H}^n$-equivalent to S, or because we are assuming that $S^* = \mathcal{R}(S)$ and we have Lemma 2.2, the fact that S^* is essentially disconnecting U_i implies that S is essentially disconnecting U_i, a contradiction. Having proved the claim, for each $i \in I$ with $|U_i| > 0$ there is a unique $\sigma(i) \in J$ such that $|U_i \Delta U^*_{\sigma(j)}| = 0$. This completes the proof. □

3. Homotopic spanning on generalized soap films (Theorem 1.3)

The goal of this section is proving Theorem 1.3, and, actually, to obtain an even more general result. Let us recall that the objective of Theorem 1.3 was to reformulate the homotopic spanning property for a Borel set S, in the case when S is locally $\mathcal{H}^n$-finite, in terms of unions of boundaries of induced essential partitions. We shall actually need this kind of characterization also for sets S of the more general form $S = K \cup E^{(1)}$, where $(K, E) \in \mathcal{K}_{\mathrm{B}}$. For an illustration of the proposed characterization of homotopic spanning on this type of sets, see Figure 3.1.

Theorem 3.1 (Homotopic spanning for generalized soap films)**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is a closed set in $\mathbb{R}^{n+1}$, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, K is a Borel set locally $\mathcal{H}^n$-finite in Ω, and E is of locally finite perimeter in Ω such that $\Omega \cap \partial^* E$ is $\mathcal{H}^n$-contained in K, then the set*

$$S = \mathcal{R}(K) \cup E^{(1)} \tag{3.1}$$

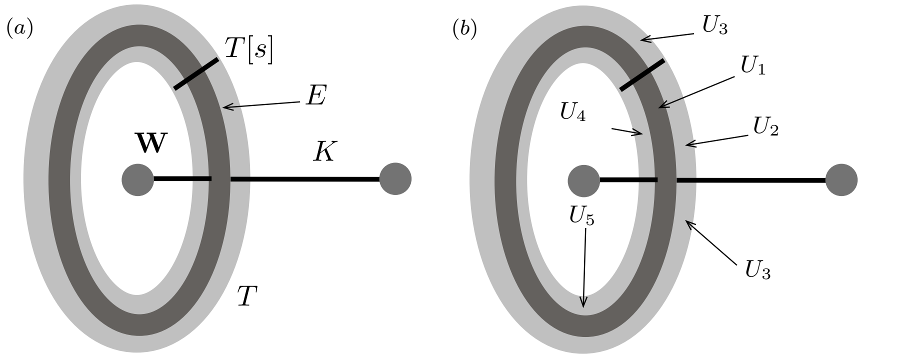

FIGURE 3.1. In panel (a) we have depicted a pair (K,E) where E is a tube inside T and K consists of the union of the boundary of E and the *non*-spanning set S of Figure 1.6-(a). Notice that K is not $\mathcal{C}$-spanning, if we see things from the point of view of Definition A, since it misses every loop γ contained in the interior of E; while, of course, $K\cup E$ is $\mathcal{C}$-spanning because E has been added. In panel (b) we have depicted the essential partition $\{U_i\}_{i=1}^5$ of T induced by $K\cup T[s]$. Notice that $E=U_1$, therefore no $\partial^*U_i\cap\partial^*U_j$ $\mathcal{H}^1$-containis $T[s]\cap E$. In particular, $T[s]\cap E$ (which $\mathcal{H}^1$-equivalent to $T[s]\setminus E^{(0)}$) is not $\mathcal{H}^1$-contained in $\mathrm{UBEP}(K\cup T[s];T)$, and we see again, this time from the point of view of Definition B as reformulated in Theorem 1.3, that K is not $\mathcal{C}$-spanning. As stated in Theorem 3.1, from the viewpoint of Definition B it is only the $\mathcal{H}^1$-containment of $T[s]\cap E^{(0)}$ into $\mathrm{UBEP}(K\cup T[s];T)$ that establishes the $\mathcal{C}$-spanning property of $K\cup E$: and this $\mathcal{H}^1$-containment indeed holds, since $T[s]\cap E^{(0)}=T[s]\setminus \mathrm{cl}\,(E)$ is $\mathcal{H}^1$-contained in the union of $\partial^*U_2\cap\partial^*U_3$ and $\partial^*U_4\cap\partial^*U_5$.

is $\mathcal{C}$-spanning $\mathbf{W}$ *if and only if, for every* $(\gamma,\Phi,T)\in\mathcal{T}(\mathcal{C})$ *and* $\mathcal{H}^1$*-a.e.* $s\in\mathbb{S}^1$,

$$T[s]\cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K\cup T[s];T)\,. \tag{3.2}$$

Remark 3.2. An immediate corollary of Theorem 3.1 is that if K is $\mathcal{H}^n$-finite and $(K,E)\in\mathcal{K}_{\rm B}$ then $K\cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ if and only if $\mathcal{R}(K)\cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$. Indeed, $\mathcal{R}(K\cup T[s])=\mathcal{R}(K)\cup T[s]$, so that, by (1.13), $\mathrm{UBEP}(K\cup T[s])=\mathrm{UBEP}(\mathcal{R}(K)\cup T[s])$.

Proof of Theorem 1.3. This is Theorem 3.1 with $E=\varnothing$. □

Proof of Theorem 3.1. Step one: We prove the following claim: If S essentially disconnects G into $\{G_1,G_2\}$ and $H\subset G$ satisfies

$$\min\{|H\cap G_1|\,,\,|H\cap G_2|\}>0\,, \tag{3.3}$$

then S essentially disconnects H into $H\cap G_1$ and $H\cap G_2$. Indeed, if $x\in H^{(1)}$, then $x\in\partial^e(H\cap G_i)$ if and only if $x\in\partial^e G_i$ $(i=1,2)$. Hence $H^{(1)}\cap\partial^e(G_1\cap H)\subset H^{(1)}\cap\partial^e G_1\subset G^{(1)}\cap\partial^e G_1$, which, by (3.3) and our assumption on S and G, gives the desired conclusion.

Step two: Taking from now on S, K and E as in the statement we preliminary notice that if $(\gamma,\Phi,T)\in\mathcal{T}(\mathcal{C})$, $s\in\mathbb{S}^1$, and $\{U_i\}_i$ is the essential partition of T induced by $(\mathcal{R}(K)\cup T[s])$, then

$$T\cap\partial^*E\overset{\mathcal{H}^n}{\subset}T\cap\bigcup_i\partial^*U_i\,. \tag{3.4}$$

Indeed, since $\Omega\cap\partial^*E$ is $\mathcal{H}^n$-contained in $\mathcal{R}(K)$, if a Borel set G is such that $|G\cap E|\,|G\setminus E|>0$ then, by step one, $\mathcal{R}(K)$ essentially disconnects G. In particular, since, for each i, $\mathcal{R}(K)\cup T[s]$ does not essentially disconnect U_i, we find that, for each i,

$$\text{either } U_i^{(1)}\subset E^{(0)}\qquad\text{or } U_i^{(1)}\subset E^{(1)}\,. \tag{3.5}$$

Clearly, (3.5) immediately implies (3.4).

Step three: We prove the "only if" part of the statement, that is, given $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ and $s \in \mathbb{S}^1$, we assume that

$$\begin{aligned} &\text{for } \mathcal{H}^n\text{-a.e. } x \in T[s]\,, \\ &\exists \text{ a partition } \{T_1, T_2\} \text{ of } T \text{ with } x \in \partial^e T_1 \cap \partial^e T_2\,, \\ &\text{and s.t. } \mathcal{R}(K) \cup E^{(1)} \cup T[s] \text{ essentially disconnects } T \text{ into } \{T_1, T_2\}\,, \end{aligned} \tag{3.6}$$

and then prove that

$$T[s] \cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \textstyle\bigcup_i \partial^* U_i\,, \tag{3.7}$$

where $\{U_i\}_i$ is the essential partition of T induced by $\mathcal{R}(K) \cup T[s]$. To this end, arguing by contradiction, we suppose that for some $s \in \mathbb{S}^1$, there is $G \subset T[s] \cap E^{(0)}$ with $\mathcal{H}^n(G) > 0$ and such that $G \cap \cup_i \partial^* U_i = \varnothing$. In particular, there is an index i such that $\mathcal{H}^n(G \cap U_i^{(1)}) > 0$, which, combined with (3.5) and $G \subset E^{(0)}$, implies

$$U_i^{(1)} \subset E^{(0)}\,. \tag{3.8}$$

Now by (3.6) and $\mathcal{H}^n(G \cap U_i^{(1)}) > 0$, we can choose $x \in G \cap U_i^{(1)}$ such that $\mathcal{R}(K) \cup E^{(1)} \cup T[s]$ essentially disconnects T into some $\{T_1, T_2\}$ such that $x \in \partial^e T_1 \cap \partial^e T_2$. Then, $\{U_i \cap T_1, U_i \cap T_2\}$ is a non-trivial partition of U_i, so that, by step one and (3.8), $\mathcal{R}(K) \cup T[s]$ essentially disconnects U_i into $\{U_i \cap T_1, U_i \cap T_2\}$. This contradicts the defining property (2.2) of essential partitions, and concludes the proof.

Step four: We prove the "if" part of the statement. More precisely, given $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ and $s \in \mathbb{S}^1$, we assume that (3.7) holds at s, and then proceed to prove that (3.6) holds at s. We first notice that, since $\{E^{(1)}, E^{(0)}, \partial^* E\}$ is a partition of Ω modulo $\mathcal{H}^n$, it is enough to prove (3.6) for $\mathcal{H}^n$-a.e. $x \in T[s] \cap (E^{(1)} \cup E^{(0)} \cup \partial^* E)$.

If $x \in T[s] \cap \partial^* E$, then by letting $T_1 = T \cap E$ and $T_2 = T \setminus E$ we obtain a partition of T such that $x \in T \cap \partial^* E = T \cap \partial^* T_1 \cap \partial^* T_2 \subset \partial^e T_1 \cap \partial^e T_2$, and such that $\partial^* E$ essentially disconnects T into $\{T_1, T_2\}$. Since $\Omega \cap \partial^* E$ is $\mathcal{H}^n$-contained in $\mathcal{R}(K)$, we deduce (3.6).

If $x \in T[s] \cap E^{(0)}$, then, thanks to (3.7) and denoting by $\{U_i\}_i$ the essential partition of T induced by $(\mathcal{R}(K) \cup T[s])$, there is an index i such that $x \in T \cap \partial^* U_i$. Setting $T_1 = U_i$ and $T_2 = T \setminus U_i$, we have that $T \cap \partial^* U_i$ (which contains x) is in turn contained into $\partial^e T_1 \cap \partial^e T_2 \cap T$. Since the latter set is non-empty, $\{T_1, T_2\}$ is a non-trivial partition of T. Moreover, by definition of essential partition,

$$T^{(1)} \cap \partial^e T_1 \cap \partial^e T_2 = T \cap \partial^e U_i \overset{\mathcal{H}^n}{\subset} \mathcal{R}(K) \cup T[s]\,,$$

so that $\mathcal{R}(K) \cup T[s]$ essentially disconnects T, and (3.6) holds.

Finally, if $x \in T[s] \cap E^{(1)}$, we let $s_1 = s$, pick $s_2 \neq s$, denote by $\{I_1, I_2\}$ the partition of $\mathbb{S}^1$ defined by $\{s_1, s_2\}$, and set

$$T_1 = \Phi(I_1 \times B_1^n) \cap E\,, \qquad T_2 = \Phi(I_2 \times B_1^n) \cup \Big(\Phi(I_1 \times B_1^n) \setminus E\Big)\,.$$

This is a Borel partition of T, and using the fact that $x \in E^{(1)}$, we compute

$$|T_1 \cap B_r(x)| = |\Phi(I_1 \times B_1^n) \cap E \cap B_r(x)| = |\Phi(I_1 \times B_1^n) \cap B_r(x)| + \mathrm{o}(r^{n+1}) = \frac{|B_r(x)|}{2} + \mathrm{o}(r^{n+1})\,.$$

Therfore $x \in \partial^e T_1 \cap \partial^e T_2$, and by standard facts about reduced boundaries [Mag12, Chapter 16],

$$\partial^e T_1 \cap \partial^e T_2 \cap T^{(1)} \overset{\mathcal{H}^n}{\subset} \partial^* T_1 \cap T^{(1)} \overset{\mathcal{H}^n}{\subset} \big(\partial^* E \cup ((T[s_1] \cup T[s_2]) \cap E^{(1)})\big) \cap T^{(1)}\,.$$

Since $\Omega \cap \partial^* E$ is $\mathcal{H}^n$-contained in $\mathcal{R}(K)$, we have shown (3.6). □

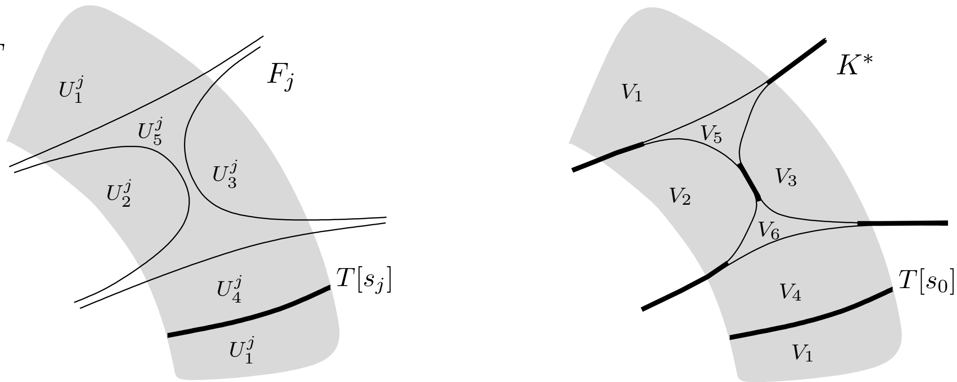

FIGURE 4.1. The situation in the proof of Theorem 4.1 in the basic case when $K_j = \Omega \cap \partial^* F_j$. The essential partition of T induced by $K_j \cup T[s_j]$ is denoted by $\{U_i^j\}_i$. The limit partition $\{U_i\}_i$ of $\{U_i^j\}_i$ may fail to be the essential partition of T induced by $K^* = T \cap \cup_i \partial^* U_i$, since some of the U_i may be essentially disconnected. In the picture, denoting by $\{V_k\}_k$ the essential partition of T induced by K^*, we have $U_5 = V_5 \cup V_6 = T \cap F$. We also notice, in reference to the notation set in (4.6), that $X_1^j = \{5\}$ and $X_0^j = \{1, 2, 3, 4\}$.

4. The fundamental closure theorem for homotopic spanning conditions

In Theorem 1.3 and Theorem 3.1 we have presented two reformulations of the homotopic spanning condition in terms of $\mathcal{H}^n$-containment into union of boundaries of essential partitions. The goal of this section is discussing the closure of such reformulations, and provide a statement (Theorem 4.1 below) which will lie at the heart of the closure theorems proved in Section 5.

Theorem 4.1 (Basic closure theorem for homotopic spanning). *Let $\mathbf{W} \subset \mathbb{R}^{n+1}$ be closed and let $\mathcal{C}$ be a spanning class for $\mathbf{W}$. Let us assume that:*

(a): *K_j are $\mathcal{H}^n$-finite Borel subsets of Ω with $\mathcal{H}^n \llcorner K_j \stackrel{*}{\rightharpoonup} \mu$ as Radon measures in Ω;*

(b): *$(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, $\{s_j\}_j$ is a sequence in $\mathbb{S}^1$ with $s_j \to s_0$ as $j \to \infty$;*

(c): *if $\{U_i^j\}_i$ denotes the essential partition of T induced by $K_j \cup T[s_j]$, then there is a limit partition $\{U_i\}_i$ of $\{U_i^j\}_i$ in the sense of (2.8) in Lemma 2.3;*

Under these assumptions, if $\mu(T[s_0]) = 0$, $F_j, F \subset \Omega$ are sets of finite perimeter with $F_j \to F$ as $j \to \infty$ and such that, for every j, $\Omega \cap \partial^* F_j$ is $\mathcal{H}^n$-contained in K_j and

$$T[s_j] \cap F_j^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } K_j^* \,, \tag{4.1}$$

then

$$T[s_0] \cap F^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } K^* \,, \tag{4.2}$$

where we have set

$$K_j^* = \mathrm{UBEP}(K_j \cup T[s_j]; T) = T \cap \bigcup_i \partial^* U_i^j \,, \qquad K^* = T \cap \bigcup_i \partial^* U_i \,. \tag{4.3}$$

Remark 4.2. Notice that $\{U_i\}_i$ may fail to be the essential partition of T induced by K^* (which is the "optimal" choice of a Borel set potentially inducing $\{U_i\}_i$ on T): indeed, some of the sets U_i may fail to be essentially connected, even though $U_i^j \to U_i$ as $j \to \infty$ and every U_i^j, as an element of an essential partition, is necessarily essentially connected; see Figure 4.1.

Proof of Theorem 4.1. Step one: We start by showing that, for each j and i such that $|U_i^j|>0$, we have

$$\text{either} \quad (U_i^j)^{(1)}\subset F_j^{(1)}\,, \qquad \text{or} \quad (U_i^j)^{(1)}\subset F_j^{(0)}\,, \tag{4.4}$$

and for each i such that $|U_i|>0$,

$$\text{either} \quad U_i^{(1)}\subset F^{(1)}\,, \qquad \text{or} \quad U_i^{(1)}\subset F^{(0)}\,. \tag{4.5}$$

Postponing for the moment the proof of (4.4) and (4.5), let us record several consequences of these inclusions. First, if we set

$$X_1^j=\big\{i:|U_i^j|>0\,,\,(U_i^j)^{(1)}\subset F_j^{(1)}\big\}\,, \qquad X_0^j=\big\{i:|U_i^j|>0\,,\,(U_i^j)^{(1)}\subset F_j^{(0)}\big\}\,, \tag{4.6}$$

$$X_1=\big\{i:|U_i|>0\,,\,U_i^{(1)}\subset F^{(1)}\big\}\,, \qquad X_0=\big\{i:|U_i|>0\,,\,U_i^{(1)}\subset F^{(0)}\big\}\,, \tag{4.7}$$

then, thanks to (4.4) and (4.5), we have

$$X^j:=\{i:|U_i^j|>0\}=X_0^j\cup X_1^j\,, \qquad X:=\{i:|U_i|>0\}=X_0\cup X_1\,. \tag{4.8}$$

Combining (4.4) and (4.5) with $F_j\to F$ and $U_i^j\to U_i$, we find that for every $i\in X$, there is $J_i\in\mathbb{N}$ such that, for every $m\in\{0,1\}$,

$$\text{if } i\in X_m, \text{ then } i\in X_m^j \text{ for all } j\ge J_i. \tag{4.9}$$

Lastly, $\{U_i^j\}_{i\in X_1^j}$ is a Lebesgue partition of $T\cap F_j$, and thus, by Federer's theorem (1.37),

$$T\cap F_j^{(1)}\overset{\mathcal{H}^n}{\subset}\bigcup_{i\in X_1^j}(U_i^j)^{(1)}\cup\partial^*U_i^j\,, \qquad T\cap\partial^*F_j\overset{\mathcal{H}^n}{\subset}T\cap\bigcup_{i\in X_1^j}\partial^*U_i^j\ \subset\ T\cap K_j^*\,. \tag{4.10}$$

To prove (4.4) *and* (4.5): Since $\{U_i^j\}_i$ is the essential partition of T induced by $K_j\cup T[s_j]$ and $K_j^*=\mathrm{UBEP}(K_j\cup T[s_j];T)$, we have

$$K_j^* \text{ is } \mathcal{H}^n\text{-contained in } K_j\cup T[s_j]\,, \qquad \forall j\,, \tag{4.11}$$

$$K_j\cup T[s_j] \text{ does not essentially disconnect } U_i^j\,, \qquad \forall i,j\,. \tag{4.12}$$

Since $\Omega\cap\partial^*F_j$ is $\mathcal{H}^n$-contained in $K_j\cup T[s_j]$, the combination of (4.12) with Federer's theorem (1.37) gives (4.4). The combination of $|U_i^j\Delta U_i|\to 0$ as $j\to\infty$ with (4.4) gives (4.5).

Step two: We reduce the proof of (4.2) to that of

$$\mathcal{H}^n(U_i^{(1)}\cap T[s_0])=0\,, \qquad \forall i\in X_0\,. \tag{4.13}$$

Indeed, $\{U_i^{(1)}:i\in X_0\}\cup\{F^{(0)}\cap\partial^*U_i:i\in X_0\}$ is an $\mathcal{H}^n$-partition of $T\cap F^{(0)}$. In particular, $T\cap F^{(0)}$ is $\mathcal{H}^n$-contained in $\cup_{i\in X_0}U_i^{(1)}\cup\partial^*U_i$, so that, should (4.13) hold, then $T[s_0]\cap F^{(0)}$ would be $\mathcal{H}^n$-contained in $\cup_{i\in X_0}\partial^*U_i$, and thus in K^*, thus proving (4.2).

Step three: We change variables from T to[10] $Y=\Phi^{-1}(T)=\mathbb{S}^1\times B_1^n$. We set $Y[s]=\Phi^{-1}(T[s])=\{s\}\times B_1^n$ for the s-slice of Y, and

$$Y_i=\Phi^{-1}(U_i)\,, \qquad Y_i^j=\Phi^{-1}(U_i^j)\,, \qquad W_i=Y\setminus Y_i\,, \qquad W_i^j=Y\setminus Y_i^j\,, \tag{4.14}$$

Since Φ is a diffeomorphism, by [KMS22a, Lemma A.1] and the area formula we have that

$$\partial^*\Phi^{-1}(H)=\Phi^{-1}(\partial^*H)\,, \qquad (\Phi^{-1}(H))^{(m)}=\Phi^{-1}(H^{(m)})\,,m\in\{0,1\}\,, \tag{4.15}$$

for every set of finite perimeter $H\subset T$; in particular, setting

$$M_j=\Phi^{-1}(F_j\cap T)\,, \qquad M=\Phi^{-1}(F\cap T)\,,$$

[10] Here we identify $\mathbb{S}^1$ with $\mathbb{R}/(2\pi\mathbb{Z})$ and, with a slight abuse of notation, denote by $\mathcal{L}^{n+1}$ the "Lebesgue measure on $\mathbb{S}^1\times B_1^n$", which we use to define sets of finite perimeter and points of density in $\mathbb{S}^1\times B_1^n$.

by Federer's theorem (1.37), we see that (4.1) is equivalent

$$Y[s_j] \text{ is } \mathcal{H}^n\text{-contained in } \bigcup_i \partial^* Y_i^j \cup M_j^{(1)} \cup \partial^* M_j \,, \tag{4.16}$$

By (4.10) and (4.15), we may rewrite (4.16) as

$$Y[s_j] \text{ is } \mathcal{H}^n\text{-contained in } \bigcup_{i\in\mathbb{N}} \partial^* Y_i^j \cup \bigcup_{i\in X_1^j} (Y_i^j)^{(1)} \,. \tag{4.17}$$

Similarly, $Y_i^{(1)} = \Phi^{-1}(U_i^{(1)})$ for every i, and thus (4.13) is equivalent to

$$\mathcal{H}^n(Y_i^{(1)} \cap Y[s_0]) = 0 \,, \qquad \forall i \in X_0 \,. \tag{4.18}$$

We are thus left to prove that (4.17) implies (4.18).

To this end, let us denote by $\mathbf{p}$ the projection of $Y = \mathbb{S}^1 \times B_1^n$ onto B_1^n, and consider the sets

$$G_i = \mathbf{p}\big(Y_i^{(1)} \cap Y[s_0]\big) \,, \qquad G_i^* = G^* \cap G_i \,,$$

corresponding to the set $G^* \subset B_1^n$ with $\mathcal{H}^n(B_1^n \setminus G^*) = 0$ defined as follows:

(i) denoting by $H_y = \{s \in \mathbb{S}^1 : (s,y) \in H\}$ the "circular slice of $H \subset Y$ above y", if $y \in G^*$, $j \in \mathbb{N}$, k is an index for the partitions $\{Y_k\}_k$ and $\{Y_k^j\}$, and $H \in \{Y_k, W_k, Y_k^j, W_k^j\}$, then H_y is a set of finite perimeter in $\mathbb{S}^1$ with

$$H_y \overset{\mathcal{H}^1}{=} (H_y)^{(1)_{\mathbb{S}^1}} \,, \qquad \partial^*_{\mathbb{S}^1}(H_y) \overset{\mathcal{H}^0}{=} (\partial^* H)_y \,, \tag{4.19}$$

(and thus with $\partial^*_{\mathbb{S}^1}(H_y) = (\partial^* H)_y$); this is a standard consequence of the slicing theory for sets of finite perimeter, see, e.g., [BCF13, Theorem 2.4] or [Mag12, Remark 18.13];

(ii) for every $y \in G^*$ and $j \in \mathbb{N}$,

$$(s_j, y) \in \bigcup_{k\in\mathbb{N}} \partial^* Y_k^j \cup \bigcup_{k\in X_1^j} (Y_k^j)^{(1)} \,; \tag{4.20}$$

this is immediate from (4.17);

(iii) for every $y \in G^*$, and k an index for the partitions $\{Y_k\}_k$ and $\{Y_k^j\}$,

$$\lim_{j\to\infty} \mathcal{H}^1((Y_k)_y \Delta (Y_k^j)_y) = 0 \,; \tag{4.21}$$

this is immediate from Fubini's theorem and $Y_k^j \to Y_k$ as $j \to \infty$;

(iv) for every $y \in G^*$,

$$\sum_k \mathcal{H}^0((\partial^* Y_k^j)_y) < \infty; \tag{4.22}$$

indeed, by applying in the order the coarea formula, the area formula and (2.3) we find

$$\begin{aligned} \sum_k \int_{B_1^n} \mathcal{H}^0((\partial^* Y_k^j)_y)\, d\mathcal{H}^n &\le \sum_k P(Y_k^j; Y) \le (\mathrm{Lip}\Phi^{-1})^n \sum_k P(U_k^j; T) \\ &\le 2\,(\mathrm{Lip}\Phi^{-1})^n\, \mathcal{H}^n(K_j \cup T[s_j]) \,. \end{aligned}$$

Now, let us pick $y \in G_i^*$. Since $y \in G_i$ implies $(s_0, y) \in Y_i^{(1)}$, and $Y_i^{(1)} \cap \partial^* Y_i = \varnothing$, we find $(s_0, y) \notin \partial^* Y_i$, i.e. $s_0 \notin (\partial^* Y_i)_y$. By $y \in G^*$, we have $(\partial^* Y_i)_y = \partial^*_{\mathbb{S}^1}(Y_i)_y$, so that

$$s_0 \notin \partial^*_{\mathbb{S}^1}(Y_i)_y \,. \tag{4.23}$$

Since $(Y_i)_y$ has finite perimeter, $\partial^*_{\mathbb{S}^1}(Y_i)_y$ is a finite set, and so (4.23) implies the existence of an open interval $\mathcal{A}_y \subset \mathbb{S}^1$, containing s_0, $\mathcal{H}^1$-contained either in $(Y_i)_y$ or in $(W_i)_y$, and such that

$$\partial_{\mathbb{S}^1} \mathcal{A}_y \subset (\partial^* Y_i)_y = \partial^*_{\mathbb{S}^1}(W_i)_y \,. \tag{4.24}$$

We claim that there is $G_i^{**} \subset G_i^*$, with full $\mathcal{H}^n$-measure in G_i^* (and thus in G_i), such that

$$\mathcal{A}_y \text{ is } \mathcal{H}^1\text{-contained in } (Y_i)_y\,, \qquad \forall y \in G_i^{**}\,. \tag{4.25}$$

Indeed, let us consider the countable decomposition $\{G_{i,m}^*\}_{m=1}^\infty$ of G_i^* given by

$$G_{i,m}^* = \Big\{ y \in G_i^* : \operatorname{dist}\big(\{s_0\}, \partial_{\mathbb{S}^1}\mathcal{A}_y\big) \in \big[1/(m+1), 1/m\big) \Big\} \subset B_1^n\,,$$

and let

$$Z_{i,m} = \big\{ y \in G_{i,m}^* : \mathcal{A}_y \text{ is } \mathcal{H}^1\text{-contained in } (W_i)_y \big\}\,.$$

If $\mathcal{H}^n(Z_{i,m}) > 0$, then there is $y^* \in Z_{i,m}^{(1)}$, so that $\mathcal{H}^n(Z_{i,m} \cap B_r^n(y^*)) = \omega_n\, r^n + \mathrm{o}(r^n)$. Therefore, if $r < 1/(m+1)$ and $B_r^1(s_0)$ denotes the open interval of center s_0 and radius r inside $\mathbb{S}^1$, then

$$\mathcal{L}^{n+1}\big(Y_i \cap \big(B_r^1(s_0) \times B_r^n(y^*)\big)\big) = \int_{B_r^n(y^*)} \mathcal{H}^1(B_r^1(s_0) \cap (Y_i)_y)\, d\mathcal{H}_y^n$$

$$= \int_{Z_{i,m}\cap B_r^n(y^*)} \mathcal{H}^1(B_r^1(s_0) \cap (Y_i)_y)\, d\mathcal{H}_y^n + \mathrm{o}(r^{n+1}) = \mathrm{o}(r^{n+1})$$

where in the last identity we have used the facts that $y \in Z_{i,m} \cap B_r^n(y^*)$, $s_0 \in \mathcal{A}_y$, and $r < 1/(m+1)$ to conclude that $B_r^1(s_0)$ is $\mathcal{H}^1$-contained in $(W_i)_y$; in particular, $(s_0, y^*) \in Y_i^{(0)}$, against the fact that $Z_{i,m} \subset G_i (= \mathbf{p}(Y[s_0] \cap Y_i^{(1)}))$. We have thus proved that each $Z_{i,m}$ is $\mathcal{H}^n$-negligible, and therefore that there is $G_i^{**} \subset G_i^*$ and $\mathcal{H}^n$-equivalent to G_i^*, such that (4.25) holds true.

Having proved (4.25), we now notice that, by (4.20), $y \in G_i^*$ implies

$$s_j \in \bigcup_{k\in\mathbb{N}} (\partial^* Y_k^j)_y \cup \bigcup_{k \in X_1^j} \big((Y_k^j)^{(1)}\big)_y = \bigcup_k \partial^*_{\mathbb{S}^1} (Y_k^j)_y \cup \bigcup_{k\in X_1^j} \big((Y_k^j)_y\big)^{(1)_{\mathbb{S}^1}}\,. \tag{4.26}$$

If (4.26) holds because $s_j \in \partial^*_{\mathbb{S}^1}(Y_k^j)_y$ for some k, then, thanks to (4.22) there must $k' \neq k$ such that $s_j \in \partial^*_{\mathbb{S}^1}(Y_{k'}^j)_y$ too; since either k or k' must be different from i, we conclude that $s_i \in \partial^*_{\mathbb{S}^1}(Y_{k(i)}^j)_y$ for some $k(i) \neq i$; if, instead, (4.26) holds because $s_j \in \big((Y_k^j)_y\big)^{(1)_{\mathbb{S}^1}}$ for some $k \in X_1^j$, then we can recall that, thanks to (4.9), $i \in X_0^j$ for every $j \geq J_i$, and thus $i \neq k$; in summary, for each $y \in G_i^*$,

$$\text{if } j \geq J_i, \text{ then } \exists k(j) \neq i \text{ s.t. } s_j \in \partial^*_{\mathbb{S}^1}(Y_{k(j)}^j)_y \cup \big((Y_{k(j)}^j)_y\big)^{(1)_{\mathbb{S}^1}}\,. \tag{4.27}$$

With the goal of obtaining a lower bound on the relative perimeters of the sets Y_i^j in a neighborhood of G_i (see (4.31) below), we now consider $y \in G_i^{**}$, and pick $r > 0$ such that $\operatorname{cl} B_r^1(s_0) \subset \mathcal{A}_y$. Correspondingly, since $s_j \to s_0$ and (4.27) holds, we can find $J^* = J^*(i, y, r) \geq J_i$ such that, for $j \geq J^*$,

$$s_j \in B_r^1(s_0) \cap \big[\partial^*_{\mathbb{S}^1}(Y_{k(j)}^j)_y \cup \big((Y_{k(j)}^j)_y\big)^{(1)_{\mathbb{S}^1}}\big] \subset \mathcal{A}_y \cap \big[\partial^*_{\mathbb{S}^1}(Y_{k(j)}^j)_y \cup \big((Y_{k(j)}^j)_y\big)^{(1)_{\mathbb{S}^1}}\big]\,. \tag{4.28}$$

Now, by (4.21), $k(j) \neq i$, and $\mathcal{A}_y \overset{\mathcal{H}^1}{\subset} (Y_i)_y$, we have

$$\lim_{j\to\infty} \mathcal{H}^1(\mathcal{A}_y \cap (Y_{k(j)}^j)_y) = 0\,. \tag{4.29}$$

Since, by (4.19), $(Y_{k(j)}^j)_y$ is $\mathcal{H}^1$-equivalent to a finite union of intervals, (4.28) implies the existence of an open interval $\mathcal{I}_y^j$ such that

$$s_j \in \operatorname{cl}_{\mathbb{S}^1} \mathcal{I}_y^j\,, \qquad \mathcal{I}_y^j \overset{\mathcal{H}^1}{\subset} (Y_{k(j)}^j)_y\,, \qquad \partial_{\mathbb{S}^1} \mathcal{I}_y^j \subset (\partial^* Y_{k(j)}^j)_y \subset (\partial^* W_i^j)_y\,, \tag{4.30}$$

which, due to (4.28) and (4.29), must satisfy

$$\lim_{j\to\infty} \operatorname{diam}\big(\mathcal{I}_y^j\big) = 0\,.$$

In particular,

$$\partial_{\mathbb{S}^1}\,\mathcal{I}_y^j \subset B_r^1(s_0)\,, \qquad \forall j \geq J^*\,,$$

and thus, by the last inclusion in (4.30),

$$\mathcal{H}^0\big(B_r^1(s_0)\cap \partial_{\mathbb{S}^1}^*(W_i^j)_y\big) \geq \mathcal{H}^0(B_r^1(s_0)\cap \partial_{\mathbb{S}^1}\mathcal{I}_j^y) \geq 2\,,$$

whenever $j \geq J^*$. Since $y \in G_i^{**}$ and $r > 0$ were arbitrary, by the coarea formula and Fatou's lemma,

$$\begin{aligned}
\liminf_{j\to\infty} P(W_i^j; B_r^1(s_0)\times G_i^{**}) \;&\geq\; \liminf_{j\to\infty}\int_{G_i^{**}} \mathcal{H}^0\big(B_r^1(s_0)\cap \partial_{\mathbb{S}^1}^*(W_i^j)_y\big)\, d\mathcal{H}_y^n \\
&\geq\; 2\,\mathcal{H}^n(G_i^{**}) = 2\,\mathcal{H}^n(G_i)\,.
\end{aligned} \tag{4.31}$$

Now, since $\partial^* W_i^j = \partial^* Y_i^j = \Phi^{-1}(\partial^* U_i^j)$, by (4.11) we have

$$Y \cap \textstyle\bigcup_i \partial^* W_i^j \text{ is } \mathcal{H}^n\text{-contained in } Y[s_j]\cup\Phi^{-1}\big(T\cap K_j\big)\,,$$

which implies, for every j large enough to have $s_j \in B_r^1(s_0)$,

$$\begin{aligned}
&P(W_i^j; B_r^1(s_0)\times G_i^{**}) \\
&\leq \mathcal{H}^n\big(Y[s_j]\cap(B_r^1(s_0)\times G_1^{**})\big) + \mathcal{H}^n\big(\Phi^{-1}(T\cap K_j)\cap(B_r^1(s_0)\times B_1^n)\big) \\
&= \mathcal{H}^n(G_i^{**}) + \mathcal{H}^n\big(\Phi^{-1}(T\cap K_j)\cap(B_r^1(s_0)\times B_1^n)\big) \\
&\leq \mathcal{H}^n(G_i) + \operatorname{Lip}(\Phi^{-1})^n\,\mathcal{H}^n\big(K_j\cap\Phi(B_r^1(s_0)\times B_1^n)\big)\,.
\end{aligned} \tag{4.32}$$

By combining (4.31) with (4.32) we conclude that for every $r > 0$

$$\mathcal{H}^n(G_i) \leq \operatorname{Lip}(\Phi^{-1})^n\,\mu\big(\Phi(\operatorname{cl}(B_r^1(s_0))\times B_1^n)\big)\,, \tag{4.33}$$

By $\mu(T[s_0]) = 0$, if we let $r \to 0^+$ in (4.33), we conclude that $\mathcal{H}^n(G_i) = 0$. Now, since $G_i = \mathbf{p}\big(Y_i^{(1)}\cap Y[s_0]\big)$, we have

$$\mathcal{H}^n\big(Y_i^{(1)}\cap Y[s_0]\big) = \mathcal{H}^n(G_i)\,, \tag{4.34}$$

thus proving (4.18), and hence the theorem. □

5. Direct Method on generalized soap films (Theorem 1.4)

In Section 5.1 we prove Theorem 1.4, while in Section 5.2 we notice the changes to that argument that are needed to prove a different closure theorem that will be crucial in the companion papers [MNR23a, MNR23b]. In particular, Section 5.2 will not be needed for the other main results of this paper (although it is included here since it is definitely easier to understand in this context).

5.1. **Proof of Theorem 1.4.** Let us first of all recall the setting of the theorem. We are given a closed set $\mathbf{W}$ in $\mathbb{R}^{n+1}$, a spanning class $\mathcal{C}$ for $\mathbf{W}$, and a sequence $\{(K_j, E_j)\}_j$ in $\mathcal{K}_{\rm B}$ such that

$$\sup_j \mathcal{H}^n(K_j) < \infty\,, \tag{5.1}$$

and, for some Borel set E and Radon measures $\mu_{\rm bk}$ and $\mu_{\rm bd}$ in Ω, it holds that $E_j \stackrel{\rm loc}{\to} E$ and

$$\mathcal{H}^n \llcorner (\Omega\cap\partial^* E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j)\cap E_j^{(0)}) \stackrel{*}{\rightharpoonup} \mu_{\rm bk}\,, \tag{5.2}$$

$$\mathcal{H}^n \llcorner (\Omega\cap\partial^* E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j)\setminus\partial^* E_j) \stackrel{*}{\rightharpoonup} \mu_{\rm bd}\,, \tag{5.3}$$

as $j \to \infty$. In this setting we want to prove that the sets

$$K_{\rm bk} := \big(\Omega\cap\partial^*E\big)\cup\Big\{x\in\Omega\cap E^{(0)}:\theta^n_*(\mu_{\rm bk})(x)\ge 2\Big\}\,, \tag{5.4}$$

$$K_{\rm bd} := \big(\Omega\cap\partial^*E\big)\cup\Big\{x\in\Omega\setminus\partial^*E:\theta^n_*(\mu_{\rm bd})(x)\ge 2\Big\}\,, \tag{5.5}$$

are such that $(K_{\rm bk},E),(K_{\rm bd},E)\in\mathcal{K}_{\rm B}$ and

$$\mu_{\rm bk} \ge \mathcal{H}^n\llcorner(\Omega\cap\partial^*E)+2\,\mathcal{H}^n\llcorner(K_{\rm bk}\cap E^{(0)})\,, \tag{5.6}$$

$$\mu_{\rm bd} \ge \mathcal{H}^n\llcorner(\Omega\cap\partial^*E)+2\,\mathcal{H}^n\llcorner(K_{\rm bd}\setminus\partial^*E)\,, \tag{5.7}$$

with

$$\liminf_{j\to\infty}\mathcal{F}_{\rm bk}(K_j,E_j)\ge\mathcal{F}_{\rm bk}(K_{\rm bk},E)\,,\qquad \liminf_{j\to\infty}\mathcal{F}_{\rm bd}(K_j,E_j)\ge\mathcal{F}_{\rm bd}(K_{\rm bd},E)\,; \tag{5.8}$$

and that the closure statements

$$\text{if } K_j\cup E_j^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for every } j, \tag{5.9}$$

$$\text{then } K_{\rm bk}\cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \tag{5.10}$$

and

$$\text{if } K_j \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for every } j, \tag{5.11}$$

$$\text{then } K_{\rm bd} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \tag{5.12}$$

hold true.

Proof of Theorem 1.4. By $\Omega\cap\partial^*E\subset K_{\rm bk}\cap K_{\rm bd}$ we have $(K_{\rm bk},E),(K_{\rm bd},E)\in\mathcal{K}_{\rm B}$. By [Mag12, Theorem 6.4], $\theta^n_*(\mu_{\rm bk})\ge 2$ on $K_{\rm bk}\cap E^{(0)}$ implies $\mu_{\rm bk}\llcorner(K_{\rm bk}\cap E^{(0)})\ge 2\mathcal{H}^n\llcorner(K_{\rm bk}\cap E^{(0)})$, and, similarly, we have $\mu_{\rm bd}\llcorner(K_{\rm bd}\setminus\partial^*E)\ge 2\,\mathcal{H}^n\llcorner(K_{\rm bd}\setminus\partial^*E)$. Since, by the lower semicontinuity of distributional perimeter, we have $\min\{\mu_{\rm bk},\mu_{\rm bd}\}\ge\mathcal{H}^n\llcorner(\partial^*E\cap\Omega)$, (5.6), (5.7) and (5.8) follow. We are thus left to prove that if either (5.9) or (5.11) holds, then (5.10) or (5.12) holds respectively. We divide the proof into three parts, numbered by Roman numerals.

I. Set up of the proof: Fixing from now on a choice of $(\gamma,\Phi,T)\in\mathcal{T}(\mathcal{C})$ against which we want to test the $\mathcal{C}$-spanning properties (5.10) and (5.12), we introducing several key objects related to (γ,Φ,T).

Introducing s_0: Up to extracting subsequences, let μ be the weak-star limit of $\mathcal{H}^n\llcorner K_j$, and set

$$J=\{s\in\mathbb{S}^1:\mu(T[s])=0\}\,, \tag{5.13}$$

so that $\mathcal{H}^1(\mathbb{S}^1\setminus J)=0$. We fix $s_0\in J$.

Introducing s_j, $\{U_i^j\}_i$, and K_j^*: For $\mathcal{H}^1$-a.e. $s\in\mathbb{S}^1$ it holds that $\mathcal{H}^n(K_j\cap T[s])=0$ for every j and (thanks to Theorem 1.3/Theorem 3.1) the essential partition $\{U_i^j[s]\}_i$ induced on T by $K_j\cup T[s]$ is such that

$$T[s]\cap E_j^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_j\cup T[s];T)\,, \qquad \text{(if (5.9) holds)}\,,$$

$$T[s] \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_j\cup T[s];T)\,, \qquad \text{(if (5.11) holds)}\,.$$

Therefore we can find a sequence $s_j\to s_0$ as $j\to\infty$ such that

$$\mathcal{H}^n(K_j\cap T[s_j])=0 \qquad \forall j\,, \tag{5.14}$$

and, denoting by $\{U_i^j\}_i$ the essential partition of T induced by $K_j\cup T[s_j]$ (i.e. $U_i^j=U_i^j[s_j]$), and setting for brevity

$$K_j^*=\mathrm{UBEP}(K_j\cup T[s_j];T)=T\cap\bigcup_i\partial^*U_i^j\,, \tag{5.15}$$

we have

$$T[s_j]\cap E_j^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } K_j^*\,, \qquad \text{(if (5.9) holds)}\,, \tag{5.16}$$

$$T[s_j] \text{ is } \mathcal{H}^n\text{-contained in } K_j^*\,, \qquad \text{(if (5.11) holds)}\,. \tag{5.17}$$

Introducing $\{U_i\}_i$ *and* K^**:* By (5.1), Lemma 2.3, and up to extract a subsequence we can find a Lebesgue partition $\{U_i\}_i$ of T such that,

$$\{U_i\}_i \text{ is the limit of } \{\{U_i^j\}_i\}_j \text{ in the sense specified by (2.8)}\,. \tag{5.18}$$

Correspondingly we set

$$K^* = T\cap\bigcup_i \partial^* U_i\,. \tag{5.19}$$

Having introduced s_0, s_j, $\{U_i^j\}_i$, K_j^*, $\{U_i\}_i$, and K^*, we notice that if (5.9) holds, then we can apply Theorem 4.1 with $F_j = E_j$ and find that

$$T[s_0]\cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } K^*\,, \qquad \text{(if (5.9) holds)}\,; \tag{5.20}$$

if, instead, (5.11) holds, then Theorem 4.1 can be applied with $F_j = F = \varnothing$ to deduce

$$T[s_0] \text{ is } \mathcal{H}^n\text{-contained in } K^*\,, \qquad \text{(if (5.11) holds)}\,. \tag{5.21}$$

We now make the following claim:

Claim: We have

$$K^*\setminus(T[s_0]\cup E^{(1)}) \text{ is } \mathcal{H}^n\text{-contained in } K_{\rm bk}\,, \tag{5.22}$$

$$K^*\setminus T[s_0] \text{ is } \mathcal{H}^n\text{-contained in } K_{\rm bd}\,. \tag{5.23}$$

The rest of the proof of the theorem is then divided in two parts: the conclusion follows from the claim, and the proof of the claim.

II. Conclusion of the proof from the claim: *Proof that* (5.11) *implies* (5.12): By $\mathcal{H}^1(\mathbb{S}^1\setminus J)=0$, the arbitrariness of $s_0\in J$, and that of $(\gamma,\Phi,T)\in\mathcal{T}(\mathcal{C})$, thanks to Theorem 1.3 we can conclude that $K_{\rm bd}$ is $\mathcal{C}$-spanning $\mathbf{W}$ by showing that

$$T[s_0] \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_{\rm bd}\cup T[s_0];T). \tag{5.24}$$

Now, since $\{U_i\}_i$ is a Lebesgue partition of T induced by K^* (in the very tautological sense that K^* is defined as $T\cap\cup_i\partial^*U_i$!) and, by (5.23) in claim one, K^* is $\mathcal{H}^n$-contained in $K_{\rm bd}\cup T[s_0]$, by Theorem 2.1-(a) we have that if $\{Z_i\}_i$ is the essential partition of T induced by $K_{\rm bd}\cup T[s_0]$, then $\cup_i\partial^*U_i$ is $\mathcal{H}^n$-contained in $\cup_i\partial^*Z_i$: therefore, by definition of K^* and by definition of UBEP, we have that

$$K^* \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}\big(K_{\rm bd}\cup T[s_0];T\big)\,. \tag{5.25}$$

By combining (5.25) with (5.21) we immediately deduce (5.24) and conclude.

Proof that (5.9) *implies* (5.10): Thanks to Theorem 3.1 it suffices to prove that

$$T[s_0]\cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_{\rm bk}\cup T[s_0];T)\,. \tag{5.26}$$

By (5.20), the proof of (5.26) can be reduced to that of

$$K^*\cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_{\rm bk}\cup T[s_0];T)\,. \tag{5.27}$$

Now, let us consider the Lebesgue partition of T defined by $\{V_k\}_k = \{U_i\setminus E\}_i\cup\{T\cap E\}$. By [Mag12, Theorem 16.3] we easily see that for each i

$$E^{(0)}\cap\partial^*U_i \overset{\mathcal{H}^n}{\subset} \partial^*(U_i\setminus E) \overset{\mathcal{H}^n}{\subset} \big(E^{(0)}\cap\partial^*U_i\big)\cup\partial^*E\,, \tag{5.28}$$

which combined with $T\cap\partial^*(T\cap E)=T\cap\partial^*E\subset K_{\rm bk}$ and with (5.22) in claim one, gives

$$T\cap\bigcup_k\partial^*V_k \;=\; (T\cap\partial^*E)\cup\Big\{T\cap\bigcup_i\partial^*(U_i\setminus E)\Big\} \overset{\mathcal{H}^n}{\subset} (T\cap\partial^*E)\cup\big(E^{(0)}\cap K^*\big)$$

$$\overset{\mathcal{H}^n}{\subset} \quad (T\cap\partial^*E)\cup\big(K^*\setminus E^{(1)}\big)\overset{\mathcal{H}^n}{\subset} K_{\rm bk}\cup T[s_0]\,. \tag{5.29}$$

By (5.29) we can exploit Theorem 2.1-(a) to conclude that

$$T\cap\bigcup_k\partial^*V_k \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K_{\rm bk}\cup T[s_0];T)\,. \tag{5.30}$$

By the first inclusion in (5.28), $E^{(0)}\cap K^*$ is $\mathcal{H}^n$-contained in $T\cap\bigcup_k\partial^*V_k$, therefore (5.30) implies (5.27), as required. We are thus left to prove the two claims.

III. Proof of the claim: We finally prove that $K^*\setminus(T[s_0]\cup E^{(1)})$ is $\mathcal{H}^n$-contained in $K_{\rm bk}$ (that is (5.22)), and that $K^*\setminus T[s_0]$ is $\mathcal{H}^n$-contained in $K_{\rm bd}$ (that is (5.23)).

To this end, repeating the argument in the proof of Theorem 4.1 with $F_j=E_j$ and $F=E$ we see that, if we set $X_m^j=\{i:(U_i^j)^{(1)}\subset E_j^{(m)}\}$ and $X_m=\{i:U_i^{(1)}\subset E^{(m)}\}$ for $m\in\{0,1\}$ (see (4.6) and (4.7)), then

$$X^j:=\{i:|U_i^j|>0\}=X_0^j\cup X_1^j\,,\qquad X:=\{i:|U_i|>0\}=X_0\cup X_1\,; \tag{5.31}$$

and, moreover, for every i there is $j(i)$ such that $i\in X_m$ implies $i\in X_m^j$ for every $j\ge j(i)$. Thanks to (5.31) we easily see that $K_j^*=T\cap\cup_i\partial^*U_i^j$ can be decomposed as

$$K_j^*\overset{\mathcal{H}^n}{=}\bigcup_{(i,k)\in X_0^j\times X_0^j,i\ne j}M_{ik}^j\cup\bigcup_{(i,k)\in X_1^j\times X_1^j,i\ne j}M_{ik}^j\cup\bigcup_{(i,k)\in X_0^j\times X_1^j}M_{ik}^j\,, \tag{5.32}$$

where $M_{ik}^j=T\cap\partial^*U_i^j\cap\partial^*U_k^j$ (an analogous decomposition of K^* holds as well, and will be used in the following, but is not explicitly written for the sake of brevity). We now prove that

$$M_{ik}^j\subset E_j^{(0)}\,,\qquad\forall i,k\in X_0^j\,,i\ne k\,, \tag{5.33}$$

$$M_{ik}^j\subset\partial^eE_j\,,\qquad\forall i\in X_0^j\,,k\in X_1^j\,, \tag{5.34}$$

$$M_{ik}^j\subset E_j^{(1)}\,,\qquad\forall i,k\in X_1^j\,,i\ne k\,. \tag{5.35}$$

To prove (5.33) *and* (5.35): if $i\ne k$, $i,k\in X_0^j$, and $x\in M_{ik}^j$, then (by $|U_i^j\cap U_k^j|=0$) U_i^j and U_k^j blow-up two complementary half-spaces at x, an information that combined with the $\mathcal{L}^{n+1}$-inclusion of $U_i^j\cup U_k^j$ in $\mathbb{R}^{n+1}\setminus E_j$ implies

$$|B_r(x)|+\mathrm{o}(r^{n+1})=|B_r(x)\cap U_i^j|+|B_r(x)\cap U_k^j|\le|B_r(x)\setminus E_j|\,,$$

that is, $x\in E_j^{(0)}$, thus proving (5.33); the proof of (5.35) is analogous.

To prove (5.34): if $i\in X_0^j$, $k\in X_1^j$, and $x\in M_{ik}^j$, then

$$|B_r(x)\cap E_j|\ge|B_r(x)\cap U_k^j|=\frac{|B_r(x)|}{2}+\mathrm{o}(r^{n+1})\,,$$

$$|B_r(x)\setminus E_j|\ge|B_r(x)\cap U_i^j|=\frac{|B_r(x)|}{2}+\mathrm{o}(r^{n+1})\,,$$

so that $x\notin E_j^{(0)}$ and $x\notin E_j^{(1)}$, i.e. $x\in\partial^eE_j$, that is (5.34).

With (5.33)–(5.35) at hand, we now prove that

$$T\cap\partial^*E_j\overset{\mathcal{H}^n}{=}\bigcup_{(i,k)\in X_0^j\times X_1^j}M_{ik}^j\,, \tag{5.36}$$

$$K_j^*\cap E_j^{(0)}\overset{\mathcal{H}^n}{=}\bigcup_{(i,k)\in X_0^j\times X_0^j,k\ne i}M_{ik}^j\,. \tag{5.37}$$

(Analogous relations hold with K^* and E in place of K_j^* and E_j.)

To prove (5.36): By $\partial^* E_j \subset \partial^e E_j$ and (4.4) we find $\partial^* E_j \cap (U_i^j)^{(1)} = \varnothing$ for every i, j; hence, since $\{(U_i^j)^{(1)}\}_i \cup \{\partial^* U_i^j\}_i$ is an $\mathcal{H}^n$-partition of T, and by repeatedly applying (5.33), (5.34) and (5.35), we find

$$\bigcup_{(i,k)\in X_0^j\times X_1^j} M_{ik}^j \overset{\mathcal{H}^n}{\subset} T\cap\partial^* E_j \overset{\mathcal{H}^n}{=} \bigcup_i (T\cap\partial^* E_j\cap\partial^* U_i^j) \overset{\mathcal{H}^n}{=} \bigcup_{i,k} M_{ik}^j\cap\partial^* E_j$$
$$\overset{\mathcal{H}^n}{=} \bigcup_{(i,k)\in X_0^j\times X_1^j} M_{ik}^j\cap\partial^* E_j\,,$$

which gives (5.36).

To prove (5.37): By (5.33), (5.34), and (5.35), M_{ik}^j has empty intersection with $E_j^{(0)}$ unless $i, k \in X_0^j$, in which case M_{ik}^j is $\mathcal{H}^n$-contained in $E_j^{(0)}$: hence,

$$\bigcup_{(i,k)\in X_0^j\times X_0^j,k\neq i} M_{ik}^j \overset{\mathcal{H}^n}{\subset} K_j^*\cap E_j^{(0)} = \bigcup_{(i,k)\in X_0^j\times X_0^j,k\neq i} E_j^{(0)}\cap M_{ik}^j\,,$$

that is (5.37).

With (5.36) and (5.37) at hand, we now prove the following perimeter formulas: for every open set $A\subset T$ and every j,

$$\sum_{i\in X_0^j} P(U_i^j;A) = \mathcal{H}^n\big(A\cap\partial^* E_j\big) + 2\,\mathcal{H}^n\big(A\cap K_j^*\cap E_j^{(0)}\big)\,, \tag{5.38}$$
$$\sum_{i\in X_1^j} P(U_i^j;A) = \mathcal{H}^n\big(A\cap\partial^* E_j\big) + 2\,\mathcal{H}^n\big(A\cap K_j^*\cap E_j^{(1)}\big)\,. \tag{5.39}$$

Analogously, for $\alpha = 0, 1$,

$$\sum_{i\in X_\alpha} P(U_i;A) = \mathcal{H}^n\big(A\cap\partial^* E\big) + 2\,\mathcal{H}^n\big(A\cap K^*\cap E^{(\alpha)}\big)\,. \tag{5.40}$$

To prove (5.38) *and* (5.39): Indeed, by (5.36) and (5.37),

$$\begin{aligned}\sum_{i\in X_0^j} P(U_i^j;A) &= \sum_{(i,k)\in X_0^j\times X_1^j} \mathcal{H}^n(A\cap M_{ik}^j) + \sum_{i\in X_0^j}\sum_{k\in X_0^j\setminus\{i\}} \mathcal{H}^n(A\cap M_{ik}^j)\\ &= \mathcal{H}^n\Big(\bigcup_{(i,k)\in X_0^j\times X_1^j} A\cap M_{ik}^j\Big) + 2\,\mathcal{H}^n\Big(\bigcup_{(i,k)\in X_0^j\times X_0^j,i\neq k} A\cap M_{ik}^j\Big)\\ &= \mathcal{H}^n(A\cap\partial^* E) + 2\,\mathcal{H}^n\big(A\cap K_j^*\cap E_j^{(0)}\big)\,,\end{aligned}$$

that is (5.38). The proof of (5.39) is analogous (since (5.39) is (5.38) applied to the complements of the E_j's – recall indeed that $\Omega\cap\partial^* E_j = \Omega\cap\partial^*(\Omega\setminus E_j)$).

Conclusion of the proof of (5.22) *in the claim*: We want to prove that $K^*\setminus(T[s_0]\cup E^{(1)})$ is $\mathcal{H}^n$-contained in $K_{\rm bk}$. Since $\{E^{(0)}, E^{(1)}, \partial^* E\}$ is an $\mathcal{H}^n$-partition of Ω, and $\Omega\cap\partial^* E$ is contained in $K_{\rm bk}$, looking back at the definition (5.4) of $K_{\rm bk}$ it is enough to show that

$$\theta_*^n(\mu_{\rm bk})(x) \ge 2 \text{ for } \mathcal{H}^n\text{-a.e. } x\in(K^*\cap E^{(0)})\setminus T[s_0]\,. \tag{5.41}$$

To this end, we begin noticing that, if Y_0 is an arbitrary finite subset of X_0, then there is $j(Y_0)$ such that $Y_0\subset X_0^j$ for every $j\ge j(Y_0)$; correspondingly,

$$\sum_{i\in Y_0} P(U_i;A) \le \liminf_{j\to\infty}\sum_{i\in Y_0} P(U_i^j;A) \le \liminf_{j\to\infty}\sum_{i\in X_0^j} P(U_i^j;A)\,.$$

By arbitrariness of Y_0, (5.40) with $\alpha=0$, (5.38), and (4.11) (notice that the $\mathcal{H}^n$-containment of the $\mathcal{H}^n$-rectifiable set K_j^* into $K_j\cup T[s_0]$ is equivalent to its $\mathcal{H}^n$-containment in $\mathcal{R}(K_j\cup T[s_j])=\mathcal{R}(K_j)\cup T[s_j])$ we conclude that, if $A\subset T$ is open and such that $\mathrm{cl}\,(A)\cap T[s_0]=\varnothing$, so that $A\cap T[s_j]=\varnothing$ for j large enough, then

$$
\begin{aligned}
&\mathcal{H}^n\big(A\cap\partial^*E\big)+2\,\mathcal{H}^n\big(A\cap K^*\cap E^{(0)}\big)\\
&=\sum_{i\in X_0}P(U_i;A)\le\liminf_{j\to\infty}\sum_{i\in X_0^j}P(U_i^j;A)\\
&=\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap K_j^*\cap E_j^{(0)}\big)\\
&\le\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap\big(\mathcal{R}(K_j)\cup T[s_j]\big)\cap E_j^{(0)}\big)\\
&=\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap\mathcal{R}(K_j)\cap E_j^{(0)}\big)\le\mu_{\rm bk}(\mathrm{cl}\,(A))\,,
\end{aligned}
\tag{5.42}
$$

where we have used the definition (5.2) of $\mu_{\rm bk}$. Now, if $x\in(K^*\cap E^{(0)})\setminus T[s_0]$, then we we can apply (5.42) with $A=B_s(x)$ and $s>0$ such that $\mathrm{cl}\,(B_s(x))\cap T[s_0]=\varnothing$, together with the fact that $x\in E^{(0)}$ implies $\mathcal{H}^n(B_s(x)\cap\partial^*E)=\mathrm{o}(s^n)$ as $s\to0^+$, to conclude that

$$
\mu_{\rm bk}(\mathrm{cl}\,(B_s(x)))\ge 2\,\mathcal{H}^n\big(B_s(x)\cap K^*\cap E^{(0)}\big)+\mathrm{o}(s^n)\,,\qquad\text{as }s\to0^+\,. \tag{5.43}
$$

Since $K^*\cap E^{(0)}$ is an $\mathcal{H}^n$-rectifiable set, and thus $\mathcal{H}^n\big(B_s(x)\cap K^*\cap E^{(0)}\big)=\omega_n\,s^n+\mathrm{o}(s^n)$ for $\mathcal{H}^n$-a.e. $x\in K^*\cap E^{(0)}$, we deduce (5.41) from (5.43).

Conclusion of the proof of (5.23) *in the claim*: We want to prove the $\mathcal{H}^n$-containment of $K^*\setminus T[s_0]$ in $K_{\rm bd}$. As in the proof of (5.22), combining Federer's theorem (1.37) with the definition (5.5) of $K_{\rm bd}$, we are left to prove that

$$
\theta^n_*(\mu_{\rm bd})(x)\ge 2\ \text{for}\ \mathcal{H}^n\text{-a.e.}\ x\in K^*\setminus(T[s_0]\cup\partial^*E)\,. \tag{5.44}
$$

As proved in (5.42), if $A\subset T$ is open and such that $\mathrm{cl}\,(A)\cap T[s_0]=\varnothing$, then by exploiting (5.38) and (5.40) with $\alpha=0$ we have

$$
\begin{aligned}
&\mathcal{H}^n\big(A\cap\partial^*E\big)+2\,\mathcal{H}^n\big(A\cap K^*\cap E^{(0)}\big)\\
&\le\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap\mathcal{R}(K_j)\cap E_j^{(0)}\big)\,;
\end{aligned}
\tag{5.45}
$$

the same argument, this time based on (5.39) and (5.40) with $\alpha=1$, also gives

$$
\begin{aligned}
&\mathcal{H}^n\big(A\cap\partial^*E\big)+2\,\mathcal{H}^n\big(A\cap K^*\cap E^{(1)}\big)\\
&\le\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap\mathcal{R}(K_j)\cap E_j^{(1)}\big)\,;
\end{aligned}
\tag{5.46}
$$

and, finally, since $\Omega\setminus\partial^*E$ is $\mathcal{H}^n$-equivalent to $\Omega\cap(E^{(0)}\cup E^{(1)})$, the combination of (5.45) and (5.46) gives

$$
\begin{aligned}
&\mathcal{H}^n\big(A\cap\partial^*E\big)+2\,\mathcal{H}^n\big(A\cap K^*\setminus\partial^*E\big)\\
&\le\liminf_{j\to\infty}\mathcal{H}^n\big(A\cap\partial^*E_j\big)+2\,\mathcal{H}^n\big(A\cap\mathcal{R}(K_j)\setminus\partial^*E_j\big)\le\mu_{\rm bd}(\mathrm{cl}\,(A))\,,
\end{aligned}
\tag{5.47}
$$

where we have used the definition (5.3) of $\mu_{\rm bd}$. Now, for $\mathcal{H}^n$-a.e. $x\in K^*\setminus(T[s_0]\cup\partial^*E)$ we have $\mathcal{H}^n(B_r(x)\cap\partial^*E)=\mathrm{o}(r^n)$ and $\mathcal{H}^n(B_r(x)\cap K^*\setminus\partial^*E)=\omega_n\,r^n+\mathrm{o}(r^n)$ as $r\to0^+$, as well as $\mathrm{cl}\,(B_r(x))\cap T[s_0]=\varnothing$ for r small enough, so that (5.47) with $A=B_r(x)$ readily implies (5.44). The proof of the claim, and thus of the theorem, is now complete. □

5.2. **A second closure theorem.** We now present a variant of the main arguments presented in this section and alternative closure theorem to Theorem 1.4. As already noticed, this second closure theorem, Theorem 5.1 below, will play a role only in the companion paper [MNR23a], where Plateau's laws will be studied in the relation to the Allen–Cahn equation, so that this section can be omitted on a first reading focused on Gauss' capillarity theory alone.

To introduce Theorem 1.4, let us consider the following question: given an $\mathcal{H}^n$-finite set S which is $\mathcal{C}$-spanning $\mathbf{W}$, *what parts of S are essential to its $\mathcal{C}$-spanning property*? We already know from Lemma 2.2 that the unrectifiable part of S is not necessary, since $\mathcal{R}(S)$ is also $\mathcal{C}$-spanning. However, some parts of $\mathcal{R}(S)$ could be discarded too – indeed rectifiable sets can be "porous at every scale", and thus completely useless from the point of view of achieving $\mathcal{C}$-spanning. To make an example, consider the rectifiable set $P \subset \mathbb{R}^2$ obtained by removing from $[0,1]$ all the intervals $(q_i - \varepsilon_i, q_i + \varepsilon_i)$ where $\{q_i\}_i$ are the rational numbers in $[0,1]$ and $2\sum_i \varepsilon_i = \varepsilon$ for some given $\varepsilon \in (0,1)$: it is easily seen that P is a rectifiable set with positive $\mathcal{H}^1$-measure in $\mathbb{R}^2$, contained in $\mathbb{R} \times \{0\}$, which fails to essentially disconnect any stripe of the form $(a,b) \times \mathbb{R}$ with $(a,b) \subset\subset (0,1)$. Intuitively, if a set like P stands as an isolated portion of S, then $\mathcal{R}(S) \setminus P$ should still be $\mathcal{C}$-spanning.

We can formalize this idea as follows. Denoting as usual $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$, we consider the open covering $\{\Omega_k\}_k$ of Ω defined by

$$\{\Omega_k\}_k = \{B_{r_{mh}}(x_m)\}_{m,h}\,, \tag{5.48}$$

where $\{x_m\}_m = \mathbb{Q}^{n+1} \cap \Omega$ and $\{r_{mh}\}_h = \mathbb{Q} \cap (0, \mathrm{dist}(x_m, \partial\Omega))$. For every $\mathcal{H}^n$-finite set S we define the **essential spanning part of S in Ω** as the Borel set

$$\mathrm{ESP}(S) = \bigcup_k \mathrm{UBEP}(S;\Omega_k) = \bigcup_k \Big\{\Omega_k \cap \bigcup_i \partial^* U_i[\Omega_k]\Big\}\,,$$

if $\{U_i[\Omega_k]\}_i$ denotes the essential partition of Ω_k induced by S. Since each $\mathrm{UBEP}(S;\Omega_k)$ is a countable union of reduced boundaries and is $\mathcal{H}^n$-contained in the $\mathcal{H}^n$-finite set S, we see that $\mathrm{ESP}(S)$ is always $\mathcal{H}^n$-rectifiable. The idea is that by following the unions of boundaries of essential partitions induced by S over smaller and smaller balls we are capturing all the parts of S that may potentially contribute to achieve a spanning condition with respect to $\mathbf{W}$. Thinking about Figure 1.5: the tendrils of S appearing in panel (a) and not captured by $\mathrm{UBEP}(S;U)$, will eventually be included into $\mathrm{ESP}(S)$ by considering UBEP's of S relative to suitable subsets of U. Another way to visualize the construction of $\mathrm{ESP}(S)$ is noticing that if $B_r(x) \subset B_s(x) \subset \Omega$, then

$$B_r(x) \cap \mathrm{UBEP}(S; B_s(x)) \subset \mathrm{UBEP}(S; B_r(x))\,,$$

which points to the monotonicity property behind the construction of $\mathrm{ESP}(S)$. Intuitively, we expect that

$$\text{if } S \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}, \text{ then } \mathrm{ESP}(S) \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \tag{5.49}$$

(where $\mathcal{C}$ is an arbitrary spanning class for $\mathbf{W}$). This fact will proved in a moment as a particular case of Theorem 5.1 below.

Next, we introduce the notion of convergence behind our second closure theorem. Consider a sequence $\{S_j\}_j$ of Borel subsets of Ω such that $\sup_j \mathcal{H}^n(S_j) < \infty$. If we denote by $\{U_i^j[\Omega_k]\}_i$ the essential partition induced on Ω_k by S_j, then a diagonal argument based on Lemma 2.3 shows the existence of a (not relabeled) subsequence in j, and, for each k, of a Borel partition $\{U_i[\Omega_k]\}_i$ of Ω_k such that $\{U_i^j[\Omega_k]\}_i$ converges to $\{U_i[\Omega_k]\}_i$ as $j \to \infty$ in the sense specified by (2.8). Since $\mathrm{UBEP}(S_j;\Omega_k) = \Omega_k \cap \bigcup_i \partial^* U_i^j[\Omega_k]$, we call any set S of the form[11]

$$S = \bigcup_k \Big\{ \Omega_k \cap \bigcup_i \partial^* U_i[\Omega_k] \Big\} , \tag{5.50}$$

a **subsequential partition limit of** $\{S_j\}_j$ **in** Ω. Having in mind (5.49), it is natural to ask if the following property holds:

$$\begin{aligned} &\text{if } S_j \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for each } j\,,\\ &\text{and } S \text{ is a subsequential partition limit of } \{S_j\}_j \text{ in } \Omega\,,\\ &\text{then } S \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,. \end{aligned} \tag{5.51}$$

Our next theorem implies both (5.49) and (5.51) as particular cases (corresponding to be taking $E_j = \varnothing$ and, respectively, $K_j = S$ and $K_j = S_j$ for every j).

Theorem 5.1 (Closure theorem for subsequential partition limits)**.** *Let* $\mathbf{W}$ *be a closed set in* $\mathbb{R}^{n+1}$*,* $\mathcal{C}$ *a spanning class for* $\mathbf{W}$*, and* $\{(K_j, E_j)\}_j$ *a sequence in* $\mathcal{K}_{\rm B}$ *such that* $\sup_j \mathcal{H}^n(K_j) < \infty$ *and* $K_j \cup E_j^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for every* j*.*

If S_0 *and* E_0 *are, respectively, a subsequential partition limit of* $\{K_j\}_j$ *in* Ω *and an* L^1*-subsequential limit of* $\{E_j\}_j$ *(corresponding to a same not relabeled subsequence in* j*), and we set*

$$K_0 = (\Omega \cap \partial^* E_0) \cup S_0 ,$$

then $(K_0, E_0) \in \mathcal{K}_{\rm B}$ *and* $K_0 \cup E_0^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*.*

Proof. Since $\Omega \cap \partial^* E_0 \subset K_0$ by definition of K_0 we trivially have $(K_0, E_0) \in \mathcal{K}_{\rm B}$. Aiming to prove that $K_0 \cup E_0^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, we fix $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, and define s_0, s_j, $\{U_i^j\}_i$ and $\{U_i\}_i$ exactly as in part I of the proof of Theorem 1.4. Thanks to Theorem 4.1 and by arguing as in part II of the proof of Theorem 1.4, we have reduced to prove that

$$K^* \setminus (T[s_0] \cup E^{(1)}) \text{ is } \mathcal{H}^n\text{-contained in } K_0\,. \tag{5.52}$$

By Federer's theorem (1.37) and since $\Omega \cap \partial^* E \subset K_0$ it is enough to prove

$$(K^* \cap E^{(0)}) \setminus T[s_0] \text{ is } \mathcal{H}^n\text{-contained in } S_0\,,$$

and, thanks to the construction of S_0, we shall actually be able to prove

$$K^* \setminus T[s_0] \text{ is } \mathcal{H}^n\text{-contained in } S_0\,. \tag{5.53}$$

To this end let us pick k such that $\Omega_k \subset\subset T$ and $\Omega_k \cap T[s_0] = \emptyset$. Then, for $j \ge j(k)$, we have $\Omega_k \cap T[s_j] = \varnothing$, so that

$$\Omega_k \cap \mathrm{UBEP}\big(K_j \cup T[s_j]; T\big) \subset \mathrm{UBEP}\big(K_j \cup T[s_j]; \Omega_k\big) = \mathrm{UBEP}\big(K_j; \Omega_k\big)\,.$$

Since $\{U_i^j\}_i$ is the essential partition of T induced by $K_j \cup T[s_j]$, if $\{U_m^j[\Omega_k]\}_m$ is the essential partition of Ω_k induced by K_j, we have just claimed that, for every i and $j \ge j(k)$,

$$\Omega_k \cap \partial^* U_i^j \subset \Omega_k \cap \bigcup_m \partial^* U_m^j[\Omega_k]\,. \tag{5.54}$$

Since $\{U_m^j[\Omega_k]\}_m$ is a Lebesgue partition of Ω_k into essentially connected sets, by (5.54) the indecomposable components of $\Omega_k \cap U_i^j$ must belong to $\{U_m^j[\Omega_k]\}_m$. In other words, for each i and each $j \ge j(k)$ there is $M(k,i,j)$ such that

$$\Omega_k \cap U_i^j = \bigcup_{m \in M(k,i,j)} U_m^j[\Omega_k]\,.$$

[11]The limit partition $\{U_i[\Omega_k]\}_i$ appearing in (5.50) may not be the essential partition induced by S on Ω_k since the individual $U_i[\Omega_k]$, arising as L^1-limits, may fail to be essentially connected. This said, $\{U_i[\Omega_k]\}_i$ is automatically a partition of Ω_k induced by S_0.

As a consequence of $U_i^j \to U_i$ and of $U_m^j[\Omega_k] \to U_m[\Omega_k]$ as $j \to \infty$ we find that, for a set of indexes $M(k,i)$, it must be

$$\Omega_k \cap U_i = \bigcup_{m \in M(k,i)} U_m[\Omega_k]\,,$$

and therefore

$$\Omega_k \cap \partial^* U_i \overset{\mathcal{H}^n}{\subset} \bigcup_{m \in M(k,i)} \partial^* U_m[\Omega_k] \subset S_0\,.$$

Since we have proved this inclusion for every i and for every k such that $\Omega_k \subset\subset T$ with $\Omega_k \cap T[s_0] = \emptyset$, it follows that $K^* \setminus T[s_0]$ is $\mathcal{H}^n$-contained in S_0, that is (5.53). □

6. Existence of minimizers and convergence to Plateau's problem (Theorem 1.5)

In this section we prove two main results: the first one (Theorem 6.1) concerns the equivalence of Harrison–Pugh Plateau's problem ℓ with its measure theoretic reformulation $\ell_{\rm B}$ (see (1.21)); the second (Theorem 1.5) is a very refined version of Theorem 1.5.

Theorem 6.1 (Existence for $\ell_{\rm B}$ and $\ell = \ell_{\rm B}$)**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is closed, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and the Harrison–Pugh formulation of the Plateau problem*

$$\ell = \inf\big\{\mathcal{H}^n(S) : S \text{ is a closed subset } \Omega,\ S \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\big\}$$

is finite, then the problem

$$\ell_{\rm B} = \inf\big\{\mathcal{H}^n(S) : S \text{ is a Borel subset } \Omega,\ S \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\big\}$$

admits minimizers, and given any minimizer S for $\ell_{\rm B}$, there exists relatively closed S^* which is $\mathcal{H}^n$-equivalent to S and a minimizer for ℓ. In particular, $\ell = \ell_{\rm B}$.

Theorem 6.2 (Theorem 1.5 refined)**.** *If $\mathbf{W}$ is a compact set in $\mathbb{R}^{n+1}$ and $\mathcal{C}$ is a spanning class for $\mathbf{W}$ such that $\ell < \infty$, then for every $v > 0$ there exist minimizers (K,E) of $\Psi_{\rm bk}(v)$. Moreover,*

(i): *if (K_*, E_*) is a minimizer of $\Psi_{\rm bk}(v)$, then there is $(K,E) \in \mathcal{K}$ such that K is $\mathcal{H}^n$-equivalent to K^*, E is Lebesgue equivalent to E_*, (K,E) is a minimizer of $\Psi_{\rm bk}(v)$, both E and K are bounded, $K \cup E$ is $\mathcal{C}$-spanning $\mathbf{W}$, $K \cap E^{(1)} = \varnothing$, and there is $\lambda \in \mathbb{R}$ such that*

$$\lambda \int_{\Omega \cap \partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\Omega \cap \partial^* E} {\rm div}^K X \, d\mathcal{H}^n + 2 \int_{K \cap E^{(0)}} {\rm div}^K X \, d\mathcal{H}^n\,, \qquad (6.1)$$
$$\forall X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1}) \quad \text{with } X \cdot \nu_\Omega = 0 \text{ on } \partial\Omega\,,$$

and there are positive constants $c = c(n)$ and $r_1 = r_1(K,E)$ such that

$$|E \cap B_\rho(y)| \le (1-c)\,\omega_{n+1}\,\rho^{n+1}\,, \qquad (6.2)$$

for every $y \in \Omega \cap \partial E$ and $\rho < \min\{r_1, {\rm dist}(y,\mathbf{W})\}$; under the further assumption that $\partial\mathbf{W}$ is C^2, then there is positive $r_0 = r_0(n, \mathbf{W}, |\lambda|)$ such that

$$\mathcal{H}^n(K \cap B_r(x)) \ge c\, r^n \qquad (6.3)$$

for every $x \in {\rm cl}\,(K)$ and $r < r_0$;

(ii): *if (K_j, E_j) is a sequence of minimizers for $\Psi_{\rm bk}(v_j)$ with $v_j \to 0^+$, then there exists a minimizer S of ℓ such that, up to extracting subsequences, as Radon measures in Ω,*

$$\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_j) + 2\,\mathcal{H}^n \llcorner (K_j \cap E_j^{(0)}) \overset{*}{\rightharpoonup} 2\mathcal{H}^n \llcorner S\,, \qquad (6.4)$$

as $j \to \infty$. In particular, $\Psi_{\rm bk}(v) \to 2\,\ell = \Psi_{\rm bk}(0)$ as $v \to 0^+$.

Proof of Theorem 6.1. By Theorem A.1, if $\ell < \infty$, then $\ell_B < \infty$. Let now $\{S_j\}_j$ be a minimizing sequence for ℓ_B, then $\{(S_j, \varnothing)\}_j$ is a sequence in $\mathcal{K}_B$ satisfying (5.1). By Theorem 1.4, we find a Borel set S which is $\mathcal{C}$-spanning $\mathbf{W}$ and is such that

$$2 \liminf_{j\to\infty} \mathcal{H}^n(S_j) = \liminf_{j\to\infty} \mathcal{F}_{\rm bk}(S_j, \varnothing) \geq \mathcal{F}_{\rm bk}(S, \varnothing) = 2\,\mathcal{H}^n(S)\,.$$

This shows that S is a minimizer of ℓ_B. By Lemma 2.2, S is $\mathcal{H}^n$-rectifiable, for, otherwise, $\mathcal{R}(S)$ would be admissible for ℓ_B and have strictly less area than S. We conclude the proof by showing that up to modifications on a $\mathcal{H}^n$-null set, S is relatively closed in Ω (and thus is a minimizer of ℓ too). Indeed the property of being $\mathcal{C}$-spanning $\mathbf{W}$ is preserved under diffeomorphism f with $\{f \neq {\rm id}\} \subset\subset \Omega$. In particular, $\mathcal{H}^n(S) \leq \mathcal{H}^n(f(S))$ for every such f, so that the multiplicity one rectifiable varifold $V_S = \mathbf{var}\,(S, 1)$ associated to S is stationary. By a standard application of the monotonicity formula, we can find S^* $\mathcal{H}^n$-equivalent to S such that S^* is relative closed in Ω. Since $\mathcal{H}^n(S) = \mathcal{H}^n(S^*)$ and $\mathcal{C}$-spanning is preserved under $\mathcal{H}^n$-null modifications, we conclude the proof. □

Proof of Theorem 6.2. Step one: We prove conclusion (i). To this end, let $(K_*, E_*) \in \mathcal{K}_B$ be a minimizer of $\Psi_{\rm bk}(v)$. Clearly, $(\mathcal{R}(K_*), E_*) \in \mathcal{K}_B$ is such that $\mathcal{R}(K_*) \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ (thanks to Theorem 3.1/Remark 3.2) and $\mathcal{F}_{\rm bk}(\mathcal{R}(K_*), E_*) \leq \mathcal{F}_{\rm bk}(K_*, E_*)$. In particular, $(\mathcal{R}(K_*), E_*)$ is a minimizer of $\Psi_{\rm bk}(v)$, and energy comparison between $(\mathcal{R}(K_*), E_*)$ and $(\mathcal{R}(K_*) \setminus E_*^{(1)}, E_*)$ (which is also a competitor for $\Psi_{\rm bk}(v)$) proves that

$$\mathcal{H}^n(\mathcal{R}(K_*) \cap E_*^{(1)}) = 0\,. \tag{6.5}$$

Since "$\mathcal{C}$-spanning $\mathbf{W}$" is preserved under diffeomorphisms, by a standard first variation argument (see, e.g. [KMS22a, Appendix C]) wee see that $(\mathcal{R}(K_*), E_*)$ satisfies (6.1) for some $\lambda \in \mathbb{R}$. In particular, the integer n-varifold $V = {\rm var}(\mathcal{R}(K_*), \theta)$, with multiplicity function $\theta = 2$ on $\mathcal{R}(K_*) \cap E_*^{(0)}$ and $\theta = 1$ on $\Omega \cap \partial^* E_*$, has bounded mean curvature in Ω, and thus satisfies $\|V\|(B_r(x)) \geq c(n)\, r^n$ for every $x \in K$ and $r < \min\{r_0, {\rm dist}(x, \mathbf{W})\}$, where $r_0 = r_0(n, |\lambda|)$ and, by definition,

$$K := \Omega \cap {\rm spt} V\,.$$

In particular, since (6.5) implies $\|V\| \leq 2\,\mathcal{H}^n \llcorner \mathcal{R}(K^*)$, we conclude (e.g. by [Mag12, Corollary 6.4]) that K is $\mathcal{H}^n$-equivalent to $\mathcal{R}(K_*)$, and is thus $\mathcal{H}^n$-rectifiable and relatively closed in Ω. Now let

$$E = \big\{x \in \Omega : \exists\, r < {\rm dist}(x, \mathbf{W}) \text{ s.t. } |E_* \cap B_r(x)| = |B_r(x)|\big\}\,,$$

so that, trivially, E is an open subset of Ω with $E \subset E_*^{(1)}$. By applying (1.35) to E_*, and by noticing that if $x \in \Omega \setminus E$ then $|E_* \cap B_r(x)| < |B_r(x)|$ for every $r > 0$, and that if $x \in \Omega \cap {\rm cl}\,(E)$ then $|E_* \cap B_r(x)| > 0$ for every $r > 0$, we see that

$$\Omega \cap \partial E \;\subset\; \big\{x \in \Omega : 0 < |E_* \cap B_r(x)| < |B_r(x)|\ \forall r > 0\big\} \;=\; \Omega \cap {\rm cl}\,(\partial^* E_*)\,. \tag{6.6}$$

Since $\|V\| \geq \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_*)$ and $\mathcal{H}^n(B_r(x) \cap \partial^* E) = \omega_n\, r^n + {\rm o}(r^n)$ as $r \to 0^+$ for every $x \in \Omega \cap \partial^* E$, we see that $\Omega \cap \partial^* E_* \subset \Omega \cap {\rm spt}\|V\| = K$, and since K is relatively closed in Ω, we have $\Omega \cap {\rm cl}\,(\partial^* E_*) \subset K$, and so $\Omega \cap \partial E \subset K$. In particular, E is of finite perimeter, and thus by applying (1.35) to E,

$$\Omega \cap {\rm cl}\,(\partial^* E) \;=\; \big\{x \in \Omega : 0 < |E \cap B_r(x)| < |B_r(x)|\ \forall r > 0\big\} \;\subset\; \Omega \cap \partial E\,. \tag{6.7}$$

Finally, if there is $x \in (\Omega \cap E_*^{(1)}) \setminus E$, then it must be $0 < |E_* \cap B_r(x)| < |B_r(x)|$ for every $r > 0$, and thus $x \in \Omega \cap {\rm cl}\,(\partial^* E_*) \subset K$. However, we *claim* that for every $x \in \Omega \cap {\rm cl}\,(\partial^* E_*)$ and $r < \min\{r_*, {\rm dist}(x, \mathbf{W})\}$ (with $r_* = r_*(K_*, E_*)$) it holds

$$|B_r(x) \cap E_*| \leq (1 - c)\, \omega_{n+1}\, r^{n+1}\,, \tag{6.8}$$

in contradiction with $x \in E^{(1)}$; this proves that $\Omega \cap E_*^{(1)} \subset E$, and thus that E_* and E are Lebesgue equivalent. Combining the latter information with (6.6) and (6.7) we conclude that $\Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K$ and conclude the proof of $(K, E) \in \mathcal{K}$ – conditional to proving (6.8).

To prove (6.8), let us fix $x \in \Omega \cap \mathrm{cl}\,(\partial^* E_*)$ and set $u(r) = |B_r(x) \setminus E_*|$, so that, for a.e. $r > 0$ we have

$$u'(r) = \mathcal{H}^n(E_*^{(0)} \cap \partial B_r(x))\,, \qquad P(B_r(x) \setminus E_*) = u'(r) + P(E_*; B_r(x))\,. \tag{6.9}$$

Since $|E_*| = v > 0$, we have $\mathcal{H}^n(\Omega \cap \partial^* E_*) > 0$, therefore there must be $y_1, y_2 \in \Omega \cap \partial^* E_*$ with $|y_1 - y_2| > 4r_*$ for some r_* depending on E_*. In particular there is $i \in \{1, 2\}$ such that $B_{r_*}(x) \cap B_{r_*}(y_i) = \varnothing$, and we set $y = y_i$. Since $y_i \in \Omega \cap \partial^* E_*$, there is $w_* > 0$ and smooth maps $\Phi : \Omega \times (-w_*, w_*) \to \Omega$ such that $\Phi(\cdot, w)$ is a diffeomorphism of Ω with $\{\Phi(\cdot, w) \neq \mathrm{Id}\,\} \subset\subset B_{r_*}(y)$, and

$$|\Phi(E_*, w)| = |E_*| - w\,, \qquad P(\Phi(E_*, w); B_{r_*}(y)) \le P(E_*, B_{r_*}(y))(1 + 2\,|\lambda|\,|w|)\,, \tag{6.10}$$

for every $|w| < w_*$. We then consider r_1 such that $|B_{r_1}| < w_*$, so that for every $r < \min\{r_1, \mathrm{dist}(x, \mathbf{W})\}$ we have $0 \le u(r) < w_*$, and thus we can define

$$(K_r, E_r) = \Big(\Phi^{u(r)}\big(K \cup \partial B_r(x)\big), \Phi^{u(r)}\big(E_* \cup B_r(x)\big)\Big)\,.$$

Since $\Phi^{u(r)}$ is a diffeomorphism, we have $\Omega \cap \partial^* E_r \subset K_r$, and by the first relation in (6.10) and $\Phi^{u(r)} = \mathrm{Id}$ on $\Omega \setminus B_{r_*}(y)$, we get

$$|E_r| - |E| = |B_r(x)| - |B_r(x) \cap E_*| + |\Phi^{u(r)}(E_*) \cap B_{r_*}(y)| - |E_* \cap B_{r_*}(y)| = u(r) - u(r) = 0\,.$$

Hence $\mathcal{F}_{\mathrm{bk}}(K_*, E_*) \le \mathcal{F}_{\mathrm{bk}}(K_r, E_r)$, from which we deduce

$$\begin{aligned} &P(E; B_r(x)) + P(E; B_{r_*}(y)) + 2\,\mathcal{H}^n(K_* \cap E_*^{(0)} \cap B_r(x)) \\ &\le \mathcal{H}^n(B_r(x) \cap E^{(0)}) + P(\Phi^{u(r)}(E_*); B_{r_*}(y)) \le u'(r) + P(E_*, B_{r_*}(y))(1 + 2\,|\lambda|\,|w|)\,; \end{aligned}$$

where we have used (6.9) and (6.10); by adding up $u'(r)$ on both sides of the inequality, and using (6.9) again, we find that

$$c(n)\, u(r)^{n/(n+1)} \le P(B_r(x) \setminus E_*) \le 2\, u'(r) + 2\,|\lambda|\,\Psi_{\mathrm{bk}}(v)\, u(r)\,,$$

for a.e. $r < \min\{r_1, \mathrm{dist}(x, \mathbf{W})\}$; since, by (6.6), $x \in \Omega \cap \mathrm{cl}\,(\partial^* E_*)$ implies $u(r) > 0$ for every $r > 0$, we can apply a standard ODE argument to conclude that (6.8) holds true.

We now prove the remaining assertions in statement (i). First of all, when $\partial\mathbf{W}$ is C^2, we can argue similarly to [KMS22b, Theorem 4.1] to deduce from the modified monotonicity formula of Kagaya and Tonegawa [KT17] that the area lower bound in (6.3) holds for every $x \in \mathrm{cl}\,(K)$ and every $r < r_0$. The validity of the volume upper bound in (6.2) is immediate from (6.8) and the Lebesgue equivalence of E_* and E. The monotonicity formula for V combined with $\mathcal{H}^n(\Omega \cap K) < \infty$ implies of course that V has bounded support. Having proved that K is bounded, $|E| < \infty$ and $\Omega \cap \partial E \subset K$ imply that E is bounded too. Since $\mathcal{R}(K_*)$ and K are $\mathcal{H}^n$-equivalent, we have that $K \cup E_*^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$. It turns out that $K \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ too, since E and E_* are Lebesgue equivalent *and* of finite perimeter, therefore such that $E^{(1)}$ and $E_*^{(1)}$ are $\mathcal{H}^n$-equivalent. In fact, on noticing that $\Omega \cap (E^{(1)} \setminus E) \subset \Omega \cap \partial E \subset K$, we see that $K \cup E^{(1)} = K \cup E$, so that $K \cup E$ is $\mathcal{C}$-spanning $\mathbf{W}$, as claimed.

Finally, we prove that $K \cap E^{(1)} = \varnothing$. We first notice that, since $E \subset \Omega$ is open and $K = \Omega \cap \mathrm{spt}\, V$ with $\|V\| \le 2\,\mathcal{H}^n \llcorner \mathcal{R}(K^*)$, if $K \cap E \neq \emptyset$, then $\mathcal{H}^n(\mathcal{R}(K_*) \cap E) > 0$; and since $E \subset E_*^{(1)}$ by construction, we arrive at a contradiction with (6.5). Hence, $K \cap E = \varnothing$. Now, if $x \in K \cap E^{(1)}$, then, by (6.2), $x \notin \Omega \cap \partial E$; combining this with $K \cap E = \varnothing$, we find $K \cap E^{(1)} \subset \Omega \setminus \mathrm{cl}\,(E) \subset E^{(0)}$, and thus $K \cap E^{(1)} = \varnothing$.

Step two: For every $v_1 \ge 0$ and $v_2 > 0$ we have

$$\Psi_{\mathrm{bk}}(v_1 + v_2) \le \Psi_{\mathrm{bk}}(v_1) + (n+1)\,\omega_{n+1}^{1/(n+1)}\, v_2^{n/(n+1)}\,. \tag{6.11}$$

Since $\Psi_{\rm bk}(0) = 2\,\ell < \infty$, (6.11) implies in particular that $\Psi_{\rm bk}(v) < \infty$ for every $v > 0$ (just take $v_1 = 0$ and $v_2 = v$).

Indeed, let (K_1, E_1) be a competitor in $\Psi_{\rm bk}(v_1)$ and let $\{B_{r_j}(x_j)\}_j$ be a sequence of balls with $|x_j| \to \infty$ and $|E_1 \cup B_{r_j}(x_j)| = v_1 + v_2$ for every j. Setting for the sake of brevity $B_j = B_{r_j}(x_j)$, sine $\partial^*(E_1 \cup B_j)$ is $\mathcal{H}^n$-contained in $(\partial^* E_1) \cup \partial B_j$ we have that (K_2, E_2), with $K_2 = K_1 \cup \partial B_j$ and $E_2 = E_1 \cup B_j$, is a competitor of $\Psi_{\rm bk}(v_1 + v_2)$. Since $\partial B_j \cap E_2^{(0)} = \varnothing$ implies $E_2^{(0)} \subset E_1^{(0)} \setminus \partial B_j$, we find that

$$
\begin{aligned}
\Psi_{\rm bk}(v_1 + v_2) &\le 2\,\mathcal{H}^n\big(K_2 \cap E_2^{(0)}\big) + \mathcal{H}^n(\Omega \cap \partial^* E_2)\\
&\le 2\,\mathcal{H}^n(K_1 \cap E_1^{(0)} \setminus \partial B_j) + \mathcal{H}^n(\Omega \cap \partial^* E_1) + \mathcal{H}^n(\partial B_j)\\
&\le \mathcal{F}_{\rm bk}(K_1, E_1) + (n+1)\,\omega_{n+1}^{1/(n+1)}\,|B_j|^{n/(n+1)}\,.
\end{aligned}
$$

Since $|x_j| \to \infty$, $|E_1| = v_1$, and $|E_1 \cup B_{r_j}(x_j)| = v_1 + v_2$ imply $|B_j| \to v_2$, we conclude by arbitrariness of (K_1, E_1).

Step three: Now let $\{(K_j, E_j)\}_j$ be a minimizing sequence for $\Psi_{\rm bk}(v)$. Since $\Psi_{\rm bk}(v) < \infty$, assumption (5.1) of Theorem 1.4 holds. Therefore there is $(K, E) \in \mathcal{K}_{\rm B}$ with $K \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ and such that, up to extracting subsequences,

$$
\lim_{j\to\infty} |(E_j \Delta E) \cap B_R| = 0 \quad \forall R > 0\,, \qquad \liminf_{j\to\infty} \mathcal{F}_{\rm bk}(K_j, E_j) \ge \mathcal{F}_{\rm bk}(K, E)\,; \tag{6.12}
$$

actually, to be more precise, if μ denotes the weak-star limit of $\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j) \cap E_j^{(0)})$ in Ω, then

$$
\mu \ge 2\,\mathcal{H}^n \llcorner (K \cap E^{(0)}) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E)\,. \tag{6.13}
$$

We *claim* that

$$
(K, E) \text{ is a minimizer of } \Psi_{\rm bk}(|E|)\,.
$$

(Notice that, at this stage of the argument, we are not excluding that $v^* := v - |E|$ is positive, nor that $|E| = 0$.) Taking into account (6.11), to prove the claim it suffices to show that

$$
\Psi_{\rm bk}(v) \ge \mathcal{F}_{\rm bk}(K, E) + (n+1)\,\omega_{n+1}^{1/(n+1)}\,(v^*)^{n/(n+1)}\,. \tag{6.14}
$$

To see this, we start noticing that, given any sequence $\{r_j\}_j$ with $r_j \to \infty$, by (6.12) and (6.13) we have that

$$
E_j \cap B_{r_j} \stackrel{\rm loc}{\to} E\,, \qquad |E_j \setminus B_{r_j}| \to v^*\,, \qquad \text{as } j \to \infty\,, \tag{6.15}
$$

$$
\liminf_{j\to\infty}\, 2\,\mathcal{H}^n\big(\mathcal{R}(K_j) \cap E_j^{(0)} \cap B_{r_j}\big) + \mathcal{H}^n(B_{r_j} \cap \partial^* E_j) \ge \mathcal{F}_{\rm bk}(K, E)\,, \tag{6.16}
$$

Moreover, since $|E_j| < \infty$, we can choose $r_j \to \infty$ so that $\mathcal{H}^n(E_j^{(1)} \cap \partial B_{r_j}) \to 0$, while, taking into account that $P(E_j \setminus B_{r_j}) = \mathcal{H}^n(E_j^{(1)} \cap \partial B_{r_j}) + \mathcal{H}^n((\partial^* E_j) \setminus B_{r_j})$, we have

$$
\begin{aligned}
\mathcal{F}_{\rm bk}(K_j, E_j) &\ge 2\,\mathcal{H}^n\big(\mathcal{R}(K_j) \cap E_j^{(0)} \cap B_{r_j}\big) + \mathcal{H}^n(B_{r_j} \cap \partial^* E_j)\\
&\quad + P(E_j \setminus B_{r_j}) - \mathcal{H}^n(E_j^{(1)} \cap \partial B_{r_j})\,.
\end{aligned}
$$

By combining these facts with (6.15), (6.16), and the Euclidean isoperimetric inequality, we conclude that

$$
\Psi_{\rm bk}(v) = \lim_{j\to\infty} \mathcal{F}_{\rm bk}(K_j, E_j) \ge \mathcal{F}_{\rm bk}(K, E) + (n+1)\,\omega_{n+1}^{1/(n+1)} \lim_{j\to\infty} |E_j \setminus B_{r_j}|^{n/(n+1)}\,,
$$

that is (6.14).

Step four: We prove the existence of minimizers in $\Psi_{\rm bk}(v)$, $v>0$. By step three, there is $(K,E)\in\mathcal{K}_{\rm B}$ such that $K\cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, (K,E) is a minimizer of $\Psi_{\rm bk}(|E|)$ and, combining (6.11) and (6.14),

$$\Psi_{\rm bk}(v)=\Psi_{\rm bk}(|E|)+(n+1)\,\omega_{n+1}^{1/(n+1)}\,(v-|E|)^{n/(n+1)}\,. \tag{6.17}$$

Since (K,E) is a minimizer in $\Psi_{\rm bk}(|E|)$, by step one we can assume that K is $\mathcal{H}^n$-rectifiable and that both K and E are bounded. We can thus find $B_r(x_0)\subset\subset\Omega$ such that $|B_r(x_0)|=v-|E|$, $|B_r(x_0)\cap E|=0$, and $\mathcal{H}^n(K\cap B_r(x_0))=0$. In this way $(K_*,E_*)=(K\cup\partial B_r(x_0),E\cup B_r(x_0))\in\mathcal{K}_{\rm B}$ is trivially $\mathcal{C}$-spanning $\mathbf{W}$ and such that $|E_*|=v$, and thus is a competitor for $\Psi_{\rm bk}(v)$. At the same time,

$$\mathcal{F}_{\rm bk}(K_*,E_*)=\mathcal{F}_{\rm bk}(K,E)+(n+1)\,\omega_{n+1}^{1/(n+1)}\,(v-|E|)^{n/(n+1)}$$

so that, by (6.17), (K_*,E_*) is a minimizer of $\Psi_{\rm bk}(v)$. Having proved that minimizers of $\Psi_{\rm bk}(v)$ do indeed exist, a further application of step one completes the proof of statement (i).

Step five: We finally prove statement (ii). Let us consider a sequence $v_j\to0^+$ and corresponding minimizers (K_j,E_j) of $\Psi_{\rm bk}(v_j)$. By (6.11) with $v_1=0$ and $v_2=v_j$ we see that $\{(K_j,E_j)\}_j$ satisfies the assumptions of Theorem 1.4. Since $|E_j|=v_j\to0$, setting $\mu_j=\mathcal{H}^n\llcorner(\Omega\cap\partial^*E_j)+2\,\mathcal{H}^n\llcorner(\mathcal{R}(K_j)\cap E_j^{(0)})$, the conclusion of Theorem 1.4 is that there are a Radon measure μ in Ω and a Borel set K such that K is $\mathcal{C}$-spanning $\mathbf{W}$ and $\mu_j\overset{*}{\rightharpoonup}\mu$ for a Radon measure μ in Ω such that $\mu\ge2\,\mathcal{H}^n\llcorner K$. Thanks to (6.11) we thus have

$$\begin{aligned}2\,\ell&=\lim_{j\to\infty}\Psi_{\rm bk}(0)+(n+1)\,\omega_{n+1}^{1/(n+1)}\,v_j^{n/(n+1)}\ge\liminf_{j\to\infty}\Psi_{\rm bk}(v_j)\\&=\liminf_{j\to\infty}\mathcal{F}_{\rm bk}(K_j,E_j)\ge\mathcal{F}_{\rm bk}(K,\emptyset)=2\,\mathcal{H}^n(K)\ge2\,\ell\,.\end{aligned}$$

We conclude that $\Psi_{\rm bk}(v_j)\to2\,\ell$, K is a minimizer of ℓ, and $\mu=2\,\mathcal{H}^n\llcorner K$, thus completing the proof of the theorem. □

Proof of Theorem 1.5. The identity (1.22) is proved in Theorem 6.1. Conclusions (i), (ii), and (iii) are proved in Theorem 6.2. □

7. Equilibrium across transition lines in wet soap films (Theorem 1.6)

We finally prove Theorem 1.6. We shall need two preliminary lemmas:

Lemma 7.1 (Representation of $\mathcal{F}_{\rm bk}$ via induced partitions)**.** *If $U\subset\Omega$ is a set of finite perimeter, $(K,E)\in\mathcal{K}_{\rm B}$ is such that $\mathcal{F}_{\rm bk}(K,E)<\infty$, and $\{U_i\}_i$ is a Lebesgue partition of $U\setminus E$ induced by K, then each U_i has finite perimeter, and, setting $K^*=\bigcup_i\partial^*U_i$, we have*

$$\mathcal{F}_{\rm bk}(K,E;U^{(1)})=\sum_i\mathcal{H}^n(U^{(1)}\cap\partial^*U_i)+2\,\mathcal{H}^n\big(U^{(1)}\cap(K\setminus K^*)\cap E^{(0)}\big)\,; \tag{7.1}$$

see Figure 7.1.

Proof. For each i, ∂^eU_i is contained in $(\partial^eU)\cup(\partial^eE)\cup(U\setminus E)^{(1)}$, where both ∂^eU and ∂^eE are $\mathcal{H}^n$-finite being U and E of finite perimeter, and where $(U\setminus E)^{(1)}\cap\partial^eU_i$ is $\mathcal{H}^n$-contained in K by assumption. Now, $(U\setminus E)^{(1)}\subset\mathbb{R}^{n+1}\setminus E^{(1)}$, so that

$$\mathcal{H}^n\big((U\setminus E)^{(1)}\cap\partial^eU_i\big)\le\mathcal{H}^n(K\setminus E^{(1)})\le\mathcal{F}_{\rm bk}(K,E)<\infty\,.$$

This shows that, for each i, U_i is a set of finite perimeter. As a consequence $\{U\cap E\}\cup\{U_i\}_i$ is a Caccioppoli partition of U, so that, by (1.46),

$$2\,\mathcal{H}^n\Big(U^{(1)}\cap\Big[\partial^*(U\cap E)\cup K^*\Big]\Big)=\mathcal{H}^n\big(U^{(1)}\cap\partial^*(U\cap E)\big)+\sum_i\mathcal{H}^n(U^{(1)}\cap\partial^*U_i)\,, \tag{7.2}$$

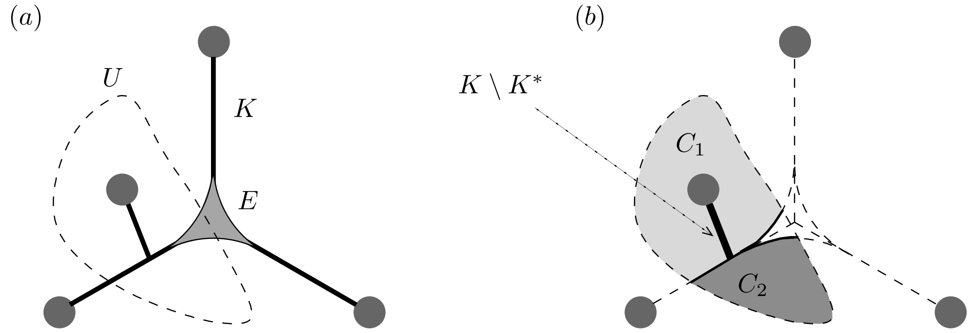

FIGURE 7.1. The situation in Lemma 7.1: (a) a depiction of the left hand side of (7.1), where $K\setminus\partial^*E$ is drawn with a bold line to indicate that, in the computation of $\mathcal{F}_{\rm bk}(K,E;U^{(1)})=\mathcal{H}^n(U^{(1)}\cap\partial^*E)+2\,\mathcal{H}^n(U^{(1)}\cap K\setminus\partial^*E)$, it is counted with multiplicity 2; (b) a depiction of the right hand side of (7.1), where $K\setminus K^*$ is drawn with a bold line to indicate that it has to be counted with multiplicity 2.

with $K^*=\bigcup_i\partial^*U_i$. Now, thanks to (1.40), (1.41), and the inclusion in (1.46), we have

$$U^{(1)}\cap\partial^*(U\cap E)\overset{\mathcal{H}^n}{=}U^{(1)}\cap\partial^*E\overset{\mathcal{H}^n}{\subset}U^{(1)}\cap K^*\,,$$

which combined with (7.2) gives

$$2\,\mathcal{H}^n(U^{(1)}\cap K^*)=\mathcal{H}^n\big(U^{(1)}\cap\partial^*E\big)+\sum_i\mathcal{H}^n(U^{(1)}\cap\partial^*U_i)\,. \tag{7.3}$$

Therefore, using in order

$$U^{(1)}\cap\partial^*E\overset{\mathcal{H}^n}{\subset}U^{(1)}\cap K^*\,,\qquad K^*\overset{\mathcal{H}^n}{\subset}K\,,\qquad \mathcal{H}^n(K^*\cap E^{(1)})=0\,,$$

and Federer's theorem (1.37), we obtain

$$\begin{aligned}\mathcal{F}_{\rm bk}(K,E;U^{(1)}) &= \mathcal{H}^n(U^{(1)}\cap\partial^*E)+2\,\mathcal{H}^n(U^{(1)}\cap K\cap E^{(0)})\\ &= 2\,\mathcal{H}^n(U^{(1)}\cap K^*\cap\partial^*E)-\mathcal{H}^n(U^{(1)}\cap\partial^*E)\\ &\quad +2\,\mathcal{H}^n(U^{(1)}\cap K^*\cap E^{(0)})+2\,\mathcal{H}^n(U^{(1)}\cap(K\setminus K^*)\cap E^{(0)})\\ &= 2\,\mathcal{H}^n(U^{(1)}\cap K^*)-\mathcal{H}^n(U^{(1)}\cap\partial^*E)+2\,\mathcal{H}^n(U^{(1)}\cap(K\setminus K^*)\cap E^{(0)})\\ &= \sum_i\mathcal{H}^n(U^{(1)}\cap\partial^*U_i)+2\,\mathcal{H}^n(U^{(1)}\cap(K\setminus K^*)\cap E^{(0)})\,,\end{aligned}$$

where in the last identity we have used (7.3). □

The next lemma is a slight reformulation of [DLGM17a, Lemma 10] and [DLDRG19, Lemma 4.1].

Lemma 7.2. *If* $\mathbf{W}$ *is closed,* $\mathcal{C}$ *is a spanning class for* $\mathbf{W}$*,* S *is relatively closed in* Ω *and* $\mathcal{C}$*-spanning* $\mathbf{W}$*, and* $B\subset\Omega$ *is an open ball, then for any* $\gamma\in\mathcal{C}$ *we either have* $\gamma(\mathbb{S}^1)\cap(S\setminus B)\neq\varnothing$*, or* $\gamma(\mathbb{S}^1)$ *has non-empty intersection with at least two connected components of* $B\setminus S$*. In particular, it intersects the boundaries of both components.*

We are now ready for the proof of Theorem 1.6.

Proof of Theorem 1.6. The opening part of the statement of Theorem 1.6 is Theorem 6.2-(i), therefore we can directly consider a minimizer $(K,E)\in\mathcal{K}$ of $\Psi_{\rm bk}(v)$ such that both E and K are bounded, $K\cup E$ is $\mathcal{C}$-spanning $\mathbf{W}$, and

$$K\cap E^{(1)}=\varnothing\,, \tag{7.4}$$

and begin by proving the existence of a closed set $\Sigma\subset K$ closed such that (i): $\Sigma=\varnothing$ if $1\le n\le 6$, Σ is locally finite in Ω if $n=7$, and $\mathcal{H}^s(\Sigma)=0$ for every $s>n-7$

if $n \ge 8$; (ii): $(\partial^* E)\setminus\Sigma$ is a smooth hypersurface with constant mean curvature; (iii) $K\setminus(\mathrm{cl}\,(E)\cup\Sigma)$ is a smooth minimal hypersurface; (iv)$_\alpha$: if $x\in[\Omega\cap(\partial E\setminus\partial^*E)]\setminus\Sigma$, then there are $r>0$, $\nu\in\mathbb{S}^n$, $u_1,u_2\in C^{1,\alpha}(\mathbf{D}_r^\nu(x);(-r/4,r/4))$ ($\alpha\in(0,1/2)$ arbitrary) such that $u_1(x)=u_2(x)=0$, $u_1\le u_2$ on $\mathbf{D}_r^\nu(x)$, $\{u_1<u_2\}$ and $\mathrm{int}\{u_1=u_2\}$ are both non-empty, and

$$\begin{aligned}\mathbf{C}_r^\nu(x)\cap K &= \cup_{i=1,2}\big\{y+u_i(y)\,\nu : y\in\mathbf{D}_r^\nu(x)\big\}\,, &(7.5)\\ \mathbf{C}_r^\nu(x)\cap\partial^*E &= \cup_{i=1,2}\big\{y+u_i(y)\nu : y\in\{u_1<u_2\}\big\}\,, &(7.6)\\ \mathbf{C}_r^\nu(x)\cap E &= \big\{y+t\,\nu : y\in\{u_1<u_2\}\,, u_1(x)<t<u_2(x)\big\}\,. &(7.7)\end{aligned}$$

(The sharp version of conclusion (iv), that is conclusion (iv)$_\alpha$ with $\alpha=1$, and conclusion (v), will be proved in the final step five of this proof.) The key step to prove conclusions (i)–(iv)$_\alpha$ is showing the validity of the following claim.

Claim: There exist positive constants Λ and r_0 such that if $B_{2r}(x)\subset\subset\Omega$, then, denoting by $\{U_j\}_j$ the open connected components of $B_{2r}(x)\setminus(E\cup K)$,

$$B_r(x)\cap K = B_r(x)\cap\cup_j\partial U_j\,, \tag{7.8}$$

$$\#\{i : B_r(x)\cap U_j\ne\varnothing\}<\infty\,, \tag{7.9}$$

$$B_{2\,r}(x)\cap\mathrm{cl}\,(\partial^*U_j) = B_{2\,r}(x)\cap\partial U_j\,, \tag{7.10}$$

$$P(U_j;B_r(x))\le P(V_j;B_r(x))+\Lambda\,|U_j\Delta V_j|\,, \tag{7.11}$$

whenever V_j satisfies $V_j\Delta U_j\subset\subset B_r(x)$ and $\mathrm{diam}\,(U_j\Delta V_j)<r_0$.

Deduction of (i)-(iv) from the claim: Let $\{B_{2r_i}(x_i)\}_{i\in\mathbb{N}}$ be a countable family of balls, locally finite in Ω, such that $B_{2r_i}(x_i)\subset\subset\Omega$ and $\Omega=\cup_i B_{r_i}(x_i)$. Setting for brevity

$$\Omega_i = B_{r_i}(x_i)\,,$$

by (7.9) there are finitely many connected components $\{U_j^i\}_{j=1}^{J_i}$ of $B_{2r_i}(x_i)\setminus(E\cup K)$ such that $U_j^i\cap\Omega_i\ne\varnothing$. Thanks to (7.11), we deduce from [Mag12, Theorem 28.1] that, if we set $\Sigma_j^i=\Omega_i\cap(\partial U_j^i\setminus\partial^*U_j^i)$, then $\Omega_i\cap\partial^*U_j^i$ is a $C^{1,\alpha}$-hypersurface for every $\alpha\in(0,1/2)$, and Σ_j^i is a closed set that satisfies the dimensional estimates listed in conclusion (i). In particular, if we set

$$\Sigma=\cup_{i\in\mathbb{N}}\cup_{j=1}^{J_i}\Sigma_j^i\,, \tag{7.12}$$

then $\Sigma\subset K$ thanks to $\Sigma_j^i\subset\Omega_i\cap\partial U_j^i$ and to (7.8), and conclusion (i) holds by the local finiteness of the covering $\{B_{2r_i}(x_i)\}_i$ of Ω and from $J_i<\infty$ for every i. Before moving to prove the remaining conclusions, we first notice that (7.8) gives

$$\begin{aligned}\Omega_i\cap K\setminus\Sigma &= \Omega_i\cap\cup_{j=1}^{J_i}\partial U_j^i\setminus\Sigma\\ &\subset \Omega_i\cap\cup_{j=1}^{J_i}(\partial U_j^i\setminus\Sigma_j^i) \;=\; \Omega_i\cap\cup_{j=1}^{J_i}\partial^*U_j^i\,; &(7.13)\end{aligned}$$

second, we notice that, since K is $\mathcal{H}^n$-finite,

$$\{E\cap\Omega_i, U_i^j\cap\Omega_i\}_{j=1}^{J_i}\text{ is a Caccioppoli partition of }\Omega_i\,; \tag{7.14}$$

finally, we recall that, by (1.23), for every $X\in C_c^1(\Omega;\mathbb{R}^{n+1})$ it holds

$$\lambda\int_{\partial^*E}X\cdot\nu_E\,d\mathcal{H}^n=\int_{\partial^*E}\mathrm{div}^K X\,d\mathcal{H}^n+2\int_{K\cap E^{(0)}}\mathrm{div}^K X\,d\mathcal{H}^n\,. \tag{7.15}$$

To prove conclusion (ii): Given $x\in\Omega\cap\partial^*E\setminus\Sigma$, there is $i\in\mathbb{N}$ such that $x\in\Omega_i\cap\partial^*E$. By $\Omega\cap\partial^*E\subset K$ and by (7.13) there is $j(x)\in\{1,...,J_i\}$ such that $x\in\partial^*U_{j(x)}^i$. By (7.14), we can use (1.47) and $x\in\Omega\cap\partial^*E\cap\partial^*U_{j(x)}^i$ to deduce that

$$x\notin\cup_{j\ne j(x)}\partial^*U_j^i\,. \tag{7.16}$$

Let $r > 0$ be such that $B_r(x) \cap \partial^* U^i_{j(x)}$ is a C^1-hypersurface. Since Σ contains $\cup_j \partial U^i_j$ and (7.10) holds, (7.16) implies that there is $r > 0$ such that

$$B_r(x) \subset\subset \Omega_i \setminus \Sigma\,, \qquad B_r(x) \cap \cup_j \partial U^i_j = B_r(x) \cap \partial U^i_{j(x)} = B_r(x) \cap \partial^* U^i_{j(x)}\,. \tag{7.17}$$

Since $B_r(x) \cap \cup_{j \neq j(x)} \partial U^i_j = \varnothing$ and $B_r(x) \cap U^i_{j(x)} \neq \varnothing$, we also have that

$$B_r(x) \cap \cup_j U^i_j = B_r(x) \cap U^i_{j(x)}\,,$$

and thus, by (7.14), that $\{E \cap B_r(x), U^i_{j(x)} \cap B_r(x)\}$ is an $\mathcal{H}^n$-partition of $B_r(x)$. In particular, $B_r(x) \cap \partial^* E = B_r(x) \cap \partial^* U^i_{j(x)}$: intersecting with $B_r(x)$ in (7.13) and taking into account (7.17), we conclude that

$$\begin{aligned} B_r(x) \cap K &= B_r(x) \cap [\Omega_i \cap K \setminus \Sigma] \subset B_r(x) \cap [\Omega_i \cap \cup_{j=1}^{J_i} \partial^* U^i_j] = B_r(x) \cap \partial^* U^i_{j(x)} \\ &= B_r(x) \cap \partial^* E\,, \end{aligned}$$

and (7.15) implies that, for every $X \in C^1_c(B_r(x); \mathbb{R}^{n+1})$,

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \mathrm{div}^K X \, d\mathcal{H}^n\,. \tag{7.18}$$

Hence, $\partial^* E$ can be represented, locally in $B_r(x)$, as the graph of distributional solutions of class $C^{1,\alpha}$ to the constant mean curvature equation. By Schauder's theory, $B_r(x) \cap \partial^* E$ is a smooth hypersurface whose mean curvature with respect to ν_E is equal to λ thanks to (7.18).

To prove conclusions (iii) and (iv): Let us now pick $x \in K \setminus (\Sigma \cup \partial^* E)$ and let $i \in \mathbb{N}$ be such that $x \in \Omega_i \cap K$. Let $i \in \mathbb{N}$ be such that $x \in \Omega_i$. By (7.13) there is $j(x) \in \{1, ..., J_i\}$ such that $x \in \partial^* U^i_{j(x)}$. By (7.14) and by (1.47), either $x \in \partial^* E$ (which is excluded from the onset), or there is $k(x) \neq j(x)$ such that $x \in \partial^* U^i_{k(x)}$. We have thus proved that

$$x \in \partial^* U^i_{j(x)} \cap \partial^* U^i_{k(x)}\,, \qquad x \notin \cup_{j \neq j(x), k(x)} \partial^* U^i_j\,. \tag{7.19}$$

To prove conclusion (iii) we notice that if we are in the case when $x \in K \setminus (\Sigma \cup \partial E) = K \setminus (\Sigma \cup \mathrm{cl}\,(E))$ (thanks to $K \cap E = \varnothing$), then $x \notin \mathrm{cl}\,(E)$ implies that, for some $r > 0$, $B_r(x) \cap (\Sigma \cup \mathrm{cl}\,(E)) = \emptyset$. In particular, by (7.14) and (7.19), $\{B_r(x) \cap U^i_{j(x)}, B_r(x) \cap U^i_{k(x)}\}$ is an $\mathcal{H}^n$-partition of $B_r(x)$, and by (7.13)

$$B_r(x) \cap K = B_r(x) \cap \partial^* U^i_{j(x)} = B_r(x) \cap \partial^* U^i_{k(x)}\,,$$

is a $C^{1,\alpha}$-hypersurface. Under these conditions, (7.15) boils down to

$$\int_K \mathrm{div}^K X \, d\mathcal{H}^n = 0\,, \qquad \forall X \in C^1_c(B_r(x); \mathbb{R}^{n+1})\,, \tag{7.20}$$

so that K can be represented, locally in $B_r(x)$, as the graph of distributional solutions to the minimal surfaces equation of class $C^{1,\alpha}$. By Schauder's theory, $B_r(x) \cap K$ is a smooth minimal surface.

To finally prove conclusion (iv), let us assume that $x \in \Omega \cap (\partial E \setminus \partial^* E) \setminus \Sigma$. In this case (7.14) and (7.19) do not imply that $\{B_r(x) \cap U^i_{j(x)}, B_r(x) \cap U^i_{k(x)}\}$ is an $\mathcal{H}^n$-partition of $B_r(x)$; actually, by $\Omega \cap \partial E = \Omega \cap \mathrm{cl}\,(\partial^* E)$, the fact that $x \in \partial E$ implies that $B_s(x) \cap \partial^* E \neq \emptyset$ for every $s > 0$, so that $|B_s(x) \cap E| > 0$ for every $s > 0$, and the situation is such that, for every $s < r$,

$$\{B_s(x) \cap U^i_{j(x)}, B_s(x) \cap U^i_{k(x)}, B_s(x) \cap E\} \text{ is an } \mathcal{H}^n\text{-partition of } B_s(x) \tag{7.21}$$

with all three sets in the partition having positive measure.

Now, by the first inclusion in (7.19), there exists $\nu \in \mathbb{S}^n$ such that, up to further decrease the value of r and for some $u_1, u_2 \in C^{1,\alpha}(\mathbf{D}_r^\nu(x); (-r/4, r/4))$ with $u_1(x) = u_2(x) = 0$ and $\nabla u_1(x) = \nabla u_2(x) = 0$ it must hold

$$\mathbf{C}_r^\nu(x) \cap U^i_{j(x)} = \big\{y + t\,\nu : y \in \mathbf{D}_r^\nu(x)\,, t > u_2(y)\big\}\,,$$
$$\mathbf{C}_r^\nu(x) \cap U^i_{k(x)} = \big\{y + t\,\nu : y \in \mathbf{D}_r^\nu(x)\,, t < u_1(y)\big\}\,.$$

By $U^i_{j(x)} \cap U^i_{k(x)} = \varnothing$ we have $u_1 \le u_2$ on $\mathbf{D}_r^\nu(x)$, so that (7.21) gives

$$\mathbf{C}_r^\nu(x) \cap E = \big\{y + t\,\nu : y \in \{u_1 < u_2\}\,, u_1(y) < t < u_2(y)\big\}\,,$$

and $\{u_1 < u_2\}$ is non-empty. Again by (7.19) and (7.13) we also have that

$$\begin{aligned}
\mathbf{C}_r^\nu(x) \cap K &= \cup_{k=1}^2 \big\{y + u_k(y)\,\nu : y \in \mathbf{D}_r^\nu(x)\big\}\,,\\
\mathbf{C}_r^\nu(x) \cap \partial^* U^i_{j(x)} \cap \partial^* U^i_{k(x)} &= \big\{y + u_1(y)\,\nu : y \in \mathbf{D}_r^\nu(x) \cap \{u_1 = u_2\}\big\}\,,\\
\mathbf{C}_r^\nu(x) \cap \partial^* E &= \cup_{k=1}^2 \big\{y + u_k(y)\,\nu : y \in \mathbf{D}_r^\nu(x) \cap \{u_1 < u_2\}\big\}\,.
\end{aligned}$$

This completes the proof of conclusion (iv)$_\alpha$.

Proof of the claim: Assuming without loss of generality that $x = 0$, we want to find Λ and r_0 positive such that if $B_{2r} \subset\subset \Omega$, then, denoting by $\{U_j\}_j$ the open connected components of $B_{2r} \setminus (E \cup K)$, we have

$$B_r \cap K = B_r \cap \cup_j \partial U_j\,, \tag{7.22}$$
$$\#\big\{j : B_r \cap U_j \ne \varnothing\big\} < \infty\,, \tag{7.23}$$
$$B_{2\,r} \cap \mathrm{cl}\,(\partial^* U_j) = B_{2\,r} \cap \partial U_j\,, \tag{7.24}$$

and that $P(U_j; B_r) \le P(V_j; B_r) + \Lambda\,|U_j \Delta V_j|$ whenever V_j satisfies $V_j \Delta U_j \subset\subset B_r$ and $\mathrm{diam}\,(U_j \Delta V_j) < r_0$.

Step one: We prove that

$$K \cap \mathrm{int}\, U_j^{(1)} = \varnothing\,, \qquad \mathrm{int}\, U_j^{(1)} = U_j \quad \forall j\,. \tag{7.25}$$

To this end, we begin by noticing that, for every j,

$$B_{2\,r} \cap \partial U_j \;\subset\; B_{2\,r} \cap K\,, \tag{7.26}$$
$$U_j \;\subset\; \mathrm{int}(U_j^{(1)}) \;\subset\; B_{2\,r} \cap \mathrm{cl}\, U_j \;\subset\; B_{2\,r} \cap (U_j \cup K)\,, \tag{7.27}$$
$$B_{2\,r} \cap \partial[\mathrm{int}(U_j^{(1)})] \;\subset\; B_{2\,r} \cap K\,. \tag{7.28}$$

Indeed, for every k and j, $U_k \cap U_j = \varnothing$ with U_k and U_j open gives $U_k \cap \partial U_j = \varnothing$, so that $B_{2r} \cap \partial U_j \subset B_{2r} \setminus \cup_k U_k = B_{2\,r} \cap (E \cup K) = B_{2\,r} \cap K$ thanks to the fact that $E \cap \partial U_j = \varnothing$ (as $U_j \cap E = \varnothing$). Having proved (7.26), one easily deduces the third inclusion in (7.27), while the first two are evident. Finally, from (7.27), and since K is closed, we find

$$B_{2\,r} \cap \mathrm{cl}\,\big(\mathrm{int}(U_j^{(1)})\big) \subset B_{2\,r} \cap (\mathrm{cl}\,(U_j) \cup K)\,,$$

so that subtracting $\mathrm{int}(U_j^{(1)})$, and recalling that $U_j \subset \mathrm{int}(U_j^{(1)})$ we find

$$B_{2\,r} \cap \partial[\mathrm{int}(U_j^{(1)})] \subset B_{2\,r} \cap (K \cup \partial U_j)$$

and deduce (7.28) from (7.26).

Next, we claim that,

$$\text{if } K_* = K \setminus \textstyle\bigcup_j \mathrm{int}\, U_j^{(1)}, \text{ then } (K_*, E) \in \mathcal{K} \text{ and } K_* \cup E \text{ is } \mathcal{C}\text{-spanning}\,. \tag{7.29}$$

To prove that $(K_*, E) \in \mathcal{K}$, the only assertion that is not immediate is the inclusion $\Omega \cap \partial E \subset K_*$. To prove it we notice that if $z \in \mathrm{int}\, U_j^{(1)}$, then $B_s(z) \subset \mathrm{int}\, U_j^{(1)}$ for some $s > 0$, so that $U_j \cap E = \varnothing$ gives $|E \cap B_s(z)| = 0$. Since E is open this implies $B_s(z) \cap E = \varnothing$, hence $z \notin \partial E$.

To prove that $E \cup K_*$ is $\mathcal{C}$-spanning: Since $E \cup K_*$ is relatively closed in Ω, it suffices to verify that for arbitrary $\gamma \in \mathcal{C}$, $(K_* \cup E) \cap \gamma \neq \varnothing$. Since $K \setminus B_{2r} = K_* \setminus B_{2r}$, we directly assume that $(K \cup E) \cap (\gamma \setminus B_{2r}) = \varnothing$. Since $K \cup E$ is $\mathcal{C}$-spanning $\mathbf{W}$, by Lemma 7.2, there are two distinct connected components U_j and U_k of $B_{2r} \setminus (K \cup E)$ such that there is $\gamma(\mathbb{S}^1) \cap B_{2\,r} \cap (\partial U_j) \cap (\partial U_k) \neq \varnothing$. We conclude by showing that

$$B_{2\,r} \cap (\partial U_j) \cap (\partial U_k) \subset K_* \,, \qquad \forall j \neq k \,. \tag{7.30}$$

Indeed any point in $B_{2r} \cap (\partial U_j) \cap (\partial U_k)$ is an accumulation point for both U_j and U_k, and thus, by (7.27), for both $\mathrm{int} U_j^{(1)}$ and $\mathrm{int} U_k^{(1)}$. Since $U_j \cap U_k = \emptyset$ implies $(\mathrm{int} U_j^{(1)}) \cap (\mathrm{int} U_k^{(1)}) = \emptyset$, an accumulation point for both $\mathrm{int} U_j^{(1)}$ and $\mathrm{int} U_k^{(1)}$ must lie in $[\partial(\mathrm{int} U_j^{(1)})] \cap [\partial(\mathrm{int} U_k^{(1)})]$. We thus deduce (7.30) from (7.28), and complete the proof of (7.29).

To deduce (7.25) *from* (7.29)*, and complete step one*: By (7.29), (K_*, E) is admissible in $\Psi_{\rm bk}(v)$. Since (K, E) is a minimizer of $\Psi_{\rm bk}(v)$, we conclude that $\mathcal{H}^n(K \setminus K_*) = 0$. Would there be $z \in \mathrm{int}(U_j^{(1)}) \cap K$ for some j, then by (6.3), and with $\rho > 0$ such that $B_\rho(z) \subset \mathrm{int}(U_j^{(1)})$, we would find

$$c\,\rho^n \leq \mathcal{H}^n(K \cap B_\rho(z)) \leq \mathcal{H}^n(K \cap \mathrm{int}(U_j^{(1)})) \leq \mathcal{H}^n(K \setminus K_*) = 0 \,.$$

This shows that $K \cap \mathrm{int}(U_j^{(1)}) = \varnothing$. Using this last fact in combination with $\mathrm{int}(U_j^{(1)}) \subset B_{2\,r} \cap (U_j \cap K)$ from (7.27) we conclude that $\mathrm{int}(U_j^{(1)}) \subset U_j$, and thus that $\mathrm{int}(U_j^{(1)}) = U_j$ by the first inclusion in (7.27).

Step two: We prove (7.24), i.e. $B_{2\,r} \cap \mathrm{cl}\,(\partial^* U_j) = B_{2\,r} \cap \partial U_j$. The $\subset$ inclusion is a general fact, see (1.35). To prove the reverse inclusion we recall, again from (1.35), that $z \in B_{2\,r} \cap \mathrm{cl}\,(\partial^* U_j)$ if and only if $0 < |B_\rho(z) \cap U_j| < |B_\rho|$ for every $\rho > 0$. Now, if $z \in B_{2\,r} \cap \partial U_j$, then clearly, being U_j open, we have $|U_j \cap B_\rho(z)| > 0$ for every $\rho > 0$; moreover, should $|B_\rho(z) \cap U_j| = |B_\rho|$ hold for some ρ, then we would have $z \in \mathrm{int}(U_j^{(1)})$, and thus $z \in U_j$ by (7.25), a contradiction.

Step three: We prove, for each j, the $\mathcal{H}^n$-equivalence of $\partial^* U_j$ and ∂U_j, that is

$$\mathcal{H}^n(B_{2\,r} \cap \partial U_j \setminus \partial^* U_j) = 0 \,. \tag{7.31}$$

By a standard argument [Mag12, Theorem 21.11] it will suffice to prove the existence of $r_0 > 0$ and $\alpha, \beta \in (0, 1/2)$ (depending on n) such that, for each j and each $z \in B_{2\,r} \cap \partial U_j$, it holds

$$\alpha\,|B_\rho| \leq |B_\rho(z) \cap U_j| \leq (1 - \beta)|B_\rho| \,, \tag{7.32}$$

for every $\rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2\,r})\}$.

Proof of the lower bound in (7.32): Since diffeomorphic images of $\mathcal{C}$-spanning sets are $\mathcal{C}$-spanning, a standard argument using diffeomorphic volume fixing variations shows the existence of positive constants Λ and r_0 such that if $(K', E') \in \mathcal{K}_{\rm B}$, $K' \cup (E')^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, and $(K' \Delta K) \cup (E' \Delta E) \subset\subset B_\rho(z)$ for some $\rho < r_0$ and $B_\rho(z) \subset\subset B_{2\,r}$, then

$$\mathcal{F}_{\rm bk}(K, E) \leq \mathcal{F}_{\rm bk}(K', E') + \Lambda\,|E \Delta E'| \,. \tag{7.33}$$

We claim that we can apply (7.33) with

$$E' = E \cup \big(B_\rho(z) \cap \mathrm{cl}\,U_j\big) \,, \quad K' = \big(K \cup (U_j^{(1)} \cap \partial B_\rho(z)\big) \setminus (E')^{(1)} \,, \tag{7.34}$$

where $\rho < r_0$, $B_\rho(z) \subset\subset B_{2\,r}$, and

$$\mathcal{H}^n\big(\partial B_\rho(z) \cap [\partial^* E \cup \partial^* U_j]\big) = \mathcal{H}^n(K \cap \partial B_\rho(z)) = 0 \,. \tag{7.35}$$

Indeed, $K' \cup (E')^{(1)}$ contains $K \cup E^{(1)}$, thus $K \cup E$ being E open, and is thus $\mathcal{C}$-spanning. To check that $(K', E') \in \mathcal{K}_{\rm B}$, we argue as follows. First, we notice that $\mathcal{H}^n(\{\nu_E =$

$\nu_{B_\rho(z)\cap\mathrm{cl}\,(U_j)}\}) = 0$, since it is $\mathcal{H}^n$-contained in the union of $\partial B_\rho(z)\cap\partial^* E$ and $\{\nu_E = \nu_{\mathrm{cl}\,(U_j)}\}$, that are $\mathcal{H}^n$-negligible by (7.35) and by the fact that $\nu_E = -\nu_{\mathrm{cl}\,(U_j)}$ $\mathcal{H}^n$-a.e. on $\partial^* E\cap\partial^*\mathrm{cl}\,(U_j)$ thanks to $|E\cap\mathrm{cl}\,(U_j)| = 0$. By $\mathcal{H}^n(\{\nu_E = \nu_{B_\rho(z)\cap\mathrm{cl}\,(U_j)}\}) = 0$ and (1.39) we thus have

$$\Omega\cap\partial^* E' \overset{\mathcal{H}^n}{=} \Omega\cap\big\{\big[E^{(0)}\cap\partial^*\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)\big]\cup\big[\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)^{(0)}\cap\partial^* E\big]\big\}\,. \tag{7.36}$$

Since U_j is Lebesgue equivalent to $\mathrm{cl}\,(U_j)$ (indeed, $B_{2\,r}\cap\partial U_j\subset K$), we have $U_j^{(1)} = [\mathrm{cl}\,(U_j)]^{(1)}$ and $\partial^*[\mathrm{cl}\,(U_j)] = \partial^* U_j$, so that (1.40) and (7.35) give

$$\begin{aligned}\partial^*\big(B_\rho(z)\cap\mathrm{cl}\,(U_j)\big) &\overset{\mathcal{H}^n}{=} \big\{[\mathrm{cl}\,(U_j)]^{(1)}\cap\partial B_\rho(z)\big\}\cup\big\{B_\rho(x)\cap\partial^*[\mathrm{cl}\,(U_j)]\big\}\,,\\ &= \big(U_j^{(1)}\cap\partial B_\rho(z)\big)\cup\big(B_\rho(x)\cap\partial^* U_j\big)\subset\big(U_j^{(1)}\cap\partial B_\rho(z)\big)\cup K\,,\end{aligned} \tag{7.37}$$

by $B_{2\,r}\cap\partial U_j\subset K$. By (7.36) and $\mathcal{H}^n((E')^{(1)}\cap\partial^* E') = 0$ we thus find that

$$\Omega\cap\partial^* E'\cap\partial^*\big(B_\rho(z)\cap\mathrm{cl}\,(U_j)\big)\overset{\mathcal{H}^n}{\subset} K'\,. \tag{7.38}$$

Moreover, by $\Omega\cap\partial^* E\subset\Omega\cap\partial E\subset K$ and

$$(\partial^* E)\cap\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)^{(0)}\subset E^{(1/2)}\cap\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)^{(0)}\subset\mathbb{R}^{n+1}\setminus(E')^{(1)}\,,$$

we find $(\partial^* E)\cap\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)^{(0)}\subset K\setminus(E')^{(1)}\subset K'$, which combined with (7.38) finally proves the $\mathcal{H}^n$-containment of $\Omega\cap\partial^* E'$ into K', and thus $(K',E')\in\mathcal{K}_{\mathrm{B}}$. We have thus proved that (K',E') as in (7.34) is admissible into (7.33). Since $\mathcal{F}_{\mathrm{bk}}(K,E;\partial B_\rho(z)) = 0$ by (7.35) and $\mathcal{F}_{\mathrm{bk}}(K,E;A) = \mathcal{F}_{\mathrm{bk}}(K',E';A)$ if $A = \Omega\setminus\mathrm{cl}\,(B_\rho(z))$, we deduce from (7.33) that

$$\mathcal{F}_{\mathrm{bk}}(K,E;B_\rho(z))\le\mathcal{F}_{\mathrm{bk}}(K',E';\mathrm{cl}\,(B_\rho(z))) + \Lambda\,|E\Delta E'|\,. \tag{7.39}$$

To exploit (7.39), we first notice that $\{B_\rho(z)\cap U_k\}_k$ is a Lebesgue partition of $B_\rho(z)\setminus E$ with $B_\rho(z)^{(1)}\cap\partial^*(B_\rho(z)\cap U_k) = B_\rho(z)\cap\partial^* U_k$ for every k, so that, by Lemma 7.1,

$$\mathcal{F}_{\mathrm{bk}}(K,E;B_\rho(z)) = 2\,\mathcal{H}^n\Big(B_\rho(z)\cap E^{(0)}\cap\Big(K\setminus\bigcup_k\partial^* U_k\Big)\Big) + \sum_k P(U_k;B_\rho(z))\,. \tag{7.40}$$

Similarly, $\{B_\rho(z)\cap U_k\}_{k\neq j}$ is a Lebesgue partition of $B_\rho(z)\setminus E'$, so that again by Lemma 7.1 we find

$$\begin{aligned}\mathcal{F}_{\mathrm{bk}}(K',E';B_\rho(z)) &= 2\,\mathcal{H}^n\Big(B_\rho(z)\cap(E')^{(0)}\cap\Big(K'\setminus\bigcup_{k\neq j}\partial^* U_k\Big)\Big) + \sum_{k\neq j} P(U_k;B_\rho(z))\\ &= 2\,\mathcal{H}^n\Big(B_\rho(z)\cap(E')^{(0)}\cap\Big(K\setminus\bigcup_k\partial^* U_k\Big)\Big) + \sum_{k\neq j} P(U_k;B_\rho(z))\end{aligned} \tag{7.41}$$

where in the last identity we have used that, by (7.34), we have $B_\rho(z)\cap(E')^{(0)}\cap\partial^* U_j = 0$ and $B_\rho(z)\cap K'\cap(E')^{(0)} = B_\rho(z)\cap K\cap(E')^{(0)}$. Combining (7.39), (7.40), (7.41) and the fact that $(E')^{(0)}\subset E^{(0)}$, we find that

$$P(U_j;B_\rho(z))\le\mathcal{F}_{\mathrm{bk}}\big(K',E';\partial B_\rho(z)\big) + \Lambda\,|B_\rho(z)\cap U_j|\,. \tag{7.42}$$

The first term in $\mathcal{F}_{\mathrm{bk}}\big(K',E';\partial B_\rho(z)\big)$ is $P(E';\partial B_\rho(z))$: taking into account $\mathcal{H}^n(\partial^* E\cap\partial B_\rho(z)) = 0$, by (7.36) and the second identity in (7.37) we find

$$\begin{aligned}P(E';\partial B_\rho(z)) &= \mathcal{H}^n\big(\partial B_\rho(z)\cap E^{(0)}\cap\partial^*\big(B_\rho(z)\cap\mathrm{cl}\,U_j\big)\big)\\ &= \mathcal{H}^n(E^{(0)}\cap U_j^{(1)}\cap\partial B_\rho(z)) = \mathcal{H}^n(U_j^{(1)}\cap\partial B_\rho(z))\,,\end{aligned}$$

while for the second term in $\mathcal{F}_{\mathrm{bk}}\big(K',E';\partial B_\rho(z)\big)$, by $\mathcal{H}^n(K\cap\partial B_\rho(z)) = 0$,

$$\mathcal{H}^n(K'\cap(E')^{(0)}\cap\partial B_\rho(z)) = \mathcal{H}^n((E')^{(0)}\cap U_j^{(1)}\cap\partial B_\rho(z)) = 0$$

since $(E')^{(0)} \subset (B_\rho(z)\cap \mathrm{cl}\,(U_j))^{(0)}$ and $B_\rho(z)\cap\mathrm{cl}\,(U_j)$ has positive Lebesgue density at points in $U_j^{(1)} \cap \partial B_\rho(z)$. Having thus proved that $\mathcal{F}_{\rm bk}\big(K', E'; \partial B_\rho(z)\big) = \mathcal{H}^n(U_j^{(1)} \cap \partial B_\rho(z))$, we conclude from (7.42) that

$$P(U_j; B_\rho(z)) \le \mathcal{H}^n(U_j^{(1)} \cap \partial B_\rho(z)) + \Lambda\,|B_\rho(z) \cap U_j|\,,$$

for a.e. $\rho < r_0$. Since $z \in B_{2\,r}\cap\partial U_j = B_{2\,r}\cap \mathrm{cl}\,(\partial^* U_j)$ and (1.35) imply that $|B_\rho(z)\cap U_j| > 0$ for every $\rho > 0$, a standard argument (see, e.g. [Mag12, Theorem 21.11]) implies that, up to further decrease the value of r_0 depending on Λ, and for some constant $\alpha = \alpha(n) \in (0, 1/2)$, the lower bound in (7.32) holds true.

Proof of the upper bound in (7.32): We argue by contradiction that, no matter how small $\beta \in (0, 1/2)$ is, we can find j, $z \in B_{2\,r} \cap \partial U_j$, and $\rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2\,r})\}$, such that

$$|B_\rho(z) \cap U_j| > (1 - \beta)\,|B_\rho|\,. \tag{7.43}$$

We first notice that for every $k \ne j$ it must be $B_{\rho/2}(z) \cap \partial U_k = \varnothing$: indeed if $w \in B_{\rho/2}(z) \cap \partial U_k$ for some $k \ne j$, then by the lower bound in (6.2) and by (7.43) we find

$$\alpha\,|B_{\rho/2}| \le |U_k \cap B_{\rho/2}(w)| \le |B_\rho(z) \setminus U_j| < \beta\,|B_\rho|$$

which gives a contradiction if $\beta < \alpha/2^{n+1}$. By $B_{\rho/2}(z) \cap \partial U_k = \varnothing$ it follows that

$$B_{\rho/2}(z) \subset \mathrm{cl}\,(U_j) \cup \mathrm{cl}\,(E)\,. \tag{7.44}$$

Let us now set

$$E' = E \setminus B_{\rho/2}(z)\,, \qquad K' = \big(K \setminus B_{\rho/2}(z)\big) \cup \big(E^{(1)} \cap \partial B_{\rho/2}(z)\big)\,. \tag{7.45}$$

By (1.41), if $\mathcal{H}^n(\partial^* E \cap \partial B_{\rho/2}) = 0$, then $(K', E') \in \mathcal{K}$, since $(\Omega \setminus B_{\rho/2}(z)) \cap \partial^* E \subset K \setminus B_{\rho/2}(z) \subset K'$ implies

$$\Omega \cap \partial^* E' \overset{\mathcal{H}^n}{=} \Omega \cap \big\{\big((\partial^* E) \setminus B_{\rho/2}(z)\big) \cup \big(E^{(1)} \cap \partial B_{\rho/2}(z)\big)\big\} \subset K'\,.$$

Moreover $K' \cup (E^{(1)})'$ is $\mathcal{C}$-spanning $\mathbf{W}$ since it contains $(K \cup E) \setminus B_{\rho/2}(z)$, and

$$(K \cup E) \setminus B_{\rho/2}(z) \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,. \tag{7.46}$$

Indeed, if $\gamma \in \mathcal{C}$ and $\gamma(\mathbb{S}^1) \cap (K \cup E) \setminus B_{\rho/2}(z) = \emptyset$, then by applying Lemma 7.2 to $S = K \cup E$ and $B = B_{2\,r}$ we see that either $\gamma(\mathbb{S}^1) \cap (K \cup E) \setminus B_{2\,r} \ne \varnothing$ (and thus $\gamma(\mathbb{S}^1)\cap(K\cup E)\setminus B_{\rho/2}(z) \ne \varnothing$ by $B_{\rho/2}(z) \subset B_r$), or there are $k \ne h$ such that $\gamma(\mathbb{S}^1)\cap\partial U_k \ne \varnothing$ and $\gamma(\mathbb{S}^1)\cap\partial U_h \ne \varnothing$. Up to possibly switch k and h, we have that $k \ne j$, so that (7.44) implies that $\varnothing \ne \gamma(\mathbb{S}^1) \cap \partial U_k = \gamma(\mathbb{S}^1) \cap \partial U_k \setminus B_{\rho/2}(z)$, where the latter set is contained in $K \setminus B_{\rho/2}(z)$ by (7.22) and $B_{\rho/2}(z) \subset B_r$. This proves (7.46).

We can thus plug the competitor (K', E') defined in (7.45) into (7.39), and find

$$\mathcal{F}_{\rm bk}(K, E; B_{\rho/2}(z)) \le \mathcal{F}_{\rm bk}\big(K', E'; \mathrm{cl}\,(B_{\rho/2}(z))\big) + \Lambda\,|E \cap B_{\rho/2}(z)|\,,$$

for every $\rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2\,r})\}$ such that $\mathcal{H}^n(K \cap \partial B_{\rho/2}(z)) = 0$. Now, by Lemma 7.1 and by (7.44) we have

$$\mathcal{F}_{\rm bk}(K, E; B_{\rho/2}(z)) \ge P(U_j; B_{\rho/2}(z)) = P(E; B_{\rho/2}(z))\,,$$

while (1.40) gives

$$\mathrm{cl}\,(B_{\rho/2}/z) \cap K' \overset{\mathcal{H}^n}{=} \mathrm{cl}\,(B_{\rho/2}/z) \cap \partial^* E' \overset{\mathcal{H}^n}{=} E^{(1)} \cap \partial B_{\rho/2}(z)\,,$$

thus proving that, for a.e. $\rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2\,r})\}$,

$$P(E; B_{\rho/2}(z)) \le \mathcal{H}^n(E^{(1)} \cap B_{\rho/2}(z)) + \Lambda\,|E \cap B_{\rho/2}(z)|\,.$$

Since $z \in B_{2r} \cap \partial U_j$ and $B_{\rho/2}(z) \cap \partial^* U_j = B_{\rho/2}(z) \cap \partial^* E$, by (1.35) we see that $|E \cap B_{\rho/2}(z)| > 0$ for every $\rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2r})\}$. By a standard argument, up to further decrease the value of r_0, we find that for some $\alpha' = \alpha'(n)$ it holds

$$|E \cap B_{\rho/2}(z)| \geq \alpha' \, |B_{\rho/2}| \,, \qquad \forall \rho < \min\{r_0, \mathrm{dist}(z, \partial B_{2r})\} \,,$$

and since $|E \cap B_{\rho/2}(z)| = |B_{\rho/2}(z) \setminus U_j|$ this give a contradiction with (7.43) up to further decrease the value of β.

Step three: We prove (7.22) and (7.23). The lower bound in (7.32) implies (7.23), i.e., $J = \#\{j : U_j \cap B_r \neq \varnothing\} < \infty$. Next, by $B_{2r} \cap \partial U_j \subset K$ (last inclusion in (7.27)), to prove (7.22) it suffices to show that

$$K \cap B_r \subset \cup_{j=1}^J \partial U_j \,. \tag{7.47}$$

Now, if $z \in K \cap B_r$, then by $K \cap E = \varnothing$ we have either $z \in K \setminus \mathrm{cl}\,(E)$ or $z \in B_r \cap \partial E$, and, in the latter case, $|E \cap B_\rho(z)| \leq (1-c)\,|B_\rho|$ for every $\rho < \min\{r_0, \mathrm{dist}(z, \partial \mathbf{W})\}$ thanks to (6.2). Therefore, in both cases, z is an accumulation point for $(\cup_{j=1}^J U_j)^{(1)} \cap B_r$. Since J is finite, there must be at least one j such that $z \in \mathrm{cl}\,(U_j)$ – hence $z \in \partial U_j$ thanks to $K \cap U_j = \varnothing$.

Before moving to the next step, we also notice that

$$\mathcal{F}_{\mathrm{bk}}(K, E; B_r) = \sum_{j=1}^J P(U_j; B_r) \,. \tag{7.48}$$

Indeed, by (7.22), (7.23), and (7.31) we have

$$K \cap B_r = B_r \cap \cup_{j=1}^J \partial U_j \overset{\mathcal{H}^n}{=} B_r \cap \cup_{j=1}^J \partial^* U_j \,, \tag{7.49}$$

so that, in the application of Lemma 7.1, i.e. in (7.40), the multiplicity 2 terms vanishes, and we find (7.48).

Step four: In this step we consider a set of finite perimeter V_1 such that, for some $B := B_\rho(z) \subset B_r$ with $\rho < r_0$ and $\mathcal{H}^n(K \cap \partial B) = 0$, we have

$$U_1 \Delta V_1 \subset\subset B \,. \tag{7.50}$$

We then define a pair of Borel sets (K', E') as

$$E' = \big(E \setminus B\big) \cup \big[B \cap \big(V_1 \Delta (E \cup U_1)\big)\big] \,, \tag{7.51}$$

$$K' = \big(K \setminus B\big) \cup \big[B \cap \big(\partial^* V_1 \cup \partial^* U_2 \cup \cdots \cup \partial^* U_J\big)\big] \,, \tag{7.52}$$

and show that $(K', E') \in \mathcal{K}_{\mathrm{B}}$, $K' \cup (E')^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, and

$$\mathcal{F}_{\mathrm{bk}}(K', E') - \mathcal{F}_{\mathrm{bk}}(K, E) \leq P(V_1; B) - P(U_1; B) \,. \tag{7.53}$$

As a consequence of (7.53), (7.33) and $|E \Delta E'| = |U_1 \Delta V_1|$, we find of course that $P(U_1; \Omega) \leq P(V_1; \Omega) + \Lambda\,|U_1 \Delta V_1|$, thus showing that U_1 is a (Λ, r_0)-perimeter minimizer in Ω.

Proving that $(K', E') \in \mathcal{K}_{\mathrm{B}}$ is immediately reduced to showing that $B \cap \partial^* E'$ is $\mathcal{H}^n$-contained in $B \cap (\partial^* V_1 \cup \partial^* U_2 \cup \cdots \cup \partial^* U_J)$ thanks to $\mathcal{H}^n(K \cap \partial B) = 0$. Now, on taking into account that, by (1.39) and (1.41), $\partial^*(X \cup Y)$ and $\partial^*(X \setminus Y)$ are both $\mathcal{H}^n$-contained in $(\partial^* X) \cup (\partial^* Y)$, and thus $\partial^*(X \Delta Y)$ is too, we easily see that

$$B \cap \partial^* E' = B \cap \partial^*[V_1 \Delta (E \cup U_1)] \overset{\mathcal{H}^n}{\subset} (B \cap \partial^* V_1) \cup (B \cap \partial^*(E \cup U_1)) \,.$$

However, $B \cap (E \cup U_1) = B \setminus (\cup_{j=2}^J U_j)$, so that $\partial^* X = \partial^*(\mathbb{R}^{n+1} \setminus X)$ gives

$$B \cap \partial^*(E \cup U_1) = B \cap \partial^*(\cup_{j=2}^J U_j) \overset{\mathcal{H}^n}{\subset} B \cap \cup_{j \geq 2} \partial^* U_j \,,$$

where we have used again the $\mathcal{H}^n$-containment of $\partial^*(X \cup Y)$ in $(\partial^* X) \cup (\partial^* Y)$. This proves that $(K', E') \in \mathcal{K}_{\mathrm{B}}$.

To prove that $K'\cup(E')^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, we show that the set S defined by

$$S=\big((K\cup E)\setminus B\big)\cup\big(\mathrm{cl}\,(B)\cap\cup_{j\ge2}\partial U_j\big)\,,$$

is $\mathcal{H}^n$-contained in $K'\cup(E')^{(1)}$ and is $\mathcal{C}$-spanning $\mathbf{W}$.

To prove that S is $\mathcal{H}^n$-contained in $K'\cup(E')^{(1)}$, we start by noticing that $(K\cup E)\setminus\mathrm{cl}\,(B)$ is $\mathcal{H}^n$-equivalent to $(K\cup E^{(1)}\cup\partial^*E)\setminus\mathrm{cl}\,(B)\subset K\cup E^{(1)}$ (by $(K,E)\in\mathcal{K}_{\rm B}$), whereas $|(E\Delta E')\setminus B|=0$ implies $(E^{(1)}\Delta(E')^{(1)})\setminus\mathrm{cl}\,(B)=\varnothing$: hence $S\setminus\mathrm{cl}\,(B)$ if $\mathcal{H}^n$-contained in $K'\cup(E')^{(1)}$. Next, by (7.31) and by definition of K',

$$S\cap B=B\cap\cup_{j\ge2}\partial U_j\overset{\mathcal{H}^n}{=}B\cap\cup_{j\ge2}\partial^*U_j\subset K'\,.$$

Finally, by $\mathcal{H}^n(K\cap\partial B)=0$, (7.26), and Federer's theorem, $(S\cap\partial B)\setminus K$ is $\mathcal{H}^n$-equivalent to $(E^{(1)}\cap\partial B)\setminus K$, where $E^{(1)}\cap A=(E')^{(1)}\cap A$ in an open neighborhood A of ∂B thanks to $U_1\Delta V_1\subset\subset B$.

To prove that S is $\mathcal{C}$-spanning $\mathbf{W}$, since S is relatively closed in Ω and thanks to Theorem A.1, we only need to check that $S\cap\gamma(\mathbb{S}^1)\ne\varnothing$ for every $\gamma\in\mathcal{C}$. Since $(K\cup E)\cap\gamma(\mathbb{S}^1)\ne\varnothing$ for every $\gamma\in\mathcal{C}$, this is immediate unless γ is such that $S\cap\gamma(\mathbb{S}^1)\setminus B=\varnothing$; in that case, however, Lemma 7.2 implies the existence of $j\ne k$ such that $\gamma(\mathbb{S}^1)\cap B\cap\partial U_j$ and $\gamma(\mathbb{S}^1)\cap B\cap\partial U_k$ are both non-empty. Since either $j\ge2$ or $k\ge2$, we conclude by (7.26) that $\gamma(\mathbb{S}^1)\cap B\cap K'\ne\varnothing$, thus completing the proof.

We are thus left to prove the validity of (7.53). Keeping (7.48) and $\mathcal{F}_{\rm bk}(K',E';B)\le\mathcal{F}_{\rm bd}(K',E';B)$ into account, this amounts to showing that

$$\mathcal{F}_{\rm bd}(K',E';B)=\mathcal{H}^n(B\cap\partial^*E')+2\,\mathcal{H}^n\big(B\cap K'\setminus\partial^*E'\big)=P(V_1;B)+\sum_{j=2}^J P(U_j;B)\,.\quad(7.54)$$

To this end we notice that by (1.44) and $B\cap E'=B\cap[V_1\Delta(E\cup U_1)]$ we have

$$\begin{aligned}B\cap\partial^*E'&\overset{\mathcal{H}^n}{=}B\cap\big\{\partial^*V_1\cup\partial^*(E\cup U_1)\big\}\\&\overset{\mathcal{H}^n}{=}B\cap\big\{(\partial^*V_1)\cup(U_1^{(0)}\cap\partial^*E)\cup(E^{(0)}\cap\partial^*U_1)\big\}\,,\end{aligned}$$

where we have used (1.39) and $\mathcal{H}^n(\{\nu_E=\nu_{U_1}\})=0$ (as $E\cap U_1=\varnothing$). By (1.46) and (1.47), since $\{B\cap E,B\cap U_j\}_{j=1}^N$ is a Caccioppoli partition of B, we have

$$U_1^{(0)}\cap\partial^*E=(\partial^*E)\cap\bigcup_{j\ge2}(\partial^*U_j)\,,\qquad E^{(0)}\cap\partial^*U_1=(\partial^*U_1)\cap\bigcup_{j\ge2}(\partial^*U_j)\,,$$

so that

$$\begin{aligned}B\cap\partial^*E'&\overset{\mathcal{H}^n}{=}B\cap\Big\{(\partial^*V_1)\cup\Big([(\partial^*E)\cup(\partial^*U_1)]\cap\bigcup_{j\ge2}(\partial^*U_j)\Big)\Big\}\,,\\B\cap(K'\setminus\partial^*E')&\overset{\mathcal{H}^n}{=}B\cap\Big(\bigcup_{j\ge2}\partial^*U_j\Big)\setminus\big[(\partial^*E)\cup(\partial^*U_1)\big]\,.\end{aligned}$$

We thus find

$$\begin{aligned}&\mathcal{H}^n(B\cap\partial^*E)+2\,\mathcal{H}^n(B\cap(K'\setminus\partial^*E'))\\&=P(V_1;B)+2\,\mathcal{H}^n\Big(\Big(\bigcup_{j\ge2}\partial^*U_j\Big)\setminus(\partial^*E\cup\partial^*U_1)\Big)+\mathcal{H}^n\Big(\Big(\bigcup_{j\ge2}\partial^*U_j\Big)\cap(\partial^*E\cup\partial^*U_1)\Big)\\&=P(V_1;B)+\sum_{j\ge2}P(U_j;B)\,,\end{aligned}$$

that is (7.54).

Step five: In this final step we prove conclusions (iv) and (v). To this end we fix $x\in[\Omega\cap(\partial E\setminus\partial^*E)]\setminus\Sigma$, and recall that, by conclusion (iv)_α, there are $r>0$, $\nu\in\mathbb{S}^n$,

$u_1, u_2 \in C^{1,\alpha}(\mathbf{D}_r^\nu(x); (-r/4, r/4))$ ($\alpha \in (0, 1/2)$ arbitrary) such that $u_1(x) = u_2(x) = 0$, $u_1 \le u_2$ on $\mathbf{D}_r^\nu(x)$, $\{u_1 < u_2\}$ and $\mathrm{int}\{u_1 = u_2\}$ are both non-empty, and

$$\mathbf{C}_r^\nu(x) \cap K = \cup_{i=1,2}\big\{y + u_i(y)\,\nu : y \in \mathbf{D}_r^\nu(x)\big\}\,, \tag{7.55}$$

$$\mathbf{C}_r^\nu(x) \cap \partial^* E = \cup_{i=1,2}\big\{y + u_i(y)\nu : y \in \{u_1 < u_2\}\big\}\,, \tag{7.56}$$

$$\mathbf{C}_r^\nu(x) \cap E = \big\{y + t\,\nu : y \in \{u_1 < u_2\}\,, u_1(x) < t < u_2(x)\big\}\,. \tag{7.57}$$

We claim that (u_1, u_2) has the minimality property

$$\mathcal{A}(u_1, u_2) \le \mathcal{A}(w_1, w_2) := \int_{\mathbf{D}_r^\nu(x)} \sqrt{1 + |\nabla w_1|^2} + \sqrt{1 + |\nabla w_2|^2}\,, \tag{7.58}$$

among all pairs (w_1, w_2) with $w_1, w_2 \in \mathrm{Lip}(\mathbf{D}_r^\nu(x); (-r/2, r/2))$ that satisfy

$$\begin{cases} w_1 \le w_2\,, & \text{on } \mathbf{D}_r^\nu(x)\,, \\ w_k = u_k\,, & \text{on } \partial\mathbf{D}_r^\nu(x),\ k = 1, 2\,, \end{cases} \qquad \int_{\mathbf{D}_r^\nu(x)} w_2 - w_1 = \int_{\mathbf{D}_r^\nu(x)} u_2 - u_1\,. \tag{7.59}$$

Indeed, starting from a given a pair (w_1, w_2) as in (7.59), we can define $(K' \cap \mathbf{C}_r^\nu(x), E' \cap \mathbf{C}_r^\nu(x))$ by replacing (u_1, u_2) with (w_1, w_2) in (7.55) and (7.57), and then define $(K', E') \in \mathcal{K}_{\rm B}$ by setting $K' \setminus \mathbf{C}_r^\nu(x) = K \setminus \mathbf{C}_r^\nu(x)$ and $E' \setminus \mathbf{C}_r^\nu(x) = E \setminus \mathbf{C}_r^\nu(x)$. Since $\partial\mathbf{C}_r^\nu \setminus (K' \cup E') = \partial\mathbf{C}_r^\nu \setminus (K \cup E)$ it is easily seen (by a simple modification of Lemma 7.2 where balls are replaced by cylinders) that (K', E') is $\mathcal{C}$-spanning $\mathbf{W}$. Since $|E'| = |E|$, the minimality of (K, E) in $\Psi_{\rm bk}(v)$ implies that $\mathcal{F}_{\rm bk}(K, E) \le \mathcal{F}_{\rm bk}(K', E')$, which readily translates into (7.58).

Recalling that both $A_0 = \mathrm{int}\{u_1 = u_2\}$ and $A_+ = \{u_1 < u_2\}$ are non-empty open subsets of $\mathbf{D}_r^\nu(x)$, and denoting by $\mathrm{MS}(u)[\varphi] = \int_{\mathbf{D}_r^\nu(x)} \nabla\varphi \cdot [(\nabla u)/\sqrt{1 + |\nabla u|^2}]$ the distributional mean curvature operator, we find that

$$\begin{aligned} \mathrm{MS}(u_1) + \mathrm{MS}(u_2) = 0\,, &\qquad \text{on } \mathbf{D}_r^\nu(x)\,, \\ \mathrm{MS}(u_k) = 0\,, &\qquad \text{on } A_0 \text{ for each } k = 1, 2\,, \\ \mathrm{MS}(u_2) = -\mathrm{MS}(u_1) = \lambda\,, &\qquad \text{on } A_+\,, \end{aligned} \tag{7.60}$$

for some constant $\lambda \in \mathbb{R}$; in particular, $u_1, u_2 \in C^\infty(A_0) \cap C^\infty(A_+)$. We notice that it must be

$$\lambda < 0\,. \tag{7.61}$$

Indeed, arguing by contradiction, should it be that $\lambda \ge 0$, then by (7.60) we find $\mathrm{MS}(u_2) \ge 0$ and $\mathrm{MS}(u_1) \le 0$ on A_+. Since A_+ is open an non-empty, there is an open ball $B \subset A_+$ such that $\partial B \cap \partial A_+ = \{y_0\}$. Denoting by x_0 the center of B and setting $\nu = (x_0 - y_0)/|x_0 - y_0|$, by $u_1 \le u_2$, $u_1(y_0) = u_2(y_0)$ and $u_k \in C^1(\mathbf{D}_r^\nu(x))$ we find that $\nabla u_1(y_0) = \nabla u_2(y_0)$. At the same time, by applying Hopf's lemma in B at y_0, we see that since $\mathrm{MS}(u_2) \ge 0$ and $\mathrm{MS}(u_1) \le 0$ on B, it must be $\nu \cdot \nabla u_2(y_0) < 0$ and $\nu \cdot \nabla u_1(y_0) > 0$, against $\nabla u_1(y_0) = \nabla u_2(y_0)$.

By (7.60), (7.61), and $u_2 \ge u_1$ on $\mathbf{D}_r^\nu(x)$ we can apply the sharp regularity theory for the double membrane problem developed in [Sil05, Theorem 5.1] and deduce that $u_1, u_2 \in C^{1,1}(\mathbf{D}_r^\nu(x))$. Next we notice that, for every $\varphi \in C_c^\infty(A_+)$, and setting $u_+ = u_2 - u_1$,

$$2\,\lambda \int_{A_+} \varphi = \mathrm{MS}(u_2)[\varphi] - \mathrm{MS}(u_1)[\varphi] = \int_{A_+} \mathrm{A}(x)[\nabla u_+] \cdot \nabla\varphi\,,$$

where we have set, with $f(z) = \sqrt{1 + |z|^2}$,

$$\mathrm{A}(x) = \int_0^1 \nabla^2 f\big(s\,\nabla u_2(x) + (1 - s)\,\nabla u_1(x)\big)\,ds\,.$$

In particular, $u_+ \in C^{1,1}(\mathbf{D}_r^\nu(x))$ is a non-negative distributional solution of

$$\mathrm{div}\,(\mathrm{A}(x)\nabla u_+) = -2\,\lambda\,, \qquad \text{on } A_+\,,$$

with a strictly positive right-hand side (by (7.61)) and with $\mathrm{A} \in \mathrm{Lip}(A_+;\mathbb{R}^{n\times n}_{\rm sym})$ uniformly elliptic. We can thus apply the regularity theory for free boundaries developed in [FGS15, Theorem 1.1, Theorem 4.14] to deduce that

$$\mathrm{FB} = \mathbf{D}^\nu_r(x) \cap \partial\{u_+ = 0\} = \mathbf{D}^\nu_r(x) \cap \partial\{u_2 = u_1\}\,,$$

can be partitioned into sets Reg and Sing such that Reg is relatively open in FB and such that for every $z \in \mathrm{Reg}$ there are $r > 0$ and $\beta \in (0,1)$ such that $B_r(x) \cap \mathrm{FB}$ is a $C^{1,\beta}$-embedded $(n-1)$-dimensional manifold, and such that $\mathrm{Sing} = \cup_{k=0}^{n-1}\mathrm{Sing}_k$ is relatively closed in FB, with each Sing_k locally $\mathcal{H}^k$-rectifiable in $\mathbf{D}^\nu_r(x)$. Since, by (7.56),

$$\mathbf{C}^\nu_r(x) \cap (\partial E \setminus \partial^* E) = \big\{y + u_1(y)\,\nu : y \in \mathrm{FB}\big\}$$

and $u_1 \in C^{1,1}(\mathbf{D}^\nu_r(x))$, we conclude by a covering argument that $\Omega \cap (\partial E \setminus \partial^* E)$ has all the required properties, and complete the proof of the theorem. □

8. Equilibrium across transition lines in wet foams (Theorem 1.7)

Proof of Theorem 1.7. Let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $(K_*, E_*) \in \mathcal{K}_{\rm foam}$. We can find $(K, E) \in \mathcal{K}$ such that K is $\mathcal{H}^n$-equivalent to K_*, E Lebesgue equivalent to E_*, and $K \cap E^{(1)} = \varnothing$ by repeating with minor variations the considerations made in step one of the proof of Theorem 6.2 (we do not have to worry about the $\mathcal{C}$-spanning condition, but have to keep track of the volume constraint imposed for each U_i, which can be done by using the volume-fixing variations for clusters from [Mag12, Part IV]). In proving the regularity part of the statement, thanks to Theorem 2.1-(a) we can directly work with balls $B \subset\subset \Omega$ having radius less than r_0 (with r_0 as in (1.33)), and consider the open connected components $\{U_i\}_i$ of B induced by $K \cup E$. Using Lemma 7.1 and, again, volume-fixing variation techniques in place of the theory of homotopic spanning, we can proceed to prove analogous statement to (7.8), (7.9), (7.10), and (7.11), thus proving the (Λ, r_0)-minimality of each U_i in B. The claimed $C^{1,\alpha}$-regularity of each U_i outside of a closed set Σ with the claimed dimensional estimates follows then from De Giorgi's theory of perimeter minimality [DG60, Tam82, Mag12]. □

Appendix A. Equivalence of homotopic spanning conditions

In Theorem A.1 we prove that, when S is a closed set, the notion of "S is $\mathcal{C}$-spanning $\mathbf{W}$" introduced in Definition B boils down to the one in Definition A. We then show that the property of being $\mathcal{C}$-spanning is stable under reduction to the rectifiable part of a Borel set, see Lemma 2.2.

Theorem A.1. *Given a closed set $\mathbf{W} \subset \mathbb{R}^{n+1}$, a spanning class $\mathcal{C}$ for $\mathbf{W}$, and a set S relatively closed in Ω, the following two properties are equivalent:*

(i): *for every $\gamma \in \mathcal{C}$, we have $S \cap \gamma(\mathbb{S}^1) \neq \varnothing$;*

(ii): *for every $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ and for $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$, we have*

$$\begin{aligned}&\textit{for } \mathcal{H}^n\textit{-a.e. } x \in T[s]\,,\\ &\exists \textit{ a partition } \{T_1, T_2\} \textit{ of } T \textit{ with } x \in \partial^e T_1 \cap \partial^e T_2\,,\\ &\textit{and s.t. } S \cup T[s] \textit{ essentially disconnects } T \textit{ into } \{T_1, T_2\}\,.\end{aligned} \tag{A.1}$$

In particular, S is $\mathcal{C}$-spanning $\mathbf{W}$ according to Definition A if and only if it does so according to Definition B.

Remark A.2 (x-dependency of $\{T_1, T_2\}$)**.** In the situation of Figure 1.4 it is clear that the same choice of $\{T_1, T_2\}$ can be used to check the validity of (A.1) at every $x \in T[s]$. One may thus wonder if it could suffice to reformulate (A.1) so that the partition $\{T_1, T_2\}$ is independent of x. The simpler example we are aware of and that shows this simpler definition would not work is as follows. In $\mathbb{R}^3$, let $\mathbf{W}$ be a closed δ-neighborhood of a

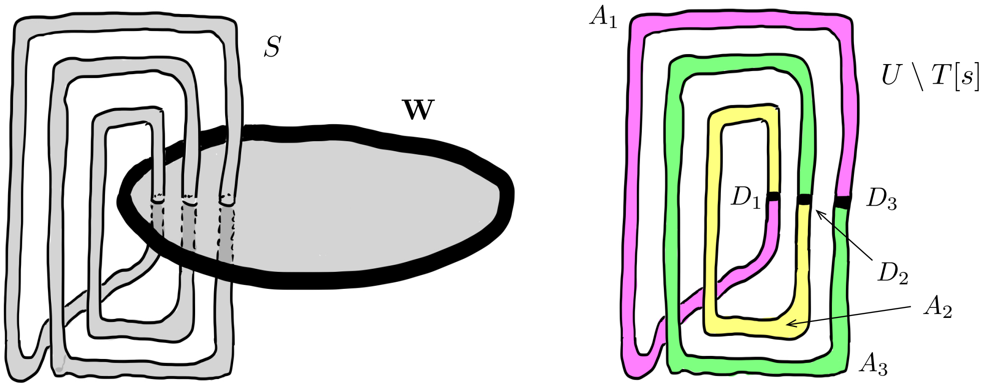

FIGURE A.1. The situation in Remark A.2. The components A_1, A_2 and A_3 (depicted in purple, yellow, and green respectively) of $U \setminus T[s]$ are bounded by the three disks $\{D_i\}_{i=1}^3$ (depicted as boldface segments).

circle Γ, let U be the open δ-neighborhood of a loop with link number *three* (or higher *odd* number) with respect to $\mathbf{W}$, let K be the disk spanned by Γ, and let $S = \Omega \cap [(K \setminus U) \cup \partial U]$, see Figure A.1. Now consider a "test tube" T which compactly contains U and is such that, for every s, $U \cap T[s]$ consists of three disks $\{D_i\}_{i=1}^3$. Since $U \subset T$, the property "$S \cup T[s]$ essentially disconnects T into $\{T_1, T_2\}$ in such a way that $T[s] \subset T \cap \partial^e T_1 \cap \partial^e T_2$" would immediately imply "$U \cap (S \cup T[s]) = U \cap T[s]$ essentially disconnects $T \cap U = U$ into $\{U_1, U_2\}$ with $U \cap T[s] \subset U \cap \partial^e U_1 \cap \partial^e U_2$", where $U_i = T_i \cap U$ (see step one in the proof of Theorem 3.1 for a formal proof of this intuitive assertion). However, the latter property does not hold. To see this, denoting by $\{A_i\}_{i=1}^3$ the three connected components of $U \setminus T[s]$, we would have $U_1 = A_i \cup A_j$ and $U_2 = A_k$ for some choice of $i \neq j \neq k \neq i$, whereas, independently of the choice made, $U \cap \partial^e U_1 \cap \partial^e U_2$ always fails to contain one of the disks $\{D_i\}_{i=1}^3$: for example, if $U_1 = A_1 \cup A_2$ and $U_2 = A_3$, then $U \cap \partial^e U_1 \cap \partial^e U_2 = D_2 \cup D_3$, and D_1 is entirely missed. We conclude that the set S just constructed, although clearly $\mathcal{C}$-spanning $\mathbf{W}$ in terms of Definition A, fails to satisfy the variant of (A.1) where a same partition $\{T_1, T_2\}$ is required to work for $\mathcal{H}^n$-a.e. choice of $x \in T[s]$.

Proof of Theorem A.1. Step one: We prove that (ii) implies (i). Indeed, if there is $\gamma \in \mathcal{C}$ such that $S \cap \gamma(\mathbb{S}^1) = \varnothing$, then, S being closed, we can find $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ such that $\operatorname{dist}(S, T) > 0$. By (ii), there is $s \in \mathbb{S}^1$ such that $S \cup T[s]$ essentially disconnects T. By $\operatorname{dist}(S, T) > 0$ we see that $(S \cup T[s]) \cap T = T[s]$, so that $T[s]$ essentially disconnects T, a contradiction.

Step two: We now prove that (i) implies (ii). To this end we consider an arbitrary $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ and aim at proving the existence of J of full $\mathcal{H}^1$-measure in $\mathbb{S}^1$ such that, if $s \in J$, then (A.1) holds.

This is trivial, with $J = \mathbb{S}^1$, if $|S \cap T| = |T|$. Indeed, in this case, we have $T = S^{(1)} \cap T$, that, combined with S being closed, implies $T = S \cap T$. In particular, $S \cup T[s] = T$ for every $s \in \mathbb{S}^1$, and since, trivially, T essentially disconnects T, the conclusion follows.

We thus assume that $|S \cap T| < |T|$: in particular,

$$U = T \setminus S$$

is a non-empty, open set, whose connected components are denoted by $\{U_i\}_{i \in I}$ (I a countable set). By the Lebesgue points theorem, $\mathcal{L}^{n+1}$-a.e. $x \in T$ belongs either to $U^{(0)}$ or to U. Then, by the smoothness of Φ and by the area formula, we can find a set J of full

$\mathcal{H}^1$-measure in $\mathbb{S}^1$ such that

$$\mathcal{H}^n\big(T[s]\setminus(U^{(0)}\cup U)\big)=0\,,\qquad\forall s\in J\,. \tag{A.2}$$

In particular, given $s\in J$, we just need to prove (A.1) when either $x\in T[s]\cap U^{(0)}$ or $x\in T[s]\cap U$. Before examining these two cases we also notice that we can further impose on J that

$$\mathcal{H}^n\Big(T[s]\cap\Big[\partial^e U\cup\partial^e S\cup\big(U^{(1)}\setminus U\big)\cup\bigcup_{i\in I}\big(U_i^{(1)}\setminus U_i\big)\Big]\Big)=0\,,\qquad\forall s\in J\,. \tag{A.3}$$

Indeed, again by the Lebesgue points theorem, the sets $\partial^e U$, $\partial^e S$, $U^{(1)}\setminus U$, and $\cup_{i\in I}U_i^{(1)}\setminus U_i$ are all $\mathcal{L}^{n+1}$-negligible.

Case one, $x\in T[s]\cap U^{(0)}$: To fix ideas, notice that $U^{(0)}\neq\varnothing$ implies $|S\cap T|>0$, and in particular S has positive Lebesgue measure. Given an arbitrary $s'\in J\setminus\{s\}$ we denote by $\{I_1,I_2\}$ the partition of $\mathbb{S}^1$ bounded by $\{s,s'\}$, and then consider the Borel sets

$$T_1=\Phi(I_1\times B_1^n)\cap S\,,\qquad T_2=\Phi(I_2\times B_1^n)\cup\Big(\Phi(I_1\times B_1^n)\setminus S\Big)\,.$$

We first notice that $\{T_1,T_2\}$ is a non-trivial partition of T: Indeed $|T_1|>0$ since x has density $1/2$ for $\Phi(I_1\times B_1^n)$ and (by $x\in U^{(0)}$) density 1 for $S\cap T$; at the same time $|T_2|=|T\setminus T_1|\geq|T\setminus S|>0$. Next, we claim that

$$T^{(1)}\cap\partial^e T_1\cap\partial^e T_2 \text{ is } \mathcal{H}^n\text{-contained in } S\,. \tag{A.4}$$

Indeed, since $\Phi(I_1\times B_1^n)$ is an open subset of T with $T\cap\partial[\Phi(I_1\times B_1^n)]=T[s]\cup T[s']$, and since $\partial^e T_1$ coincides with $\partial^e S$ inside the open set $\Phi(I_1\times B_1^n)$, we easily see that

$$\begin{aligned}T^{(1)}\cap\partial^e T_1\cap\partial^e T_2 &= T\cap\partial^e T_1=T\cap\partial^e\big(\Phi(I_1\times B_1^n)\cap S\big)\\ &\subset \big(\Phi(I_1\times B_1^n)\cap\partial^e S\big)\cup\Big(\big(T[s]\cup T[s']\big)\setminus S^{(0)}\Big)\,.\end{aligned}$$

Now, on the one hand, by $\mathcal{H}^n(\partial^e S\cap(T[s]\cup T[s']))=0$ (recall (A.3)), it holds

$$\big(T[s]\cup T[s']\big)\setminus S^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } T\cap S^{(1)}\,;$$

while, on the other hand, by $\Omega\cap\partial^e S\subset\Omega\cap\partial S\subset\Omega\cap S$ (since S is closed in Ω) and by $\Phi(I_1\times B_1^n)\subset T\subset\Omega$, we also have that $\Phi(I_1\times B_1^n)\cap\partial^e S\subset T\cap S$; therefore

$$T^{(1)}\cap\partial^e T_1\cap\partial^e T_2 \text{ is } \mathcal{H}^n\text{-contained in } T\cap(S\cup S^{(1)})=T\cap S\,,$$

where we have used that S is closed to infer $S^{(1)}\subset S$. Having proved (A.4) and the non-triviality of $\{T_1,T_2\}$, we conclude that S (and, thus, $S\cup T[s]$) essentially disconnects T into $\{T_1,T_2\}$. We are left to prove that $x\in T\cap\partial^e T_1\cap\partial^e T_2$. To this end, we notice that $x\in T[s]\cap(T\setminus S)^{(0)}$ and $\Phi(I_1\times B_1^n)\subset T$ imply

$$|T_1\cap B_r(x)|=|\Phi(I_1\times B_1^n)\cap S\cap B_r(x)|=|\Phi(I_1\times B_1^n)\cap B_r(x)|+\mathrm{o}(r^{n+1})=\frac{|B_r(x)|}{2}+\mathrm{o}(r^{n+1})\,,$$

so that $x\in(T_1)^{(1/2)}\subset\partial^e T_1$; since $T\cap\partial^e T_1=T\cap\partial^e T_1\cap\partial^e T_2$ and $x\in T$ we conclude the proof in the case when $x\in T[s]\cap U^{(0)}$.

Case two, $x\in T[s]\cap U$: In this case there exists $i\in I$ such that $x\in U_i$, and, correspondingly, we claim that

$$\begin{aligned}&\exists\{V_1,V_2\} \text{ a non-trivial Borel partition of } U_i\setminus T[s]\,,\\ &\text{s.t. } x\in\partial^e V_1\cap\partial^e V_2 \text{ and } T\cap(\partial V_1\cup\partial V_2)\subset S\cup T[s]\,.\end{aligned} \tag{A.5}$$

Given the claim, we conclude by setting $T_1=V_1$ and $T_2=V_2\cup(T\setminus U_i)$. Indeed, since $V_2\cap U_i=T_2\cap U_i$ with U_i open implies $U_i\cap\partial^e V_1=U_i\cap\partial^e T_1$, we deduce from (A.5) that

$$x\in U_i\cap\partial^e V_1\cap\partial^e V_2=U_i\cap\partial^e T_1\cap\partial^e T_2\,;$$

at the same time, $S\cup T[s]$ essentially disconnects T into $\{T_1,T_2\}$ since, again by (A.5),

$$T^{(1)}\cap\partial^e T_1\cap\partial^e T_2=T\cap\partial^e T_1=T\cap\partial^e V_1\subset T\cap\partial V_1\subset S\cup T[s]\,.$$

We are thus left to prove (A.5). To this end, let us choose $r(x)>0$ small enough to have that $B_{r(x)}(x)\subset U_i$, and that $B_{r(x)}(x)\setminus T[s]$ consists of exactly two connected components $\{V_1^x,V_2^x\}$; in this way,

$$x\in(V_1^x)^{(1/2)}\cap(V_2^x)^{(1/2)}\,. \tag{A.6}$$

Next, we define

$$\begin{aligned}V_1&=\text{the connected component of } U_i\setminus T[s]\text{ containing } V_1^x\,,\\ V_2&=U_i\setminus(T[s]\cup V_1)\,.\end{aligned}$$

Clearly $\{V_1,V_2\}$ is a partition of $U_i\setminus T[s]$, and, thanks to $\partial V_1\cup\partial V_2\subset T[s]\cup\partial U_i$, we have

$$T\cap(\partial V_1\cup\partial V_2)\subset T\cap(T[s]\cup\partial U_i)\subset S\cup T[s]\,.$$

Therefore (A.5) follows by showing that $|V_1|\,|V_2|>0$. Since V_1 contains the connected component V_1^x of $B_{r(x)}(x)\setminus T[s]$, which is open and non-empty, we have $|V_1|>0$. Arguing by contradiction, we assume that

$$|V_2|=|U_i\setminus(T[s]\cup V_1)|=0\,.$$

Since V_1 is a connected component of the open set $U_i\setminus T[s]$ this implies that

$$U_i\setminus T[s]=V_1\,.$$

Let $x_1\in V_1^x$ and $x_2\in V_2^x$ (where V_1^x and V_2^x are the two connected components of $B_{r(x)}(x)\setminus T[s]$). Since V_1 is connected and $\{x_1,x_2\}\subset U_i\setminus T[s]=V_1$, there is a smooth embedding γ_1 of $[0,1]$ into V_1 with $\gamma_1(0)=x_1$ and $\gamma_1(1)=x_2$. Arguing as in [DLGM17b, Proof of Lemma 10, Step 2] using Sard's theorem, we may modify γ_1 by composing with a smooth diffeomorphism such that the modified γ_1 intersects $\partial B_{r(x)}(x)$ transversally at finitely many points. Thus $\gamma_1([0,1])\setminus\operatorname{cl}B_{r(x)}(x)$ is partitioned into finitely many curves $\gamma_1((a_i,b_i))$ for disjoint arcs $(a_i,b_i)\subset[0,1]$. Since $B_{r(x)}(x)\setminus T[s]$ is disconnected into V_1^x and V_2^x and γ_1 is disjoint from $T[s]$, there exists i such that, up to interchanging V_1^x and V_2^x, $\gamma(a_i)\in\operatorname{cl}V_1^x\cap\partial B_{r(x)}(x)$ and $\gamma(b_i)\in\operatorname{cl}V_2^x\cap\partial B_{r(x)}(x)$. Let us call $\tilde{\gamma_1}$ the restriction of γ_1 to $[a_i,b_i]$. Next, we choose a smooth embedding γ_2 of $[0,1]$ into $B_{r(x)}(x)$ such that $\gamma_2(0)=\tilde{\gamma}_1(a_i)$, $\gamma_2(1)=\tilde{\gamma}_1(b_i)$, and $\gamma_2([0,1])$ intersects $T[s]\cap B_{r(x)}(x)$ at exactly one point, denoted by $x_{12}=\gamma_2(t_0)$, with

$$\gamma_2'(t_0)\neq0\,. \tag{A.7}$$

Since $\tilde{\gamma_1}((a_i,b_i))\cap\operatorname{cl}B_{r(x)}(x)=\varnothing$ and $\gamma_2([0,1])\subset\operatorname{cl}B_r(x)$, we can choose γ_2 so that the concatenation of γ_1 and γ_2 defines a smooth embedding γ_* of $\mathbb{S}^1$ into $U_i\subset T$. Up to reparametrizing we may assume that $\gamma_*(1)=x_{12}$. Since $\gamma_1([0,1])\subset V_1$ and $V_1\cap(S\cup T[s])=\varnothing$, we have that

$$\gamma_*(\mathbb{S}^1)\cap(S\cup T[s])=\gamma_2([0,1])\cap(S\cup T[s])=\{x_{12}\}\subset T[s]\cap B_{r(x)}(x)\,. \tag{A.8}$$

A first consequence of (A.8) is that $\gamma_*(\mathbb{S}^1)\cap S=\varnothing$. Similarly, the curve $\gamma_{**}:\mathbb{S}^1\to\Omega$ defined via $\gamma_{**}(t)=\gamma_*(\bar t)$ ($t\in\mathbb{S}^1$) where the bar denotes complex conjugation, has the same image as γ_* and thus satisfies $\gamma_{**}(\mathbb{S}^1)\cap S=\varnothing$ as well. Therefore, in order to obtain a contradiction with $|V_2|=0$, it is enough to prove that either $\gamma_*\in\mathcal{C}$ or $\gamma_{**}\in\mathcal{C}$. To this end we are now going to prove that one of γ_* or γ_{**} is homotopic to γ in T (and thus in Ω), where γ is the curve from the tube $(\gamma,\Phi,T)\in\mathcal{T}(\mathcal{C})$ considered at the start of the argument.

Indeed, let $\mathbf{p}:\mathbb{S}^1\times B_1^n\to\mathbb{S}^1$ denote the canonical projection $\mathbf{p}(t,x)=t$, and consider the curves $\sigma_*=\mathbf{p}\circ\Phi^{-1}\circ\gamma_*:\mathbb{S}^1\to\mathbb{S}^1$ and $\sigma_{**}=\mathbf{p}\circ\Phi^{-1}\circ\gamma_{**}$. By (A.8), $\sigma_*^{-1}(\{s\})=\{1\}$, and 1 is a regular point of σ_* by (A.7) and since Φ is a diffeomorphism. Similarly, $\sigma_{**}^{-1}(\{s\})=\{1\}$

and 1 is a regular point of σ_{**}. Now by our construction of γ_{**}, exactly one of γ_* or γ_{**} is orientation preserving at 1 and the other is orientation reversing. So we may compute the winding numbers of σ_* and σ_{**} via (see e.g. [Mil97, pg 27]):

$$\deg\sigma_* = \mathrm{sgn}\,\det D\sigma_*(1) = -\mathrm{sgn}\,\det D\sigma_{**}(1) = -\deg\sigma_{**} \in \{+1,-1\}\,.$$

If we define $\sigma = \mathbf{p}\circ\Phi^{-1}\circ\gamma$, then σ has winding number 1, and so is homotopic in $\mathbb{S}^1$ to whichever of σ_* or σ_{**} has winding number 1. Since Φ is a diffeomorphism of $\mathbb{S}^1\times B_1^n$ into Ω, we conclude that γ is homotopic relative to Ω to one of γ_* or γ_{**}, and, thus, that $\gamma^*\in\mathcal{C}$ or $\gamma_{**}\in\mathcal{C}$ as desired. □

Appendix B. Convergence of every minimizing sequence of $\Psi_{\rm bk}(v)$

In proving Theorem 1.5 we have shown that every minimizing sequence $\{(K_j,E_j)\}_j$ of $\Psi_{\rm bk}(v)$ has a limit (K,E) such that, denoting by $B^{(w)}$ a ball of volume w, it holds

$$\Psi_{\rm bk}(v)=\Psi_{\rm bk}(|E|)+P(B^{(v-|E|)})\,,\qquad \Psi_{\rm bk}(|E|)=\mathcal{F}_{\rm bk}(K,E)\,,$$

with both K and E bounded. In particular, minimizers of $\Psi_{\rm bk}(v)$ can be constructed in the form $(K\cup\partial B^{(v-|E|)}(x),E\cup B^{(v-|E|)}(x))$ provided x is such that $B^{(v-|E|)}(x)$ is disjoint from $K\cup E\cup\mathbf{W}$. This argument, although sufficient to prove the existence of minimizers of $\Psi_{\rm bk}(v)$, it is not sufficient to prove the convergence of every minimizing sequence of $\Psi_{\rm bk}(v)$, i.e., to exclude the possibility that $|E|<v$. This is done in the following theorem at the cost of assuming the C^2-regularity of $\partial\Omega$. This result will be important in the companion paper [MNR23a].

Theorem B.1. *If $\mathbf{W}$ is the closure of a bounded open set with C^2-boundary, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and $\ell<\infty$, then for every $v>0$ and every minimizing sequence $\{(K_j,E_j)\}_j$ of $\Psi_{\rm bk}(v)$ there is a minimizer (K,E) of $\Psi_{\rm bk}(v)$ such that K is $\mathcal{H}^n$-rectifiable and, up to extracting subsequences and as $j\to\infty$,*

$$E_j\to E\,,\qquad \mu_j\stackrel{*}{\rightharpoonup}\mathcal{H}^n\llcorner(\Omega\cap\partial^*E)+2\,\mathcal{H}^n\llcorner(K\cap E^{(0)})\,,\tag{B.1}$$

where $\mu_j=\mathcal{H}^n\llcorner(\Omega\cap\partial^*E_j)+2\,\mathcal{H}^n\llcorner(\mathcal{R}(K_j)\cap E_j^{(0)})$.

Proof. By step three in the proof of Theorem 6.2, there is $(K,E)\in\mathcal{K}_{\rm B}$ satisfying (B.1) and such that K and E are bounded, (K,E) is a minimizer of $\Psi_{\rm bk}(|E|)$, K is $\mathcal{H}^n$-rectifiable, and $|E|\le v$; moreover, if $v>|E|$, then there is $x\in\mathbb{R}^{n+1}$ such that $B^{(v-|E|)}(x)$ is disjoint from $K\cup E\cup\mathbf{W}$ and $(K',E')=(K\cup\partial B^{(v-|E|)}(x),E\cup B^{(v-|E|)}(x))$ is a minimizer of $\Psi_{\rm bk}(v)$. We complete the proof by deriving a contradiction with the $v^*=v-|E|>0$ case. The idea is to relocate $B^{(v^*)}(x)$ to save perimeter by touching $\partial\mathbf{W}$ or ∂E; see Figure B.1.

First of all, we claim that $K=\Omega\cap\partial E$. If not, since (K,E) and (K',E') respectively are minimizers of $\Psi_{\rm bk}(|E|)$ and $\Psi_{\rm bk}(v)$, then there are $\lambda,\lambda'\in\mathbb{R}$ such that (K,E) and (K',E') respectively satisfy (6.1) with λ and λ'. By localizing (6.1) for (K',E') at points in $\Omega\cap\partial^*E$ we see that it must be $\lambda=\lambda'$; by localizing at points in $\partial B^{(v-|E|)}(x)$, we see that λ is equal to the mean curvature of $\partial B^{(v-|E|)}(x)$, so that $\lambda>0$; by arguing as in the proof of [KMS21, Theorem 2.9] (see [Nov23] for the details), we see that if $K\setminus(\Omega\cap\partial E)\ne\varnothing$, then $\lambda\le0$, a contradiction.

Having established that $K=\Omega\cap\partial E$, we move an half-space H compactly containing $\mathrm{cl}\,(E)\cup\mathbf{W}$ until the boundary hyperplane ∂H first touches $\mathrm{cl}\,(E)\cup\mathbf{W}$. Up to rotation and translation, we can thus assume that $H=\{x_{n+1}>0\}$ and

$$0\in\mathrm{cl}\,(E)\cup\mathbf{W}\subset\mathrm{cl}\,(H)\,.\tag{B.2}$$

We split (B.2) into two cases, $0\in\Omega\cap\partial E$ and $0\in\mathbf{W}$, that are then separately discussed for the sake of clarity. In both cases we set $x=(x',x_{n+1})\in\mathbb{R}^n\times\mathbb{R}\equiv\mathbb{R}^{n+1}$, and set

$$\mathbf{C}_\delta\ =\ \{x:x_{n+1}\in(0,\delta)\,,|x'|<\delta\}\,,$$

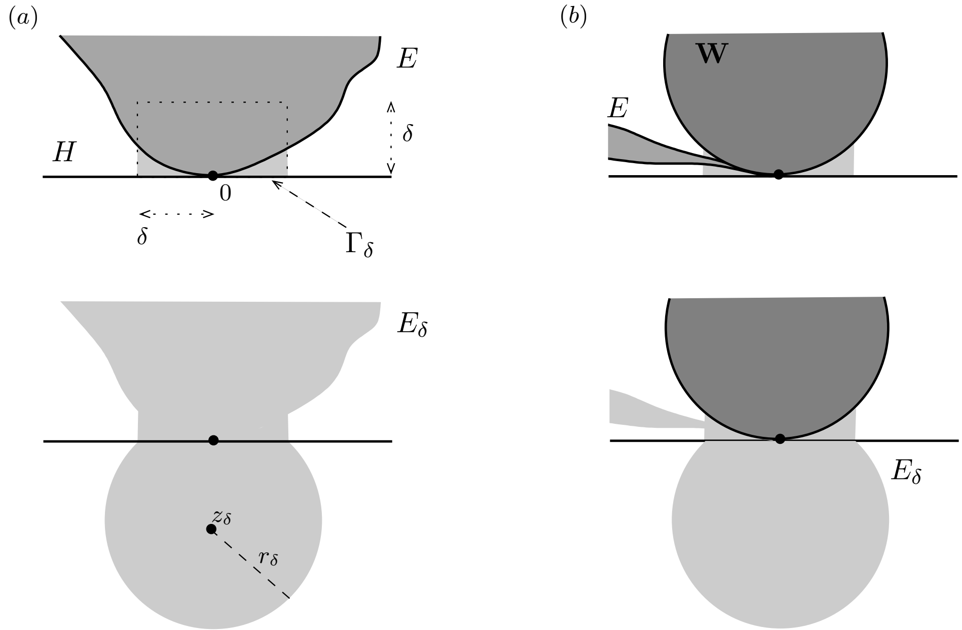

FIGURE B.1. (a): the construction of E_δ when $0 \in \Omega \cap \partial E$; (b) the construction of E_δ when $0 \in \mathbf{W}$.

$$
\begin{aligned}
\mathbf{L}_\delta &= \{x : |x'| = \delta, x_{n+1} \in (0,\delta)\}\,,\\
\mathbf{T}_\delta &= \{x : x_{n+1} = \delta\,, |x'| < \delta\}\,,\\
\mathbf{D}_\delta &= \{x : x_{n+1} = 0\,, |x'| < \delta\}\,,
\end{aligned}
$$

for every $\delta > 0$.

Case one, $0 \in \Omega \cap \partial E$: In this case, by the maximum principle [DM19, Lemma 3], (6.1), and the Allard regularity theorem, we can find $\delta_0 > 0$ and $u \in C^2(\mathbf{D}_{\delta_0};[0,\delta_0])$ with $u(0) = 0$ and $\nabla u(0) = 0$ such that $\mathbf{C}_{\delta_0} \subset\subset \Omega$ and

$$
\begin{aligned}
E \cap \mathbf{C}_{\delta_0} &= \big\{x \in \mathbf{C}_{\delta_0} : \delta_0 > x_{n+1} > u(x')\big\}\,,\\
(\partial E) \cap \mathbf{C}_{\delta_0} &= \big\{x \in \mathbf{C}_{\delta_0} : x_{n+1} = u(x')\big\}\,.
\end{aligned}
\tag{B.3}
$$

Since $0 \le u(x') \le C\,|x'|^2$ for some $C = C(E)$, if we set

$$
\Gamma_\delta = \Big\{x \in \mathbf{C}_\delta : 0 < x_{n+1} < u(x')\Big\}\,, \qquad \delta \in (0,\delta_0)\,,
\tag{B.4}
$$

then we have

$$
|\Gamma_\delta| \;\le\; C\,\delta^{n+2}\,,
\tag{B.5}
$$

$$
P\big(\Gamma_\delta;\mathbf{L}_\delta\big) \;\le\; C\,\delta^{n+1}\,.
\tag{B.6}
$$

We then set

$$
E_\delta = E \cup \Gamma_\delta \cup \big(B_{r_\delta}(z_\delta) \setminus H\big)\,,
\tag{B.7}
$$

see Figure B.1-(a), where $r_\delta > 0$ and $z_\delta \in \mathbb{R}^{n+1} \setminus \mathrm{cl}\,(H)$ are uniquely determined by requiring that, first,

$$
\mathrm{cl}\,(B_{r_\delta}(z_\delta)) \cap \partial H = \partial \mathbf{C}_\delta \cap \partial H = \big\{x : x_{n+1} = 0\,, |x'| \le \delta\big\}\,,
\tag{B.8}
$$

and, second, that

$$
|E_\delta| = v\,.
\tag{B.9}
$$

To see that this choice is possible, we first notice that, since $E \cap \Gamma_\delta = \varnothing$, (B.9) is equivalent to

$$
\big|B_{r_\delta}(z_\delta) \setminus H\big| = v - |E| - |\Gamma_\delta| = v^* - |\Gamma_\delta|\,.
\tag{B.10}
$$

Taking (B.5) into account we see that (B.8) and (B.10) uniquely determine $z_\delta \in \mathbb{R}^{n+1}$ and $r_\delta > 0$ as soon as δ_0 is small enough to guarantee $v^* - |\Gamma_{\delta_0}| > 0$. In fact, by (B.5), $v^* - |\Gamma_\delta| \to v^* > 0$ with $\mathcal{H}^n(\partial \mathbf{C}_\delta \cap \partial H) \to 0$ as $\delta \to 0^+$, so that, up to further decrease δ_0, we definitely have $z_\delta \notin H$, and

$$\Big| r_\delta - \Big(\frac{v^*}{\omega_{n+1}}\Big)^{1/(n+1)} \Big| \le C\,\delta^{n+2}\,, \tag{B.11}$$

where $C = C(E, n, v^*)$.

We now use the facts that $K \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ and that $E \subset E_\delta$ to prove that

$$(K_\delta, E_\delta) = ((\Omega \cap \partial^* E_\delta) \cup (K \cap E_\delta^{(0)}), E_\delta) \tag{B.12}$$

is such that $K_\delta \cup E_\delta^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ (and thus is admissible in $\Psi_{\rm bk}(v)$ by (B.9)). To this end, it is enough to show that

$$K \cup E^{(1)} \overset{\mathcal{H}^n}{\subset} K_\delta \cup E_\delta^{(1)}\,. \tag{B.13}$$

Indeed, by $E \subset E_\delta$ and Federer's theorem (1.37) we have

$$E^{(1)} \subset E_\delta^{(1)}\,, \qquad E_\delta^{(0)} \subset E^{(0)}\,, \qquad E^{(1)} \cup \partial^* E \overset{\mathcal{H}^n}{\subset} E_\delta^{(1)} \cup \partial^* E_\delta\,. \tag{B.14}$$

(Notice indeed that $\partial^* E \subset E^{(1/2)} \subset \mathbb{R}^{n+1} \setminus E_\delta^{(0)}$). Next, using in order Federer's theorem (1.37), (B.14) and $K \subset \Omega$, and the definition of K_δ, we have

$$E^{(1)} \cup (K \setminus E_\delta^{(0)}) \overset{\mathcal{H}^n}{=} E^{(1)} \cup [K \cap (\partial^* E_\delta \cup E_\delta^{(1)})] \overset{\mathcal{H}^n}{\subset} E_\delta^{(1)} \cup (\Omega \cap \partial^* E_\delta) \subset E_\delta^{(1)} \cup K_\delta\,.$$

But $K \cap E_\delta^{(0)} \subset K_\delta$ by definition, which combined with the preceding containment completes the proof of (B.13). Having proved that (K_δ, E_δ) is admissible in $\Psi_{\rm bk}(v)$, we have

$$\mathcal{F}_{\rm bk}(K, E) + P(B^{(v^*)}) = \Psi_{\rm bk}(v) \le \mathcal{F}_{\rm bk}(K_\delta, E_\delta)\,. \tag{B.15}$$

By (B.15), the definition of K_δ, and (B.14), we find

$$\begin{aligned} P(E;\Omega) + 2\,\mathcal{H}^n(K \cap E^{(0)}) + P(B^{(v^*)}) &\le P(E_\delta;\Omega) + 2\,\mathcal{H}^n(K_\delta \cap E_\delta^{(0)}) \\ &\le P(E_\delta;\Omega) + 2\,\mathcal{H}^n(K \cap E_\delta^{(0)}) \le P(E_\delta;\Omega) + 2\,\mathcal{H}^n(K \cap E^{(0)})\,, \end{aligned}$$

from which we deduce

$$P(E;\Omega) + P(B^{(v^*)}) \le P(E_\delta;\Omega)\,. \tag{B.16}$$

We now notice that E_δ coincides with E in the open set $\Omega \cap H \setminus {\rm cl}\,(\mathbf{C}_\delta)$, and with $B_{r_\delta}(z_\delta)$ in the open set $\mathbb{R}^{n+1} \setminus {\rm cl}\,(H)$, so that

$$\begin{aligned} \Big(\Omega \cap H \setminus {\rm cl}\,(\mathbf{C}_\delta)\Big) \cap \partial^* E_\delta &= \Big(\Omega \cap H \setminus {\rm cl}\,(\mathbf{C}_\delta)\Big) \cap \partial^* E\,, \\ \big(\Omega \setminus {\rm cl}\,(H)\big) \cap \partial^* E_\delta &= \big(\partial B_{r_\delta}(z_\delta)\big) \setminus {\rm cl}\,(H)\,, \end{aligned}$$

and (B.16) is equivalent to

$$\begin{aligned} &P\big(E; \Omega \cap (\partial H \cup {\rm cl}\,(\mathbf{C}_\delta)\big) + P(B^{(v^*)}) \\ &\le P\big(E_\delta; \Omega \cap (\partial H \cup {\rm cl}\,(\mathbf{C}_\delta)\big) + P(B_{r_\delta}(z_\delta); \mathbb{R}^{n+1} \setminus {\rm cl}\,(H))\,. \end{aligned} \tag{B.17}$$

In fact, it is easily proved that $(\partial^* E) \cap (\partial H) \setminus {\rm cl}\,(\mathbf{C}_\delta) = (\partial^* E_\delta) \cap (\partial H) \setminus {\rm cl}\,(\mathbf{C}_\delta)$ (which is evident from Figure B.1), so that (B.17) readily implies

$$P(B^{(v^*)}) \le P\big(E_\delta; \Omega \cap {\rm cl}\,(\mathbf{C}_\delta)\big) + P(B_{r_\delta}(z_\delta); \mathbb{R}^{n+1} \setminus {\rm cl}\,(H))\,. \tag{B.18}$$

Now, $\mathbf{C}_\delta \subset\subset \Omega$. Moreover, by (B.3), we have that $\mathbf{T}_\delta$ (the top part of $\partial \mathbf{C}_\delta$) is contained in $E^{(1)} \subset E_\delta^{(1)}$, and is thus $\mathcal{H}^n$-disjoint from $\partial^* E_\delta$. Similarly, again by (B.3) we have $E \cup \Gamma_\delta = \mathbf{C}_\delta$, and thus $\mathbf{D}_\delta \subset (E \cup \Gamma_\delta)^{(1/2)}$; at the same time, by (B.8) we have $\mathbf{D}_\delta \subset$

$(B_{r_\delta}(z_\delta)\setminus H)^{(1/2)}$; therefore $\mathbf{D}_\delta\subset E_\delta^{(1)}$, and thus $\mathbf{D}_\delta$ is $\mathcal{H}^n$-disjoint from $\partial^* E_\delta$. Finally, again by $E\cup\Gamma_\delta=\mathbf{C}_\delta$ we see that $P(E_\delta;\mathbf{C}_\delta)=0$. Therefore, in conclusion,

$$P\big(E_\delta;\Omega\cap\mathrm{cl}\,(\mathbf{C}_\delta)\big)=P(E_\delta;\mathbf{L}_\delta)=P(\Gamma_\delta;\mathbf{L}_\delta)\le C\,\delta^{n+1}\,, \tag{B.19}$$

where we have used again first (B.3), and then (B.6). Combining (B.18)-(B.19) we get

$$P(B^{(v^*)})\le P(B_{r_\delta}(z_\delta);\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(H))+C\,\delta^{n+1}\,. \tag{B.20}$$

Finally, by (B.8), (B.5), and (B.11) we have

$$P(B_{r_\delta}(z_\delta);\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(H))\le P(B^{(v^*)})-C(n)\,\delta^n\,;$$

by combining this estimate with (B.20), we reach a contradiction for δ small enough.

Case two, $0\in\mathbf{W}$: In this case, by the C^2-regularity of $\partial\Omega$ we can find $\delta_0>0$ and $u\in C^2(\mathbf{D}_{\delta_0};[0,\delta_0])$ with $u(0)=0$ and $\nabla u(0)=0$ such that

$$\begin{aligned}\mathbf{W}\cap\mathbf{C}_{\delta_0}&=\big\{x\in\mathbf{C}_{\delta_0}:\delta_0>x_{n+1}>u(x')\big\}\,,\\(\partial\Omega)\cap\mathbf{C}_\delta&=\big\{x\in\mathbf{C}_{\delta_0}:x_{n+1}=u(x')\big\}\,.\end{aligned} \tag{B.21}$$

We have $0\le u(x')\le C\,|x'|^2$ for every $|x'|<\delta_0$ (and some $C=C(\mathbf{W})$), so that defining Γ_δ as in (B.4) we still obtain (B.5) and (B.6). We then define E_δ, r_δ, and z_δ, as in (B.7), (B.8) and (B.9). Notice that now E and Γ_δ may not be disjoint (see Figure B.1-(b)), therefore (B.9) is not equivalent to (B.10), but to

$$\big|B_{r_\delta}(z_\delta)\setminus H\big|=v-|E|-|\Gamma_\delta\setminus E|=v^*-|\Gamma_\delta\setminus E|\,.$$

This is still sufficient to repeat the considerations based on (B.8) and (B.5) proving that r_δ and z_δ are uniquely determined, and satisfy (B.11). We can repeat the proof that (K_δ,E_δ) defined as in (B.12) is admissible in $\Psi_{\rm bk}(v)$ (since that proof was based only on the inclusion $E\subset E_\delta$), and thus obtain (B.16). The same considerations leading from (B.16) to (B.18) apply in the present case too, and so we land on

$$P(B^{(v^*)})\le P\big(E_\delta;\Omega\cap\mathrm{cl}\,(\mathbf{C}_\delta)\big)+P(B_{r_\delta}(z_\delta);\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(H))\,. \tag{B.22}$$

Now, by (B.21), $\mathbf{T}_\delta$ is contained in $\mathbf{W}$, so that $P(E_\delta;\mathbf{T}_\delta)=0$. At the same time, if $x=(x',0)\in\mathbf{D}_\delta\cap\Omega$, then $u(x')>0$, and thus $x\in(E_\delta\cap H)^{(1/2)}$; since, by (B.8), we also have $x\in(E_\delta\setminus H)^{(1/2)}$, we conclude that $\mathbf{D}_\delta\cap\Omega\subset E_\delta^{(1)}$, and thus that

$$P\big(E_\delta;\Omega\cap\mathrm{cl}\,(\mathbf{C}_\delta)\big)=P\big(E_\delta;\Omega\cap\mathbf{L}_\delta\big)\le\mathcal{H}^n(\Omega\cap\mathbf{L}_\delta)\le C\,\delta^{n+1}\,,$$

where we have used $0\le u(x')\le C\,|x'|^2$ for every $|x'|<\delta_0$ again. We thus deduce from (B.22) that

$$P(B^{(v^*)})\le P(B_{r_\delta}(z_\delta);\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(H))+C\,\delta^{n+1}\,,$$

and from here we conclude as in case one. □

Appendix C. An elementary lemma

In this appendix we provide a proof of Lemma 7.2. The proof is an immediate corollary of a geometric property of closed $\mathcal{C}$-spanning sets (see (C.2)-(C.3) below) first proved in $\mathbb{R}^{n+1}$ for $n\ge 2$ [DLDRG19, Lemma 4.1]. Here we extend this property to the plane. The difference between $\mathbb{R}^2$ and $\mathbb{R}^{n+1}$ for $n\ge 2$ stems from a part of the argument where one constructs a new admissible spanning curve by modifying an existing one inside a ball. Specifically, ensuring that the new curve does not intersect itself requires an extra argument in $\mathbb{R}^2$.

Lemma C.1. *Let $n \geq 1$, $\mathbf{W} \subset \mathbb{R}^{n+1}$ be closed, $\mathcal{C}$ be a spanning class for $\mathbf{W}$, $S \subset \Omega := \mathbb{R}^{n+1} \setminus \mathbf{W}$ be relatively closed and $\mathcal{C}$-spanning $\mathbf{W}$, and $B_r(x) \subset\subset \Omega$. Let $\{\Gamma_i\}_i$ be the countable family of equivalence classes of $\partial B_r(x) \setminus S$ determined by the relation:*

$$y \sim x \iff \exists \tilde{\gamma} \in C^0([0,1], \operatorname{cl} B_r(x) \setminus S) : \tilde{\gamma}(0) = y,\ \tilde{\gamma}(1) = z,\ \tilde{\gamma}((0,1)) \subset B_r(x)\,. \quad \text{(C.1)}$$

Then if $\gamma \in \mathcal{C}$, either

$$\gamma \cap (S \setminus B_r(x)) \neq \emptyset \quad \text{(C.2)}$$

or there exists a connected component σ of $\gamma \cap \operatorname{cl} B_r(x)$ which is homeomorphic to an interval and such that

$$\text{the endpoints of } \sigma \text{ belong to two distinct equivalence classes of } \partial B_r(x) \setminus S. \quad \text{(C.3)}$$

In particular, the conclusion of Lemma 7.2 holds.

Remark C.2. The planar version of Lemma C.1 allows one to extend the main existence result [DLDRG19, Theorem 2.7] to $\mathbb{R}^2$.

Proof of Lemma C.1. The proof is divided into two pieces. First we show how to deduce Lemma 7.2 from the fact that at least one of (C.2)-(C.3) holds. Then we show in $\mathbb{R}^2$ that (C.3) must hold whenever (C.2) does not, completing the lemma since the case $n \geq 2$ is contained in [DLDRG19, Lemma 4.1].

Conclusion of Lemma 7.2 from (C.2)-(C.3): We must show that either $\gamma(\mathbb{S}^1) \setminus B_r(x) \neq \varnothing$ or that it intersects at least two open connected components of $B_r(x) \setminus S$. If $\gamma(\mathbb{S}^1) \setminus B_r(x) \neq \varnothing$ we are done, so suppose that $\gamma(\mathbb{S}^1) \setminus B_r(x) = \varnothing$. Then (C.3) must be true, so that the endpoints of some arc $\sigma = \gamma((a,b)) \subset B_r(x)$ for an interval $(a,b) \subset \mathbb{S}^1$ belong to distinct equivalence classes. Choose ρ small enough so that $B_\rho(\gamma(a)) \cup B_\rho(\gamma(b)) \subset \Omega \setminus S$ and a', $b' \in I$ such that $\gamma(a') \in B_\rho(\gamma(a))$ and $\gamma(b') \in B_\rho(\gamma(b))$. If $\gamma(a')$ and $\gamma(b')$ belonged to the same open connected component of $B_r(x) \setminus S$, we would contradict (C.3), so they belong to different components as desired.

Verification of (C.2)-(C.3) *in* $\mathbb{R}^2$: As in [DLGM17a, Lemma 10], we may reduce to the case where γ intersects $\partial B_r(x)$ transversally at finitely many points $\{\gamma(a_k)\}_{k=1}^K \cup \{\gamma(b_k)\}_{k=1}^K$ such that $\gamma \cap B_r(x) = \cup_k \gamma((a_k, b_k))$ and $\{[a_k, b_k]\}_k$ are mutually disjoint closed arcs in $\mathbb{S}^1$. If (C.2) holds we are done, so we assume that

$$\gamma \cap S \setminus B_r(x) = \varnothing \quad \text{(C.4)}$$

and prove (C.3). Note that each pair $\{\gamma(a_k), \gamma(b_k)\}$ bounds two open arcs in $\partial B_r(x)$; we make a choice now as follows. Choose $s_0 \in \partial B_r(x) \setminus \cup_k \{\gamma(a_k), \gamma(b_k)\}$. Based on our choice of s_0, for each k there is a unique open arc $\ell_k \subset \partial B_r(x)$ such that $\partial_{\partial B_r(x)} \ell_k = \{\gamma(a_k), \gamma(b_k)\}$ and $s_0 \notin \operatorname{cl}_{\partial B_r(x)} \ell_k$. We claim that

$$\text{if } k \neq k', \text{ then either } \ell_k \subset\subset \ell_{k'} \text{ or } \ell_{k'} \subset\subset \ell_k\,. \quad \text{(C.5)}$$

To prove (C.5): We consider simple closed curves γ_k with images $\gamma((a_k, b_k)) \cup \operatorname{cl}_{\partial B_r(x)} \ell_k$. By the Jordan curve theorem, each γ_k defines a connected open subset U_k of $B_r(x)$ with $\partial U_k \cap \partial B_r(x) = \operatorname{cl}_{\partial B_r(x)} \ell_k$. Aiming for a contradiction, if (C.5) were false, then for some $k \neq k'$, either

$$\gamma(a_k) \in \ell_{k'} \subset \operatorname{cl} U_{k'} \text{ and } \gamma(b_k) \in \partial B_r(x) \setminus \operatorname{cl}_{\partial B_r(x)} \ell_{k'} \subset \partial B_r(x) \setminus \operatorname{cl} U_{k'} \text{ or}$$
$$\gamma(b_k) \in \ell_{k'} \subset \operatorname{cl} U_{k'} \text{ and } \gamma(a_k) \in \partial B_r(x) \setminus \operatorname{cl}_{\partial B_r(x)} \ell_{k'} \subset \partial B_r(x) \setminus \operatorname{cl} U_{k'}\,;$$

in particular, $\gamma((a_k, b_k))$ has non-trivial intersection with both the open sets $U_{k'}$ and $B_r(x) \setminus \operatorname{cl} U_{k'}$. By the continuity of γ and the connectedness of (a_k, b_k), we thus deduce that $\gamma((a_k, b_k)) \cap \partial U_{k'} \neq \varnothing$. Upon recalling that $\gamma((a_k, b_k)) \subset B_r(x)$, we find $\gamma((a_k, b_k)) \cap \partial U_{k'} \cap B_r(x) = \gamma((a_k, b_k)) \cap \gamma((a_{k'}, b_{k'})) \neq \varnothing$. But this contradicts the fact that γ smoothly embeds $\mathbb{S}^1$ into Ω. The proof of (C.5) is finished.

Returning to the proof of (C.3), let us assume for contradiction that

$$\gamma(a_k) \sim \gamma(b_k) \quad \forall 1 \le k \le K\,. \tag{C.6}$$

We are going to use (C.4), (C.5), and (C.6) to create a piecewise smooth embedding $\overline{\gamma} : \mathbb{S}^1 \to \Omega$ which is a homotopic deformation of γ (and thus approximable by elements in $\mathcal{C}$) such that $\overline{\gamma} \cap S = \varnothing$. After reindexing the equivalence classes Γ_i, we may assume that $\{\Gamma_1, \dots, \Gamma_{I_\gamma}\}$ are those equivalence classes containing any pair $\{\gamma(a_k), \gamma(b_k)\}$ for $1 \le k \le K$. We will construct $\overline{\gamma}$ in steps by redefining γ on those $[a_k, b_k]$ with images under γ having endpoints belonging to the same Γ_i. For future use, let Ω_i be the equivalence classes of $B_r(x) \setminus S$ determined by the relation (C.1). Note that they are open connected components of $B_r(x) \setminus S$.

Construction corresponding to Γ_1: Relabelling in k if necessary, we may assume that $\{1, \dots, K_1\}$ for some $1 \le K_1 \le K$ are the indices such that $\{\gamma(a_k), \gamma(b_k)\} \subset \Gamma_1$. By further relabelling and applying (C.5) we may assume: first, that ℓ_1 is a "maximal" arc among $\{\ell_1, \dots, \ell_{K_1}\}$, in other words

$$\text{for given } k \in \{2, \dots K_1\}, \text{ either } \ell_1 \cap \ell_k = \varnothing \text{ or } \ell_k \subset\subset \ell_1\,; \tag{C.7}$$

and second, that for some $K_1^1 \le K_1$, $\{\ell_2, \dots, \ell_{K_1^1}\}$ are those arcs contained in ℓ_1. Since Ω_1 is open and connected, we may connect $\gamma(a_1)$ to $\gamma(b_1)$ by a smooth embedding $\overline{\gamma}_1 : [a_1, b_1] \to \operatorname{cl} B_r(x) \setminus S$ with $\overline{\gamma}_1((a_1, b_1)) \subset \Omega_1$. Also, by the Jordan curve theorem, $\ell_1 \cup \overline{\gamma}_1$ defines an open connected subset W_1 of $B_r(x)$ with $\partial W_1 \cap S = \varnothing$. Using (C.5), we now argue towards constructing pairwise disjoint smooth embeddings $\overline{\gamma}_k : [a_k, b_k] \to \Gamma_1 \cup \Omega_1$.

We first claim that

$$W_1 \setminus S \text{ is path-connected}\,. \tag{C.8}$$

To prove (C.8), consider any $y, z \in W_1 \setminus S$. Since $\Omega_1 \supset W_1 \setminus S$ is open and path-connected, we may obtain continuous $\tilde{\gamma} : [0, 1] \to \Omega_1$ connecting y and z. If $\tilde{\gamma}([0, 1]) \subset W_1 \setminus S$, we are done. Otherwise, $\varnothing \ne \tilde{\gamma} \cap (\Omega_1 \setminus (W_1 \setminus S)) = \Omega_1 \setminus W_1$, with the equality following from $\Omega_1 \cap S = \varnothing$. Combining this information with $\tilde{\gamma}(\{0, 1\}) \subset W_1 \setminus S$, we may therefore choose $[\delta_1, \delta_2] \subset (0, 1)$ to be the smallest interval such that $\tilde{\gamma}([0, 1] \setminus [\delta_1, \delta_2]) \subset W_1 \setminus S$. On (δ_1, δ_2), we redefine $\tilde{\gamma}$ using the fact that $\tilde{\gamma}(\{\delta_1, \delta_2\}) \subset \partial W_1 \cap B_r(x) = \overline{\gamma}_1((a_1, b_1))$ by letting $\tilde{\gamma}((\delta_1, \delta_2)) = \overline{\gamma}_1(I)$, where $\overline{\gamma}_1(I)$ has endpoints $\tilde{\gamma}(\delta_1)$ and $\tilde{\gamma}(\delta_2)$ and $I \subset (a_1, b_1)$. The modified $\tilde{\gamma}$ is a concatenation of continuous curves and is thus continuous; furthermore, $\tilde{\gamma}^{-1}(W_1 \setminus S) = [0, \delta_1) \cup (\delta_2, 1]$. It only remains to "push" $\tilde{\gamma}$ entirely inside $W_1 \setminus S$, which we may easily achieve by projecting $\tilde{\gamma}((\delta_1 - \varepsilon, \delta_2 + \varepsilon))$ inside $W_1 \setminus S$ for small ε using the distance function to the smooth curve $\overline{\gamma}_1(a_1, b_1) = \partial W_1 \cap B_r(x) \subset B_r(x) \setminus S$. This completes (C.8).

But now since $W_1 \setminus S$ is path-connected and open, we may connect any two points in it by a smooth embedding of $[0, 1]$, which in particular allows us to connect $\gamma(a_2)$ and $\gamma(b_2)$ by smooth embedding $\overline{\gamma}_2 : [a_2, b_2] \to \operatorname{cl} W_1 \setminus S$ with $\overline{\gamma}_2((a_2, b_2)) \subset W_1 \setminus S$. Let W_2 be the connected open subset of W_1 determined by the Jordan curve $\overline{\gamma}_2 \cup \ell_2$. Arguing exactly as in (C.8), $W_2 \setminus S$ is open and path-connected, so we can iterate this argument to obtain mutually disjoint embeddings $\overline{\gamma}_k : [a_k, b_k] \to \operatorname{cl} W_1 \setminus S \subset \Gamma_1 \cup \Omega_1$ with $\overline{\gamma}_k((a_k, b_k)) \subset \Omega_1$ for $1 \le k \le K_1^1$.

Next, let $\ell_{K_1^1+1}$ be another maximal curve with endpoints in Γ_1. The same argument as in proving (C.8) implies that $\Omega_1 \setminus \operatorname{cl} W_1$ is path-connected, and so $\gamma(a_{K_1^1+1})$, $\gamma(b_{K_1^1+1})$ may be connected by a smooth embedding $\overline{\gamma}_{K_1^1+1} : [a_{K_1^1+1}, b_{K_1^1+1}] \to (\Gamma_1 \cup \Omega_1) \setminus \operatorname{cl} W_1$, that, together with $\ell_{K_1^1+1}$, defines a connected domain $W_{K_1^1+1} \subset \Omega_1$ by the Jordan curve theorem. In addition, $W_{K_1^1+1} \cap W_1 = \varnothing$ since $(\ell_2 \cup \overline{\gamma}_{K_1^1+1}) \cap \operatorname{cl} W_1 = \varnothing$ by (C.7) and the definition of $\overline{\gamma}_{K_1^1+1}$. Repeating the whole iteration procedure for those intervals contained in $\ell_{K_1^1+1}$ and then the rest of the maximal arcs, we finally obtain mutually disjoint embeddings $\overline{\gamma}_k : [a_k, b_k] \to \Gamma_1 \cup \Omega_1$ with $\overline{\gamma}_k((a_k, b_k)) \subset \Omega_1$ as desired for $1 \le k \le K_1$.

Conclusion of the proof of (C.3): Repeating the Γ_1 procedure for $\{\Gamma_2, \dots, \Gamma_{I_\gamma}\}$ and using the mutual pairwise disjointness of Γ_i, we obtain mutually disjoint embeddings $\overline{\gamma}_k : [a_k, b_k] \to \operatorname{cl} B_r(x) \setminus S$ with $\overline{\gamma}_k((a_k, b_k)) \subset B_r(x) \setminus S$ for $1 \le k \le K_1$. We define $\overline{\gamma} : \mathbb{S}^1 \to \Omega$ by

$$\overline{\gamma}(t) = \begin{cases} \gamma(t) & t \in \mathbb{S}^1 \setminus \cup[a_k, b_k] \\ \overline{\gamma}_k(t) & t \in [a_k, b_k]\,, \ \ 1 \le k \le K\,. \end{cases}$$

Since $\overline{\gamma} = \gamma$ outside $B_r(x) \subset\subset \Omega$, $\overline{\gamma}$ is homotopic to γ relative to Ω. Furthermore, $\overline{\gamma}$ is piecewise smooth and homotopic to γ, and so it can be approximated in the C^0 norm by $\{\gamma_j\} \subset \mathcal{C}$. However, by (C.4) and the construction of $\overline{\gamma}_k$, $\overline{\gamma} \cap S = \varnothing$, which implies that $S \cap \gamma_j = \varnothing$ for large j. This contradicts the fact that S is $\mathcal{C}$-spanning $\mathbf{W}$, and so (C.3) is true. □

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