Elastic flow of curves with partial free boundary

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Abstract

We consider a curve with boundary points free to move on a line in $\mathbb{R}^2$, which evolves by the L^2–gradient flow of the elastic energy, that is a linear combination of the Willmore and the length functional. For such planar evolution problem we study the short and long–time existence. Once we establish under which boundary conditions the PDE's system is well–posed (in our case the Navier boundary conditions), employing the Solonnikov theory for linear parabolic systems in Hölder space, we show that there exists a unique flow in a maximal time interval $[0, T)$. Then, using energy methods we prove that the maximal time is actually $T = +\infty$.

Keywords: Geometric evolution, elastic energy, parabolic Hölder spaces, long-–time existence.

Mathematics Subject Classification (2020): Primary $53E40$; $35G31$, $35A01$.

Contents

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1 Introduction

In this paper we consider a geometric evolution of a curve with partially free boundary. To be more precise, we consider the gradient flow of the elastic energy under the constraint that the boundary points of the curve have to remain attached to the x–axis.

This paper fits the broad range of topics of geometric evolution of curves and surfaces where the evolution law is dictated by functions of the curvature, that recently gained more and more attention from the mathematical community, due both to their application to a variety of physical problems and to fascinating challenges in analysis and geometry.

The elastic energy of a curve is a linear combination of the L^2–norm of its curvature $\boldsymbol{k}$ (also known as one–dimensional Willmore functional) and its weighted length, namely

$$\mathcal{E}(\gamma) = \int_\gamma |\boldsymbol{\kappa}|^2 + \mu \,\mathrm{d}s$$

where $\mu > 0$.

Before passing to the evolutionary problem, we say few words about the critical points of the energy $\mathcal{E}$, known as *elasticae* or *elastic curves*. As it is explained in [46], the elasticae have been studied since Bernoulli and Euler as the elastic energy was employed as a model for the bending energy of an elastic rods. Still later, Born, in his Thesis of 1906, was the first to publish figures of elasticae plotted numerically. However, in the last decades, many authors contribute to their classification, for instance we refer to Langer and Singer [23, 24], Linnér [27], Djondjorov et al. [17] and to Langer and Singer [25], Bevilacqua, Lussardi and Marzocchi [9], the same authors with Ballarin [8], for the case of functional which depends both on the curvature and the torsion of the curve. More recently, Miura and Yoshizawa in a series of papers [34, 33, 35], give a complete classification of both clamped and pinned p–elasticae.

In this paper we aim to study the L^2–gradient flow of $\mathcal{E}$. To the best of our knowledge the problem was introduced for the first time by Polden in his PhD Thesis [40], where it is shown that given as initial datum a smooth immersion of the circle in the plane, then there exists a smooth solution to the gradient flow problem for all positive times which sub–converges to an elastica. Then, Dziuk, Kuwert and Schätzle generalized the global existence and sub–convergence result to $\mathbb{R}^n$ and derived an algorithm to treat the flow and compute several numerical examples. Later, the evolution of elastic curves has been extended and studied in detail both for closed curves (see for instance [18, 30, 40, 41]) as well as for open curves with Navier boundary conditions in [36, 37] and clamped boundary conditions in [16, 37, 26, 45]. We also recall that a slightly different problem was tackled, among others, by Wen in [47] and by Rupp and Spener in [42], where the authors analyzed the elastic flow of curves with a nonzero rotation index and clamped boundary conditions respectively, which are in both cases subjected to fix length and in [22, 38, 39] where a variety of constraints are considered. For the sake of completeness, we also mention that the L^2–gradient flow of $\int_\gamma |\boldsymbol{\kappa}|^2 \,\mathrm{d}s$ for curve subjected to fix length is studied in [12, 13, 18], indeed other fourth (or higher) order flows are analyzed, for instance, in [15, 48, 1, 2, 32, 31, 49]. Finally, we mention the survey [29] for a complete review of the literature and we recommend all the reference therein.

As already said, in this paper we let evolve a curve supposing that it remains attached to the x–axis. To derive the flow, we start by writing the associated Euler–Lagrange equations and in particular we find suitable "natural" boundary conditions for this problem (this kind of boundary conditions is also referred in the literature by Navier conditions). We thus get that the evolution can be described by solutions of a system of quasilinear fourth order with boundary conditions in (2.6), namely the attachment condition, second and third order conditions. We then introduce a class of admissible initial curves of class $C^{4+\alpha}$ with $\alpha \in (0, 1)$ which needs to be non–degenerate, in the sense that the y–component of the unit tangent vector must be positive at boundary points, and satisfy (in addition to the conditions mentioned above) an extra fourth order condition (see Definition 2.2).

Then, we establish well–posedness of the flow. More precisely, starting with a (geometrically) admissible initial curve we prove in Theorem 3.14 that there exists a unique (up to reparametrization) solution of the flow in a small time interval $[0, T]$ with $T > 0$, that can be described by a parametrization of class $C^{\frac{4+\alpha}{4},4}([0, T] \times [0, 1])$.

To do so, we choose a specific tangential velocity turning the system (2.8) into a non–degenerate parabolic boundary value problem without changing the geometric nature of the evolution (namely the *analytic problem* (3.3)). Then, we solve the *analytic problem* using a linearization procedure and a fixed point argument. The main difficulty is actually to solve the associated *linear system* (3.6), coupled with extra compatibility conditions (see Definition 3.4), employing the Solonnikov theory for linear parabolic systems in Hölder space introduced in [44], as it is shown in Theorem 3.5.

Once we have a solution for the *analytic problem*, the key point is to ensure that solving (3.3) is enough to obtain a unique solution to the original *geometric problem*. This is shown in Theorem 3.14, following the approach presented in [19] and later in [20].

The second natural step is trying to understand the long–time behaviour of the evolving curves. This leads to our main result.

Theorem 1.1. *Let γ_0 be a geometrically admissible initial curve and γ_t be a solution to the elastic flow with initial datum γ_0 in the maximal time interval $[0, T)$ with $T \in (0, \infty) \cup \{\infty\}$. Then, up to reparametrization and translation of γ_t, it follows*

$$T = \infty$$

or at least one of the following holds

- *the inferior limit of the length of γ_t is zero as $t \to T$;*
- *the inferior limit of the y–component of the unit tangent vector at boundary is zero as $t \to T$.*

Even though the structure of the proof of this result is based on an contradiction argument already present in the literature (see for instance [40, 18, 29, 14, 21]) this is the most technical part of the paper and it contains relevant novelties.
We find energy type inequalities, more precisely a bounds on the L^2–norm of second and sixth derivative of the curvature, which leads to contradict the finiteness of T. Those estimates, which involved the smallest number of derivatives, can be derived under the assumption that during the evolution the length is uniformly bounded away from zero and that the curve remains non–degenerate in a uniform sense (see Definition 4.4).
Moreover, we underline that only estimates on geometric quantities, namely the curvature, are needed. In particular, the proof itself is independent of the choice of tangential velocity

which corresponds to the very definition of the flow, where only the normal velocity is prescribed. For this reason, following [14], we reparametrize the flow in such a way that the tangential velocity linearly interpolate its values at boundary points (see condition (4.14)) and such that suitable estimates both inside and at boundary points hold. With this choice and the uniform bounds for the curvature, we are able to extend the flow smoothly up to the time T given by the short–time existence result and then restart the flow, contradicting the maximality of T.
In short, our approach combines the one presented in [14] and the other in [21], in the sense that we choose a tangential velocity as explained above and we use the minimun number of derivatives (and hence of estimates) which are needed in order to conclude the proof of Theorem 1.1.

This work is organized as follows: in the next section we formulate the geometric evolution problem for elastic curves in a precise way and we show that those curves decrease the energy $\mathcal{E}$. In Section 3 we show short–time existence of a unique smooth solution using the Solonnikov theory and a contraction argument. We also show geometric uniqueness. In the final Section 5, we prove the long–time existence result using the curvature bounds provided in Section 4.

2 The elastic flow

2.1 Preliminary definitions and notation

A regular curve is a continuous map $\gamma : [a, b] \to \mathbb{R}^2$ which is differentiable on (a, b) and such that $|\partial_x\gamma|$ never vanishes on (a, b). Without loss of generality, from now on we consider $[a, b] = [0, 1]$.

We denote by s the arclength parameter, then $\partial_s := \frac{1}{|\partial_x\gamma|}\partial_x$ and $\mathrm{d}s := |\partial_x\gamma|\,\mathrm{d}x$ are, respectively, the derivative and the measure with respect to the arclength parameter of the curve γ.

From now on, we will pass to the arclength parametrization of the curves without further comments.

If we assume that γ is a regular planar curve of class at least C^1, we can define the unit tangent vector $\tau = |\partial_x\gamma|^{-1}\partial_x\gamma$ and the unit normal vector ν as the anticlockwise rotation by $\pi/2$ of the unit tangent vector.

We introduce the operator $\partial_s^\perp$ that acts on vector fields φ defined as the normal component of $\partial_s\varphi$ along the curve γ, that is $\partial_s^\perp\varphi = \partial_s\varphi - \langle\partial_s\varphi, \partial_s\gamma\rangle\,\partial_s\gamma$. Moreover, for any vector $\psi(\cdot) \in \mathbb{R}^2$, we use the notation $(\psi(\cdot)_1$,$\psi(\cdot)_2)$ to denote the projection on the x–axis and y–axis, respectively.

Let $\mu > 0$. Then, assuming that γ is of class H^2, we denote by $\boldsymbol{\kappa} = \partial_s\tau$ the curvature vector and we define the *elastic energy with a length penalization*

$$\mathcal{E}(\gamma) = \int_\gamma |\boldsymbol{\kappa}|^2 + \mu\,\mathrm{d}s\,.$$

We recall that in the plane we can write the curvature vector as $\boldsymbol{k} = k\nu$, where k is the oriented curvature. Hence, the energy functional can be equivalently written as

$$\mathcal{E}(\gamma) = \int_\gamma k^2 + \mu \,\mathrm{d}s \,. \tag{2.1}$$

2.2 Formal derivation of the flow

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular curve of class H^2 and $\varepsilon \in \mathbb{R}$. Then, for $|\varepsilon|$ small enough, a variation $\gamma_\varepsilon = \gamma + \varepsilon\psi$ with $\psi : [0,1] \to \mathbb{R}^2$ of class H^2 is still a regular curve. By direct computations, one gets the *first variation* of $\mathcal{E}$

$$\frac{d}{d\varepsilon}\mathcal{E}(\gamma_\varepsilon)\Big|_{\varepsilon=0} = \int_\gamma 2\langle \boldsymbol{\kappa}, \partial_s^2\psi\rangle \,\mathrm{d}s + \int_\gamma (-3|\boldsymbol{\kappa}|^2 + \mu)\,\langle \tau, \partial_s\psi\rangle \,\mathrm{d}s \,. \tag{2.2}$$

As usual, we say that a regular curve γ of class H^2 is a *critical* point of $\mathcal{E}$ if for any ψ the first variation vanishes.

Lemma 2.1 (Euler–Lagrange equations). *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular curve of class H^2 parametrized proportional to arclength, that is a critical point for $\mathcal{E}$. Then the curve is smooth and on $(0,1)$ it satisfies*

$$(\partial_s^\perp)^2\boldsymbol{\kappa} + |\boldsymbol{\kappa}|^2\boldsymbol{\kappa} - \mu\boldsymbol{\kappa} = 0 \,.$$

Moreover, if the endpoints of the curve are constrained to the x–axis for $y \in \{0,1\}$ the following Navier boundary conditions are fulfilled:

$$\begin{cases} k(y) = 0 & \textit{curvature or second order conditions} \\ (-2\partial_s k(y)\nu(y) + \mu\tau(y))_1 = 0 & \textit{third order conditions}\,. \end{cases} \tag{2.3}$$

Proof. By a standard bootstrap argument one can show that critical points of $\mathcal{E}$ are actually smooth (for the reader convenience a proof of this fact is given in Proposition 6.2 in the appendix). Hence we can integrate by parts the expression (2.2) to get

$$\begin{aligned} \frac{d}{d\varepsilon}\mathcal{E}(\gamma_\varepsilon)\Big|_{\varepsilon=0} &= \int_\gamma \Big\langle 2(\partial_s^\perp)^2\boldsymbol{\kappa} + |\boldsymbol{\kappa}|^2\boldsymbol{\kappa} - \mu\boldsymbol{\kappa}, \psi \Big\rangle \,\mathrm{d}s \\ &\quad + 2\,\langle \boldsymbol{\kappa}, \partial_s\psi\rangle\big|_0^1 + \langle -2\partial_s^\perp\boldsymbol{\kappa} - |\boldsymbol{\kappa}|^2\tau + \mu\tau, \psi\rangle\Big|_0^1 \,. \end{aligned} \tag{2.4}$$

From formula (2.4), we immediately get

$$(\partial_s^\perp)^2\boldsymbol{\kappa} + |\boldsymbol{\kappa}|^2\boldsymbol{\kappa} - \mu\boldsymbol{\kappa} = 0 \,,$$

coupled with

$$2\,\langle \boldsymbol{\kappa}, \partial_s\psi\rangle\big|_0^1 + \langle -2\partial_s^\perp\boldsymbol{\kappa} - |\boldsymbol{\kappa}|^2\tau + \mu\tau, \psi\rangle\Big|_0^1 = 0 \,. \tag{2.5}$$

By the hypothesis the endpoints of the curve are constrained to the x-axis, hence

$$\gamma(0)_2 = 0 \,, \quad \gamma(1)_2 = 0 \,.$$

To maintain this property we only consider variations $\gamma_\varepsilon = \gamma + \varepsilon\psi$ where

$$\psi(0)_2 = 0 \,, \quad \psi(1)_2 = 0 \,.$$

As a consequence the boundary terms in (2.5) reduce to

$$
\begin{aligned}
&2\left\langle k\nu,\partial_s\psi\right\rangle\Big|_0^1+\left(-2\partial_s k(1)\nu(1)-k(1)^2\tau(1)+\mu\tau(1)\right)_1\psi(1)_1\\
&-\left(-2\partial_s k(0)\nu(0)-k(0)^2\tau(0)+\mu\tau(0)\right)_1\psi(0)_1=0\,.
\end{aligned}
$$

We firstly choose as test function ψ such that

$$
\partial_s\psi(0)=\nu(0)\qquad\text{and}\qquad\partial_s\psi(1)=0
$$

and then, interchanging the role of $\partial_s\psi(0)$ and $\partial_s\psi(1)$, we get boundary conditions in (2.3). □

Thanks to the previous lemma we can formally define the elastic flow of a curve with endpoints constrained to the x-axis coupling the motion equation

$$
\partial_t\gamma=-(\partial_s^\perp)^2\boldsymbol{\kappa}-|\boldsymbol{\kappa}|^2\boldsymbol{\kappa}+\mu\boldsymbol{\kappa}\,,
$$

with the following Navier boundary conditions

$$
\begin{cases}
\gamma(y)_2=0 & \text{attachment conditions}\\
k(y)=0 & \text{curvature or second order conditions}\\
\left(-2\partial_s k(y)\nu(y)+\mu\tau(y)\right)_1=0 & \text{third order conditions}
\end{cases}
\tag{2.6}
$$

for $y\in\{0,1\}$.

2.3 Definition of the geometric problem

We briefly introduce of the parabolic Hölder spaces where we set our problem (see [44] for instance).

For a function $u:[0,T]\times[0,1]\to\mathbb{R}$, for $\rho\in(0,1)$ the semi–norms $[u]_{\rho,0}$ and $[u]_{0,\rho}$ are defined as

$$
[u]_{\rho,0}:=\sup_{(t,x),(\tau,x)}\frac{|u(t,x)-u(\tau,x)|}{|t-\tau|^\rho}\,,
$$

and

$$
[u]_{0,\rho}:=\sup_{(t,x),(t,y)}\frac{|u(t,x)-u(t,y)|}{|x-y|^\rho}\,.
$$

For $l\in\{0,1,2,3,4\}$, $\alpha\in(0,1)$ the parabolic Hölder space

$$
C^{\frac{l+\alpha}{4},l+\alpha}([0,T]\times[0,1])
$$

is the space of all functions $u:[0,T]\times[0,1]\to\mathbb{R}$ that have continuous derivatives $\partial_t^i\partial_x^j u$ where $i,j\in\mathbb{N}$ are such that $4i+j\le l$ for which the norm

$$
\|u\|_{\frac{l+\alpha}{4},l+\alpha}:=\sum_{4i+j=0}^{l}\left\|\partial_t^i\partial_x^j u\right\|_\infty+\sum_{4i+j=l}\left[\partial_t^i\partial_x^j u\right]_{0,\alpha}+\sum_{0<l+\alpha-4i-j<4}\left[\partial_t^i\partial_x^j u\right]_{\frac{l+\alpha-4i-j}{4},0}
$$

is finite.

The space $C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])$ is exactly the space

$$C^{\frac{\alpha}{4}}\left([0,T];C^0([0,1])\right)\cap C^0\left([0,T];C^\alpha([0,1])\right)\,,$$

with equivalent norms.

Whenever it is useful, we identify the spaces

$$C^{\frac{l+\alpha}{4},l+\alpha}([0,T]\times\{0,1\},\mathbb{R}^m)$$

with $C^{\frac{l+\alpha}{4}}([0,T],\mathbb{R}^{2m})$ via the isomorphism $f\mapsto(f(t,0),f(t,1))^t$.

We are now in a good position to define the flow.

Definition 2.2 (Admissible initial curve)**.** A regular curve $\gamma_0:[0,1]\to\mathbb{R}^2$ is a an *admissible initial curve* for the elastic flow if

1. it admits a parametrization which belongs to $C^{4+\alpha}([0,1],\mathbb{R}^2)$ for some $\alpha\in(0,1)$;
2. it satisfies the Navier boundary conditions in (2.6): attachment, curvature and third order conditions;
3. it satisfies the *non–degeneracy condition* at boundary points, that is, there exists $\rho>0$ such that
$$(\tau_0(y))_2\geq\rho\qquad\text{for } y\in\{0,1\}\,. \tag{2.7}$$
4. it satisfies the *fourth order condition* at boundary points, that is
$$((-2\partial_s^2k_0(y)-k_0^3(y)+k_0(y))\nu_0(y))_2=0\qquad\text{for } y\in\{0,1\}\,.$$

Definition 2.3 (Solution of the geometric problem)**.** Let γ_0 be an admissible initial curve as in Definition 2.2 and $T>0$. A time dependent family of curves γ_t for $t\in[0,T]$ is a solution to the *elastic flow* with initial datum γ_0 in the maximal time interval $[0,T]$ if there exists a parametrization

$$\gamma(t,x)\in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1],\mathbb{R}^2\right)\,,$$

with γ regular, and such that for every $t\in[0,T]$, $x\in[0,1]$ the system

$$\begin{cases}(\partial_t\gamma)^\perp=\left(-2\partial_s^2k-k^3+k\right)\nu\\ \gamma(0,x)=\gamma_0(x)\,,\end{cases} \tag{2.8}$$

coupled with boundary conditions (2.6), is satisfied.

Remark 2.4. Observe that the formulation of the problem given so far involves purely geometric quantities and hence it is invariant under reparametrizations. Thus, given a solution γ of (2.8), any reparametrization of γ is still a solution to system (2.8).

Remark 2.5. In system (2.8) only the normal component of the velocity is prescribed. This does not mean that the tangential velocity is necessary zero. We can equivalently write the motion equations as

$$\partial_t\gamma=V\nu+\Lambda\tau\,, \tag{2.9}$$

where $V=-2\partial_s^2k-k^3+\mu k$ and Λ is some at least continuous function.

2.4 Energy monotonicity

We now aim to show that the first crucial property of gradient flows is fulfilled: the energy of an evolving curve decreases in time.

Lemma 2.6. *If γ satisfies (2.9), the commutation rule*

$$\partial_t \partial_s = \partial_s \partial_t + (kV - \partial_s \Lambda)\, \partial_s$$

holds. The measure $\mathrm{d}s$ evolves as

$$\partial_t(\,\mathrm{d}s) = (\partial_s \Lambda - kV)\, \mathrm{d}s\,. \tag{2.10}$$

Moreover the unit tangent vector, unit normal vector and the j–th derivatives of scalar curvature of a curve satisfy

$$\begin{aligned}
\partial_t \tau &= (\partial_s V + \Lambda k)\, \nu\,, \\
\partial_t \nu &= -\,(\partial_s V + \Lambda k)\, \tau\,, \qquad\qquad (2.11)\\
\partial_t k &= \langle \partial_t \boldsymbol{\kappa}, \nu\rangle = \partial_s^2 V + \Lambda \partial_s k + k^2 V \\
&= -2\partial_s^4 k - 5k^2 \partial_s^2 k - 6k\,(\partial_s k)^2 + \Lambda \partial_s k - k^5 + \mu\left(\partial_s^2 k + k^3\right)\,, \qquad (2.12)
\end{aligned}$$

Proof. The proof of the lemma is obtained by direct computations, we refer for instance to [29, Lemma 2.19]. □

Proposition 2.7. *Let γ_t be a solution to the elastic flow in the sense of Definition 2.3. Then*

$$\partial_t \mathcal{E}(\gamma_t) = -\int_\gamma V^2\, \mathrm{d}s\,.$$

Proof. Using the evolution laws collected in Lemma 2.6, we get

$$\begin{aligned}
\partial_t \int_\gamma k^2 + \mu\, \mathrm{d}s &= \int_\gamma 2k\partial_t k + \left(k^2 + \mu\right)(\partial_s \Lambda - kV)\, \mathrm{d}s \\
&= \int_\gamma 2k\left(\partial_s^2 V + \partial_s k \Lambda + k^2 V\right) + \left(k^2 + \mu\right)(\partial_s \Lambda - kV)\, \mathrm{d}s \\
&= \int_\gamma 2k\partial_s^2 V + k^3 V - \mu k V + \partial_s\left(\Lambda\left(k^2 + \mu\right)\right)\, \mathrm{d}s\,.
\end{aligned}$$

Integrating twice by parts the term $\int_\gamma 2k\partial_s^2 V\, \mathrm{d}s$ we obtain

$$\partial_t \int_\gamma k^2 + \mu\, \mathrm{d}s = -\int_\gamma V^2\, \mathrm{d}s + \left(2k\partial_s V - 2\partial_s k V + \Lambda(k^2 + \mu)\right)\Big|_0^1. \tag{2.13}$$

It remains to show that the contribution of the boundary term in (2.13) is zero once we assume that Navier boundary conditions hold.

Hence, since $k(y) = 0$ for $y \in \{0, 1\}$, we only need to show that

$$-2\partial_s k V + \mu \Lambda\Big|_0^1 = 0\,.$$

Recalling that $\gamma(y) = (\gamma_1(y), 0)$ and using relation (2.6) we have

$$\begin{aligned}
0 =& \langle \partial_t \gamma(y), -2\partial_s k(y)\nu(y) + \mu\tau(y)\rangle \\
=& \langle V(y)\nu(y) + \Lambda(y)\tau(y), -2\partial_s k(y)\nu(y) + \mu\tau(y)\rangle \\
=& -2\partial_s k(y) V(y) + \mu\Lambda(y)\,,
\end{aligned}$$

where $y \in \{0, 1\}$. □

3 Short–time existence

In this section we show that there exists a positive time T such that, fixed an admissible initial curve, the *elastic flow* in Definition 2.3 lives at least in $[0,T]$.

To do so, we firstly find a unique solution of the associated analytic problem defined in (3.3) using a standard linearization procedure. More precisely, we use Solonnikov theory [44] to prove the well–posedness of the linearized system and then a fixed point argument. A key point is to ensure that solving the analytic problem is enough to obtain a solution to the geometric problem (2.8) and that the solution of (2.8) is unique up to reparametrization.

3.1 Definition of the analytic problem

Let $T>0$ and $\alpha\in(0,1)$ and let us consider a time dependent family of curves parametrized by a map γ in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1])$.

We compute the normal velocity of such moving curves in terms of the parametrization:

$$
\begin{aligned}
(\partial_t\gamma)^\perp =& -2\frac{\partial_x^4\gamma}{|\partial_x\gamma|^4}+12\frac{\partial_x^3\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}+5\frac{\partial_x^2\gamma\left|\partial_x^2\gamma\right|^2}{|\partial_x\gamma|^6}+8\frac{\partial_x^2\gamma\left\langle\partial_x^3\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}-35\frac{\partial_x^2\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle^2}{|\partial_x\gamma|^8}\\
&+\left\langle 2\frac{\partial_x^4\gamma}{|\partial_x\gamma|^4}-12\frac{\partial_x^3\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}-5\frac{\partial_x^2\gamma\left|\partial_x^2\gamma\right|^2}{|\partial_x\gamma|^6}-8\frac{\partial_x^2\gamma\left\langle\partial_x^3\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}+35\frac{\partial_x^2\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle^2}{|\partial_x\gamma|^8},\tau\right\rangle\tau\\
&+\mu\frac{\partial_x^2\gamma}{|\partial_x\gamma|^2}-\left\langle\mu\frac{\partial_x^2\gamma}{|\partial_x\gamma|^2},\tau\right\rangle\tau\,.
\end{aligned}
$$

In order to obtain a well–posed parabolic PDE for the parametrization, we consider the following tangential velocity

$$
\begin{aligned}
\widetilde{\Lambda} :=\Big\langle &-2\frac{\partial_x^4\gamma}{|\partial_x\gamma|^4}+12\frac{\partial_x^3\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}+5\frac{\partial_x^2\gamma\left|\partial_x^2\gamma\right|^2}{|\partial_x\gamma|^6}+8\frac{\partial_x^2\gamma\left\langle\partial_x^3\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}\\
&-35\frac{\partial_x^2\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle^2}{|\partial_x\gamma|^8}+\mu\frac{\partial_x^2\gamma}{|\partial_x\gamma|^2},\tau\Big\rangle\,.
\end{aligned}
$$

Hence, we end up with the following motion equation

$$
\begin{aligned}
\partial_t\gamma =& V\nu+\widetilde{\Lambda}\tau\\
=& -2\frac{\partial_x^4\gamma}{|\partial_x\gamma|^4}+12\frac{\partial_x^3\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}+5\frac{\partial_x^2\gamma\left|\partial_x^2\gamma\right|^2}{|\partial_x\gamma|^6}+8\frac{\partial_x^2\gamma\left\langle\partial_x^3\gamma,\partial_x\gamma\right\rangle}{|\partial_x\gamma|^6}\\
&-35\frac{\partial_x^2\gamma\left\langle\partial_x^2\gamma,\partial_x\gamma\right\rangle^2}{|\partial_x\gamma|^8}+\mu\frac{\partial_x^2\gamma}{|\partial_x\gamma|^2}\,.
\end{aligned}
\tag{3.1}
$$

Similarly, we specify another tangential condition on the parametrization at boundary points:

$$
\langle\partial_x^2\gamma(y),\tau(y)\rangle=0 \tag{3.2}
$$

for $y\in\{0,1\}$. We notice that the extra condition in (3.2) together with the curvature condition, is equivalent to the *second order condition*

$$
\partial_x^2\gamma(y)=0
$$

for $y \in \{0,1\}$.

Definition 3.1 (Admissible initial parametrization)**.** A map $\gamma_0 : [0,1] \to \mathbb{R}^2$ is an *admissible initial parametrization* if it parametrize a regular curve with the following properties:

1. it belongs to $C^{4+\alpha}([0,1],\mathbb{R}^2)$ for some $\alpha \in (0,1)$;
2. it satisfies the Navier boundary conditions in (2.6): attachment, curvature and third order conditions;
3. it satisfies the *non–degeneracy condition* (2.7);
4. it satisfies the *fourth order condition* at boundary points, that is
$$\Big(V(0,y)\nu_0(y) + \widetilde{\Lambda}(0,y)\tau_0(y)\Big)_2 = 0 \qquad \text{for } y \in \{0,1\},$$
where ν_0, τ_0 are the normal and tangent unit vectors to γ_0.

From now on, we refer to conditions $(2)-(4)$ in Definition 3.1 as *compatibility conditions*. Moreover, we identify the curve with its parametrization without further comments.

Definition 3.2 (Solution of the analytic problem)**.** Let γ_0 be an admissible initial parametrization as in Definition 3.1. A time dependent parametrization γ_t for $t \in [0,T]$ is a solution to the *analytic elastic flow* with initial datum γ_0 in the time interval $[0,T]$ with $T > 0$ if

$$\gamma(t,x) \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1],\mathbb{R}^2\right),$$

with γ regular and such that for every $t \in [0,T]$, $x \in [0,1]$ and $y \in \{0,1\}$, satisfies

$$\begin{cases} \partial_t\gamma = V\nu + \widetilde{\Lambda}\tau = -2\frac{\partial_x^4\gamma}{|\partial_x\gamma|^4} + l.o.t. & \\ \gamma(y)_2 = 0 & \text{attachment conditions,} \\ \partial_x^2\gamma(y) = 0 & \text{second order conditions,} \\ \left(-2\partial_s k(y)\nu(y) + \mu\tau(y)\right)_1 = 0 & \text{third order conditions,} \\ \gamma(0,\cdot) = \gamma_0(\cdot) & \text{initial condition.} \end{cases} \tag{3.3}$$

3.2 Linearization

This section is devoted to proving existence and uniqueness of solutions to the linearized system associated to (3.3). To do so, we will show that the linearized system satisfies the requirements of the general theory indroduced by Solonnikov [44].

We linearize the highest order terms of the motion equation (3.1) around the initial parametrization γ_0 obtaining

$$\begin{aligned} \partial_t\gamma + \frac{2}{|\partial_x\gamma_0|^4}\partial_x^4\gamma &= \left(\frac{2}{|\partial_x\gamma_0|^4} - \frac{2}{|\partial_x\gamma|^4}\right)\partial_x^4\gamma + \widetilde{f}(\partial_x^3\gamma,\partial_x^2\gamma,\partial_x\gamma) \\ &=: f(\partial_x^4\gamma,\partial_x^3\gamma,\partial_x^2\gamma,\partial_x\gamma)\,. \end{aligned} \tag{3.4}$$

Then, after noticing that the attachment condition and the second order condition are already linear, we linearize the highest order terms of the third order condition conditions, getting

$$\begin{aligned}\left(-\frac{1}{|\partial_x\gamma_0|^3}\langle\partial_x^3\gamma,\nu_0\rangle\nu_0\right)_1 &= \left(-\frac{1}{|\partial_x\gamma_0|^3}\langle\partial_x^3\gamma,\nu_0\rangle\nu_0+\frac{1}{|\partial_x\gamma|^3}\langle\partial_x^3\gamma,\nu\rangle\nu+h(\partial_x\gamma)\right)_1\\ &=: b(\partial_x^3\gamma,\partial_x\gamma)\,.\end{aligned}\tag{3.5}$$

The linearized system associated to (3.3) is given by

$$\begin{cases}\partial_t\gamma+\frac{2}{|\partial_x\gamma_0|^4}\partial_x^4\gamma=f & \\ \gamma_2=0 & \text{attachment conditions,}\\ \partial_x^2\gamma=0 & \text{second order conditions,}\\ \left(-\frac{1}{|\partial_x\gamma_0|^3}\left\langle\partial_x^3\gamma,\nu_0\right\rangle\nu_0\right)_1=b & \text{third order conditions,}\\ \gamma(0)=\gamma_0 & \text{initial condition}\end{cases}\tag{3.6}$$

where f,b are defined in (3.4), (3.5) and we have omitted the dependence on $(t,x)\in[0,T]\times[0,1]$ in the motion equation, on $(t,y)\in[0,T]\times\{0,1\}$ for the boundary conditions and on $x\in[0,1]$ for the initial condition.

Remark 3.3. In the following, we replace the right hand side of system (3.6) with (f,b,ψ). Thus, we get the general system

$$\begin{cases}\partial_t\gamma+\frac{2}{|\partial_x\gamma_0|^4}\partial_x^4\gamma=f & \\ \gamma_2=0 & \text{attachment conditions,}\\ \partial_x^2\gamma=0 & \text{second order conditions,}\\ \left(-\frac{1}{|\partial_x\gamma_0|^3}\left\langle\partial_x^3\gamma,\nu_0\right\rangle\nu_0\right)_1=b & \text{third order conditions,}\\ \gamma(0)=\psi & \text{initial condition}\end{cases}\tag{3.7}$$

where $f\in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1],\mathbb{R}^2)$, $(b(\cdot,0),b(\cdot,1))\in C^{\frac{1+\alpha}{4}}([0,T],\mathbb{R}^2)$ and $\psi\in C^{4+\alpha}\left([0,1],\mathbb{R}^2\right)$.

Definition 3.4. [Linear compatibility conditions] Let (f,b) be a given right hand side to the linear system (3.7). A function $\psi\in C^{4+\alpha}\left([0,1],\mathbb{R}^2\right)$ satisfies the *linear compatibility conditions* with respect to (f,b) if for $y\in\{0,1\}$ there hold

$$\begin{aligned}&\psi(y)_2=0\,,\\ &\partial_x^2\psi(y)=0\,,\\ &\left(-\frac{1}{|\partial_x\gamma_0|^3}\left\langle\partial_x^3\psi(y),\nu_0(y)\right\rangle\nu_0(y)\right)_1=b(0,y),\\ &\left(\frac{2}{|\partial_x\gamma_0|^4}\partial_x^4\psi(y)-f(0,y)\right)_2=0\,.\end{aligned}$$

Theorem 3.5. *Let $\alpha\in(0,1)$ and let $T>0$. Suppose that*

- $f\in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1],\mathbb{R}^2)$;
- $(b(\cdot,0),b(\cdot,1))\in C^{\frac{1+\alpha}{4}}([0,T],\mathbb{R}^2)$;

- $\psi \in C^{4+\alpha}\left([0,1],\mathbb{R}^2\right)$;
- *ψ satisfies the linear compatibility conditions in Definition 3.4 with respect to (f,b).*

Then, the linearized problem (3.7) has a unique solution $\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1],\mathbb{R}^2)$.

Moreover, for all $T>0$ there exists a $C(T)>0$ such that the solution satisfies

$$\|\gamma\|_{\frac{4+\alpha}{4},4+\alpha} \leq C(T)\left(\|f\|_{\frac{\alpha}{4},\alpha} + \|b\|_{\frac{1+\alpha}{4}} + \|\psi\|_{4+\alpha}\right).$$

Proof. To show the result we have to prove that system (3.7) satisfies all the hypothesis of the general [44, Theorem 4.9].

First of all we write $\gamma = (u,v)$.

The two integers b,r which gives, respectively, the number of boundary and initial conditions in our case are $b=2$, $r=2$.

We write our motion equation in the form

$$\mathcal{L}\gamma = f \tag{3.8}$$

where the 2×2 matrix $\mathcal{L}$ is given by

$$\mathcal{L}(x,t,\partial_x,\partial_t) = \begin{bmatrix} \partial_t + \frac{2}{|\partial_x\gamma_0|^4}\partial_x^4 & 0 \\ 0 & \partial_t + \frac{2}{|\partial_x\gamma_0|^4}\partial_x^4 \end{bmatrix}$$

and the vector $f=(f^1,f^2)$ is the right hand side of motion equation in system (3.7).

- We firstly show that system (3.8) satisfies the parabolicity condition [44, page 8]. As in [44], we call $\mathcal{L}_0$ the principal part of the matrix $\mathcal{L}$ and we choose the integers s_k, t_j in [44, page 8] as follows: $s_k = 4$ for $k \in \{1,2\}$ and $t_j = 0$ for $j \in \{1,2\}$. Hence, we have $\mathcal{L}_0 = \mathcal{L}$ and its determinant

  $$\det\mathcal{L}_0(x,t,i\xi,p) = \left(\frac{2}{|\partial_x\gamma_0|}\xi^4 + p\right)^2$$

  is a polynomial of degree two in p with one root

  $$p = -\frac{2}{|\partial_x\gamma_0|^4}\xi^4$$

  of multiplicity two.
  Then, choosing $\delta \leq \frac{2}{|\partial_x\gamma_0|^4}$, the conditions of [44, page 8] are satisfied and the system is parabolic in the sense of Solonnikov.

- The compatibility condition at boundary points stated in [44, page 11] is equivalent to the following Lopatinskii-Shapiro condition (see [43, pages 11–15]).

  We check the condition only for the case $y=0$, the case $y=1$ can be treated analogously.

Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii-Shapiro condition at y is satisfied if every solution $\gamma \in C^4([0,\infty),\mathbb{C}^2)$ to the system of ODEs

$$\begin{cases} \lambda\gamma(x) + \frac{1}{|\partial_x\gamma_0|^4}\partial_x^4\gamma(x) = 0 \\ \gamma(y)_2 = 0 \\ \partial_x^2\gamma(y) = 0 \\ \left(\frac{1}{|\partial_x\gamma_0|^3}\left\langle \partial_x^3\gamma(y), \nu_0(y)\right\rangle \nu_0(y)\right)_1 = 0 \end{cases} \tag{3.9}$$

where $x \in [0,\infty)$, which satisfies $\lim_{x\to\infty}|\gamma(x)| = 0$ is the trivial solution.

To do so, we consider a solution γ to (3.9) such that $\lim_{x\to\infty}|\gamma(x)| = 0$. We test the motion equation by $|\partial_x\gamma_0|\,\langle\overline{\gamma}(x),\nu_0\rangle\,\nu_0$, and integrate twice by part to get

$$\begin{aligned} 0 = \lambda|\partial_x\gamma_0| \int_0^\infty |\left\langle\gamma(x),\nu_0\right\rangle|^2\,\mathrm{d}x + \frac{1}{|\partial_x\gamma_0|^3}\int_0^\infty |\left\langle\partial_x^2\gamma(x),\nu_0\right\rangle|^2\,\mathrm{d}x \\ + \frac{1}{|\partial_x\gamma_0|^3}\left\langle\overline{\gamma}(0),\nu_0\right\rangle\left\langle\partial_x^3\gamma(0),\nu_0\right\rangle - \frac{1}{|\partial_x\gamma_0|^3}\left\langle\overline{\partial_x\gamma}(0),\nu_0\right\rangle\left\langle\partial_x^2\gamma(0),\nu_0\right\rangle\,, \end{aligned} \tag{3.10}$$

where we have already used the fact that all derivatives decay to zero for x tending to infinity, due to the specific exponential form of the solutions to (3.9). We now observe that since γ_0 is an admissible initial parametrizationthe first component of ν_0 is bounded from below. That is, from the third order condition in system (3.9) it follows that $\left\langle\partial_x^3\gamma(0),\nu_0\right\rangle = 0$. Thus, this condition together with the second order condition, imply that the boundary terms in (3.10) vanish. Then, taking the real part of (3.10) and recalling that $\Re(\lambda) > 0$, we have $\langle\gamma(x),\nu_0\rangle = 0$ for all $x \in [0,\infty)$. In particular, from the attachment condition in (3.9), it follows that $\gamma(0) = 0$.
As before, testing the motion equation by $|\partial_x\gamma_0|\,\langle\overline{\gamma}(x),\tau_0\rangle\,\tau_0$ and integrating by part, we get

$$\begin{aligned} 0 = \lambda|\partial_x\gamma_0| \int_0^\infty |\left\langle\gamma(x),\tau_0\right\rangle|^2\,\mathrm{d}x + \frac{1}{|\partial_x\gamma_0|^3}\int_0^\infty |\left\langle\partial_x^2\gamma(x),\tau_0\right\rangle|^2\,\mathrm{d}x \\ + \frac{1}{|\partial_x\gamma_0|^3}\left\langle\overline{\gamma}(0),\tau_0\right\rangle\left\langle\partial_x^3\gamma(0),\tau_0\right\rangle - \frac{1}{|\partial_x\gamma_0|^3}\left\langle\overline{\partial_x\gamma}(0),\tau_0\right\rangle\left\langle\partial_x^2\gamma(0),\tau_0\right\rangle\,. \end{aligned} \tag{3.11}$$

The boundary term in (3.11) vanish due to the fact that $\gamma(0) = 0$ and the second order condition. Hence, considering again the real part of (3.11) we have that $\langle\gamma(x),\tau_0\rangle = 0$ for all $x \in [0,\infty)$. So, we conclude that $\gamma(x) = 0$ for all $x \in [0,\infty)$.

- Finally, to check the complementary condition for the initial datum stated in [44, page 12], we observe that the 2×2 matrix $[C_{\alpha j}]$ is the identity matrix. Then, choosing $\gamma_{\alpha j} = 0$ for $\alpha \in \{1,2\}$ and $j \in \{1,2\}$, we obtain $\rho_\alpha = 0$ and $C_0 = Id$.
  Moreover, the rows of the matrix $\mathcal{D}(x,p) = \hat{\mathcal{L}}_0(x,0,0,p) = pId$ are linearly independent modulo the polynomial p^2.

□

3.3 Short–time existence of the analytic problem

From now on, we fix $\alpha \in (0,1)$ and we consider an admissible initial parametrization γ_0 as in Definition 3.1, with $\|\gamma_0\|_{4+\alpha} = R$. Moreover, with a slight abuse of notation, we denote by $b(\cdot)$ the vector $(b(\cdot,0), b(\cdot,1))$ in the statement of Theorem (3.5).

Definition 3.6. For $T > 0$ we define the linear spaces

$$\begin{aligned}
\mathbb{E}_T &:= \{\gamma \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T]\times[0,1],\mathbb{R}^2) \text{ such that for } t \in [0,T]\,,\\
&\qquad \text{attachment and second order conditions hold}\}\,,\\
\mathbb{F}_T &:= \{(f,b,\psi) \in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1],\mathbb{R}^2) \times C^{\frac{1+\alpha}{4}}([0,T],\mathbb{R}^2) \times C^{4+\alpha}\left([0,1],\mathbb{R}^2\right)\\
&\qquad \text{such that the linear compatibility conditions hold}\}\,,
\end{aligned}$$

endowed with the norms

$$\begin{aligned}
\|\gamma\|_{\mathbb{E}_T} &= \|\gamma\|_{\frac{4+\alpha}{4},4+\alpha}\\
\|(f,b,\psi)\|_{\mathbb{F}_T} &= \|f\|_{\frac{\alpha}{4},\alpha} + \|b\|_{\frac{1+\alpha}{4}} + \|\psi\|_{4+\alpha}.
\end{aligned}$$

Moreover, we consider the affine spaces

$$\begin{aligned}
\mathbb{E}_T^0 &:= \{\gamma \in \mathbb{E}_T \text{ such that } \gamma_{|t=0} = \gamma_0\}\,,\\
\mathbb{F}_T^0 &:= \{(f,b) \text{ such that } (f,b,\gamma_0) \in \mathbb{F}_T\} \times \{\gamma_0\}\,.
\end{aligned}$$

Lemma 3.7. *For $T > 0$, the map $L_T : \mathbb{E}_T \to \mathbb{F}_T$ defined by*

$$L_T(\gamma) := \begin{pmatrix} \partial_t\gamma + \frac{2}{|\partial_x\gamma_0|^4}\partial_x^4\gamma \\ \left(-\frac{1}{|\partial_x\gamma_0|^3}\left\langle \partial_x^3\gamma, \nu_0\right\rangle \nu_0\right)_1 \\ \gamma_0 \end{pmatrix},$$

is a continuous isomorphism.

In the following we denote by L_T^{-1} the inverse of L_T, by B_M the open ball of radius $M > 0$ and center 0 in $\mathbb{E}_T$ and by $\overline{B_M}$ its closure.

Before proceeding we notice that, since the admissible initial parametrization $\gamma_0 : [0,1] \to \mathbb{R}^2$ is a regular curve, there exists a constant $C > 0$ such that

$$\inf_{x\in[0,1]} |\partial_x\gamma_0| \geq C\,, \tag{3.12}$$

which obviously implies that

$$\sup_{x\in[0,1]} \frac{1}{|\partial_x\gamma_0|} \leq \frac{1}{C}\,.$$

Then, as it is shown in [20], there exists a constant $\widetilde{C}$ depending on R and C, such that for every $j \in \mathbb{N}$ it holds

$$\left\|\frac{1}{|\partial_x\gamma_0|^j}\right\|_\alpha \leq \left(\frac{\|\partial_x\gamma_0\|_\alpha}{C^2}\right)^j \leq \left(\frac{R}{C^2}\right)^j \qquad \text{and} \qquad \left\|\frac{1}{|\partial_x\gamma_0|^j}\right\|_{1+\alpha} \leq \widetilde{C}(R,C)\,.$$

We also notice that these estimates are preserved during the flow. More precisely, following the proof in [20], one can show that there exists $\widetilde{T}(M,C) \in (0,1]$ such that for $T \in [0,\widetilde{T}(M,C)]$ every curve $\gamma \in \mathbb{E}_T^0 \cap B_M$ is regular and for all $t \in [0,\widetilde{T}(M,C)]$ it holds

$$\sup_{x\in[0,1]} \frac{1}{|\partial_x\gamma(t,x)|} \leq \frac{2}{C}\,.$$

Furthermore, for every $j \in \mathbb{N}$ and $y \in \{0,1\}$, we have

$$\left\| \frac{1}{|\partial_x \gamma|^j} \right\|_{\frac{\alpha}{4},\alpha} \leq \left(\frac{4M}{C^2} \right)^j \qquad \text{and} \qquad \left\| \frac{1}{|\partial_x \gamma(y)|^j} \right\|_{\frac{1+\alpha}{4}} \leq \widetilde{C}(R,C)\,.$$

Lemma 3.8. *For $T \in (0, \widetilde{T}(M,C)]$, the map $N_T(\gamma) := (N_{T,1}, N_{T,2}, \gamma_0)$ given by*

$$N_{T,1} : \begin{cases} \mathbb{E}_T^0 & \to C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1],\mathbb{R}^2), \\ \gamma & \mapsto f(\gamma) := f(\partial_x^4\gamma, \partial_x^3\gamma, \partial_x^2\gamma, \partial_x\gamma), \end{cases}$$
$$N_{T,2} : \begin{cases} \mathbb{E}_T^0 & \to C^{\frac{1+\alpha}{4}}([0,T],\mathbb{R}^2), \\ \gamma & \mapsto b(\gamma) := b(\partial_x^3\gamma, \partial_x\gamma) \end{cases}$$

where f, b are defined in (3.4), (3.5) respectively, is a well defined mapping from $\mathbb{E}_T^0$ to $\mathbb{F}_T^0$.

Proof. We have that $\gamma(t,\cdot)$ is a regular curve thanks to the discussion above, hence N_T is well defined. In order to show that $N_T(\gamma) \in \mathbb{F}_T^0$, we have to prove that γ_0 satisfies the linear compatibility conditions with respect to $(N_{T,1}, N_{T,2})$. This easily follows from the definition of $N_{T,1}, N_{T,2}$ and the fact that γ_0 is an admissible initial parametrization as in Definition 3.1. □

Definition 3.9. Let γ_0 be an admissible initial parametrization and let $C > 0$ the constant given by (3.12). For $M > 0$ and $T \in (0, \widetilde{T}(M,C)$ we define the mapping $K_T : \mathbb{E}_T^0 \to \mathbb{E}_T^0$ as $K_T := L_T^{-1} N_T$.

With a proof similar to [20, Proposition 3.28 and Proposition 3.29] one can prove the following result.

Proposition 3.10. *There exists a positive radius $M = M(R,C)$ and a positive time $\hat{T}(M) \in (0, \widetilde{T}(M,C))$ such that for all $T \in (0, \hat{T}(M)]$ the map $K_T : \mathbb{E}_T^0 \cap \overline{B_M} \to \mathbb{E}_T^0 \cap \overline{B_M}$ is well defined and it is a contraction.*

Theorem 3.11. *Let γ_0 be an admissible initial parametrization as in Definition 3.1. There exists a positive radius M and a positive time T such that the system (3.3) has a unique solution in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1]) \cap \overline{B_M}$.*

Proof. Let M and $\hat{T}(M)$ be the radius and time as in Proposition 3.10 and let $T \in (0, \hat{T}(M)]$. The solutions of (3.3) in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1]) \cap \overline{B_M}$ are the fixed points of K_T in $\mathbb{E}_T^0 \cap \overline{B_M}$. It is unique by the Banach-Caccioppoli contraction theorem as K_T is a contraction of the complete metric space $\mathbb{E}_T^0 \cap \overline{B_M}$. □

3.4 Geometric existence and uniqueness

In Theorem 3.11 we show that there exists a unique solution to the analytic problem (3.3) provided that the initial curve is admissible. In this section, we first establish a relation between geometrically admissible initial curves and admissible initial parametrizations, then we show the geometric uniqueness of the flow, in the sense that, up to reparametrization, the geometric problem (2.8) has a unique solution.

Lemma 3.12. *Suppose that γ_0 is a geometrically admissible initial curve as in Definition 2.2. Then, there exists a smooth function $\psi_0 : [0,1] \to [0,1]$ such that the reparametrization $\widetilde{\gamma}_0 = \gamma_0 \circ \psi_0$ of γ_0 is an admissible initial parametrization for the analytic problem* (3.3).

Proof. We look for a smooth map $\psi_0 : [0,1] \to [0,1]$ such that $\partial_x \psi_0(x) \neq 0$ for every $x \in [0,1]$, in order to have a map $\widetilde{\gamma}_0 = \gamma_0 \circ \psi_0 : [0,1] \to \mathbb{R}^2$ regular and of class $C^{4+\alpha}([0,1])$. Moreover, if we require that $\psi_0(y) = y$ for $y \in \{0,1\}$, then $\widetilde{\gamma}_0$ satisfies the attachment condition. Since the geometric quantities are invariant under reparametrization, also the non-degeneracy condition and the third order condition are still satisfied. In order to fulfil the second order condition $\partial_x^2 \widetilde{\gamma}_0(y) = 0$, we consider a map ψ_0 such that

$$\partial_x \psi_0(y) = 1 \qquad \text{and} \qquad \partial_x^2 \psi_0(y) = -\frac{\partial_x^2 \gamma_0(y)}{\partial_x \gamma_0(y)}$$

for $y \in \{0,1\}$. Thus, it remains to show that

$$\left(\widetilde{V}_0 \widetilde{\nu}_0 + \widetilde{T}_0 \widetilde{\tau}_0\right)_2 = 0\,.$$

As we notice above, this is equivalent to

$$\left(V_0 \nu_0 + \widetilde{T}_0 \tau_0\right)_2 = 0\,,$$

however, since γ_0 is a geometrically admissible initial curve, it is enough to prove that

$$\widetilde{T}_0 - T_0 = 0\,. \tag{3.13}$$

Thus, asking that $\partial_x^3 \psi_0(y) = 1$, we rewrite relation (3.13) as

$$g_1(\partial_x \gamma)(y) \partial_x^4 \psi_0(y) + g_2(\partial_x \gamma, \partial^2 \gamma, \partial_x^3 \gamma)(y) = 0$$

where g_1, g_2 are non-linear functions. Hence, $\partial_x^4 \psi_0(y)$ are uniquely determined for $y \in \{0,1\}$. In the end, we may choose ψ_0 to be the fourth Taylor polynomial near each boundary points, join these values up inside the interval $(0,1)$ and then make it smooth. □

Definition 3.13. Let γ_0 be a geometrically admissible initial curve as in Definition 2.2 and $T > 0$. A time dependent family of curves γ_t for $t \in [0,T)$ is a maximal solution to the elastic flow with initial datum γ_0 if it is a solution in the sense of Definition 2.3 in $[0,\hat{T}]$ for some $\hat{T} < T$ and if there does not exist a solution $\widetilde{\gamma}_t$ in $[0,\widetilde{T}]$ with $\widetilde{T} > T$ and such that $\gamma = \widetilde{\gamma}$ in $(0,T)$.

Following the arguments in [21, Lemma 5.8 and Lemma 5.9], one can show that a maximal solution to the elastic flow always exists and is unique in geometric sense, that is it is unique up to reparametrization. Hence, from now on we only consider the time T in Definition 3.13 which we call maximal time of existence and we denote by $T_{\max}$.

We notice that the following theorem is slightly different to the corresponding one in [21], where the authors firstly prove the geometric uniqueness in a "generic" time interval $[0,T]$ and then they show the existence of $T_{\max}$ using the fact that the solution is unique in a geometric sense. However, by means of an intermediate step, the result can be stated as follows.

Theorem 3.14. *[Geometric existence and uniqueness] Let γ_0 be a geometrically admissible initial curve as in Definition 2.2. Then, there exists a positive time $T_{\max}$ such that within the time interval $[0, T_{\max})$ there is a unique elastic flow γ_t in the sense of Definition 2.3.*

Proof. By Lemma 3.12 there exists a reparametrization $\widetilde{\gamma}_0$ of the geometrically admissible initial curve γ_0 which is an admissible initial parametrization in sense of Definition 3.1. Then, by Theorem 3.11 there exists a solution $\widetilde{\gamma}_t$ of system (3.3) in some maximal time interval $[0, \widetilde{T}_{\max}]$. In particular, $\widetilde{\gamma}_t$ is a solution to system (2.8).
Let us suppose that γ_t is another solution to the elastic flow in sense of Definition 2.3 in a time interval $[0, T']$ with the same geometrically admissible initial curve. We aim to show that there exists a time $T_{\max} \in (0, \max\{\widetilde{T}_{\max}, T'\})$ such that $\widetilde{\gamma}_t = \gamma_t$ for every $t \in [0, T_{\max}]$ as a curves.
To be precise, we need to construct a regular reparametrization $\psi(t,x) : [0, T_{\max}] \times [0,1] \to [0,1]$, such that the reparametrized curve $\sigma(t,x) = \gamma(t, \psi(t,x))$ is a solution to the analytic problem (3.3) and coincides with $\widetilde{\gamma}_t$ in a possibly small but positive time interval. Hence, computing the space and time derivative of $\sigma(t,x)$ as a composed function, and replacing in the evolution equation

$$\partial_t \sigma(t,x) = \frac{\partial_x^4 \sigma}{|\partial_x \sigma|^4} + l.o.t..$$

we get the following evolution equation for ψ

$$\begin{aligned}
\partial_t \psi(t,x) = & - \frac{\langle \partial_t \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle}{|\partial_x \gamma(t, \psi(t,x))|^2} + \frac{\langle \partial_x^4 \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle}{|\partial_x \gamma(t, \psi(t,x)|^6} \\
& + \frac{6 \langle \partial_x^3 \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle \partial_x^2 \psi(t,x)}{|\partial_x \gamma(t, \psi(t,x)|^6 (\partial_x \psi(t,x))^2} + \frac{3 \langle \partial_x^2 \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle (\partial_x^2 \psi(t,x))^2}{|\partial_x \gamma(t, \psi(t,x)|^6 (\partial_x \psi(t,x))^4} \\
& + \frac{4 \langle \partial_x^2 \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle \partial_x^3 \psi(t,x)}{|\partial_x \gamma(t, \psi(t,x)|^6 (\partial_x \psi(t,x))^3} + \frac{\partial_x^4 \psi(t,x)}{|\partial_x \gamma(t, \psi(t,x)|^2 (\partial_x \psi(t,x))^4} + l.o.t..
\end{aligned}$$

Taking into account the boundary conditions, we have that such parametrization has to satisfy the following boundary value problem

$$\begin{cases}
\partial_t \psi(t,x) = \frac{\partial_x^4 \psi(t,x)}{|\partial_x \gamma(t, \psi(t,x)|^2 (\partial_x \psi(t,x))^4} + g \\
\psi(t,y) = y \\
\partial_x^2 \psi(t,y) = - \frac{\langle \partial_x^2 \gamma(t, \psi(t,x)), \partial_x \gamma(t, \psi(t,x)) \rangle (\partial_x \psi)^2}{|\partial_x \gamma(t, \psi(t,x))|^2} \\
\psi(0,x) = \psi_0(x)
\end{cases} \tag{3.14}$$

for $y \in \{0,1\}$ and $t \in [0, T_{\max}]$, where the function ψ_0 is given by Lemma 3.12 and the terms in g depend on the solution ψ, $\partial_x^j \psi$ for $j \in \{1,2,3\}$ and $\partial_t \gamma$, $\partial_x^j \gamma$ for $j \in \{1,2,3,4\}$. From the computation above, it follows that the function γ and its time-space derivatives depend also on ψ. In order to remove this dependence, we consider the associated problem for the inverse of ψ, that is $\xi(t,\cdot) = \psi^{-1}(t,\cdot)$. So, the differentiation rules

$$
\begin{aligned}
\partial_z \xi(t,z) =& \partial_x \psi(t,\xi(t,z))^{-1} \\
\partial_z^2 \xi(t,z) =& -(\partial_z \xi(t,z))^3 \partial_x^2 \psi(t,\xi(t,z)) \\
\partial_z^3 \xi(t,z) =& 3\frac{(\partial_z^2 \xi(t,z))^2}{\partial_z \xi(t,z)} - (\partial_z \xi(t,z))^4 \partial_x^3 \psi(t,\xi(t,z)) \\
\partial_z^4 \xi(t,z) =& -15\frac{(\partial_z^2 \xi(t,x))^3}{(\partial_z \xi(t,x))^2} + 10\frac{\partial_z^2 \xi(t,z)\partial_z^3 \xi(t,z)}{\partial_z \xi(t,z)} - (\partial_z \xi(t,z))^5 \partial_x^4 \psi(t,\xi(t,z))
\end{aligned}
$$

yield the evolution equation

$$
\begin{aligned}
\partial_t \xi(t,z) =& -\frac{\langle \partial_t \sigma(t,z), \partial_z \sigma(t,z)\rangle}{|\partial_z \sigma(t,z)|^2}\partial_z \xi(t,z) + \frac{\langle \partial_z^4 \sigma(t,z), \partial_z \sigma(t,z)\rangle}{|\partial_z \sigma(t,z)|^6}\partial_z \xi(t,z) \\
& -\frac{6\langle \partial_z^3 \sigma(t,z), \partial_z \sigma(t,z)\rangle}{|\partial_z \sigma(t,z)|^6}\partial_z^2 \xi(t,z) + \frac{3\langle \partial_z^2 \sigma(t,z), \partial_z \sigma(t,z)\rangle}{|\partial_z \sigma(t,z)|^6}\frac{(\partial_z^2 \xi(t,z))^2}{\partial_z \xi(t,z)} \\
& + \frac{\langle \partial_z^2 \sigma(t,z), \partial_z \sigma(t,z)\rangle}{|\partial_z \sigma(t,z)|^6}\left(-4\partial_z^3 \xi(t,z) + \frac{12(\partial_z^2 \xi(t,z))^2}{\partial_z \xi(t,z)}\right) \\
& + \frac{1}{|\partial_z \sigma(t,z)|^2}\left(-\partial_z^4 \xi(t,z) + \frac{10\partial_z^2 \xi(t,z)\partial_z^3 \xi(t,z)}{\partial_z \xi(t,z)} - \frac{15(\partial_z^2 \xi(t,z))^3}{(\partial_z \xi(t,z))^2}\right) + l.o.t..
\end{aligned}
$$

Hence, we obtain the following system for ξ

$$
\begin{cases}
\partial_t \xi(t,z) = -\frac{\partial_z^4 \xi(t,z)}{|\partial_z \sigma(t,z|^2} + g \\
\xi(t,y) = y \\
\partial_z^2 \xi(t,y) = \frac{\langle \partial_z^2 \sigma(t,y), \partial_z \sigma(t,y)\rangle \partial_z \xi(t,y)}{|\partial_z \sigma(t,y)|^2} \\
\xi(0,z) = \psi_0^{-1}(z)
\end{cases}
\tag{3.15}
$$

where g is a non-linear smooth function which depends on $\partial_z\sigma, \partial_z^2\sigma, \partial_z^3\sigma, \partial_z^4\sigma, \partial_t\sigma, \partial_x\xi, \partial_x^2\xi, \partial_z^3\xi$. We observe that the resulting system for ξ has a very similar structure as (3.7): once we linearize system (3.15) and apply the linear theory developed by Solonnikov in [44], we get well–posedness. Contraction estimates similar to our previous one allows to conclude the existence and uniqueness of solution with a fixed point argument. Reversing the above argumentation yields that the inverse functions ψ solves system (3.14).
Then, σ_t is a solution to the system (3.3). Indeed, the motion equation follows from (3.14) and the geometric evolution of γ_t in normal direction. The geometric boundary conditions, that is attachment, curvature and third order conditions, are satisfied as γ_t is a solution to the geometric problem. Moreover, the boundary conditions on $\partial_x^2\psi$ in system (3.14) ensures that σ_t satisfies the second order condition.
Thus, by uniqueness of the analytic problem proved in Theorem 3.11, σ_t (that is γ_t up to reparametrization) and $\widetilde{\gamma}_t$ need to coincide on a small time interval. □

4 Curvature bounds

In this section we estimate the derivative of the curvature.

In order to simplify the notation, we introduce the following polynomials.

Definition 4.1. We denote by $\mathfrak{p}^h_\sigma(k)$ a polynomial in $k, \dots, \partial_s^h k$ with constant coefficients in $\mathbb{R}$ such that every monomial it contains is of the form

$$C \prod_{l=0}^{h} (\partial_s^l k)^{\alpha_l} \quad \text{with} \quad \sum_{l=0}^{h} (l+1)\alpha_l = \sigma\,,$$

where $\alpha_l \in \mathbb{N}$ for $l \in \{0, \dots, h\}$ and $\alpha_{l_0} \geq 1$ for at least one index l_0.

Remark 4.2. We notice that

$$\begin{aligned} \partial_s \left(\mathfrak{p}^h_\sigma(k) \right) &= \mathfrak{p}^{h+1}_{\sigma+1}(k)\,, \\ \mathfrak{p}^{h_1}_{\sigma_1}(k)\mathfrak{p}^{h_2}_{\sigma_2}(k) &= \mathfrak{p}^{\max\{h_1,h_2\}}_{\sigma_1+\sigma_2}(k)\,, \\ \mathfrak{p}^{h_1}_{\sigma}(k) + \mathfrak{p}^{h_2}_{\sigma}(k) &= \mathfrak{p}^{\max\{h_1,h_2\}}_{\sigma}(k). \end{aligned} \tag{4.1}$$

Moreover, following the arguments in [29], we get

$$\partial_t \left(\mathfrak{p}^h_\sigma(k) \right) = \mathfrak{p}^{h+4}_{\sigma+4}(k) + \Lambda \mathfrak{p}^{h+1}_{\sigma+1}(k) + \mu \mathfrak{p}^{h+2}_{\sigma+2}(k)\,. \tag{4.2}$$

Lemma 4.3. *If γ satisfies (2.9), then the j–th derivatives of scalar curvature of a curve satisfy*

$$\partial_t \partial_s^j k = -2\partial_s^{j+4} k - 5k^2 \partial_s^{j+2} k + \mu\, \partial_s^{j+2} k + \Lambda \partial_s^{j+1} k + \mathfrak{p}^{j+1}_{j+5}(k) + \mu\, \mathfrak{p}^{j}_{j+3}(k)\,. \tag{4.3}$$

Proof. For $j = 0$ we have

$$\begin{aligned} \partial_t k &= -2\partial_s^4 k - 5k^2 \partial_s^2 k - 6k\,(\partial_s k)^2 + \Lambda \partial_s k - k^5 + \mu \left(\partial_s^2 k + k^3 \right) \\ &= -2\partial_s^4 k - 5k^2 \partial_s^2 k + \mu \partial_s^2 k + \Lambda \partial_s k + \mathfrak{p}^1_5(k) + \mu \mathfrak{p}^0_4(k)\,. \end{aligned}$$

Then, we assume it is true for j and we show that it is true for $j+1$, since we have

$$\begin{aligned} \partial_t \partial_s^{j+1} k =& \partial_t \partial_s \partial_s^j k = \partial_s \partial_t \partial_s^j k + (kV - \partial_s \Lambda)\partial_s^{j+1} k \\ =& -2\partial_s^{j+5} k - 10k\partial_s k \partial_s^{j+2} k - 5k^2 \partial_s^{j+3} k + \mu \partial_s^{j+3} k + \partial_s \Lambda \partial_s^{j+1} k + \Lambda \partial_s^{j+2} k \\ &+ \mathfrak{p}^{j+2}_{j+6} + \mu \mathfrak{p}^{j+1}_{j+4} - 2k\partial_s^2 k \partial_s^{j+1} k - k^4 \partial_s^{j+1} k + \mu k^2 \partial_s^{j+1} k - \partial_s \Lambda \partial_s^{j+1} k \\ =& -2\partial_s^{j+5} k - 5k^2 \partial_s^{j+3} k + \mu\, \partial_s^{j+3} k + \Lambda \partial_s^{j+2} k + \mathfrak{p}^{j+2}_{j+6}(k) + \mu\, \mathfrak{p}^{j+1}_{j+4}(k)\,. \end{aligned}$$

By induction, formula (4.3) holds. □

4.1 Bound on $\|\partial_s^2 k\|_{L^2}$

We show that, once the following condition is satisfied, the tangential velocity goes like the normal one at least at boundary points.

Definition 4.4. Let γ_t be a maximal solution to the elastic flow in $[0, T_{\max})$. We say that γ_t satisfies the *uniform non–degeneracy condition* if there exists $\rho > 0$ such that

$$\tau_2(y) \geq \rho \tag{4.4}$$

for every $t \in [0, T_{\max})$ and $y \in \{0, 1\}$.

Lemma 4.5. *Let γ_t be a maximal solution to the elastic flow of curves subjected to boundary conditions (2.6), such that the uniform non–degeneracy condition (4.4) holds in $[0, T_{\max})$. Then, for every $t \in [0, T_{\max})$ and $y \in \{0, 1\}$, the tangential velocity is proportional to the normal velocity, that is*

$$\Lambda(y) \approx \partial_s^2 k(y) \, .$$

Proof. Since the boundary point are constrained to the x-axis, we have that

$$(\partial_t \gamma_t)_2(y) = -2\partial_s^2 k(y)\nu_2(y) + \Lambda(y)\tau_2(y) = 0$$

for $y \in \{0, 1\}$ and $t \in [0, T_{\max})$. By the fact that τ_2 (hence, ν_2) are bounded from below at boundary points, it follows

$$\Lambda(y) = 2\partial_s^2 k(y) \frac{\nu_2(y)}{\tau_2(y)} \approx \partial_s^2 k(y)$$

for $t \in [0, T_{\max})$ and $y \in \{0, 1\}$. □

Proposition 4.6. *Let γ_t be a maximal solution to the elastic flow of curves subjected to boundary conditions (2.6) with initial datum γ_0, which satisfies the* uniform non-degeneracy condition (4.4) *in the maximal time interval $[0, T_{\max})$. Then for all $t \in [0, T_{\max})$ it holds*

$$\frac{d}{dt} \int_\gamma |\partial_s^2 k|^2 \, \mathrm{d}s \leq C(\mathcal{E}(\gamma_0)) \, .$$

Proof. From formula (4.3) we have

$$\begin{aligned}
\frac{d}{dt} \int_\gamma |\partial_s^2 k|^2 \, \mathrm{d}s &= \int_\gamma 2\partial_s^2 k \partial_t \partial_s^2 k + (\partial_s^2 k)^2 (\partial_s \Lambda - kV) \, \mathrm{d}s \\
&= \int_\gamma -4\partial_s^2 k \partial_s^6 k - 10k^2 \partial_s^2 k \partial_s^4 k + 2\mu \partial_s^2 k \partial_s^4 k + 2\Lambda \partial_s^3 k \partial_s^2 k \\
&\quad + \mathfrak{p}_{10}^3(k) + \mu \mathfrak{p}_8^2(k) + (\partial_s^2 k)^2 (\partial_s \Lambda - kV) \, \mathrm{d}s \\
&= \int_\gamma -4\partial_s^2 k \partial_s^6 k - 10k^2 \partial_s^2 k \partial_s^4 k + 2\mu \partial_s^2 k \partial_s^4 k \\
&\quad + 2\Lambda \partial_s^3 k \partial_s^2 k + 2\partial_s \Lambda (\partial_s^2 k)^2 + \mathfrak{p}_{10}^3(k) + \mu \mathfrak{p}_8^2(k) \, \mathrm{d}s \, .
\end{aligned}$$

Thus, the terms involving the tangential velocity can be written as

$$\int_\gamma \partial_s \Lambda (\partial_s^2 k)^2 + 2\Lambda \partial_s^2 k \partial_s^3 k \, \mathrm{d}s = \int_\gamma \partial_s (\Lambda (\partial_s^2 k)^2) \, \mathrm{d}s = \Lambda (\partial_s^2 k)^2 \Big|_0^1,$$

moreover, integrating by parts the other terms, we get

$$\begin{aligned}
\frac{d}{dt} \int_\gamma |\partial_s^2 k|^2 \, \mathrm{d}s &= \int_\gamma -4(\partial_s^4 k)^2 - 2\mu (\partial_s^3 k)^2 + \mathfrak{p}_{10}^3(k) + \mu \mathfrak{p}_8^2(k) \, \mathrm{d}s \\
&\quad + \Lambda (\partial_s^2 k)^2 \Big|_0^1 + 4(\partial_s^3 k \partial_s^4 k - \partial_s^2 k \partial_s^5 k) \Big|_0^1 - 10k^2 \partial_s^2 k \partial_s^3 k \Big|_0^1 \\
&\quad - 12(\partial_s k)^3 \partial_s^2 k \Big|_0^1 + 2\mu \partial_s^2 k \partial_s^3 k \Big|_0^1 \, .
\end{aligned} \tag{4.5}$$

By means of Navier boundary conditions, the boundary terms in equation (4.5) reduce to

$$\Lambda(\partial_s^2 k)^2\Big|_0^1 + 4(\partial_s^3 k\partial_s^4 k - \partial_s^2 k\partial_s^5 k)\Big|_0^1 - 12(\partial_s k)^3\partial_s^2 k\Big|_0^1 + 2\mu\partial_s^2 k\partial_s^3 k\Big|_0^1. \tag{4.6}$$

We aim to lower the order of the second and third terms in (4.6). In particular, differentiating in time the condition $k(y) = 0$ using relation (2.12), we have

$$4\partial_s^3 k\partial_s^4 k = 2\Lambda\partial_s k\partial_s^3 k + 2\mu\partial_s^2 k\partial_s^3 k\,. \tag{4.7}$$

Moreover, from conditions in (2.6), it follows

$$\partial_t\langle\gamma, 2\partial_s k\nu - \mu\tau\rangle = \langle V\nu + \Lambda\tau, \partial_t(2\partial_s k\nu - \mu\tau)\rangle = 0$$

then, computing the scalar production using (4.3), we end up with

$$\begin{aligned}
0 =& -2\partial_t\partial_s kV + 2\Lambda\partial_s k\partial_s V + \mu V\partial_s V\\
=&4\partial_t\partial_s k\partial_s^2 k + 2\Lambda\partial_s k(-2\partial_s^3 k + \mu\partial_s k) - 2\mu\partial_s^2 k(-2\partial_s^3 k + \mu\partial_s k)\\
=&4\partial_s\partial_t k\partial_s^2 k - 4\partial_s\Lambda\partial_s k\partial_s^2 k + 2\Lambda\partial_s k(-2\partial_s^3 k + \mu\partial_s k)\\
&- 2\mu\partial_s^2 k(-2\partial_s^3 k + \mu\partial_s k)\\
=& -8\partial_s^2 k\partial_s^5 k - 24(\partial_s k)^3\partial_s^2 k + 4\partial_s\Lambda\partial_s k\partial_s^2 k + 4\Lambda(\partial_s^2 k)^2\\
&+ 4\mu\partial_s^2 k\partial_s^3 k - 4\partial_s\Lambda\partial_s k\partial_s^2 k\\
&+ 2\Lambda\partial_s k(-2\partial_s^3 k + \mu\partial_s k) - 2\mu\partial_s^2 k(-2\partial_s^3 k + \mu\partial_s k)\\
=& -8\partial_s^2 k\partial_s^5 k - 24(\partial_s k)^3\partial_s^2 k + 4\Lambda(\partial_s^2 k)^2 - 4\Lambda\partial_s k\partial_s^3 k\\
&+ 8\mu\partial_s^2 k\partial_s^3 k + 2\mu\Lambda(\partial_s k)^2 - 2\mu^2\partial_s k\partial_s^2 k\,,
\end{aligned}$$

that is,

$$-4\partial_s^2 k\partial_s^5 k = 12(\partial_s k)^3\partial_s^2 k - 2\Lambda(\partial_s^2 k)^2 + 2\Lambda\partial_s k\partial_s^3 k - 4\mu\partial_s^2 k\partial_s^3 k - \mu\Lambda(\partial_s k)^2 + \mu^2\partial_s k\partial_s^2 k\,. \tag{4.8}$$

Hence, replacing the terms (4.7) and (4.8) in (4.6), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\gamma |\partial_s^2 k|^2\,\mathrm{d}s =& \int_\gamma -4(\partial_s^4 k)^2 - 2\mu(\partial_s^3 k)^2 + \mathfrak{p}_{10}^3(k) + \mu\mathfrak{p}_8^2(k)\,\mathrm{d}s\\
&- \Lambda(\partial_s^2 k)^2\Big|_0^1 + 4\Lambda\partial_s k\partial_s^3 k\Big|_0^1 - \mu\Lambda(\partial_s k)^2\Big|_0^1 + \mu^2\partial_s k\partial_s^2 k\Big|_0^1.
\end{aligned}$$

We now recall that Λ is proportional to $\partial_s^2 k$ at boundary points (see Lemma 4.5), hence it follows that $\Lambda\mathfrak{p}_\sigma^h(k) = \mathfrak{p}_{\sigma+3}^{\max\{2,h\}}(k)$. Thus, we have

$$\begin{aligned}
\frac{d}{dt}\int_\gamma |\partial_s^2 k|^2\,\mathrm{d}s =& -4\|\partial_s^4 k\|_{L^2(\gamma)}^2 - 2\mu\|\partial_s^3 k\|_{L^2(\gamma)}^2 + \int_\gamma \mathfrak{p}_{10}^3(k) + \mu\mathfrak{p}_8^2(k)\,\mathrm{d}s\\
&+ \mathfrak{p}_9^3(k)\Big|_0^1 + \mu\mathfrak{p}_7^3(k)\Big|_0^1 + \mu^2\mathfrak{p}_5^2(k)\Big|_0^1.
\end{aligned}$$

By means of Lemma 4.6 and Lemma 4.7 in [29], for any $\varepsilon > 0$ we have

$$\begin{aligned}
\int_\gamma |\mathfrak{p}^3_{10}\,(k)\,|\,\mathrm{d}s \leq & \varepsilon\|\partial_s^4 k\|^2_{L^2} + C(\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + \|k\|^{\Theta_1}_{L^2}\right)\,,\\
\int_\gamma |\mathfrak{p}^2_{8}\,(k)\,|\,\mathrm{d}s \leq & \varepsilon\|\partial_s^3 k\|^2_{L^2} + C(\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + C\|k\|^{\Theta_2}_{L^2}\right)\,,\\
\mathfrak{p}^3_{9}(k)(y)| \leq & \varepsilon\|\partial_s^4 k\|^2_{L^2} + C(\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + \|k\|^{\Theta_3}_{L^2}\right)\,,\\
\mathfrak{p}^3_{7}(k)(y) \leq & \varepsilon\|\partial_s^4 k\|^2_{L^2} + C(\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + C\|k\|^{\Theta_4}_{L^2}\right)\,,\\
\mathfrak{p}^2_{5}(k)(y) \leq & \varepsilon\|\partial_s^3 k\|^2_{L^2} + C(\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + C\|k\|^{\Theta_5}_{L^2}\right)\,,
\end{aligned}$$

for some exponents $\Theta_i > 2$ and $i = 1, \ldots, 5$.
Hence, we get

$$\frac{d}{dt}\int_\gamma |\partial_s^2 k|^2\,\mathrm{d}s \leq -C\left(\|\partial_s^4 k\|^2_{L^2(\gamma)} + \mu\|\partial_s^3 k\|^2_{L^2(\gamma)}\right) + C\left(\|k^2\|^2_{L^2(\gamma)} + \|k^2\|^{\Theta}_{L^2(\gamma)}\right)$$

for some exponent $\Theta > 2$ and constant C which depend on $\ell(\gamma)$. Using the energy monotonicity proved in Proposition 2.7, we conclude that

$$\frac{d}{dt}\int_\gamma |\partial_s^2 k|^2\,\mathrm{d}s \leq C(\mathcal{E}(\gamma_0))\,.$$

□

4.2 Bound on $\|\partial_s^6 k\|_{L^2}$

We observe that, since (2.9) is a parabolic fourth order equation, after having controlled the second order derivative of the curvature, it is natural to control the sixth order derivative of the curvature. Then, using interpolation inequalities, we get estimates for all the intermediate order. Before doing that, we notice that the elastic flow of curves become instantaneously smooth. More precisely, following the proof presented in [29] in the case of closed curves (both using the so-called Angenent's parameter trick [6, 5, 11] and the classical theory of linear parabolic equations [44]), one can show that given a solution to the elastic flow in a time interval $[0, T]$, then it is smooth for positive times, in the sense that it admits a C^∞–parametrization in the interval $[\varepsilon, T]$ for every $\varepsilon \in (0, T)$.

From now on, we denote by

$$v := \partial_t\gamma = V\nu + \Lambda\tau \tag{4.9}$$

the velocity of γ. Hence, by means of integration by parts and the commutation rule in Lemma 2.6, we get the following identity

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\gamma |\partial_t^\perp v|^2\,\mathrm{d}s = & -2\int_\gamma |(\partial_s^\perp)^2(\partial_t^\perp v)|^2\,\mathrm{d}s \frac{1}{2}\int_\gamma |\partial_t^\perp v|^2(\partial_s\Lambda - kV)\,\mathrm{d}s + \int_\gamma \langle Y, \partial_t^\perp v\rangle\,\mathrm{d}s\\
& -2\langle\partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v)\rangle\Big|_0^1 + 2\langle\partial_s^\perp(\partial_t^\perp v), (\partial_s^\perp)^2(\partial_t^\perp v)\rangle\Big|_0^1\,,
\end{aligned} \tag{4.10}$$

where $Y = \partial_t^\perp(\partial_t^\perp v) + 2(\partial_s^\perp)^4(\partial_t^\perp v)$.
Before proceeding, we prove the following lemma, which gives estimates for some special family of polynomials.

Lemma 4.7. *Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a smooth regular curve. For all* $j \leq 7$*, if the polynomial* $\mathfrak{p}^j_{\sigma(j)}(k)$ *defined as in Definition 4.1 satisfies one of the following conditions:*

(i) $\sigma(j) \geq 2(l+1)$ *for all* $l \leq j$*,*

(ii) $\sigma(j) \geq 2(l+1)$ *for all* $l \leq j-1$ *and* $(j+1) \leq \sigma(j) < 2(j+1)$*,*

and

$$\sigma(j) - \sum_{l=0}^{j} \alpha_l < 15\,, \tag{4.11}$$

then there exists a constant C *and an exponent* $\Theta > 2$ *such that*

$$\int_\gamma |\mathfrak{p}^j_{\sigma(j)}(k)\,|\,\mathrm{d}s \leq \varepsilon \|\partial_s^8 k\|^2_{L^2} + C(j,\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + \|k\|^{\Theta}_{L^2}\right)\,.$$

Similarly, for all $j \leq 7$ *and*

$$\sigma'(j) - \sum_{l=0}^{j} \alpha_l < 16\,,$$

there exists a constant C *and an exponent* $\Theta' > 2$ *such that for* $y \in \{0,1\}$ *it holds*

$$|\mathfrak{p}^j_{\sigma'(j)}(k)(y)| \leq \varepsilon \|\partial_s^8 k\|^2_{L^2} + C(j,\varepsilon,\ell(\gamma))\left(\|k\|^2_{L^2} + \|k\|^{\Theta'}_{L^2}\right)\,.$$

Proof. By definition, every monomial of $\mathfrak{p}^j_{\sigma(j)}(k)$ is of the form $C\prod_{l=0}^{j}(\partial_s^l k)^{\alpha_l}$ with

$$\alpha_l \in \mathbb{N} \qquad \text{and} \qquad \sum_{l=0}^{j} \alpha_l(l+1) = \sigma(j)\,.$$

We set

$$\beta_l := \frac{\sigma(j)}{(l+1)\alpha_l}$$

for every $l \leq j$ and we take $\beta_l = 0$ if $\alpha_l = 0$. We observe that $\sum_{l \in J} \frac{1}{\beta_l} = 1$, hence by Hölder inequality, we get

$$C\int_\gamma \prod_{l=0}^{j} |\partial_s^l k|^{\alpha_l}\,\mathrm{d}s \leq C\prod_{l=0}^{j}\left(\int_\gamma |\partial_s^l k|^{\alpha_l\beta_l}\,\mathrm{d}s\right)^{\frac{1}{\beta_l}} = C\prod_{l=0}^{j}\|\partial_s^l k\|^{\alpha_l}_{L^{\alpha_l\beta_l}}\,.$$

If condition (i) holds, then $\alpha_l\beta_l \geq 2$ for every $l \in J$. Applying the Gagliardo–Nirenberg inequality (see [3] or [7], for instance) for every $l \leq j$ yields

$$\|\partial_s^l k\|_{L^{\alpha_l\beta_l}} \leq C(l,j,\alpha_l,\beta_l,\ell(\gamma))\|\partial_s^8 k\|^{\eta_l}_{L^2}\|k\|^{1-\eta_l}_{L^2} + \|k\|_{L^2}$$

where the coefficient η_l is given by

$$\eta_l = \frac{l + 1/2 - 1/(\alpha_l\beta_l)}{8} \in \left[\frac{l}{8}, 1\right)\,. \tag{4.12}$$

Then, we have

$$
\begin{aligned}
C\int_\gamma \prod_{l=0}^{j} |\partial_s^l k|^{\alpha_l}\,\mathrm{d}s &\leq C\prod_{l=0}^{j} \|\partial_s^l k\|^{\alpha_l}_{L^{\alpha_l\beta_l}}\\
&\leq C\prod_{l=0}^{j} \|k\|_{L^2}^{(1-\eta_l)\alpha_l}\left(\|\partial_s^8 k\|_{L^2}+\|k\|_{L^2}\right)^{\eta_l\alpha_l}_{L^2}\\
&= C\|k\|_{L^2}^{\sum_{l=0}^{j}(1-\eta_l)\alpha_l}\left(\|\partial_s^8 k\|_{L^2}+\|k\|_{L^2}\right)^{\sum_{l=0}^{j}\eta_l\alpha_l}_{L^2}.
\end{aligned}
$$

Moreover, from condition (4.11), we have

$$
\sum_{l=0}^{j}\eta_l\alpha_l \leq \frac{\sigma(j)-1-\sum_{l=0}^{j}\alpha_l}{8} < 2\,,
$$

that is, by means of Young's inequality with $p=\frac{2}{\sum_{l=0}^{j}\eta_l\alpha_l}$ and $q=\frac{2}{2-\sum_{l=0}^{j}\eta_l\alpha_l}$ we obtain

$$
C\int_\gamma \prod_{l=0}^{j} |\partial_s^l k|^{\alpha_l}\,\mathrm{d}s \leq \varepsilon C\left(\|\partial_s^8 k\|_{L^2}+\|k\|_{L^2}\right)^2_{L^2}+\frac{C}{\varepsilon}\|k\|^{\Theta}_{L^2} \tag{4.13}
$$

where constant C depends on j, ε, $\ell(\gamma)$ and $\Theta > 2$.

Otherwise, if condition (ii) holds, we have $1\leq \alpha_j\beta_j < 2$, that is

$$
\|\partial_s^j k\|^{\alpha_j}_{L^{\alpha_j\beta_j}} \leq \|\partial_s^j k\|^{\alpha_j}_{L^2} \leq \|\partial_s^8 k\|^{\eta_j\alpha_j}_{L^2}\|k\|^{(1-\eta_j)\alpha_j}_{L^2}+\|k\|^{\alpha_j}_{L^2}
$$

where $\eta_j=\frac{j}{8}$ and we use the fact that $\ell(\gamma)$ is bounded.

Hence, following as in the previous case, we have

$$
\begin{aligned}
C\int_\gamma \prod_{l=0}^{j} |\partial_s^l k|^{\alpha_l}\,\mathrm{d}s \leq& C\prod_{l=0}^{j-1} \|\partial_s^l k\|^{\alpha_l}_{L^{\alpha_l\beta_l}}\|\partial_s^j k\|^{\alpha_j}_{L^{\alpha_j\beta_j}}\\
\leq& C\left(\|\partial_s^8 k\|_{L^2}^{\sum_{l=0}^{j-1}\eta_l\alpha_l}\|k\|_{L^2}^{\sum_{l=0}^{j-1}(1-\eta_l)\alpha_l}+\|k\|_{L^2}^{\sum_{l=0}^{j-1}\alpha_l}\right)\\
&\left(\|\partial_s^8 k\|^{\eta_j\alpha_j}_{L^2}\|k\|^{(1-\eta_j)\alpha_j}_{L^2}+\|k\|^{\alpha_j}_{L^2}\right)\\
\leq& C\|k\|_{L^2}^{\sum_{l=0}^{j}(1-\eta_l)\alpha_l}\Big(\|\partial_s^8 k\|_{L^2}^{\sum_{l=0}^{j}\eta_l\alpha_l}+\|\partial_s^8 k\|_{L^2}^{\sum_{l=0}^{j-1}\eta_l\alpha_l}\|k\|_{L^2}^{\eta_j\alpha_j}\\
&\qquad+\|\partial_s^8 k\|_{L^2}^{\eta_j\alpha_j}\|k\|_{L^2}^{\sum_{l=0}^{j-1}\eta_l\alpha_l}+\|k\|_{L^2}^{\sum_{l=0}^{j}\eta_l\alpha_l}\Big)
\end{aligned}
$$

where for all $l\leq j-1$ the coefficient η_l is given by expression (4.12) and $\eta_j=\frac{j}{8}$. Applying again Young's inequality, since $\sum_{l=0}^{j}\eta_l\alpha_l < 2$ still holds, we obtain estimate (4.13).

The second part of lemma, comes using same arguments. □

Lemma 4.8. *Let γ_t be a maximal solution to the elastic flow of curves subjected to Navier boundary conditions (2.6), such that the* uniform non-degeneracy condition *(4.4) holds in the maximal time interval $[0, T_{\max})$. Then, for every $j\in\mathbb{N}$, it holds*

$$
\partial_s^j\Lambda(y)\mathfrak{p}_\sigma^h(k)(y) = \mathfrak{p}_{\sigma+j+3}^{\max\{h,j+2\}}(k)(y)
$$

and

$$\partial_t \Lambda(y)\mathfrak{p}_\sigma^h(k)(y) = \mathfrak{p}_{\sigma+7}^{\max\{h,6\}}(k)(y) + \mu\mathfrak{p}_{\sigma+5}^{\max\{h,4\}}(k)(y)$$

for every $t \in [0, T_{\max})$ and $y \in \{0,1\}$.

Proof. By means of Lemma 4.5 and by the fact that τ_2 is bounded from below, we have

$$\Lambda(y) = \partial_s^2 k(y)\frac{\nu_2(y)}{\tau_2(y)} = \mathfrak{p}_3^2(k)(y)$$

for $y \in \{0,1\}$. Hence, by the formula of the derivative of polynomials (4.1), it follows

$$\partial_s^j \Lambda(y) = \mathfrak{p}_{j+3}^{j+2}(k)(y)\,,$$

and thus, by properties in (4.1),

$$\partial_s^j \Lambda(y)\mathfrak{p}_\sigma^h(k)(y) = \mathfrak{p}_{j+3}^{j+2}(k)(y)\mathfrak{p}_\sigma^h(k)(y) = \mathfrak{p}_{\sigma+j+3}^{\max\{h,j+2\}}(k)(y)\,.$$

Similarly, by formula (4.2), we have

$$\partial_t \Lambda(y) = \partial_t\Big(\mathfrak{p}_3^2(k)(y)\Big) = \mathfrak{p}_7^6(k)(y) + \mu\mathfrak{p}_5^4(k)(y)$$

then,

$$\partial_t \Lambda(y)\mathfrak{p}_\sigma^h(k)(y) = \mathfrak{p}_{\sigma+7}^{\max\{h,6\}}(k)(y) + \mu\mathfrak{p}_{\sigma+5}^{\max\{h,4\}}(k)(y)\,.$$

□

From now on, for any $t \in [0, T_{\max})$, we choose the tangential velocity Λ inside the space interval $[0,1]$ as the linear interpolation between the value at the boundary points, that is

$$\Lambda(t,x) = \Lambda(t,0)\Big(1 + \frac{\Lambda(t,1)-\Lambda(t,0)}{\Lambda(t,0)}\frac{1}{\ell(\gamma)}\int_0^x |\partial_x\gamma|\,dx\Big)\,. \tag{4.14}$$

Lemma 4.9. *Let Λ be the tangential velocity defined in (4.14), there exist two constants $C_1 = C_1(\ell(\gamma))$ and $C_2 = C_2(\mathcal{E}(\gamma_0), \ell(\gamma))$ such that*

$$\begin{aligned}
|\partial_s \Lambda(t,x)| &\leq C_1(|\Lambda(t,1)| + |\Lambda(t,0)|)\,,\\
|\partial_t \Lambda(t,x)| &\leq C_2\Big[|\partial_t\Lambda(t,0)| + |\partial_t\Lambda(t,1)| + |\partial_t\Lambda(t,0)|\frac{|\Lambda(t,1)|}{|\Lambda(t,0)|}\\
&\qquad + |\Lambda(t,1)-\Lambda(t,0)|^2 + |\Lambda(t,1)-\Lambda(t,0)|\Big]
\end{aligned}$$

for $t \in [0, T_{\max})$ and $x \in [0,1]$.

Proof. From (4.14) it easily follows that

$$\partial_s \Lambda(t,x) = \frac{\Lambda(t,1)-\Lambda(t,0)}{\ell(\gamma)} \qquad \text{and} \qquad \partial_s^j \Lambda(t,x) = 0 \quad \text{for } j \geq 2.$$

Moreover, taking the time derivative, we get

$$\begin{aligned}\partial_t\Lambda(t,x) =&\partial_t\Lambda(t,0)\Big(1+\frac{\Lambda(t,1)-\Lambda(t,0)}{\Lambda(t,0)}\frac{1}{\ell(\gamma)}\int_0^x |\partial_x\gamma|\,dx\Big)\\ &+\frac{(\partial_t\Lambda(t,1)-\partial_t\Lambda(t,0))\Lambda(t,0)-(\Lambda(t,1)-\Lambda(t,0))\partial_t\Lambda(t,0)}{\Lambda(t,0)}\frac{1}{\ell(\gamma)}\int_0^x|\partial_x\gamma|\,dx\\ &-\big(\Lambda(t,1)-\Lambda(t,0)\big)\frac{1}{\ell^2(\gamma)}\frac{d(\ell(\gamma))}{dt}\int_0^x|\partial_x\gamma|\,dx\\ &+\big(\Lambda(t,1)-\Lambda(t,0)\big)\frac{1}{\ell(\gamma)}\frac{d}{dt}\int_0^x|\partial_x\gamma|\,dx\\ =&\partial_t\Lambda(t,0)\Big(1+\frac{\Lambda(t,1)-\Lambda(t,0)}{\Lambda(t,0)}\frac{1}{\ell(\gamma)}\int_0^x |\partial_x\gamma|\,dx\Big)\\ &+\frac{(\partial_t\Lambda(t,1)-\partial_t\Lambda(t,0))\Lambda(t,0)-(\Lambda(t,1)-\Lambda(t,0))\partial_t\Lambda(t,0)}{\Lambda(t,0)}\frac{1}{\ell(\gamma)}\int_0^x|\partial_x\gamma|\,dx\\ &-\frac{\big(\Lambda(t,1)-\Lambda(t,0)\big)^2}{\ell^2(\gamma)}\int_0^x|\partial_x\gamma|\,dx+\frac{\Lambda(t,1)-\Lambda(t,0)}{\ell^2(\gamma)}\int_\gamma kV\,\mathrm{d}s\int_0^x|\partial_x\gamma|\,dx\\ &+\big(\Lambda(t,1)-\Lambda(t,0)\big)\frac{1}{\ell(\gamma)}\frac{d}{dt}\int_0^x|\partial_x\gamma|\,dx\,.\end{aligned}$$

where we used relations (2.10). Hence, noticing that from interpolation and from Proposition 4.6 it follows

$$\int_\gamma kV\le C(\mathcal{E}(\gamma_0))\,,$$

we obtain the last estimate in the statement. □

Lemma 4.10. *If γ satisfies (2.9), then*

$$\begin{aligned}\partial_t^2\gamma =&\Big(4\partial_s^6k+10k^2\partial_s^4k+\mathfrak{p}_7^3(k)-4\Lambda\partial_s^3k-6\Lambda k^2\partial_sk+\Lambda^2k\\ &-4\mu\partial_s^4k+\mu\mathfrak{p}_5^2(k)+2\mu\Lambda\partial_sk+\mu^2\partial_s^2k+\mu^2\mathfrak{p}_3^0(k)\Big)\nu\\ &+\Big(\partial_t\Lambda+\mathfrak{p}_7^3(k)-2\Lambda k\partial_s^3k-3\Lambda k^3\partial_sk+\mu\mathfrak{p}_5^3(k)+\mu\Lambda k\partial_sk+\mu^2\mathfrak{p}_3^1(k)\Big)\tau\,.\end{aligned}\tag{4.15}$$

Proof. We firstly compute

$$\partial_tV=4\partial_s^6k+\mathfrak{p}_7^3(k)-2\Lambda\partial_s^3k+\Lambda\mathfrak{p}_4^1(k)-4\mu\partial_s^4k+\mu\mathfrak{p}_5^2(k)+\mu\Lambda\partial_sk+\mu^2\partial_s^2k+\mu^2\mathfrak{p}_3^0(k)$$

and

$$\begin{aligned}\partial_t\partial_sV=&4\partial_s^7k+\mathfrak{p}_8^4(k)-2\Lambda\partial_s^4k+\Lambda\mathfrak{p}_5^2(k)+\partial_s\Lambda\mathfrak{p}_4^1(k)-4\mu\partial_s^5k+\mu\mathfrak{p}_6^3(k)\\ &+\mu\Lambda\partial_s^2k+3\mu\partial_s\Lambda\partial_sk+\mu^2\partial_s^3k+\mu^2\mathfrak{p}_4^1(k)\,.\end{aligned}$$

Then, by means of Lemma 2.6, we have

$$\begin{aligned}\partial_t^2\tau=&(\partial_t\partial_sV+\partial_t\Lambda k+\Lambda\partial_tk)\nu-(\partial_sV+\Lambda k)^2\tau\\ =&\Big(4\partial_s^7k+\mathfrak{p}_8^4(k)-4\Lambda\partial_s^4k+\Lambda\mathfrak{p}_5^2(k)+\partial_s\Lambda\mathfrak{p}_4^1(k)+\Lambda^2\partial_sk+\partial_t\Lambda k-4\mu\partial_s^5k+\mu\mathfrak{p}_6^3(k)\\ &+\mu\Lambda\mathfrak{p}_3^2(k)+3\mu\partial_s\Lambda\partial_sk+\mu^2\partial_s^3k+\mu^2\mathfrak{p}_4^1(k)\Big)\nu\\ &+\Big(\mathfrak{p}_8^3(k)+\Lambda\mathfrak{p}_5^3(k)+\Lambda^2k^2+\mu\mathfrak{p}_6^3(k)+\mu\Lambda\mathfrak{p}_3^2(k)+\mu^2\mathfrak{p}_4^1(k)\Big)\tau\,.\end{aligned}\tag{4.16}$$

Similarly, differentiating in time the relation (2.11) we get

$$\partial_t^2 \nu = -(\partial_s V + \Lambda k)^2 \nu - (\partial_t \partial_s V + \partial_t \Lambda k + \Lambda \partial_t k)\tau\,,$$

that is

$$\begin{aligned}\partial_t^2 \nu =& \Big(\mathfrak{p}_8^3(k) + \Lambda \mathfrak{p}_5^3(k) + \Lambda^2 k^2 + \mu \mathfrak{p}_6^3(k) + \mu\Lambda \mathfrak{p}_3^2(k) + \mu^2 \mathfrak{p}_4^1(k)\Big)\nu \\ &- \Big(4\partial_s^7 k + \mathfrak{p}_8^4(k) - 4\Lambda\partial_s^4 k + \Lambda\mathfrak{p}_5^2(k) + \partial_s\Lambda \mathfrak{p}_4^1(k) + \Lambda^2\partial_s k + \partial_t \Lambda k - 4\mu\partial_s^5 k + \mu\mathfrak{p}_6^3(k) \\ &+ \mu\Lambda\mathfrak{p}_3^2(k) + 3\mu\partial_s\Lambda\partial_s k + \mu^2\partial_s^3 k + \mu^2\mathfrak{p}_4^1(k)\Big)\tau\,. \end{aligned} \tag{4.17}$$

Using computations (4.16) and (4.17), we obtain

$$\begin{aligned}\partial_t^2\gamma =& (\partial_t V + \Lambda(\partial_s V + \Lambda k))\nu + (\partial_t\Lambda - V(\partial_s V - \Lambda k))\tau \\ =& \Big(\partial_t V + \Lambda(-2\partial_s^3 k - 3k^2\partial_s k + \mu\partial_s k) + \Lambda^2 k\Big)\nu \\ &+ \Big(\partial_t\Lambda - (-2\partial_s^2 k - k^3 - \mu k)(-2\partial_s^3 k - 3k^3\partial_s k - \mu\partial_s k - \Lambda k)\Big)\tau \\ =& \Big(4\partial_s^6 k + 10k^2\partial_s^4 k + \mathfrak{p}_7^3(k) - 4\Lambda\partial_s^3 k - 6\Lambda k^2\partial_s k + \Lambda^2 k \\ &- 4\mu\partial_s^4 k + \mu\mathfrak{p}_5^2(k) + 2\mu\Lambda\partial_s k + \mu^2\partial_s^2 k + \mu^2\mathfrak{p}_3^0(k)\Big)\nu \\ &+ \Big(\partial_t\Lambda + \mathfrak{p}_7^3(k) - 2\Lambda k\partial_s^3 k - 3\Lambda k^3\partial_s k + \mu\mathfrak{p}_5^3(k) + \mu\Lambda k\partial_s k + \mu^2\mathfrak{p}_3^1(k)\Big)\tau\,.\end{aligned}$$

□

In the following, we show that in order to estimate the L^2–norm of $\partial_s^6 k$ it is enough to control the L^2–norm of $\partial_t^\perp v$. Hence, we start writing the boundary terms in (4.10) using the curvature and its derivative, lowering the order by means of the boundary condition.

Lemma 4.11. *Let γ_t be a family of curves moving with velocity v defined in* (4.9)*. Then,*

$$\langle \partial_s^\perp(\partial_t^\perp v), (\partial_s^\perp)^2(\partial_t^\perp v)\rangle = \mathfrak{p}_{17}^7(k) + \mathfrak{p}_{15}^7(k) + \mathfrak{p}_{13}^7(k) + \mathfrak{p}_{11}^5(k) + \mu^4\mathfrak{p}_9^4(k)\,.$$

Proof. By straightforward computations, we have that

$$\partial_s^\perp(\partial_t^\perp v) = \partial_s(\partial_t v^\perp)\nu\,,\quad (\partial_s^\perp)^2(\partial_t^\perp v) = \partial_s^2(\partial_t v^\perp)\nu$$

where $\partial_t v^\perp$ is the normal component of $\partial_t^2\gamma$, which is computed in (4.15). Hence, we compute

$$\begin{aligned}\partial_s(\partial_t v^\perp) =& 4\partial_s^7 k + \mathfrak{p}_8^4(k) - 4\Lambda\partial_s^4 k - 4\partial_s\Lambda\partial_s^3 k + \Lambda^2\partial_s k - 4\mu\partial_s^5 k + \mu\mathfrak{p}_7^4(k) \\ &+ 2\mu\Lambda\partial_s^2 k + 2\mu\partial_s\Lambda\partial_s k + \mu^2\partial_s^3 k + \mu^2\mathfrak{p}_4^1(k)\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\partial_s^2(\partial_t^\perp v) =& \mathfrak{p}_9^5(k) + 4\Lambda\partial_s\Lambda\partial_s k + \Lambda\mathfrak{p}_6^2(k) - 4\partial_s^2\Lambda\partial_s^3 k - 8\partial_s\Lambda\partial_s^4 k - \partial_t\Lambda\partial_s k \\ &+ \mu\mathfrak{p}_7^4(k) + \mu\Lambda\mathfrak{p}_4^1(k) + 2\mu\partial_s^2\Lambda\partial_s k + 4\mu\partial_s\Lambda\partial_s^2 k + \mu^2\mathfrak{p}_5^2(k)\,,\end{aligned} \tag{4.19}$$

where in relation (4.19) we substitute

$$
\begin{aligned}
4\partial_s^8 k =&\mathfrak{p}_9^5(k) + 4\Lambda\partial_s^5 k + \Lambda\mathfrak{p}_6^2(k) - \Lambda^2\partial_s^2 k - \partial_t\Lambda\partial_s k \\
&+ 4\mu\partial_s^6 k + \mu\mathfrak{p}_7^4(k) - 2\mu\Lambda\partial_s^3 k + \mu\Lambda\mathfrak{p}_4^1(k) - \mu^2\partial_s^4 k + \mu^2\mathfrak{p}_5^2(k)
\end{aligned}
$$

since we are considering Navier boundary condition. So, using expressions (4.18) and (4.19), replacing Λ and its derivatives by means of Lemma 4.8 and recalling that $\mu > 0$ is constant, we get

$$
\langle \partial_s^\perp(\partial_t^\perp v), (\partial_s^\perp)^2(\partial_t^\perp v)\rangle = \mathfrak{p}_{17}^7(k) + \mathfrak{p}_{15}^7(k) + \mathfrak{p}_{13}^7(k) + \mathfrak{p}_{11}^5(k) + \mu^4\mathfrak{p}_9^4(k)\,.
$$

□

Lemma 4.12. *Let γ_t be a family of curves moving with velocity v defined in (4.9). Then,*

$$
\begin{aligned}
\langle \partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v)\rangle =&\langle \partial_t^\perp v, 4\partial_s^9 k\nu\rangle + \langle \partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v) - 4\partial_s^9 k\nu\,\rangle \\
=&\mathfrak{p}_{17}^6(k) + \mathfrak{p}_{15}^7(k) + \mathfrak{p}_{13}^7(k) + \mathfrak{p}_{11}^7(k) + \mathfrak{p}_9^5(k) + \mathfrak{p}_7^2(k)\,.
\end{aligned}
$$

Proof. Let us analogously handle the other boundary term in (4.10). After some computations, we have that

$$
(\partial_s^3)^\perp(\partial_t^\perp v) = \partial_s^3(\partial_t v^\perp)\nu
$$

where

$$
\begin{aligned}
\partial_s^3(\partial_t v^\perp) =&4\partial_s^9 k + \mathfrak{p}_{10}^6(k) - 4\Lambda\partial_s^6 k + \partial_s\Lambda\mathfrak{p}_6^5(k) - 12\partial_s^2\Lambda\partial_s^4 k - 4\partial_s^3\Lambda\partial_s^3 k \\
&+ \Lambda^2\partial_s^3 k + 6\Lambda\partial_s\Lambda\partial_s^2 k + 6\Lambda\partial_s^2\Lambda\partial_s k - 4\mu\partial_s^7 k + \mu\mathfrak{p}_8^5(k) \\
&+ \mu 2\Lambda\partial_s^4 k + 6\mu\partial_s\Lambda\partial_s^3 k + 6\mu\partial_s^2\Lambda\partial_s^2 k + 2\mu\partial_s^3\Lambda\partial_s k + \mu^2\partial_s^5 k + \mu^2\mathfrak{p}_6^3(k)\,.
\end{aligned}
$$

As above, we aim to write the ninth order derivative as the sum of lower order derivatives. Hence, from condition (2.6), at boundary points it holds

$$
\langle \partial_t v, \partial_t^2(-2\partial_s k\nu + \mu\tau)\rangle = 0\,, \tag{4.20}
$$

where

$$
\begin{aligned}
\partial_t v =&\partial_t V\nu + V\partial_t\nu + \partial_t\Lambda\tau + \Lambda\partial_t\tau \\
=&\Big(\partial_t V + \Lambda\partial_s V\Big)\nu + (\partial_t\Lambda - V\partial_s V)\tau \\
=&\Big(4\partial_s^6 k + \mathfrak{p}_7^3(k) - 4\Lambda\partial_s^3 k + \Lambda\mathfrak{p}_4^1(k) - 4\mu\partial_s^4 k + \mu\mathfrak{p}_5^2(k) + 2\mu\Lambda\partial_s k + \mu^2\partial_s^2 k + \mu^2\mathfrak{p}_3^0(k)\Big)\nu \\
&+ \Big(\partial_t\Lambda - 4\partial_s^2 k\partial_s^3 k\Big)\tau
\end{aligned} \tag{4.21}
$$

and

$$
\begin{aligned}
\partial_t^2(-2\partial_s k\nu + \mu\tau) =& - 2\partial_t^2\partial_s k\nu - 4\partial_t\partial_s k\partial_t\nu - 2\partial_s k\partial_t^2\nu + \mu\partial_t^2\tau \\
=&\Big(- 2\partial_t^2\partial_s k - 2\partial_s k(\partial_t^2\nu)^\perp + \mu(\partial_t^2\tau)^\perp\Big)\nu \\
&+ \Big(4\partial_s V\partial_t\partial_s k - 2\partial_s k(\partial_t^2\nu)^\top + \mu(\partial_t^2\tau)^\top\Big)\tau\,.
\end{aligned}
$$

Then, after computing $\partial_t^2 \partial_s k$, we have

$$
\begin{aligned}
\partial_t^2(-2\partial_s k\nu + \mu\tau) =& \Big(-8\partial_s^9 k + \mathfrak{p}_{10}^6(k) + \Lambda\mathfrak{p}_7^6(k) + \partial_s\Lambda\mathfrak{p}_6^2(k) + \Lambda^2\mathfrak{p}_4^3(k) + (\partial_s\Lambda)^2\mathfrak{p}_2^1(k) \\
&+ \partial_t\Lambda\mathfrak{p}_3^2(k) + \mu\mathfrak{p}_8^7(k) + \mu\Lambda\mathfrak{p}_5^4(k) + \mu\partial_s\Lambda\mathfrak{p}_4^1(k) + \mu\Lambda^2\mathfrak{p}_2^1(k) \\
&+ \mu\partial_t\Lambda\mathfrak{p}_1^0(k) + \mu^2\mathfrak{p}_6^5(k) + \mu^2\Lambda\mathfrak{p}_3^2(k) + \mu^2\partial_s\Lambda\mathfrak{p}_2^1(k) + \mu^3\mathfrak{p}_4^3(k) \Big)\nu \\
&+ \Big(\mathfrak{p}_{10}^7(k) + \Lambda\mathfrak{p}_7^4(k) + \partial_s\Lambda\mathfrak{p}_6^1(k) + \Lambda^2\mathfrak{p}_4^1(k) + \partial_t\Lambda\mathfrak{p}_3^1(k) \\
&+ \mu\mathfrak{p}_8^3(k) + \mu\Lambda\mathfrak{p}_5^3(k) + \mu\partial_s\Lambda\mathfrak{p}_4^1(k) + \mu\Lambda^2\mathfrak{p}_2^0(k) \\
&+ \mu^2\mathfrak{p}_6^3(k) + \mu^2\Lambda\mathfrak{p}_3^2(k) + \mu^3\mathfrak{p}_4^1(k) \Big)\tau \,.
\end{aligned} \tag{4.22}
$$

Replacing equations (4.21) and (4.22) in the scalar product (4.20) and recalling that at boundary points Λ and its derivatives can be approximated by suitable polynomials (as it is shown in Lemma 4.8), we get

$$
\langle \partial_t v, 8\partial_s^9 k\nu \rangle = \mathfrak{p}_{17}^6(k) + \mathfrak{p}_{15}^7(k) + \mathfrak{p}_{13}^7(k) + \mathfrak{p}_{11}^7(k) + \mathfrak{p}_9^5(k) + \mathfrak{p}_7^2(k) \,.
$$

We now notice that

$$
\langle \partial_t v, \partial_s^9 k\nu \rangle = \langle \partial_t v^\perp \nu + \partial_t v^\top \tau, \partial_s^9 k\nu \rangle = \langle \partial_t^\perp v, \partial_s^9 k\nu \rangle \,,
$$

hence, we have

$$
\begin{aligned}
\langle \partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v) \rangle =& \langle \partial_t^\perp v, 4\partial_s^9 k\nu \rangle + \langle \partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v) - 4\partial_s^9 k\nu \rangle \\
=& \mathfrak{p}_{17}^6(k) + \mathfrak{p}_{15}^7(k) + \mathfrak{p}_{13}^7(k) + \mathfrak{p}_{11}^7(k) + \mathfrak{p}_9^5(k) + \mathfrak{p}_7^2(k) \,.
\end{aligned}
$$

□

Proposition 4.13. *Let γ_t be a maximal solution to the elastic flow of curves subjected to boundary conditions (2.6), with initial datum γ_0 in the maximal time interval $[0, T_{\max})$. Then for all $t \in (0, T_{\max})$ it holds*

$$
\int_\gamma |\partial_t^\perp v|^2 \,\mathrm{d}s \leq C(\mathcal{E}(\gamma_0)) \,.
$$

Proof. We start estimating the integral terms in (4.10). From equation (4.19), we have

$$
\begin{aligned}
-2\int_\gamma |(\partial_s^\perp)^2(\partial_t^\perp v)|^2 \,\mathrm{d}s = -2\int_\gamma & |4\partial_s^8 k + \mathfrak{p}_9^5(k) + \Lambda\mathfrak{p}_6^5(k) + \partial_s\Lambda\mathfrak{p}_5^4(k) \\
&+ \partial_s^2\Lambda\mathfrak{p}_4^3(k) + \Lambda^2\mathfrak{p}_3^2(k) + \Lambda\partial_s\Lambda\mathfrak{p}_2^1(k) \\
&+ \mu\mathfrak{p}_7^6(k) + \mu\Lambda\mathfrak{p}_4^3(k) + \mu\partial_s\Lambda\mathfrak{p}_3^2(k) + \mu^2\mathfrak{p}_5^4(k)|^2 \,\mathrm{d}s \,.
\end{aligned}
$$

Hence, using the simple inequalities

$$
\begin{aligned}
|a+b|^2 &\leq C\big(|a|^2 + |b|^2\big) \,, \\
|a+b|^2 &\geq (1-\varepsilon)|a|^2 - C(\varepsilon)|b|^2
\end{aligned}
$$

with $\varepsilon = \frac{1}{2}$, we get

$$
\begin{aligned}
-2\int_\gamma |(\partial_s^\perp)^2(\partial_t^\perp v)|^2\,\mathrm{d}s \leq& -\int_\gamma |4\partial_s^8 k|^2 + C\int_\gamma |\mathfrak{p}_{18}^5(k) + \Lambda^2\mathfrak{p}_{12}^5(k) + (\partial_s\Lambda)^2\mathfrak{p}_{10}^4(k) \\
&+ (\partial_s^2\Lambda)^2\mathfrak{p}_8^3(k) + \Lambda^4\mathfrak{p}_6^2(k) + \mu^2\mathfrak{p}_{14}^6(k) \\
&+ \mu^2\Lambda^2\mathfrak{p}_8^3(k) + \mu^2(\partial_s\Lambda)^2\mathfrak{p}_6^2(k) + \mu^4\mathfrak{p}_{10}^4(k)\,\mathrm{d}s \\
\leq& -16\int_\gamma |\partial_s^8 k|^2 + \int_\gamma |\mathfrak{p}_{18}^5(k) + \mathfrak{p}_{16}^4(k) + \mathfrak{p}_{14}^6(k) + \mathfrak{p}_{12}^2(k) + \mathfrak{p}_{10}^4(k)|\,\mathrm{d}s\,,
\end{aligned}
\tag{4.23}
$$

where we use the very expression of Λ in (4.14) and the estimates in Lemma 4.9.

Moreover, with same arguments, from equation (4.21) we get

$$
\begin{aligned}
\frac{1}{2}\int_\gamma |\partial_t^\perp v|^2(\partial_s\Lambda - kV)\,\mathrm{d}s =& \frac{1}{2}\int_\gamma |\partial_t^\perp v|^2(\partial_s\Lambda + 2k\partial_s^2 k + k^4 - \mu k^2)\,\mathrm{d}s \\
=& \int_\gamma |\mathfrak{p}_{18}^6(k) + \mathfrak{p}_{17}^6(k) + \mathfrak{p}_{16}^6(k) + \mathfrak{p}_{15}^6(k) + \mathfrak{p}_{14}^6(k) + \mathfrak{p}_{13}^6(k) \\
&+ \mathfrak{p}_{12}^6(k) + \mathfrak{p}_{12}^2(k) + \mathfrak{p}_{10}^4(k) + \mathfrak{p}_9^2(k) + \mathfrak{p}_8^2(k)|\,\mathrm{d}s\,.
\end{aligned}
\tag{4.24}
$$

We only need to compute the integral involving Y. By straightforward computation, we have

$$
\begin{aligned}
\partial_t(\partial_t v)^\perp =& -8\partial_s^{10}k - 20k^2\partial_s^8 k + \mathfrak{p}_{11}^7(k) + \Lambda\mathfrak{p}_8^7(k) + \Lambda^2\mathfrak{p}_5^4(k) + \partial_t\Lambda\mathfrak{p}_4^3(k) \\
&+ 12\mu\partial_s^8 k + \mu\mathfrak{p}_9^6(k) + \mu\Lambda\mathfrak{p}_6^5(k) + \mu\Lambda^2\mathfrak{p}_3^2(k) + \mu\partial_t\Lambda\mathfrak{p}_2^1(k) + \mu^2\mathfrak{p}_7^6(k) \\
&+ \mu^2\Lambda\mathfrak{p}_4^3(k) + \mu^3\mathfrak{p}_5^4(k)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_s^4(\partial_t v)^\perp =& 4\partial_s^{10}k + \mathfrak{p}_{11}^7(k) + \Lambda\mathfrak{p}_8^7(k) + \partial_s\Lambda\mathfrak{p}_7^6(k) + \partial_s^2\Lambda\mathfrak{p}_6^5(k) + \partial_s^3\Lambda\mathfrak{p}_5^4(k) + \partial_s^4\Lambda\mathfrak{p}_4^3(k) \\
&- 4\mu\partial_s^8 k + \mu\mathfrak{p}_9^6(k) + \mu\Lambda\mathfrak{p}_6^5(k) + \mu\partial_s\Lambda\mathfrak{p}_5^4(k) + \mu\partial_s^2\Lambda\mathfrak{p}_4^3(k) \\
&+ \mu\partial_s^3\Lambda\mathfrak{p}_3^2(k) + \mu\partial_s^4\Lambda\mathfrak{p}_2^1(k) + \mu^2(\mathfrak{p}_7^6(k)\,.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
Y =& \partial_t^\perp(\partial_t^\perp v) + 2(\partial_s^\perp)^4(\partial_t^\perp v) \\
=& \Big(-20k^2\partial_s^8 k + \mathfrak{p}_{11}^7(k) + \Lambda\mathfrak{p}_8^7(k) + \partial_s\Lambda\mathfrak{p}_7^6(k) + \Lambda^2\mathfrak{p}_5^4(k) + \partial_t\Lambda\mathfrak{p}_4^3(k) \\
&+ 4\mu\partial_s^8 k + \mu\mathfrak{p}_9^6(k) + \mu\Lambda\mathfrak{p}_6^5(k) + \mu\partial_s\Lambda\mathfrak{p}_5^4(k) + \mu\Lambda^2\mathfrak{p}_3^2(k) + \mu\partial_t\Lambda\mathfrak{p}_2^1(k) \\
&+ \mu^2\mathfrak{p}_7^6(k) + \mu^2\Lambda\mathfrak{p}_4^3(k) + \mu^3\mathfrak{p}_5^4(k)\Big)\nu\,.
\end{aligned}
$$

Hence, computing the scalar product $\langle Y, \partial_t^\perp v\rangle$, using the well-known Peter-Paul inequality and integrating by parts the integral $\int_\gamma \partial_s^6 k \partial_s^8 k \, \mathrm{d}s$, we have

$$\begin{aligned}\int_\gamma \langle Y, \partial_t^\perp v\rangle \, \mathrm{d}s \leq & \frac{1}{2}\int_\gamma |\partial_s^8 k|^2 \, \mathrm{d}s - 4\mu \int_\gamma |\partial_s^7 k|^2 \, \mathrm{d}s + \mathfrak{p}_{15}^7(k)\Big|_0^1 \\ & + \int_\gamma |\mathfrak{p}_{18}^7(k) + \mathfrak{p}_{17}^6(k) + \mathfrak{p}_{18}^7(k)\mathfrak{p}_{16}^7(k) + \mathfrak{p}_{15}^6(k) + \mathfrak{p}_{14}^7(k) \\ & + \mathfrak{p}_{13}^6(k) + \mathfrak{p}_{12}^6(k) + \mathfrak{p}_{11}^4(k) + \mathfrak{p}_{10}^4(k) + \mathfrak{p}_8^4(k) + \mathfrak{p}_6^2(k)| \, \mathrm{d}s \,. \qquad (4.25)\end{aligned}$$

where, as above, we estimate Λ and its derivatives by means of Lemma 4.9.

Moreover, using identities in Lemma 4.11 and Lemma 4.12, we end up with the following inequality

$$\begin{aligned}-2\langle \partial_t^\perp v, (\partial_s^\perp)^3(\partial_t^\perp v)\rangle\Big|_0^1 + 2\langle \partial_s^\perp(\partial_t^\perp v), (\partial_s^\perp)^2(\partial_t^\perp v)\rangle\Big|_0^1 \leq & |\mathfrak{p}_{17}^7(k)| + |\mathfrak{p}_{15}^7(k)| + |\mathfrak{p}_{13}^7(k)| \\ & + |\mathfrak{p}_{11}^7(k)| + |\mathfrak{p}_9^5(k)| + |\mathfrak{p}_7^2(k)| \qquad (4.26)\end{aligned}$$

Then, putting together inequalities (4.23), (4.24), (4.25) and (4.26), we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\gamma |\partial_t^\perp v|^2 \, \mathrm{d}s \leq & \int_\gamma |\mathfrak{p}_{18}^7(k) + \mathfrak{p}_{17}^7(k) + \mathfrak{p}_{16}^7(k) + \mathfrak{p}_{14}^7(k) + \mathfrak{p}_{15}^6(k) + \mathfrak{p}_{13}^6(k) + \mathfrak{p}_{12}^6(k) \\ & + \mathfrak{p}_{11}^4(k) + \mathfrak{p}_{10}^4(k) + \mathfrak{p}_8^4(k) + \mathfrak{p}_9^2(k) + \mathfrak{p}_6^2(k)| \, \mathrm{d}s \\ & + |\mathfrak{p}_{17}^7(k)| + \mu|\mathfrak{p}_{15}^7(k)| + \mu^2|\mathfrak{p}_{13}^7(k)| + \mu^3|\mathfrak{p}_{11}^7(k)| + \mu^4|\mathfrak{p}_9^5(k)| + \mu^5|\mathfrak{p}_7^2(k)|\Big|_0^1 \,.\end{aligned}$$

By means of Lemma 4.7, we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\gamma |\partial_t^\perp v|^2 \, \mathrm{d}s \leq & -C\Big(\|\partial_s^8 k\|_{L^2(\gamma)}^2 + \mu\|\partial_s^7 k\|_{L^2(\gamma)}^2\Big) + C\Big(\|k\|_{L^2(\gamma)}^2 + \|k\|_{L^2(\gamma)}^\Theta\Big) \\ \leq & C(\mathcal{E}(\gamma_0))\end{aligned}$$

for some exponent $\Theta > 2$ and constant C which depends on $\ell(\gamma)$.

Hence, by integrating, it follows

$$\int_\gamma |\partial_t^\perp v|^2 \, \mathrm{d}s \leq C(\mathcal{E}(\gamma_0)) \,.$$

□

Proposition 4.14. *Let γ_t be a maximal solution to the elastic flow of curves subjected to Navier boundary conditions with initial datum γ_0, which satisfies the* uniform non-degeneracy condition (4.4) *in the maximal time interval $[0, T_{\max})$. Then for all $t \in (0, T_{\max})$ it holds*

$$\int_\gamma |\partial_s^6 k|^2 \, \mathrm{d}s \leq C(\mathcal{E}(\gamma_0)) \,.$$

Proof. From formula (4.15) and Lemma 4.8, it follows

$$\partial_t^\perp v = \partial_t v^\perp \nu = \Big(\partial_s^6 k + \mathfrak{p}_7^4(k) + \mu\mathfrak{p}_5^4(k) + \mu^2\mathfrak{p}_3^2(k)\Big)\nu\,.$$

However, since we are assuming that μ is constant, we simply have

$$\partial_s^6 k = \partial_t v^\perp + \mathfrak{p}_7^4(k) + \mathfrak{p}_5^4(k) + \mathfrak{p}_3^2(k)$$

and by means of Peter-Paul inequality, we get

$$\int_\gamma |\partial_s^6 k|^2\,\mathrm{d}s \le \int_\gamma |\partial_t v^\perp|^2\,\mathrm{d}s + C\left(\int_\gamma |\mathfrak{p}_7^4(k)|^2\,\mathrm{d}s + \int_\gamma |\mathfrak{p}_5^4(k)|^2\,\mathrm{d}s + \int_\gamma |\mathfrak{p}_3^2(k)|^2\,\mathrm{d}s\right). \tag{4.27}$$

We now estimate separately the integrals involving the polynomials. We start considering

$$\int_\gamma |\mathfrak{p}_7^4(k)|^2\,\mathrm{d}s = \int_\gamma \Big|\prod_{l=0}^4 (\partial_s^l k)^{\alpha_l}\Big|^2\,\mathrm{d}s$$

where $\alpha_l \in \mathbb{N}$ and $\sum_{l=0}^4 (l+1)\alpha_l = 7$. So, by Hölder inequality, we get

$$\int_\gamma |\mathfrak{p}_7^4(k)|^2\,\mathrm{d}s = \int_\gamma \Big|\prod_{l=0}^4 (\partial_s^l k)^{\alpha_l}\Big|^2\,\mathrm{d}s \le \prod_{l=0}^4 \left(\int_\gamma |\partial_s^l k|^{2\alpha_l\beta_l}\,\mathrm{d}s\right)^{\frac{1}{\beta_l}} = \prod_{l=0}^4 \|\partial_s^l k\|^{2\alpha_l}_{L^{2\alpha_l\beta_l}(\gamma)}$$

where $\beta_l := \frac{7}{(l+1)\alpha_l} > 1$ if $\alpha_l \neq 0$ (if $\alpha_l = 0$ we simply have the integral of a unitary function), which clearly satisfy

$$\sum_{l=0}^4 \frac{1}{\beta_l} = 1\,.$$

Then, we estimate any of such products by the well-known interpolation inequalities (see [28], for instance),

$$\|\partial_s^l k\|_{L^{2\alpha_l\beta_l}(\gamma)} \le C\|\partial_s^6 k\|^{\sigma_l}_{L^2(\gamma)}\|k\|^{1-\sigma_l}_{L^2(\gamma)} + \|k\|_{L^2(\gamma)} \tag{4.28}$$

for some constant C depending on α_l, β_l and coefficient σ_l given by

$$\sigma_l = \frac{1}{6}\Big(l - \frac{1}{2\alpha_l\beta_l} + \frac{1}{2}\Big) \in \Big[\frac{l}{6}, 1\Big)\,.$$

Moreover, we notice that

$$\begin{aligned}\sum_{l=0}^4 2\alpha_l\sigma_l &= \sum_{l=0}^4 \frac{1}{3}\Big(\alpha_l(l+1) - \frac{1}{2\beta_l} - \frac{\alpha_l}{2}\Big)\\ &= \frac{7}{3} - \frac{1}{6} - \frac{\sum_{l=0}^4 \alpha_l}{6} < 2\,,\end{aligned}$$

where in the last inequality we use the fact that, since l, α_l are respectively the order of derivations and the exponents of the derivative in $\mathfrak{p}_7^4(k)$, it follows

$$1 < \sum_{l=0}^4 \alpha_l \le 7\,.$$

Then, multiplying together inequalities (4.28) and applying the Young inequality, we have

$$
\begin{aligned}
\int_\gamma |\mathfrak{p}_7^4(k)|^2 \,\mathrm{d}s \leq & \big(\|\partial_s^6 k\|_{L^2\gamma)} + \|k\|_{L^2(\gamma)}\big)^{\sum_{l=0}^4 2\alpha_l\sigma_l} \, \|k\|_{L^2(\gamma)}^{\sum_{l=0}^4 2\alpha_l(1-\sigma_l)} \\
\leq & \varepsilon \left(\|\partial_s^6 k\|_{L^2(\gamma)} + \|k\|_{L^2(\gamma)}\right)^2 + C(\varepsilon)\|k\|_{L^2(\gamma)}^{\Theta_1}
\end{aligned} \tag{4.29}
$$

for some exponent $\Theta_1 > 2$.

Arguing in the same way, one can check that

$$
\int_\gamma |\mathfrak{p}_5^4(k)|^2 \,\mathrm{d}s \leq \varepsilon \left(\|\partial_s^6 k\|_{L^2(\gamma)} + \|k\|_{L^2(\gamma)}\right)^2 + C(\varepsilon)\|k\|_{L^2(\gamma)}^{\Theta_2} \tag{4.30}
$$

and

$$
\int_\gamma |\mathfrak{p}_3^2(k)|^2 \,\mathrm{d}s \leq \varepsilon \left(\|\partial_s^6 k\|_{L^2(\gamma)} + \|k\|_{L^2(\gamma)}\right)^2 + C(\varepsilon)\|k\|_{L^2(\gamma)}^{\Theta_3} \tag{4.31}
$$

for some exponents $\Theta_2, \Theta_3 > 2$.
Replacing the estimates (4.29), (4.30) and (4.31) in (4.27) and moving the small part of $\|\partial_s^6 k\|_{L^2(\gamma)}^2$ on the right hand side, we have

$$
\int_\gamma |\partial_s^6 k|^2 \,\mathrm{d}s \leq \int_\gamma |\partial_t v^\perp|^2 \,\mathrm{d}s + C(\|k\|_{L^2(\gamma)}^2 + \|k\|_{L^2(\gamma)}^{\Theta}),
$$

where, as above, $\theta > 2$. Then, we conclude using Proposition 4.13 and the energy monotonicity in Proposition 2.7. □

5 Long–time existence

Theorem 5.1. *Let γ_0 be a geometrically admissible initial curve. Suppose that γ_t is a maximal solution to the elastic flow with initial datum γ_0 in the maximal time interval $[0, T_{\max})$ with $T_{\max} \in (0,\infty) \cup \{\infty\}$. Then, up to reparametrization and translation of γ_t, it follows*

$$
T_{\max} = \infty
$$

or at least one of the following holds

- $\liminf \ell(\gamma_t) \to 0$ *as* $t \to T_{\max}$*;*
- $\liminf \tau_2 \to 0$ *as* $t \to T_{\max}$ *at boundary points.*

Proof. Suppose by contradiction that the two assertions in the statement are not fulfilled and that $T_{\max}$ is finite. So, in the whole time interval $[0, T_{\max})$ the length of the curves γ_t is uniformly bounded from below away from zero and the uniform condition (4.4) is satisfied. Moreover, since the energy (2.1) decreases in time, both the L^2–norm of the curvature and the length of γ are uniformly bounded from above. Let $\varepsilon > 0$ be fixed, by means of Proposition 4.6 and Proposition 4.13 we have that

$$
\partial_s^2 k \in L^\infty([0, T_{\max}); L^2) \qquad \text{and} \qquad \partial_s^6 k \in L^\infty((\varepsilon, T_{\max}); L^2)\,.
$$

Hence, using Gagliardo–Nirenberg inequality for all $t \in [0, T_{\max})$ we get

$$
\|\partial_s^j k\|_{L^2(\gamma)} \leq C_1 \|\partial_s^6 k\|_{L^2(\gamma)}^{\sigma} \|k\|_{L^2(\gamma)}^{1-\sigma} + C_2\|k\|_{L^2(\gamma)} \leq C(\mathcal{E}(\gamma_0))\,,
$$

for every integer $j \leq 6$, with constants independent on t and for suitable exponent σ. Actually, by interpolation, we have

$$\partial_s^j k \in L^\infty((\varepsilon, T_{\max}); L^\infty)$$

for every integer $j \leq 5$. Reparametrizing the curve γ_t into $\widetilde{\gamma}_t$ with the property $|\partial_x \widetilde{\gamma}(x)| = \ell(\widetilde{\gamma})$ for every $x \in [0,1]$ and for all $t \in [0, T_{\max})$ and translating so that it remains in a ball $B_R(0)$ for every time (since its length is uniformly bounded from above), we get

- $0 < c \leq \sup_{t\in[0,T_{\max}),x\in[0,1]} |\partial_x \widetilde{\gamma}(t,x)| \leq C < \infty$,
- $0 < c \leq \sup_{t\in[0,T_{\max}),x\in[0,1]} |\widetilde{\gamma}(t,x)| \leq C < \infty$.

Hence, $\tau \in L^\infty([0,T_{\max}); L^\infty)$ and $\partial_x^j \widetilde{\gamma} \in L^\infty((\varepsilon, T_{\max}); L^\infty)$ for every integer $j \leq 7$. Then, from the observation above and the fact that $\boldsymbol{\kappa} = k\nu$, we get $\partial_s^j \boldsymbol{\kappa} \in L^\infty((\varepsilon, T_{\max}); L^\infty)$ for every integer $j \leq 5$ and $\partial_s^6 \boldsymbol{\kappa} \in L^\infty((\varepsilon, T_{\max}); L^2)$. Moreover, thanks to our choice of parametrization, we have

$$\boldsymbol{\kappa}(x) = \frac{\partial_x^2 \widetilde{\gamma}(x)}{\ell(\widetilde{\gamma})^2} \qquad \text{and} \qquad \partial_s^j \boldsymbol{\kappa}(x) = \frac{\partial_x^j \widetilde{\gamma}(x)}{\ell(\widetilde{\gamma})^j}.$$

So, it follows that $\partial_x^j \widetilde{\gamma} \in L^\infty((\varepsilon, T_{\max}); L^\infty)$ for every integer $1 \leq j \leq 7$ and $\partial_x^8 \widetilde{\gamma} \in L^\infty((\varepsilon, T_{\max}); L^2)$. Then, by Ascoli–Arzelà Theorem, there exists a curve $\gamma_{\max}$ such that

$$\lim_{t \nearrow T_{\max}} \partial_x^j \widetilde{\gamma}(x) = \partial_x^j \gamma_{\max}(x)$$

for every integer $j \leq 6$. The curve $\gamma_{\max}$ is an admissible initial curve, since by continuity of k and $\partial_s^2 k$ it fulfills the system (2.6) and uniform condition (4.4) at boundary points. Then, there exists an elastic flow $\overline{\gamma}_t \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([T_{\max}, T_{\max}+\delta) \times [0,1]; \mathbb{R}^2\right)$ with $\delta > 0$. We again reparametrize $\overline{\gamma}_t$ in $\hat{\gamma}_t$ with constant speed equal to length and we have

$$\lim_{t \searrow T_{\max}} \partial_x^j \hat{\gamma}(x) = \partial_x^j \gamma_{\max}(x)$$

for every integer $j \leq 6$.

Then,

$$\lim_{t \nearrow T_{\max}} \partial_t \widetilde{\gamma}(t,x) = \lim_{t \searrow T_{\max}} \partial_t \hat{\gamma}(t,x).$$

Thus, we found a solution to the elastic flow in $C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0, T_{\max}+\delta) \times [0,1]; \mathbb{R}^2\right)$. This obviously contradicts the maximality of $T_{\max}$. $\square$

6 Appendix

In this appendix we collect some results which were fundamental in the previous sections.

Firstly we show the smoothness of critical point of functional $\mathcal{E}$.

Lemma 6.1 ([4, Corollary 6.13, Exercise 6.7]). *Suppose that $\Omega \subset \mathbb{R}^n$ is open, $f \in L^1_{loc}(\Omega)$, $p \in (1,\infty]$, $1/p + 1/p' = 1$, $m \in \mathbb{N}_0$, and that there exists a constant C_0 such that for all $k \in \mathbb{N}_0$ with $k \leq m$ and all $\zeta \in C_c^\infty(\Omega)$*

$$\left| \int_\Omega f \partial^k \zeta \, \mathrm{d}x \right| \leq C_0 \|\zeta\|_{L^{p'}(\Omega)}.$$

Then $f \in W^{m,p}(\Omega)$ and there exists a constant $C = C(m, C_0)$ with $\|f\|_{W^{m,p}} \leq C$.

Proposition 6.2 (Regularity for critical point of $\mathcal{E}$)**.** *Suppose that γ is a critical point of $\mathcal{E}$, then γ is of class C^∞. Moreover, for all $l \in \mathbb{N}$ there exists a constant $C_l = C_l(\|\gamma\|_{H^2})$ such that*

$$\|\gamma\|_{W^{l+2,\infty}} \leq C_l(\|\gamma\|_{H^2})\,. \tag{6.1}$$

Proof. In order to show the regularity of a critical point of the elastic energy, we follow a bootstrap argument based on Lemma 6.1 (see [10] for a similar proof).

Indeed, we prove that for any $m \in \mathbb{N}_0$, $\eta : [0,1] \to \mathbb{R}$ of class C^∞ and $l \in \mathbb{N}_0$, $l \leq m$, we have

$$\int_\gamma k\partial_s^l \eta \, \mathrm{d}s \leq C(\|\gamma\|_{H^2})\|\eta\|_{L^1}\,. \tag{6.2}$$

Then, by Lemma 6.1 we conclude that $\boldsymbol{\kappa} \in W^{m,\infty}$ and $\gamma \in W^{m+2,\infty}$, where $\boldsymbol{\kappa} = k\nu$.

We start showing the assertion for $m = 1$. We recall that, since γ is a critical point of $\mathcal{E}$, it holds

$$\int_\gamma 2\langle \boldsymbol{\kappa}, \partial_s^2 \psi\rangle \, \mathrm{d}s + \int_\gamma \left(-3|\boldsymbol{\kappa}|^2 + \mu\right) \langle \tau, \partial_s \psi\rangle \, \mathrm{d}s = 0 \tag{6.3}$$

for all $\psi : [0,1] \to \mathbb{R}^2$ of class H^2 such that

$$\psi(0)_2 = 0 \qquad \text{and} \qquad \psi(1)_2 = 0\,.$$

Moreover, the fact that $\gamma \in H^2$ ensures that the L^2-norm of the curvature is bounded, that is

$$\|\boldsymbol{\kappa}\|_{L^1} \leq C\|\boldsymbol{\kappa}\|_{L^2} \leq C(\|\gamma\|_{H^2})\,.$$

We now denote by $F(\gamma, \psi)$ the second integral in (6.3), so we have

$$|F(\gamma,\psi)| \leq C(\|\boldsymbol{\kappa}\|^2_{L^2} + \mu\ell(\gamma))\|\partial_s\psi\|_{L^\infty} \leq C(\|\gamma\|_{H^2})\|\psi\|_{W^{2,1}}\,. \tag{6.4}$$

In order to show the L^∞-regularity of $\boldsymbol{\kappa}$, we consider $\eta \in C^\infty$ and we use

$$\psi(x) = \ell(\gamma)^2 \int_0^x \int_0^y \eta(t)\nu(t)\,\mathrm{d}t\,\mathrm{d}y + \ell(\gamma)^2 x \int_0^1 \int_0^y \eta(t)\nu(t)\,\mathrm{d}t\,\mathrm{d}y$$

as test function in (6.3). It clearly follows that $\psi \in H^2$ and $\partial_s^2\psi = \eta\nu$ (using the relation $|\gamma'(x)| = \ell(\gamma)$ for all $x \in [0,1]$). Then, if we replace ψ in (6.3) we have

$$\int_\gamma 2\langle \boldsymbol{\kappa}, \eta\nu\rangle \, \mathrm{d}s = -F(\gamma,\psi)$$

for all $\eta \in C^\infty$. Hence, using the estimate (6.4), we obtain

$$\int_\gamma 2\langle \boldsymbol{\kappa}, \eta\nu\rangle \, \mathrm{d}s = \int_\gamma 2k\eta\,\mathrm{d}s \leq C(\|\gamma\|_{H^2})\|\psi\|_{W^{2,1}} \leq C(\|\gamma\|_{H^2})\|\eta\|_{L^1}\,,$$

and by Lemma 6.1, we conclude that $\boldsymbol{\kappa} \in L^\infty$ (that is $\gamma \in W^{2,\infty}$) and there exists a constant $C_0 = C_0(\|\gamma\|_{H^2})$ such that

$$\|\boldsymbol{\kappa}\|_{L^\infty} \leq C_0(\|\gamma\|_{H^2})\,. \tag{6.5}$$

Arguing in the same way, we want to show that $k \in W^{1,\infty}$. For $\eta \in C^\infty$, we use

$$\psi(x) = \ell(\gamma) \int_0^x \eta(t)\nu(t)\,\mathrm{d}t + \ell(\gamma)x \int_0^1 \eta(t)\nu(t)\,\mathrm{d}t$$

as test function in (6.3). So we have $\psi \in H^2$, $\partial_s \psi = \eta\nu$ and

$$\langle \partial_s^2 \psi, \nu \rangle = \langle \partial_s \eta \nu, \nu \rangle = \partial_s \eta \,.$$

Then, relation (6.3) can be written as

$$\int_\gamma k \partial_s \eta \,\mathrm{d}s = \int_\gamma (3|\boldsymbol{\kappa}|^2 - \mu)\langle \tau, \partial_s \psi \rangle \,\mathrm{d}s \leq C(\|\gamma\|_{H^2}) \|\partial_s \psi\|_{L^1} \leq C(\|\gamma\|_{H^2}) \|\eta\|_{L^1}$$

where we used the L^∞-bound in (6.5). Then, by Lemma 6.1 it follows that $\boldsymbol{\kappa} \in W^{1,\infty}$ (that is $\gamma \in W^{3,\infty}$) and there exists a constant $C_1 = C_1(\|\gamma\|_{H^2})$ such that

$$\|\boldsymbol{\kappa}\|_{W^{1,\infty}} \leq C_1(\|\gamma\|_{H^2}) \,.$$

Once we showed the assertion for $m = 1$, we can suppose that $m \geq 2$ and that it holds for $m - 1$. So, we only need to prove the estimate (6.2) for $l = m$.

For $\eta \in C^\infty$, we use $\psi = \partial_s^{l-2} \eta \nu$ as a test function in (6.3). Hence, we have

$$\begin{aligned} \langle \partial_s^2 \varphi, \nu \rangle &= \langle \partial_s (\partial_s^{l-1} \eta \nu + \partial_s^{l-2} \eta \partial_s \nu), \nu \rangle = \partial_s^l \eta + 2 \langle \partial_s^{l-1} \eta \partial_s \nu, \nu \rangle + \langle \partial_s^{l-2} \eta \partial_s^2 \nu, \nu \rangle \\ &= \partial_s^l \eta - \langle \partial_s^{l-2} \eta \partial_s (k\tau), \nu \rangle = \partial_s^l \eta - k^2 \partial_s^{l-2} \eta \,. \end{aligned}$$

and

$$\langle \partial_s \psi, \tau \rangle = \langle \partial_s^{l-1} \eta \nu + \partial_s^{l-2} \eta \partial_s \nu, \tau \rangle = -k \partial_s^{l-2} \eta \,.$$

Replacing this relations in (6.3), we obtain

$$\int_\gamma k \partial_s^l \eta \,\mathrm{d}s = \int_\gamma k^3 \partial_s^{l-2} \eta - \int_\gamma k(3k^2 - \mu) \partial_s^{l-2} \eta \,\mathrm{d}s \,.$$

In view of the regularity already established, we may integrate by parts the terms involving derivatives of η on the right-hand side and we obtain

$$\int_\gamma k \partial_s^l \eta \,\mathrm{d}s \leq C(\|\gamma\|_{H^2}) \,.$$

Since this estimate holds for all $l \leq m$, by Lemma 6.1 we conclude that $\boldsymbol{\kappa} \in W^{m,\infty}$ and there exists a constant $C_l = C_l(\|\gamma\|_{H^2})$ such that estimate (6.1) holds. □

Acknowledgements

The author is grateful to Alessandra Pluda and Carlo Mantegazza for several discussions and helpful suggestions.

The author is partially supported by the INdAM–GNAMPA 2022 Project *Isoperimetric problems: variational and geometric aspects*.

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