SOLUTIONS OF NEUMANN PROBLEMS IN DOMAINS WITH CRACKS AND APPLICATIONS TO FRACTURE MECHANICS

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ABSTRACT. The first part of the course is devoted to the study of solutions to the Laplace equation in $\Omega \setminus K$, where Ω is a two-dimensional smooth domain and K is a compact one-dimensional subset of Ω. The solutions are required to satisfy a homogeneous Neumann boundary condition on K and a nonhomogeneous Dirichlet condition on (part of) $\partial\Omega$. The main result is the continuous dependence of the solution on K, with respect to the Hausdorff metric, provided that the number of connected components of K remains bounded. Classical examples show that the result is no longer true without this hypothesis.

Using this stability result, the second part of the course develops a rigorous mathematical formulation of a variational quasi-static model of the slow growth of brittle fractures, recently introduced by Francfort and Marigo. Starting from a discrete-time formulation, a more satisfactory continuous-time formulation is obtained, with full justification of the convergence arguments.

Keywords: stability of Neumann problems, domains with cracks, variational models, energy minimization, free-discontinuity problems, crack propagation, quasi-static evolution, brittle fractures.

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1. NEUMANN PROBLEMS IN DOMAINS WITH CRACKS

In these lectures Ω is a fixed *bounded connected open* subset of $\mathbb{R}^2$ with a *Lipschitz boundary* $\partial\Omega$, and $\mathcal{K}(\overline{\Omega})$ is the set of all compact subsets of $\overline{\Omega}$. Given $K \in \mathcal{K}(\overline{\Omega})$, we consider the differential equation $\Delta u = 0$ on $\Omega \setminus K$ with a homogeneous Neumann boundary condition on ∂K and on a part $\partial_N\Omega$ of the boundary of Ω, and with a non-homogeneous Dirichlet boundary condition on the rest of the boundary of $\Omega \setminus K$. For simplicity we assume that $\partial_N\Omega$ is a (possibly empty) relatively open subset of $\partial\Omega$, with a finite number of connected components, and we set $\partial_D\Omega := \partial\Omega \setminus \overline{\partial_N\Omega}$, which turns out to be a relatively open subset of $\partial\Omega$, with a finite number of connected components. For the applications we have in mind K will be a one-dimensional set, and will be considered as a crack in the domain Ω, but we do not need this assumption in this lecture.

If ∂K is not smooth, the variational formulation of this problems requires the *Deny-Lions space* $L^{1,2}(A)$, defined for every open set $A \subset \mathbb{R}^2$ as the space of functions $u \in L^2_{loc}(A)$ such that the distributional gradient ∇u belongs to $L^2(A;\mathbb{R}^2)$ (see [17]). The advantage of this space is that the set $\{\nabla u : u \in L^{1,2}(A)\}$ is closed in $L^2(A;\mathbb{R}^2)$ even if ∂A is irregular (for the proof we refer, e.g., to [25, Section 1.1.13]). It is well known that, if A is bounded and has Lipschitz boundary, then $L^2(A;\mathbb{R}^2)$ coincides with the usual Sobolev space $H^1(A)$ (see, e.g., [25, Corollary to Lemma 1.1.11]).

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Given $K \in \mathcal{K}(\overline{\Omega})$ and $g \in L^{1,2}(\Omega \backslash K)$, we consider the following boundary value problem:

$$
\begin{cases} \Delta u = 0 & \text{in } \Omega \backslash K\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial(\Omega \backslash K) \cap (K \cup \partial_N \Omega)\,, \\ u = g & \text{on } \partial_D \Omega \backslash K\,. \end{cases} \tag{1.1}
$$

By a solution of (1.1) we mean a function u which satisfies the following conditions:

$$
\begin{cases} u \in L^{1,2}(\Omega \backslash K)\,, \ \ u = g \ \text{ on } \ \partial_D \Omega \backslash K\,, \\ \displaystyle\int_{\Omega \backslash K} \nabla u \, \nabla z \, dx = 0 \quad \forall z \in L^{1,2}(\Omega \backslash K)\,, \ z = 0 \ \text{ on } \ \partial_D \Omega \backslash K\,, \end{cases} \tag{1.2}
$$

where the equalities on $\partial_D \Omega$ are in the sense of traces.

It is clear that problem (1.2) can be solved separately in each connected component of $\Omega \backslash K$. By the Lax-Milgram lemma there exists a unique solution in those components whose boundary meets $\partial_D \Omega \backslash K$, while on the other components the solution is given by an arbitrary constant. Thus the solution is not unique, if there is a connected component whose boundary does not meet $\partial_D \Omega \backslash K$. Note, however, that ∇u is always unique. Moreover, the map $g \mapsto \nabla u$ is linear from $L^{1,2}(\Omega \backslash K)$ into $L^2(\Omega \backslash K; \mathbb{R}^2)$ and satisfies the estimate

$$
\int_{\Omega \backslash K} |\nabla u|^2 \, dx \le \int_{\Omega \backslash K} |\nabla g|^2 \, dx\,. \tag{1.3}
$$

By standard arguments on the minimization of quadratic forms it is easy to see that u is a solution of problem (1.2) if and only if u is a solution of the minimum problem

$$
\min_v \Big\{ \int_{\Omega \backslash K} |\nabla v|^2 \, dx : v \in L^{1,2}(\Omega \backslash K)\,, \ \ v = g \ \text{ on } \ \partial_D \Omega \backslash K \Big\}\,. \tag{1.4}
$$

In these lectures, given a function $u \in L^{1,2}(\Omega \backslash K)$ for some $K \in \mathcal{K}(\overline{\Omega})$, we always extend ∇u to Ω by setting $\nabla u = 0$ a.e. on K. Note that, however, ∇u is the distributional gradient of u only in $\Omega \backslash K$, and, in general, it does not coincide in Ω with the gradient of an extension of u.

To study the continuous dependence on K of the solutions of problem (1.2) we consider the *Hausdorff distance* between two sets K_1, $K_2 \in \mathcal{K}(\overline{\Omega})$, which is defined by

$$
d_H(K_1, K_2) := \max \big\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1) \big\}\,,
$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = \operatorname{diam}(\Omega)$ if $K \neq \emptyset$. We say that (K_n) *converges to* K *in the Hausdorff metric* if $d_H(K_n, K) \to 0$. The following compactness theorem is well-known (see, e.g., [30, Blaschke's Selection Theorem]).

Theorem 1.1. *Let (K_n) be a sequence in $\mathcal{K}(\overline{\Omega})$. Then there exists a subsequence which converges in the Hausdorff metric to a set $K \in \mathcal{K}(\overline{\Omega})$.*

The following example shows that the convergence of (K_n) to K in the Hausdorff metric does not imply the convergence of the solutions of problems (1.2) relative to K_n to the solution relative to K, if we have no bound on the number of connected components of K_n. In the next lecture we will prove a convergence result under a uniform bound on the number of connected components of K_n.

Example 1.2. *Let $\Omega := (0,1) \times (-1,1)$, let $\partial_D \Omega := (0,1) \times \{-1, 1\}$, and let $g := \pm 1$ on $(0,1) \times \{\pm 1\}$. For every $n \ge 1$ let*

$$
K_n := \bigcup_{0=1}^{n-1} \Big[\frac{i}{n}, \frac{i}{n} + \frac{1}{2n} \Big] \times \{0\}\,,
$$

and let u_n be the solution of problem (1.1) relative to K_n and g. Then (K_n) converges in the Hausdorff metric to the set $K := [0,1] \times \{0\}$ and (u_n) converges in $L^2(\Omega)$ to the solution u of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial_N\Omega = \partial\Omega \backslash \overline{\partial_D\Omega}\,, \\ u = g & \text{on } \partial_D\Omega\,. \end{cases} \tag{1.5}$$

Since this solution is given explicitly by $u(x_1,x_2) := x_2$, we see that it does not satisfy the Neumann boundary condition $\partial u/\partial\nu = 0$ on K.

Proof. Let $\Omega^\pm := \{x \in \Omega : \pm x_2 > 0\}$. By (1.3) we have $\int_{\Omega^\pm} |\nabla u_n|^2 dx \le \int_\Omega |\nabla g|^2 dx$ for every n, so that, passing to a subsequence, we may assume that (u_n) converges weakly in $H^1(\Omega^+ \cup \Omega^-)$ to a function $w \in H^1(\Omega^+ \cup \Omega^-)$ such that $w = g$ on $\partial_D\Omega$. By symmetry the traces $u_n^\pm$ of u_n from $\Omega^\pm$ vanish on $K\backslash K_n$. Let χ_n be the characteristic function of $K\backslash K_n$. From the definition of K_n it follows that (χ_n) converges to $1/2$ weakly in $L^2(K,\mathcal{H}^1)$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Since the trace operator is compact from $H^1(\Omega^\pm)$ into $L^2(K,\mathcal{H}^1)$, the traces $u_n^\pm$ of u_n from $\Omega^\pm$ converge to the corresponding traces $w^\pm$ of w strongly in $L^2(K,\mathcal{H}^1)$. Therefore $(u_n^\pm\chi_n)$ converges to $w^\pm/2$ weakly in $L^1(K,\mathcal{H}^1)$. As $u_n^\pm\chi_n = 0$ on K (recall that $u_n^\pm = 0$ on $K\backslash K_n$ and $\chi_n = 0$ on K_n), we conclude that $w^\pm/2 = 0$ on K, therefore $w \in H^1(\Omega)$. By using the weak formulation (1.2) we obtain

$$\int_{\Omega\backslash K_n} \nabla u_n \, \nabla z \, dx = 0 \qquad \forall z \in H^1(\Omega)\,,\ \ z = 0 \text{ on } \partial_D\Omega\,.$$

Passing to the limit as $n \to \infty$ we obtain

$$\int_{\Omega} \nabla w \, \nabla z \, dx = 0 \qquad \forall z \in H^1(\Omega)\,,\ \ z = 0 \text{ on } \partial_D\Omega\,.$$

This implies that w coincides with the solution of problem (1.5). By uniqueness, the whole sequence (u_n) converges to u. □

Remark 1.3. The hypothesis $g := \pm 1$ on $[0,1] \times \{\pm 1\}$ was intoduced only to simplify the proof. Indeed the same result holds when g is an arbitrary function of $H^1(\Omega)$. To prove this fact, we can not use the equality $u_n = 0$ on $K\backslash K_n$, which is not true in the general case; instead we introduce the functions $\tilde u_n(x_1,x_2) := u_n(x_1,x_2) - u_n(x_1,-x_2)$ and $\tilde w(x_1,x_2) := w(x_1,x_2) - w(x_1,-x_2)$, and observe that $(\tilde u_n)$ converges to $\tilde w$ weakly in $H^1(\Omega^+ \cup \Omega^-)$. Since the traces of $\tilde u_n$ from $\Omega^\pm$ vanish on $K\backslash K_n$, arguing as before we obtain that the traces of $\tilde w$ vanish on K. This implies that $w^+ = w^-$ on K, and hence $w \in H^1(\Omega)$. The conclusion follows now as as in the previous proof.

In some cases the limit problem can contain a transmission condition, as shown by the following example, for which we refer to [16] and [27]. Note that in this case the one-dimensional measure of K_n converges to the one dimensional measure of K.

Example 1.4. *Let $\Omega := (0,1) \times (-1,1)$, let $\partial_D\Omega := (0,1) \times \{-1,1\}$, and let $g \in H^1(\Omega)$. For every $n \ge 1$ let*

$$K_n := \bigcup_{i=0}^{n-1} \Big[\frac{i}{n}, \frac{i+1}{n} - e^{-n}\Big] \times \{0\}\,,$$

and let u_n be the solution of problem (1.1) relative to K_n. Then (K_n) converges in the Hausdorff metric to the set $K := [0,1] \times \{0\}$ and (u_n) converges in $L^2(\Omega)$ to the solution

u of the problem

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega\setminus K\,, \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\setminus(\overline{\partial_D\Omega}\cup K)\,, \\
u = g & \text{on } \partial_D\Omega\,, \\
\dfrac{\partial u^\pm}{\partial \nu^\pm} = \pm\dfrac{\pi}{2}(u^- - u^+) & \text{on } K\,,
\end{cases}
\tag{1.6}
$$

where $u^\pm$ is the restriction of u to $\Omega^\pm := \{x\in\Omega : \pm x_2 > 0\}$ and $\nu^\pm$ is the outer unit normal to $\Omega^\pm$.

In the literature of homogenization theory one can find other examples where the convergence in the Hausdorff metric of K_n to K does not imply the convergence of the solutions of the Neumann problems on $\Omega\setminus K_n$ to the solution of the Neumann problem on $\Omega\setminus K$ (see, e.g., [23], [1], and [14]). These papers show also that the bound on the number of connected components of K_n, that we shall consider in the next lecture, would not be enough in dimension larger than two.

The one-dimensional Hausdorff measure $\mathcal{H}^1$ is not lower semicontinuous with respect to the convergence in the Hausdorff metric. For instance, in Example 1.2 we have $\mathcal{H}^1(K_n) = 1/2$ for every n, while for the limit set we have $\mathcal{H}^1(K) = 1$. However the lower semicontinuity holds if we have a uniform bound on the number of connected components, as shown by the following theorem.

Theorem 1.5. *Let (K_n) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to K in the Hausdorff metric. Assume that each set K_n has at most m connected components. Then K has at most m connected components and*

$$\mathcal{H}^1(K\cap U) \le \liminf_{n\to\infty} \mathcal{H}^1(K_n\cap U)$$

for every open set $U\subset\mathbb{R}^2$.

Proof. The case $m = 1$ and $U = \mathbb{R}^2$ is the Gołąb theorem, for which we refer to [19, Theorem 3.18]. For an independent proof for an arbitrary U see [26, Theorem 10.19]. The case $m > 1$ is an easy consequence of the case $m = 1$ (see [15, Corollary 3.3]). □

2. CONVERGENCE OF SOLUTIONS

Given an integer $m\ge 1$ and a constant $\lambda \ge 0$, let $\mathcal{K}_m^\lambda(\overline{\Omega})$ be the set of all compact subsets K of $\overline{\Omega}$, with $\mathcal{H}^1(K)\le\lambda$, having at most m connected components.

In this lecture we give the main ideas of the proof of the following theorem. The complete proof can be found in [15, Section 5].

Theorem 2.1. *Let $m\ge 1$ and $\lambda\ge 0$, let (K_n) be a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, and let (g_n) be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_v \Big\{ \int_{\Omega\setminus K_n} |\nabla v|^2\,dx : v\in L^{1,2}(\Omega\setminus K_n)\,,\ \ v = g_n \ \text{ on } \partial_D\Omega\setminus K_n \Big\}\,, \tag{2.1}$$

and let u be a solution of the minimum problem

$$\min_v \Big\{ \int_{\Omega\setminus K} |\nabla v|^2\,dx : v\in L^{1,2}(\Omega\setminus K)\,,\ \ v = g \ \text{ on } \partial_D\Omega\setminus K \Big\}\,. \tag{2.2}$$

Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

This result is related to those obtained by A. Chambolle and F. Doveri in [12] and by D. Bucur and N. Varchon in [8], [9], and [10], which deal with the case of a pure Neumann boundary condition. Since we impose a Dirichlet boundary condition on $\partial_D\Omega\setminus K_n$ and a Neumann boundary condition on the rest of the boundary, our results can not be deduced easily from these papers, so we give an independent proof, which uses the duality argument which appears also in [10].

To focus on the main ideas of the proof, we consider only the case $m = 1$ and Ω simply connected. Moreover we assume that $K \cap \partial\Omega = \emptyset$, to avoid minor difficulties arising at the boundary. The technicalities needed to avoid these simplifying hypotheses can be found in [15].

First of all, we want to construct the harmonic conjugate of u_n. Let R be the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$.

Definition 2.2. *We say that a function $v \in H^1(\Omega)$ is equal to a constant c on a set $K \in \mathcal{K}(\overline{\Omega})$ if there exists a sequence (v_n) in $C^1(\overline{\Omega})$ converging to v strongly in $H^1(\Omega)$ and such that each v_n is equal c in a neighbourhood of K.*

Remark 2.3. It is possible to prove that, if

$$\lim_{r\to 0} \operatorname*{ess\,sup}_{B_r(x)} |v - c| = 0 \qquad \forall x \in K\,,$$

then $v = c$ on K in the sense of the previous definition (it is enough to adapt the proof of [7, Théorème IX.17] or to apply [22, Theorem 4.5]).

Theorem 2.4. *Let K be a connected compact set contained in Ω and let u be a solution of problem (1.2). Then there exists a function $v \in H^1(\Omega)$ such that $\nabla v = R\nabla u$ a.e. on Ω. Moreover v is constant on K and on each connected component of $\partial_N\Omega$ (according to Definition 2.2).*

Proof. If $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \nabla u\, \nabla\varphi\, dx = \int_{\Omega\setminus K} \nabla u\, \nabla\varphi\, dx = 0\,, \tag{2.3}$$

where the first equality follows from our convention $\nabla u = 0$ a.e. in K, while the second one follows from (1.2). Equality (2.3) means that $\operatorname{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$, hence $\operatorname{rot}(R\nabla u) = 0$ in $\mathcal{D}'(\Omega)$. As Ω is simply connected and has a Lipschitz boundary, there exists $v \in H^1(\Omega)$ such that $\nabla v = R\nabla u$ a.e. in Ω.

Since $\partial u/\partial\nu = 0$ on $\partial_N\Omega$, the tangential derivative of v (which is equal to the normal derivative of u) vanishes on $\partial_N\Omega$, and this implies that v is constant on each connected component of $\partial_N\Omega$.

If K has a non-empty interior and a smooth boundary, then v is constant a.e. on the interior of K, since $\nabla v = 0$ a.e. on K. Therefore v is constant on K according to Definition 2.2.

The case of a general $K \in \mathcal{K}(\overline{\Omega})$ can be obtained by approximating K by a decreasing sequence of compact sets with non-empty interior and a smooth boundary (we refer to [15, Theorem 4.2] for the details). □

Proof of Theorem 2.1. Note that u is a minimum point of (2.2) if and only if it satisfies (1.2); similarly, u_n is a minimum point of (2.1) if and only if it satisfies (1.2) with K and g replaced by K_n and g_n.

Taking $v := g_n$ in the functional to be minimized, we obtain that the sequence (∇u_n) is bounded in $L^2(\Omega;\mathbb{R}^2)$. Therefore, passing to a subsequence, (∇u_n) converges weakly in $L^2(\Omega;\mathbb{R}^2)$ to a function ψ. As $K \in \mathcal{K}_m^\lambda(\overline{\Omega})$ by Theorem 1.5, and hence $\operatorname{meas}(K) = 0$, is easy to see that there exists a function $u^* \in L^{1,2}(\Omega\setminus K)$, with $u^* = g$ on $\partial_D\Omega$, such that $\psi = \nabla u^*$ a.e. on Ω (we are assuming here that $K \cap \partial_D\Omega = \emptyset$, see [15, Lemma 4.1] for the details).

We will prove that

$$\nabla u^* = \nabla u \ \text{ a.e. in } \Omega \backslash K\,. \tag{2.4}$$

As the limit does not depend on the subsequence, this implies that the whole sequence (∇u_n) converges to ∇u weakly in $L^2(\Omega;\mathbb{R}^2)$. Taking $u_n - g_n$ and $u - g$ as test functions in the equations satisfied by u_n and u, we obtain

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega \nabla u_n \nabla g_n \, dx\,, \qquad \int_\Omega |\nabla u|^2 dx = \int_\Omega \nabla u \, \nabla g \, dx\,.$$

As $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$ and $\nabla g_n \to \nabla g$ strongly in $L^2(\Omega;\mathbb{R}^2)$, from the previous equalities we obtain that $\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^2)}$ converges to $\|\nabla u\|_{L^2(\Omega;\mathbb{R}^2)}$, which implies the strong convergence of the gradients in $L^2(\Omega;\mathbb{R}^2)$.

By the uniqueness of the gradients of the solutions, to prove (2.4) it is enough to show that u^* is a solution of (1.2). This will be done by considering, for each u_n, its harmonic conjugate v_n given by Theorem 2.4. By adding a suitable constant, we may assume that $\int_\Omega v_n \, dx = 0$ for every n. Since $\nabla v_n = R\,\nabla u_n$ a.e. on Ω, we deduce that (∇v_n) converges to $R\,\nabla u^*$ weakly in $L^2(\Omega;\mathbb{R}^2)$, and by the Poincaré inequality (v_n) converges weakly in $H^1(\Omega)$ to a function v which satisfies $\nabla v = R\,\nabla u^*$ a.e. on Ω.

Let us prove that v is constant on K according to Definition 2.2. This is trivial if K reduces to a point. If K has more than one point, then $\liminf_n \operatorname{diam}(K_n) > 0$; since the sets K_n are connected, we obtain also $\liminf_n \operatorname{cap}(K_n,\Omega) > 0$, where the capacity $\operatorname{cap}(K_n,\Omega)$ of K_n with respect to Ω is defined by

$$\operatorname{cap}(K_n,\Omega) := \inf_z \Big\{ \int_\Omega |\nabla z|^2 dx : z \in C^1_c(\Omega)\,, \ \ z \geq 1 \ \text{ on } \ K_n \Big\}\,.$$

As $v_n = c_n$ on K_n for suitable constants c_n (see Theorem 2.4), using the Poincaré inequality (see, e.g., [33, Corollary 4.5.3]) it follows that $(v_n - c_n)$ is bounded in $H^1(\Omega)$, hence the sequence (c_n) is bounded, and therefore, passing to a subsequence, we may assume that (c_n) converges to a suitable constant c.

Let us fix a constant $R > 0$ with $R < \operatorname{diam}(K)/2$ and $R < \operatorname{dist}(K,\partial\Omega)$. Since $\Delta v_n = 0$ on $\Omega \backslash K_n$, by Maz'ya's estimate (see [24, Theorem 1]) there exist two constants $M > 0$ and $\beta > 0$, independent of n, x_n, and r, such that for every $x_n \in K_n$ and every $r \in (0,R)$

$$\operatorname*{ess\,sup}_{B_r(x_n)} |v_n - c_n| \leq M \exp\Big(-\beta \int_r^R \gamma_n(x_n,\rho)\, \frac{d\rho}{\rho} \Big)\,, \tag{2.5}$$

where $\gamma_n(x_n,\rho) := \operatorname{cap}(K_n \cap B_\rho(x_n), B_{2\rho}(x_n))$. For n large we have $\operatorname{diam}(K_n) > 2R$, so that for every $x_n \in K_n$ and every $\rho \in (0,R)$ we have $K_n \cap \partial B_\rho(x_n) \neq \emptyset$. As K_n is connected, there exists a constant $\alpha > 0$, independent of n, x_n, and ρ, such that $\gamma_n(x_n,\rho) = \operatorname{cap}(K_n \cap B_\rho(x_n), B_{2\rho}(x_n)) \geq \alpha$. Therefore (2.5) yields

$$\operatorname*{ess\,sup}_{B_r(x_n)} |v_n - c| \leq M \Big(\frac{r}{R}\Big)^{\alpha\beta} + |c_n - c|$$

for every $x_n \in K_n$ and every $r \in (0,R)$.

Let us fix $x \in K$ and a sequence $x_n \in K_n$ converging to x. For every $\rho \in (0,r)$ we have $B_\rho(x) \subset B_r(x_n)$ for n large enough, hence

$$\operatorname*{ess\,sup}_{B_\rho(x)} |v_n - c| \leq M \Big(\frac{r}{R}\Big)^{\alpha\beta} + |c_n - c|$$

for n large enough. Passing to the limit first as $n \to \infty$ and then as $\rho \to r$ we get

$$\operatorname*{ess\,sup}_{B_r(x)} |v - c| \leq M \Big(\frac{r}{R}\Big)^{\alpha\beta}\,. \tag{2.6}$$

As $r \to 0$ we obtain that v is equal to c on K (see Remark 2.3).

On the other hand, every v_n is constant on each connected component of $\partial_N\Omega$ (see Theorem 2.4). Since $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, we conclude that v is constant on each connected component of $\partial_N\Omega$.

Therefore there exists a sequence (w_n) in $C^1(\overline{\Omega})$ converging to v strongly in $H^1(\Omega)$, such that each w_n is constant in a neighbourhood of K and in a neighbourhood of each connected component of $\partial_N\Omega$.

Let $z \in L^{1,2}(\Omega\backslash K)$ with $z = 0$ on $\partial_D\Omega$., and let $\varphi_n \in C^1(\overline{\Omega})$ with $\varphi_n = 1$ on $\mathrm{supp}(\nabla w_n)$ and $\varphi_n = 0$ on a neighbourhood of $K \cup \partial_N\Omega$. As $\mathrm{div}(R\nabla w_n) = 0$ in Ω and $z\,\varphi_n \in H^1_0(\Omega)$, we have

$$\int_{\Omega\backslash K} R\,\nabla w_n\,\nabla z\,dx = \int_{\Omega\backslash K} R\,\nabla w_n\,\nabla(z\,\varphi_n)\,dx = 0\,.$$

Since $R\,\nabla u^* = \nabla v$ a.e. on Ω, passing to the limit as $n \to \infty$ we obtain

$$\int_{\Omega\backslash K} \nabla u^*\,\nabla z\,dx = -\int_{\Omega\backslash K} R\,\nabla v\,\nabla z\,dx = 0\,,$$

which shows that u^* is a solution of (1.2). □

3. A QUASI-STATIC MODEL FOR BRITTLE FRACTURES

Since the pioneering work of A. Griffith [21], the growth of a brittle fracture is considered to be the result of the competition between the energy spent to increase the crack and the corresponding release of bulk energy. This idea is the basis of the celebrated Griffith's criterion for crack growth (see, e.g., [31]), and is used to study the crack propagation along a preassigned path. The actual path followed by the crack is often determined by using different criteria (see, e.g., [18], [31], [32]).

Recently G.A. Francfort and J.-J. Marigo [20] proposed a variational model for the quasi-static growth of brittle fractures, based on Griffith's theory, where the interplay between bulk and surface energy determines also the crack path.

The purpose of this and of the next lecture is to give a precise mathematical formulation of a variant of this model in the *two-dimensional case*, and to prove an existence result for the *quasi-static evolution of a fracture* by using the *time discretization method* proposed in [20].

To simplify the mathematical description of the model, we consider only *linearly elastic homogeneous isotropic materials*, with Lamé coefficients λ and μ. We restrict our analysis to the case of an *anti-plane shear*, where the reference configuration is an infinite cylinder $\Omega\times\mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, and the displacement has the special form $(0, 0, u(x_1, x_2))$ for every $(x_1, x_2, y) \in \Omega\times\mathbb{R}$. We assume also that the cracks have the form $K\times\mathbb{R}$, where K is a compact set in $\overline{\Omega}$. In this case the notions of bulk energy and surface energy refer to a finite portion of the cylinder determined by two cross sections separated by a unit distance. The *bulk energy* is given by

$$\frac{\mu}{2}\int_{\Omega\backslash K} |\nabla u|^2 dx\,, \tag{3.1}$$

while the *surface energy* is given by

$$k\,\mathcal{H}^1(K)\,, \tag{3.2}$$

where k is a constant which depends on the toughness of the material, and $\mathcal{H}^1$ is the *one-dimensional Hausdorff measure*, which coincides with the ordinary length in case K is a rectifiable arc. For simplicity we take $\mu = 2$ and $k = 1$ in (3.1) and (3.2).

As in the previous lectures Ω is a fixed *bounded connected open* subset of $\mathbb{R}^2$ with *Lipschitz boundary*. Following [20], we fix an open subset $\partial_D\Omega$ of $\partial\Omega$, on which we want to prescribe a *Dirichlet boundary condition* for the displacement u. As in the previous lectures we assume that $\partial_D\Omega$ has a *finite number of connected components*.

Given a function g on $\partial_D\Omega$, we consider the boundary condition $u = g$ on $\partial_D\Omega\backslash K$. We can not prescribe a Dirichlet boundary condition on $\partial_D\Omega \cap K$, because the boundary displacement is not transmitted through the crack, if the crack touches the boundary. Assuming that *the fracture is traction free* (and, in particular, without friction), the displacement u in $\Omega\backslash K$ is obtained by *minimizing (3.1) under the boundary condition $u = g$ on $\partial_D\Omega\backslash K$*. The *total energy* relative to the boundary displacement g and to the crack determined by K is therefore

$$\mathcal{E}(g, K) = \min_v \Big\{ \int_{\Omega\backslash K} |\nabla v|^2 dx + \mathcal{H}^1(K) : v \in L^{1,2}(\Omega),\ v = g \text{ on } \partial_D\Omega\backslash K \Big\}. \tag{3.3}$$

The existence of a minimizer has been proved in the first lecture.

In the theory developed in [20] a crack with finite surface energy is any compact subset K of $\overline{\Omega}$ with $\mathcal{H}^1(K) < +\infty$. For technical reasons, due to the hypotheses of Theorem 2.1, we propose a variant of this model, where we prescribe an a priori bound on the number of connected components of the cracks. Without this restriction, some convergence arguments used in the proof of our existence result are not justified by the present development of the mathematical theories related to this subject. Given an integer $m \geq 1$, let $\mathcal{K}^f_m(\overline{\Omega})$ be the set of all compact subsets K of $\overline{\Omega}$, with $\mathcal{H}^1(K) < +\infty$, having at most m connected components.

We begin by describing a discrete-time model of *quasi-static irreversible evolution of a fracture* under the action of a *time dependent boundary displacement* $g(t)$, $0 \leq t \leq 1$. As usual, we assume that $g(t)$ can be extended to a function, still denoted by $g(t)$, which belongs to the Sobolev space $H^1(\Omega)$. For simplicity, we assume also that $g(0) = 0$.

Given a time step $\delta > 0$, for every integer $i \geq 0$ we set $t^\delta_i := i\delta$ and $g^\delta_i := g(t^\delta_i)$. The fracture K^δ_i at time t^δ_i is defined inductively in the following way: for $i = 0$ we set $K^\delta_0 := K_0$, while for $i \geq 1$ K^δ_i is any minimizer of the problem

$$\min_K \big\{ \mathcal{E}(g^\delta_i, K) : K \in \mathcal{K}^f_m(\overline{\Omega}),\ K \supset K^\delta_{i-1} \big\}. \tag{3.4}$$

Lemma 3.1. *There exists a solution of the minimum problem (3.4).*

Proof. By definition $K^\delta_0 := K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Assume by induction that $K^\delta_{i-1} \in \mathcal{K}^f_m(\overline{\Omega})$ and let λ be a constant such that $\lambda > \mathcal{E}(g^\delta_i, K^\delta_{i-1})$. Consider a minimizing sequence (K_n) of problem (3.4). We may assume that $K_n \in \mathcal{K}^\lambda_m(\overline{\Omega})$ for every n. By the Compactness Theorem 1.1, passing to a subsequence, we may assume that (K_n) converges in the Hausdorff metric to some compact set K containing K^δ_{i-1}. For every n let u_n be a solution of the minimum problem (3.3) which defines $\mathcal{E}(g^\delta_i, K_n)$. By Theorem 2.1 (∇u_n) converges strongly in $L^2(\Omega;\mathbb{R}^2)$ to ∇u, where u is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g^\delta_i, K)$. By Theorem 1.5 we have $K \in \mathcal{K}_m(\overline{\Omega})$ and $\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(K_n) \leq \lambda$, hence $K \in \mathcal{K}^\lambda_m(\overline{\Omega})$. As $\|\nabla u\| = \lim_n \|\nabla u_n\|$, we conclude that $\mathcal{E}(g^\delta_i, K) \leq \liminf_n \mathcal{E}(g^\delta_i, K_n)$. Since (K_n) is a minimizing sequence, this proves that K is a solution of the minimum problem (3.4). $\square$

Let u^δ_i be a solution of the minimum problem (3.3) which defines $\mathcal{E}(g^\delta_i, K^\delta_i)$. Then the pair (u^δ_i, K^δ_i) minimizes the sum of the bulk and surface energy among all $K \in \mathcal{K}^f_m(\overline{\Omega})$ with $K \supset K^\delta_{i-1}$ and among all $u \in L^{1,2}(\Omega\backslash K)$ with $u = g$ on $\partial_D\Omega\backslash K$.

In order to pass to a continuous-time model, given $T > 0$ we define the step functions $K_\delta \colon [0, T] \to \mathcal{K}^f_m(\overline{\Omega})$ and $u_\delta \colon [0, T] \to L^2_{\mathrm{loc}}(\Omega)$ by setting

$$K_\delta(t) := K^\delta_i \qquad \text{and} \qquad u_\delta(t) := u^\delta_i \qquad \text{for } t^\delta_i \leq t < t^\delta_{i+1}. \tag{3.5}$$

Our purpose is to pass to the limit as $\delta \to 0$. To this aim we use the following result, which is the analogue of the Helly theorem for compact valued increasing functions.

Theorem 3.2. *Let (K_n) be a sequence of increasing functions from $[0, T]$ into $\mathcal{K}(\overline{\Omega})$, i.e., $K_n(s) \subset K_n(t)$ for $0 \leq s < t \leq T$. Then there exist a subsequence, still denoted by (K_n),* *and an increasing function* $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$*, such that* $K_n(t) \to K(t)$ *in the Hausdorff metric for every* $t \in [0,T]$*.*

To prove Theorem 3.2 we use the following result, which extends another well known property of real valued monotone functions.

Lemma 3.3. *Let* K_1*,* $K_2\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ *be two increasing functions such that*

$$K_1(s) \subset K_2(t) \qquad \text{and} \qquad K_2(s) \subset K_1(t) \tag{3.6}$$

for every s*,* $t \in [0,T]$ *with* $s < t$*. Let* Θ *be the set of points* $t \in [0,T]$ *such that* $K_1(t) = K_2(t)$*. Then* $[0,T] \backslash \Theta$ *is at most countable.*

Proof. For $i = 1,2$, consider the functions $f_i\colon \overline{\Omega} \times [0,T] \to \mathbb{R}$ defined by $f_i(x,t) := \operatorname{dist}(x, K_i(t))$, with the convention that $\operatorname{dist}(x,\emptyset) = \operatorname{diam}(\Omega)$. Then the functions $f_i(\cdot,t)$ are Lipschitz continuous with constant 1 for every $t \in [0,T]$, and the functions $f_i(x,\cdot)$ are non-increasing for every $x \in \overline{\Omega}$.

Let D be a countable dense subset of $\overline{\Omega}$. For every $x \in D$ there exists a countable set $N_x \subset [0,T]$ such that $f_i(x,\cdot)$ are continuous at every point of $[0,T] \backslash N_x$. By (3.6) we have $f_1(x,s) \geq f_2(x,t)$ and $f_2(x,s) \geq f_1(x,t)$ for every $x \in \overline{\Omega}$ and every s, $t \in [0,T]$ with $s < t$. This implies that $f_1(x,t) = f_2(x,t)$ for every $x \in D$ and every $t \in [0,T] \backslash N_x$. Let N be the countable set defined by $N := \bigcup_{x\in D} N_x$, and let $t \in [0,T] \backslash N$. Then $f_1(x,t) = f_2(x,t)$ for every $x \in D$, and, by continuity, for every $x \in \overline{\Omega}$, which yields $K_1(t) = K_2(t)$. This proves that $[0,T] \backslash N \subset \Theta$, hence $[0,T] \backslash \Theta \subset N$. □

Proof of Theorem 3.2. Let D be a countable dense subset of $(0,T)$. Using the Compactness Theorem 1.1 and a diagonal argument, we find a subsequence, still denoted by (K_n), and an increasing function $K\colon D \to \mathcal{K}(\overline{\Omega})$, such that $K_n(t) \to K(t)$ in the Hausdorff metric for every $t \in D$. Let $K^-\colon (0,T] \to \mathcal{K}(\overline{\Omega})$ and $K^+\colon [0,T) \to \mathcal{K}(\overline{\Omega})$ be the increasing functions defined by

$$K^-(t) := \operatorname{cl}\Big(\bigcup_{s<t,\ s\in D} K(s) \Big) \qquad \text{for } 0 < t \leq T\,,$$

$$K^+(t) := \bigcap_{s>t,\ s\in D} K(s) \qquad \text{for } 0 \leq t < T\,,$$

where cl denotes the closure. Let Θ be the set of points $t \in [0,T]$ such that $K^-(t) = K^+(t)$. As K^- and K^+ satisfy (3.6), by Lemma 3.3 the set $[0,T] \backslash \Theta$ is at most countable.

Since $K^-(t) \subset K(t) \subset K^+(t)$ for every $t \in D$, we have $K(t) = K^-(t) = K^+(t)$ for every $t \in \Theta \cap D$. For every $t \in \Theta \backslash D$ we define $K(t) := K^-(t) = K^+(t)$. To prove that $K_n(t) \to K(t)$ for a given $t \in \Theta \backslash D$, by the Compactness Theorem 1.1 we may assume that $K_n(t)$ converges in the Hausdorff metric to a set K^*. For every s_1, $s_2 \in D$, with $s_1 < t < s_2$, by monotonicity we have $K(s_1) \subset K^* \subset K(s_2)$. As K^* is closed, this implies $K^-(t) \subset K^* \subset K^+(t)$, therefore $K_n(t) \to K(t)$ by the definitions of Θ and $K(t)$.

Since $[0,T] \backslash (\Theta \cup D)$ is at most countable, by a diagonal argument we find a further subsequence, still denoted by (K_n), and a function $K\colon [0,T] \backslash (\Theta \cup D) \to \mathcal{K}(\overline{\Omega})$, such that $K_n(t) \to K(t)$ in the Hausdorff metric for every $t \in [0,T] \backslash (\Theta \cup D)$.

Therefore $K_n(t) \to K(t)$ for every $t \in [0,T]$, and this implies that $K(\cdot)$ is increasing on $[0,T]$. □

The following result on the continuity of compact valued increasing maps will be used in the next lecture. Its proof is similar to the proof of Theorem 3.2.

Proposition 3.4. *Let* $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ *be an increasing function, and let* $K^-\colon (0,T] \to \mathcal{K}(\overline{\Omega})$ *and* $K^+\colon [0,T) \to \mathcal{K}(\overline{\Omega})$ *be the functions defined by*

$$K^-(t) := \operatorname{cl}\big(\textstyle\bigcup_{s<t} K(s) \big) \qquad \text{for } 0 < t \leq T\,, \tag{3.7}$$

$$K^+(t) := \textstyle\bigcap_{s>t} K(s) \qquad \text{for } 0 \leq t < T\,, \tag{3.8}$$

where cl *denotes the closure. Then*

$$K^-(t) \subset K(t) \subset K^+(t) \qquad \text{for } 0 < t < T\,. \tag{3.9}$$

Let Θ *be the set of points* $t \in (0,T)$ *such that* $K^-(t) = K^+(t)$. *Then* $[0,T]\backslash\Theta$ *is at most countable, and* $K(t_n) \to K(t)$ *in the Hausdorf metric for every* $t \in \Theta$ *and every sequence* (t_n) *in* $[0,T]$ *converging to* t.

Proof. It is clear that $K^+(\cdot)$ and $K^-(\cdot)$ are increasing and satisfy (3.6). Therefore $[0,T]\backslash\Theta$ is at most countable by Lemma 3.3.

Let us fix $t \in \Theta$ and a sequence (t_n) in $[0,T]$ converging to t. By the Compactness Theorem 1.1 we may assume that $K(t_n)$ converges in the Hausdorff metric to a set K^*. For every $s_1,\, s_2 \in [0,T]$, with $s_1 < t < s_2$, we have $K(s_1) \subset K(t_n) \subset K(s_2)$ for n large enough, hence $K(s_1) \subset K^* \subset K(s_2)$. As K^* is closed this implies $K^-(t) \subset K^* \subset K^+(t)$, therefore $K^* = K(t)$ by (3.9) and by the definition of Θ. ☐

According to Theorem 3.2, there exist a sequence (δ_k) converging to 0 and an increasing function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ such that, for every $t \in [0,T]$, $K_\delta(t) \to K(t)$ in the Hausdorff metric as δ tends to 0 along this sequence. In the next lecture we will prove the main properties of the function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ obtained in this way, which represents the continuous-time evolution of the fracture. To simplify the notation, when we write $\delta \to 0$ we always mean that δ tends to 0 along the sequence (δ_k) considered above.

4. PROPERTIES OF THE CONTINUOUS-TIME MODEL

In this final lecture we prove some properties of the function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ defined as the limit, as $\delta \to 0$, of the functions $K_\delta\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ given by the discrete-time model. This function represents the quasi-static evolution of the crack in our continuous-time model.

To prove these results, we assume that the function $t \mapsto g(t)$, which gives the imposed boundary displacement on $\partial_D\Omega$, is *absolutely continuous* from $[0,T]$ into $H^1(\Omega)$. Its time derivative is a Bochner integrable function from $[0,T]$ into $H^1(\Omega)$, which will be denoted by $\dot g(t)$. For the main properties of absolutely continuous functions with values in a Hilbert space we refer, e.g., to [6, Appendix].

We begin with a crucial estimate for the solutions of the discrete-time problems. Here and in the rest of the lecture $(\cdot|\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\Omega;\mathbb{R}^2)$.

Lemma 4.1. *There exists a positive function* $\omega(\delta)$, *converging to zero as* $\delta \to 0$, *such that*

$$\|\nabla u_j^\delta\|^2 + \mathcal{H}^1(K_j^\delta) \le \|\nabla u_i^\delta\|^2 + \mathcal{H}^1(K_i^\delta) + 2\int_{t_i^\delta}^{t_j^\delta} (\nabla u_\delta(t)|\nabla \dot g(t))\, dt + \omega(\delta) \tag{4.1}$$

for $0 \le i < j$ *with* $t_j^\delta \le T$.

Proof. Let us fix an integer r with $i \le r < j$. From the absolute continuity of g we have

$$g_{r+1}^\delta - g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \dot g(t)\, dt\,,$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$. This implies that

$$\nabla g_{r+1}^\delta - \nabla g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \nabla \dot g(t)\, dt\,, \tag{4.2}$$

where the integral is a Bochner integral for functions with values in $L^2(\Omega;\mathbb{R}^2)$.

As $u_r^\delta + g_{r+1}^\delta - g_r^\delta \in L^{1,2}(\Omega\backslash K_r^\delta)$ and $u_r^\delta + g_{r+1}^\delta - g_r^\delta = g_{r+1}^\delta$ on $\partial_D\Omega\backslash K_r^\delta$, we have

$$\mathcal{E}(g_{r+1}^\delta, K_r^\delta) \le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta)\,. \tag{4.3}$$

By the minimality of u^δ_{r+1} and by (3.4) we have

$$\|\nabla u^\delta_{r+1}\|^2 + \mathcal{H}^1(K^\delta_{r+1}) = \mathcal{E}(g^\delta_{r+1}, K^\delta_{r+1}) \le \mathcal{E}(g^\delta_{r+1}, K^\delta_r)\,. \tag{4.4}$$

From (4.2), (4.3), and (4.4) we obtain

$$\begin{aligned}\|\nabla u^\delta_{r+1}\|^2 + \mathcal{H}^1(K^\delta_{r+1}) &\le \|\nabla u^\delta_r + \nabla g^\delta_{r+1} - \nabla g^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) \le \\ &\le \|\nabla u^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) + 2\int_{t^\delta_r}^{t^\delta_{r+1}} (\nabla u^\delta_r|\nabla \dot g(t))\,dt + \Big(\int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(t)\|\,dt\Big)^2 \le \\ &\le \|\nabla u^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) + 2\int_{t^\delta_r}^{t^\delta_{r+1}} (\nabla u_\delta(t)|\nabla \dot g(t))\,dt + \sigma(\delta)\int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(t)\|\,dt\,,\end{aligned}$$

where

$$\sigma(\delta) := \max_{0\le r,\, t^\delta_r < T} \int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(t)\|\,dt \longrightarrow 0$$

by the absolute continuity of the integral. Iterating now this inequality for $i \le r < j$ we get (4.1) with $\omega(\delta) := \sigma(\delta)\int_0^1 \|\nabla \dot g(t)\|\,dt$. □

Lemma 4.2. *There exists a constant λ, depending only on g and K_0, such that*

$$\|\nabla u^\delta_i\| \le \lambda \quad \textit{and} \quad \mathcal{H}^1(K^\delta_i) \le \lambda \tag{4.5}$$

for every $\delta > 0$ and for every $i \ge 0$ with $t^\delta_i \le T$.

Proof. As $v := g^\delta_i$ is admissible for the problem (3.3) which defines $\mathcal{E}(g^\delta_i, K^\delta_i)$, by the minimality of u^δ_i we have $\|\nabla u^\delta_i\| \le \|\nabla g^\delta_i\|$, hence $\|\nabla u_\delta(t)\| \le \|\nabla g_\delta(t)\|$ for every $t \in [0,T]$. As $t \mapsto g(t)$ is absolutely continuous with values in $H^1(\Omega)$, the function $t \mapsto \|\nabla \dot g(t)\|$ is integrable on $[0,T]$ and there exists a constant $C > 0$ such that $\|\nabla g(t)\| \le C$ for every $t \in [0,T]$. This implies the former inequality in (4.5). The latter inequality follows now from Lemma 4.1 and from the inequality $\|\nabla u^\delta_0\|^2 + \mathcal{H}^1(K^\delta_0) \le \mathcal{H}^1(K_0)$, which is an obvious consequence of the minimality of u^δ_0 and of the fact that $g^\delta_0 = g(0) = 0$. □

Lemma 4.3. *Let λ be the constant in Lemma 4.2. Then $K_\delta(t) \in \mathcal{K}^\lambda_m(\overline\Omega)$ and $K(t) \in \mathcal{K}^\lambda_m(\overline\Omega)$ for every $t \in [0,T]$.*

Proof. By Lemma 4.2 we have $\mathcal{H}^1(K_\delta(t)) \le \lambda$ for every $t \in [0,T]$ and every $\delta > 0$. By Theorem 1.5 this implies $K(t) \in \mathcal{K}^\lambda_m(\overline\Omega)$ for every $t \in [0,T]$. □

For every $t \in [0,T]$ let $u(t)$ be a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(t), K(t))$.

Lemma 4.4. *For every $t \in [0,T]$ we have $\nabla u_\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$.*

Proof. As $u_\delta(t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g_\delta(t), K_\delta(t))$, and $g_\delta(t) \to g(t)$ strongly in $H^1(\Omega)$, the conclusion follows from Theorem 2.1 and Lemma 4.3. □

The following lemma shows the minimality of the set $K(t)$ for the functional $\mathcal{E}(g(t),\cdot)$ with respect to sets K containing $K(t)$.

Lemma 4.5. *For every $t \in [0,T]$ we have*

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K) \quad \forall\, K \in \mathcal{K}^f_m(\overline\Omega)\,,\ K \supset K(t)\,. \tag{4.6}$$

Proof. Let us fix $t \in [0,T]$ and $K \in \mathcal{K}^f_m(\overline\Omega)$ with $K \supset K(t)$. Since $K_\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$, it is possible to construct a a sequence (K_δ) in $\mathcal{K}^f_m(\overline\Omega)$, converging to K in the Hausdorff metric, such that $K_\delta \supset K_\delta(t)$ and $\mathcal{H}^1(K_\delta \setminus K_\delta(t)) \to \mathcal{H}^1(K \setminus K(t))$ as $\delta \to 0$. By Lemma 4.2 this implies that $\mathcal{H}^1(K_\delta)$ is bounded as $\delta \to 0$. The main difficulty in the construction of K_δ is the constraint on the number of connected components. The proof of the details is quite long, but elementary, and is given in [15, Lemma 3.5].

Let v_δ and v be solutions of the minimum problems (3.3) which define $\mathcal{E}(g_\delta(t), K_\delta)$ and $\mathcal{E}(g(t), K)$, respectively. By Theorem 2.1 $\nabla v_\delta \to \nabla v$ strongly in $L^2(\Omega; \mathbb{R}^2)$. The minimality of $K_\delta(t)$ expressed by (3.4) gives $\mathcal{E}(g_\delta(t), K_\delta(t)) \le \mathcal{E}(g_\delta(t), K_\delta)$, which implies $\|\nabla u_\delta(t)\|^2 \le \|\nabla v_\delta\|^2 + \mathcal{H}^1(K_\delta \backslash K_\delta(t))$. Passing to the limit as $\delta \to 0$ and using Lemma 4.4 we get $\|\nabla u(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1(K \setminus K(t))$. Adding $\mathcal{H}^1(K(t))$ to both sides we obtain (4.6). □

We can now pass to the limit in Lemma 4.1.

Lemma 4.6. *For every s, t with $0 \le s < t \le T$*

$$\|\nabla u(t)\|^2 + \mathcal{H}^1(K(t)) \le \|\nabla u(s)\|^2 + \mathcal{H}^1(K(s)) + 2\int_s^t (\nabla u(\tau)|\nabla \dot g(\tau)) d\tau\,. \tag{4.7}$$

Proof. Let us fix s, t with $0 \le s < t \le T$. Given $\delta > 0$ let i and j be the integers such that $t_i^\delta \le s < t_{i+1}^\delta$ and $t_j^\delta \le t < t_{j+1}^\delta$. Let us define $s_\delta := t_i^\delta$ and $t_\delta := t_j^\delta$. Applying Lemma 4.1 we obtain

$$\|\nabla u_\delta(t)\|^2 + \mathcal{H}^1(K_\delta(t) \backslash K_\delta(s)) \le \|\nabla u_\delta(s)\|^2 + 2\int_{s_\delta}^{t_\delta} (\nabla u_\delta(\tau)|\nabla \dot g(\tau))\, d\tau + \omega(\delta)\,, \tag{4.8}$$

with $\omega(\delta)$ converging to zero as $\delta \to 0$. By Lemma 4.4 for every $\tau \in [0, T]$ we have $\nabla u_\delta(\tau) \to \nabla u(\tau)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ as $\delta \to 0$, and by Lemma 4.2 we have $\|\nabla u_\delta(\tau)\| \le \lambda$ for every $\tau \in [0, T]$.

Given $\varepsilon > 0$, let $K^\varepsilon(s) := \{x \in \overline{\Omega} : \mathrm{dist}(x, K(s)) \le \varepsilon\}$. As $K_\delta(s) \subset K^\varepsilon(s)$ for δ small enough, we have $K_\delta(t) \backslash K^\varepsilon(s) \subset K_\delta(t) \backslash K_\delta(s)$. Applying Theorem 1.5 with $U = \mathbb{R}^2 \backslash K^\varepsilon(s)$ we get

$$\mathcal{H}^1(K(t) \backslash K^\varepsilon(s)) \le \liminf_{\delta \to 0}\, \mathcal{H}^1(K_\delta(t) \backslash K^\varepsilon(s)) \le \liminf_{\delta \to 0}\, \mathcal{H}^1(K_\delta(t) \backslash K_\delta(s))\,.$$

Passing to the limit as $\varepsilon \to 0$ we obtain

$$\mathcal{H}^1(K(t) \backslash K(s)) \le \liminf_{\delta \to 0}\, \mathcal{H}^1(K_\delta(t) \backslash K_\delta(s))\,.$$

Passing now to the limit in (4.8) as $\delta \to 0$ we obtain (4.7). □

We are now in a position to prove the absolute continuity of the function $t \mapsto \mathcal{E}(g(t), K(t))$ and to compute its derivative.

Lemma 4.7. *The function $t \mapsto \mathcal{E}(g(t), K(t))$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t)|\nabla \dot g(t)) \qquad \textit{for a.e. } t \in [0, T]\,. \tag{4.9}$$

Proof. Let $0 \le s < t \le T$. From the previous lemma we get

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \le 2\int_s^t (\nabla u(\tau)|\nabla \dot g(\tau))\, d\tau\,. \tag{4.10}$$

On the other hand, by Lemma 4.5 we have $\mathcal{E}(g(s), K(s)) \le \mathcal{E}(g(s), K(t))$. It is easy to see that the Frechet differential $d\mathcal{E}(g, K)$ of $\mathcal{E}(g, K)$ (with respect to g) is given by

$$d\mathcal{E}(g, K)\, h = 2\int_{\Omega \backslash K} \nabla u_g \nabla h\, dx\,, \tag{4.11}$$

where u_g is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g, K)$. Therefore we have

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2\int_s^t (\nabla u(\tau, t)|\nabla \dot g(\tau))\, d\tau\,,$$

where $u(\tau,t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(\tau),K(t))$. Together with the inequality $\mathcal{E}(g(s),K(s)) \le \mathcal{E}(g(s),K(t))$, this implies

$$\mathcal{E}(g(t),K(t)) - \mathcal{E}(g(s),K(s)) \ge 2\int_s^t (\nabla u(\tau,t)|\nabla \dot g(\tau))\, d\tau\,. \tag{4.12}$$

Since there exists a constant C such that $\|\nabla u(\tau)\| \le \|\nabla g(\tau)\| \le C$ and $\|\nabla u(\tau,t)\| \le \|\nabla g(\tau)\| \le C$ for $s \le \tau \le t$, from (4.10) and (4.12) we obtain

$$\big|\mathcal{E}(g(t),K(t)) - \mathcal{E}(g(s),K(s))\big| \le 2\,C\int_s^t \|\nabla \dot g(\tau)\|\, d\tau\,,$$

which proves that the function $t \mapsto \mathcal{E}(g(t),K(t))$ is absolutely continuous.

As $\nabla u(\tau,t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ when $\tau \to t$, if we divide (4.10) and (4.12) by $t-s$, and take the limit as $s \to t-$ we obtain (4.9). □

The result of the previous lemma can be expressed equivalently in the following way.

Lemma 4.8. *The function $t \mapsto \mathcal{E}(g(t),K(t))$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = 0 \quad \textit{for a.e. } t \in [0,T]\,. \tag{4.13}$$

Proof. Let Θ be the set defined in Proposition 3.4. By (4.11) and by Theorem 2.1 the differential $d\mathcal{E}(g,K(s))$ tends to $d\mathcal{E}(g(t),K(t))$ as (g,s) tends to $(g(t),t)$ in $H^1(\Omega)\times\mathbb{R}$. Let us fix a point t in Θ such that the function $s \mapsto \mathcal{E}(g(s),K(s))$ is differentiable at $s=t$ and t is a Lebesgue point of $\dot g$. For every $s \in [0,T]$ we have

$$\begin{aligned} &\mathcal{E}(g(s),K(s)) - \mathcal{E}(g(t),K(t)) = \\ &= \mathcal{E}(g(s),K(s)) - \mathcal{E}(g(t),K(s)) + \mathcal{E}(g(t),K(s)) - \mathcal{E}(g(t),K(t)) = \\ &= \int_t^s d\mathcal{E}(g(\tau),K(s))\,\dot g(\tau)\, d\tau + \mathcal{E}(g(t),K(s)) - \mathcal{E}(g(t),K(t))\,. \end{aligned}$$

Dividing by $s-t$ and taking the limit as $s \to t$ we obtain

$$\frac{d}{ds}\mathcal{E}(g(s),K(s))\Big|_{s=t} = d\mathcal{E}(g(t),K(t))\,\dot g(t) + \frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t}\,.$$

The conclusion follows from (4.11) and Lemma 4.7. □

The properties of the function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ are summarized by the following theorem, which is an immediate consequence of Lemmas 4.3, 4.5, 4.7, and 4.8.

Theorem 4.9. *Let $m \ge 1$, let $g \in AC([0,T];H^1(\Omega))$, and let $K_0 \in \mathcal{K}_m^f(\overline{\Omega})$. Then the function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ introduced at the end of the last lecture satisfies the following properties:*

(a) $$K(0) = K_0\,,$$

(b) $$K_0 \subset K(s) \subset K(t) \ \textit{ and } \ K(t) \in \mathcal{K}_m^f(\overline{\Omega}) \ \textit{ for } \ 0 \le s \le t \le T\,,$$

(c) $$\textit{for } \ 0 \le t \le T \quad \mathcal{E}(g(t),K(t)) \le \mathcal{E}(g(t),K) \quad \forall\, K \in \mathcal{K}_m^f(\overline{\Omega}),\ \ K \supset K(t)\,,$$

(d) $$t \mapsto \mathcal{E}(g(t),K(t)) \ \textit{ is absolutely continuous on } [0,T]\,,$$

(e) $$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = 0 \quad \textit{for a.e. } t \in [0,T]\,.$$

Moreover every function $K\colon [0,T] \to \mathcal{K}_m^f(\overline{\Omega})$ which satisfies (a)–(e) satisfies also

(f) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2(\nabla u(t)|\nabla \dot g(t)) \qquad \textit{for a.e. } t \in [0,T]\,,$$

where $u(t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(t),K(t))$.

In our continuous-time model, the function $K\colon [0,T] \to \mathcal{K}^f_m(\overline{\Omega})$ represents the quasi-static irreversible evolution of the crack starting from K_0 (condition (a)) under the action of the boundary displacement $g(t)$. Condition (b) reflects the *irreversibility of the evolution* and the *absence of a healing process*. Condition (c) is a unilateral minimality condition. Condition (e) says that, for almost every $t \in [0,T]$, the total energy $s \mapsto \mathcal{E}(g(t), K(s))$ is stationary at $s = t$. Conditions (c) and (e) together lead to Griffith's analysis of the energy balance in our model, and, under some very mild regularity assumptions on the cracks, allow to express the classical Griffith's criterion for crack growth in terms of the stress intensity factors at the tips of the cracks (see [15, Section 8]).

We underline that, although we can not exclude that the surface energy $\mathcal{H}^1(K(t))$ may present some jump discontinuities in time (see [20, Section 4.3]), in our result *the total energy is always an absolutely continuous function of time* by condition (d).

If $\partial_D\Omega$ is sufficiently smooth, we can integrate by parts the right hand side of (f) and, taking into account the Euler equation satisfied by $u(t)$, we obtain

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2\int_{\partial_D\Omega\setminus K(t)} \frac{\partial u(t)}{\partial \nu}\,\dot{g}(t)\,d\mathcal{H}^1 \qquad \text{for a.e. } t \in [0,T], \tag{4.14}$$

where ν is the outer unit normal to $\partial\Omega$. Since the right hand side of (4.14) is the power of the force exerted on the boundary to obtain the displacement $g(t)$ on $\partial_D\Omega\setminus K(t)$, equality (4.14) expresses the *conservation of energy* in our quasi-static model, where all kinetic effects are neglected.

The discrete-time model described in the previous lecture turns out to be a useful tool for the proof of the existence of a solution $K(t)$ of the problem considered in Theorem 4.9, and provides also an effective way for the numerical approximation of this solution (see [5]), since many algorithms have been developed for the numerical solution of minimum problems of the form (3.4) (see, e.g., [2], [28], [29], [3], [11], [4]).

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