NEW WEIGHTED INEQUALITIES ON TWO–MANIFOLDS

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ABSTRACT. We establish a new class of L^2-weighted elliptic estimates on smooth two-manifolds for a family of weights satisfying an equation with explicit constants. This family includes weights that are comparable to the product of positive powers of the geodesic distance to a given collection of points. Our primary motivation is to derive estimates related to a weighted Hodge decomposition for one-forms.

1. INTRODUCTION

1.1. **Motivation.** This article is primarily motivated by the *weighted Hodge decomposition* of one-forms on a Riemannian two-manifold M^2 (Lemma 1.1) with boundary. Some weights under consideration take the following form:

$$\omega(x) \sim \Pi_{k=1}^{N} d_M(x, x_k)^{\alpha_k} .$$

Here $\{x_k\}_{k=1}^N \subset M^2$ is a collection of points, and $d_M(x,y)$ is the geodesic distance on M^2 with $\alpha_k > 0$.

This article, along with [5, 4], forms the first part of a trilogy and establishes the main analytical tools and estimates used in the proof of quantitative stability for Yang–Mills–Higgs instantons in [5]. We present these results in a more general setting, as we believe they may be useful in other contexts.

To motivate our approach, we first extend the classical Hodge decomposition to the weighted setting. The standard Hodge decomposition of a one-form A seeks the nearest closed (or co-closed) form to A in a variational sense. In the weighted case, consider the toy model with weight $|x|^2$ and the following variational problem:

$$\inf_{\phi \in C_c^\infty(M^2)} \int_{B_1^2} |x|^2 |A - \star d\phi|^2 , \tag{1.1}$$

A natural function space for this problem is:

$$X = \{\phi \in C_c^\infty(B_1^2) : \int_{B_1^2} |x|^2 |d\phi|^2 < \infty\} ,$$

equipped with a weighted inner product:

$$\langle \phi_1, \phi_2 \rangle_X := \int_{B_1^2} |x|^2 \langle d\phi_1, d\phi_2 \rangle .$$

Taking the completion $\overline{X}$ under the induced norm gives a natural framework for solving (1.1) via the direct method in the calculus of variations.

A key result, following from a special case of the Caffarelli-Kohn-Nirenberg (CKN) interpolation inequalities [2], states that:

$$\forall f \in C_c^\infty(\mathbb{R}^2) : \quad \int_{\mathbb{R}^2} |f|^2 \leq \int_{\mathbb{R}^2} |x|^2 |df|^2 ,$$

As shown in [3], these inequalities admit a geometric interpretation under the log-polar transformation

$$B_1^2 \ni x \rightsquigarrow (-\log(|x|), \theta) \in [0, \infty) \times S^1 .$$

with $u = |x| f$ which transforms the weighted term into a Sobolev norm on the infinite cylinder:

$$\int_{\mathbb{R}^2} |x|^2 |df|^2 = \int_{S^1 \times [0,\infty)} |u|^2 + |du|^2 \, d\mathrm{vol}_{S^1 \times [0,\infty)} .$$

Using weak lower semicontinuity in Sobolev spaces—either on the cylinder or directly via CKN inequalities—we obtain a minimizer ϕ of (1.1). The associated Euler–Lagrange equation is

$$d(|x|^2 (A - \star d\phi)) = 0 ,$$

implying that $A - \star d\phi$ is closed. Since it has zero trace, it follows that:

$$|x| A = |x| \star d\phi + |x|^{-1} d\xi .$$

for some compactly supported function ξ. This is the *weighted Hodge decomposition*, which satisfies the following orthogonality relation:

$$\int_{B_1^2} |x|^2 |A|^2 = \int_{B_1^2} |x|^2 |d\phi|^2 + |x|^{-2} |d\xi|^2 .$$

Comparing the standard Hodge decomposition $A = \star dp + dq$ with the weighted one, we obtain the key estimate (among other results):

$$\int_{B_1^2} |x|^{2+2\varepsilon} |d(\phi - p)|^2 \leq C \varepsilon^{-2} \int_{B_1^2} |x|^{-2} |d\xi|^2 .$$

In fact we show that the co-closed part of the weighted and the standard decomposition are L^2-close.

1.2. **General formulation and examples of weights.** In this article, we extend this heuristic to all weights satisfying a weak formulation (Definition 2.1) of the differential equation

$$\omega^2 \Delta_g \log(\omega) = -\kappa(x)\omega^2 \,, \tag{1.2}$$

where ω is a positive weight in $W^{1,2}(M^2)$, and Δ_g is the Laplace–Beltrami operator on a smoothm, connected Riemannian two–manifold (M^2, g) (with boundary).

This formulation allows us to handle weights that vanish at multiple points, with the advantage that the proofs rely on careful but elementary integration by parts, yielding uniform constants.

We mention a few examples:

- For any bounded open subset $\Omega \subset \mathbb{R}^2$ and a weight ω as follows:

$$\omega(x) = \Pi_{i=1}^N |x - x_i|^{\alpha_i} \text{ for } x_1, \dots, x_N \in \Omega \subset \mathbb{R}^2 \text{ and } \alpha_1, \dots, \alpha_N > 0\,. \tag{1.3}$$

- For any bounded open domain $\Omega \subset \mathcal{M}^2$ of a smooth two manifold, let $\mathcal{G}_p$ be the green's function for Ω centered on p and ω as follows:

$$\omega(x) = \Pi_{i=1}^N e^{-\alpha_i \mathcal{G}_{p_i}(x)} \text{ for } p_1, \dots, p_N \in \Omega \subset \mathcal{M}^2 \text{ and } \alpha_1, \dots, \alpha_N > 0\,. \tag{1.4}$$

Note that the weights of (1.4) are comparable:

$$\mathbb{C}^{-1} Pi_{i=1}^N d_M(x, x_i)^{p_i} \le \omega(x) \le C Pi_{i=1}^N d_M(x, x_i)^{p_i}\,,$$

where $d_M(x, y)$ is the geodesic distance on M between x, y.

Our results improve upon Caffarelli-Kohn-Nirenberg inequalities [2] in two dimensions by proving estimates for a broader class of weights, including those vanishing at multiple points. Notably, these weights do not belong to any Muckenhoupt class but instead resemble Carleman-type estimates in a different regime [1]. A similar strategy has been explored in the radial case in [6].

1.3. **Main results.** Let $\Omega \subset \mathcal{M}^2$ be a smooth open connected domain and let λ_1 be the first Dirichlet eigenvalue of the Laplace-Beltrami operator on Ω.

The central result is the following estimate regarding the weighted Hodge decomposition:

Lemma 1.1. Let $(\mathcal{M}^2, g)$ be a Riemannian two–manifold and let $\Omega \in \mathcal{M}^2$ be a smooth open domain and ω is a weight as in Definition 2.1 with $\kappa = 0$.

Any smooth one-form $A \in C_c^\infty(\bigwedge^1 \Omega)$ has a Hodge decomposition and a *weighted Hodge decomposition* as follows:

$$A = \star d\xi_1 + d\xi_2 \text{ and } \omega A = \star \omega d\phi_1 + \omega^{-1} d\phi_2\,,$$

for 4 compactly supported functions $\xi_1, \xi_2, \phi_1, \phi_2$. Moreover for any $0 \le \varepsilon \le C$ we have the estimates:

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\Omega)} \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \|\omega^{-1} d\phi_2\|^2_{L^2(\Omega)}\,.$$

This estimates is a corollary of the following inequalities:

First we provide a generalization of Caffarelli-Kohn-Nirenberg interpolation inequalities in two dimensions:

Theorem 1.2. Let $(\mathcal{M}^2, g)$ be a smooth two–manifold and a weight ω as in Definition 2.1 and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega |\nabla \omega|^2 |f|^2 d\mathrm{vol}_g \le \int_\Omega \omega^2 |\nabla f|^2 d\mathrm{vol}_g\,, \tag{1.5}$$

provided that $\kappa \le \lambda_1$.

In the next theorem we provide a homogeneous elliptic estimate:

Theorem 1.3. Let $(\mathcal{M}^2, g)$ be a smooth two–manifold and a weight ω as in Definition 2.1 and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^2 |\nabla f|^2 d\mathrm{vol}_g \le \tau^{-1} \int_\Omega 2\frac{\omega^4}{|\nabla \omega|^2} |\Delta_g f|^2 + 5|\nabla\omega|^2 |f|^2 d\mathrm{vol}_g\,, \tag{1.6}$$

provided that $-\frac{\lambda_1}{8}(2-\tau) \le \kappa \le \lambda_1$ for some $0 \le \tau \le 2$.

Theorem 1.4 is the main ingredient used in the proof of the Lemma 1.1 on the weighted Hodge decomposition. We break the homogeneity to remove the term $|\nabla\omega| f$ from the right hand side, thereby introducing a constant on the right hand side as follows:

Theorem 1.4. Let $(\mathcal{M}^2, g)$ be a smooth two–manifold and a weight ω in Definition 2.1 with $\kappa = 0$ and $\varepsilon \ge 0$ and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^{2+2\varepsilon} |\nabla f|^2 d\mathrm{vol}_g \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \int_\Omega \frac{\omega^4}{|\nabla\omega|^2} |\Delta_g f|^2 d\mathrm{vol}_g\,, \tag{1.7}$$

with the bound $C \le \frac{8\varepsilon^2 + 5(1+\varepsilon)^4}{8(1+\varepsilon)^2}$ which is comparable to $\frac{5}{8}$ as $\varepsilon \to 0$.

Note that the Laplace-Beltrami operator Δ_g on functions $u \in W^{1,2}(\mathcal{M}, g)$ is defined by the duality relation below:

$$\int_\Omega -\Delta_g uv \, d\mathrm{vol}_g = \int_\Omega \langle \nabla u, \nabla v \rangle \, d\mathrm{vol}_g \,, \text{ for all } v \in W_0^{1,2}(\Omega).$$

In a special case, Theorem 1.2 and 1.4 provide weighted elliptic estimates for the weight $\omega = |x|^\alpha$:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha-1)} |f|^2 \le \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2\alpha} |\nabla f|^2 \,,$$

$$\int_{\mathbb{R}^2} |x|^{2\alpha} |\nabla f|^2 \le \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2(\alpha+2)} |\Delta f|^2 + \alpha^2 \int_{\mathbb{R}^2} \frac{5}{2} |x|^{2(\alpha-1)} |f|^2 \,,$$

$$\int_{B_1} |x|^{2(\alpha+\varepsilon)} |\nabla f|^2 \le C(\varepsilon\alpha)^{-2} \int_{B_1} |x|^{2(\alpha+1)} |\Delta f|^2 \,,$$

provided that $\alpha > 0$.

The methods throughout the paper are inspired by [2] and [3] and are quiet elementary and only use Stokes theorem. A crucial part of our proof, equation (2.5), uses Lemma 2.2 which is an identity about symmetric matrices in two dimensions which does not hold in other dimensions.

Remark. In the case of unbounded domains (e.g. $\mathcal{M}^2 = \mathbb{R}^2$) we set $\lambda_1 = 0$ in Theorem 1.2 and 1.3.

Remark. Theorem 1.2 and 1.3 also work for the case of closed two–manifolds $\Omega = \mathcal{M}^2$ with the assumption that $\int_\Omega \omega f \, d\mathrm{vol}_g = 0$. However Theorem 1.4 is a trivial statement for closed manifolds since $\kappa = 0$, meaning:

$$\Delta_g \omega^2 = 4|d\omega|^2 \ge 0 \,,$$

which are just constants on close manifolds.

2. The Proof

Definition 2.1. The weak formulation of (1.2) for a weight $\omega \in W^{1,2}(\mathcal{M}^2)$ is as follows: For any smooth test function $\phi \in C_c^\infty(\mathcal{M}^2)$ we have that:

$$\int_\Omega (4|\nabla\omega|^2 - 2\kappa\omega^2)\phi - \omega^2 \Delta_g \phi \, d\mathrm{vol}_g = 0 \,.$$

To prove Theorem 1.2 to 1.4 we use Stokes theorem to relate the integral of a carefully chosen positive term, to the difference of the right and the left hand side of (1.5) to (1.7).

Proof of Theorem 1.2. We begin with the identity below:

$$0 \leq \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g = \int_\Omega |\omega \nabla f + \nabla\omega f|^2 \, d\mathrm{vol}_g$$
$$= \int_\Omega \omega^2 |\nabla f|^2 + |\nabla\omega|^2 |f|^2 + 2\langle \omega\nabla\omega, \nabla f f\rangle \, d\mathrm{vol}_g \,.$$

After completing the derivative for the cross term and using Definition 2.1 we see that:

$$\int_\Omega 2\langle \omega\nabla\omega, \nabla f f\rangle \, d\mathrm{vol}_g = \int_\Omega -\frac{\omega^2}{2}\Delta_g(f^2) \, d\mathrm{vol}_g = \int_\Omega (\kappa\omega^2 - 2|\nabla\omega|^2)|f|^2 \, d\mathrm{vol}_g \,.$$

Then we use $\kappa \leq \lambda_1$ to estimate:

$$\int_\Omega \kappa\omega^2 |f|^2 \, d\mathrm{vol}_g \leq \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \,.$$

Finally we conclude that:

$$0 \leq \int_\Omega \omega^2 |\nabla f|^2 - |\nabla\omega|^2 |f|^2 \, d\mathrm{vol}_g \,.$$

$\square$

Proof of Theorem 1.3. Similarly we begin by integrating a positive term:

$$0 \leq \int_\Omega \Big|\frac{\omega^2}{|\nabla\omega|}\Delta_g f + |\nabla\omega| f\Big|^2 \, d\mathrm{vol}_g = \int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 2\omega^2 f \Delta_g f + |\nabla\omega|^2 |f|^2 \, d\mathrm{vol}_g \,.$$

By Stokes theorem for the cross term and Definition 2.1 we get that:

$$\int_\Omega 2\omega^2 f \Delta_g f \, d\mathrm{vol}_g = \int_\Omega -2\omega^2 |\nabla f|^2 + (4|\nabla\omega|^2 - 2\kappa\omega^2)|f|^2 \, d\mathrm{vol}_g \,.$$

Since the assumption for an unbounded domain is $\kappa = 0$ the proof follows immediately. Otherwise by the assumption $-\kappa \leq \lambda_1(\frac{1}{4} - \frac{\tau}{8})$ we see that:

$$\int_\Omega -2\kappa |\omega f|^2 \, d\mathrm{vol}_g \leq \lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\omega f|^2 \, d\mathrm{vol}_g \,.$$

By the characterization of the first eigenvalue of the Laplace-Beltrami operator Δ_g we see that:

$$\lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega \omega^2 |f|^2 \, d\mathrm{vol}_g \leq (\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \,.$$

Since $\kappa \leq \lambda_1$, Theorem 1.2 applies and we get that:

$$(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\nabla(\omega f)|^2 \, d\mathrm{vol}_g \leq (2 - \tau) \int_\Omega \omega^2 |\nabla f|^2 \, d\mathrm{vol}_g \,.$$

Finally putting the estimates together, we conclude that:

$$0 \leq \int_\Omega 2\frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 5|\nabla\omega|^2|f|^2 - \tau\omega^2|\nabla f|^2 d\mathrm{vol}_g \,.$$

□

In the proof of Theorem 1.4 we deal with the weighted hessian matrix $\omega^2\nabla^2\log(\omega)$ and by the condition (1.2) we know that it is a two dimensional symmetric trace-free matrix. The following lemma uses this structure and it is essential in the proof of Theorem 1.4:

Lemma 2.2. Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, namely $A^T = A$. Then we have that for any two real vectors $b, c \in \mathbb{R}^2$:

$$2\langle A : b\otimes c\rangle\langle b, c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2 = \mathrm{trace}(A)\langle b, c^\perp\rangle^2\,, \tag{2.1}$$

where $\langle : \rangle$ is the matrix element-wise inner product and $c^\perp$ is the perpendicular vector to c.

Proof. We first calculate the expression above in dimension n. Since A is symmetric, it has n distinct perpendicular eigen-vectors e_i with real eigenvalues μ_i. Then setting $b_i = \langle b, e_i\rangle$ and $c_i = \langle c, e_i\rangle$ we compute:

$$\begin{aligned}&2\langle A : b\otimes c\rangle\langle b, c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2\\ &= \sum_{1\le i,j\le n} \mu_i (a_i c_j - c_i a_j)^2\,.\end{aligned}$$

In the case $n = 2$:

$$2\langle A : b\otimes c\rangle\langle b, c\rangle - \langle A : b\otimes b\rangle|c|^2 - (A : c\otimes c)|b|^2 = \mathrm{trace}(A)(b_1c_2 - c_1b_2)^2\,.$$

□

Proof of Theorem 1.4. First we integrate a carefully chosen positive term of the form below:

$$\begin{aligned}0 &\le \int_\Omega \Big|\frac{\omega^2}{|\nabla\omega|}\Delta_g f + 2\omega\langle\frac{\nabla\omega}{|\nabla\omega|}, \nabla f\rangle + 2|\nabla\omega| f\Big|^2 \, d\mathrm{vol}_g\\ &= \int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 + 4\omega^2\langle\frac{d\omega}{|\nabla\omega|}, \nabla f\rangle^2 + 4|\nabla\omega|^2|f|^2 \qquad (2.2)\\ &\quad + 4\frac{\omega^3}{|\nabla\omega|^2}\langle\nabla\omega, \nabla f\rangle\Delta_g f + 4\omega^2\Delta_g f f + 8\langle\omega\nabla\omega, f\nabla f\rangle \, d\mathrm{vol}_g\,. \qquad (2.3)\end{aligned}$$

Then for the first cross term in (2.3) we calculate by Stokes theorem and (1.2) (with the weak formulation in Definition 2.1) and the assumption $\kappa \ge 0$

that:

$$\begin{aligned}
&\int_\Omega 4\frac{\omega^3}{|\nabla\omega|^2}\langle\nabla\omega,\nabla f\rangle\Delta_g f\ d\mathrm{vol}_g\\
&=\int_\Omega 2\mathrm{div}_g(\frac{\omega^3}{|\nabla\omega|^2}d\omega)|\nabla f|^2-4\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega):\nabla f\otimes\nabla f\ d\mathrm{vol}_g\\
&=\int_\Omega(4\omega^2-4\frac{\omega^4}{|\nabla\omega|^4}\nabla^2(\log(\omega)):\nabla\omega\otimes\nabla\omega)|\nabla f|^2 \qquad (2.4)\\
&\quad-4\langle\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega):\nabla f\otimes\nabla f\rangle\ d\mathrm{vol}_g\,.
\end{aligned}$$

The last line follows from:

$$\begin{aligned}
2\mathrm{div}_g(\frac{\omega^3}{|\nabla\omega|^2}d\omega)&=2\frac{\omega^3}{|\nabla\omega|^2}\Delta\omega+6\omega^2-4\frac{\omega^3}{|\nabla\omega|^4}\nabla^2\omega:\nabla\omega\otimes\nabla\omega\\
&=4\omega^2+2\frac{\omega^4}{|\nabla\omega|^2}\Delta(\log(\omega))-4\frac{\omega^4}{|\nabla\omega|^4}\langle\nabla^2\log(\omega):\nabla\omega\otimes\nabla\omega\rangle\\
&=4\omega^2-4\frac{\omega^4}{|\nabla\omega|^4}\langle\nabla^2\log(\omega):\nabla\omega\otimes\nabla\omega\rangle\,.
\end{aligned}$$

Here we used the following identity:

$$\omega\nabla^2\omega=\omega^2\nabla^2\log(\omega)+\nabla\omega\otimes\nabla\omega\,,$$

for the second and third term in (2.4). We get that:

(2.5)

$$\begin{aligned}
&-4\langle\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega):\nabla f\otimes\nabla f\rangle-4\frac{\omega^4}{|\nabla\omega|^4}\langle\nabla^2\log(\omega):\nabla\omega\otimes\nabla\omega\rangle|\nabla f|^2\\
&=-8\omega^2\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2+4\frac{\omega^4}{|\nabla\omega|^4}\big[2\langle\nabla^2\log(\omega):\nabla\omega\otimes\nabla f\rangle\langle\nabla\omega,\nabla f\rangle\\
&-\langle\nabla^2\log(\omega):\nabla f\otimes\nabla f\rangle|\nabla\omega|^2-\langle\nabla^2\log(\omega):\nabla\omega\otimes\nabla\omega\rangle|\nabla f|^2\big]\,.
\end{aligned}$$

We apply Lemma 2.2 with:

$$A=\omega^2\nabla^2\log(\omega)\,,\quad b=\frac{\nabla\omega}{|\nabla\omega|}\quad\text{and } c=\nabla f\,,$$

and $\mathrm{trace}(A)=\omega^2\Delta_g\log(\omega)=0$ to see that:

$$\begin{aligned}
-4\langle\nabla(\frac{\omega^3}{|\nabla\omega|^2}\nabla\omega):\nabla f\otimes\nabla f\rangle-4\frac{\omega^4}{|\nabla\omega|^4}\langle\nabla^2(\log(\omega)):\nabla\omega\otimes\nabla\omega\rangle|\nabla f|^2&\\
=-8\omega^2\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2\,.&
\end{aligned}$$

For the second and third cross term in (2.3) we see that:

$$\int_\Omega 4\omega^2\Delta_g f f+8\langle\omega\nabla\omega,f\nabla f\rangle\ d\mathrm{vol}_g=\int_\Omega-4\omega^2|\nabla f|^2\ d\mathrm{vol}_g\,.$$

Then putting the estimates together we see that:

$$4\int_\Omega \omega^2\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - |\nabla\omega|^2|f|^2\,d\mathrm{vol}_g \le \int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2\,d\mathrm{vol}_g\,. \tag{2.6}$$

Using (1.2) with $\kappa = 0$ we get that for (2.6):

$$\begin{aligned}
&\int_\Omega \omega^2\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - |\nabla\omega|^2|f|^2\,d\mathrm{vol}_g\\
=&\int_\Omega |\omega\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega|f|^2\,d\mathrm{vol}_g\\
\ge&(\sup_\Omega\omega)^{-2\varepsilon}\int_\Omega \omega^{2\varepsilon}|\omega\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega|f|^2\,d\mathrm{vol}_g\,. &(2.7)
\end{aligned}$$

Notice that $\omega^{1+\varepsilon}$ also satisfies (1.2) weakly in the case of $\kappa = 0$, so we compute (2.7) as follows:

$$\begin{aligned}
&\int_\Omega \omega^{2\varepsilon}|\omega\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle + |\nabla\omega|f|^2\,d\mathrm{vol}_g\\
=&\int_\Omega \omega^{2+2\varepsilon}\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 + \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 + 2\omega^{1+2\varepsilon}\langle\nabla\omega,\nabla f\rangle f\,d\mathrm{vol}_g\\
=&\int_\Omega \omega^{2+2\varepsilon}\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 + \omega^{2\varepsilon}|\nabla\omega|^2|f|^2 - \Delta_g(\frac{\omega^{2+2\varepsilon}}{2+2\varepsilon})|f|^2\,d\mathrm{vol}_g\\
=&\int_\Omega \langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|\nabla\omega|^2|f|^2\,d\mathrm{vol}_g\,. &(2.8)
\end{aligned}$$

Notice that for $\omega^{1+\varepsilon}$ we have:

$$\begin{aligned}
0\le&\int_\Omega \omega^{2\varepsilon}|\omega\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle + (1+\varepsilon)|\nabla\omega|f|^2\,d\mathrm{vol}_g\\
=&\int_\Omega \omega^{2+2\varepsilon}\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle + (1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2\\
&\qquad + 2(1+\varepsilon)\omega^{1+2\varepsilon}\langle\nabla\omega,\nabla f\rangle f\,d\mathrm{vol}_g\\
=&\int_\Omega \omega^{2+2\varepsilon}\langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle - (1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2\,d\mathrm{vol}_g\,.
\end{aligned}$$

We expand the square $(1+\varepsilon)^2$ to get a lower bound for (2.8):

$$\int_\Omega \langle\nabla f,\frac{\nabla\omega}{|\nabla\omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|\nabla\omega|^2|f|^2\,d\mathrm{vol}_g \ge \varepsilon^2\int_\Omega \omega^{2\varepsilon}|\nabla\omega|^2|f|^2\,d\mathrm{vol}_g\,.$$

and we get a preliminary inequality as follows:

$$\int_\Omega \omega^{2\varepsilon}|\nabla\omega|^2|f|^2\,d\mathrm{vol}_g \le \frac{(\sup_\Omega\omega)^{2\varepsilon}}{4\varepsilon^2}\int_\Omega \frac{|\omega|^4}{|\nabla\omega|^2}|\Delta_g f|^2\,d\mathrm{vol}_g\,. \tag{2.9}$$

Then we use Theorem 1.3 for $\omega^{1+\varepsilon}$ and $\kappa=0$ and $\tau=2$ to see that:

$$\int_\Omega 2\omega^{2+2\varepsilon}|\nabla f|^2 \, d\mathrm{vol}_g \leq \int_\Omega 2\frac{\omega^{4+2\varepsilon}}{(1+\varepsilon)^2|\nabla\omega|^2}|\Delta_g f|^2 + 5(1+\varepsilon)^2\omega^{2\varepsilon}|\nabla\omega|^2|f|^2 \, d\mathrm{vol}_g \,.$$

Finally we use (2.9) to conclude that:

$$\int_\Omega \omega^{2+2\varepsilon}|\nabla f|^2 \, d\mathrm{vol}_g \leq \Big(\frac{8\varepsilon^2+5(1+\varepsilon)^4}{8(1+\varepsilon)^2}\Big)\frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2}\int_\Omega \frac{\omega^4}{|\nabla\omega|^2}|\Delta_g f|^2 \, d\mathrm{vol}_g \,.$$

□

Remark 2.3. In the case of $\mathcal{M}^2 = B_1^2(0) \subset \mathbb{R}^2$ and $\omega = |x|$ after the log-polar transformation $B_1^2 \to \mathbb{R}^+ \times S^1 = \mathcal{C}$ by the map $t = -\log(|x|)$ and $\theta = \arctan(\frac{y}{x})$ or equivalently a conformal change of metric with the factor $\frac{1}{|x|^2}$ and defining $f = |x|^{-1}u$ for $f \in C_1^\infty(B_1^2(0))$ we can see that:

$$\int_{B_1^2(0)} |\nabla\omega|^2|f|^2 = \int_{\mathcal{C}} |u|^2 d\mathrm{vol}_{\mathcal{C}}\,, \tag{2.10}$$

$$\int_{B_1^2(0)} \omega^2|\nabla f|^2 = \int_{\mathcal{C}} |\nabla u|^2 + |u|^2 d\mathrm{vol}_{\mathcal{C}}\,,$$

$$\int_{B_1^2(0)} \frac{\omega^4}{|\nabla\omega|^2}|\nabla f|^2 = \int_{\mathcal{C}} |\Delta u + 2\partial_t u + u|^2 d\mathrm{vol}_{\mathcal{C}}\,. \tag{2.11}$$

After squaring and integrating by parts we see that (2.11) becomes:

$$\int_{B_1^2(0)} \frac{\omega^4}{|\nabla\omega|^2}|\nabla f|^2 = \int_{\mathcal{C}} |\partial_{tt}u|^2 + |\partial_{t\theta}u|^2 + 2|\partial_t u|^2 + |\partial_{\theta\theta}u + u|^2 \,.$$

We can see that if $u(t,\theta) = \sin(\theta)$ then (2.11) vanishes however (2.10) does not vanish so the term $|\nabla\omega| f$ on the right hand side of (1.6) is necessary. However the extra ε in the power

$$\int_{B_1^2(0)} \omega^{2+2\varepsilon}|\nabla f|^2 = \int_{\mathcal{C}} (|\nabla u|^2 + |u|^2)e^{-2\varepsilon t} d\mathrm{vol}_{\mathcal{C}}\,,$$

compactifies the domain $\mathbb{R}^+ \times S^1$ with a total measure of ε^{-2}. This provides some insight on Theorem 1.4 and the constants in (1.7).

We conclude the paper with the proof of the weighted Hodge decomposition estimates:

Proof of Lemma 1.1. We consider the two variational problems below:

$$\inf_{\xi \in C_c^\infty(\Omega)} \int_\Omega |A - \star d\xi|^2 \, d\mathrm{vol}_g \text{ and } \inf_{\phi \in C_c^\infty(\Omega)} \int_\Omega \omega^2|A - \star d\phi|^2 \, d\mathrm{vol}_g \,. \tag{2.12}$$

Let $W_0^{1,2}(\omega^2,\Omega)$ be the completion of $C_c^\infty(\Omega)$ under the ω^2-weighted norm

$$\|u\|_{W_0^{1,2}(\omega^2,\Omega)} = \int_\Omega \omega^2(|u|^2 + |du|^2)\,.$$

By Theorem 1.2 we see that

$$C^{-1}\|u\|_{W^{1,2}(\omega^2,\Omega)} \leq \|\omega u\|_{W^{1,2}(\Omega)} \leq C\|u\|_{W^{1,2}(\omega^2,\Omega)}\,,$$

and by the equivalence of the norms, the family of functions $\{u : \omega u \in W_0^{1,2}(\Omega)\}$ is equivalent to $W_0^{1,2}(\omega^2,\Omega)$ the existence of minimizers of (2.12) follows from convexity and the direct method in the calculus of variations. The Euler Lagrange equations for minimizers tell us that

$$\star d(A - \star d\xi_1) = 0 \Rightarrow \text{ there exists } \xi_2 \text{ such that } A - \star d\xi_1 = d\xi_2 \text{ and}$$
$$\star d(\omega^2(A - \star d\phi_1)) = 0 \Rightarrow \text{ there exists } \phi_2 \text{ such that } \omega^2(A - \star d\phi_1) = d\phi_2\,.$$

in the sense of distributions. Then with a direct application of Theorem 1.4

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} \leq C\frac{(\sup_{\mathcal{M}^2}\omega)^{2\varepsilon}}{\varepsilon^2}\|\frac{\omega^2}{|d\omega|}\Delta_g(\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)}$$

and

$$\|\frac{\omega^2}{|d\omega|}\Delta_g(\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} = \|\frac{\omega^2}{|d\omega|}d(\omega^{-2}d\phi_2 - d\xi_1)\|^2_{L^2(\mathcal{M}^2)} = 4\|\omega^{-1}d\phi_2\|^2_{L^2(\mathcal{M}^2)}\,.$$

we conclude the proof. □

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