On the shape of hypersurfaces with boundary which have zero fractional mean curvature

Fumihiko Onoue*

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Abstract

We consider hypersurfaces with boundary in $\mathbb{R}^N$ that are the critical points of the fractional area introduced by Paroni, Podio-Guidugli, and Seguin in [27]. In particular, we study the shape of such hypersurfaces in several simple settings. First we show that the critical points whose boundary is an $(N-2)$-sphere coincide with $(N-1)$-balls. Second we show that the critical points whose boundary is the union of two parallel $(N-2)$-spheres do not coincide with two parallel $(N-1)$-balls. Moreover, the interior of the critical points does not intersect the boundary of the convex hull of the two $(N-2)$-spheres, while it can happen in the situation considered by Dipierro, Onoue, and Valdinoci in [14]. We also obtain a quantitative bound which may tell us how different the critical points are from the two $(N-1)$-balls. Finally, in the same setting as in the second case, we show that, if the two parallel boundaries are far away from each other, then the critical points are disconnected and, if the two parallel boundaries are close to each other, then the boundaries are in the same connected component of the critical points when $N \geq 3$. Moreover, by computing the fractional mean curvature of a cone with the same boundaries as those of the critical points, we also obtain that the interior of the critical points does not touch the cone if the critical points are contained in either the inside or the outside of the cone.

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*Technische Universität München, Bolzmannstrasse 3, 85748 Garching, Germany.
E-mail: `fumihiko.onoue@tum.de`

1 Introduction

Fractional minimal surfaces without boundary were first investigated by Caffarelli, Roquejoffre, and Savin in [6] and, since then, this topic has attracted many authors to study their geometric properties as an analogy of classical minimal surfaces. Roughly speaking, a fractional (or nonlocal) minimal surface without boundary is given as the boundary of a set which minimizes an energy functional defined by the pointwise interaction of a set and its complement. The typical interaction taken into account is scaling and translation invariant with some polynomial decay. Precisely, if $s \in (0,\, 1)$ and Ω is an open set with smooth boundary, one of such standard energies of a set $E \subset \mathbb{R}^N$ relative to Ω is the so-called fractional perimeter in Ω and is defined by

$$P_s(E;\Omega) \coloneqq \int_{E\cap\Omega}\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega} \frac{dx\,dy}{|x-y|^{N+s}} \tag{1.1}$$

where we denote by E^c the complement of E. With this notion, we say that a set $E \subset \mathbb{R}^N$ is a minimizer of P_s relative to Ω if it holds that

$$P_s(E;\Omega') \le P_s(F;\Omega)$$

for any open bounded set $\Omega' \subset \Omega$ and any $F \subset \mathbb{R}^N$ with $F \setminus \Omega' = E \setminus \Omega'$. The existence and regularity of such minimizers was shown by Caffarelli, Roquejoffre, and Savin in [6]. Moreover, they showed in [6] that, if a set $E \subset \mathbb{R}^N$ is a minimizer of P_s, then the following Euler-Lagrange equation holds in the viscosity sense:

$$\int_{\mathbb{R}^N} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-x|^{N+s}}\,dy = 0 \tag{1.2}$$

for $x \in \partial E$. The integral in (1.2) is intended in the Cauchy principal value sense. This can be regarded as a nonlocal counterpart of the classical minimal surface equation and the left-hand side in (1.2) is the so-called fractional mean curvature on the boundary ∂E. Dipierro, Savin, and Valdinoci in particular have revealed many properties which classical minimal surfaces cannot possess (see, for instance, [16, 17] for the detail). In addition, many authors have studied the fractional(nonlocal) minimal surfaces or minimal graphs for more than a decade since the fractional(nonlocal) minimal surfaces appear in many other topics in which a long-range interaction is involved (see [11, 30]). For further discussions about the geometric features of fractional(nonlocal) minimal surfaces without boundary, we refer to [2, 4, 5, 7, 9, 12–15, 18, 19].

Quite recently, motivated by some mathematical modelling of thin elastic structures, Paroni, Podio-Guidugli, and Seguin in [27] introduced a new notion of fractional areas and fractional mean curvatures for smooth manifolds which are not necessarily closed in the following way: let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\mathcal{M} \subset \Omega$ be any $(N-1)$-dimensional compact smooth manifold with or without boundary. Then the fractional area of $\mathcal{M}$ relative to Ω is defined by

$$\mathrm{Per}_s(\mathcal{M};\Omega) \coloneqq c_N \iint_{\mathcal{X}(\mathcal{M})} \frac{\max\{\chi_\Omega(x),\, \chi_\Omega(y)\}}{|x-y|^{N+s}}\,dx\,dy \tag{1.3}$$

where c_N is some positive dimensional constant and $\mathcal{X}(\mathcal{M})$ is a set of all pairs $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N$ such that the segment $[x;\, y]$ with two end points x and y has an odd number of cross intersections with $\mathcal{M}$ and $[x;\, y]$ is not tangent to $\mathcal{M}$. Note that the presence of the term $\max\{\chi_\Omega(x),\, \chi_\Omega(y)\}$ in (1.3) is necessary to ensure that the integral converges whenever $\partial\mathcal{M} \neq \emptyset$.

As is explained in [27], if a $(N-1)$-dimensional smooth manifold $\mathcal{M}$ satisfies $\mathcal{M} = \partial E$ for some set $E \subset \mathbb{R}^N$, then two notions (1.1) and (1.3) are equivalent, i.e., it holds that

$$\mathrm{Per}_s(\mathcal{M};\Omega) = P_s(E;\Omega).$$

Interestingly, Paroni, Podio-Guidugli, and Seguin also proved in [27, Theorem 3.3] that $(1-s)\mathrm{Per}_s(\mathcal{M};\Omega) \to \mathcal{H}^{N-1}(\mathcal{M})$ as $s \uparrow 1$ for a compact $(N-1)$-dimensional C^1 manifold $\mathcal{M}$ contained in a bounded domain Ω, as it happens for P_s in (1.1) (see [1, 8]). See [25, 28, 31] for further discussions on Per_s.

This manuscript is devoted to develop the theory of the fractional area Per_s for manifolds with boundary. In particular, we aim to investigate the shape and topology of critical points of Per_s. Here the critical point of Per_s is defined by a smooth manifold such that the first variation of Per_s vanishes with respect to a perturbation associated with the unit normal vector of that manifold (in the sequel, we will call this perturbation "normal variations"). The authors in [27] obtained a necessary and sufficient condition for the vanishing of the first variation for manifolds as follows: let $\mathcal{M}$ be an orientable compact smooth manifold with or without boundary and assume that $\mathcal{M}$ is contained in a bounded domain $\Omega \subset \mathbb{R}^N$. Then it holds that

$$\delta\mathrm{Per}_s(\mathcal{M};\Omega) = 0 \quad \iff \quad H_{\mathcal{M},s}(z) = 0 \quad \text{for any } z \in \mathcal{M}. \tag{1.4}$$

Here we denote by $\delta\mathrm{Per}_s(\mathcal{M};\Omega)$ the first variation of $\mathcal{M}$ under normal variations and $H_{\mathcal{M},s}$ is the fractional mean curvature associated with Per_s which is defined by

$$H_{\mathcal{M},s}(z) \coloneqq c_N \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i(z)}(y) - \chi_{\mathcal{A}_e(z)}(y)}{|y-z|^{N+s}}\, dy$$

for any $z \in \mathcal{M}$ where c_N is as in (1.3) and the sets $\mathcal{A}_i(z)$ and $\mathcal{A}_e(z)$ are defined by

$$\begin{aligned} \mathcal{A}_i(z) &\coloneqq \{y \in \mathbb{R}^N \mid \text{either } (z,y) \in \mathcal{X}(\mathcal{M})\,\&\,(z-y)\cdot\nu_{\mathcal{M}}(z) < 0 \\ &\qquad\qquad \text{or } (z,y) \notin \mathcal{X}(\mathcal{M})\,\&\,(z-y)\cdot\nu_{\mathcal{M}}(z) > 0\} \qquad (1.5) \\ \mathcal{A}_e(z) &\coloneqq \{y \in \mathbb{R}^N \mid \text{either } (z,y) \in \mathcal{X}(\mathcal{M})\,\&\,(z-y)\cdot\nu_{\mathcal{M}}(z) > 0 \\ &\qquad\qquad \text{or } (z,y) \notin \mathcal{X}(\mathcal{M})\,\&\,(z-y)\cdot\nu_{\mathcal{M}}(z) < 0\}. \qquad (1.6) \end{aligned}$$

The sets $\mathcal{A}_i(z)$ and $\mathcal{A}_e(z)$ can be regarded as the "interior" and "exterior" of $\mathcal{M}$ relative to the point z, respectively, and these sets are determined uniquely once the unit normal vector of $\mathcal{M}$ at z is specified. See [27] for more discussions on the notions. Note that, if a manifold is not orientable, then the unit normal vector of the manifold cannot be determined uniquely and neither can the "interior" $\mathcal{A}_i$ and "exterior" $\mathcal{A}_e$. Moreover, in this paper, we require the $C^{1,\alpha}$-regularity with $\alpha > s$ of hypersurfaces so that the fractional mean curvatures are finite everywhere.

The study of critical points or fractional minimal surfaces with boundary can be related to the classical problem on free boundary minimal surfaces in differential geometry. One of the main topics in the problem is to determine the shape of a manifold Σ (embedded or immersed) in another smooth manifold $\mathcal{S}$ such that Σ minimizes its area in $\mathcal{S}$ and $\partial\Sigma \subset \partial\mathcal{S}$ with some topological constraints. The study of this classical problem was first considered by R. Courant in [10] in 1940 and, since then, a lot of authors have been intensively working on this topic. See, for instance, [22, 24, 26, 29, 32] for the detail. We also refer the readers to two surveys: [20] for classical works and [23] for more recent results. The references here are obviously not exhausted.

As an analogy of the classical free boundary minimal surfaces, it is natural to consider a fractional(nonlocal) version of free boundary minimal surfaces; however, the nonlocal version is not understood so far because, to our knowledge, suitable notions of fractional areas for manifolds with boundary had not been considered until Paroni, Podio-Guidugli, and Seguin in [27] introduced the notion of Per_s in (1.3). To tackle the nonlocal version of the free boundary minimal surface problem, it is important to understand the geometric properties of critical points of Per_s.

Given the importance of critical points of Per_s from the above perspective, it is desirable to develop some intuition about their geometric features. To do this, since it is quite difficult to have explicit solutions which entirely describe critical points or minimizers of Per_s, it is often convenient to study simplified cases in which the boundary of the critical points has some special characteristics. In this paper, we basically consider three cases: the first is that the boundary of critical points in $\mathbb{R}^N$ lies in a hyperplane and is homeomorphic to $\mathbb{S}^{N-2}$ (our result is also true if the boundary is not always homeomorphic to $\mathbb{S}^{N-2}$). The second is that the boundary is the union of two distinct parallel and co-axial manifolds each of which lies in a hyperplane, is homeomorphic to $\mathbb{S}^{N-2}$, and has distance of d from another boundary. The last is that the distance d is sufficiently large or sufficiently small.

Our first goal in this paper is to determine the shape of critical points of Per_s whose boundary lies on a hyperplane. Precisely, we first define a set $\mathcal{C} \subset \mathbb{R}^N$ as

$$\mathcal{C} \coloneqq \mathcal{G} \times \mathbb{R} \tag{1.7}$$

where $\mathcal{G}$ is a non-empty bounded open subset of $\{x_N = 0\}$ with a smooth boundary. Then we define an $(N-2)$-dimensional smooth manifold Γ as

$$\Gamma_0 \coloneqq \partial\mathcal{C} \cap \{x_N = 0\} \, (= \partial\mathcal{G} \times \{0\}). \tag{1.8}$$

Assume that $\mathcal{M} \subset \mathbb{R}^N$ is an orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold with $\partial\mathcal{M} = \Gamma_0$ and that $\mathcal{M}$ is a critical point of Per_s. Note that the orientability of $\mathcal{M}$ implies the orientability of $\partial\mathcal{M} = \Gamma_0$. Then, as our first theorem, we aim to rigorously prove that $\mathcal{M}$ must coincide with $\mathcal{C} \cap \{x_N = 0\}$, as we can intuitively expect this to be true.

Theorem 1.1. *Let $s \in (0, 1)$. Let Γ_0 be as in* (1.8)*. Let $\mathcal{M}$ be an orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold with $\partial\mathcal{M} = \Gamma_0$. If $\mathcal{M}$ is a critical point of Per_s under normal variations, then $\mathcal{M}$ is a hyperplane lying on $\{x_N = 0\}$, i.e.,*

$$\mathcal{M} = \overline{\mathcal{C}} \cap \{x_N = 0\} \, (= \overline{\mathcal{G}} \times \{0\}).$$

Our second goal in this paper is to study the shape of critical points of Per_s whose boundary consists of two disjoint components. The problem setting in the second theorem is as follows: we define two distinct compact $(N-2)$-dimensional smooth manifolds Γ_1 and Γ_2 by

$$\Gamma_1 \coloneqq \partial\mathcal{C} \cap \{x_N = h_1\} \quad \text{and} \quad \Gamma_2 \coloneqq \partial\mathcal{C} \cap \{x_N = h_2\}, \tag{1.9}$$

where $\mathcal{C}$ is as in (1.7) and h_1 and h_2 are given constants with $h_2 < h_1$. Then a critical point exhibit a different shape from a hyperplane. Precisely we prove

Theorem 1.2. *Let $s \in (0, 1)$. Let Γ_1 and Γ_2 be as in* (1.9) *and let $\mathcal{M}$ be an orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold with $\partial\mathcal{M} = \Gamma_1 \cup \Gamma_2$. Assume that $\mathcal{C}$ is convex where $\mathcal{C}$ is as in* (1.7)*. If $\mathcal{M}$ is a critical point of Per_s under normal variations, then any connected component of $\mathcal{M}$ is neither C_1 nor C_2 where we define*

$$C_1 := \overline{\mathcal{C}} \cap \{x_N = h_1\} \quad \textit{and} \quad C_2 := \overline{\mathcal{C}} \cap \{x_N = h_2\}.$$

In particular, $\mathcal{M} \neq C_1 \cup C_2$ Moreover, $\mathcal{M} \setminus \partial\mathcal{M}$ does not intersect $\partial\mathcal{C} = \partial\mathcal{G} \times \mathbb{R}$.

We remark that, by using a cone whose boundary is $\Gamma_1 \cup \Gamma_2$ as in Theorem 1.2 with $h_1 = 1$ and $h_2 = -1$, we can further detect how the critical points behave. See Subsection 3.2 of Section 3 for the detail.

Our third goal is to further study the shape and, in particular, the topology of critical points of Per_s in the same situation as the one in Theorem 1.2. Precisely, taking Γ_1 and Γ_2 as in Theorem 1.2 with $d := h_1 - h_2 > 0$, we will see what critical points of Per_s under normal variations look like in terms of connectedness if d is sufficiently large or sufficiently small.

To reach the third goal, we first show the following lemma which somehow tells us how different critical points are from hyperplanes.

Lemma 1.3. *Let $s \in (0, 1)$ and $d > 0$. Let Γ_1 and Γ_2 be as in* (1.9) *with $h_1 = 0$ and $h_2 = -d$. Assume that $\mathcal{C}$ is convex where $\mathcal{C}$ is as in* (1.7)*. Then there exists a constant $\varepsilon_0 > 0$, depending only on N, s, and d, such that the following holds: let $\mathcal{M}$ be an orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold with $\partial\mathcal{M} = \Gamma_1 \cup \Gamma_2$. If $\mathcal{M}$ is a critical point of Per_s under normal variations, then a set enclosed by $\mathcal{M}$ and the union of $\mathcal{C} \cap \{x_N = 0\}$ and $\mathcal{C} \cap \{x_N = -d\}$ contains two half-balls*

$$B^-_{\varepsilon_0}(0) := \{x \in B_{\varepsilon_0}(0) \mid x_N < 0\} \quad \textit{and} \quad B^+_{\varepsilon_0}(p_d) := \{x \in B_{\varepsilon_0}(p_d) \mid x_N > -d\}$$

where $p_d := (0, -d) \in \mathbb{R}^{N-1} \times \mathbb{R}$.

To favor the intuition, a sketch of our critical points is given in Figure 1.

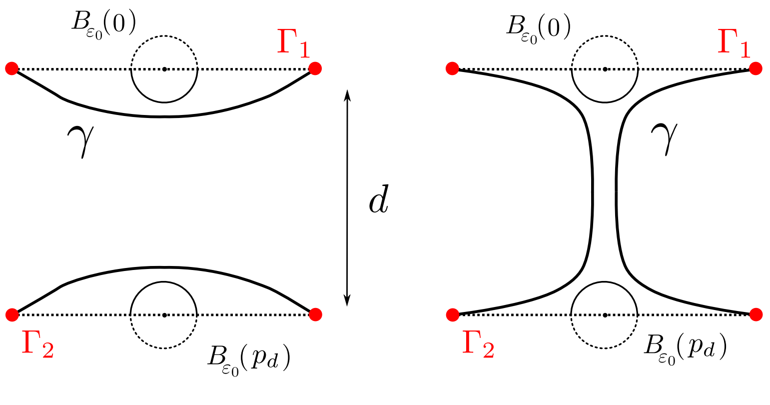

Figure 1: Two possible situations in dimension 2 in Theorem 1.3 in which the 'interior" or "exterior" of the critical point γ with $\partial\gamma = \Gamma_1 \cup \Gamma_2$ contains two half-balls.

As a result of Lemma 1.3, we prove that, if the distance d between two parallel and co-axial boundaries is sufficiently small, then any critical point is connected in the sense that the two boundaries are in the same connected component when $N \geq 3$. Moreover, when $N = 2$, any critical point is disconnected and its two distinct connected components should look like the right-hand side of Figure 1 with $0 < d \ll 1$.

Precisely, our third theorem is as follows.

Theorem 1.4. *Let $s \in (0, 1)$. Let Γ_1 and Γ_2 be as in Lemma 1.3. Assume that $\mathcal{C}$ is convex where $\mathcal{C}$ is as in* (1.7)*. Then there exists $d_0 = d_0(N, s) > 0$ such that the following holds: for any $d \in (0, d_0)$, we take any orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold $\mathcal{M} \subset \mathbb{R}^N$ with $\partial\mathcal{M} = \Gamma_1 \cup \Gamma_2$. If $\mathcal{M}$ is a critical point of Per_s under normal variations, then Γ_1 and Γ_2 are in the same connected component of $\mathcal{M}$ if $N \geq 3$ and $\mathcal{M}$ is disconnected if $N = 2$.*

Moreover, when $N = 2$, there exist two distinct connected components $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ such that $\operatorname{dist}(\mathcal{M}_1, \mathcal{M}_2) \geq c$ *with some constant $c > 0$, depending only on N and s, and $\partial\mathcal{M}_i$ intersects both Γ_1 and Γ_2 for each $i \in \{1, 2\}$.*

As a counterpart of Theorem 1.4, we prove that, if the distance d between two parallel and co-axial boundaries is sufficiently large, then any critical point is disconnected in any dimensions and it should look like the left-hand side of Figure 1 with $d \gg 1$.

Our last theorem is as follows.

Theorem 1.5. *Let $s \in (0, 1)$. Let Γ_1 and Γ_2 be as in Lemma 1.3. Assume that $\mathcal{C}$ is convex where $\mathcal{C}$ is as in* (1.7)*. Then there exists $d_1 = d_1(N, s) > 0$ such that the following holds: we assume that, for any $d > d_1$, $\mathcal{M} \subset \mathbb{R}^N$ is any orientable compact $(N-1)$-dimensional $C^{1,\alpha}$ manifold with $\partial\mathcal{M} = \Gamma_1 \cup \Gamma_2$. If $\mathcal{M}$ is a critical point of Per_s under normal variations, then $\mathcal{M}$ is disconnected.*

Moreover, there exist two disjoint connected components $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ such that $\partial\mathcal{M}_i = \Gamma_i$ for any $i \in \{1, 2\}$.

The topological properties in Theorem 1.4 and 1.5 could be expected to be true because Dipierro, Valdinoci, and the author of this paper obtained similar results in [14] on the topology of fractional minimal surfaces without boundary in the similar situations. On one hand, they showed that minimizers of P_s in a given cylinder coincides with the cylinder itself for sufficiently small d where d is the distance between two disjoint parallel and co-axial external(boundary) data. On the other hand, they showed that minimizers of P_s in the cylinder are disconnected for sufficiently large d.

Interestingly, however, we show in Theorem 1.2 that the critical points (not necessarily fractional area-minimizing) cannot touch the boundary of the cylinder $\mathcal{C}$ no mater what distance two parallel and co-axial boundaries have, while it is shown in [14] that minimizers of P_s in a cylinder favorably stick to the boundary of the cylinder if $N = 2$ and d is large or if $N \geq 2$ and d is small. Moreover, our results together with Remark 4.1 of Section 4 possibly indicate that critical points of Per_s with two nearby parallel and co-axial compact boundaries might develop necks of catenoids, while this is not the case with fractional minimal surfaces considered in [14]. We remark that the existence of fractional minimal catenoids without boundary in $\mathbb{R}^3$ was shown by Dávila, Del Pino, and Wei in [13] if s is close to 1.

The organization of this paper is as follows: in Section 2, we prove Theorem 1.1 by "sliding" a hyperplane until it touches critical points (see the proof of Theorem 1.1 for the detail). In Section 3, we first give the proof of Theorem 1.2 and then we study further properties of critical points of Per_s, computing the fractional mean curvature of a cone passing through the boundary of critical points. In Section 4, we first give the proof of Lemma 1.3 by constructing a suitable barrier and then, by using this lemma, we prove

Theorem 1.4. Moreover, in Section 4, we also prove Theorem 1.5 by means of the "sliding method" (see Section 4 for the detail).

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The idea of the proof is inspired by the so-called sliding method introduced by Dipierro, Savin, and Valdinoci in [16]. They developed this method in order to investigate the shape of fractional(nonlocal) minimal surfaces (see also [14, 15, 17] for further discussions).

We proceeds with the proof in the following way: we slide a hyperplane, parallel to $\mathcal{C} \cap \{x_N = 0\}$, from below or above until it touches $\mathcal{M}$ and assume by contradiction that there exists a touching point in $(\mathcal{C} \cap \{x_N = 0\})^c$. At the touching point q, we obtain the Euler-Lagrange equation (1.4). Then, taking into account all the contributions from the "interior" $\mathcal{A}_i(q)$ and the "exterior" $\mathcal{A}_e(q)$ of $\mathcal{M}$, we can observe that the contribution from either $\mathcal{A}_i(q)$ or $\mathcal{A}_e(q)$ turns out to be strictly larger than that from the other region. This contradicts the Euler-Lagrange equation.

Proof of Theorem 1.1. We first define a hyperplane $H_\lambda := \{(x', x_N) \mid x_N = \lambda\}$ and two half-spaces

$$H_\lambda^+ := \{(x', x_N) \mid x_N > \lambda\} \quad \text{and} \quad H_\lambda^- := \{(x', x_N) \mid x_N < \lambda\} \tag{2.1}$$

for $\lambda \in \mathbb{R}$. We set $P_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ as the reflection map with respect to H_λ for $\lambda \in \mathbb{R}$ and set $x_\lambda := P_\lambda(x)$ for any $x \in \mathbb{R}^N$. Moreover, we denote by $C_{\Gamma_0}(q)$ a (filled) cone with vertex q whose boundary passes through Γ_0, that is, $\{|x'| = a,\, x_N = 0\} \cap \partial C_{\Gamma_0}(q) = \Gamma_0$. We further set $C^\lambda_{\Gamma_0}(q) := P_\lambda(C_{\Gamma_0}(q))$.

Now let $\mathcal{M} \subset \mathbb{R}^N$ be the critical point chosen in Theorem 1.1. The minimizer $\mathcal{M}$ is bounded. Hence, we can slide the hyperplane H_λ from below until it touches the minimizer $\mathcal{M}$. Our result in Theorem 1.1 states that this touching does not occur in $H_0^- \cup H_0^+$ and thus, we assume by contradiction that there exist a constant $\lambda_0 < 0$ and a point $q \in \mathcal{M} \cap \Omega$ such that

$$T_q\mathcal{M} = H_{\lambda_0} \quad \text{and} \quad H^-_{\lambda_0} \cap \mathcal{M} = \emptyset$$

where $T_q\mathcal{M}$ is a tangent space of $\mathcal{M}$ at q. Due to the symmetry of our setting, we can conduct the same argument that we will show below in the case that we slide the hyperplane from above and the touching occurs in H_0^+. Hence, it is sufficient to show the proof in the case that the touching occurs in H_0^-. See also Figure 2 for the situation that we consider in dimension 2.

Since $\mathcal{M}$ is an orientable compact critical point of Per_s, which means the vanishing of the first variation of Per_s at $\mathcal{M}$, and since $q \in \mathcal{M}$, we obtain, from (1.4), that

$$0 = H_{\mathcal{M},s}(q) := c_N \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i(q)}(y) - \chi_{\mathcal{A}_e(q)}(y)}{|y-q|^{N+s}}\, dy \tag{2.2}$$

where the sets $\mathcal{A}_e(q)$ and $\mathcal{A}_i(q)$ are defined as in (1.5) and (1.6). We consider all the contributions from $\mathcal{A}_e(q)$ and $\mathcal{A}_i(q)$ in detail and show that the singular integral in the right-hand side of (2.2) is strictly positive, which is a contradiction.

Indeed, since $C_{\Gamma_0}(q) \subset H^+_{\lambda_0}$ and H_{λ_0} is tangential to $\mathcal{M}$, we have that $P_{\lambda_0}(\mathcal{A}_e(q)) \subset H^-_{\lambda_0} \subset \mathcal{A}_i(q)$. This implies that $\mathbb{R}^N = \mathcal{A}_e(q) \cup P_{\lambda_0}(\mathcal{A}_e(q)) \cup \mathcal{A}_i(q) \setminus P_{\lambda_0}(\mathcal{A}_e(q))$, up to

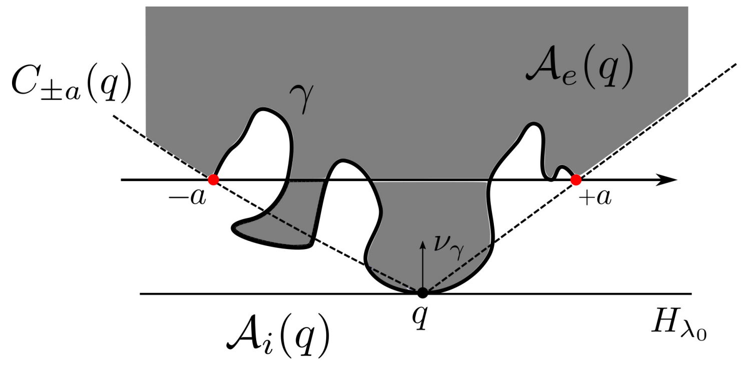

Figure 2: The situation in dimension 2 in which the critical point $\mathcal{M} = \gamma$ is a $C^{1,\alpha}$ curve with $\partial\gamma = \{\pm a\}$ where $\pm a \coloneqq (\pm a,\, 0)$. The set $\mathcal{A}_e(q)$ is shown in dark gray, the set $\mathcal{A}_i(q)$ in white. The dashed lines represent the boundary of the cone $C_{\pm a}(q)$.

negligible sets, and thus we can compute the fractional mean curvature $H_{\mathcal{M},s}$ at q as follows:

$$
\begin{aligned}
c_N^{-1} H_{\mathcal{M},s}(q) &= \int_{\mathcal{A}_e(q)} \frac{\chi_{\mathcal{A}_i(q)}(y) - \chi_{\mathcal{A}_e(q)}(y)}{|y-q|^{N+s}}\, dy + \int_{P_{\lambda_0}(\mathcal{A}_e(q))} \frac{\chi_{\mathcal{A}_i(q)}(y) - \chi_{\mathcal{A}_e(q)}(y)}{|y-q|^{N+s}}\, dy \\
&\quad + \int_{\mathcal{A}_i(q)\setminus P_{\lambda_0}(\mathcal{A}_e(q))} \frac{\chi_{\mathcal{A}_i(q)}(y) - \chi_{\mathcal{A}_e(q)}(y)}{|y-q|^{N+s}}\, dy \\
&= \int_{\mathcal{A}_e(q)} \frac{-1}{|y-q|^{N+s}}\, dy + \int_{P_{\lambda_0}(\mathcal{A}_e(q))} \frac{1}{|y-q|^{N+s}}\, dy \\
&\quad + \int_{\mathcal{A}_i(q)\setminus P_{\lambda_0}(\mathcal{A}_e(q))} \frac{1}{|y-q|^{N+s}}\, dy.
\end{aligned} \tag{2.3}
$$

From the change of variables $y \mapsto P_{\lambda_0}(y)$ and the definition of P_{λ_0}, we have

$$
\int_{P_{\lambda_0}(\mathcal{A}_e(q))} \frac{1}{|y-q|^{N+s}}\, dy = \int_{\mathcal{A}_e(q)} \frac{1}{|y-q|^{N+s}}\, dy. \tag{2.4}
$$

Moreover, we have that the volume of the set $\mathcal{A}_i(q) \setminus P_{\lambda_0}(\mathcal{A}_e(q))$ is not zero because

$$
\mathcal{A}_i(q) \setminus P_{\lambda_0}(\mathcal{A}_e(q)) \supset \Omega^c \cap H_{\lambda_0} \cap C_{\Gamma_0}(q)^c \supset B_{\frac{|\lambda_0|}{100}}(p),
$$

where $p = (p',\, \lambda_0) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and some $p' \in \Omega^c$ with $|p'| > |\lambda_0| + a$. See also Figure 3 for illustration in dimension 2. From (2.3) and (2.4), we obtain

$$
\begin{aligned}
0 &= \int_{\mathcal{A}_e(q)} \frac{-1}{|y-q|^{N+s}} dy + \int_{\mathcal{A}_e(q)} \frac{1}{|y-q|^{N+s}} dy + \int_{\mathcal{A}_i(q)\setminus P_{\lambda_0}(\mathcal{A}_e(q))} \frac{1}{|y-q|^{N+s}} dy \\
&= \int_{\mathcal{A}_i(q)\setminus P_{\lambda_0}(\mathcal{A}_e(q))} \frac{1}{|y-q|^{N+s}} dy > 0,
\end{aligned}
$$

which is a contradiction.

$\square$

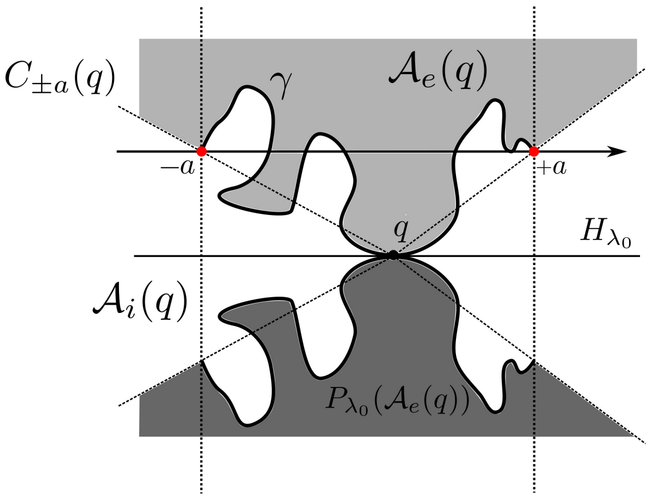

Figure 3: The same situation as in Figure 2. The reflection $P_{\lambda_0}(\mathcal{A}_e(q))$ of $\mathcal{A}_e(q)$ is shown in dark gray, the set $\mathcal{A}_e(q)$ in light gray.

3 Shape of Critical Points with Two Disjoint Compact Boundaries

In this section, we first give the proof of Theorem 1.2 and then we further show some properties of critical points of Per_s and compute the fractional mean curvature of cones.

3.1 Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2. The idea of the proof is basically the same as the one in the proof of Theorem 1.1. The convexity assumption on $\mathcal{C}$ is necessary for us to use the sliding method.

Proof of Theorem 1.2. We first define

$$H^{+}_{\Gamma_i} \coloneqq \{(x',\, x_N) \mid x_N > h_i\}, \quad H^{-}_{\Gamma_i} \coloneqq \{(x',\, x_N) \mid x_N < h_i\}$$

for each $i \in \{1\,2\}$. Notice that

$$\partial H^{+}_{\Gamma_i} \cap \partial \mathcal{C} = \partial H^{-}_{\Gamma_i} \cap \partial \mathcal{C} = \Gamma_i \quad \text{and} \quad \partial H^{+}_{\Gamma_i} \cap \mathcal{C} = \partial H^{-}_{\Gamma_i} \cap \mathcal{C} = C_i$$

for each $i \in \{1,\, 2\}$.

Let $\mathcal{M} \subset \mathbb{R}^N$ be the critical point chosen in Theorem 1.2. By using the same argument as we show in the proof of Theorem 1.1, we obtain that $\mathcal{M}$ cannot exist in the regions $H^{-}_{\Gamma_2}$ and $H^{+}_{\Gamma_1}$, that is, $\mathcal{M} \cap (H^{-}_{\Gamma_2} \cup H^{+}_{\Gamma_1}) = \emptyset$.

We now show that any connected component of $\mathcal{M}$ cannot be either C_1 or C_2. To see this, we assume by contradiction that there exists a connected component $\mathcal{M}_1$ of $\mathcal{M}$ such that $\mathcal{M}_1$ coincides with C_1. Taking any $q \in \mathcal{M}_1$, we have that the cone C_{q,Γ_2} of vertex q whose boundary passes through Γ_2 is contained in $H^{-}_{\Gamma_1}$. By choosing a proper orientation of $\mathcal{M}$, we can have that $H^{+}_{\Gamma_1} \subset \mathcal{A}_e(q)$ and $\mathcal{A}_i(q) \subset H^{-}_{\Gamma_1}$ where the sets $\mathcal{A}_e(q)$ and $\mathcal{A}_i(q)$ are defined as in (1.5) and (1.6), respectively. See Figure 4 for the situation in dimension 2.

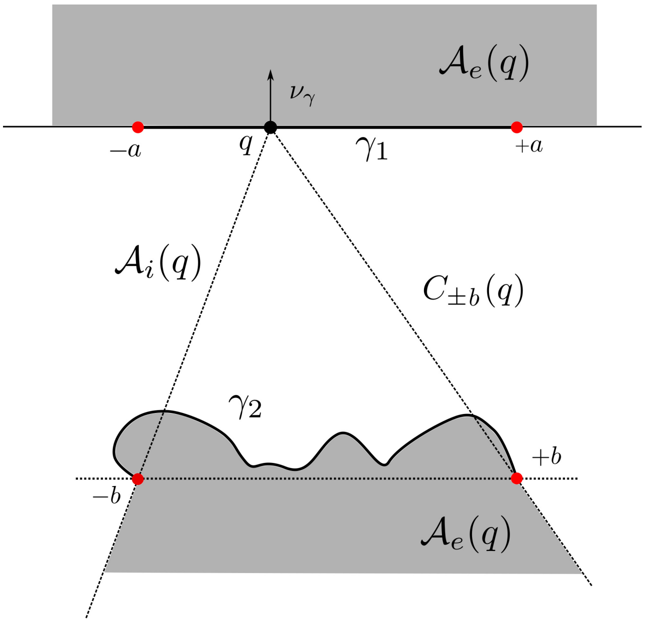

Figure 4: The situation in dimension 2 in which each component $\mathcal{M}_i = \gamma_i$ of the critical point $\mathcal{M} = \gamma$ for $i \in \{1, 2\}$ is a $C^{1,\alpha}$ curve with $\partial\gamma_i = \Gamma_i$ where $\Gamma_1 = \{\pm a\}$ and $\Gamma_2 = \{\pm b\}$. The set $\mathcal{A}_e(q)$ is shown in gray, the set $\mathcal{A}_i(q)$ in white.

Since $\mathcal{M}$ is a critical point of Per_s, from (1.4), we have that

$$0 = H_{\mathcal{M},s}(q) = c_N \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i(q)}(y) - \chi_{\mathcal{A}_e(q)}(y)}{|y-q|^{N+s}}\, dy. \tag{3.1}$$

Now, by employing the same argument we show in the proof of Theorem 1.1, we obtain that

$$\begin{aligned} c_N^{-1} H_{\mathcal{M},s}(q) &= \int_{\mathcal{A}_e(q)\cap H^+_{\Gamma_1}} \frac{-1}{|y-q|^{N+s}}\, dy + \int_{\mathcal{A}_e(q)\cap H^-_{\Gamma_1}} \frac{-1}{|y-q|^{N+s}}\, dy \\ &\quad + \int_{\mathcal{A}_i(q)} \frac{1}{|y-q|^{N+s}}\, dy \\ &\le \int_{B_{1/2}(-\lambda e_N)} \frac{-1}{|y-q|^{N+s}}\, dy < 0 \end{aligned}$$

because $B_{1/2}(-\lambda e_N) \subset \mathcal{A}_e(q) \cap H^-_{\Gamma_2}$ where $\lambda > \max\{|x-z| \mid x \in C_2,\, z \in \mathcal{M}\} + 1$. This contradicts (3.1). Therefore, we conclude that the first claim is valid.

To prove the rest of the claim, we can argue in the same way as in the proof of the first claim. Indeed, we slide any hyperplane parallel to the x_N-axis from right to left or from left to right until it touches the boundary of the cylinder $\mathcal{C}$. If there is no touching point, from the convexity of $\mathcal{C}$, we obtain that the critical point $\mathcal{M}$ is strictly contained in $\mathcal{C}$ except for its boundary. Thus, we assume by contradiction that there exists a touching point q of $\mathcal{M}$ in the complement of $\overline{\mathcal{C}}$. Then, by choosing a proper orientation of $\mathcal{M}$, we can show that the contribution from $\mathcal{A}_e(q)$ relative to the touching point q is strictly larger (or smaller) than that from $\mathcal{A}_i(q)$, respectively, as we see in the proof of the first claim. This contradicts that the fractional mean curvature vanishes at the touching point q. Therefore, we conclude the proof of Theorem 1.2.

□

3.2 Further Study on Critical Points and Cones

In this subsection, we study more the shape of critical points of Per_s in the same situation as in Theorem 1.2 with $h_1 = 1$ and $h_2 = -1$.

First, we investigate the shape of critical points in dimension 2. To see this, we divide $\mathbb{R}^2$ into four regions, that is, we define four regions C_0^t, C_0^b, C_0^r, and C_0^ℓ by

$$\begin{aligned} C_0^t &\coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > |x_1|\}, \\ C_0^b &\coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 < -|x_1|\}, \\ C_0^r &\coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid -|x_1| < x_2 < |x_1|, \, 0 < x_1\}, \\ \text{and} \quad C_0^\ell &\coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid -|x_1| < x_2 < |x_1|, \, x_1 < 0\}, \end{aligned}$$

respectively. Moreover, we set

$$C_0 \coloneqq (\partial C_0^t \cup \partial C_0^b) \cap \{(x_1, x_2) \mid |x_2| \le 1\}. \tag{3.2}$$

Notice that $\partial C_0 = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are given in Theorem 1.2 with $h_1 = 1$ and $h_2 = -1$ in $\mathbb{R}^2$. From the definition of Γ_1 and Γ_2, we have that $\Gamma_1 = \{(\pm 1, 1)\}$ and $\Gamma_2 = \{(\pm 1, -1)\}$.

Now we prove that the fractional mean curvature of the cone C_0 vanishes at regular points, i.e.,

$$H_{C_0, s}(z) = 0 \tag{3.3}$$

for any $z \in C_0 \setminus \partial C_d$ with $z \neq 0$. Indeed, let $z \in C_0 \setminus \{0, (\pm 1, 1), (\pm 1, -1)\}$ and, by symmetry, we may assume that $z = (z_1, z_2)$ satisfies $-1 < z_1 < 0$ and $0 < z_2 < 1$. Then, from the definition of the "interior" $\mathcal{A}_i(z)$ and the "exterior" $\mathcal{A}_e(z)$ of the cone C_0 and by taking a suitable orientation of $C_0 \setminus \{0\}$, we may obtain that

$$\mathcal{A}_i(z) = \Big(([z, (1, 1)]^- \cap [z, (1, -1)]^+) \setminus \overline{C_0^t} \Big) \cup \big(([z, (-1, 1)]^- \cap [z, (-1, -1)]^+) \cup C_0^\ell \big)$$

and

$$\mathcal{A}_e(z) = \big(([z, (-1, 1)]^+ \cap [z, (1, 1)]^+) \cup C_0^t \big) \cup \Big(([z, (-1, -1)]^- \cap [z, (1, -1)]^-) \setminus \overline{C_0^\ell} \Big)$$

where we denote by $[p, q]$ the straight line passing through $p, q \in \mathbb{R}^2$ with $p \neq q$ and we define $[p, q]^+$ and $[p, q]^-$ by the upper part and the lower part of the region separated by the straight line $[p, q]$, respectively.

Now, because of the symmetry of the cone C_0, we readily observe that, in dimension 2, the sets $\mathcal{A}_i(z)$ and $\mathcal{A}_e(z)$ are equivalent to each other in the sense that $T(\mathcal{A}_i(z)) = \mathcal{A}_e(z)$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometric map such that $\frac{x + T(x)}{2} \in \{(x_1, x_2) \mid x_2 = x_1\}$ for any $x \in \mathbb{R}^2$. By definition, we notice that $T(z) = z$.

Therefore, from the change of variables $x \mapsto T(x)$ and , we obtain that

$$\begin{aligned} c_N^{-1} H_{C_0, s}(z) &= \int_{\mathcal{A}_i(z)} \frac{1}{|y - z|^{2+s}} \, dy - \int_{\mathcal{A}_e(z)} \frac{1}{|y - z|^{2+s}} \, dy \\ &= \int_{\mathcal{A}_i(z)} \frac{1}{|y - z|^{2+s}} \, dy - \int_{\mathcal{A}_i(z)} \frac{1}{|T(y) - T(z)|^{2+s}} \, dy \\ &= 0. \end{aligned}$$

By combining this fact with Theorem 1.2, we can prove the following proposition.

Proposition 3.1. *Let $N = 2$ and $s \in (0, 1)$. Let Γ_1 and Γ_2 be as in Theorem 1.2 with $h_1 = 1$ and $h_2 = -1$. Let $\gamma \subset \mathbb{R}^2$ be an orientable compact $C^{1,\alpha}$ curve with $\partial\gamma = \Gamma_1 \cup \Gamma_2$. Assume that $\mathcal{C} = \{(x_1, x_2) \mid |x_1| < 1\}$ where $\mathcal{C}$ is as in (1.7). If γ is a critical point of Per_s under normal variations, then γ is not contained in either $\overline{C_0^t \cup C_0^b}$ or $\overline{C_0^r \cup C_0^\ell}$ whenever $(\gamma \setminus \partial\gamma) \cap (C_0 \setminus \{0\}) \neq \emptyset$.*

Remark 3.2. We may observe, by combining Proposition 3.1 with Theorem 1.2, that the possible shape of minimizers of Per_s in dimension 2 whose boundary is $\Gamma_1 \cup \Gamma_2$ is depicted in Figure 5.

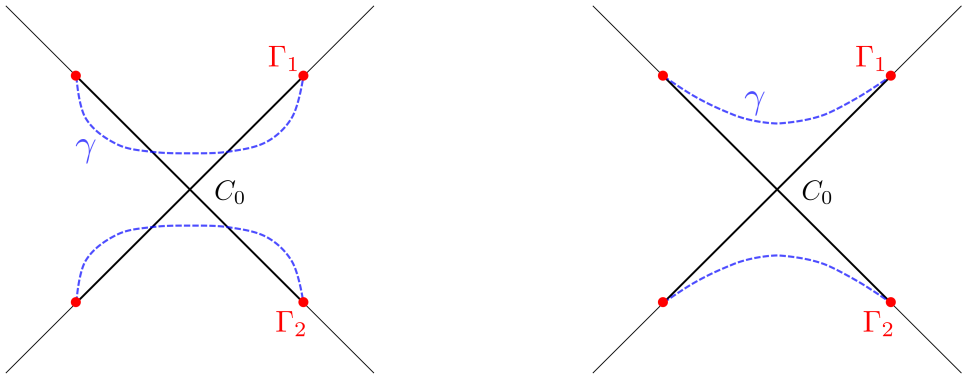

Figure 5: Possible minimizers γ of Per_s in dimension 2 with $\partial\gamma = \Gamma_1 \cup \Gamma_2$ is shown with dashed lines. On the right, γ does not intersect with C_0 except at their boundaries Γ_1 and Γ_2.

Proof. Let $\gamma \subset \gamma$ be as in Proposition 3.1 and we assume that $(\gamma \setminus \partial\gamma) \cap (C_0 \setminus \{0\}) \neq \emptyset$. We argue by contradiction that either $\gamma \subset \overline{C_0^t \cup C_0^b}$ or $\gamma \subset \overline{C_0^r \cup C_0^\ell}$ holds. Due to the symmetry of C_0, it is sufficient to consider the case that $\gamma \subset \overline{C_0^t \cup C_0^b}$ holds. From the assumption, we can choose a point $z \in (\gamma \setminus \partial\gamma) \cap (C_0 \setminus \{0\})$.

Now, by choosing a proper orientation, we consider the "interior" and "exterior" of γ and C_0 at the touching point z. To see this, we set the interior and exterior at $q \in \eta$ of a curve $\eta \subset \mathbb{R}^2$ as $\mathcal{A}_i^\eta(z)$ and $\mathcal{A}_e^\eta(z)$, respectively. Then, from the smoothness of the critical point γ and the assumption that $\gamma \subset C_0^t \cup C_0^b$, we obtain, by taking a suitable orientation of γ and C_0, that

$$|\mathcal{A}_e^{C_0}(z) \setminus \mathcal{A}_e^{\gamma}(z)| = |\mathcal{A}_i^{\gamma}(z) \setminus \mathcal{A}_i^{C_0}(z)| \neq 0 \tag{3.4}$$

and

$$|\mathcal{A}_e^{\gamma}(z) \setminus \mathcal{A}_e^{C_0}(z)| = |\mathcal{A}_i^{C_0}(z) \setminus \mathcal{A}_i^{\gamma}(z)| = 0. \tag{3.5}$$

Here, from Theorem 1.2, we have used the fact that all the critical points of Per_s in our situation are contained in the box $\{(x_1, x_2) \mid |x_1| < 1, |x_2| < 1\}$. See also Figure 6 for our situation.

Hence, since γ is a critical point of Per_s, we have that

$$H_{\gamma,s}(z) = 0.$$

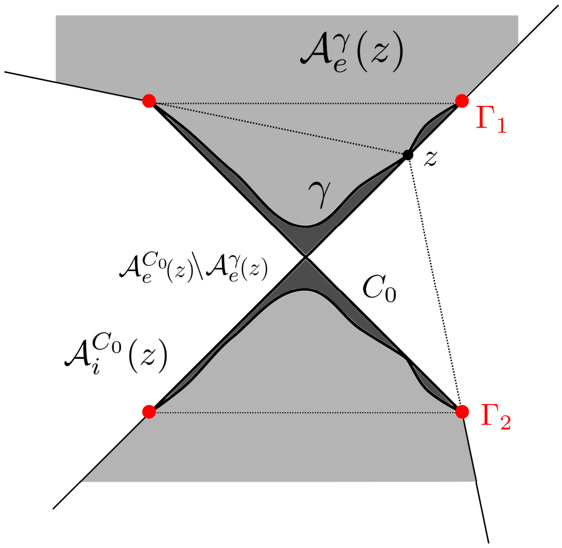

Figure 6: The situation of the critical point γ and the touching point z in which γ is included in $C_0^t \cup C_0^b$ with $\partial\gamma = \Gamma_1 \cup \Gamma_2$. The set $\mathcal{A}_e^\gamma(z)$ is shown in light gray, the set $\mathcal{A}_i^{C_0}(z)$ in white, and the set $\mathcal{A}_e^{C_0}(z) \setminus \mathcal{A}_e^\gamma(z)$ in dark gray.

From (3.3), (3.4), and (3.5), we have

$$\begin{aligned}
0 = H_{\gamma,s}(z) &= H_{\gamma,s}(z) - H_{C_0,s}(z) \\
&= \int_{\mathbb{R}^2} \frac{\chi_{\mathcal{A}_i^\gamma(z)}(y) - \chi_{\mathcal{A}_i^{C_0}(z)}(y) + \chi_{\mathcal{A}_e^{C_0}(z)}(y) - \chi_{\mathcal{A}_e^\gamma(z)}(y)}{|y-z|^{2+s}}\, dy \\
&= \int_{\mathcal{A}_i^\gamma(z)\setminus\mathcal{A}_i^{C_0}(z)} \frac{1}{|y-z|^{2+s}}\, dy - \int_{\mathcal{A}_i^{C_0}(z)\setminus\mathcal{A}_i^\gamma(z)} \frac{1}{|y-z|^{2+s}}\, dy \\
&\quad + \int_{\mathcal{A}_e^{C_0}(z)\setminus\mathcal{A}_e^\gamma(z)} \frac{1}{|y-z|^{2+s}}\, dy - \int_{\mathcal{A}_e^\gamma(z)\setminus\mathcal{A}_e^{C_0}(z)} \frac{1}{|y-z|^{2+s}}\, dy \\
&= \int_{\mathcal{A}_i^\gamma(z)\setminus\mathcal{A}_i^{C_0}(z)} \frac{1}{|y-z|^{2+s}}\, dy + \int_{\mathcal{A}_e^{C_0}(z)\setminus\mathcal{A}_e^\gamma(z)} \frac{1}{|y-z|^{2+s}}\, dy > 0,
\end{aligned} \tag{3.6}$$

which is a contradiction. Therefore we obtain the claim. □

Remark 3.3. We briefly consider the situation of Theorem 1.2 with $h_1 = d$ and $h_2 = -d$ for $d \neq 1$ and $d > 0$ and see what kind of shape the critical points in dimension 2 look like. Notice that we have treated the case of $d = 1$ in Proposition 3.1.

Assume that $h_1 = d$ and $h_2 = -d$ for $d > 0$. We define a cone C_d of vertex 0 by

$$C_d \coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| = d|x_1|,\ |x_2| \le d\}. \tag{3.7}$$

Notice that $\partial C_d = \Gamma_1 \cup \Gamma_2$. By slightly modifying the argument for showing that $H_{C_0,s} = 0$ on $C_0 \setminus (\partial C_0 \cup \{0\})$ and taking a proper orientation, we can show that the fractional mean curvature $H_{C_d,s}(z)$ of C_d is either positive or negative for any $z \in C_d \setminus \partial C_d$ with $z \neq 0$. Then, again by slightly modifying the argument in the proof of Proposition 3.1, we obtain the same result as in Proposition 3.1 even for any $d \neq 1$.

We next prove the same result as Proposition 3.1 in higher dimensions. To see this, we also show that the fractional mean curvature of a cone passing through $\Gamma_1 \cup \Gamma_2$ is either positive or negative everywhere except at its vertex in higher dimensions. The idea of the proof is the same as that in the proof of Proposition 3.1. We first give some notations. We define a bounded tube D_0 and a unbounded cone $\widetilde{C}_0$ by

$$
\begin{aligned}
D_0 &\coloneqq \{(x',\, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x'| < 1,\, -1 < x_N < 1\} \\
\widetilde{C}_0 &\coloneqq \{(x',\, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x_N| > |x'|\}.
\end{aligned}
$$

Moreover, we set $C_0^N \coloneqq \partial \widetilde{C}_0 \cap \{(x',\, x_N) \mid |x_N| \leq 1\}$ and decompose $\widetilde{C}_0$ into two parts $\widetilde{C}_0^+$ and $\widetilde{C}_0^-$ which are defined by

$$
\begin{aligned}
\widetilde{C}_0^+ &\coloneqq \{(x',\, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > |x'|\} \\
\widetilde{C}_0^- &\coloneqq \{(x',\, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N < -|x'|\}.
\end{aligned}
$$

Notice that C_0^N coincides with C_0 given in (3.2) if $N = 2$ and $\partial C_0^N = \Gamma_1 \cup \Gamma_2$.

Proposition 3.4. *Let $N \geq 3$ and $s \in (0,\, 1)$. Let Γ_1 and Γ_2 be as in Theorem 1.2 with $h_1 = 1$ and $h_2 = -1$. Let $\mathcal{M} \subset \mathbb{R}^N$ be an orientable compact $C^{1,\alpha}$ manifold with $\partial \mathcal{M} = \Gamma_1 \cup \Gamma_2$. Assume that $\mathcal{C} = \{(x',\, x_N) \mid |x'| < 1\}$ where $\mathcal{C}$ is as in (1.7). If $\mathcal{M}$ is a critical point of Per_s under normal variations, then $\mathcal{M}$ is not contained in either $\overline{\widetilde{C}_0^+ \cup \widetilde{C}_0^-}$ or $D_0 \setminus (\overline{\widetilde{C}_0^+ \cup \widetilde{C}_0^-})$ whenever $(\mathcal{M} \setminus \partial \mathcal{M}) \cap (C_0^N \setminus \{0\}) \neq \emptyset$.*

Proof. The proof is similar to that of Proposition 3.1 and we here show a rough sketch of the proof. Let $\mathcal{M}$ be the critical point selected in Proposition 3.4. We assume that $(\mathcal{M} \setminus \partial \mathcal{M}) \cap (C_0^N \setminus \{0\})$ is not empty and we choose a point $z \in (\mathcal{M} \setminus \partial \mathcal{M}) \cap (C_0^N \setminus \{0\})$. Suppose by contradiction that either

$$
\mathcal{M} \subset \overline{\widetilde{C}_0^+ \cup \widetilde{C}_0^-} \quad \text{or} \quad \mathcal{M} \subset D_0 \setminus (\overline{\widetilde{C}_0^+ \cup \widetilde{C}_0^-})
$$

holds. First, by choosing a proper orientation, we show that

$$
H_{C_0^N, s}(z) > 0. \tag{3.8}
$$

Indeed, if we take the unit normal vector $\nu_{C_0^N}(z)$ of the cone C_0^N at z in such a way that the direction is towards $\widetilde{C}_0$, then the "interior" $\mathcal{A}_i^{C_0^N}(z)$ and "exterior" $\mathcal{A}_e^{C_0^N}(z)$ can be defined as

$$
\mathcal{A}_i^{C_0^N}(z) = \mathbb{R}^N \setminus \left(\mathcal{A}_e^{C_0^N}(z) \cup C_0^N \right)
$$

and

$$
\mathcal{A}_e^{C_0^N}(z) = (\widetilde{C}_0 \cap \{(x',\, x_N) \mid |x_N| \leq 1\}) \cup ((C_{\Gamma_1}(z) \cup C_{\Gamma_2}(z)) \cap \{(x',\, x_N) \mid |x_N| \geq 1\})
$$

where $C_{\Gamma_i}(z)$ is defined by a (filled) cone of vertex z passing through Γ_i for each $i \in \{1, 2\}$. Now we take a hyperplane H_z which is tangent to $\partial \widetilde{C}_0$ and passes through z and define the reflection map T_{H_z} with respect to H_z. From the definitions of C_0^N, $\mathcal{A}_i^{C_0^N}(z)$, and $\mathcal{A}_e^{C_0^N}(z)$, we have

$$
T_{H_z}(\mathcal{A}_e^{C_0^N}(z)) \subset \mathcal{A}_i^{C_0^N}(z) \quad \text{and} \quad \left| \mathcal{A}_i^{C_0^N}(z) \setminus T_{H_z}(\mathcal{A}_e^{C_0^N}(z)) \right| \neq 0.
$$

Since T_{H_z} is an isometry and $T_{H_z}(z) = z$, we obtain the following:

$$\begin{aligned}
H_{C_0^N,s}(z) = &\int_{\mathcal{A}_i^{C_0^N}(z)\setminus T_{H_z}(\mathcal{A}_e^{C_0^N}(z))} \frac{dx}{|x-z|^{N+s}} + \int_{T_{H_z}(\mathcal{A}_e^{C_0^N}(z))} \frac{dx}{|x-z|^{N+s}} \\
&- \int_{\mathcal{A}_e^{C_0^N}(z)} \frac{dx}{|x-z|^{N+s}} \\
= &\int_{\mathcal{A}_i^{C_0^N}(z)\setminus T_{H_z}(\mathcal{A}_e^{C_0^N}(z))} \frac{dx}{|x-z|^{N+s}} + 0 > 0,
\end{aligned} \tag{3.9}$$

which implies (3.8).

Now, since $\mathcal{M}$ is a critical point of Per_s, we have the Euler-Lagrange equation

$$H_{\mathcal{M},s}(z) = 0.$$

Thus, taking the unit normal vector $\nu_{\mathcal{M}}(z)$ of $\mathcal{M}$ at z as $\nu_{C_0^N}(z)$, we can have the following computation:

$$\begin{aligned}
0 &= H_{\mathcal{M},s}(z) - H_{C_0^N,s}(z) + H_{C_0^N,s}(z) \\
&= 2\int_{\mathcal{A}_e^{C_0^N}(z)\setminus \mathcal{A}_e^{\mathcal{M}}(z)} \frac{1}{|x-z|^{N+s}}\,dx - 2\int_{\mathcal{A}_e^{\mathcal{M}}(z)\setminus \mathcal{A}_e^{C_0^N}(z)} \frac{1}{|x-z|^{N+s}}\,dx + H_{C_0^N,s}(z).
\end{aligned} \tag{3.10}$$

From the assumption, we can observe that

$$\left|\mathcal{A}_e^{C_0^N}(z) \setminus \mathcal{A}_e^{\mathcal{M}}(z)\right| > 0 \quad \text{and} \quad \left|\mathcal{A}_e^{\mathcal{M}}(z) \setminus \mathcal{A}_e^{C_0^N}(z)\right| = 0.$$

Therefore, from (3.8) and (3.10), we reach a contradiction. $\square$

4 Topology of Critical Points

In this section, we investigate the topology of critical points with two parallel and co-axial boundaries and prove Theorem 1.4 and 1.5.

Before proving our main theorems of this section, we show Lemma 1.3. The idea of the proof is to construct a small barrier, whose fractional mean curvature is strictly positive or negative, and to "slide" the barrier until it touches the critical point. The construction of the barrier is inspired by the one shown in [18]. See also [14, Proof of Proposition 4.1]. In the sequel, without loss of generality, we may assume that $\mathcal{C} = \{(x',\, x_N) \mid |x'| < 1\}$ where $\mathcal{C}$ is as in (1.7) for simplicity.

Proof of Lemma 1.3. We first fix $\varepsilon \in (0,\, 1)$ so small that $\delta = \delta(\varepsilon) := (-\log\varepsilon)^{-1/2} < \frac{1}{2}$ and we define a smooth bump function $w_\varepsilon : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$w_\varepsilon(x') := \begin{cases} -\exp\left(-\frac{1}{\delta^2 - |x'|^2}\right) & \text{for } |x'| < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $w_\varepsilon \in C^\infty(\mathbb{R}^{N-1})$, $w_\varepsilon(x') = 0$ for $|x'| = \delta$, $w_\varepsilon(0) = -\varepsilon$, and

$$\lim_{\varepsilon\downarrow 0} \phi(\varepsilon) := \lim_{\varepsilon\downarrow 0} \|\nabla'^2 w_\varepsilon\|_{C^0} = 0. \tag{4.1}$$

If necessary, we may choose ε in such a way that $\phi(\varepsilon) < 1$. Note that, since ϕ is an increasing function in a neighborhood $I_\phi \subset [0, 1)$ of the origin, its inverse function ϕ^{-1} exists in a neighborhood $J_\phi \subset [0, 1)$ of the origin. We then set

$$r(\varepsilon) := (2(N-1)\phi(\varepsilon))^{-1} \quad \text{and} \quad d(\varepsilon) := 2r(\varepsilon). \tag{4.2}$$

Moreover, we define a positive constant ε_d as

$$\varepsilon_d := \begin{cases} \phi^{-1}((2(N-1)d)^{-1}) & \text{if } (2(N-1)d)^{-1} \in J_\phi \\ (\text{any positive constant in } J_\phi) & \text{if } (2(N-1)d)^{-1} \notin J_\phi. \end{cases}$$

By definition, we observe that $r(\varepsilon_d) \geq d$ and ε_d can be chosen independently of d if $d < (2(N-1))^{-1}$ since $J_\phi \subset [0, 1)$.

In addition, we choose a smooth function $v_\varepsilon : \mathbb{R}^{N-1} \to \mathbb{R}$ such that v_ε is radially symmetric, $0 \leq v_\varepsilon(x') \leq 1$ for $x' \in \mathbb{R}^{N-1}$, and $\operatorname{spt} v_\varepsilon \subset B'_{1/8}(0)$ where we denote by $B'_r(0)$ an open ball centered at the origin of radius r in $\mathbb{R}^{N-1}$. In particular, we choose v_ε in such a way that its subgraph $\{(x', x_N) \mid 0 \leq x_N \leq v_\varepsilon(x')\}$ of v_ε contains a cylinder of height $\phi(\varepsilon)^\beta < 1$ for $\beta \in (0, s)$ with the base of radius $\frac{1}{16}$. Then we define a function $\widetilde{w}_\varepsilon : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$\widetilde{w}_\varepsilon(x') := \begin{cases} w_\varepsilon(x') & \text{for } |x'| < \delta \\ 0 & \text{for } \delta \leq |x'| < \frac{5}{8} \\ v_\varepsilon(x' - b') & \text{for } \frac{5}{8} \leq |x'| < \frac{7}{8} \\ 0 & \text{for } |x'| \geq \frac{7}{8} \end{cases}$$

where $b' \in \mathbb{R}^{N-1}$ is any point with $|b'| = \frac{3}{4}$. Notice that $\widetilde{w}_\varepsilon$ is smooth in $\mathbb{R}^{N-1}$.

Now we construct a barrier against $\widetilde{\mathcal{M}}^{\varepsilon,t}$, i.e., an orientable compact $(N-1)$-dimensional piecewise smooth manifold $\widetilde{\mathcal{M}}^{\varepsilon,t}$ in the following way: first, taking any $t \in (0, \varepsilon]$, we define four sets

$$\begin{aligned} \mathcal{M}_1^{\varepsilon,t} &:= \{(x', x_N) \mid |x'| \leq 1, \, x_N = \widetilde{w}_\varepsilon(x') + t\}, \\ \text{and} \quad \mathcal{M}_2^{\varepsilon,t} &:= \{(x', x_N) \mid |x'| \leq 1, \, x_N = -d(\varepsilon) + t\} \end{aligned}$$

where $d(\varepsilon) := 2r(\varepsilon)$. Then we define our barrier as $\widetilde{\mathcal{M}}^{\varepsilon,t} := \mathcal{M}_1^{\varepsilon,t} \cup \mathcal{M}_2^{\varepsilon,t}$. By construction, we can easily see that $\widetilde{\mathcal{M}}^{\varepsilon,t}$ is an orientable compact $(N-1)$-dimensional smooth manifold with $\partial \mathcal{M}_1^{\varepsilon} = \Gamma_1^{\varepsilon,t}$ and $\partial \mathcal{M}_2^{\varepsilon,t} = \Gamma_2^{\varepsilon,t}$ where we define

$$\Gamma_1^{\varepsilon,t} := \mathcal{C} \cap \{x_N = t\} \quad \text{and} \quad \Gamma_2^{\varepsilon,t} := \mathcal{C} \cap \{x_N = -d(\varepsilon) + t\}.$$

We next construct another barrier in which the small bump associated with v_ε is removed from $\widetilde{\mathcal{M}}^{\varepsilon,t}$. First, for any $t \in (0, \varepsilon]$, we define a manifold $\mathcal{M}_3^{\varepsilon,t}$ as the graph of w_ε, i.e.,

$$\mathcal{M}_3^{\varepsilon,t} := \{(x', x_N) \mid |x'| < 1, \, x_N = w_\varepsilon(x') + t\}$$

and, then, define the second barrier as $\mathcal{M}^{\varepsilon,t} := \mathcal{M}_3^{\varepsilon,t} \cup \mathcal{M}_2^{\varepsilon,t}$. Notice that $\partial \mathcal{M}^{\varepsilon,t} = \Gamma_1^{\varepsilon,t} \cup \Gamma_2^{\varepsilon,t}$.

We now show, up to orientation, that the fractional mean curvature of $\widetilde{\mathcal{M}}^{\varepsilon,t}$ is negative on the graph of w_ε. Let $q \in \mathcal{M}_1^{\varepsilon,t}$ be any point such that $|q'| < \delta(\varepsilon)$ where we set $q = (q', q_N)$. We now define $C_{\Gamma_i^{\varepsilon,t}}(q)$ by a (filled) cone of vertex q whose boundary passes

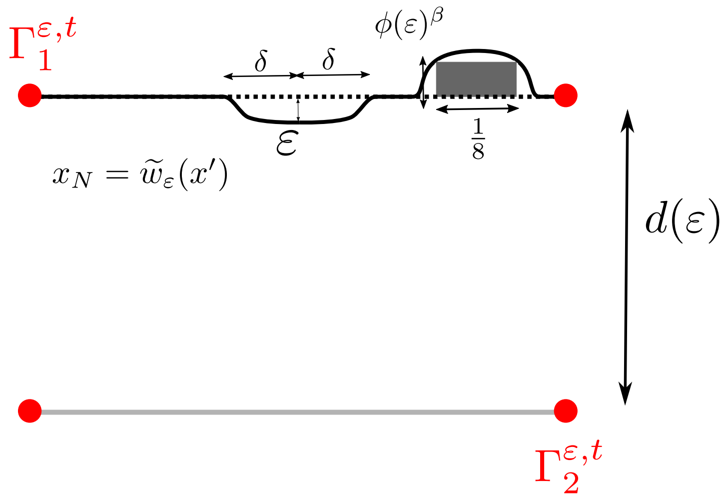

Figure 7: The barrier $\widetilde{\mathcal{M}}^{\varepsilon,t} = \mathcal{M}_1^{\varepsilon,t} \cup \mathcal{M}_2^{\varepsilon,t}$ associated with a function $\widetilde{w}_\varepsilon$ in dimension 2. The graph of $\widetilde{w}_\varepsilon$ in $\{|x'| < 1\}$ is depicted with black lines and the cylinder in dark gray.

through $\Gamma_i^{\varepsilon,t}$ for $i \in \{1, 2\}$. Then, up to orientation, we can choose the interior and exterior of $\widetilde{\mathcal{M}}^{\varepsilon,t}$ at q as

$$\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) := \mathbb{R}^N \setminus \left(\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) \cup \widetilde{\mathcal{M}}^{\varepsilon,t}\right)$$

and

$$\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) := \left(C_{\Gamma_2^{\varepsilon,t}}(q) \cap \{(x', x_N) \mid x_N < -d(\varepsilon) + t\}\right) \cup \{(x', x_N) \mid x_N > \widetilde{w}_\varepsilon^q(x')\},$$

respectively, where we define a function $\widetilde{w}_\varepsilon^q : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$\widetilde{w}_\varepsilon^q(x') := \begin{cases} \widetilde{w}_\varepsilon(x') & \text{for } |x'| < 1 \\ (\text{the graph function of } \partial C_{\Gamma_1^{\varepsilon,t}}(q)) & \text{for } |x'| \geq 1. \end{cases}$$

We now compute the fractional mean curvature $H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q)$ at q of $\widetilde{\mathcal{M}}^{\varepsilon,t}$. From the definition of the fractional mean curvature and by the change of variables, we have

$$\begin{aligned} -H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) &= \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x) - \chi_{\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x)}{|x|^{N+s}}\,dx \\ &= \int_{B_r'(0)\times(-r,r)} \frac{\chi_{\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x) - \chi_{\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x)}{|x|^{N+s}}\,dx \\ &\quad + \int_{(B_r'(0)\times(-r,r))^c} \frac{\chi_{\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x) - \chi_{\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(q-x)}{|x|^{N+s}}\,dx \\ &=: (I) + (II) \end{aligned} \tag{4.3}$$

where we set $r := r(\varepsilon)$ where $r(\varepsilon)$ is as in (4.2).

We first compute (I). Thanks to the choice of r and the construction of $\widetilde{\mathcal{M}}^{\varepsilon,t}$, we observe that

$$(B_r'(q) \times (-r, r)) \cap \left(C_{\Gamma_2^{\varepsilon,t}}(q) \cap \{(x', x_N) \mid x_N < -d(\varepsilon) + t\}\right) = \emptyset.$$

Thus we can represent the set $\partial \mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)$ in $B'_r(0) \times (-r, r)$ as the graph of $\widetilde{w}^q_\varepsilon$. By doing the similar computation to the one in [3, Section 3], we obtain

$$
\begin{aligned}
(I) &= -2 \int_{B'_r(0)} F\left(\frac{\widetilde{w}^q_\varepsilon(q') - \widetilde{w}^q_\varepsilon(q' - x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}} \\
&= - \int_{B'_r(0)} F\left(\frac{\widetilde{w}^q_\varepsilon(q') - \widetilde{w}^q_\varepsilon(q' - x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}} \\
&\quad - \int_{B'_r(0)} F\left(\frac{\widetilde{w}^q_\varepsilon(q') - \widetilde{w}^q_\varepsilon(q' + x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}} \\
&= \int_{B'_r(0)} F\left(\frac{-\widetilde{w}^q_\varepsilon(q') + \widetilde{w}^q_\varepsilon(q' - x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}} \\
&\quad - \int_{B'_r(0)} F\left(\frac{\widetilde{w}^q_\varepsilon(q') - \widetilde{w}^q_\varepsilon(q' + x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}}
\end{aligned} \tag{4.4}
$$

where we set

$$
F(t) := \int_0^t \frac{1}{(1+\sigma^2)^{\frac{N+s}{2}}} \, d\sigma
$$

for any $t \in \mathbb{R}$. Note that we have used the change of variables $x' \mapsto -x'$ in the second equality of (4.4) and the fact that F is odd in the last equality of (4.4). By definition, we have that $\widetilde{w}^q_\varepsilon(q') = w_\varepsilon(q')$ and $\widetilde{w}^q_\varepsilon \geq w_\varepsilon$ in $\mathbb{R}^{N-1}$. Since F is increasing, we derive from (4.4) that

$$
\begin{aligned}
(I) &\geq \int_{B'_r(0)} F\left(\frac{-w_\varepsilon(q') + w_\varepsilon(q' - x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}} \\
&\quad - \int_{B'_r(0)} F\left(\frac{w_\varepsilon(q') - w_\varepsilon(q' + x')}{|x'|} \right) \frac{dx'}{|x'|^{N-1+s}}.
\end{aligned} \tag{4.5}
$$

Now, by using the fundamental theorem of calculus in (4.4), we obtain

$$
(I) \geq - \int_{B'_r(0)} \int_0^1 F'\left(a(x', q', \lambda)\right) d\lambda \frac{2w_\varepsilon(q') - w_\varepsilon(q' + x') - w_\varepsilon(q' - x')}{|x'|^{N+s}} \, dx' \tag{4.6}
$$

where we set $a(x', q', \lambda)$ as

$$
a(x', q', \lambda) := \lambda \frac{w_\varepsilon(q') - w_\varepsilon(q' + x')}{|x'|} + (1-\lambda) \frac{-w_\varepsilon(q') + w_\varepsilon(q' - x')}{|x'|}
$$

for $x', q' \in \mathbb{R}^{N-1}$ and $\lambda \in [0, 1]$. Thus, by using again the fundamental theorem of calculus, we have

$$
(I) \geq - \int_{B'_r(0)} \int_0^1 \frac{|\nabla' w_\varepsilon(q' + \rho x') - \nabla' w_\varepsilon(q' - \rho x')|}{|x'|^{N-1+s}} \, d\rho \, dx'.
$$

Since w_ε is smooth in $\mathbb{R}^{N-1}$, we then have

$$
(I) \geq -2 \|\nabla'^2 w_\varepsilon\|_{C^0} \int_{B'_r(0)} \frac{dx'}{|x'|^{N-2+s}} = - \frac{2\omega_{N-2}}{1-s} \|\nabla'^2 w_\varepsilon\|_{C^0} \, r^{1-s}. \tag{4.7}
$$

Now we compute (II) in the following way: since $B_r(0) \subset B'_r(0) \times (-r, r) \subset \mathbb{R}^N$, we have

$$(II) \geq -\int_{B_r^c(0)} \frac{dx}{|x|^{N+s}} = -\frac{\omega_{N-1}}{s} r^{-s}. \tag{4.8}$$

Therefore, from (4.7) and (4.8), we obtain

$$-H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) \geq -\left(c_1 \, \|\nabla'^2 w_\varepsilon\|_{C^0} \, r^{1-s} + c_2 \, r^{-s}\right) \tag{4.9}$$

where c_1 and c_2 are defined as

$$c_1 := \frac{2\omega_{N-2}}{1-s} \quad \text{and} \quad c_2 := \frac{\omega_{N-1}}{s},$$

respectively. From (4.2), it holds that the right-hand side of (4.9) takes the maximum at $r = r(\varepsilon) \in (0, \, d(\varepsilon))$. Hence we finally obtain, from (4.9), that

$$-H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) \geq -c \, \|\nabla'^2 w_\varepsilon\|_{C^0}^s = -c \, \phi(\varepsilon)^s \tag{4.10}$$

where we set the constant $c = c(N, s) > 0$ as

$$c = c(N, s) := \frac{(2(N-1))^s \omega_{N-1}}{s(1-s)}.$$

Next we compute the fractional mean curvature $H_{\mathcal{M}^{\varepsilon,t},s}(q)$ at q by using Estimate (4.10) of the fractional mean curvature $H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q)$ at q. Indeed, from the construction of $\mathcal{M}^{\varepsilon,t}$ and $\widetilde{\mathcal{M}}^{\varepsilon,t}$, we have that, by choosing a proper orientation, $\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) \subset \mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q)$ and thus we obtain

$$\begin{aligned}
-H_{\mathcal{M}^{\varepsilon,t},s}(q) &= -H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) \\
&\quad + \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q)}(x) - \chi_{\mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(x) + \chi_{\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(x) - \chi_{\mathcal{A}_i^{\mathcal{M}^{\varepsilon,t}}(q)}(x)}{|x-q|^{N+s}} \, dx \\
&= -H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) + \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q) \setminus \mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)}(x) + \chi_{\mathcal{A}_i^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) \setminus \mathcal{A}_i^{\mathcal{M}^{\varepsilon,t}}(q)}(x)}{|x-q|^{N+s}} \, dx \\
&= -H_{\widetilde{\mathcal{M}}^{\varepsilon,t},s}(q) + 2 \int_{\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q) \setminus \mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)} \frac{1}{|x-q|^{N+s}} \, dx.
\end{aligned} \tag{4.11}$$

Recalling that $\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q) \setminus \mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q)$ contains the subgraph $\{0 \leq x_N \leq v_\varepsilon(x' - b')\}$ and the subgraph contains the cylinder of height $\phi(\varepsilon)^\beta$ with the base of radius $1/16$, we have

$$\left| \mathcal{A}_e^{\mathcal{M}^{\varepsilon,t}}(q) \setminus \mathcal{A}_e^{\widetilde{\mathcal{M}}^{\varepsilon,t}}(q) \right| \geq c' \, \phi(\varepsilon)^\beta.$$

where a constant $c' = c'(N) > 0$ depends only on N. Moreover, we observe that the distance between q and the cylinder is less than, at most, $2 + \phi(\varepsilon)^\beta$ and this is bounded from above by some constant depending only on N, s, and β. Hence from (4.10) and (4.11) and by recalling the choice of ϕ, we obtain

$$\begin{aligned}
-H_{\mathcal{M}^{\varepsilon,t},s}(q) &\geq -c \, \phi(\varepsilon)^s + \frac{2c'}{(2 + \phi(\varepsilon)^\beta)^{N+s}} \, \phi(\varepsilon)^\beta \\
&\geq -c \, \phi(\varepsilon)^s + c'' \phi(\varepsilon)^\beta \\
&= \phi(\varepsilon)^\beta \left(-c \, \phi(\varepsilon)^{s-\beta} + c''\right)
\end{aligned} \tag{4.12}$$

where $c'' > 0$ is a constant depending only on N, s, and β. Since $0 < \beta < s$ and $\phi(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, we choose $\varepsilon_1 = \varepsilon_1(N, s, \beta) \in I_\phi \cap (0, \frac{1}{100})$ so small that the right-hand side of (4.12) is positive for any $\varepsilon \in (0, \varepsilon_1]$. Therefore, from (4.12), we obtain that $H_{\mathcal{M}^{\varepsilon,t},s}(q) < 0$ for $\varepsilon \in (0, \varepsilon_1]$.

Now we set $\varepsilon_2 \coloneqq \min\{\varepsilon_1, \varepsilon_d\}$. Since $r(\varepsilon_d) \geq d$ and $\delta(\varepsilon_2) < \frac{1}{2}$, we may observe that $d(\varepsilon_2) \geq d$ and $\mathcal{M}_3^{\varepsilon_2,t} \cap \Gamma_1 = \emptyset$ for any $t \in (0, \varepsilon_2]$. For our convenience, we denote ε_2 by ε in the sequel.

We then slide the barrier $\mathcal{M}^{\varepsilon,t}$ from above, i.e., we vary the parameter t stating at ε until $\mathcal{M}^{\varepsilon,t}$ touches the critical point $\mathcal{M}$. To prove the claim, we assume by contradiction that there exists $t_1 \in (0, \varepsilon]$ such that $\mathcal{M} \cap \mathcal{M}_3^{\varepsilon,t_1} \neq \emptyset$ and $\mathcal{M} \cap \mathcal{M}_3^{\varepsilon,t} = \emptyset$ for any $t \in (t_1, \varepsilon]$. We pick up a point $q_{\varepsilon,t_1} \in \mathcal{M} \cap \mathcal{M}_3^{\varepsilon,t_1}$. Notice that

$$\{(x', x_N) \mid -d < x_N < 0\} \cap \mathcal{M}_2^{\varepsilon,t_1} = \emptyset$$

since $d(\varepsilon) \geq d$. See Figure 8 to favor the intuition in dimension 2.

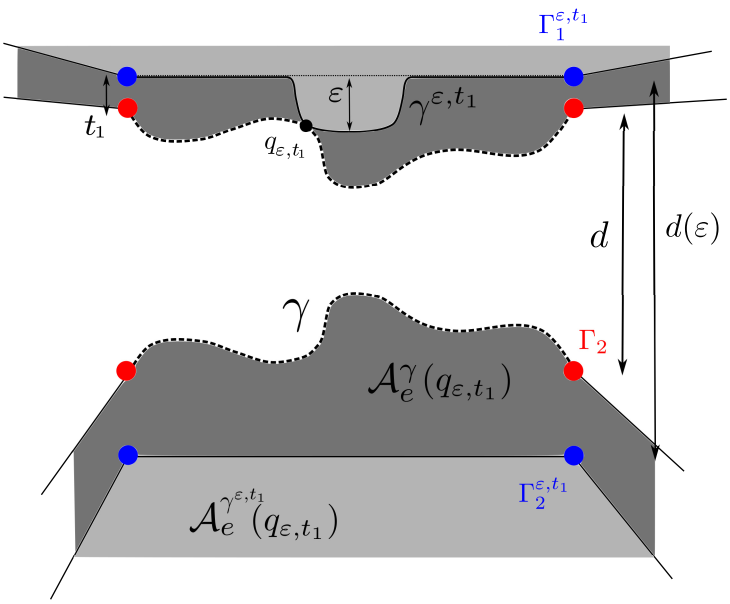

Figure 8: The critical point γ depicted with dashed lines and the barrier γ^{ε,t_1} with black line. γ touches γ^{ε,t_1} at q_{ε,t_1} from above. The exterior $\mathcal{A}_e^{\gamma^{\varepsilon,t_1}}(q_{\varepsilon,t_1})$ of γ^{ε,t_1} is depicted in light gray and the exterior $\mathcal{A}_e^{\gamma}(q_{\varepsilon,t_1})$ of γ in both light and dark gray.

Since $\mathcal{M}$ is a critical point of Per_s under normal variations, we obtain

$$H_{\mathcal{M},s}(q_{\varepsilon,t_1}) = 0.$$

From Theorem 1.2 and the above argument, we obtain that the touching point $q_{\varepsilon,t_1} \coloneqq (q'_{\varepsilon,t_1}, q^N_{\varepsilon,t_1}) \in \mathbb{R}^{N-1} \times \mathbb{R}$ satisfies $|q'_{\varepsilon,t_1}| < \delta(\varepsilon)$ and thus $H_{\mathcal{M}^{\varepsilon,t_1},s}(q_{\varepsilon,t_1}) < 0$. Moreover, from the construction of $\mathcal{M}_1^{\varepsilon,t_1}$, we have, by choosing a proper orientation, that

$$|\mathcal{A}_e^{\mathcal{M}}(q_{\varepsilon,t_1}) \setminus \mathcal{A}_e^{\mathcal{M}^{\varepsilon,t_1}}(q_{\varepsilon,t_1})| > 0 \quad \text{and} \quad |\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t_1}}(q_{\varepsilon,t_1}) \setminus \mathcal{A}_e^{\mathcal{M}}(q_{\varepsilon,t_1})| = 0.$$

Therefore, we obtain

$$
\begin{aligned}
0 &= H_{\mathcal{M},s}(q_{\varepsilon,t_1}) - H_{\mathcal{M}^{\varepsilon,t_1},s}(q_{\varepsilon,t_1}) + H_{\mathcal{M}^{\varepsilon,t_1},s}(q_{\varepsilon,t_1}) \\
&< \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q_{\varepsilon,t_1})}(x) - \chi_{\mathcal{A}_i^{\mathcal{M}^{\varepsilon,t_1}}(q_{\varepsilon,t_1})}(x) + \chi_{\mathcal{A}_e^{\mathcal{M}^{\varepsilon,t_1}}(q_{\varepsilon,t_1})}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q_{\varepsilon,t_1})}(x)}{|x - q_{\varepsilon,t_1}|^{N+s}}\, dx + 0 \\
&= -2 \int_{\mathcal{A}_e^{\mathcal{M}}(q_{\varepsilon,t_1}) \setminus \mathcal{A}_e^{\mathcal{M}^{\varepsilon,t_1}}(q_{\varepsilon,t_1})} \frac{1}{|x - q_{\varepsilon,t_1}|^{N+s}}\, dx < 0,
\end{aligned} \tag{4.13}
$$

which is a contradiction. We thus conclude that we can slide the barrier $\mathcal{M}^{\varepsilon,t}$ until the boundary $\Gamma_1^{\varepsilon,t} = \partial\mathcal{M}_3^{\varepsilon,t}$ coincides with the boundary $\Gamma_1 = \partial\mathcal{M}_1$. By symmetry, we can slide the barrier from below and do the same argument.

Therefore we obtain that two open half-balls of radius ε_2 are contained in a set enclosed by $\mathcal{M}$ and the union of $\mathcal{C} \cap \{x_N = 0\}$ and $\mathcal{C} \cap \{x_N = -d\}$.

$\square$

As a consequence of Lemma 1.3, we now prove Theorem 1.4.

Proof of Theorem 1.4. Assume that ε_2 and $\widetilde{\mathcal{M}}^{\varepsilon,t}$ are given in the proof of Lemma 1.3 for $\varepsilon \in (0, \varepsilon_2]$ and $t \in (0, \varepsilon]$. From the definition of ε_2, we can choose $d' > 0$ so small that $d' < (2(N-1))^{-1}$ and that ε_2 can be chosen independently of d for any $d \in (0, d')$. Moreover, if necessary, we may assume that $\varepsilon_2\, \phi(\varepsilon_2) < (2(N-1))^{-1}$, which is still independent of d.

Let $\mathcal{M}$ be the critical point chosen in Theorem 1.4. We set $d_0 \coloneqq \min\{d', \varepsilon_2\}$. From the choice of ϕ and ε_2, we have that $d_0\, \phi(d_0) < (2(N-1))^{-1}$. Then we observe that $d(\varepsilon_2) - t = ((N-1)\phi(\varepsilon_2))^{-1} - t > d$ for any $t \in (0, \varepsilon_2]$ and thus we have that

$$
\Gamma_2^{\varepsilon_2,t} \cap \{(x', x_N) \mid -d < x_N < 0\} = \emptyset
$$

for any $t \in (0, \varepsilon_2]$ and any $d < d_0$.

Now, by Lemma 1.3, we find that we can slide the barrier $\mathcal{M}^{\varepsilon_2,t}$ until the parameter t reaches 0. Thus, by combining this with Theorem 1.2, we obtain that

$$
\begin{aligned}
\mathcal{M} &\subset \big(\overline{\mathcal{C}} \setminus \{(x', x_N) \mid |x'| < \varepsilon_2\}\big) \cap \{(x', x_N) \mid -d \le x_N \le 0\} \\
&= \{(x', x_N) \mid \varepsilon_2 \le |x'| \le 1,\ -d \le x_N \le 0\}
\end{aligned}
$$

for any $d < d_0$.

If $N = 2$, then, since Γ_i consists of two distinct points for $i \in \{1, 2\}$, by a simple geometric argument, we conclude that the critical point $\mathcal{M}$ is disconnected for any $d \in (0, d_0)$. Moreover, from the construction of the barrier, we obtain that there exist two connected components $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ such that $\operatorname{dist}(\mathcal{M}_1, \mathcal{M}_2) \ge \varepsilon_2$ and $\mathcal{M}_i$ intersects both Γ_1 and Γ_2 for each $i \in \{1, 2\}$ at its boundary (see also Remark 4.1).

If $N \ge 3$, then, by using a homology theory, we conclude that Γ_1 and Γ_2 are in the same connected component of the critical point $\mathcal{M}$ for any $d \in (0, d_0)$ (see [21]). Indeed, we assume by contradiction that there exists a connected component $\mathcal{M}_0$ of $\mathcal{M}$ with $\partial\mathcal{M}_0 = \Gamma_1$. Taking the fundamental class $[\Gamma_1] \in H_{N-2}(\Gamma_1)$ where $H_k(\mathcal{S})$ is the k-th homology group of $\mathcal{S}$, we may have that the image in $H_{N-2}(\mathcal{M}_0)$ of $[\Gamma_1]$ by the induced map of homology from the inclusion $i : \Gamma_1 \to \mathcal{M}_0$ does not vanish because $\mathcal{M}_0 \subset \mathcal{M} \subset A_{\varepsilon_2}$ and A_{ε_2} deformation-retracts to $\Gamma_1 \simeq \mathbb{S}^{N-2}$ where

$$
A_{\varepsilon_2} \coloneqq \{(x', x_N) \mid \varepsilon_2 \le |x'| \le 1, -d \le x_N \le 0\}.
$$

However, since $[\Gamma_1]$ is the boundary of $[\mathcal{M}_0]$ and by using an exact homology sequence of the pair $(\mathcal{M}_0,\, \Gamma_1)$, we obtain the contradiction. □

Remark 4.1. Combining Remark 3.3 with Lemma 1.3 and Theorem 1.4, we may observe that two possible critical points of Per_s in dimension 2 whose boundary is $\Gamma_1 \cup \Gamma_2 = \{(\pm 1,\, d),\, (\pm 1,\, -d)\}$ are depicted in Figure 9.

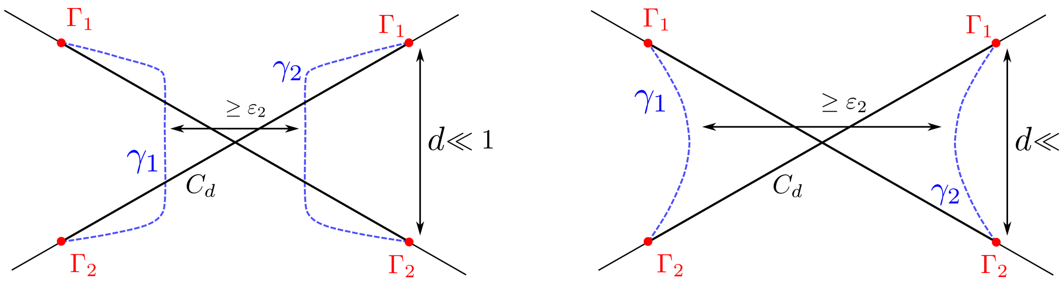

Figure 9: Two possible critical points γ of Per_s in dimension 2 with $\partial\gamma = \Gamma_1 \cup \Gamma_2$ are shown with dashed lines and the cone C_d defined in (3.7) is shown with crossed lines. On the right, γ does not intersect with C_d except at their boundaries Γ_1 and Γ_2. In both figures, two distinct connected components γ_1 and γ_2 of γ are placed at mutually positive distance of at least $\varepsilon_2 > 0$.

Finally in this section, we prove Theorem 1.5. The idea of the proof is basically the same as the one in [14, Theorem 1.2], i.e., we use the "sliding method" that is developed by Dipierro, Savin, and Valdinoci in [15–17].

Proof of Theorem 1.5. Let $\mathcal{M}$ be the critical point selected in Theorem 1.5 and we set $\Gamma \coloneqq \Gamma_1 \cup \Gamma_2$.

Given $t \in \mathbb{R}$ and $\alpha \in (0,\, 1)$, we consider the open ball $B_{d^\alpha/2}(p_{t,d})$ where $p_{t,d} \coloneqq (te'_1,\, \frac{-d}{2}) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $e'_1 \coloneqq (1,\, 0, \cdots, 0) \in \mathbb{R}^{N-1}$. Here we take d conveniently large so that $d - d^\alpha > 100$. Then we slide the ball from left to right until it touches the critical point $\mathcal{M}$, which means that we vary t from $t = -\infty$ to $t = +\infty$. Note that $B_{d^\alpha/2}(p_{t,d}) \subset \mathcal{C}^c$ for $|t| > 1 + d^\alpha/2$ and $B_{d^\alpha/2}(p_{t,d}) \cap \Gamma = \emptyset$ for any t. To prove the claim, we suppose by contradiction that there exists $t_0 \in \mathbb{R}$ such that $\overline{B}_{d^\alpha/2}(p_{t,d}) \cap \mathcal{M} = \emptyset$ for $t < t_0$ and $\partial B_{d^\alpha/2}(p_{t_0,d}) \cap \mathcal{M} \neq \emptyset$.

We choose a point $q \in \partial B_{d^\alpha/2}(p_{t_0,d}) \cap \mathcal{M}$. Note that, due to Theorem 1.2, $q \in \mathcal{C}$. By the Euler-Lagrange equation, we have that

$$H_{\mathcal{M},s}(q) = 0. \tag{4.14}$$

Moreover, by choosing a proper orientation, we can choose the interior $\mathcal{A}_i^{\mathcal{M}}(q)$ and exterior $\mathcal{A}_e^{\mathcal{M}}(q)$ at q of $\mathcal{M}$ in such a way that

$$B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \subset \mathcal{A}_i^{\mathcal{M}}(q) \quad \text{and} \quad \mathcal{A}_e^{\mathcal{M}}(q) = \mathbb{R}^N \setminus \left(\mathcal{A}_i^{\mathcal{M}}(q) \cup \mathcal{M}\right). \tag{4.15}$$

We now consider the symmetric ball of $B_{d^\alpha/2}(p_{t_0,d})$ with respect to q and we denote it by $\widetilde{B} \coloneqq B_{d^\alpha/2}(\widetilde{p}_{t_0,d})$ where $\widetilde{p}_{t_0,d} \coloneqq p_{t_0,d} + 2(q - p_{t_0,d})$.

We define a cylinder S_d as

$$S_d \coloneqq \{(x',\, x_N) \mid |x'| < 2,\, -d < x_N < 0\}.$$

Notice that $\mathcal{C} \cap \{(x', x_N) \mid -d < x_N < 0\} \subset S_d$ and $\mathcal{M} \subset S_d$ thanks to Theorem 1.2. From the symmetry of the balls, we have

$$\int_{S_d \cap B_{\frac{d^\alpha}{2}}(p_{t_0,d})} \frac{dx}{|x-q|^{N+s}} = \int_{S_d \cap \widetilde{B}} \frac{dx}{|x-q|^{N+s}}$$

and therefore, from (4.15),

$$\begin{aligned}
\int_{S_d} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx &= \int_{S_d \cap B_{\frac{d^\alpha}{2}}(p_{t_0,d})} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx \\
&\quad + \int_{S_d \cap \widetilde{B}} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx \\
&\quad + \int_{S_d \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx \\
&\geq \int_{S_d \cap B_{\frac{d^\alpha}{2}}(p_{t_0,d})} \frac{dx}{|x-q|^{N+s}} - \int_{S_d \cap \widetilde{B}} \frac{dx}{|x-q|^{N+s}} \\
&\quad - \int_{S_d \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{dx}{|x-q|^{N+s}} \\
&\geq - \int_{S_d \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{dx}{|x-q|^{N+s}}. \qquad (4.16)
\end{aligned}$$

By employing the result in [15, Lemma 3.1] with $R = d^\alpha/2$ and $\lambda = d^{-\frac{\alpha}{2}}$, we obtain

$$\int_{B_{d^{\frac{\alpha}{2}}}(q) \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{dx}{|x-q|^{N+s}} \leq C_0\, d^{-\frac{1+s}{2}\alpha}$$

where $C_0 > 0$ is a constant depending only on N and s. As a consequence, we obtain

$$\begin{aligned}
\int_{S_d \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{dx}{|x-q|^{N+s}} &\leq \int_{B_{d^{\frac{\alpha}{2}}}(q) \setminus \left(B_{\frac{d^\alpha}{2}}(p_{t_0,d}) \cup \widetilde{B}\right)} \frac{dx}{|x-q|^{N+s}} \\
&\quad + \int_{S_d \setminus B_{d^{\frac{\alpha}{2}}}(q)} \frac{dx}{|x-q|^{N+s}} \\
&\leq C_0\, d^{-\frac{1+s}{2}\alpha} + \int_{\mathbb{R}^N \setminus B_{d^{\frac{\alpha}{2}}}(q)} \frac{dx}{|x-q|^{N+s}} \\
&\leq C_0\, d^{-\frac{1+s}{2}\alpha} + C_1\, d^{-\frac{s}{2}\alpha} \leq C_2\, d^{-\frac{s}{2}\alpha} \qquad (4.17)
\end{aligned}$$

where $C_2 \coloneqq C_0 + C_1$ is a constant depending only on N and s. From (4.16) and (4.17), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx &= \int_{S_d} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx \\
&\quad + \int_{S_d^c} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx \\
&\geq -C_2\, d^{-\frac{s}{2}\alpha} + \int_{S_d^c} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\, dx. \qquad (4.18)
\end{aligned}$$

Now we consider the contributions from $\mathcal{A}_i^{\mathcal{M}}(q)$ and $\mathcal{A}_e^{\mathcal{M}}(q)$ in S_d^c. We now define $C_\Gamma(q)$ by a (filled) cone of vertex q whose boundary passes through Γ. Moreover we define $C_{S_d}(q)$ by a (filled) cone of vertex q whose boundary passes through

$$\partial S_d \cap \{(x',\, x_N) \mid x_N = 0\} \quad \text{and} \quad \partial S_d \cap \{(x',\, x_N) \mid x_N = -d\}.$$

From the definitions of S_d and Γ, we observe that

$$C_\Gamma(q) \subset C_{S_d}(q).$$

We now set $\widehat{C}_\Gamma(q) \coloneqq C_\Gamma(q) \cap \{(x',\, x_N) \mid x_N > 0 \text{ or } x_N < -d\}$. We then rotate $\widehat{C}_\Gamma(q)$ by angle $\pi/2$ or $-\pi/2$ with respect to the straight line parallel to the x_1-axis passing trough q (if $N = 2$, then we just rotate $\widehat{C}_\Gamma(q)$ by angle $\pi/2$ or $-\pi/2$ with respect to q). Since we choose d so large that $d - d^\alpha > 100$, we obtain that

$$R(\widehat{C}_\Gamma(q)) \subset S_d^c \cap \mathcal{A}_i^{\mathcal{M}}(q) \cap C_{S_d}(q)^c$$

where $R(\widehat{C}_\Gamma(q))$ is an image of $\widehat{C}_\Gamma(q)$ by the rotation map $R : \mathbb{R}^N \to \mathbb{R}^N$ in the above. See Figure 10 for an intuitive understanding. Then, observing that $R(q) = q$ and

$$S_d^c \cap \mathcal{A}_e^{\mathcal{M}}(q) = \widehat{C}_\Gamma(q)$$

and by a change of variables, we have

$$\int_{R(\widehat{C}_\Gamma(q))} \frac{dx}{|x-q|^{N+s}} = \int_{\widehat{C}_\Gamma(q)} \frac{dx}{|x-q|^{N+s}} = \int_{S_d^c \cap \mathcal{A}_e^{\mathcal{M}}(q)} \frac{dx}{|x-q|^{N+s}}.$$

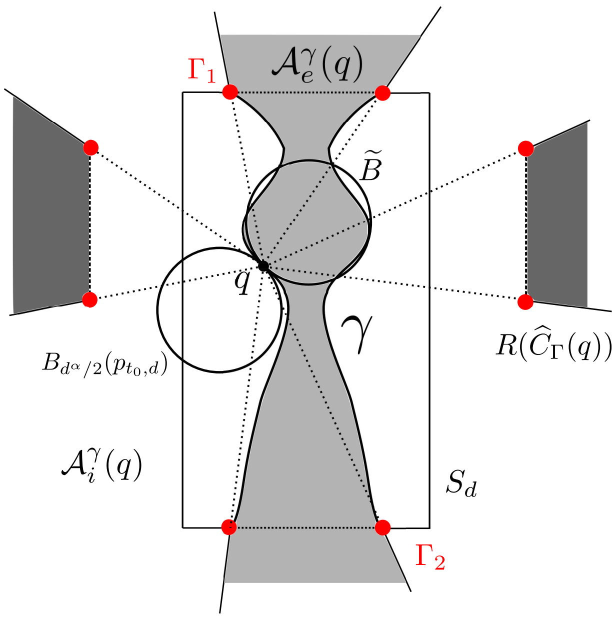

Figure 10: The touching between the ball $B_{d^\alpha/2}(p_{t_0,d})$ and its symmetric ball $\widetilde{B}$ at q. The image of the set $\widehat{C}_\Gamma(q)$ by the rotation map R is depicted in dark gray and the set $\mathcal{A}_e^\gamma(q)$ in light gray.

From the definitions of S_d and the rotation map R, we can choose an open ball outside S_d and $R(\widehat{C}_\Gamma(q))$ but close to q, i.e., we have

$$B_1(q+5e_1) \subset \left(S_d^c \cap \mathcal{A}_i^{\mathcal{M}}(q)\right) \setminus R(\widehat{C}_\Gamma(q))$$

where we set $e_1 \coloneqq (1, 0, \cdots, 0) \in \mathbb{R}^N$. Thus, we obtain

$$\begin{aligned}
\int_{S_d^c} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\,dx &= \int_{S_d^c \cap \mathcal{A}_i^{\mathcal{M}}(q)} \frac{dx}{|x-q|^{N+s}} - \int_{\widehat{C}_\Gamma(q)} \frac{dx}{|x-q|^{N+s}} \\
&\geq \int_{S_d^c \cap \mathcal{A}_i^{\mathcal{M}}(q) \cap R(\widehat{C}_\Gamma(q))} \frac{dx}{|x-q|^{N+s}} \\
&\quad + \int_{\left(S_d^c \cap \mathcal{A}_i^{\mathcal{M}}(q)\right) \setminus R(\widehat{C}_\Gamma(q))} \frac{dx}{|x-q|^{N+s}} \\
&\quad - \int_{\widehat{C}_\Gamma(q)} \frac{dx}{|x-q|^{N+s}} \\
&\geq \int_{B_1(q+5e_1)} \frac{dx}{|x-q|^{N+s}} \\
&= \int_{B_1(5e_1)} \frac{dx}{|x|^{N+s}} =: C_3 > 0
\end{aligned}$$

where C_3 depends only on N and s. This with (4.18) leads to

$$H_{\mathcal{M},s}(q) = \int_{\mathbb{R}^N} \frac{\chi_{\mathcal{A}_i^{\mathcal{M}}(q)}(x) - \chi_{\mathcal{A}_e^{\mathcal{M}}(q)}(x)}{|x-q|^{N+s}}\,dx \geq -C_2\, d^{-\frac{s}{2}\alpha} + C_3.$$

Therefore, there exists $d_1 = d_1(N,s) > 0$ such that $H_{\mathcal{M},s}(q) > 0$ for any $d > d_1$ and this contradicts the Euler-Lagrange equation (4.14). $\square$

Remark 4.2. From Lemma 1.3 and the choice of ε_2 in the proof of Lemma 1.3, we also obtain that, for sufficiently large d, a set enclosed by $\mathcal{M}$ and the union of $\mathcal{C} \cap \{x_N = 0\}$ and $\mathcal{C} \cap \{x_N = -d\}$ contains two half-balls of radius $\varepsilon_2 \approx \phi^{-1}(d^{-1})$ where ϕ^{-1} is as in the proof of Lemma 1.3.

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References

[1] L. Ambrosio, G. De Philippis, and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math. **134** (2011), no. 3-4, 377–403. MR2765717

[2] B. Baronowitz, S. Dipierro, and E. Valdinoci, *The stickiness property for antisymmetric nonlocal minimal graphs*, Discrete Contin. Dyn. Syst. **43** (2023), no. 3-4, 1006–1025. MR4548844

[3] B. Barrios, A. Figalli, and E. Valdinoci, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 3, 609–639. MR3331523

[4] C. Bucur, S. Dipierro, L. Lombardini, and E. Valdinoci, *Minimisers of a fractional seminorm and nonlocal minimal surfaces*, Interfaces Free Bound. **22** (2020), no. 4, 465–504. MR4184583

[5] C. Bucur, L. Lombardini, and E. Valdinoci, *Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **36** (2019), no. 3, 655–703. MR3926519

[6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144. MR2675483

[7] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (2011), no. 1-2, 203–240. MR2782803

[8] ______, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (2011), no. 1-2, 203–240. MR2782803

[9] ______, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. **248** (2013), 843–871. MR3107529

[10] R. Courant, *The existence of minimal surfaces of given topological structure under prescribed boundary conditions*, Acta Math. **72** (1940), 51–98. MR2478

[11] M. Cozzi, S. Dipierro, and E. Valdinoci, *Planelike interfaces in long-range Ising models and connections with nonlocal minimal surfaces*, J. Stat. Phys. **167** (2017), no. 6, 1401–1451. MR3652519

[12] M. Cozzi and L. Lombardini, *On nonlocal minimal graphs*, Calc. Var. Partial Differential Equations **60** (2021), no. 4, Paper No. 136, 72. MR4279395

[13] J. Dávila, M. del Pino, and J. Wei, *Nonlocal s-minimal surfaces and Lawson cones*, J. Differential Geom. **109** (2018), no. 1, 111–175. MR3798717

[14] S. Dipierro, F. Onoue, and E. Valdinoci, *(Dis)connectedness of nonlocal minimal surfaces in a cylinder and a stickiness property*, Proc. Amer. Math. Soc. **150** (2022), no. 5, 2223–2237. MR4392355

[15] S. Dipierro, O. Savin, and E. Valdinoci, *Graph properties for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, Art. 86, 25. MR3516886

[16] ______, *Boundary behavior of nonlocal minimal surfaces*, J. Funct. Anal. **272** (2017), no. 5, 1791–1851. MR3596708

[17] ______, *Nonlocal minimal graphs in the plane are generically sticky*, Comm. Math. Phys. **376** (2020), no. 3, 2005–2063. MR4104542

[18] ______, *Boundary continuity of nonlocal minimal surfaces in domains with singularities and a problem posed by Borthagaray, Li, and Nochetto*, arXiv, 2023.

[19] ______, *A strict maximum principle for nonlocal minimal surfaces*, arXiv, 2023.

[20] S. Hildebrandt, *Free boundary problems for minimal surfaces and related questions*, 1986, pp. S111–S138. Frontiers of the mathematical sciences: 1985 (New York, 1985). MR861485

[21] (https://math.stackexchange.com/users/1225249/igotyourpoint), *Connectedness of manifolds lying in a cylinder with a hole.* URL:https://math.stackexchange.com/q/4774713 (version: 2023-09-24).

[22] H. Lewy, *On mimimal surfaces with partially free boundary*, Comm. Pure Appl. Math. **4** (1951), 1–13. MR52711

[23] M. M.-c. Li, *Free boundary minimal surfaces in the unit ball: recent advances and open questions*, Proceedings of the International Consortium of Chinese Mathematicians 2017, [2020] ©2020, pp. 401–435. MR4251121

[24] W. H. Meeks III and S. T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) **112** (1980), no. 3, 441–484. MR595203

[25] C. Mihaila and B. Seguin, *A definition of fractional k-dimensional measure: bridging the gap between fractional length and fractional area*, arXiv, 2023.

[26] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 1–19. MR784101

[27] R. Paroni, P. Podio-Guidugli, and B. Seguin, *On the nonlocal curvatures of surfaces with or without boundary*, Commun. Pure Appl. Anal. **17** (2018), no. 2, 709–727. MR3733825

[28] ———, *On a notion of nonlocal curvature tensor*, arXiv, 2022.

[29] A. Ros, *Stability of minimal and constant mean curvature surfaces with free boundary*, Mat. Contemp. **35** (2008), 221–240. MR2584186

[30] O. Savin and E. Valdinoci, Γ*-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **29** (2012), no. 4, 479–500. MR2948285

[31] B. Seguin, *A fractional notion of length and an associated nonlocal curvature*, J. Geom. Anal. **30** (2020), no. 1, 161–181. MR4058510

[32] B. Smyth, *Stationary minimal surfaces with boundary on a simplex*, Invent. Math. **76** (1984), no. 3, 411–420. MR746536