ON THE MONOTONICITY OF NON-LOCAL PERIMETER OF CONVEX BODIES

FLAVIA GIANNETTI AND GIORGIO STEFANI

Abstract. Under mild assumptions on the kernel $K \ge 0$, the non-local K-perimeter P_K satisfies the monotonicity property on nested convex bodies, i.e., if $A \subset B \subset \mathbb{R}^n$ are two convex bodies, then $P_K(A) \le P_K(B)$. In this note, we prove quantitative lower bounds on the difference of the K-perimeters of A and B in terms of their Hausdorff distance, provided that K satisfies suitable symmetry properties.

1. Introduction

1.1. **Monotonicity property.** Let $n \ge 2$. If $A \subset B \subset \mathbb{R}^n$ are two nested convex bodies, that is, compact convex sets with non-empty interior, then

$$P(A) \le P(B). \tag{1.1}$$

Here $P(E) = \mathscr{H}^{n-1}(\partial E)$ is the Euclidean perimeter of the convex body $E \subset \mathbb{R}^n$ and $\mathscr{H}^d$ is the d-dimensional Hausdorff measure, $d \in [0, n]$.

The monotonicity property (1.1) is well known and dates back to the ancient Greeks (Archimedes took it as a postulate in his work on the sphere and the cylinder [1, p. 36]). Inequality (1.1) follows from the *Cauchy formula* for the area surface of convex bodies [6, §7], the monotonicity property of *mixed volumes* [6, §8], the Lipschitz property of the projection on a convex closed set [8, Lem. 2.4] and, finally, from the fact that the perimeter is decreased by intersection with half-spaces [30, Ex. 15.13].

The monotonicity property (1.1) also holds for the *anisotropic* (*Wulff*) *perimeter*

$$P_\Phi(E) = \int_{\partial E} \Phi(\nu_E(x))\, \mathrm{d}\mathscr{H}^{n-1}(x) \tag{1.2}$$

Date: September 20, 2023.

2020 *Mathematics Subject Classification.* Primary 52A20. Secondary 52A40.

Key words and phrases. Convex body, non-local perimeter, monotonicity, Hausdorff distance, Schwartz symmetrization.

Acknowledgements. The authors are members of INdAM–GNAMPA. The first-named author is a member of UMI–TAA group and has received funding from INdAM under the INdAM–GNAMPA 2023 Project *Su alcuni problemi di regolarità del Calcolo delle Variazioni con convessità degenere* (grant agreement No. CUP_E53C22001930001) and from the FRA Project 2020 *Regolarità per minimi di funzionali ampiamente degeneri* (Project code 000022). The second-named author has received funding from INdAM under the INdAM–GNAMPA 2023 Project *Problemi variazionali per funzionali e operatori non-locali* (grant agreement No. CUP_E53C22001930001), and has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 945655). The present research started during a visit of the second-named author at the Dipartimento di Matematica e Applicazioni 'Renato Caccioppoli' of the Università degli Studi di Napoli 'Federico II'. The second-named author wishes to thank the department for its kind hospitality.

of a convex body $E \subset \mathbb{R}^n$, where $\nu_E \colon \partial E \to \mathbb{S}^{n-1}$, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, is the inner unitary normal of E (defined for $\mathscr{H}^{n-1}$-a.e. $x \in \partial E$) and $\Phi \colon \mathbb{R}^n \to [0, +\infty]$ is positively 1-homogeneous and convex (see [30, Ch. 20]). Similarly to (1.1), the monotonicity property of (1.2) is a consequence of the anisotropic *Cauchy formula*, of the monotonicity property of *mixed volumes* [6, §7,§8], and of the fact that (1.2) is decreased by intersection with half-spaces [30, Rem. 20.3].

The monotonicity property of perimeters has gained increasing attention in recent years, see [5, 18, 20, 25, 31, 33] for some applications and related results. A current active line of research concerns quantitative formulations of the monotonicity property. Lower bounds on the deficit $\delta(B, A) = P(B) - P(A)$ in terms of the *Hausdorff distance* $h(A, B)$ between A and B (see [34, Sec. 1.8] for the definition) have been obtained for $n = 2, 3$ in [10, 11, 27] (also see the survey [23]) and for any $n \geq 2$ in [35]. Actually, the main result of [35] establishes a quantitative lower bound also for (1.2), provided that Φ possess suitable symmetry properties. We stress that the inequalities proved in [10, 11, 27, 35] are sharp, in the sense that they are equalities at least in one non-trivial case. Quantitative monotonicity inequalities for the perimeter can be applied to achieve lower bounds on the minimal number of convex components of a non-convex set [12, 24, 27].

1.2. **Main results.** The aim of the present note is to study the monotonicity property of non-local perimeter functionals, also in its quantitative form.

Given a non-negative measurable *kernel* $K \colon \mathbb{R}^n \to [0, +\infty]$, the associated *non-local K-perimeter* of a measurable set $E \subset \mathbb{R}^n$ is defined as

$$P_K(E) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\chi_E(x) - \chi_E(y)| \, K(x - y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.3}$$

A prominent example of such non-local functional is the *s-fractional perimeter*, $s \in (0, 1)$,

$$P_s(E) = \int_E \int_{E^c} \frac{\mathrm{d}x \, \mathrm{d}y}{|x - y|^{n+s}}, \tag{1.4}$$

induced by the *fractional kernel* $K_s = |\cdot|^{-n-s}$, see [19]. The K-perimeter (1.3) has attracted considerable attention in recent years, see [3, 4, 9, 13–17, 22, 26, 29, 32].

Due to the definition in (1.3), it is not restrictive to assume that

$$K(-x) = K(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.5}$$

Moreover, since we need the K-perimeter to be finite on convex bodies, we assume that

$$\int_{\mathbb{R}^n} \min\{1, |x|\} \, K(x) \, \mathrm{d}x < +\infty. \tag{1.6}$$

Actually, condition (1.6) yields that $P_K(E) < +\infty$ whenever $P(E) < +\infty$ and $|E| < +\infty$, see Lemma 2.6 below (where $|E| = \mathscr{H}^n(E)$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$). We tacitly assume (1.5) and (1.6) throughout the paper.

The monotonicity property of non-local perimeters was first proved for (1.4) in [21, Lem. B.1] and then for (1.3) in [4, Cor. 2.40] (we also refer to [14, Rem. 2.4] for the monotonicity of the localized functional $P_K(\,\cdot\,; B_R)$ for $R > 0$, see (2.1) below for the precise definition). Actually, the monotonicity results proved in [4, 14, 21] require further assumptions on K in addition to (1.5) and (1.6). This is due to the fact that [4, 14, 21] employ the monotonicity property of P_K for convex sets which may be unbounded.

Our first main result is the following theorem, showing that assumptions (1.5) and (1.6) guarantee the monotonicity property of P_K on nested convex bodies.

Theorem 1.1 (K-monotonicity)**.** *Let $n \geq 2$. If $A \subset B \subset \mathbb{R}^n$ are two convex bodies, then*

$$P_K(A) \leq P_K(B). \tag{1.7}$$

The proof of Theorem 1.1 follows the same strategy of the corresponding results in [4, 14, 21], but with some simplifications due to the boundedness on the involved sets. Although the proof of (1.7) may be known to experts, we briefly outline it below to keep the present note as self-contained as possible.

Our second main aim is to provide a quantitative version of Theorem 1.1 by proving a lower bound on the non-local deficit $\delta_K(B, A) = P_K(B) - P_K(A)$ in terms of the Hausdorff distance $h(A, B)$ between A and B. Our strategy is modeled on the approach adopted in [35] for dealing with (1.2) and, essentially, requires that the kernel K possess suitable symmetry properties. Precisely, we assume that $x \mapsto K(x)$ is symmetric-decreasing with respect to the component of $x \in \mathbb{R}^n$ which is orthogonal to some fixed direction $\nu \in \mathbb{S}^{n-1}$.

Definition 1.2 ($\nu^\perp$-symmetric decreasing kernel)**.** Let $n \geq 2$. We say that the kernel K is *$\nu^\perp$-symmetric decreasing* for some $\nu \in \mathbb{S}^{n-1}$ if there exists a measurable function $k_\nu \colon [0, +\infty)^2 \to [0, +\infty]$ such that $r \mapsto k_\nu(r, t)$ is decreasing for all $t \geq 0$ and

$$K(x) = k_\nu(|x - (x \cdot \nu)\nu|, |x \cdot \nu|) \quad \text{for } x \in \mathbb{R}^n.$$

If K is $\nu^\perp$-symmetric decreasing for some $\nu \in \mathbb{S}^{n-1}$, then it also satisfies (1.5) and, moreover, it is also $(-\nu)^\perp$-symmetric decreasing with $k_{-\nu} = k_\nu$. In particular, Definition 1.2 applies to *radially-symmetric decreasing* kernels, i.e., $K(x) = \phi(|x|)$, $x \in \mathbb{R}^n$, for some decreasing $\phi \colon [0, +\infty] \to [0, +\infty]$. Indeed, in this case, one chooses

$$k_\nu(r, t) = \phi\left(\sqrt{r^2 + t^2}\right) \quad \text{for } r, t \geq 0,$$

for any $\nu \in \mathbb{S}^{n-1}$. Note that this holds for P_s in (1.4), with $\phi(r) = r^{-n-s}$ for $r > 0$.

With this notation at disposal, our second main result reads as follows. Here and in the rest of the paper, we let $\langle \nu \rangle = \{t\nu : t \in \mathbb{R}\} \subset \mathbb{R}^n$.

Theorem 1.3 (Quantitative K-monotonicity)**.** *Let $n \geq 2$ and let K be $\nu^\perp$-symmetric decreasing for some $\nu \in \mathbb{S}^{n-1}$ as in Definition 1.2. There exists a function $f_{K,\nu} \colon [0, +\infty)^3 \to [0, +\infty)$ with the following property. If $A \subset B \subset \mathbb{R}^n$ are two convex bodies with Hausdorff distance $h(A, B) = |a - b|$ achieved for some $a \in A$ and $b \in B$ such that $a - b \in \langle \nu \rangle$, then*

$$P_K(A) + f_{K,\nu}\left(h(A, B), \mathscr{H}^{n-1}(B \cap \partial H), |B \cap H|\right) \leq P_K(B), \tag{1.8}$$

where $H = \{x \in \mathbb{R}^n : (b - a) \cdot (x - a) \leq 0\}$.

The function $f_{K,\nu}$ in Theorem 1.3 is implicit, due to the fact that, contrary to the local case [35], the non-locality of the functional (1.3) prevents us to compute the lower bound on the K-perimeter deficit explicitly. Nonetheless, the proof of Theorem 1.3 yields a partial characterization of optimal configurations. In particular, inequality (1.8) is sharp, in the sense that it is an equality at least in one non-trivial case.

The strategy of the proof of Theorem 1.3 is inspired by [26, 35]. We first reduce the given convex bodies to more symmetric ones by performing a *Schwartz symmetrization* (done orthogonally to the given $\nu \in \mathbb{S}^{n-1}$), thanks to the symmetry of K ensured by Definition 1.2, and then we exploit a compactness argument.

As a consequence of Theorem 1.3, in the radially-symmetric decreasing case, we get Corollary 1.4 below. Here and in the following, we let $P_\phi = P_{\phi(|\cdot|)}$ for any decreasing function $\phi\colon [0,+\infty] \to [0,+\infty]$, we let $t^+ = \max\{t,0\}$ be the positive part of $t \in \mathbb{R}$ and, given $h, r \geq 0$, we let

$$\mathcal{C}_r^h = \bigcup_{x \in D_r} \{(1-t)h\mathrm{e}_n + tx : t \in [0,1]\}$$

be the (closed) right circular cone with base $D_r = \{x \in \mathbb{R}^n : |x| \leq r,\ x_n = 0\}$ and height h in the direction $\mathrm{e}_n = (0,\dots,0,1) \in \mathbb{S}^{n-1}$. Finally, we let $h = h(A,B)$ denote the Hausdorff distance between A and B and, as usual, $\omega_{n-1} = \mathscr{H}^{n-1}(\mathbb{S}^{n-1})$.

Corollary 1.4 (Quantitative ϕ-monotonicity)**.** *Let $n \geq 2$. If $A \subset B \subset \mathbb{R}^n$ are two convex bodies, then*

$$P_\phi(A) + \Big(P_\phi(\mathcal{C}_r^h) - \sigma_\phi \max\Big\{\tfrac{1}{2}P(B \cap H), |B \cap H|\Big\}\Big)^+ \leq P_\phi(B) \tag{1.9}$$

where $\sigma_\phi = \int_{\mathbb{R}^n} \min\{1, |x|\}\, \phi(|x|)\,\mathrm{d}x$,

$$r = \sqrt[n-1]{\frac{\mathscr{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \qquad H = \{x \in \mathbb{R}^n : (b-a)\cdot(x-a) \leq 0\},$$

and $a \in A$ and $b \in B$ such that $|a-b| = h(A,B)$.

Inequality (1.9) is apparently worse than (1.8). However, the lower bound given by (1.9) is more explicit than the one given by (1.8). In fact, one is only left to estimate $P_\phi(\mathcal{C}_r^h)$, which may be explicitly done for a specific choice of ϕ.

A special instance of Corollary 1.4 is the fractional case. In fact, exploiting the *fractional isoperimetric inequality* (see [21] and the references therein)

$$P_s(E) \geq c_{n,s}^{\mathrm{iso}}\, |E|^{\frac{n-s}{n}}, \qquad c_{n,s}^{\mathrm{iso}} = \frac{P_s(B_1)}{|B_1|^{\frac{n-s}{n}}}, \tag{1.10}$$

valid for any measurable $E \subset \mathbb{R}^n$, we get the following result.

Corollary 1.5 (Quantitative s-monotonicity)**.** *Let $n \geq 2$. If $A \subset B \subset \mathbb{R}^n$ are two convex bodies, then*

$$P_s(A) + \left(c_{n,s}^{\mathrm{iso}} \left(\tfrac{\omega_{n-1}}{n}\, h\, r^{n-1}\right)^{1-\frac{s}{n}} - \tfrac{2^{2-s}\, n\, \omega_n}{s\,(1-s)}\, P(B \cap H)^s\, |B \cap H|^{1-s}\right)^+ \leq P_s(B) \tag{1.11}$$

where $h, r \geq 0$, $a \in A$, $b \in B$ and $H \subset \mathbb{R}^n$ are as in Corollary 1.4.

Inequality (1.11) is worse than (1.9), since we used (1.10) to estimate the term $P_s(\mathcal{C}_r^h)$ from below in terms of $|\mathcal{C}_r^h|$ which, in turn, can be explicitly computed from h, r. Anyway, the lower bound we get is more explicit than the one given by (1.9).

1.3. **The 1D case.** In closing, let us comment on the special case $n = 1$.

The monotonicity property (1.1) becomes trivial for $n = 1$, since $P(A) = P(B) = 2$ for any two (not necessarily nested) segments $A, B \subset \mathbb{R}$. Instead, the case $n = 1$ becomes non-trivial in the non-local setting. For example, in the fractional case, due to the scaling and translation invariance of (1.4), we have

$$P_s(B) - P_s(A) = c_s\left(|B|^{1-s} - |A|^{1-s}\right) \tag{1.12}$$

for any two (not necessarily nested) segments $A, B \subset \mathbb{R}$, where $c_s = P_s((0,1))$.

With (1.12) in mind, we have the following result, which is inspired by [4, Lem. 2.31].

Proposition 1.6 (Case $n = 1$)**.** *Assume that $\phi\colon [0,+\infty) \to (0,+\infty)$ satisfies*

$$\inf_{t>0} \frac{R^2\phi(Rt) - r^2\phi(rt)}{\phi(t)} \ge \psi(R) - \psi(r) \quad \textit{for all } R \ge r \ge 0, \tag{1.13}$$

for some increasing function $\psi\colon [0,+\infty) \to [0,+\infty)$. If $A, B \subset \mathbb{R}$ are two segments with $|A| \le |B|$, then, setting $c_\phi = P_\phi((0,1))$,

$$P_\phi(A) + c_\phi\Big(\psi(|B|) - \psi(|A|)\Big) \le P_\phi(B). \tag{1.14}$$

Note that the assumption in (1.13) implies that ϕ is decreasing. In the fractional case, $\phi(r) = r^{-1-s}$ for $r > 0$, so $\psi(r) = r^{1-s}$ for $r \ge 0$, and we recover (1.12).

2. Proofs of the statements

2.1. Proof of Theorem 1.1.

Theorem 1.1 is a consequence of the following result.

Proposition 2.1 (Intersection with convex sets)**.** *If $E \subset \mathbb{R}^n$ is a convex body, then $P_K(E \cap C) \le P_K(E)$ for any convex set $C \subset \mathbb{R}^n$.*

The proof of Proposition 2.1 exploits the *local minimality* of half-spaces, see Lemma 2.2 below. This latter result was proved first for (1.4) in [2, Prop. 17] and then for (1.3) in [32, Th. 1] (also see [9, Cor. 2.5] and [4, Lem. 2.31]). Here and below, for measurable sets $E, A \subset \mathbb{R}^n$, we let

$$P_K(E; A) = \left(\int_{E\cap A}\int_{E^c\cap A} + \int_{E\cap A}\int_{E^c\cap A^c} + \int_{E\cap A^c}\int_{E^c\cap A}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.1}$$

be the *K-perimeter of E relative to A*.

Lemma 2.2 (Local minimality of half-spaces)**.** *Let $R > 0$. If $H \subset \mathbb{R}^n$ is a half-space, then $P_K(H; B_R) \le P_K(E; B_R)$ for any $E \subset \mathbb{R}^N$ such that $E \setminus B_R = H \setminus B_R$.*

Proof of Proposition 2.1. Our strategy is a simplification of the one used for the proof of [4, Th. 2.29] (see also [21, Lem. B.1] and [14, Rem. 2.4]). Since $C \subset \mathbb{R}^n$ is convex, we can find a sequence of half-spaces $(H_k)_{k\in\mathbb{N}}$ such that the sets

$$C_N = \bigcap_{k=1}^{N} H_k, \quad N \in \mathbb{N},$$

satisfy $|C_N \triangle C| \to 0^+$ as $N \to +\infty$. Therefore, thanks to the lower semicontinuity property of P_K (see [4, Lem. 2.10] for instance), we can assume that $C = H$ is a half-space. We can hence write

$$P_K(E) - P_K(E \cap H) = \left(\int_E\int_{E^c} - \int_{E\cap H}\int_{(E\cap H)^c}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \left(\int_{E\cap H} + \int_{E\setminus H} \right) \int_{E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \left(\int_{E^c} + \int_{E\setminus H} \right) \int_{E\cap H} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \left(\int_{E^c} - \int_{E\cap H} \right) \int_{E\setminus H} K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now let $R > 0$ be such that $E \subset B_R$. Defining $F = E \cup H$, one easily checks that $E \subset F$, $E \setminus H = F \setminus H$, $F \cap H = H$ and $F \setminus B_R = H \setminus B_R$. A plain computation then yields

$$\left(\int_{E^c} - \int_{E\cap H} \right) \int_{E\setminus H} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \ge \left(\int_{F^c} - \int_{F\cap H} \right) \int_{F\setminus H} K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

We now observe that the right-hand side of the above inequality can be rewritten as $P_K(F; B_R) - P_K(H; B_R)$ exploiting the definition in (2.1). Therefore, thanks to Lemma 2.2, we get that

$$P_K(E) - P_K(E \cap H) \ge P_K(F; B_R) - P_K(H; B_R) \ge 0,$$

yielding the conclusion. □

2.2. **Proof of Theorem 1.3.** We begin by adapting [30, Sec. 19.2] (which corresponds to the choice $\nu = \mathrm{e}_1$ in what follows) to our setting.

Let $\nu \in \mathbb{S}^{n-1}$ be fixed. For any $x \in \mathbb{R}^n$, we set

$$x'_\nu = x - (x \cdot \nu)\nu \quad \text{and} \quad x_\nu = (x \cdot \nu)\nu.$$

We naturally identify $x_\nu \in \langle \nu \rangle$ and $x'_\nu \in \langle \nu \rangle^\perp$, where $\langle \nu \rangle^\perp$ is the linear space orthogonal to $\langle \nu \rangle$, with points in $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively. In particular, with a slight abuse of notation, we write $x = x'_\nu + x_\nu = (x'_\nu, x_\nu)$ for any $x \in \mathbb{R}^n$.

Definition 2.3 (Schwartz ν-symmetrization)**.** Given $E \subset \mathbb{R}^n$ with $|E| < +\infty$, we let

$$E^\nu_t = \left\{ y \in \langle \nu \rangle^\perp : y + t\nu \in E \right\}, \quad t \in \mathbb{R}, \tag{2.2}$$

be the *slice of E orthogonal to $\nu \in \mathbb{S}^{n-1}$*. We hence let

$$E^{\#\nu} = \left\{ x \in \mathbb{R}^n : \omega_{n-1} |x'_\nu|^{n-1} \le \mathscr{H}^{n-1}\left(E^\nu_{x_\nu} \right) \right\} \tag{2.3}$$

be the *Schwartz ν-symmetrization of E*.

The set $E^{\#\nu}$ is measurable, with slice $(E^{\#\nu})^\nu_t$ equal to an open ball such that

$$\mathscr{H}^{n-1}((E^{\#\nu})^\nu_t) = \mathscr{H}^{n-1}(E^\nu_t)$$

for each $t \in \mathbb{R}$. Hence $|E^{\#\nu}| = |E|$ by Fubini–Tonelli's Theorem.

The following result is a non-local analog of the Schwartz inequality $P(E) \ge P(E^{\#\nu})$ for the Eulidean perimeter (see [30, Th. 19.11]). Actually, inequality (2.4) below is a special case of [7, Lem. 3.2], but we give a direct and simpler proof of it via the well-known *Riesz rearrangement inequality* (see [28, Ch. 3] for a detailed presentation). Here and in the rest of the paper, we let

$$L_K(E, F) = \int_E \int_F K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

be the *K-interaction functional* between the two measurable sets $E, F \subset \mathbb{R}^n$.

Lemma 2.4 (Non-local Schwartz ν-inequality)**.** *Let $n \geq 2$ and let K be $\nu^{\perp}$-symmetric decreasing kernel for some $\nu \in \mathbb{S}^{n-1}$ as in Definition 1.2. If $E, F \subset \mathbb{R}^n$ are such that $|E|, |F| < +\infty$, then*

$$L_K(E, F) \leq L_K(E^{\#\nu}, F^{\#\nu}). \tag{2.4}$$

Moreover, if $P_K(E) < +\infty$, then also $P_K(E^{\#\nu}) < +\infty$ with

$$P_K(E) \geq P_K(E^{\#\nu}). \tag{2.5}$$

Proof. Let k_ν be the function given by Definition 1.2. By Tonelli's Theorem, and recalling the definition of slice in (2.2), we can write

$$L_K(E, F) = \int_{\mathbb{R}} \int_{\mathbb{R}} L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, F^\nu_{y_\nu}) \,\mathrm{d}x_\nu \,\mathrm{d}y_\nu, \tag{2.6}$$

where

$$L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, F^\nu_{y_\nu}) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \chi_{E^\nu_{x_\nu}}(x'_\nu)\, \chi_{F^\nu_{y_\nu}}(y'_\nu)\, k_\nu(|x'_\nu - y'_\nu|, |x_\nu - y_\nu|) \,\mathrm{d}x'_\nu \,\mathrm{d}y'_\nu. \tag{2.7}$$

Since $|E|, |F| < +\infty$, we also have $\mathscr{H}^{n-1}(E^\nu_{x_\nu}), \mathscr{H}^{n-1}(F^\nu_{y_\nu}) < +\infty$ for a.e. $x_\nu, y_\nu \in \langle \nu \rangle$. Since the function $z \mapsto k_\nu(|z|, |x_\nu - y_\nu|)$ is radially-symmetric decreasing by Definition 1.2, by Riesz rearrangement inequality [28, Th. 3.4] we infer that

$$L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, F^\nu_{y_\nu}) \leq L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(D[E^\nu_{x_\nu}], D[F^\nu_{y_\nu}]) \quad \text{for a.e. } x_\nu, y_\nu \in \langle \nu \rangle. \tag{2.8}$$

Here $D[E^\nu_{x_\nu}]$ and $D[F^\nu_{y_\nu}]$ are the closed $(n-1)$-dimensional discs in $\langle \nu \rangle^\perp$ centered at the origin with $(n-1)$-dimensional volume equal to $\mathscr{H}^{n-1}(E^\nu_{x_\nu})$ and $\mathscr{H}^{n-1}(F^\nu_{y_\nu})$, respectively. Integrating (2.8), and again using Tonelli's Theorem and (2.3) and (2.6), we get (2.4). Similarly, choosing $F = E$ in (2.6), we can write

$$P_K(E) = \int_{\mathbb{R}} \int_{\mathbb{R}} L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, (E^c)^\nu_{y_\nu}) \,\mathrm{d}x_\nu \,\mathrm{d}y_\nu. \tag{2.9}$$

Since $P_K(E) < +\infty$, we must have

$$L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, (E^c)^\nu_{y_\nu}) < +\infty \quad \text{for a.e. } x_\nu, y_\nu \in \langle \nu \rangle. \tag{2.10}$$

Now, for any fixed $x_\nu, y_\nu \in \langle \nu \rangle$, we can write

$$k_\nu(|z|, |x_\nu - y_\nu|) = \int_0^{+\infty} \chi_{\{k_\nu(|\cdot|, |x_\nu - y_\nu|) > t\}}(z) \,\mathrm{d}t, \quad \text{for } z \in \langle \nu \rangle^\perp.$$

Therefore, for a.e. $x_\nu, y_\nu \in \langle \nu \rangle$, we can decompose

$$L_{k_\nu(|\cdot|, |x_\nu - y_\nu|)}(E^\nu_{x_\nu}, (E^c)^\nu_{y_\nu}) = \int_0^{+\infty} L_{\chi_{\{k_\nu(|\cdot|, |x_\nu - y_\nu|) > t\}}}(E^\nu_{x_\nu}, (E^c)^\nu_{y_\nu}) \,\mathrm{d}t, \tag{2.11}$$

where, as in (2.7), we have

$$\begin{aligned} &L_{\chi_{\{k_\nu(|\cdot|, |x_\nu - y_\nu|) > t\}}}(E^\nu_{x_\nu}, (E^c)^\nu_{y_\nu}) \\ &\qquad = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \chi_{E^\nu_{x_\nu}}(x'_\nu)\, \chi_{(E^c)^\nu_{y_\nu}}(y'_\nu)\, \chi_{\{k_\nu(|\cdot|, |x_\nu - y_\nu|) > t\}}(x'_\nu - y'_\nu) \,\mathrm{d}x'_\nu \,\mathrm{d}y'_\nu. \end{aligned}$$

We now observe that, for a.e. $x_\nu, y_\nu \in \langle \nu \rangle$ and for any $t > 0$, the set

$$\left\{ z \in \langle \nu \rangle^\perp : k_\nu(|z|, |x_\nu - y_\nu|) > t \right\}$$

is an $(n-1)$-dimensional disc (possibly empty or the entire subspace $\langle\nu\rangle^\perp$), with

$$\mathscr{H}^{n-1}\left(\left\{z\in\langle\nu\rangle^\perp : k_\nu(|z|,|x_\nu-y_\nu|)>t\right\}\right)<+\infty \tag{2.12}$$

for a.e. $t>0$. Indeed, if this is not the case, then, for some $S\subset\mathbb{R}$ with $\mathscr{H}^1(S)>0$,

$$\begin{aligned}L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},(E^c)^\nu_{y_\nu}) &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\chi_{E^\nu_{x_\nu}}(x'_\nu)\,\chi_{(E^c)^\nu_{y_\nu}}(y'_\nu)\,\mathrm{d}x'_\nu\,\mathrm{d}y'_\nu\\ &= \mathscr{H}^{n-1}(E^\nu_{x_\nu})\,\mathscr{H}^{n-1}((E^c)^\nu_{y_\nu})=+\infty\end{aligned}$$

for all $t\in S$, which, together with (2.11), contradicts (2.10). Now, for any $t>0$ yielding (2.12), we have $\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}\in L^1(\langle\nu\rangle^\perp,\mathscr{H}^{n-1})$. Hence, we can decompose

$$\begin{aligned}L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},(E^c)^\nu_{y_\nu}) &= L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},\langle\nu\rangle^\perp)-L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},E^\nu_{y_\nu})\\ &= \mathscr{H}^{n-1}(E^\nu_{x_\nu})\,\mathscr{H}^{n-1}(\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\})-L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},E^\nu_{y_\nu}).\end{aligned}$$

Since $\mathscr{H}^{n-1}(E^\nu_{x_\nu})=\mathscr{H}^{n-1}(D[E^\nu_{x_\nu}])$ by (2.3) and

$$L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},E^\nu_{y_\nu})\le L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(D[E^\nu_{x_\nu}],D[E^\nu_{y_\nu}])$$

by (2.8) (applied with $E=F$), again recalling (2.3) we readily get that

$$L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},(E^c)^\nu_{y_\nu})\ge L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}((E^{\#\nu})^\nu_{x_\nu},((E^{\#\nu})^c)^\nu_{y_\nu}).$$

Integrating back in $t>0$ and recalling (2.11), we get

$$L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}(E^\nu_{x_\nu},(E^c)^\nu_{y_\nu})\ge L_{\chi_{\{k_\nu(|\cdot|,|x_\nu-y_\nu|)>t\}}}((E^{\#\nu})^\nu_{x_\nu},((E^{\#\nu})^c)^\nu_{y_\nu}).$$

Finally, integrating back in $x_\nu,y_\nu\in\langle\nu\rangle$ and recalling (2.9), we get (2.5). $\square$

In the proof of Theorem 1.3, we will use the following notation. Given $p\in\mathbb{R}^n$ and a (non-empty) set $S\subset\mathbb{R}^n$, we define the cones with vertex p and base S

$$\mathcal{C}(p,S)=\bigcup_{s\in S}\{p+t(s-p):t\in[0,1]\},\quad \mathcal{C}_\infty(p,S)=\bigcup_{s\in S}\{p+t(s-p):t\ge 0\}.$$

Note that, if S is convex, then also $\mathcal{C}(p,S)$ and $\mathcal{C}_\infty(p,S)$ are convex. Moreover, if S is bounded, then also $\mathcal{C}(p,S)$ is bounded. Finally, given $p\in\mathbb{R}^n$, $r\ge 0$ and $\nu\in\mathbb{S}^{n-1}$, we let

$$D^\nu_r(p)=p+D^\nu_r(0),\quad D^\nu_r(0)=\left\{x\in\mathbb{R}^n : x\in\langle\nu\rangle^\perp,\ |x|\le r\right\},$$

be the closed $(n-1)$-dimensional disc centered at p, with radius r, and orthogonal to ν.

Proof of Theorem 1.3. Let $\nu\in\mathbb{S}^{n-1}$, $a\in A$, $b\in B$, $h=h(A,B)=|a-b|$ and $H\subset\mathbb{R}^n$ be as in the statement. Since $A\subset B$ are compact convex sets, we have that

$$h(A,B)=|a-b|=\max_{y\in B}\min_{x\in A}|x-y|.$$

Consequently, a is the orthogonal projection of b on A. By definition of H and by minimality of the projection, the closed hyperplane ∂H is a supporting one for A in a. As a consequence, we must have that $A\subset B\cap H$.

Step 1: reduction to symmetric sets. Let $E=B\cap H$ and $C=\mathcal{C}(b,B\cap\partial H)$ be the (bounded and closed) cone with vertex b and base $B\cap\partial H$. Note that E, C and $E\cup C$ are convex bodies. Since $A\subset E$ and $E\cup C\subset B$, we have

$$\delta_K(B,A)\ge\delta_K(B,E)\ge\delta_K(E\cup C,E),$$

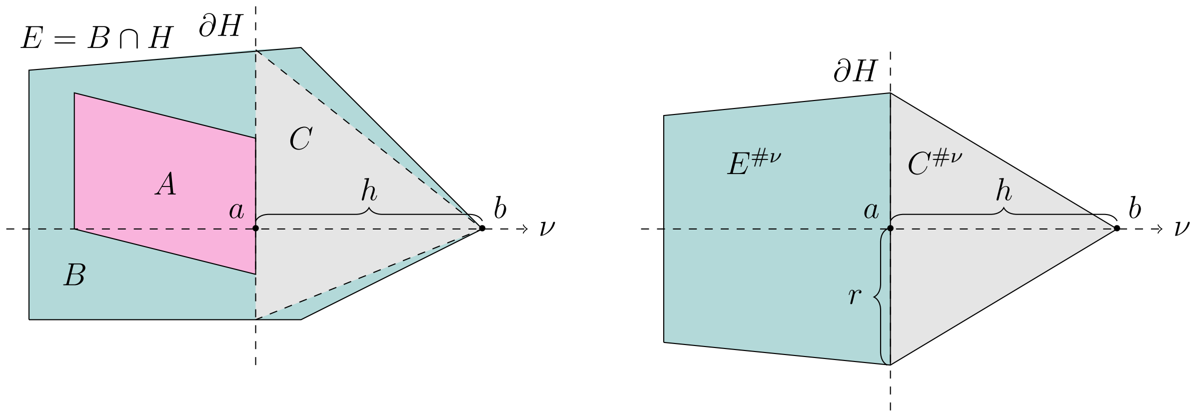

Figure 1. Reduction to symmetric sets in the proof of Theorem 1.3: an initial configuration (left) and its symmetrization (right).

with equality if $A = E$ and $B = E \cup C$. Since $|E \cap C| = 0$ and $E^c = C \cup (E^c \setminus C) = C \cup (E^c \cap C^c)$, we can write

$$\begin{aligned} P_K(E \cup C) - P_K(E) &= L_K(E \cup C, E^c \cap C^c) - L_K(E, E^c) \\ &= L_K(E, E^c \cap C^c) + L_K(C, E^c \cap C^c) - L_K(E, C) - L_K(E, E^c \cap C^c) \\ &= L_K(C, E^c \cap C^c) - L_K(E, C) \\ &= L_K(C, C^c \setminus E) - L_K(E, C) \\ &= L_K(C, C^c) - L_K(C, E) - L_K(E, C) \\ &= P_K(C) - 2L_K(E, C). \end{aligned} \tag{2.13}$$

We now apply Lemma 2.4 to the convex bodies E and C, so that

$$L_K(E, C) \le L_K(E^{\#\nu}, C^{\#\nu}) \quad \text{and} \quad P_K(C) \ge P_K(C^{\#\nu}).$$

As a consequence, reading the chain of equalities in (2.13) backwards, we get that

$$\delta_K(B, A) \ge \delta_K(E \cup C, E) \ge \delta_K(E^{\#\nu} \cup C^{\#\nu}, E^{\#\nu}) = P_K(C^{\#\nu}) - 2L_K(E^{\#\nu}, C^{\#\nu}). \tag{2.14}$$

In particular, setting $A^* = E^{\#\nu}$ and $B^* = E^{\#\nu} \cup C^{\#\nu}$, we proved that

$$\delta_K(B, A) \ge \delta_K(B^*, A^*),$$

with equality obviously if $A = A^*$ and $B = B^*$, reducing the proof of Theorem 1.3 to the case of the symmetric convex bodies A^*, B^*.

Step 2: description of $C^{\#\nu}$. We now note that $C^{\#\nu}$ is uniquely determined by $a - b$ and $\mathscr{H}^{n-1}(B \cap \partial H)$. Indeed, since by definition

$$(x - a) \cdot \nu = 0 \quad \text{for all } x \in \partial H,$$

the half-space H is preserved under Schwartz ν-symmetrization, $H^{\#\nu} = H$. In particular, we recognize that $E^{\#\nu} = B^{\#\nu} \cap H$ and hence $C^{\#\nu} = \mathcal{C}(b, E^{\#\nu} \cap \partial H) = \mathcal{C}(b, B^{\#\nu} \cap \partial H)$. Moreover, since $a \in E^{\#\nu}$ is still the orthogonal projection of $b \in C^{\#\nu}$ on $E^{\#\nu}$, ∂H is a supporting hyperplane for $E^{\#\nu}$ in a and thus, since $E^{\#\nu} \subset H$, we conclude that

$$h = h(A, B) = h(E, E \cup C) = h(E^{\#\nu}, E^{\#\nu} \cup C^{\#\nu}).$$

Finally, by definition of Schwartz symmetrization in (2.3), we have

$$\mathscr{H}^{n-1}(B \cap \partial H) = \mathscr{H}^{n-1}(B^{\#\nu} \cap \partial H) = \mathscr{H}^{n-1}(E^{\#\nu} \cap \partial H).$$

In conclusion, we get that $C^{\#\nu} = \mathcal{C}(b, D^\nu_r(a))$, where

$$r = \sqrt[n-1]{\frac{\mathscr{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \tag{2.15}$$

see Figure 1. Arguing similarly, also the (unbounded and closed) cone $C^{\#\nu}_\infty = \mathcal{C}_\infty(b, B^{\#\nu} \cap \partial H)$ is uniquely determined as $C^{\#\nu}_\infty = \mathcal{C}_\infty(b, D^\nu_r(a))$.

Step 3: description of $E^{\#\nu}$. By definition, each slice $(E^{\#\nu})^\nu_t$ with $t \in \mathbb{R}$ is a (possibly empty) bounded and closed $(n-1)$-dimensional disc. More precisely, from now on assuming $(b-a) \cdot \nu \geq 0$ without loss of generality, we have that

$$E^{\#\nu} = \bigcup_{t \in [0, d_E]} D^\nu_{R_{E^{\#\nu}}(t)}(a - t\nu)$$

for some $d_E \in [0, +\infty)$ and some concave function $R_{E^{\#\nu}} \colon [0, d_E] \to [0, +\infty)$ such that $R_{E^{\#\nu}}(0) = r$ as in (2.15). Moreover, recalling the definition in (2.3),

$$|E^{\#\nu}| = |E| = |B \cap H|.$$

Finally, by construction, $E^{\#\nu} \subset \overline{C^{\#\nu}_\infty \setminus C^{\#\nu}}$, which equivalently rewrites as

$$R_{E^{\#\nu}}(t) \leq \frac{r}{h}\, t + r \quad \text{for } t \in [0, d_{E^{\#\nu}}].$$

Step 4: construction of a family of symmetric sets. We now set $w = |E|$ for brevity. We let $\mathcal{F}$ be the family of convex bodies $F \subset \mathbb{R}^n$ such that $|F| = w$ and

$$F = \bigcup_{t \in [0, d_F]} D^\nu_{R_F(t)}(a - t\nu) \tag{2.16}$$

for some $d_F \in [0, +\infty)$ and some concave function $R_F \colon [0, d_F] \to [0, +\infty)$ with $R_F(0) = r$ as in (2.15) and

$$R_F(t) \leq \frac{r}{h}\, t + r \quad \text{for } t \in [0, d_F]. \tag{2.17}$$

Note that $F^{\#\nu} = F$ and $F \subset \overline{C^{\#\nu}_\infty \setminus C^{\#\nu}}$ for any $F \in \mathcal{F}$. We now claim that

$$\sup_{F \in \mathcal{F}} d_F \leq \frac{n w r^{1-n}}{\omega_{n-1}}. \tag{2.18}$$

Indeed, given any $F \in \mathcal{F}$, we have $q_F = a - d_F \nu \in F$ by (2.16). Consequently, since $F \in \mathcal{F}$ is a convex body, the (bounded and closed) convex cone $\mathcal{C}(q_F, D^\nu_r(a))$ is contained in F, so that $|\mathcal{C}(q_F, D^\nu_r(a))| \leq |F|$, which means that

$$\frac{\omega_{n-1}}{n}\, r^{n-1} d_F \leq w, \tag{2.19}$$

proving the claimed (2.18). Combining (2.16), (2.17) and (2.18), we get that

$$F \subset \overline{\mathcal{C}(b, D^\nu_{R_w}(q_w)) \setminus C^{\#\nu}} \tag{2.20}$$

where $d_w = \frac{n w r^{1-n}}{\omega_{n-1}}$, $q_w = a - d_w \nu$ and $R_w = \frac{r}{h}\, d_w + r$, see Figure 2. We conclude by claiming that the family $\mathcal{F}$, endowed with the Hausdorff distance $h(\cdot, \cdot)$, is a compact

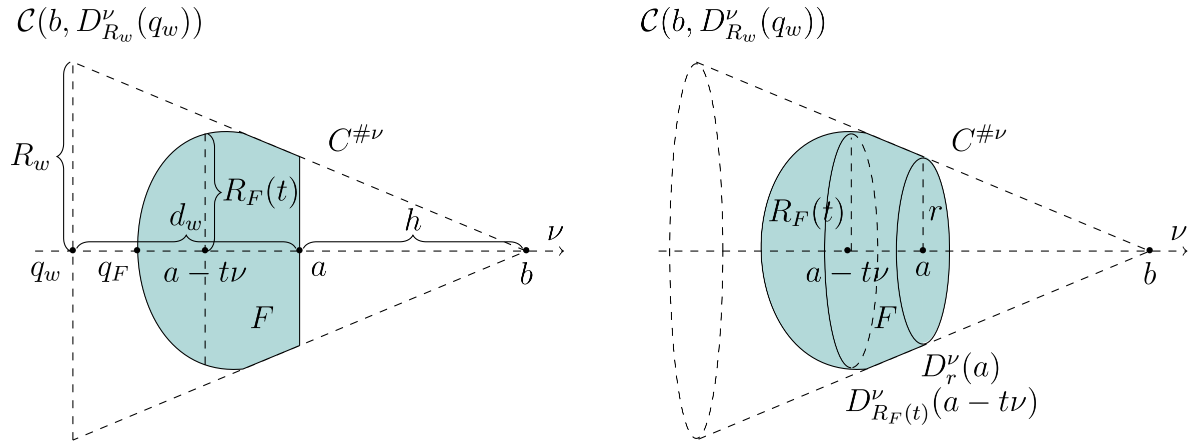

Figure 2. An element of the family $\mathcal{F}$ constructed in the proof of Theorem 1.3: its main lengths (left) and its circular sections orthogonal to $\nu \in \mathbb{S}^{n-1}$ (right).

metric space. Indeed, if $(F_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ is such that $h(F_j, \bar{F}) \to 0^+$ as $j \to +\infty$ for some $\bar{F} \subset \mathbb{R}^n$, then $\bar{F}$ is a convex body by Blaschke's Selection Theorem (see [34, Th. 1.8.7] for instance). Since $\sup_{j\in\mathbb{N}} P(F_j) < +\infty$ by (2.20), up to a subsequence, we also get that $\chi_{F_j} \to \chi_{\bar{F}}$ in $L^1(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$ as $j \to +\infty$ (see [30, Th. 12.26] for example). As a consequence, the limit convex body $\bar{F}$ still satisfies (2.20) and $|\bar{F}| = w$. Moreover, thanks to (2.19), up to a subsequence, we may assume that $d_{F_j} \to \bar{d}$ monotonically as $j \to +\infty$, for some $\bar{d} \in [0, d_w]$. Therefore, recalling (2.2) and since

$$|F_j \triangle \bar{F}| = \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big((F_j \triangle \bar{F})^\nu_t\Big)\,\mathrm{d}t = \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big((F_j)^\nu_t \triangle (\bar{F})^\nu_t\Big)\,\mathrm{d}t$$

for all $j \in \mathbb{N}$, again up to a subsequence, we get that $\mathcal{H}^{n-1}\Big((F_j)^\nu_t \triangle (\bar{F})^\nu_t\Big) \to 0^+$ as $j \to +\infty$ for a.e. $t \in \mathbb{R}$, proving that $\bar{F}$ satisfies (2.16) for $d_{\bar{F}} = \bar{d}$ and a convex function $R_{\bar{F}} \colon [0, d_{\bar{F}}] \to [0, +\infty)$ as in (2.17). We hence get that $\bar{F} \in \mathcal{F}$, yielding the claim.

Step 5: definition of $f_{K,\nu}$. By (2.14) and since $E^{\#\nu} \in \mathcal{F}$ by Step 3, we get that

$$\delta_K(B, A) \geq P_K(C^{\#\nu}) - 2L_K(E^{\#\nu}, C^{\#\nu}) \geq P_K(C^{\#\nu}) - 2\sup_{F\in\mathcal{F}} L_K(F, C^{\#\nu}). \tag{2.21}$$

Now consider the maximization problem

$$m = \sup_{F\in\mathcal{F}} L_K(F, C^{\#\nu}). \tag{2.22}$$

Since $F \subset (C^{\#\nu})^c$ for any $F \in \mathcal{F}$ by Step 4, we can trivially estimate

$$m \leq L_K((C^{\#\nu})^c, C^{\#\nu}) = P_K(C^{\#\nu}) < +\infty.$$

Hence let $(F_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ be any sequence such that $L_K(F_j, C^{\#\nu}) \to m$ as $j \to +\infty$. By the compactness of the metric space $(\mathcal{F}, h)$ proved in Step 4, up to a subsequence, we find some $M \in \mathcal{F}$ such that $h(F_j, M) \to 0^+$ as $j \to +\infty$. Since also $\sup_{j\in\mathbb{N}} P(F_j) < +\infty$ by (2.20), up to a subsequence, we also get that $\chi_{F_j} \to \chi_M$ in $L^1(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$ as $j \to +\infty$. By Fatou's Lemma, we hence infer that

$$m \leq L_K(M, C^{\#\nu}) \leq \liminf_{j\to+\infty} L_K(F_j, C^{\#\nu}) = m, \tag{2.23}$$

yielding that $M \in \mathcal{F}$ is a maximizer of (2.22). Now we observe that

$$P_K(C^{\#\nu}) - 2L_K(M, C^{\#\nu}) \geq 0.$$

Indeed, arguing exactly as in (2.13), we check that

$$P_K(C^{\#\nu}) - 2L_K(M, C^{\#\nu}) = P_K(M \cup C^{\#\nu}) - P(M), \tag{2.24}$$

which is non-negative by Theorem 1.1, since $M \cup C^{\#\nu}$ is a convex body by construction of the family $\mathcal{F}$. Now, again by the definition of $\mathcal{F}$ in Step 4, the value $m = m(h, r, w)$ of the maximization problem (2.22) is uniquely determined in terms of h, r and w. In addition, thanks to Step 3, $P_K(C^{\#\nu})$ is uniquely determined by h and r. We hence define

$$f_{K,\nu}(h, \omega_{n-1} r^{n-1}, w) = P_K(C^{\#\nu}) - 2\, m(h, r, w) = P_K(C^{\#\nu}) - 2L_K(M, C^{\#\nu}), \tag{2.25}$$

yielding (1.8) thanks to (2.21), (2.22) and (2.23). Finally, in virtue of the above construction and of (2.24) and (2.25), the equality in (1.8) is attained by the sets $A = M$ and $B = M \cup C^{\#\nu}$, where M is any solution of the maximization problem (2.22). □

Remark 2.5 (On the maximization problem (2.22))**.** With the same notation of the proof of Theorem 1.3, we can rewrite

$$L_K(F, C^{\#\nu}) = \int_F g^K_{C^{\#\nu}}(x)\,\mathrm{d}x, \quad F \in \mathcal{F}, \tag{2.26}$$

where $g^K_{C^{\#\nu}} \colon \mathbb{R}^n \to [0, +\infty]$ is given by

$$g^K_{C^{\#\nu}}(x) = (K * \chi_{C^{\#\nu}})(x) = \int_{\mathbb{R}^n} K(x-y)\,\chi_{C^{\#\nu}}(y)\,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^n.$$

Hence problem (2.22) can be equivalently interpreted as the maximization of the $g^K_{C^{\#\nu}}$-*potential energy* (2.26) among convex bodies $F \in \mathcal{M}$.

2.3. **Proof of Corollaries 1.4 and 1.5.** The following result is a simple interpolation estimate: the first part of the statement follows from [3, Prop. 2.2], while the second part is an easy refinement. We leave the plain details to the reader.

Lemma 2.6 (Interpolation)**.** *If $E \subset \mathbb{R}^n$ is a convex body, then*

$$P_K(E) \leq \max\left\{\tfrac{1}{2}\, P(E), |E|\right\} \int_{\mathbb{R}^n} \min\{1, |x|\}\, K(x)\,\mathrm{d}x. \tag{2.27}$$

In particular, in the fractional case $s \in (0,1)$,

$$P_s(E) \leq \frac{2^{1-s}\, n\,\omega_n}{s\,(1-s)}\, P(E)^s\, |E|^{1-s}. \tag{2.28}$$

Proof of Corollary 1.4. Since P_ϕ is invariant by rotations, we can apply Theorem 1.3 for $\nu = \mathrm{e}_n$, so that $C^{\#\mathrm{e}_n} = \mathcal{C}^h_r$ by Step 2 of the proof of Theorem 1.3. By Steps 1 and 3 of the proof of Theorem 1.3, and thanks to (2.27) in Lemma 2.6, we can estimate

$$\begin{aligned} L_K(E^{\#\mathrm{e}_n}, \mathcal{C}^h_r) &\leq L_K(E^{\#\mathrm{e}_n}, (E^{\#\mathrm{e}_n})^c) = P_K(E^{\#\mathrm{e}_n}) \\ &\leq \max\left\{\tfrac{1}{2}\, P(E^{\#\mathrm{e}_n}), |E^{\#\mathrm{e}_n}|\right\} \int_{\mathbb{R}^n} \min\{1, |x|\}\, \phi(|x|)\,\mathrm{d}x. \end{aligned}$$

The conclusion thus follows by observing that $|E^{\#\mathrm{e}_n}| = |B \cap H|$ thanks to the definition in (2.3) and also that $P(E^{\#\mathrm{e}_n}) \leq P(B \cap H)$ by the Schwartz inequality for the Euclidean perimeter (see [30, Th. 19.11]). □

Proof of Corollary 1.5. We argue as in the proof of Corollary 1.4. We just notice that

$$P_s(C^{\#e_n}) = P_s(\mathcal{C}_h^r) \geq c_{n,s}^{\mathrm{iso}}\, |\mathcal{C}_r^h|^{\frac{n-s}{n}} = c_{n,s}^{\mathrm{iso}} \left(\tfrac{\omega_{n-1}}{n}\, h\, r^{n-1}\right)^{1-\frac{s}{n}}$$

thanks to (1.10) and that

$$L_s(E^{\#e_n}, \mathcal{C}_r^h) \leq P_s(E^{\#e_n}) \leq \frac{2^{1-s}\, n\, \omega_n}{s\,(1-s)}\, P(E^{\#e_n})^s\, |E^{\#e_n}|^{1-s} \leq \frac{2^{1-s}\, n\, \omega_n}{s\,(1-s)}\, P(E)^s\, |E|^{1-s}$$

by (2.28) in Lemma 2.6, readily yielding the conclusion. □

2.4. **Proof of Proposition 1.6.** We conclude by dealing with the case $n = 1$.

Proof of Proposition 1.6. Since P_ϕ is invariant by translation, we can assume that $A = (0, |A|)$ and $B = (0, |B|)$ in $\mathbb{R}$. We now observe that, by a simple change of variables,

$$P_\phi(A) = \int_{(0,|A|)} \int_{(0,|A|)^c} \phi(|x-y|)\, \mathrm{d}x\, \mathrm{d}y = |A|^2 \int_{(0,1)} \int_{(0,1)^c} \phi(|A|\, |\xi - \eta|)\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

Therefore, by (1.13), we can estimate

$$\begin{aligned} P_\phi(B) - P_\phi(A) &= \int_{(0,1)} \int_{(0,1)^c} \Big(|B|^2 \phi(|B|\, |\xi-\eta|) - |A|^2 \phi(|A|\, |\xi - \eta|) \Big)\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &\geq \int_{(0,1)} \int_{(0,1)^c} \Big(\psi(|B|) - \psi(|A|) \Big) \phi(|\xi - \eta|)\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &= \Big(\psi(|B|) - \psi(|A|) \Big)\, P_\phi((0,1)), \end{aligned}$$

yielding the conclusion. □

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(F. Giannetti) Dipartimento di Matematica ed Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy
Email address: giannett@unina.it

(G. Stefani) Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, 34136 Trieste, Italy
Email address: gstefani@sissa.it or giorgio.stefani.math@gmail.com