TOPOLOGICAL SINGULARITIES ARISING FROM FRACTIONAL-GRADIENT ENERGIES

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Abstract. We prove that, on a planar regular domain, suitably scaled functionals of Ginzburg–Landau type, given by the sum of quadratic fractional Sobolev seminorms and a penalization term vanishing on the unitary sphere, Γ-converge to vortex-type energies with respect to the flat convergence of Jacobians. The compactness and the Γ-$\liminf$ follow by comparison with standard Ginzburg–Landau functionals depending on Riesz potentials. The Γ-$\limsup$, instead, is achieved via a direct argument by joining a finite number of vortex-like functions suitably truncated around the singularity.

1. Introduction

1.1. **Classical framework.** Let $\Omega \subset \mathbb{R}^2$ be a non-empty, connected, simply connected, bounded open set with Lipschitz boundary. Given a scale parameter $\varepsilon > 0$ and an additional parameter $\lambda > 0$, the *Ginzburg–Landau functionals*

$$\mathrm{GL}_{\varepsilon,\lambda}(\,\cdot\,;\Omega)\colon H^1(\Omega;\mathbb{R}^2) \to [0,\infty]$$

are defined as

$$\mathrm{GL}_{\varepsilon,\lambda}(v;\Omega) = \frac{1}{|\log\varepsilon|}\int_\Omega |Dv|^2\,\mathrm{d}x + \frac{\lambda}{\varepsilon^2|\log\varepsilon|}\int_\Omega \left(|v|^2-1\right)^2\mathrm{d}x \tag{1.1}$$

for $v \in H^1(\Omega;\mathbb{R}^2)$. One may prescribe a trace constraint at the boundary by imposing that $v|_{\partial\Omega} = g$ for some fixed boundary datum $g\colon \partial\Omega \to \mathbb{S}^1$ with (topological) degree $d = \deg(g,\partial\Omega) \in \mathbb{Z}$, in which case the functionals in (1.1) are restricted to the subspace

$$H^1_g(\Omega;\mathbb{R}^2) = \left\{v \in H^1(\Omega;\mathbb{R}^2) : v|_{\partial\Omega} = g\right\}.$$

Much effort has been devoted to understanding the asymptotic behavior of the minimizers v_ε of the functionals in (1.1) as the scale parameter vanishes. In the limit as $\varepsilon \to 0^+$, the minimizers v_ε of the functionals in (1.1) develop *vortex-like singularities* of the form $\frac{x-x_i}{|x-x_i|}$ (possibly, up to a fixed rotation) for $|d|$ points x_i's in $\overline{\Omega}$.

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After the works [18, 19, 23, 24] (we also refer to the monograph [8] and to [2] for a more detailed presentation of the problem, and to [1] for the higher-dimensional setting), the picture is nowadays well understood. The Γ-*convergence* of the functionals in (1.1) as $\varepsilon \to 0^+$ is related to the *flat* (or 1-*Wasserstein*) *convergence* of the Jacobians $\mathrm{Jac}(v_\varepsilon) = \det(Dv_\varepsilon)$ of their minimizers v_ε to an *atomic* measure $\mu \in \mathcal{X}(\overline{\Omega})$, where

$$\mathcal{X}(\overline{\Omega}) = \left\{ \sum_{i=1}^{N} d_i \delta_{x_i} : d_i \in \mathbb{Z} \text{ and } x_i \in \overline{\Omega} \text{ for } i = 1, \dots, N, \text{ with } N \in \mathbb{N} \right\}. \tag{1.2}$$

In more precise terms, we can state the following result. For a detailed presentation of the notion of Γ-*convergence*, we refer to the monographs [5, 15]. For the definition of *flat convergence*; i.e., in the dual norm with respect to Lipschitz functions, see Section 2.2.

Theorem 1.1 (Compactness and Γ-convergence of Ginzburg–Landau energies)**.** *Let $\Omega \subset \mathbb{R}^2$ be a non-empty, connected, simply connected, bounded open set with Lipschitz boundary, $g \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{S}^1)$ with $d = \deg(g|_{\partial\Omega}, \partial\Omega) \in \mathbb{Z}$ and $\lambda > 0$.*

(i) (Compactness) *If $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_g(\Omega; \mathbb{R}^2)$, with $\varepsilon_k \to 0^+$ as $k \to \infty$, is such that*

$$\sup_{k\in\mathbb{N}} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k}; \Omega) < \infty,$$

then there exists a subsequence $(v_{\varepsilon_{k_j}})_{j\in\mathbb{N}}$ and $\mu \in \mathcal{X}(\overline{\Omega})$ such that $\mu(\overline{\Omega}) = d$ and

$$\mathrm{Jac}(v_{\varepsilon_{k_j}})\, \mathscr{L}^2 \xrightarrow{\mathrm{flat}(\overline{\Omega})} \mu \quad \text{as } j \to \infty.$$

(ii) (Γ-$\liminf$ inequality) *If $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_g(\Omega; \mathbb{R}^2)$, with $\varepsilon_k \to 0^+$ as $k \to \infty$, is such that*

$$\mathrm{Jac}(v_{\varepsilon_k})\, \mathscr{L}^2 \xrightarrow{\mathrm{flat}(\overline{\Omega})} \mu \quad \text{as } k \to \infty \tag{1.3}$$

for some $\mu \in \mathcal{X}(\overline{\Omega})$, then $\mu(\overline{\Omega}) = d$ and

$$\liminf_{k\to\infty} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k}; \Omega) \geq 2\pi |\mu|(\overline{\Omega}).$$

(iii) (Γ-$\limsup$ inequality) *If $\mu \in \mathcal{X}(\overline{\Omega})$ is such that $\mu(\overline{\Omega}) = d$, then there exists a sequence $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_g(\Omega; \mathbb{R}^2)$, with $\varepsilon_k \to 0^+$ as $k \to \infty$, such that (1.3) holds and*

$$\limsup_{k\to\infty} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k}; \Omega) \leq 2\pi |\mu|(\overline{\Omega}).$$

1.2. **Fractional framework.** In the present work, we investigate the validity of a fractional analog of Theorem 1.1, in which the differential operator in the L^2 energy in (1.1) is replaced by the *fractional (Riesz) s-gradient*

$$D^s u(x) = (1-s)\, \frac{2^{s-1}}{\pi}\, \frac{\Gamma\left(\frac{3+s}{2}\right)}{\Gamma\left(\frac{3-s}{2}\right)} \int_{\mathbb{R}^2} \frac{(u(y) - u(x)) \otimes (y - x)}{|y - x|^{3+s}}\, \mathrm{d}y, \quad x \in \mathbb{R}^2, \tag{1.4}$$

for $s \in (0,1)$, where Γ stands for *Euler's Gamma function*. The integro-differential operator in (1.4) has gained considerable interest in recent years, leading to a rapidly growing number of works concerning the development of a fractional analog of classical calculus and of its applications. For a non-comprehensive account on the existing literature, we refer to [3, 4, 7, 12–14, 20, 26–29] and the references therein.

Our main aim is to replace (1.1) with functionals of the form

$$\mathrm{GL}_{\varepsilon_s,\lambda}(u;\Omega) = \frac{1}{|\log \varepsilon_s|}\int_\Omega |D^s u|^2 \,\mathrm{d}x + \frac{\lambda}{\varepsilon_s^2|\log\varepsilon_s|}\int_\Omega (|u|^2-1)^2\,\mathrm{d}x \tag{1.5}$$

for some scale parameter $\varepsilon_s > 0$, depending on s, such that $\varepsilon_s \to 0^+$ as $s \to 1^-$, in such a way that the analog of Theorem 1.1 holds.

In order to determine ε_s, we observe that the s-gradient in (1.4) can be equivalently presented as the gradient of the *Riesz potential* of order $1-s$; that is, $D^s u = Dv$, where

$$v(x) = I_{1-s}u(x) = \frac{1-s}{1+s}\,\frac{2^{s-1}}{\pi}\,\frac{\Gamma\big(\frac{3+s}{2}\big)}{\Gamma\big(\frac{3-s}{2}\big)}\int_{\mathbb{R}^2}\frac{u(y)}{|y-x|^{1+s}}\,\mathrm{d}y, \quad x\in\mathbb{R}^2. \tag{1.6}$$

Therefore, since $I_{1-s}u$ tends to u as $s \to 1^-$, we expect that, along a family of minimizers u_s, the functionals in (1.5) can be reasonably approximated as

$$\mathrm{GL}_{\varepsilon_s,\lambda}(u_s;\Omega) \sim \frac{1}{|\log\varepsilon_s|}\int_\Omega |Dv_s|^2\,\mathrm{d}x + \frac{\lambda}{\varepsilon_s^2|\log\varepsilon_s|}\int_\Omega(|v_s|^2-1)^2\,\mathrm{d}x \sim \mathrm{GL}_{\varepsilon_s,\lambda}(v_s;\Omega) \tag{1.7}$$

with $v_s = I_{1-s}u_s$ a family of 'almost minimizers' of the functionals in (1.1). Assuming that u_s and v_s are uniformly bounded and supported in a bounded neighborhood of Ω, we expect that the size of the error in the approximation (1.7) should be not larger than

$$\frac{1}{\varepsilon_s^2|\log\varepsilon_s|}\int_\Omega \Big|(|v_s|^2-1)^2 - (|u_s|^2-1)^2\Big|\,\mathrm{d}x \lesssim \frac{(1-s)^2}{\varepsilon_s^2|\log\varepsilon_s|}\,[u_s]^2_{H^s} \tag{1.8}$$

(we refer to Lemma 3.2 in Section 3.1 below for the precise computations), where

$$[u]^2_{H^s} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|u(y)-u(x)|^2}{|y-x|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \tag{1.9}$$

is the *Sobolev–Slobodeckij H^s-energy* of u. Since the L^2 norm of the s-gradient in (1.4) is proportional to the energy in (1.9),

$$\int_{\mathbb{R}^2}|D^s u|^2\,\mathrm{d}x = \frac{4^s}{2\pi}\,\frac{s\,\Gamma(1+s)}{\Gamma(2-s)}\,(1-s)\,[u]^2_{H^s} \tag{1.10}$$

(e.g., see [27, Rem. 2.3], as well as Proposition 2.8 below), we get that the size of the error in the approximation (1.7) should be at most

$$\frac{(1-s)}{\varepsilon_s^2|\log\varepsilon_s|}\int_{\mathbb{R}^2}|D^s u_s|^2\,\mathrm{d}x \sim \frac{(1-s)}{\varepsilon_s^2|\log\varepsilon_s|}\int_{\mathbb{R}^2}|Dv_s|^2\,\mathrm{d}x.$$

Therefore, in order to re-absorb such an error, we have to require that its size is comparable to the one of the other term in $\mathrm{GL}_{\varepsilon_s,\lambda}(v_s)$; that is,

$$\frac{(1-s)}{\varepsilon_s^2|\log\varepsilon_s|}\int_{\mathbb{R}^2}|Dv_s|^2\,\mathrm{d}x \sim \frac{1}{|\log\varepsilon_s|}\int_{\mathbb{R}^2}|Dv_s|^2\,\mathrm{d}x,$$

from which we get that $\varepsilon_s \sim \sqrt{1-s}$.

1.3. **Statement of the main result.** Although rather naive, the approximation in (1.7) leads to the correct fractional analog of (1.1). In view of (1.9) and of the identification in (1.10), as customary we set

$$H^s(\mathbb{R}^2;\mathbb{R}^2) = \left\{u \in L^2(\mathbb{R}^2;\mathbb{R}^2) : [u]^2_{H^s} < \infty\right\}$$

for $s \in (0,1)$. Therefore, given $\lambda > 0$, we define the *Ginzburg–Landau fractional s-energies*

$$\mathrm{GL}^s_\lambda(\,\cdot\,;\Omega) \colon H^s(\mathbb{R}^2;\mathbb{R}^2) \to [0,\infty)$$

by letting

$$\mathrm{GL}^s_\lambda(u;\Omega) = \frac{1}{|\log(1-s)|}\int_{\mathbb{R}^2} |D^s u|^2 \,\mathrm{d}x + \frac{\lambda}{(1-s)|\log(1-s)|}\int_\Omega (|u|^2-1)^2 \,\mathrm{d}x \tag{1.11}$$

for $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$. By (1.10), we can equivalently write

$$\mathrm{GL}^s_\lambda(u;\Omega) = \frac{1-s}{|\log(1-s)|}\frac{2}{\pi}\,[u]^2_{H^s} + \frac{\lambda}{(1-s)|\log(1-s)|}\int_\Omega (|u|^2-1)^2 \,\mathrm{d}x$$

for $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$. We note that analogous scalar energies (but with different scaling regimes) are related to a non-local approach to phase-transition problems, see [16, 17, 25].

In order to state the fractional analog of Theorem 1.1, we need to introduce a suitable boundary condition, which—due to the non-locality of (1.4)—has to be prescribed on $\mathbb{R}^2\setminus\Omega$ instead of just on $\partial\Omega$. In addition, in view of the underlying approximation argument in (1.7), we require that the boundary datum is bounded and supported in a bounded neighborhood of Ω. Hence, the set of boundary data we consider is defined as

$$\mathcal{B}_\Omega = \left\{g \in H^1(\mathbb{R}^2;\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2;\mathbb{R}^2) : \begin{array}{c} g \text{ has compact support and} \\ |g| = 1 \text{ in an open neighborhood of } \partial\Omega \end{array}\right\}. \tag{1.12}$$

Thus, given $g \in \mathcal{B}_\Omega$ and $L \in [\|g\|_{L^\infty(\mathbb{R}^2)},\infty)$, for $s \in (0,1)$ we define the spaces

$$H^s_g(\Omega;B_L) = \left\{u \in H^s(\mathbb{R}^2;\mathbb{R}^2) : u = g \text{ on } \mathbb{R}^2\setminus\Omega \text{ and } \|u\|_{L^\infty} \le L\right\}. \tag{1.13}$$

In view of the representation of the s-gradient in (1.4) via the Riesz potential in (1.6), the approximation in (1.7), and the analogy with Theorem 1.1, we are naturally led to define the *fractional Jacobian* of a function $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$ as the usual Jacobian of the function $v = I_{1-s}u$ for all $s \in (0,1)$, letting

$$\mathrm{Jac}^s(u) = \det(D^s u) = \det(Dv) = \mathrm{Jac}(v). \tag{1.14}$$

The convergence of $\mathrm{Jac}^s(u)$ as $s \to 1^-$ is then defined according to the customary definition of flat convergence.

With the above notation in force, our main result states as follows.

Theorem 1.2 (Compactness and Γ-convergence of fractional Ginzburg–Landau energies)**.** *Let $\Omega \subset \mathbb{R}^2$ be a non-empty, connected, simply connected, bounded open set with Lipschitz boundary, $g \in \mathcal{B}_\Omega$ with $d = \deg(g|_{\partial\Omega},\partial\Omega) \in \mathbb{Z}$, $L \in [\|g\|_{L^\infty},\infty)$ and $\lambda > 0$.*

(i) (Compactness) *If $u_{s_k} \in H^{s_k}_g(\Omega;B_L)$, with $s_k \to 1^-$ as $k \to \infty$, is such that*

$$\mathrm{GL}^{s_k}_\lambda(u_{s_k};\Omega) < \infty,$$

then there exists a subsequence $(u_{s_{k_j}})_{j\in\mathbb{N}}$ and $\mu \in \mathcal{X}(\overline{\Omega})$ such that $\mu(\overline{\Omega}) = d$ and

$$\mathrm{Jac}^{s_{k_j}}(u_{s_{k_j}}) \xrightarrow{\mathrm{flat}(\overline{\Omega})} \pi\mu \quad \text{as } j \to \infty.$$

(ii) (Γ-lim inf inequality) *If* $u_{s_k} \in H^{s_k}_g(\Omega; B_L)$, *with* $s_k \to 1^-$ *as* $k \to \infty$, *is such that*

$$\mathrm{Jac}^{s_k}(u_{s_k}) \xrightarrow{\mathrm{flat}(\overline{\Omega})} \pi\mu \quad \text{as } k \to \infty \tag{1.15}$$

for some $\mu \in \mathcal{X}(\overline{\Omega})$, *then* $\mu(\overline{\Omega}) = d$ *and*

$$\liminf_{k\to\infty} \mathrm{GL}^{s_k}_\lambda(u_{s_k}; \Omega) \geq \pi|\mu|(\overline{\Omega}).$$

(iii) (Γ-lim sup inequality) *If* $\mu \in \mathcal{X}(\overline{\Omega})$ *is such that* $\mu(\overline{\Omega}) = d$, *then there exists a sequence* $u_{s_k} \in H^{s_k}_g(\Omega; B_L)$, *with* $s_k \to 1^-$ *as* $k \to \infty$, *such that* (1.15) *holds and*

$$\limsup_{k\to\infty} \mathrm{GL}^{s_k}_\lambda(u_{s_k}; \Omega) \leq \pi|\mu|(\overline{\Omega}).$$

1.4. **Strategy of proof.** The proof of Theorem 1.2 is split in two parts.

On the one side, claims (i) and (ii) of Theorem 1.2 follow by comparison with the local setting (recall Theorem 1.1) via a rigorous formulation of the approximation argument sketched in (1.7). The overall idea is to show that, if the fractional Ginzburg–Landau s_k-energy of u_{s_k} is uniformly bounded, then the integer ε_k-Ginzburg–Landau energy of $v_{\varepsilon_k} = I_{1-s_k} u_{s_k}$, with $\varepsilon_k = \sqrt{1-s_k}$, is also uniformly bounded and, actually, we have that

$$\mathrm{GL}^{s_k}_\lambda(u_{s_k}; \Omega) \geq c_k \, \mathrm{GL}_{\varepsilon_k,\Lambda}(v_{\varepsilon_k}; \Omega)$$

for some $c_k > 0$ such that $c_k \to \frac{1}{2}$ as $k \to \infty$ and $\Lambda > 0$ which does not depend on k. From Theorem 1.1 we hence infer compactness and the Γ-lim inf inequality for $\mathrm{GL}^{s_k}_\lambda(u_{s_k}; \Omega)$ thanks to the fact that, due to the definition in (1.14) and (1.6), we have

$$\mathrm{Jac}^{s_k}(u_{s_k}) = c'_k \, \mathrm{Jac}(v_{\varepsilon_k}) \tag{1.16}$$

for some $c'_k > 0$ such that $c'_k \to 1$ as $k \to \infty$. The fact that $\mu(\overline{\Omega}) = d$ follows from an integration-by-parts argument, roughly exploiting the fact that, recalling (1.16),

$$\mathrm{Jac}^{s_k}(u_{s_k}) = c'_k \, \mathrm{Jac}(v_{s_k}) = c'_k \, \mathrm{curl}\, j(v_{s_k}) \sim \mathrm{curl}\, j(g) = \mathrm{Jac}(v_{s_k})$$

in a neighborhood U of $\partial\Omega$, where $j(v_{s_k}) = v_{s_k} \times Dv_{s_k}$, since

$$\|D^{s_k} u_{s_k}\|_{L^2(\mathbb{R}^2)} \, \|v_{s_k} - g\|_{L^2(U\setminus\Omega)} \lesssim \mathrm{GL}^{s_k}_\lambda(u_{s_k}; \Omega) \sqrt{1-s_k} |\log(1-s_k)| \to 0^+, \quad \text{as } k \to \infty,$$

in view of the definition in (1.11) and of the approximation in (1.7) and (1.8).

On the other side, concerning claim (iii) in Theorem 1.2, the recovery sequence $(u_{s_k})_{k\in\mathbb{N}}$ is built according to the classical cut-off approach around a vortex singularity. Precisely, we first consider the case in which μ is given by

$$\mu = \sum_{i=1}^{N} d_i \delta_{x_i}, \tag{1.17}$$

with $N \in \mathbb{N}$, $x_i \in \Omega$ and $d_i \in \{-1, 1\}$ such that

$$\sum_{i=1}^{N} d_i = \deg(g|_{\partial\Omega}, \partial\Omega), \tag{1.18}$$

and construct a recovery sequence of the form

$$u_{s_k} = \begin{cases} \eta_k(\,\cdot\, - x_i)\, \dfrac{(\,\cdot\, - x_i)^{d_i}}{|\,\cdot\, - x_i|} & \text{around } x_i, \text{ for each } i = 1, \dots, N, \\ g & \text{outside } \Omega, \\ \widehat{u} & \text{everywhere else,} \end{cases} \tag{1.19}$$

where η_k is a suitable cut-off near the origin depending on s_k, $x^d = \begin{pmatrix} x_1 \\ (-1)^d x_2 \end{pmatrix}$ for $x \in \mathbb{R}^2$ and $d \in \{-1, 1\}$, and $\widehat{u}$ is an H^1 function with values in $\mathbb{S}^1$ which does not depend on s_k, but only on the position of the x_i's in Ω, interpolating the vortices $\frac{(\,\cdot\,-x_i)^{d_i}}{|\,\cdot\,-x_i|}$, $i = 1, \dots, N$, and the datum g (the existence of such a function $\widehat{u}$ is guaranteed by the condition (1.18)). Once the special case (1.17) is achieved, the general situation can be then established via a routine diagonal approximation argument.

1.5. **Comments.** We conclude this introduction with some comments on our main result.

An important difference between Theorem 1.1 and Theorem 1.2 appears in the domains of definition of the corresponding energies. In our Theorem 1.2, we only consider H^s functions attaining the datum $g \in \mathcal{B}_\Omega$ outside Ω which are bounded by some constant $L \ge \|g\|_{L^\infty}$. We do not know if this additional constraint—which we need in the error estimate in (1.8)—is merely technical and can be removed. We nevertheless notice that such constraint does not seem to be essential in our approach. Indeed, the fractional energies in (1.11) are decreased by truncation (see Corollary 2.9 below) and the L^∞ norm of the recovery sequence constructed in point (iii) of Theorem 1.2 is bounded by $\|g\|_{L^\infty}$.

Another relevant point concerning the definition in (1.13) is whether Theorem 1.2 may be achieved with respect to the smaller space

$$H^s_g(\Omega; \mathbb{S}^1) = \Big\{ u \in H^s(\mathbb{R}^2; \mathbb{R}^2) : u = g \text{ on } \mathbb{R}^2 \setminus \Omega \text{ and } |u| = 1 \text{ on } \Omega \Big\} \tag{1.20}$$

for $s \in (0, 1)$, simplifying the non-local energies in (1.11) to

$$\mathrm{GL}^s(u; \Omega) = \frac{1}{|\log(1-s)|} \int_{\mathbb{R}^2} |D^s u| \, \mathrm{d}x. \tag{1.21}$$

While the compactness and the Γ-$\liminf$ follow as in the proof of Theorem 1.2 up to minor changes, the Γ-$\limsup$ remains unclear. As done in [30], one may directly take $\frac{x^d}{|x|}$, with $x \neq 0$ and $d \in \{1, -1\}$, as a building block for the recovery sequence; that is, like in (1.19), but avoiding the truncation near the singularity. However, the recovery sequence $\widetilde{u}_s \in H^1_g(\Omega; \mathbb{S}^1)$ constructed in this way satisfies

$$\mathrm{GL}^s(\widetilde{u}_s; \Omega) \lesssim \frac{1}{(1-s)|\log(1-s)|} \quad \text{for } s \in (0, 1). \tag{1.22}$$

We do not know if the bound in (1.22) is optimal but, at least, it seems to agree with the one given by [30, Th. 2.6] when (formally) rephrased to our setting. Indeed, given a kernel $\rho \ge 0$, the energy considered in [30] is defined as

$$F_\varepsilon(u) = \frac{1}{\varepsilon^4 |\log \varepsilon|} \int_\Omega \int_\Omega \rho\Big(\frac{|x-y|}{\varepsilon}\Big) |u(x) - u(y)|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{1.23}$$

Choosing $\rho(r) = r^{-2-2s}$ for $r > 0$ (note that this kernel is not admissible for [30, Th. 2.6]) and $\varepsilon = \sqrt{1-s}$, the functional in (1.23) becomes asymptotically equivalent as $s \to 1^-$ to

$$u \mapsto \frac{1}{|\log(1-s)|} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y. \tag{1.24}$$

In virtue of (1.9) and (1.10), and ignoring the reminder terms coming from the interactions between Ω and $\mathbb{R}^2 \setminus \Omega$, the energy in (1.24) differs from the one in (1.21) exactly by a factor $(1-s)$, which is precisely the additional factor appearing in the bound (1.22).

The above considerations suggest that a more refined analysis of the energy (1.22) in the space in (1.20) is required in order to properly understand the Γ-limit as $s \to 1^-$, and in particular whether vortex singularities are ruled out for our functionals with the strict constraint $|u| = 1$.

We finally note that functionals in (1.11) can be equivalently defined for $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. In this case, Theorem 1.2 must be stated in terms of $(n-2)$-dimensional integral currents. We do not address the details here, but we refer to [1] for proper statements and for the extension of the construction of recovery sequences for arbitrary $n \geq 2$.

1.6. **Organization of the paper.** The main notation and definitions, plus some basic results concerning the Jacobian, the degree, the fractional gradient and the Riesz potential, are given in Section 2. The proof of Theorem 1.2 is then detailed across Section 3.

2. Preliminaries

In this section, we recall the main notation and collect some preliminary results.

2.1. **General notation.** Throughout the paper, the ambient space is $\mathbb{R}^2$. The norm of a point $x = \binom{x_1}{x_2} \in \mathbb{R}^2$ is given by $|x|^2 = x_1^2 + x_2^2$, while the norm of a matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is given by $|A|^2 = A_{11}^2 + A_{12}^2 + A_{21}^2 + A_{22}^2$. We let $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ be the standard unitary circle and we let $\Omega \subset \mathbb{R}^2$ denote an open set, possibly satisfying further topological and/or regularity assumptions.

2.2. **Lipschitz functions, measures and flat norm.** We let $\mathrm{Lip}(\Omega)$ be the space of real-valued *Lipschitz functions* on Ω, endowed with the norm

$$\|\varphi\|_{\mathrm{Lip}} = \|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)}, \quad \text{for } \varphi \in \mathrm{Lip}(\Omega),$$

where $\nabla\varphi = \begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \end{pmatrix}$ as customary, and we let $\mathrm{Lip}_c(\Omega)$ be its subspace of functions with compact support in Ω.

We let $\mathcal{M}_b(\overline{\Omega})$ be the space of *Radon measures* on $\overline{\Omega}$ with finite total variation. We let

$$\|\mu\|_{\mathrm{flat}(\Omega)} = \sup\left\{ \int_\Omega \varphi \,\mathrm{d}\mu : \varphi \in \mathrm{Lip}_c(\Omega),\ \|\varphi\|_{\mathrm{Lip}} \leq 1 \right\},$$

and, analogously,

$$\|\mu\|_{\mathrm{flat}(\overline{\Omega})} = \sup\left\{ \int_\Omega \varphi \,\mathrm{d}\mu : \varphi \in \mathrm{Lip}(\Omega),\ \|\varphi\|_{\mathrm{Lip}} \leq 1 \right\},$$

be the *flat norm* of $\mu \in \mathcal{M}_b(\overline{\Omega})$ on Ω and $\overline{\Omega}$, respectively. Consequently, given $\mu_k, \mu \in \mathcal{M}_b(\overline{\Omega})$, $k \in \mathbb{N}$, we write $\mu_k \xrightarrow{\mathrm{flat}(\Omega)} \mu$ (respectively, $\mu_k \xrightarrow{\mathrm{flat}(\overline{\Omega})} \mu$) if $\|\mu_n - \mu\|_{\mathrm{flat}(\Omega)} \to 0$ (respectively, $\|\mu_k - \mu\|_{\mathrm{flat}(\overline{\Omega})} \to 0$) as $k \to \infty$.

2.3. **Sobolev functions, Jacobian and degree.** We let

$$H^1(\Omega;\mathbb{R}^2)=\left\{v\in L^2(\Omega;\mathbb{R}^2):Dv\in L^2(\Omega;\mathbb{R}^{2\times 2})\right\}$$

be the standard *Sobolev space* on Ω, endowed with the norm

$$\|v\|_{H^1}=\|v\|_{L^2(\Omega)}+\|Dv\|_{L^2(\Omega)},\quad \text{for } v\in H^1(\Omega;\mathbb{R}^2),$$

see [21, Ch. 11] for a detailed introduction. Here we note that

$$Dv=\begin{pmatrix}\partial_{x_1}v_1 & \partial_{x_2}v_1\\ \partial_{x_1}v_2 & \partial_{x_2}v_2\end{pmatrix}$$

and that $\|Dv\|^2_{L^2(\Omega)}=\||Dv|\|^2_{L^2(\Omega)}$, where $|Dv|$ is defined as in Section 2.1.

The *Jacobian* of $v\in H^1(\Omega;\mathbb{R}^2)$ is defined as

$$\mathrm{Jac}(v)=\det(Dv)=\partial_{x_1}v_1\partial_{x_2}v_2-\partial_{x_2}v_1\partial_{x_1}v_2\in L^1(\Omega). \tag{2.1}$$

In particular, we tacitly identify $\mathrm{Jac}(v)$ with the Radon measure $\mathrm{Jac}(v)\mathscr{L}^2\in\mathcal{M}_b(\overline{\Omega})$. For a (historical) presentation of the Jacobian of Sobolev functions, we refer to the survey [11].

We now recall some elementary properties of the Jacobian that will be useful in the sequel. The first one concerns H^1 functions with values in $\mathbb{S}^1$ (e.g., see [11, Sec. 3]).

Lemma 2.1. *If $v\in H^1(\Omega;\mathbb{R}^2)$ is such that $|v|=1$ on Ω, then $\mathrm{Jac}(v)=0$ on Ω.*

Proof. Differentiating $|v|^2=1$, we get that $v\in\ker(Du)$, proving that $\det(Dv)=0$. □

In order to state the second result on the Jacobian, we need the following notation.

Definition 2.2. Given $v,w\in H^1(\Omega;\mathbb{R}^2)$, we define the *current*

$$j(v,w)=v\times Dw=v_1Dw_2-v_2Dw_1\in L^1(\Omega;\mathbb{R}^2).$$

In particular, we let $j(v)=j(v,v)\in L^1(\Omega;\mathbb{R}^2)$.

The next result relates the Jacobian of H^1 functions to the current j in Definition 2.2 and also shows that the curl of j is zero.

Lemma 2.3. *If $v,w\in H^1(\Omega;\mathbb{R}^2)$, then*

$$\int_\Omega \mathrm{Jac}(v)\,\varphi\,\mathrm{d}x=\frac{1}{2}\int_\Omega j(v)\times\nabla\varphi\,\mathrm{d}x \tag{2.2}$$

and

$$\int_\Omega (j(v,w)-j(w,v))\times\nabla\varphi\,\mathrm{d}x=0 \tag{2.3}$$

whenever $\varphi\in\mathrm{Lip}_c(\Omega)$.

Proof. If $v,w\in C^2(\Omega;\mathbb{R}^2)$, then we can write

$$\mathrm{Jac}(v)=\frac{1}{2}\,\mathrm{curl}\,j(v)\quad\text{and}\quad j(v,w)-j(w,v)=\nabla(v\times w),$$

from which we get (2.2) and (2.3). In the general case, the validity of (2.2) and (2.3) follows by approximating $v,w\in H^1(\Omega;\mathbb{R}^2)$ with smooth functions. □

Lemma 2.3 can be exploited to estimate the flat distance between the Jacobians of two H^1 functions, as follows (note that this is a particular instance of [10, Th. 1(i)]).

Proposition 2.4. *If $v, w \in H^1(\Omega;\mathbb{R}^2)$, then*

$$\|\operatorname{Jac}(v) - \operatorname{Jac}(w)\|_{\mathrm{flat}(\Omega)} \le 2\,\|v-w\|_{L^2(\Omega)}\,(\|Dv\|_{L^2(\Omega)} + \|Dw\|_{L^2(\Omega)}).$$

Proof. Given $v, w \in H^1(\Omega;\mathbb{R}^2)$, by Definition 2.2 we can estimate

$$|j(v) - j(w)| \le |j(v, v-w)| + |j(w, v-w)| \le 4\,|v-w|\,(|Dv| + |Dw|).$$

From (2.2), by Hölder's inequality we hence get that

$$\left|\int_\Omega (\operatorname{Jac}(v) - \operatorname{Jac}(w))\,\varphi \,\mathrm{d}x\right| \le 2\,\|v-w\|_{L^2(\Omega)}\,(\|Dv\|_{L^2(\Omega)} + \|Dw\|_{L^2(\Omega)})\|\nabla\varphi\|_{L^\infty(\Omega)}$$

whenever $\varphi \in \operatorname{Lip}_c(\Omega)$, yielding the conclusion. □

Last, but not least, we recall the following result linking the Jacobian of Sobolev functions to their (*topological*) *degree*. For a more detailed account, we refer to the survey [9].

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^2$ be a non-empty, connected, simply connected, bounded open set with Lipschitz boundary. If $u \in H^1(\Omega;\mathbb{R}^2)$ is such that $|u| = 1$ on $\partial\Omega$, then the (topological) degree $\deg(u|_{\partial\Omega}, \partial\Omega) \in \mathbb{Z}$ of the trace $u|_{\partial\Omega}$ of u on $\partial\Omega$ satisfies*

$$\deg(u|_{\partial\Omega}, \partial\Omega) = \frac{1}{\pi}\int_\Omega \operatorname{Jac}(u)\,\mathrm{d}x. \tag{2.4}$$

2.4. **Ginzburg–Landau energies.** As briefly recalled in Section 1.1, given $\varepsilon, \lambda > 0$, the *Ginzburg–Landau functionals* on a non-empty open set Ω,

$$\mathrm{GL}_{\varepsilon,\lambda}(\,\cdot\,;\Omega)\colon H^1(\Omega;\mathbb{R}^2) \to \mathbb{R},$$

are defined as

$$\mathrm{GL}_{\varepsilon,\lambda}(u;\Omega) = \frac{1}{|\log\varepsilon|}\int_\Omega |Du|^2\,\mathrm{d}x + \frac{\lambda}{\varepsilon^2|\log\varepsilon|}\int_\Omega (|u|^2-1)^2\,\mathrm{d}x \tag{2.5}$$

for $u \in H^1(\Omega;\mathbb{R}^2)$. The following result rephrases Theorem 1.1 in the case no boundary condition is imposed on $\partial\Omega$ (again, refer to [2, 8, 18, 19, 23, 24] for the proof, and see [1] for the higher-dimensional setting). Here and in below, as in (1.2), we let

$$\mathcal{X}(A) = \left\{\mu = \sum_{i=1}^N d_i\,\delta_{x_i} : d_i \in \mathbb{Z} \text{ and } x_i \in A \text{ for } i = 1,\dots,N, \text{ with } N \in \mathbb{N}\right\}$$

be the collections of *atomic* measures on a non-empty set $A \subset \mathbb{R}^2$.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a non-empty, connected, simply connected, bounded open set with Lipschitz boundary and $\lambda > 0$.*

(i) (Compactness) *If $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^2)$, with $\varepsilon_k \to 0^+$ as $k \to \infty$, is such that*

$$\sup_{k\in\mathbb{N}} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k};\Omega) < \infty,$$

then there exists a subsequence $(v_{\varepsilon_{k_j}})_{j\in\mathbb{N}}$ and $\mu \in \mathcal{X}(\overline{\Omega})$ such that

$$\operatorname{Jac}(v_{\varepsilon_{k_j}}) \xrightarrow{\mathrm{flat}(\overline{\Omega})} \mu \quad \text{as } j \to \infty.$$

(ii) (Γ-lim inf inequality) *If* $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_g(\Omega;\mathbb{R}^2)$, *with* $\varepsilon_k \to 0^+$ *as* $k\to\infty$, *is such that*

$$\mathrm{Jac}(v_{\varepsilon_k}) \xrightarrow{\mathrm{flat}(\overline{\Omega})} \mu \quad \text{as } k\to\infty \tag{2.6}$$

for some $\mu \in \mathcal{X}(\overline{\Omega})$, *then*

$$\liminf_{k\to\infty} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k};\Omega) \ge 2\pi|\mu|(\overline{\Omega}).$$

(iii) (Γ-lim sup inequality) *If* $\mu \in \mathcal{X}(\overline{\Omega})$, *then there exists a sequence* $(v_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_g(\Omega;\mathbb{R}^2)$, *with* $\varepsilon_k \to 0^+$ *as* $k\to\infty$, *such that* (2.6) *holds and*

$$\limsup_{k\to\infty} \mathrm{GL}_{\varepsilon_k,\lambda}(v_{\varepsilon_k};\Omega) \le 2\pi|\mu|(\overline{\Omega}).$$

2.5. **Fractional Sobolev functions.** We let

$$H^s(\mathbb{R}^2;\mathbb{R}^2) = \left\{u \in L^2(\Omega;\mathbb{R}^2) : [u]_{s,\Omega} < \infty\right\}$$

be the *fractional Sobolev space* of order $s\in(0,1)$, endowed with the norm

$$\|u\|_{H^s} = \|u\|_{L^2} + [u]_{s,\Omega}, \quad \text{for } u \in H^s(\Omega;\mathbb{R}^2),$$

where

$$[u]^2_{s,\Omega} = \int_\Omega\int_\Omega \frac{|u(y)-u(x)|^2}{|y-x|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.7}$$

see [22, Ch. 6] for a detailed introduction. If $\Omega=\mathbb{R}^2$, then we simply write $[u]_s = [u]_{s,\mathbb{R}^2}$.

For future convenience, we recall the following result, which corresponds to [6, Lem. 3]. Here and below, for $\tau>0$, we let

$$\triangle_\tau = \left\{(x,y)\in\mathbb{R}^2 : |x-y|\le\tau\right\} \subset \mathbb{R}^2. \tag{2.8}$$

Lemma 2.7. *Let* $\Omega\subset\mathbb{R}^2$ *be a measurable set. If* $u\in L^2(\Omega)$, *then*

$$\iint_{(\Omega\times\Omega)\setminus\triangle_\tau} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{4\pi\|u\|^2_{L^2(\Omega)}}{s\tau^{2s}}$$

for $\tau>0$ *and* $s\in(0,1)$.

Proof. By the definition in (2.8), we can estimate

$$\iint_{(\Omega\times\Omega)\setminus\triangle_\tau} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \le 4\int_\Omega |u(x)|^2 \int_{\mathbb{R}^2\setminus B_\tau} \frac{\mathrm{d}h}{|h|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y = \frac{4\pi\|u\|^2_{L^2(\Omega)}}{s\tau^{2s}}$$

whenever $\tau>0$ and $s\in(0,1)$. □

2.6. **Fractional gradient.** By combining [27, Rem. 2.3] with [7, Cor. 1] (also see the discussion in [13, Sec. 3.9]), we can equivalently define

$$H^s(\mathbb{R}^2;\mathbb{R}^2) = \left\{u\in L^2(\mathbb{R}^2;\mathbb{R}^2) : D^s u \in L^2(\mathbb{R}^2;\mathbb{R}^{2\times2})\right\}$$

for $s\in(0,1)$, where, as briefly recalled in (1.4),

$$D^s u(x) = (1-s)\,\frac{2^{s-1}}{\pi}\,\frac{\Gamma\left(\frac{3+s}{2}\right)}{\Gamma\left(\frac{3-s}{2}\right)}\int_{\mathbb{R}^2} \frac{(u(y)-u(x))\otimes(y-x)}{|y-x|^{3+s}}\,\mathrm{d}y, \quad x\in\mathbb{R}^2, \tag{2.9}$$

is the *fractional* (*Riesz*) *s-gradient* of u. Note that the s-gradient in (2.9) is well defined for sufficiently regular functions ($u\in \mathrm{Lip}_c(\mathbb{R}^2;\mathbb{R}^2)$ would suffice, see the discussion in [13, Sec. 2.2] for instance), while, for $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$, the operator in (2.9) is defined in the distributional sense via (fractional) integration-by-parts, see [13, Def. 3.19].

Here we just recall the following result, which relates the L^2 norm of the s-gradient in (2.9) with the fractional seminorm in (2.7). Here and below, as in Section 2.3, we set $\||D^s u\|^2_{L^2} = \||D^s u|\|^2_{L^2}$, where $|D^s u|$ is as defined in Section 2.1.

Proposition 2.8. *If $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$, then*

$$\|D^s u\|^2_{L^2} = (1-s)\, c_s\, [u]^2_s, \tag{2.10}$$

where $c_s = \dfrac{4^s}{2\pi}\,\dfrac{s\,\Gamma(1+s)}{\Gamma(2-s)} > 0$ *satisfies* $\displaystyle\lim_{s\to 1^-} c_s = \frac{2}{\pi}$.

Proof. By density, we may assume that $u \in C^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$ without loss of generality. In this case, formula (2.10) follows either by applying Fourier's transform, or by observing that

$$(1-s)\, c_s\, [u]^2_s = \int_{\mathbb{R}^2} u\,(-\Delta)^s u\,\mathrm{d}x = -\int_{\mathbb{R}^2} u\,\operatorname{div}^s(D^s u)\,\mathrm{d}x = \int_{\mathbb{R}^2} |D^s u|^2\,\mathrm{d}x,$$

where $(-\Delta)^s$ is the *fractional Laplacian* of order s and div^s is the *fractional divergence* of order s (the dual operator of (2.9)), see [13, Lem. 2.5 and Sec. 3.10]. □

From Proposition 2.8, we deduce that the fractional Ginzburg–Landau energies in (1.11) are decreased by truncation. In order to precisely state this result, we need to introduce some notation. Given $L > 0$, we let

$$T_L(t) = \max\{-L, \min\{t, L\}\}, \quad \text{for } t \in \mathbb{R},$$

and we set

$$T_L(x) = \begin{pmatrix} T_L(x_1) \\ T_L(x_2) \end{pmatrix}, \quad \text{for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

Note that $|T_L(x)| \le |x|$ for all $x \in \mathbb{R}^2$ by definition.

Corollary 2.9. *If $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$, then $T_L(u) \in H^s(\mathbb{R}^2;\mathbb{R}^2)$ and*

$$\mathrm{GL}^s_\lambda(T_L(u)) \le \mathrm{GL}^s_\lambda(u) \tag{2.11}$$

for all $s \in (0,1)$ and $L \ge 1$.

Proof. Since $T_L\colon \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, from Proposition 2.8 we get that

$$\|D^s(T_L(u))\|^2_{L^2} = (1-s)\, c_s\, [T_L(u)]^2_s \le (1-s)\, c_s\, [u]^2_s = \|D^s u\|^2_{L^2}$$

whenever $L > 0$. Moreover, for $L \ge 1$, we also have that $(|T_L(u)|^2 - 1)^2 \le (|u|^2 - 1)^2$, from which the inequality in (2.11) follows. □

2.7. **Riesz potential.** As observed in [27, Th. 1.2] (also see [13, Sec. 2.3]), the s-gradient in (2.9) can be rewritten as $D^s = D I_{1-s}$, where

$$I_{1-s} u(x) = \frac{1}{\gamma_s} \int_{\mathbb{R}^2} \frac{u(y)}{|y-x|^{1+s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^2, \tag{2.12}$$

is the *Riesz potential* of order $1-s$ in $\mathbb{R}^2$, with

$$\gamma_s = 2^{1-s}\pi\,\frac{1+s}{1-s}\,\frac{\Gamma\left(\frac{3-s}{2}\right)}{\Gamma\left(\frac{3+s}{2}\right)}. \tag{2.13}$$

Precisely, considering the differential operators in the distributional sense, we have that

$$D^s u = D I_{1-s} u \text{ in } L^2(\mathbb{R}^2;\mathbb{R}^{2\times 2}) \tag{2.14}$$

for $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$. Therefore, if $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$, then

$$v = I_{1-s} u \in H^1(\mathbb{R}^2;\mathbb{R}^2) \tag{2.15}$$

and so, recalling (2.1), we may generalize the notion of Jacobian to fractional Sobolev functions as follows.

Definition 2.10 (Fractional Jacobian)**.** Given $s \in (0,1)$, the *fractional (Riesz) s-Jacobian* of $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$ is defined as

$$\mathrm{Jac}^s(u) = \mathrm{Jac}(I_{1-s}u) \in L^1(\mathbb{R}^2).$$

In particular, we tacitly identify $\mathrm{Jac}^s(u)$ with the Radon measure $\mathrm{Jac}^s(u)\mathscr{L}^2 \in \mathcal{M}_b(\mathbb{R}^2)$ whenever $s \in (0,1)$ and $u \in H^s(\mathbb{R}^2;\mathbb{R}^2)$.

Remark 2.11. Although not needed in the present work, Definition 2.2, Lemma 2.3, and Proposition 2.4 can be naturally analogously reformulated in the fractional setting.

3. Proof of Theorem 1.2

The rest of the paper is dedicated to the proof of Theorem 1.2, which is split across Sections 3.2, 3.3 and 3.5. From now on, with the notation introduced in (1.12), we fix a boundary datum $g \in \mathcal{B}_\Omega$ and a bounded open neighborhood $U \subset \mathbb{R}^2$ of $\partial\Omega$ such that $|g| = 1$ on U. We hence let $L \in [\|g\|_{L^\infty(\mathbb{R}^2)}, \infty)$, we let $A = U \cup \Omega$ and $R > 0$ be such that $\operatorname{supp} \bar{u} \subset A_R$, where

$$A_R = \bigcup_{x \in A} B_R(x),$$

and we work in the fractional Sobolev space $H^s_g(\Omega; B_L)$ defined in (1.13).

3.1. A truncated Riesz potential.

We begin with the following definition, which provides a truncated version of the Riesz potential defined in (2.12).

Definition 3.1 (Truncated Riesz potential)**.** For $s \in (0,1)$, the *truncated Riesz potential* $I^R_{1-s}u \colon \mathbb{R}^2 \to \mathbb{R}^2$ of $u \in H^s_g(\Omega; B_L)$ is defined as

$$I^R_{1-s}u(x) = \frac{1-s}{2\pi R^{1-s}} \int_{B_R(x)} \frac{u(y)}{|x-y|^{1+s}}\, \mathrm{d}y, \quad \text{for } x \in \mathbb{R}^2. \tag{3.1}$$

Let $u \in H^s_g(\Omega; B_L)$ be fixed, and set $v = I^R_{1-s}u$ as in Definition 3.1 above for brevity. Recalling (2.12), and owing to the fact that $\operatorname{supp} g \subset A$, we can write

$$v(x) = \frac{(1-s)\gamma_s}{2\pi R^{1-s}}\, I_{1-s}u(x), \quad \text{for } x \in A, \tag{3.2}$$

with $\gamma_s > 0$ as in (2.13) such that

$$\lim_{s \to 1^-} \frac{(1-s)\gamma_s}{2\pi R^{1-s}} = 1. \tag{3.3}$$

Moreover, by combining (2.14) and (2.15) with (3.2), we get that $v \in H^1(A; B_M)$, with

$$Dv = \frac{(1-s)\gamma_s}{2\pi R^{1-s}}\, D^s u \quad \text{in } L^2(A;\mathbb{R}^{2\times 2}) \tag{3.4}$$

and

$$\mathrm{Jac}^s(u) = \left(\frac{2\pi R^{1-s}}{(1-s)\gamma_s}\right)^2 \mathrm{Jac}(v) \quad \text{on } A. \tag{3.5}$$

We now exploit Definition 3.1 in two ways. We first provide a quantitative formulation of the error in the approximation argument carried in (1.7) and (1.8).

Lemma 3.2 (L^2 comparison). *If $u \in H^s_{\bar u}(\Omega; B_L)$ then $v = I^R_{1-s}u$ as in Definition 3.1 satisfies*

$$\int_A \left(|v|^2 - |u|^2\right)^2 \,\mathrm{d}x \le 4L^2 \int_A |u-v|^2 \,\mathrm{d}x \le \frac{(1-s)^2 L^2}{\pi R^{1-2s}} [u]_s^2 \tag{3.6}$$

for any $s \in (0,1)$.

Proof. Since

$$u(x) = \frac{1-s}{2\pi R^{1-s}} \int_{B_R(x)} \frac{u(x)}{|x-y|^{1+s}} \,\mathrm{d}y, \quad \text{for } x \in \mathbb{R}^2,$$

we can estimate

$$|u+v|^2 = \left|\frac{1-s}{2\pi R^{1-s}} \int_{B_R(x)} \frac{u(x)+u(y)}{|x-y|^{1+s}} \,\mathrm{d}y\right|^2 \le 4\|u\|^2_{L^\infty(A_R)} \le 4L^2$$

for any $x \in A$, so that

$$\left(|v(x)|^2 - |u(x)|^2\right)^2 \le |v(x)+u(x)|^2 \, |v(x)-u(x)|^2 \le 4L^2 \, |v(x)-u(x)|^2$$

for any $x \in A$, giving the first inequality in (3.6) by integrating on A. For the second inequality in (3.6), we just need to observe that, by Jensen's inequality,

$$|v(x)-u(x)|^2 = \left|\frac{1-s}{2\pi R^{1-s}} \int_{B_R(x)} \frac{u(x)-u(y)}{|x-y|^{1+s}} \,\mathrm{d}y\right|^2 \le \frac{(1-s)^2}{\pi R^{1-2s}} \int_{B_R(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}y$$

for any $x \in A$, and the conclusion follows by integrating on A again. □

Additionally, we compare the Ginzburg–Landau energies in (2.5) with their fractional counterparts in (1.11). As mentioned in Section 1.4, Lemma 3.3 below will play a crucial role in the proof of claims (i) and (ii) of Theorem 1.2.

Lemma 3.3 (Energy comparison). *If $u \in H^s_{\bar u}(\Omega; B_L)$, then $v = I^R_{1-s}u$ as in Definition 3.1 satisfies*

$$\mathrm{GL}^s_\lambda(u;\Omega) \ge \frac{4\pi R^{1-2s} c_s}{(1-s)|\log(1-s)|} \int_A |u-v|^2 \,\mathrm{d}x, \tag{3.7}$$

$$\mathrm{GL}^s_\lambda(u;\Omega) \ge \frac{\sqrt{4\pi R^{1-2s} c_s}}{\sqrt{1-s}\,|\log(1-s)|} \, \|D^s u\|_{L^2(\mathbb{R}^2)} \, \|v-u\|_{L^2(A)}, \tag{3.8}$$

$$\mathrm{GL}^s_\lambda(u;\Omega) \ge \frac{\eta}{2}\left(\frac{2\pi R^{1-s}}{(1-s)\gamma_s}\right)^2 \mathrm{GL}_{\sqrt{1-s},\,\Lambda_{s,\lambda,\eta}}(v;A), \tag{3.9}$$

for $s \in (0,1)$, $\lambda > 0$ and $\eta \in (0,1)$, where

$$\Lambda_{s,\lambda,\eta} = \frac{1-\eta}{2\eta}\left(\frac{(1-s)\gamma_s}{2\pi R^{1-s}}\right)^2 m_{s,\lambda}, \quad m_{s,\lambda} = \min\left\{\frac{c_s \pi R^{1-2s}}{L^2}, \lambda\right\}, \tag{3.10}$$

and $c_s, \gamma_s > 0$ are as in (2.10) and (2.13), respectively.

Proof. By Proposition 2.8 and the first inequality in (3.6) in Lemma 3.2, we get that

$$\mathrm{GL}^s_\lambda(u;\Omega) \ge \frac{(1-s)\,c_s}{|\log(1-s)|}\,[u]^2_s \ge \frac{4\pi R^{1-2s} c_s}{(1-s)|\log(1-s)|}\int_A |u-v|^2\,\mathrm{d}x,$$

proving (3.7). Combining (3.7) with the inequality

$$\mathrm{GL}^s_\lambda(u;\Omega) \ge \frac{1}{|\log(1-s)|}\int_{\mathbb{R}^2} |D^s u|^2\,\mathrm{d}x,$$

we get (3.8). Owing to the fact that $u = g$ and $|g| = 1$ on $A \setminus \Omega$, Proposition 2.8 and the second inequality in (3.6) in Lemma 3.2, we can estimate

$$\begin{aligned}
\mathrm{GL}^s_\lambda(u;\Omega) &= \frac{(1-s)\,c_s}{|\log(1-s)|}\,[u]^2_s + \frac{\lambda}{(1-s)|\log(1-s)|}\int_A \left(|u|^2-1\right)^2\mathrm{d}x\\
&\ge \frac{(1-s)\,c_s}{|\log(1-s)|}\,\frac{\pi R^{1-2s}}{(1-s)^2L^2}\int_A \left(|v|^2-|u|^2\right)^2\mathrm{d}x\\
&\quad + \frac{\lambda}{(1-s)|\log(1-s)|}\int_A \left(|u|^2-1\right)^2\mathrm{d}x\\
&\ge \frac{m_{s,\lambda}}{2\,(1-s)|\log(1-s)|}\int_A \left(|v|^2-1\right)^2\mathrm{d}x.
\end{aligned} \tag{3.11}$$

Therefore, by (3.11) and (3.4), we get that

$$\begin{aligned}
&\mathrm{GL}^s_\lambda(u) = \eta\,\mathrm{GL}^s_\lambda(u) + (1-\eta)\,\mathrm{GL}^s_\lambda(u)\\
&\ge \frac{\eta}{|\log(1-s)|}\int_A |D^s u|^2\,\mathrm{d}x + \frac{(1-\eta)\,m_{s,\lambda}}{2\,(1-s)|\log(1-s)|}\int_A \left(|v|^2-1\right)^2\mathrm{d}x\\
&\ge \frac{\eta}{2}\left(\frac{2\pi R^{1-s}}{(1-s)\gamma_s}\right)^2 \frac{1}{|\log\sqrt{1-s}|}\int_A |Dv|^2\,\mathrm{d}x + \frac{(1-\eta)\,m_{s,\lambda}}{4\,(1-s)|\log\sqrt{1-s}|}\int_A \left(|v|^2-1\right)^2\mathrm{d}x,
\end{aligned}$$

for any $\eta \in (0,1)$, from which (3.9) follows due to the definition in (2.5). □

For future convenience, completing the definitions in (3.10) in Lemma 3.3, we set

$$\Lambda_{1,\lambda,\eta} = \lim_{s\to 1^-} \Lambda_{s,\lambda,\eta} = \frac{1-\eta}{2\eta}\,m_{1,\lambda}, \tag{3.12}$$

for any $\lambda > 0$ and $\eta \in (0,1)$, where

$$m_{1,\lambda} = \lim_{s\to 1^-} m_{s,\lambda} = \min\left\{\frac{1}{RL^2}, \lambda\right\}. \tag{3.13}$$

3.2. **Proof of claim (i) of Theorem 1.2.** Let $u_{s_k} \in H^{s_k}_g(\Omega; B_L)$, with $s_k \to 1^-$ as $k \to \infty$, be such that

$$\sup_{k\in\mathbb{N}} \mathrm{GL}^{s_k}_\lambda(u_{s_k};\Omega) < \infty. \tag{3.14}$$

Following Definition 3.1, we define $v_{s_k} = I^R_{1-s_k} u_{s_k}$ for $k \in \mathbb{N}$. By combining (3.3) and (3.13) with (3.9) in Lemma 3.3, from (3.14) we infer that

$$\sup_{k\in\mathbb{N}} \mathrm{GL}_{\sqrt{1-s_k},\,C}(v_{s_k}; A) < \infty$$

for some $C > 0$ which does not depend on k. Thus, by Theorem 2.6(i), we find $\mu \in \mathcal{X}(A)$ such that, up to passing to a subsequence (which we do not relabel),

$$\operatorname{Jac}(v_{s_k}) \xrightarrow{\text{flat}(A)} \pi\mu \quad \text{as } k \to \infty. \tag{3.15}$$

Thanks to (3.5), we can estimate

$$\|\operatorname{Jac}^{s_k}(u_{s_k}) - \pi\mu\|_{\text{flat}(A)} \le \left|1 - \left(\frac{2\pi R^{1-s_k}}{(1-s_k)\gamma_{s_k}}\right)^2\right| \|\operatorname{Jac}(v_{s_k})\|_{\text{flat}(A)} + \|\operatorname{Jac}(v_{s_k}) - \pi\mu\|_{\text{flat}(A)},$$

so that, by (3.3) and (3.15), we deduce that

$$\operatorname{Jac}^{s_k}(u_{s_k}) \xrightarrow{\text{flat}(A)} \pi\mu \quad \text{as } k \to \infty. \tag{3.16}$$

Since $\Omega \Subset A$ by construction, from (3.16) in particular we get that

$$\operatorname{Jac}^{s_k}(u_{s_k}) \xrightarrow{\text{flat}(\overline{\Omega})} \pi\mu \quad \text{as } k \to \infty.$$

We now prove that $\|\mu\|_{\text{flat}(A\setminus\Omega)} = 0$, so that $\operatorname{supp}\mu \subset \overline{\Omega}$. To this aim, we observe that

$$\pi\|\mu\|_{\text{flat}(A\setminus\Omega)} \le \|\operatorname{Jac}^{s_k}(u_{s_k}) - \pi\mu\|_{\text{flat}(A)} + \|\operatorname{Jac}^{s_k}(u_{s_k})\|_{\text{flat}(A\setminus\Omega)}. \tag{3.17}$$

In view of (3.16), we just need to deal with the second term in the right-hand side. Owing to (3.5), Lemma 2.1 (since $|g| = 1$ on $A \setminus \Omega$), Proposition 2.4 and (3.4), we have that

$$\begin{aligned}
\|\operatorname{Jac}^{s_k}(u_{s_k})\|_{\text{flat}(A\setminus\Omega)} &= \left(\tfrac{2\pi R^{1-s_k}}{(1-s_k)\gamma_s}\right)^2 \|\operatorname{Jac}(v_s)\|_{\text{flat}(A\setminus\Omega)} \\
&= \left(\tfrac{2\pi R^{1-s_k}}{(1-s_k)\gamma_{s_k}}\right)^2 \|\operatorname{Jac}(v_{s_k}) - \operatorname{Jac}(g)\|_{\text{flat}(A\setminus\Omega)} \\
&\le C\left(\tfrac{2\pi R^{1-s_k}}{(1-s_k)\gamma_{s_k}}\right)^2 \|v_{s_k} - g\|_{L^2(A\setminus\Omega)} \left(\|Dv_{s_k}\|_{L^2(A\setminus\Omega)} + \|Dg\|_{L^2(A\setminus\Omega)}\right) \\
&\le C\left(\tfrac{2\pi R^{1-s_k}}{(1-s_k)\gamma_{s_k}}\right)^2 \|v_{s_k} - g\|_{L^2(A\setminus\Omega)} \left(\tfrac{(1-s_k)\gamma_s}{2\pi R^{1-s}} \|D^{s_k}u_{s_k}\|_{L^2(\mathbb{R}^2)} + \|Dg\|_{L^2(\mathbb{R}^2)}\right).
\end{aligned} \tag{3.18}$$

By combining (3.8) in Lemma 3.3 with the fact that $u_{s_k} = g$ on $A \setminus \Omega$, we find that

$$v_{s_k} \to g \quad \text{in } L^2(A \setminus \Omega) \text{ as } k \to \infty. \tag{3.19}$$

Therefore, by exploiting (3.19) in combination with (3.14) and (3.8), we get that

$$\lim_{k\to\infty} \|D^{s_k}u_{s_k}\|_{L^2(\mathbb{R}^2)} \|v_{s_k} - g\|_{L^2(A\setminus\Omega)} = 0, \tag{3.20}$$

which, together with (3.17) and (3.18), yields that $\|\mu\|_{\text{flat}(A\setminus\Omega)} = 0$ and thus $\operatorname{supp}\mu \subset \overline{\Omega}$. Finally, we show that

$$\mu(\overline{\Omega}) = \deg(g|_{\partial\Omega}, \partial\Omega). \tag{3.21}$$

To this end, let $\varphi \in \operatorname{Lip}_c(A)$ be such that $\varphi = 1$ on $\overline{\Omega}$. According to Definition 2.2, we can decompose

$$j(v_{s_k}) = j(v_{s_k}, v_{s_k}) = j(v_{s_k} - g, v_{s_k}) + j(g, v_{s_k})$$

for $k \in \mathbb{N}$. Since $\nabla\varphi = 0$ on Ω by definition, from (3.20) we deduce that

$$\limsup_{k\to\infty} \left|\int_A j(v_{s_k} - g, v_{s_k}) \times \nabla\varphi \, \mathrm{d}x\right| \le \operatorname{Lip}(\varphi) \lim_{k\to\infty} \|v_{s_k} - g\|_{L^2(A\setminus\Omega)} \|Dv_{s_k}\|_{L^2(A)} = 0.$$

Thus, by combining this with (3.15), (2.2) and (2.3) in Lemma 2.3, and (3.19), we get

$$\begin{aligned}\pi\int_A \varphi\,\mathrm{d}\mu &= \lim_{k\to\infty}\int_A \operatorname{Jac}(v_{s_k})\,\varphi\,\mathrm{d}x = \lim_{k\to\infty}\int_A j(v_{s_k})\times\nabla\varphi\,\mathrm{d}x = \lim_{k\to\infty}\int_A j(g,v_{s_k})\times\nabla\varphi\,\mathrm{d}x\\ &= \lim_{k\to\infty}\int_A j(v_{s_k},g)\times\nabla\varphi\,\mathrm{d}x = \int_A j(g)\times\nabla\varphi\,\mathrm{d}x = \pi\int_A \operatorname{Jac}(g)\,\varphi\,\mathrm{d}x.\end{aligned}$$

Since $\varphi = 1$ on $\overline{\Omega}$, $\operatorname{supp}\mu\subset\overline{\Omega}$ and $|g| = 1$ on $A\setminus\Omega$, by Lemmas 2.1 and 2.5 we hence get

$$\mu(\overline{\Omega}) = \frac{1}{\pi}\int_A \varphi\,\mathrm{d}\mu = \frac{1}{\pi}\int_A \operatorname{Jac}(g)\,\varphi\,\mathrm{d}x = \frac{1}{\pi}\int_\Omega \operatorname{Jac}(g)\,\mathrm{d}x = \deg(g|_{\partial\Omega},\partial\Omega),$$

proving (3.21) and thus yielding the conclusion. □

3.3. **Proof of claim (ii) of Theorem 1.2.** Let $u_{s_k}\in H^{s_k}_g(\Omega;B_L)$, with $s_k\to 1^-$ as $k\to\infty$, be such that $\operatorname{Jac}^{s_k}(u_{s_k})\xrightarrow{\operatorname{flat}(\overline{\Omega})}\pi\mu$ as $s\to 1^-$ for some $\mu\in\mathcal{X}(\overline{\Omega})$. Letting $v_{s_k} = I^R_{1-s_k}u_{s_k}$ as in Definition 3.1, and repeating the argument of the proof of Theorem 1.2(i), we get

$$\mu(\overline{\Omega}) = \deg(\bar{u}|_{\partial\Omega},\partial\Omega)$$

and

$$\operatorname{Jac}^{s_k}(v_{s_k})\xrightarrow{\operatorname{flat}(A)}\pi\nu\quad\text{as } k\to\infty,$$

for some $\nu\in\mathcal{X}(A)$ such that $\operatorname{supp}\nu\subset\overline{\Omega}$ and $\nu|_{\overline{\Omega}} = \mu$. Therefore, owing to (3.9) in Lemma 3.3, (3.3), (3.12) and Theorem 2.6(ii), we can estimate

$$\begin{aligned}\liminf_{k\to\infty}\mathrm{GL}^{s_k}_\lambda(u_{s_k};\Omega) &\ge \liminf_{k\to\infty}\frac{\eta}{2}\left(\frac{2\pi R^{1-s_k}}{(1-s_k)\gamma_{s_k}}\right)^2 \mathrm{GL}_{\sqrt{1-s_k},\Lambda_{s_k,\lambda,\eta}}(v_{s_k};A)\\ &= \frac{\eta}{2}\liminf_{k\to\infty}\mathrm{GL}_{\sqrt{1-s_k},\Lambda_{s_k,\eta,\lambda}}(v_{s_k};A)\\ &\ge \frac{\eta}{2}\liminf_{k\to\infty}\mathrm{GL}_{\sqrt{1-s_k},\frac{\Lambda_{1,\lambda,\eta}}{2}}(v_{s_k};A)\\ &\ge \eta\,\pi|\nu|(A) = \eta\,\pi|\mu|(\overline{\Omega}),\end{aligned}$$

for any $\eta\in(0,1)$, yielding the conclusion. □

3.4. **Truncated vortex.** The proof of point (iii) of Theorem 1.2 requires some preliminaries. We begin by introducing the following definition.

Definition 3.4 (Cut-off function)**.** Given $0\le r<R<\infty$, we define the *cut-off function* $\eta_{r,R}\colon\mathbb{R}^2\to[0,1]$ by letting

$$\eta_{r,R}(t) = \begin{cases} 0 & \text{if } |x|\in[0,r],\\ \dfrac{|x|-r}{R-r} & \text{if } |x|\in[r,R],\\ 1 & \text{if } |x|\ge R.\end{cases}\tag{3.22}$$

In the following result, we collect some properties of the cut-off function $\eta_{r,R}$ given in Definition 3.4 that will be useful below.

Lemma 3.5. *Given $0 < r < R < \infty$, the function $\eta_{r,R}$ in Definition 3.4 satisfies*

$$|\eta_{r,R}| \le 1, \quad \mathrm{Lip}(\eta_{r,R}) = \frac{1}{R-r}, \tag{3.23}$$

$$\iint_{(B_R\times B_R)\cap\triangle_\tau} \frac{|\eta_{r,R}(x)-\eta_{r,R}(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{\pi^2 R^2}{(R-r)^2}\,\frac{\tau^{2-2s}}{1-s}, \tag{3.24}$$

$$\iint_{(B_\rho\setminus B_r)\times(B_\rho\setminus B_r)\cap\triangle_\tau} \frac{|\eta_{r,R}(x)-\eta_{r,R}(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{\pi^2((R+\tau)^2-r^2)}{(R-r)^2}\,\frac{\tau^{2-2s}}{1-s} \tag{3.25}$$

and

$$\int_{B_\varrho} \left(\eta_{r,R}^2 - 1\right)^2 \mathrm{d}x \le \pi R^2 \tag{3.26}$$

for $s \in (0,1)$, $\tau \in (0,r)$ and $\rho > R+\tau$.

In the proof of Lemma 3.5, we will need the following elementary estimates.

Lemma 3.6. *Let $n \in \mathbb{N}$. If $x, y \in \mathbb{R}^n \setminus \{0\}$, then*

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \le \frac{2}{|x|}\,|x-y| \tag{3.27}$$

and

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 \le \frac{|x|}{|y|}\left(\frac{|y-x|^2}{|x|^2} - \frac{|x\cdot(y-x)|^2}{|x|^4} + \frac{|y-x|^3}{|x|^3}\right). \tag{3.28}$$

Proof. By the triangular inequality, we can estimate

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \le \left|\frac{x}{|x|} - \frac{y}{|x|}\right| + \left|\frac{y}{|x|} - \frac{y}{|y|}\right| \le \frac{2}{|x|}\,|x-y|,$$

proving (3.27). To prove (3.28), instead, we observe that

$$\begin{aligned}
\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 &= 2\left(1 - \frac{x\cdot y}{|x|\,|y|}\right) = 2\left(1 - \frac{x\cdot(y-x)}{|x|\,|y|} - \frac{|x|}{|y|}\right) = 2\,\frac{|x|}{|y|}\left(\frac{|y|}{|x|} - \frac{x\cdot(y-x)}{|x|^2} - 1\right)\\
&= 2\,\frac{|x|}{|y|}\left(\sqrt{\left|\frac{x}{|x|} + \frac{y-x}{|x|}\right|^2} - 1 - \frac{x\cdot(y-x)}{|x|^2}\right)\\
&= 2\,\frac{|x|}{|y|}\left(\sqrt{1 + 2\,\frac{x\cdot(y-x)}{|x|^2} + \frac{|y-x|^2}{|x|^2}} - 1 - \frac{x\cdot(y-x)}{|x|^2}\right).
\end{aligned}$$

Thanks to the elementary inequality

$$\sqrt{1+t} \le 1 + \frac{1}{2}\,t - \frac{1}{8}\,t^2 \quad \text{for } t \ge 0,$$

we thus get that

$$\begin{aligned}
\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 &\le 2\,\frac{|x|}{|y|}\left(\frac{1}{2}\,\frac{|y-x|^2}{|x|^2} - \frac{1}{8}\left(2\,\frac{x\cdot(y-x)}{|x|^2} + \frac{|y-x|^2}{|x|^2}\right)^2\right)\\
&\le 2\,\frac{|x|}{|y|}\left(\frac{1}{2}\,\frac{|y-x|^2}{|x|^2} - \frac{1}{2}\,\frac{|x\cdot(y-x)|^2}{|x|^2} - \frac{1}{2}\,\frac{(x\cdot(y-x))\,|y-x|^2}{|x|^4}\right)\\
&\le \frac{|x|}{|y|}\left(\frac{|y-x|^2}{|x|^2} - \frac{|x\cdot(y-x)|^2}{|x|^2} + \frac{|y-x|^3}{|x|^3}\right),
\end{aligned}$$

yielding (3.28) and concluding the proof. □

Proof of Lemma 3.5. From the definition in (3.22), we get (3.23) and thus

$$\int_{B_\varrho} \left(\eta_{r,R}^2 - 1\right)^2 \mathrm{d}x = \int_{B_R} \left(\eta_{r,R}^2 - 1\right)^2 \mathrm{d}x \le |B_R|,$$

giving (3.26). To prove (3.24), we observe that

$$\int_{(B_R \times B_R) \cap \triangle_\tau} \frac{|\eta_{r,R}(|x|) - \eta_{r,R}(|y|)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \le \int_{B_R} \int_{B_\tau} \frac{|\eta_{r,R}(|x|) - \eta_{r,R}(|x+h|)|^2}{|h|^{2+2s}} \,\mathrm{d}h\,\mathrm{d}x$$
$$\le \mathrm{Lip}(\eta_{r,R})^2 \int_{B_R} \int_{B_\tau} \frac{|h|^2}{|h|^{2+2s}} \,\mathrm{d}h\,\mathrm{d}x = \frac{\pi^2 R^2}{(R-r)^2} \frac{\tau^{2-2s}}{1-s},$$

while, to prove (3.25), we note that

$$x \in B_\varrho \setminus B_{R+\tau} \implies |x+h| > R \quad \text{for } h \in B_\tau,$$

and thus we can estimate

$$\iint_{(B_\rho \setminus B_r) \times (B_\rho \setminus B_r) \cap \triangle_\tau} \frac{|\eta_{r,R}(x) - \eta_{r,R}(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y$$
$$\le \int_{B_\tau} \frac{1}{|h|^{2+2s}} \int_{B_\varrho \setminus B_r} |\eta_{r,R}(x) - \eta_{r,R}(x+h)|^2 \,\mathrm{d}x\,\mathrm{d}h$$
$$= \int_{B_\tau} \frac{1}{|h|^{2+2s}} \int_{B_{R+\tau} \setminus B_r} |\eta_{r,R}(|x|) - \eta_{r,R}(|x+h|)|^2 \,\mathrm{d}x\,\mathrm{d}h \le \frac{\pi^2((R+\tau)^2 - r^2)}{(R-r)^2} \frac{\tau^{2-2s}}{1-s},$$

and the proof is complete. □

We can introduce the notion of *truncated vortex* by exploiting the cut-off function $\eta_{r,R}$ given in Definition 3.4.

Definition 3.7 (Truncated vortex)**.** Given $0 \le r < R < \infty$ and $d \in \{-1, 1\}$, we let $\upsilon_{d,r,R} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the *truncated vortex* defined as

$$\upsilon_{d,r,R}(x) = \eta_{r,R}(|x|) \frac{x^d}{|x|}, \quad \text{for } x \in \mathbb{R}^2, \tag{3.29}$$

where $\eta_{r,R}$ is as in Definition 3.4 and $x^d = \left(\begin{smallmatrix} x_1 \\ (-1)^d x_2 \end{smallmatrix}\right)$ for all $x \in \mathbb{R}^2$.

We now collect several properties of the truncated vortex introduced in Definition 3.7. On the one hand, we have the following result.

Lemma 3.8. *Given $0 \le r < R < \infty$ and $d \in \{-1, 1\}$, the truncated vortex $\upsilon_{d,r,R}$ in Definition 3.7 satisfies*

$$\upsilon_{d,r,R} \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^2), \tag{3.30}$$

with

$$|D\upsilon_{d,r,R}(x)| \le \frac{C}{|x|} \eta_{r,R}(|x|), \quad x \in \mathbb{R}^2, \tag{3.31}$$

where $C > 0$ is a numerical constant. In addition, it holds that

$$\int_{\mathbb{R}^2} |\upsilon_{d,r,R} - \upsilon_{d,0,R}|^2 \,\mathrm{d}x \le 4\pi R^2, \tag{3.32}$$

$$\mathrm{Jac}(\upsilon_{d,0,R}) = \pi d \frac{\chi_{B_R}}{|B_R|} \tag{3.33}$$

and

$$\int_{B_\varrho} |D\upsilon_{d,0,R}|^2 \,\mathrm{d}x \le C \log\left(\frac{\varrho}{R}\right) \tag{3.34}$$

for $\varrho > R$, where $C > 0$ is a numerical constant.

Proof. From (3.29) in Definition 3.7, we get that $\|\upsilon_{d,r,R}\|_{L^\infty} \le 1$ and a direct computation yields (3.31), from which we get (3.30). The validity of (3.32) is a consequence of (3.22) in Definition 3.4. Finally, from the fact that

$$D\upsilon_{d,0,R}(x) = \frac{\chi_{B_R}}{R} J_d + \frac{\chi_{B_R^c}}{|x|}\left(J_d - \frac{x \otimes x^d}{|x|^2}\right), \quad x \in \mathbb{R}^2 \setminus \{0\},$$

where we have set

$$J_d = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^d \end{pmatrix}, \quad d \in \{1, -1\},$$

we infer (3.33) and (3.34), concluding the proof. □

On the other hand, similarly to Lemma 3.5, we have the following result.

Lemma 3.9. *Given $0 < r < R < \infty$ and $d \in \{-1, 1\}$, the truncated vortex $\upsilon_{d,r,R}$ in Definition 3.7 satisfies*

$$\iint_{(B_R \times B_R) \cap \triangle_\tau} \frac{|\upsilon_{d,r,R}(x) - \upsilon_{d,r,R}(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \le \left(\frac{8\pi^2(R^2 - r^2)}{r^2} + \frac{2\pi^2 R^2}{(R-r)^2}\right)\frac{\tau^{2-2s}}{1-s}, \tag{3.35}$$

$$\begin{aligned} \iint_{((B_\varrho \setminus B_r) \times (B_\varrho \setminus B_r)) \cap \triangle_\tau} & \frac{|\upsilon_{d,r,R}(x) - \upsilon_{d,r,R}(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ & \le (1+\varepsilon)\left(\frac{\pi^2 r}{r-\tau} \log\left(\frac{\varrho}{r}\right) \frac{\tau^{2-2s}}{1-s} + \frac{4\pi^2(\varrho - r)}{\varrho\,(r-\tau)} \frac{\tau^{3-2s}}{3-2s}\right) \\ & \quad + \left(1 + \frac{1}{\varepsilon}\right) \frac{\pi^2((R+\tau)^2 - r^2)}{(R-r)^2} \frac{\tau^{2-2s}}{1-s} \end{aligned} \tag{3.36}$$

and

$$\int_{B_\varrho} \left(|\upsilon_{d,r,R}|^2 - 1\right)^2 \mathrm{d}x \le \pi R^2 \tag{3.37}$$

for $s \in (0,1)$, $\tau \in (0, r)$, $\varrho > R + \tau$ and $\varepsilon > 0$.

Proof. We have that

$$|\upsilon_{d,r,R}(x) - \upsilon_{d,r,R}(y)| \le \eta_{r,R}(|x|) \left|\frac{x}{|x|} - \frac{y}{|y|}\right| + |\eta_{r,R}(|x|) - \eta_{r,R}(|y|)|. \tag{3.38}$$

Thanks to (3.27) in Lemma 3.6, we can estimate

$$\begin{aligned} \iint_{(B_R \times B_R) \cap \triangle_\tau} \eta_{r,R}^2(|x|) \frac{\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y & \le \int_{((B_R \setminus B_r) \times B_R) \cap \triangle_\tau} \frac{4}{|x|^2} \frac{|x-y|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ & \le \frac{4}{r^2} \int_{B_R \setminus B_r} \int_{B_\tau} \frac{|h|^2}{|h|^{2+2s}} \,\mathrm{d}h\,\mathrm{d}x = \frac{4\pi^2(R^2 - r^2)}{r^2} \frac{\tau^{2-2s}}{1-s}. \end{aligned} \tag{3.39}$$

Therefore, by combining (3.38) and (3.39) with (3.24) in Lemma 3.5, we get that

$$\iint_{(B_R \times B_R) \cap \triangle_\tau} \frac{|\upsilon_{d,r,R}(x) - \upsilon_{d,r,R}(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \le \left(\frac{8\pi^2(R^2 - r^2)}{r^2} + \frac{2\pi^2 R^2}{(R-r)^2}\right)\frac{\tau^{2-2s}}{1-s},$$

proving (3.35). We now deal with (3.36). In view of the elementary inequality

$$(a+b)^2 \le (1+\varepsilon)\, a^2 + \left(1+\frac{1}{\varepsilon}\right) b^2 \quad \text{for } a,b\ge 0,\ \varepsilon>0,$$

we can revisit (3.38) as

$$|\upsilon_{d,r,R}(x)-\upsilon_{d,r,R}(y)|^2 \le (1+\varepsilon)\, \eta_{r,R}(|x|)^2 \left|\frac{x}{|x|}-\frac{y}{|y|}\right|^2 + \left(1+\frac{1}{\varepsilon}\right) |\eta_{r,R}(|x|)-\eta_{r,R}(|y|)|^2 \tag{3.40}$$

whenever $\varepsilon>0$. We now observe that

$$\begin{aligned}\iint_{((B_\varrho\setminus B_r)\times(B_\varrho\setminus B_r))\cap\triangle_\tau} &\eta_{r,R}^2(|x|)\, \frac{\left|\frac{x}{|x|}-\frac{y}{|y|}\right|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le \int_{B_\tau}\frac{1}{|h|^{2+2s}}\int_{B_\varrho\setminus B_r}\left|\frac{x}{|x|}-\frac{x+h}{|x+h|}\right|^2 \mathrm{d}x\,\mathrm{d}h.\end{aligned} \tag{3.41}$$

Since $\tau<r$, we have that

$$|x+h|\ge|x|-|h|\ge r-\tau>0 \quad \text{for } x\in B_\varrho\setminus B_r,\ h\in B_\tau$$

and therefore we can estimate

$$\frac{|x|}{|x+h|}=\frac{1}{\left|\frac{x}{|x|}+\frac{h}{|x|}\right|}\le\frac{1}{1-\frac{|h|}{|x|}}\le\frac{1}{1-\frac{\tau}{r}}=\frac{r}{r-\tau} \quad \text{for } x\in B_\varrho\setminus B_r,\ h\in B_\tau.$$

Thus, thanks to (3.28) in Lemma 3.6, we get that

$$\begin{aligned}\int_{B_\tau}&\frac{1}{|h|^{2+2s}}\int_{B_\varrho\setminus B_r}\left|\frac{x}{|x|}-\frac{x+h}{|x+h|}\right|^2 \mathrm{d}x\,\mathrm{d}h \\ &\le \int_{B_\tau}\frac{1}{|h|^{2+2s}}\int_{B_\varrho\setminus B_r}\frac{|x|}{|x+h|}\left(\frac{|h|^2}{|x|^2}-\frac{|x\cdot h|^2}{|x|^4}+\frac{|h|^3}{|x|^3}\right)\mathrm{d}x\,\mathrm{d}h \\ &\le \frac{r}{r-\tau}\int_{B_\tau}\frac{1}{|h|^{2+2s}}\int_{B_\varrho\setminus B_r}\frac{|h|^2}{|x|^2}-\frac{|x\cdot h|^2}{|x|^4}+\frac{|h|^3}{|x|^3}\,\mathrm{d}x\,\mathrm{d}h.\end{aligned}$$

At this point, we make the following observation. Given $h\in B_\tau$, we can find a rotation matrix $\mathcal{R}\in\mathrm{SO}(2)$ such that $h=\mathcal{R}\mathrm{e}_1$. Therefore, we can compute

$$\int_{B_\varrho\setminus B_r}\frac{|x\cdot h|^2}{|x|^4}\,\mathrm{d}x=\int_{B_\varrho\setminus B_r}\frac{|x\cdot \mathrm{e}_1|^2}{|x|^4}\,\mathrm{d}x=\int_{B_\varrho\setminus B_r}\frac{x_1^2}{|x|^4}\,\mathrm{d}x$$

and thus, by replacing e_1 with e_2, we get that

$$\int_{B_\varrho\setminus B_r}\frac{|x\cdot h|^2}{|x|^4}\,\mathrm{d}x=\frac12\int_{B_\varrho\setminus B_r}\frac{x_1^2}{|x|^4}\,\mathrm{d}x+\frac12\int_{B_\varrho\setminus B_r}\frac{x_2^2}{|x|^4}\,\mathrm{d}x=\frac12\int_{B_\varrho\setminus B_r}\frac{|x|^2}{|x|^4}\,\mathrm{d}x=\pi\,\log\left(\frac{\varrho}{r}\right).$$

As a consequence, we have that

$$\int_{B_\tau}\frac{1}{|h|^{2+2s}}\int_{B_\varrho\setminus B_r}\frac{|h|^2}{|x|^2}-\frac{|x\cdot h|^2}{|x|^4}+\frac{|h|^3}{|x|^3}\,\mathrm{d}x\,\mathrm{d}h=\pi^2\log\left(\frac{\varrho}{r}\right)\frac{\tau^{2-2s}}{1-s}+4\pi^2\left(\frac1r-\frac1\varrho\right)\frac{\tau^{3-2s}}{3-2s},$$

from which we deduce that

$$\int_{B_\tau} \frac{1}{|h|^{2+2s}} \int_{B_\varrho \setminus B_r} \left| \frac{x}{|x|} - \frac{x+h}{|x+h|} \right|^2 \mathrm{d}x\, \mathrm{d}h \le \frac{\pi^2\, r}{r-\tau} \log\left(\frac{\varrho}{r}\right) \frac{\tau^{2-2s}}{1-s} + \frac{4\pi^2(\varrho - r)}{\varrho\,(r-\tau)} \frac{\tau^{3-2s}}{3-2s}. \tag{3.42}$$

Thus, by combining (3.40), (3.41) and (3.42) with (3.25) in Lemma 3.9, we get that

$$\begin{aligned} \iint_{((B_\varrho \setminus B_r)\times(B_\varrho \setminus B_r))\cap \triangle_\tau} & \frac{|v_{d,r,R}(x) - v_{d,r,R}(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\, \mathrm{d}y \\ & \le (1+\varepsilon)\left(\frac{\pi^2\, r}{r-\tau} \log\left(\frac{\varrho}{r}\right) \frac{\tau^{2-2s}}{1-s} + \frac{4\pi^2(\varrho - r)}{\varrho\,(r-\tau)} \frac{\tau^{3-2s}}{3-2s} \right) \\ & \quad + \left(1 + \frac{1}{\varepsilon}\right) \frac{\pi^2((R+\tau)^2 - r^2)}{(R-r)^2} \frac{\tau^{2-2s}}{1-s} \end{aligned}$$

whenever $\varepsilon > 0$, yielding (3.36). The validity of (3.37) follows by combining the definition in (3.29) with (3.26). The proof is complete. □

3.5. **Proof of claim (iii) of Theorem 1.2.** Below, in order to avoid heavy notation, we will frequently adopt the following shorthand. Given $x_1, \dots, x_N \in \mathbb{R}^2$ for some $N \in \mathbb{N}$, any $\ell > 0$ and any non-empty open set $V \subset \mathbb{R}^2$, we let

$$\widehat{V}_\ell = V \setminus \left(\bigcup_{i=1}^N \overline{B_\ell(x_i)} \right). \tag{3.43}$$

Proof of claim (iii) of Theorem 1.2. We divide the proof in three steps.

Step 1. Let $\mu \in \mathcal{X}(\overline{\Omega})$ be given by

$$\mu = \sum_{i=1}^N d_i\, \delta_{x_i} \tag{3.44}$$

for some $N \in \mathbb{N}$, $x_i \in \Omega$ and $d_i \in \{-1, 1\}$, for each $i \in \{1, \dots, N\}$, such that

$$\mu(\overline{\Omega}) = \mu(\Omega) = \sum_{i=1}^N d_i = \deg(g|_{\partial\Omega}, \partial\Omega). \tag{3.45}$$

In this step, we construct $u_s \in H^s_g(\Omega; B_L)$ for $s \in (0,1)$ such that

$$\limsup_{s \to 1^-} \mathrm{GL}^s_\lambda(u_s) \le \pi |\mu|(\overline{\Omega}) = \pi N \tag{3.46}$$

whenever $\lambda \in (0, \infty)$.

To define u_s, we need to introduce some parameters. We let

$$\tau_s = \sqrt{1-s} \quad \text{and} \quad M_s = \sqrt[4]{|\log(1-s)|} \quad \text{for } s \in (0,1). \tag{3.47}$$

In view of the assumption made in (3.44), we define

$$\overline{r} = \min\left\{ \mathrm{dist}(x_i, \mathbb{R}^2 \setminus \Omega),\ \frac{1}{2}|x_i - x_j| : i,j \in \{1, \dots, N\},\ i \ne j \right\} \in (0, \infty) \tag{3.48}$$

and we fix

$$r < \frac{\overline{r}}{2}. \tag{3.49}$$

Without loss of generality, we tacitly work with $s \in (0,1)$ sufficiently close to 1 so that

$$(M_s+2)\,\tau_s < \frac{r}{2}, \tag{3.50}$$

which is always possibile in virtue of the definitions in (3.47).

We can now define u_s. With the notation of Definition 3.7, we define

$$u_{s,i} = v_{d_i,M_s\tau_s,(M_s+1)\tau_s}(\,\cdot\, - x_i) \quad \text{for each } i \in \{1,\dots,N\}. \tag{3.51}$$

By well-known results (e.g., see [8, Th. I.4]), due to (3.45), we can find $\widehat{u} \in H^1(\widehat{\Omega}_r;\mathbb{S}^1)$ (where $\widehat{\Omega}_r$ is defined using the shorthand (3.43) with points $x_1,\dots,x_N$ given by (3.44) and $\ell = r$ as fixed in (3.49)) such that

$$\widehat{u} = \begin{cases} u_{s,i} & \text{on } \partial B_r(x_i),\ \text{for each } i \in \{1,\dots,N\},\\ g & \text{on } \partial\Omega. \end{cases} \tag{3.52}$$

Since, by (3.51) and by Definition 3.1, for each $i \in \{1,\dots,N\}$, we have

$$u_{s,i} = \frac{(\,\cdot\, - x_i)^{d_i}}{|\,\cdot\, - x_i|} \quad \text{in an open neighborhood of } \partial B_r(x_i),$$

the function $\widehat{u}$ does not depend on s. We thus define $u_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting

$$u_s = \begin{cases} u_{s,i} & \text{in } \overline{B_r(x_i)},\ \text{for each } i \in \{1,\dots,N\},\\ \widehat{u} & \text{in } \widehat{\Omega}_r,\\ g & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{3.53}$$

We observe that $u_s \in H^s_g(\Omega; B_L)$, since, by (3.51) and by (3.30) in Lemma 3.8, and by (3.52), we have that $u_s \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ with $\|u_s\|_{L^\infty} \le \|g\|_{L^\infty}$.

We now detail the proof of (3.46). We begin by observing that, owing to (3.53), the fact that $|u_s| = |\widehat{u}| = 1$ on $\widehat{\Omega}_r$ and $|u_s| = |u_{s,i}| = 1$ on $B_r(x_i) \setminus B_{(M_s+1)\tau_s}(x_i)$ for each $i \in \{1,\dots,N\}$, and to (3.37) in Lemma 3.9, we can estimate

$$\begin{aligned}\int_\Omega \left(|u_s|^2-1\right)^2 \mathrm{d}x &= \int_{\widehat{\Omega}_r} \left(|\widehat{u}|^2-1\right)^2 \mathrm{d}x + \sum_{i=1}^N \int_{B_r(x_i)} \left(|u_{s,i}|^2-1\right)^2 \mathrm{d}x\\ &= N\int_{B_{(M_s+1)\tau_s}} \left(|v_{d_i,M_s\tau_s,(M_s+1)\tau_s}|^2-1\right)^2 \mathrm{d}x \le N\pi\,(M_s+1)^2\,\tau_s^2,\end{aligned}$$

from which, in virtue of the definitions in (3.47), we get that

$$\limsup_{s\to1^-} \int_\Omega (|u_s|^2-1)^2\,\mathrm{d}x = 0.$$

Hence, to show (3.46), by (2.10) in Proposition 2.8, we just need to prove that

$$\limsup_{s\to1^-} \frac{(1-s)}{|\log(1-s)|}\,[u_s]_s^2 \le \frac{\pi^2 N}{2}. \tag{3.54}$$

Recalling the notation introduced in (2.8), by Lemma 2.7 and (3.53), we can estimate

$$\begin{aligned}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y &\le \frac{4\pi\|u_s\|_{L^2}^2}{s\,(1-s)^s}\\ &\le \frac{4\pi}{s\,(1-s)^s}\left(\|g\|_{L^2}^2+\|\widehat{u}\|_{L^2(\widehat{\Omega}_r)}^2+N\pi r^2\right),\end{aligned}$$

from which we deduce that

$$\lim_{s\to1^-}\frac{(1-s)}{|\log(1-s)|}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{3.55}$$

Hence, by combining (3.54) with (3.55), the validity of (3.46) reduces to

$$\limsup_{s\to1^-}\frac{(1-s)}{|\log(1-s)|}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\cap\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y\le\frac{\pi^2N}{2}. \tag{3.56}$$

Our aim is now to estimate the integral

$$\iint_{\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

To do so, we let

$$Q_s(V,W)=\iint_{(V\times W)\cap\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

for any two measurable sets $V, W\subset\mathbb{R}^2$. Since we can write $\mathbb{R}^2=V_{s,1}\cup V_{s,2}\cup V_{s,3}$, with pairwise disjoint union, where

$$V_{s,1}=\bigcup_{i=1}^N B_{(M_s+1)\tau_s}(x_i),\quad V_{s,2}=\bigcup_{i=1}^N B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\quad\text{and}\quad V_{s,3}=\widehat{(\mathbb{R}^2)}_r,$$

and $\widehat{(\mathbb{R}^2)}_r$ is as in (3.43) (with $V=\mathbb{R}^2$ and $\ell=r$), we can decompose

$$\iint_{\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y=\sum_{j,k=1}^3 Q_s(V_{s,j},V_{s,k}).$$

We now deal with each possible pair. We begin by observing that, if $x\in B_{(M_s+1)\tau_s}(x_i)$ and $y\in B_{(M_s+1)\tau_s}(x_h)$ for some $i,h\in\{1,\dots,N\}$ with $i\neq h$, then

$$|x-y|>|x_i-x_h|-2(M_s+1)\tau_s\ge 2\bar r-2(M_s+1)\tau_s>2r-2(M_s+1)\tau_s>\tau_s,$$

because of (3.48), (3.49) and (3.50), so that

$$Q_s(V_{s,1},V_{s,1})=\sum_{i=1}^N Q_s\left(B_{(M_s+1)\tau_s}(x_i),B_{(M_s+1)\tau_s}(x_i)\right).$$

Next, we notice that

$$Q_s(V_{s,1},V_{s,2})=Q_s(\widetilde V_{s,1},V_{s,2})\quad\text{and}\quad Q_s(V_{s,2},V_{s,1})=Q_s(V_{s,2},\widetilde V_{s,1}),$$

where we set

$$\widetilde V_{s,1}=\bigcup_{i=1}^N B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i).$$

Moreover, if $x\in B_r(x_i)$ and $y\in B_{(M_s+1)\tau_s}(x_h)$ for some $i,h\in\{1,\dots,N\}$ with $i\neq h$, then

$$|x-y|>|x_i-x_h|-r-(M_s+1)\tau_s\ge 2\bar r-r-(M_s+1)\tau_s>r-(M_s+1)\tau_s>\tau_s,$$

again because of (3.48), (3.49) and (3.50), so that

$$Q_s(V_{s,1},V_{s,2})=\sum_{i=1}^N Q_s\left(B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i), B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\right)$$

and, similarly,

$$Q_s(V_{s,2},V_{s,1})=\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i), B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i)\right).$$

As above, if $x\in B_r(x_i)$ and $y\in B_r(x_h)$ for some $i,h\in\{1,\dots,N\}$ with $i\neq h$, then

$$|x-y|>|x_i-x_h|-2r\geq 2\bar{r}-2r>4r-2r>\tau_s,$$

again because of (3.48), (3.49) and (3.50), so that

$$Q_s(V_{s,2},V_{s,2})=\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i), B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\right).$$

We hence infer that

$$\begin{aligned}
&Q_s(V_{s,1},V_{s,2})+Q_s(V_{s,2},V_{s,1})+Q_s(V_{s,2},V_{s,2})\\
&\quad=\sum_{i=1}^N Q_s\left(B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i), B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\right)\\
&\qquad+\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i), B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i)\right)\\
&\qquad+\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i), B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\right)\\
&\quad=\sum_{i=1}^N Q_s\left(B_{(M_s+1)\tau_s}(x_i)\setminus B_{M_s\tau_s}(x_i), B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i)\right)\\
&\qquad+\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{(M_s+1)\tau_s}(x_i), B_r(x_i)\setminus B_{M_s\tau_s}(x_i)\right)\\
&\quad\leq\sum_{i=1}^N Q_s\left(B_r(x_i)\setminus B_{M_s\tau_s}(x_i), B_r(x_i)\setminus B_{M_s\tau_s}(x_i)\right).
\end{aligned}$$

Finally, if $x\in B_{(M_s+1)\tau_s}(x_i)$ and $y\in\mathbb{R}^2\setminus B_r(x_h)$ for some $i,h\in\{1,\dots,N\}$, then

$$|x-y|\geq r-(M_s+1)\tau_s>\tau_s$$

by (3.50), so that

$$Q_s(V_{s,1},V_{s,3})=Q_s(V_{s,3},V_{s,1})=0.$$

Moreover, if $x\in V_{2,s}$ and $y\in V_{3,s}$, then $(x,y)\in\triangle_{\tau_s}$ only in the case in which

$$x\in B_r(x_i)\setminus B_{r-\tau_s}(x_i)\quad\text{and}\quad y\in\mathbb{R}^2\setminus B_r(x_i)$$

for some $i \in \{1, \dots, N\}$. We therefore get that

$$
\begin{aligned}
Q_s(V_{2,s}, V_{3,s}) + Q_s(V_{3,s}, V_{2,s}) + Q_s(V_{3,s}, V_{3,s}) &= Q_s\left(\bigcup_{i=1}^N B_r(x_i) \setminus B_{r-\tau_s}(x_i), \widehat{(\mathbb{R}^2)}_r\right) \\
&\quad + Q_s\left(\widehat{(\mathbb{R}^2)}_r, \bigcup_{i=1}^N B_r(x_i) \setminus B_{r-\tau_s}(x_i)\right) + Q_s\left(\widehat{(\mathbb{R}^2)}_r, \widehat{(\mathbb{R}^2)}_r\right) \\
&= Q_s\left(\bigcup_{i=1}^N B_r(x_i) \setminus B_{r-\tau_s}(x_i), \widehat{(\mathbb{R}^2)}_r\right) + Q_s\left(\widehat{(\mathbb{R}^2)}_r, \widehat{(\mathbb{R}^2)}_{r-\tau_s}\right) \\
&\leq Q_s\left(\widehat{(\mathbb{R}^2)}_{r-\tau_s}, \widehat{(\mathbb{R}^2)}_{r-\tau_s}\right).
\end{aligned}
$$

By combining all the above estimates, we thus conclude that

$$
\begin{aligned}
\iint_{\triangle_{\tau_s}} \frac{|u_s(x) - u_s(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x \,\mathrm{d}y \leq \sum_{i=1}^N Q_s\left(B_{(M_s+1)\tau_s}(x_i), B_{(M_s+1)\tau_s}(x_i)\right) \\
+ \sum_{i=1}^N Q_s\left(B_r(x_i) \setminus B_{M_s\tau_s}(x_i), B_r(x_i) \setminus B_{M_s\tau_s}(x_i)\right) \\
+ Q_s\left(\widehat{(\mathbb{R}^2)}_{r-\tau_s}, \widehat{(\mathbb{R}^2)}_{r-\tau_s}\right).
\end{aligned}
\tag{3.57}
$$

We estimate each term in the right-hand side of (3.57) separately. Recalling (3.51), by (3.35) in Lemma 3.9 we have that

$$
Q_s\left(B_{(M_s+1)\tau_s(x_i)}, B_{(M_s+1)\tau_s(x_i)}\right) \leq \frac{CM_s^2}{1-s} \tag{3.58}
$$

for each $i \in \{1, \dots, N\}$, where $C > 0$ is a numerical constant. Moreover, by (3.36) in Lemma 3.9, we similarly get that

$$
\begin{aligned}
&Q_s\left(B_r(x_i) \setminus B_{M_s\tau_s}(x_i), B_r(x_i) \setminus B_{M_s\tau_s}(x_i)\right) \\
&\quad \leq (1+\varepsilon)\,\pi^2 \,\frac{M_s\,(1-s)^{1-s}}{M_s - 1}\, \frac{\log r - \log M_s + \frac{1}{2}|\log(1-s)|}{1-s} + \frac{CM_s}{\varepsilon\,(1-s)}
\end{aligned}
\tag{3.59}
$$

for each $i \in \{1, \dots, N\}$ and any $\varepsilon > 0$, where $C > 0$ is a numerical constant. Finally, by the Fundamental Theorem of Calculus, Jensen's inequality and (3.50), we have that

$$
\begin{aligned}
Q_s\left(\widehat{(\mathbb{R}^2)}_{r-\tau_s}, \widehat{(\mathbb{R}^2)}_{r-\tau_s}\right) &\leq \int_{B_{\tau_s}} \frac{1}{|h|^{2+2s}} \int_{\widehat{(\mathbb{R}^2)}_{r-\tau_s}} |u_s(x+h) - u_s(x)|^2 \,\mathrm{d}x \,\mathrm{d}h \\
&\leq \int_{B_{\tau_s}} \frac{|h|^2}{|h|^{2+2s}} \int_{\widehat{(\mathbb{R}^2)}_{r-\tau_s}} \left(\int_0^1 |Du_s(x+th)| \,\mathrm{d}t\right)^2 \mathrm{d}x \,\mathrm{d}h \\
&\leq \int_{B_{\tau_s}} \frac{1}{|h|^{2s}} \int_{\widehat{(\mathbb{R}^2)}_{r-\tau_s}} \int_0^1 |Du_s(x+th)|^2 \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}h \\
&\leq \int_{B_{\tau_s}} \frac{1}{|h|^{2s}} \int_{\widehat{(\mathbb{R}^2)}_{r-2\tau_s}} |Du_s(z)|^2 \,\mathrm{d}z \,\mathrm{d}h \\
&\leq \frac{C}{1-s} \int_{\widehat{(\mathbb{R}^2)}_{r/2}} |Du_s(z)|^2 \,\mathrm{d}z,
\end{aligned}
$$

where $C > 0$ is a numerical constant. We also observe that, by (3.53), we can write

$$\int_{\widehat{(\mathbb{R}^2)}_{r/2}} |Du_s(z)|^2 \,\mathrm{d}z = \int_{\mathbb{R}^2\setminus\Omega} |D\bar{u}(z)|^2 \,\mathrm{d}z + \int_{\widehat{\Omega}_r} |D\widehat{u}(z)|^2 \,\mathrm{d}z + \sum_{i=1}^{N} \int_{B_r(x_i)\setminus B_{r/2}(x_i)} |Du_{s,i}|^2 \,\mathrm{d}x,$$

with, thanks to (3.31) and again (3.50),

$$\int_{B_r(x_i)\setminus B_{r/2}(x_i)} |Du_{s,i}|^2 \,\mathrm{d}x \le C \int_{B_r\setminus B_{r/2}} \frac{1}{|x|^2} \,\mathrm{d}x \le 4C\pi,$$

where $C > 0$ is a numerical constant. Therefore, we have that

$$Q_s\Big(\widehat{(\mathbb{R}^2)}_{r-\tau_s}, \widehat{(\mathbb{R}^2)}_{r-\tau_s}\Big) \le \frac{C}{1-s}, \tag{3.60}$$

where $C > 0$ does not depend on s. Hence, by combining (3.57), (3.58), (3.59) and (3.60), and by recalling (3.47), we infer that

$$\limsup_{s\to 1^-} \frac{(1-s)}{|\log(1-s)|} \iint_{\triangle_{\tau_s}} \frac{|u_s(x)-u_s(y)|^2}{|x-y|^{2+2s}} \,\mathrm{d}x\,\mathrm{d}y \le (1+\varepsilon)\,\frac{\pi^2 N}{2}$$

whenever $\varepsilon > 0$, proving (3.56) and thus (3.46).

Step 2. Let μ be as in (3.44) and let u_s be as in (3.53). In this step, we prove that

$$\mathrm{Jac}^s(u_s) \xrightarrow{\mathsf{flat}(\overline{\Omega})} \pi\mu \quad \text{as } s \to 1^-. \tag{3.61}$$

Thanks to (3.3) and (3.5), to show (3.61) we just need to prove that

$$\mathrm{Jac}(v_s) \xrightarrow{\mathsf{flat}(\overline{\Omega})} \pi\mu \quad \text{as } s \to 1^-, \tag{3.62}$$

where, following Definition 3.1, $v_s = I^R_{1-s} u_s$ for $s \in (0,1)$. To prove (3.62), in turn, we need to introduce the auxiliary function $\widetilde{u}_s \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\widetilde{u}_s = \begin{cases} \upsilon_{d_i,0,(M_s+1)\tau_s}(\,\cdot\, - x_i) & \text{in } \overline{B_r(x_i)}, \text{ for each } i \in \{1,\dots,N\},\\ \widehat{u} & \text{in } \widehat{\Omega}_r,\\ g & \text{in } \mathbb{R}^2\setminus\Omega. \end{cases} \tag{3.63}$$

As for u_s, we have that $\widetilde{u}_s \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ and $\|\widetilde{u}_s\|_{L^\infty} \le \|g\|_{L^\infty}$. In addition, by Lemma 2.1 (since $|\widetilde{u}_s| = 1$ on $\widehat{(\mathbb{R}^2)}_r$) and by (3.33) in Lemma 3.8, we have

$$\mathrm{Jac}(\widetilde{u}_s) = \sum_{i=1}^{N} \mathrm{Jac}(\upsilon_{d_i,0,(M_s+1)\tau_s}(\,\cdot\, - x_i))\,\chi_{B_{(M_s+1)\tau_s}(x_i)} = \pi \sum_{i=1}^{N} d_i\,\frac{\chi_{B_{(M_s+1)\tau_s}(x_i)}}{|B_{(M_s+1)\tau_s}(x_i)|},$$

from which, recalling (3.44), we deduce that

$$\mathrm{Jac}(\widetilde{u}_s) \xrightarrow{\mathsf{flat}(A)} \pi\mu \quad \text{as } s \to 1^-.$$

We can hence achieve (3.62) by proving that

$$\lim_{s\to 1^-} \|\,\mathrm{Jac}(v_s) - \mathrm{Jac}(\widetilde{u}_s)\|_{\mathsf{flat}(A)} = 0. \tag{3.64}$$

To this end, we exploit Proposition 2.4. On the one hand, by (3.7) in Lemma 3.3 and (3.46), and by (3.63), (3.32) in Lemma 3.8, and (3.47), we can estimate

$$\|v_s - \widetilde{u}_s\|_{L^2(A)} \le \|v_s - u_s\|_{L^2(A)} + \|u_s - \widetilde{u}_s\|_{L^2(A)} \le C\sqrt{(1-s)\,|\log(1-s)|}, \tag{3.65}$$

where $C > 0$ is a constant which does not depend on s. On the other hand, by (3.3), (3.4) and (3.46), and by (3.63) and (3.34), we also have that

$$\|Dv_s\|_{L^2(A)} + \|D\widetilde{u}_s\|_{L^2(A)} \le C\sqrt{|\log(1-s)|}, \tag{3.66}$$

where $C > 0$ is a constant which does not depend on s. Hence, by Proposition 2.4, the validity of (3.64) follows by combining (3.65) and (3.66).

At this point, thanks to Steps 1 and 2 above, we proved (iii) in Theorem 1.2 for any $\mu \in \mathcal{X}(\overline{\Omega})$ as in (3.44) such that $\mu(\overline{\Omega}) = \deg(g|_{\partial\Omega}, \partial\Omega)$.

Step 3. We now prove (iii) in Theorem 1.2 in full generality. Let $\mu \in \mathcal{X}(\overline{\Omega})$ be such that $\mu(\overline{\Omega}) = \deg(g|_{\partial\Omega}, \partial\Omega)$. We can find a sequence $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{X}(\overline{\Omega})$ as in (3.44), that is,

$$\mu_k(\overline{\Omega}) = \deg(g|_{\partial\Omega}, \partial\Omega) \quad \text{and} \quad \mu_k = \sum_{i=1}^{N_k} d_{i,k}\,\delta_{x_{i,k}} \quad \text{for each } k \in \mathbb{N},$$

with $N_k \in \mathbb{N}$, $x_{i,k} \in \Omega$ and $d_{i,k} \in \{-1, 1\}$ for each $i \in \{1, \dots, N_k\}$, such that

$$\mu_k \xrightarrow{\text{flat}(\overline{\Omega})} \mu \quad \text{as } k \to \infty.$$

In particular, we have that $|\mu_k|(\Omega) \to |\mu|(\overline{\Omega})$ as $k \to \infty$. By Step 1, for each $k \in \mathbb{N}$ we can find $u_{s,k} \in H^s_g(\Omega; B_L)$ such that

$$\limsup_{s\to 1^-} \mathrm{GL}^s_\lambda(u_{s,k}) \le \pi|\mu_k|(\Omega)$$

whenever $\lambda > 0$. Thus, given any sequence $(s_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $s_j \to 1^-$ as $j \to \infty$, by a diagonal argument we can extract a subsequence $(s_{j_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathrm{GL}^{s_{j_k}}_\lambda(u_{s_{j_k},k}) \le \lim_{k\to\infty} \pi|\mu_k|(\Omega) = \pi|\mu|(\overline{\Omega}),$$

concluding the proof. □

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