SOME PROPERTIES OF THE DISTANCE FUNCTION AND A CONJECTURE OF DE GIORGI

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ABSTRACT. In the paper [2] Ennio De Giorgi conjectured that any compact n–dimensional regular submanifold M of $\mathbb{R}^{n+m}$, moving by the gradient of the functional

$$\int_M 1 + |\nabla^k \eta^M|^2 \, d\mathcal{H}^n \,,$$

where η^M is the square of the distance function from the submanifold M and $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+m}$, does not develop singularities in finite time provided k is large enough, depending on the dimension n.
We prove this conjecture by means of the analysis of the geometric properties of the high derivatives of the distance function from a submanifold of the Euclidean space. In particular, we show some relations with the second fundamental form and its covariant derivatives of independent interest.

1. INTRODUCTION

In the paper [2] (see also [3, Section 5]) Ennio De Giorgi stated the following conjecture (Congettura 2, Pag. 267):

Any compact n–dimensional regular submanifold M of $\mathbb{R}^{n+m}$ moving by the gradient of the functional

$$\mathcal{G}_k = \int_M 1 + |\nabla^k \eta^M|^2 \, d\mathcal{H}^n \,,$$

where η^M is the square of the distance function from M and $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+m}$, does not develop singularities in finite time if $k > n + 1$.

We make some preliminary comments before proceeding with the analysis.

The regular submanifold M can be described with an embedding $\varphi : M \to \mathbb{R}^{n+m}$ which induces a metric tensor g on M, by pulling back the standard scalar product of $\mathbb{R}^{n+m}$, turning (M, g) in a smooth Riemannian manifold isometrically embedded in $\mathbb{R}^{n+m}$ via the map φ. Then, we let μ and ∇ to be respectively the canonical volume measure, which coincides with the Hausdorff measure restricted to the image $\varphi(M)$, and the covariant differentiation operator on (M, g).

Date: October 6, 2003.
1991 *Mathematics Subject Classification.* Primary 53A07; Secondary 53A55.
Key words and phrases. Distance function, second fundamental form, gradient flow.

Despite of the use of the same symbol, the iterated gradient which appears in the functional $\mathcal{G}_k$ is not such covariant differentiation but the standard k–order differential in the canonical basis of $\mathbb{R}^{n+m}$.

When $k \geq 3$ the variational gradient flow associated to the functional $\mathcal{G}_k$ is governed by a parabolic system of order higher than 2, precisely of order $2k-2$ (see [1]), hence, maximum principle and comparison theorems are not available. This means that initially embedded submanifolds can possibly develop self–intersections during the flow. Consequently, looking for more "robust" functionals $\mathcal{G}_k$ in order to deal also with *immersed*–only submanifolds $\varphi : M \to \mathbb{R}^{n+m}$, two problems naturally arise, one is the difference (because of the possible multiplicities) between the Hausdorff measure on the image $\varphi(M)$ and the canonical volume measure μ on (M, g) which can be overcome substituting $\mathcal{H}^n$ with this latter, the second point is the non smoothness of the distance function near the points of self–intersection.
If a compact smooth submanifold is embedded, the square of the distance function from M turns out to be locally smooth so the computation of the derivatives gives no trouble. This is no more true when M is only immersed, hence, in such situation, at any $p \in M$ we will consider the derivatives of the distance function from an *embedded* image $\varphi(U_p)$ of a local neighborhood $U_p \subset M$ of the point p.
Thus, we redefine the functionals as follows

$$\mathcal{G}_k(\varphi) = \int_M 1 + |\nabla^k \eta^M|^2 \, d\mu \,, \tag{1.1}$$

on the space of smooth immersions $\varphi : M \to \mathbb{R}^{n+m}$ of a compact n–dimensional manifold M, where η^M is the square of the distance function from $\varphi(M)$ (keeping in mind the previous discussion at the points of self intersections) and μ is the canonical volume measure associated to the Riemannian manifold (M, g), with $g = \varphi^*\langle \, , \, \rangle_{\mathbb{R}^{n+m}}$.
Then, De Giorgi's conjecture can be restated as follows,

Conjecture 1.1. Any initial n–dimensional smooth submanifold $\varphi : M \to \mathbb{R}^{n+m}$ evolves by the gradient of the functional $\mathcal{G}_k$ without developing singularities in finite time, if $k > n+1$.

We will see that actually the weaker hypothesis $k > [n/2] + 2$, where $[n/2]$ denotes the integer part of $n/2$, is sufficient.

In order to show such conjecture, we work out some properties, of independent interest, about the high derivatives of the square of the distance function from a submanifold, in particular their relation with the covariant derivatives of the second fundamental form. This is the goal of the first part of the paper which can be seen as a continuation of the analysis carried out in [1].
Then, in Section 3 we show that these properties imply a priori estimates on the Sobolev constants of the evolving manifolds, which allow us to follow the method used in [6] to prove the regularity of the flow associated to the functionals

$$\mathcal{F}_k(\varphi) = \int_M 1 + |\nabla^k \nu|^2 \, d\mu \,,$$

where ν is the normal vector field of a hypersurface M in the Euclidean space.

We conclude the paper discussing the subsequent open problem of De Giorgi, again stated in [2] (see also [3]), about the singular approximation of the motion by mean curvature with these smooth higher order flows.

2. The Squared Distance Function from a Manifold

We denote with $e_1, \dots, e_{n+m}$ the canonical basis of $\mathbb{R}^{n+m}$ and with $\langle\, , \,\rangle$ its standard scalar product.

We let $M \subset \mathbb{R}^{n+m}$ be a smooth, compact, n–dimensional, regular submanifold without boundary, then T_xM and $N_xM \subset \mathbb{R}^{n+m}$ are, respectively, the tangent space and the normal space to M at $x \in M \subset \mathbb{R}^{n+m}$.

The *distance function* $d^M(x)$ and the *squared distance function* $\eta^M(x)$ from M are simply given by

$$d^M(x) = \inf_{y \in M} |x - y| \qquad \text{and} \qquad \eta^M(x) = [d^M(x)]^2$$

for any $x \in \mathbb{R}^{n+m}$ (we will drop the superscript M when no ambiguity is possible). In this section we recall some facts from [1] about the distance function and we establish some new relations between the high derivatives of η^M and the second fundamental form of M.

Since M is smooth, embedded and compact, there exists an open neighborhood $\Omega \subset \mathbb{R}^{n+m}$ of M such that d^M is smooth in $\Omega \setminus M$ and η^M is smooth in all Ω.

Clearly, η^M and $\nabla\eta^M(x) = 0$ at every point of $x \in M$, moreover, for every $x \in \Omega$ we have that $x - \nabla\eta^M(x)/2$ is *the unique* point in M of minimum distance from x (the *projection* of x on M), that we denote with $\pi^M(x)$.

Another nice property of the squared distance is that, for every $x \in M$ the Hessian matrix $\nabla^2\eta^M(x)$ is twice the matrix of orthogonal projection onto the normal space N_xM. We will denote respectively with X^M and $X^\perp$ the projections of a vector X on the tangent and normal space of M.

Let $x \in M$ and $X, Y \in T_xM$, the vector valued *second fundamental form* of M at the point x is given by

$$\mathrm{B}(X, Y) = \Big(\frac{\partial Y(x)}{\partial X}\Big)^\perp$$

where we extended locally the two vectors X, Y to tangent vector fields on M (the derivative is well defined since X is a tangent vector at x).

If $\{\nu_\alpha\}_{\alpha=1,\dots,m}$ is a local basis of the normal bundle we have clearly

$$\mathrm{B}(X, Y) = -\sum_{\alpha=1}^{m} \Big\langle \frac{\partial \nu_\alpha(x)}{\partial X}, Y \Big\rangle \nu_\alpha\,.$$

We will see B as a bilinear map from $T_xM \times T_xM$ to $\mathbb{R}^{n+m}$, hence, as a family of $n+m$ bilinear forms $\mathrm{B}^k = \langle \mathrm{B}, e_k\rangle : T_xM \times T_xM \to \mathbb{R}^{n+m}$. Moreover, we consider B acting also on vectors of $\mathbb{R}^{n+m}$, not necessarily tangent, by setting $\mathrm{B}(V, W) = \mathrm{B}(V^M, W^M) \in$

$N_xM \subset \mathbb{R}^{n+m}$ for every pair V, $W \in \mathbb{R}^{n+m}$. With such definition, $\mathrm{B}^k_{ij} = \langle \mathrm{B}(e_i, e_j), e_k\rangle$.
It is well known that B is a symmetric bilinear form and its trace is the *mean curvature* of components $\mathrm{H}^k = \sum_j \mathrm{B}^k_{jj}$.

We introduce now the function

$$A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2},$$

smooth as η^M in the neighborhood Ω of M, and we set

$$A^M_{i_1\dots i_k}(x) = \frac{\partial^k A^M(x)}{\partial x_{i_1}\dots\partial x_{i_k}}$$

for the derivatives of A^M at every point $x \in \Omega$.
The following Proposition (see [1] for the proof) shows the first connection between the second fundamental form and the function A^M (or equivalently, the squared distance function).

Proposition 2.1. *The following relations hold,*

- *For any $x \in \Omega$, the point $\nabla A^M(x)$ is the projection point $\pi^M(x)$.*
- *If $x \in M$, then $\nabla^2 A^M(x)$ is the matrix of orthogonal projection on T_xM.*
- *For every $x \in M$,*

$$\begin{aligned} \mathrm{B}^k_{ij} &= A^M_{ijs}(\delta_{ks} - A^M_{ks}) \\ A^M_{ijk} &= \mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki} \\ \mathrm{H}^k &= \sum_i A^M_{iik} \end{aligned}$$

We define now the k–derivative tensor $A^k(x)$ working on the k–uple of vectors $X_i \in \mathbb{R}^{n+m}$, where $X_i = X^j_i e_j$, as follows

$$A^k(x)(X_1, \dots, X_k) = A^M_{i_1\dots i_k}(x) X^{i_1}_1 \dots X^{i_k}_k,$$

notice that the tensors A^k are symmetric.
By sake of simplicity, we dropped the superscript M on A^k, by the same reason, we will also avoid to indicate the point $x \in M$ in the sequel.

Our goal is to express A^k in terms of covariant derivatives of B.

Proposition 2.2. *For every $k \geq 2$ and for every $s \in \{0, \dots k\}$ there exists a family $p^{k,s}_{j_1\dots j_{k-s}}$ of symmetric polynomial tensors of type $(s, 0)$ on M, where $j_1, \dots, j_{k-s} \in \{1, \dots, n+m\}$, which are contractions of the second fundamental form* B *and its covariant derivatives with the metric tensor g, such that*

$$A^k(X_1, \dots, X_s, N_1, \dots, N_{k-s}) = p^{k,s}_{j_1\dots j_{k-s}}(X_1, \dots, X_s) N^{j_1}_1 \dots N^{j_{k-s}}_{k-s}$$

for every s–uple of tangent vectors X_h *and* $(k-s)$*–uple of normal vectors* N_h *in* $\mathbb{R}^{n+m}$*.*
Moreover, the tensors $p^{k,s}_{j_1\dots j_{k-s}}$ *are invariant by exchange of the j–indices and the maximum order of differentiation of* B *which appears in every* $p^{k,s}_{j_1\dots j_{k-s}}$ *is at most* $k-3$*. Considering the tangent plane at any point* $x\in M$ *also as a subset of* $\mathbb{R}^{n+m}$*, the polynomials tensors* $p^{k,s}_{j_1\dots j_{k-s}}$ *are expressed in the coordinate basis of the Euclidean space as follows*

$$p^{k,s}_{j_1\dots j_{k-s}}(X_1,\dots,X_s)N_1^{j_1}\dots N_{k-s}^{j_{k-s}} = p^{k,s}_{j_1\dots j_{k-s},i_1\dots i_s}X_1^{i_1}\dots X_s^{i_s}N_1^{j_1}\dots N_{k-s}^{j_{k-s}}\,.$$

Then, a family of tensors satisfying the above properties can be defined recursively according to the following formulas

$$p^{2,0}_{j_1j_2} = p^{2,1}_{j_1,i_1} = 0\,,\qquad p^{2,2}_{i_1i_2} = \delta_{i_1i_2} \tag{2.1}$$

$$p^{k,0}_{j_1\dots j_k} = p^{k,1}_{j_1\dots j_{k-1},i_1} = 0 \qquad \textit{for every } k\geq 2 \tag{2.2}$$

$$\begin{aligned} p^{k+1,s}_{j_1\dots j_{k-s+1},i_0i_1\dots i_{s-1}} &= (\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}} \qquad \textit{if } s<k+1\\ &\quad-\sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0}\\ &\quad-\sum_{h=1}^{s-1} p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^{r}_{i_0i_h}\\ &\quad+\sum_{h=1}^{k-s+1} p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0} \end{aligned} \tag{2.3}$$

$$p^{k+1,k+1}_{i_0i_1\dots i_{k+1}} = \nabla p^{k,k}_{i_0i_1\dots i_k} - \sum_{h=1}^{k} p^{k,k-1}_{r,i_1\dots i_{h-1}i_{h+1}\dots i_k}\mathrm{B}^r_{i_0i_h}\,. \tag{2.4}$$

Proof. If $k=2$ we have immediately

$$A^2(N_1,N_2)=0,\qquad A^2(X_1,N_1)=0,\qquad A^2(X_1,X_2)=X_1^iX_2^i=\delta_{i_1i_2}X_1^{i_1}X_2^{i_2}$$

since X_1 and X_2 are tangent and A^2 is the projection on the tangent space. Hence, formula (2.1) follows.
We argue now by induction on $k\geq 2$. When $s=0$ the value $A^k(N_1,\dots,N_k)(x)$ depends only on the function A^M on the m–dimensional normal subspace to M at x, and on this subspace A^M is identically zero, hence the first equality in (2.2) is proved.
Suppose now that $s\in\{1,\dots,k+1\}$, we extend the vectors $X_h\in T_xM$ and $N_h\in N_xM$

to a family of local vector fields, respectively tangent and normal to M, then

$$\begin{aligned}
A^{k+1}(X_0, X_1, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}) &= \frac{\partial}{\partial X_0}\Big(A^k(X_1, \dots, X_{s-1}, N_1, \dots, N_{k-s+1})\Big) \\
&\quad - \sum_{h=1}^{s-1} A^k\Big(X_1, \dots X_{h-1}, \frac{\partial X_h}{\partial X_0}, X_{h+1}, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}\Big) \\
&\quad - \sum_{h=1}^{k-s+1} A^k\Big(X_1, \dots, X_{s-1}, N_1, \dots, \frac{\partial N_h}{\partial X_0}, \dots, N_{k-s+1}\Big)
\end{aligned}$$

where the last line is not present in the special case $s = k+1$ and the second line is not present if $s = 1$. In this last case, we have

$$A^{k+1}(X_0, N_1, \dots, N_k) = \frac{\partial}{\partial X_0}\Big(A^k(N_1, \dots, N_k)\Big) - \sum_{h=1}^{k} A^k\Big(N_1, \dots, \frac{\partial N_h}{\partial X_0}, \dots, N_k\Big) = 0$$

since the first term of the right member is zero by the first equality in (2.2) and, after decomposing $\frac{\partial N_h}{\partial X_0}$ in tangent and normal part, the tangent term is zero by induction and the normal term is zero for (2.2) again. This shows the second equality in (2.2).
So we suppose $1 < s < k+1$, by the inductive hypothesis,

$$A^k(X_1, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}) = p^{k,s-1}_{j_1 \dots j_{k-s+1}}(X_1, \dots, X_{s-1}) N_1^{j_1} \dots N_{k-s+1}^{j_{k-s+1}}$$

thus, differentiating along X_0, which is a tangent field, we obtain

$$\begin{aligned}
A^{k+1}(X_0,&X_1, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}) \\
&= \frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1 \dots j_{k-s+1}}(X_1, \dots, X_{s-1}) N_1^{j_1} \dots N_{k-s+1}^{j_{k-s+1}}\Big) \\
&\quad - \sum_{h=1}^{s-1} A^k\Big(X_1, \dots, \Big(\frac{\partial X_h}{\partial X_0}\Big)^M, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}\Big) \\
&\quad - \sum_{h=1}^{s-1} A^k\Big(X_1, \dots, \Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp, \dots, X_{s-1}, N_1, \dots, N_{k-s+1}\Big) \\
&\quad - \sum_{h=1}^{k-s+1} A^k\Big(X_1, \dots, X_{s-1}, N_1, \dots, \Big(\frac{\partial N_h}{\partial X_0}\Big)^M, \dots, N_{k-s+1}\Big) \\
&\quad - \sum_{h=1}^{k-s+1} A^k\Big(X_1, \dots, X_{s-1}, N_1, \dots, \Big(\frac{\partial N_h}{\partial X_0}\Big)^\perp, \dots, N_{k-s+1}\Big).
\end{aligned}$$

We use now the symmetry of A^k and we substitute recursively $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ to A^k, according to the number of tangent vectors inside A^k,

$$
\begin{aligned}
&A^{k+1}(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&\qquad = \frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})\Big)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad + \sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\frac{\partial N_h^{j_h}}{\partial X_0}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{s-1} p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,\nabla_{X_0}X_h,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{s-1} p^{k,s-2}_{rj_1\cdots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^{\perp}\Big]^{r}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{k-s+1} p^{k,s}_{j_1\cdots j_{h-1}j_{h+1}\cdots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^{M}\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^{\perp}\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}
$$

Adding the first and the third line we get the covariant derivative of $p^{k,s-1}_{j_1\cdots j_{k-s+1}}$, adding the second and the last line we get

$$
\begin{aligned}
&A^{k+1}(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&\qquad = \nabla p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad + \sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^{M}\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{s-1} p^{k,s-2}_{rj_1\cdots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^{\perp}\Big]^{r}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\qquad\quad - \sum_{h=1}^{k-s+1} p^{k,s}_{j_1\cdots j_{h-1}j_{h+1}\cdots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^{M}\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}
$$

Taking now into account that

$$\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^{M}\Big]^{r} = -\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_h^{j_h} \text{ and } \Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^{\perp}\Big]^{r} = \mathrm{B}^{r}_{i_0i_h}X_0^{i_0}X_h^{i_h}$$

we get

$$
\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1}}(X_1,\dots,X_{s-1})\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\mathrm{B}^{r}_{i_0i_h}X_0^{i_0}X_h^{i_h}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1}}(X_1,\dots,X_{s-1},\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}e_r)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}
$$

Then, expressing the tensors in coordinates, we have

$$
\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=(\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}}X_0^{i_0}X_1^{i_1}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^{r}_{i_0i_h}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,,
\end{aligned}
$$

which is formula (2.3).
In the special case $s=k+1$, to get formula (2.4), we just have to repeat the computations dropping all the lines containing sums like $\sum_{h=1}^{k-s+1}\dots$, which are not present.
Finally, assuming inductively that the polynomial tensors $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ are symmetric in the j–indices and contain covariant derivatives of B only up to the order $k-3$, also the claims about the symmetry and the order of the derivatives of B follow.

□

Example 2.3. We compute some $p^{k,s}$ as a consequence of this proposition.

(1) When $k=2$ we saw that

$$
p^{2,0}_{j_1j_2}=0,\qquad p^{2,1}_{j_1}=0,\qquad p^{2,2}=g\,.
$$

(2) When $k = 3$ we have

$$
\begin{aligned}
&p^{3,0}_{j_1 j_2 j_3} = 0, \qquad p^{3,1}_{j_1 j_2} = 0,\\
&p^{3,2}_{j_1, i_1 i_2} = p^{2,2}_{i_2 r}\mathrm{B}^{j_1}_{r i_1} = \mathrm{B}^{j_1}_{i_1 i_2}\\
&p^{3,3}_{i_1 i_2 i_3} = (\nabla p^{2,2})_{i_1 i_2 i_3} + p^{2,1}_{r, i_2}\mathrm{B}^{r}_{i_1 i_3} + p^{2,1}_{r, i_3}\mathrm{B}^{r}_{i_1 i_2} = 0
\end{aligned}
$$

that is,

$$p^{3,2}_{j_1} = \mathrm{B}^{j_1} \text{ and } p^{3,3} = 0\,.$$

(3) When $k = 4$ we have

$$
\begin{aligned}
p^{4,0}_{j_1 j_2 j_3 j_4} &= 0, \qquad p^{4,1}_{j_1 j_2 j_3} = 0,\\
p^{4,2}_{j_1 j_2, i_1 i_2} &= p^{3,2}_{j_1, i_1 r}\mathrm{B}^{j_2}_{r i_2} + p^{3,2}_{j_2, i_1 r}\mathrm{B}^{j_1}_{r i_1} = \mathrm{B}^{j_1}_{i_1 r}\mathrm{B}^{j_2}_{r i_2} + \mathrm{B}^{j_2}_{i_2 r}\mathrm{B}^{j_1}_{r i_1}\,,\\
p^{4,3}_{j_1, i_1 i_2 i_3} &= (\nabla p^{3,2}_{j_1})_{i_1 i_2 i_3} + p^{3,2}_{r, i_2 i_3}\mathrm{B}^{j_1}_{r i_1} = (\nabla p^{3,2}_{j_1})_{i_1 i_2 i_3} + \mathrm{B}^{r}_{i_2 i_3}\mathrm{B}^{j_1}_{r i_1} = (\nabla \mathrm{B}^{j_1})_{i_1 i_2 i_3}
\end{aligned}
$$

since we contracted a normal vector with a tangent one,

$$p^{4,4}_{i_1 i_2 i_3 i_4} = -\,p^{3,2}_{r}{}_{i_3 i_4}\mathrm{B}^{r}_{i_1 i_2} - p^{3,2}_{r}{}_{i_2 i_4}\mathrm{B}^{r}_{i_1 i_3} - p^{3,2}_{r}{}_{i_2 i_3}\mathrm{B}^{r}_{i_1 i_4} = -\mathrm{B}^{r}_{i_3 i_4}\mathrm{B}^{r}_{i_1 i_2} - \mathrm{B}^{r}_{i_2 i_4}\mathrm{B}^{r}_{i_1 i_3} - \mathrm{B}^{r}_{i_2 i_3}\mathrm{B}^{r}_{i_1 i_4}\,.$$

Proposition 2.2 allows us to write A^k in terms of the tensors $p^{k,s}$ and the projections on the tangent and normal spaces (hence contracting with the scalar product of $\mathbb{R}^{n+m}$), so we get the following corollary.

Corollary 2.4. *For every $k \geq 3$ the symmetric tensor A^k can be expressed as a polynomial tensor in* B *and its covariant derivatives, contracted with the scalar product of $\mathbb{R}^{n+m}$.*
The maximum order of differentiation of B *which appears in A^k is $k-3$. Precisely, the only tensors among the $p^{k,s}$ containing such highest derivative are $p^{k,k-1}_{j_1}$, given by*

$$p^{k,k-1}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1} + \mathrm{LOT}\,.$$

where we denoted with LOT (lower order terms) *a polynomial term containing only derivatives of* B *up to the order k-4.*

Proof. Looking at the tensors with the derivative of B of maximum order among the $p^{k,s}_{j_1 \dots j_{k-s}}$, by formula (2.3) and the fact that the only non zero polynomials $p^{3,s}_{j_1 \dots j_{3-s}, i_1 \dots i_s}$ are $p^{3,2}_{j_1, i_1 i_2} = \mathrm{B}^{j_1}_{i_1 i_2}$ (see Example 2.3), it is clear that they come from the derivative $\nabla p^{k-1,k-2}_{j_1}$. Iterating the argument, the leading term in $p^{k,k-1}_{j_1}$ is given by $\nabla^{k-3}p^{3,2}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1}$. $\square$

Remark 2.5. We can see in Example 2.3 that when $k = 3$ and 4, the lower order term which appears above is zero, actually, when $k \geq 5$ this is no more true.

The decomposition of A^k in its tangent and normal components is very useful in studying the norm of A^k, which is the main quantity we are interested in.

Fixing at a point $x \in M$ an orthonormal basis $\{e_1, \dots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$ such that $\{e_1, \dots, e_n\}$ is a basis of T_xM, we have obviously

$$\begin{aligned}
|A^k|^2 &= \sum_{1\leq i_1,\dots,i_k\leq n+m} [A^k(e_{i_1}, \dots, e_{i_k})]^2 \\
&\geq \sum_{\substack{1\leq i_1,i_2\leq n\\ n<i_3,\dots,i_k\leq n+m}} [A^k(e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_k})]^2 \\
&\geq \sum_{n<j\leq n+m}\sum_{1\leq i_1,i_2\leq n} [A^k(e_{i_1}, e_{i_2}, e_j, \dots, e_j)]^2 \\
&= \sum_{n<j\leq n+m}\sum_{1\leq i_1,i_2\leq n} [p^{k,2}_{j\dots j,i_1i_2}]^2\,,
\end{aligned}$$

that is,

$$|A^k|^2 \geq \sum_{n<j\leq n+m} |p^{k,2}_{j\dots j}|^2\,.$$

We analyse this last term by means of formula (2.3). We have $p^{2,2} = g$ and for every $k \geq 2$,

$$p^{k+1,2}_{j\dots j,i_0i_1} = \sum_{h=1}^{k-1} p^{k,2}_{j\dots j,i_1r}\mathrm{B}^j_{ri_0} = (k-1)\, p^{k,2}_{j\dots j,i_1r}\mathrm{B}^j_{ri_0}\,.$$

Then, by induction, it is easy to see that

$$p^{k,2}_{j\dots j,i_0i_1} = (k-2)!\,\mathrm{B}^j_{i_0r_1}\mathrm{B}^j_{r_1r_2}\dots\mathrm{B}^j_{r_{k-3}i_1}$$

hence, as the bilinear form B^j is symmetric, denoting with λ^j_s its eigenvalues at the point $x \in M$, we conclude

$$|p^{k,2}_{j\dots j}|^2 = [(k-2)!]^2\sum_{s=1}^{n}(\lambda^j_s)^{2(k-2)} \geq \widetilde{C}|\mathrm{B}^j|^{2k-4}\,.$$

Coming back to our estimate,

$$|A^k|^2 \geq \widetilde{C}\sum_{n<j\leq n+m}|\mathrm{B}^j|^{2k-4} \geq C\Big(\sum_{n<j\leq n+m}|\mathrm{B}^j|^2\Big)^{k-2} = C|\mathrm{B}|^{2k-4}\,.$$

Proposition 2.6. *The following estimate holds,*

$$|A^k|^2 \geq C|\mathrm{B}|^{2k-4}$$

where C is an universal constant depending only on k, n and m.

3. Flows by Geometric Functionals

The very first step in proving De Giorgi's Conjecture 1.1 is to see that any initial submanifold actually moves smoothly by the gradient of the functional $\mathcal{G}_k$ in (1.1), at least for some small time.

In the paper [1], Theorem 4.5 and Theorem 5.9, it has been shown that the *first variation* of the functional $\mathcal{G}_k$ is given by

$$E_{\mathcal{G}_k} = \Big(-\mathrm{H} + 2k(-1)^{k-1}\overbrace{\Delta^M \circ \Delta^M \circ \ldots \circ \Delta^M}^{(k-2)\text{–times}}\,\mathrm{H}\Big)^{\perp} + h^j(A^M)e_j^{\perp}$$

where the functions $h^j(A^M)$ are polynomials in the derivatives of A^M up to the order $2k-2$.
By means of Corollary 2.4 we can express the terms $h^j(A^M)$ as polynomials $q^j(\mathrm{B})$ obtained contracting B and its covariant derivatives up to the order $2k-5$ with the scalar product of $\mathbb{R}^{n+m}$.
To get a solution of the geometric evolution problem for the initial submanifold $\psi : M \to \mathbb{R}^{n+m}$, we look then for a smooth function $\varphi : M \times [0,T) \to \mathbb{R}^{n+m}$ such that

(1) the map $\varphi_t = \varphi(\cdot, t) : M \to \mathbb{R}^{n+m}$ is an immersion for every $t \in [0,T)$;
(2) $\varphi_0(p) = \varphi(p,0) = \psi(p)$ for every $p \in M$;
(3) the following parabolic system is satisfied

$$\frac{\partial\varphi}{\partial t} = \mathrm{H} + 2k(-1)^k\Big(\overbrace{\Delta^{M_t} \circ \Delta^{M_t} \circ \ldots \circ \Delta^{M_t}}^{(k-2)\text{–times}}\,\mathrm{H}\Big)^{\perp} + q^j(\mathrm{B})e_j^{\perp}\,.$$

Here we denoted with Δ^{M_t} the Laplacian of the Riemannian manifolds $M_t = (M, g_t)$, where g_t is the metric induced on M by the map φ_t.
We say that a solution φ_t is the flow by the gradient of the functional $\mathcal{G}_k$ of the initial submanifold ψ.

By means of a slight extension of Polden's Theorem in [5] (see [4] for details), there exists for some positive time a unique smooth evolution φ_t of any initial smooth submanifold M. Our aim now is to show that under suitable hypotheses, such a flow actually remains smooth for every time.

Theorem 3.1. *If the differentiation order k is strictly larger than $\left[\frac{n}{2}\right]+2$, then the flow by the gradient of $\mathcal{G}_k$ of any initial n–dimensional submanifold is smooth for every positive time.*
Moreover, as $t \to +\infty$, the evolving submanifolds φ_t sub–converges (up to reparametrization and translation) to a smooth critical point of the functional $\mathcal{G}_k$.

Since the flow φ_t is variational, the value of the functional $\mathcal{G}_k$ is monotone non increasing in time, hence it is bounded by its value on the initial submanifold. This implies that, for all the evolving submanifolds,

$$\mathrm{Vol}(M_t) + \int_M |A^k|^2\, d\mu_t \leq C\,.$$

Hence, by means of Proposition 2.6 we get

$$\mathrm{Vol}(M_t) + \int\limits_M |\mathrm{B}|^{2k-4}\, d\mu_t \leq C$$

for a constant C independent of time.
Since when $k > \left[\frac{n}{2}\right] + 2$ we have $2k - 4 \geq n + 1$ we conclude that

$$\mathrm{Vol}(M_t) + \|\mathrm{H}\|_{L^{n+1}(\mu_t)} \leq C \tag{3.1}$$

uniformly in time, for a constant C depending only on the initial submanifold.

By the results of Sections 5 and 6 in [6], the above a priori bound implies the following time–independent interpolation inequalities for functions and tensors on M.

Proposition 3.2. *As long as the flow φ_t exists, for every smooth covariant tensor $T = T_{i_1 \ldots i_l}$ we have the inequalities*

$$\|\nabla^j T\|_{L^p(\mu_t)} \leq\ C\, \|T\|^a_{W^{s,q}(\mu_t)} \|T\|^{1-a}_{L^r(\mu_t)}\,,$$

for all $j \in [0, s]$, $p, q, r \in [1, +\infty)$ and $a \in [j/s, 1]$ with the compatibility condition

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{q} - \frac{s}{n}\right) + \frac{1-a}{r}\,.$$

If such condition gives a negative value for p, the inequality holds for every $p \in [1, +\infty)$ on the left side.
The constant C depends on the dimensions, the orders of differentiation, the exponents of the involved norms and the value of $\mathcal{G}_k$ on the initial submanifold, but it is independent of time.

Another consequence of inequality (3.1) is an uniform lower bound on the *Volume* of M_t (see the end of Section 5 in [6]), thus

$$0 < c \leq \mathrm{Vol}(M_t) \leq C < +\infty\,,$$

again with a couple of constants c and C independent of time.

These a priori estimates allow us to forget the "geometry" of the evolving submanifolds which, as it is well known, influences the Sobolev constants (hence also the ones involved in the interpolation inequalities). Thus, we can proceed with the estimates on the relevant quantities that are the L^2 norms of the second fundamental form and its derivatives.

From this point on, the rest of the proof follows step by step Sections 7 and 8 in [6], so we only give a sketch, referring to such paper for the details.

Differentiating in time the L^2 integrals $\int_M |\nabla^s \mathrm{B}|^2\, d\mu_t$ during the flow, after some computations, one gets

$$\frac{d}{dt}\int_M |\nabla^s \mathrm{B}|^2\, d\mu_t \leq -3\int_M |\nabla^{s+k-1}\mathrm{B}|^2\, d\mu_t + \int_M |q_{2(s+k)}(\mathrm{B})| + |q_{2(s+2)}(\mathrm{B})|\, d\mu_t$$

where $q_{2(s+k)}(\mathrm{B})$ is a polynomial term not containing derivatives of B of order higher than $s+k-2$, such that any of its monomials $Q_j(\mathrm{B})$ has the following structure,

$$Q_j(\mathrm{B}) = \nabla^{i_1}\mathrm{B} * \cdots * \nabla^{i_N}\mathrm{B}\,,$$

where the symbol $*$ means contraction with the scalar product of $\mathbb{R}^{n+m}$, and $2(s+k) = \sum_{l=1}^{N}(i_l+1)$; the term $q_{2(s+2)}(\mathrm{B})$ is analogous (see [6] for more details).
Then, we can estimate them as follows

$$|Q_j| \leq \prod_{i=0}^{s+k-2} |\nabla^i \mathrm{B}|^{\alpha_{ji}} \qquad \text{with} \qquad \sum_{i=0}^{s+k-2} \alpha_{ji}(i+1) = 2(s+k)\,.$$

Then,

$$\begin{aligned}
\int_M |Q_j|\,d\mu_t &\leq \int_M \prod_{i=0}^{s+k-2} |\nabla^i \mathrm{B}|^{\alpha_{ji}}\,d\mu_t \\
&\leq \prod_{i=0}^{s+k-2} \left(\int_M |\nabla^i \mathrm{B}|^{\alpha_{ji}\gamma_i}\,d\mu_t\right)^{\frac{1}{\gamma_i}} \\
&= \prod_{i=0}^{s+k-2} \|\nabla^i \mathrm{B}\|^{\alpha_{ij}}_{L^{\alpha_{ji}\gamma_i}(\mu_t)}
\end{aligned}$$

where the γ_i are arbitrary positive values such that $\sum 1/\gamma_i = 1$.
Interpolating now every factor of this product between $\|\mathrm{B}\|_{W^{s+k-1,2}(\mu_t)}$ and $\|\mathrm{B}\|_{L^{n+1}(\mu_t)}$ to some powers, by means of Proposition 3.2, it is possible to conclude

$$\int_M |Q_j|\,d\mu_t \leq \varepsilon_j \int_M |\nabla^{s+k-1}\mathrm{B}|^2\,d\mu_t + C$$

for arbitrarily small constants $\varepsilon_j > 0$. Repeating this argument for all the monomials Q_j and choosing suitable ε_j whose sum is less than one,

$$\frac{d}{dt}\int_M |\nabla^s \mathrm{B}|^2\,d\mu_t \;\leq\; -2\int_M |\nabla^{s+k-1}\mathrm{B}|^2\,d\mu_t + C + \int_M |q_{2(s+2)}(\mathrm{B})|\,d\mu_t$$

with a constant C independent of time.
Dealing analogously with the other polynomial term $q_{2(s+2)}(\mathrm{B})$, we finally get

$$\frac{d}{dt}\int_M |\nabla^s \mathrm{B}|^2\,d\mu_t \;\leq\; -\int_M |\nabla^{s+k-1}\mathrm{B}|^2\,d\mu_t + C \leq\; -C\int_M |\nabla^s \mathrm{B}|^2\,d\mu_t + C\,,$$

where in the second passage we used Poincarè inequality, which also follows by Proposition 3.2.
Then, a simple ODE argument gives

$$\int_M |\nabla^s \mathrm{B}|^2\,d\mu_t \leq C_s$$

for every $s \in \mathbb{N}$, with some constants C_s dependent only on the initial submanifold. Again, via Proposition 3.2, one can now pass from this family of Sobolev estimates to time–independent pointwise bounds on all the covariant derivatives of B.

Once we got these latter, the smoothness for every positive time and the sub–convergence follow by standard arguments about geometric flows.

We underline the two key points where the properties of the distance function play a role. First, when the order k is large enough, the estimate $|A^k| \geq C_k |\mathrm{B}|^{2k-4}$ implies the a priori estimates (3.1) leading to the geometry–independent interpolation inequalities of Proposition 3.2.

Secondly, the structure of $|A^k|^2$, (in particular the leading term, once expressed in terms of the second fundamental form) produces a nice first variation (computed in [1]) giving rise to a well behaved parabolic problem.

We conclude the paper mentioning the subsequent open problem suggested by Ennio De Giorgi in the same paper [2] (Osservazione 2 and Congettura 3, Pag. 267).

If $k > [n/2] + 2$ it is easy to see, by the same proof, that for every $\varepsilon > 0$ all the flows φ_t^ε, associated to the functionals

$$\mathcal{G}_k^\varepsilon = \int_M 1 + \varepsilon |\nabla^k \eta^M|^2 \, d\mu \,,$$

of a common initial n–dimensional submanifold, are smooth for every positive time. Then, a natural question is the following:

When the parameter ε goes to zero, the flows φ_t^ε converge, in some sense, to the flow associated to the limit functional which is simply the Volume *functional?*
That is, do they converge to the motion by mean curvature *of the initial submanifold?*

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