Optimal quantization with branched optimal transport distances

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We consider the problem of optimal approximation of a target measure by an atomic measure with N atoms, in branched optimal transport distance. This is a new branched transport version of optimal quantization problems. New difficulties arise, since in classical semi-discrete optimal transport with Wasserstein distance, the interfaces between cells associated with neighboring atoms have Voronoï structure and satisfy an explicit description. This description is missing for our problem, in which the cell interfaces are thought to have fractal boundary. We study the asymptotic behaviour of optimal quantizers for absolutely continuous measures as the number N of atoms grows to infinity. We compute the limit distribution of the corresponding point clouds and show in particular a branched transport version of Zador's theorem. Moreover, we establish uniformity bounds of optimal quantizers in terms of separation distance and covering radius of the atoms, when the measure is d-Ahlfors regular. A crucial technical tool is the uniform in N Hölder regularity of the landscape function, a branched transport analog to Kantorovich potentials in classical optimal transport.

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Contents

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Notation

General notations

$\#A$	Cardinal of a set A
$B_r^d(x)$	Open ball with center x and radius $r > 0$ in $\mathbb{R}^d$ (superscript d will ofter be dropped)
$Q_\delta^d(x)$	$:= x + \delta[-1/2, 1/2]^d$, closed cube (superscript d will ofter be dropped)
Q_1^d	$:= Q_1^d(0)$
$\mathscr{L}^d$	Lebesgue measure on $\mathbb{R}^d$
ω_d	$:= \mathscr{L}^d(B_1(0))$
$\mathscr{M}(X), \mathscr{M}_c(X)$	Space of finite signed measures (resp. with compact support) over a Polish space X
$\mathscr{M}^+(X), \mathscr{M}_c^+(X)$	Subset of finite positive measures (resp. with compact support) over a Polish space X
$\mathscr{P}(X)$	Space of probability measures over X
$\\|\mu\\|$	$:= \sup\{\int \phi \,\mathrm{d}\mu : \phi \in \mathscr{C}_b(X), \\|\phi\\|_\infty \leq 1\}$, norm (total mass) of a measure μ
$\xrightharpoonup{\mathscr{C}_b'}$	Narrow convergence of measures, in the duality with continuous and bounded functions
$\xrightharpoonup{\mathscr{C}_0'}$	Weak convergence of measures, in the duality with functions $\mathscr{C}_0 = \overline{\mathscr{C}_c}$
$\mu \wedge \nu$	Greatest submeasure of both μ and ν
$\mu \perp \nu$	μ and ν are mutually singular
$\rho_{\mathrm{ac}}, \rho_{\mathrm{s}}$	Density and singular part of the Radon-Nikodym decomposition of $\rho = \rho_{\mathrm{ac}}\mathscr{L}^d + \rho_{\mathrm{s}}$

We use the conventions $0^b = +\infty$ if $b < 0$, $0^0 = 1$, and $+\infty \times 0 = 0$. With a slight abuse, when $\rho \ll \mathscr{L}^d$, we will denote by ρ as well its density with respect to $\mathscr{L}^d$.

Notations on branched optimal transport (from Section 2.1)

Γ^d	$:= \mathrm{Lip}_1(\mathbb{R}^+, \mathbb{R}^d)$, space of 1-Lipschitz curves endowed with the (metrizable) topology of uniform convergence on compact sets

$\mathbf{TP}^d$	space of traffic plans on $\mathbb{R}^d$
$\mathbf{TP}(\mu^-,\mu^+)$	set of traffic plans $\mathbf{P}$ on $\mathbb{R}^d$ with initial and final marginals μ^- and μ^+ respectively
$\mathbf{P}_n \xrightharpoonup{\star} \mathbf{P}$	weak-$\star$ convergence in $\mathscr{M}^+(\Gamma^d)$ in the duality with $\mathscr{C}(\Gamma^d)$
$\Theta_{\mathbf{P}}(x)$	$:= \int_{\Gamma^d} \#\gamma^{-1}(\{x\})\,\mathrm{d}\mathbf{P}(\gamma)$, multiplicity at x w.r.t. $\mathbf{P}$
$\Sigma_{\mathbf{P}}$	$:= \{x : \Theta_{\mathbf{P}}(x) > 0\}$, network associated with $\mathbf{P}$
$\mathbb{M}^\alpha(\mathbf{P})$	$:= \int_{\Sigma_{\mathbf{P}}} \Theta_{\mathbf{P}}(x)^\alpha \mathscr{H}^1(x)$, α-mass of a rectifiable traffic plan $\mathbf{P}$

In all the paper, α will denote a fixed number in $(0,1)$. When needed, we will explicitly assume that $\alpha \in (1-1/d,1)$ where d is the ambient dimension, in which case we will denote by $\beta = \beta(\alpha,d)$ the exponent

$$\beta := 1 + d\alpha - d = d\left(\alpha - (1-1/d)\right) \in (0,1).$$

1 Introduction

In this work we study for the first time the asymptotics and uniformity properties of optimal quantization with interactions given via branched optimal transport distances, which we will also call, for brevity, *branched quantization*. The fields of branched optimal transport and optimal quantization both have a large variety of applications but have not been connected before. We give a very short review and motivations of both, after which we point out why building a connection is interesting to explore.

1.1 Branched optimal transport and optimal quantization motivations

Branched optimal transport (or branched transport for short) is an umbrella term for a class of optimization problems, related to classical optimal transport, in which mass particles are assumed to interact (as opposed to traveling independently) while moving from a source to a target distribution. The interaction favours the transportation of particles in a grouped way by lowering the transportation cost, which is justified in many practical situations by an *economy of scale*. A consequence of this assumption is that the particles' paths form a one-dimensional network with a branched structure. The most common model assumes a cost of the form $\ell \times m^\alpha$ to move a group of particles of total mass m over a distance ℓ, where $\alpha \in (0,1)$, so that the cost is a concave power of the mass. This problem was first introduced in [Gil67], in a discrete setting, to optimize communications network, and was extended to two different continuous settings in [Xia03; MSM03] (both are actually equivalent [PS06; Peg17b]). For an introduction to the theory of branched transport we refer to the book [BCM09]. In pure mathematics, H-mass minimization over 1-dimensional flat chains (or more generally mass minimization amongst 1-dimensional flat G-chains) with fixed boundary provides versions of branched transport, see e.g. [Xia04; PS06; MM16] and the fundamental results in [Fle66; Whi99a; Whi99b; DH03]. We also mention [PZ18] for a first result on the classification of groups G that produce branching. This last work, together with the classification of homotopy groups of spheres and the classification [HR08] indicates that branched transport costs must commonly appear in connections between vortices of nonlinear Sobolev maps. This has recently led to important insights into weak density results such as [Bet20]. Branched transport is also connected to size-minimization [DH03] and the Steiner problem [MM16], network transport systems and urban planning [Dur06; But+09; BW16], superconductivity [Con+18], traffic flow optimization [ILB21], models of tree roots and branches [BS18; BPS19; BGS22], models of river systems [RR01], amongst others. Finding the optimal branched transport map is in general NP-hard (while for classical optimal transport the complexity is $O(n^3\log n)$ for $n-$point masses), therefore computational approximations are an interesting direction of research, see for instance [OS11; Mon17; BO20; BOO18; BOO21]. Finally, we refer to [DS19; LSW22b; LSW22a; CDM19; CDM21; Col+22; CMS23] for the most recent developments of branched transport theory.

Classical optimal quantization deals with the question of how to discretize a given positive measure $\nu \in \mathscr{M}_c^+(X)$ (in which $X = \mathbb{R}^d$ or X is a more general metric space), in such a way that the discrete $N-$point approximant $\nu_N \in \mathscr{M}^+(\mathbb{R}^d)$ is at minimum distance according to a distance or cost $c : X \times X \to (0, \infty)$. Usually, it is a power of the distance over $X = \mathbb{R}^d$, i.e. $c(x, y) = |x - y|^p, p \geq 1$, inducing the classical p-Wasserstein distance over probability measures that is widely in optimal transport. The quantization problem can then be reformulated as a semi-discrete optimal transport problem, see e.g. [Mér11], which in the case of a uniform target density reduces to the study of Voronoidal tessellations and power diagrams [AHA98; DFG99]. We refer to general reference books [GL00], [GG92] for an overview of the quantization problem, and to [Fej59; Zad63; Zad82; Ger79] for the first historical references. Applications of optimal quantization range from clustering [Sax+17], [Oka+00], to signal processing [GG92], to numerical integration and quadrature [Pag98], to material science [BPT14; BC21; BPR23], to spatial economy [BS72; MB02], where optimal quantization is often referred to as optimal location [BJM02; BJM11; Bra+09; BSS13]. Asymptotics and continuum limits of the problem as the number of discretization points tends to infinity have been studied for the classical optimal quantization problems by Zador [Zad82], by Bouchitté, Jimenez and Mahadevan [BJM02; BJM11], who introduced a Γ-convergence approach, and by Gruber [Gru04] who also provides geometric information on optimal configurations.

Further problems that are not directly formulated as a quantization problem but can also be seen as generalizations of the problem in other directions, appear in minimization of energies of a large number of "charges" under Riesz-Coulomb interactions: see the book [BHS19], and the crystallization survey [LB15]. We mention the related problems of optimal unconstrained polarization [HPS22], jellium equidistribution [PR18], amongst others.

In this work, we focus on the case where the cost underlying the optimal quantization problem is given by a branched transport cost. The motivations for formulating this new problem come both from mathematically interesting new difficulties, and from its relevance for mathematical modelling and its potential applications.

Mathematically, the most important difficulty with the optimal quantization problem via branched transport, is that the regularity of interfaces is not known (rather thought fractal), and the interfaces do not satisfy an explicit condition. This makes branched quantization much more challenging than classical optimal quantization, and required us to give replacements for the main steps in the proofs known in the classical Wasserstein setting. We expect that our approach may allow to study some classes of problems involving random interfaces as well, since we do not make direct use of properties of the shapes of the interfaces in our estimates.

In terms of modelling and applicability, in many clustering tasks the choice of classical distances is only due to their being *a simple first choice* and computationally easy to handle. However, complex clustering tasks are better approximated via hierarchical tree-like clustering structures, such as those formed by branched transport networks. Many biological models such as the study of plant root competition (a natural extension to models such as [BS18]) would directly lead to branched quantization formulations. The same goes for supply chains modelling, in which several sources have to be optimized in order to supply a target density of users: in an urban area relying on a transportation network, branched transport distances (or other H-mass generallizations) are much more realistic than classical Wasserstein distances.

1.2 Main results

In this section we give simplified statements of our asymptotics and uniformity results for optimal branched quantizers, using a minimal amount of definitions. For full definitions and background results, see Section 2.

Loosely speaking, a traffic plans $\mathbf{P}$ between probability measures μ, ν, is a suitable measure over $1-$Lipschitz curves transporting μ to ν. For $\alpha \in (0,1)$ we define the α-mass $\mathbb{M}^\alpha(\mathbf{P})$ as the integral of the α-th power of the transported mass flux $\Theta_{\mathbf{P}}$ (called *multiplity*), over the network $\Sigma_{\mathbf{P}}$ induced by $\mathbf{P}$, when the latter is 1-rectifiable (see full definition in Section 2.1). It is indeed proportional to $m^\alpha \times \ell$ when moving a total mass m over a distance ℓ. Then for a given $N \geq 1$ we consider the *branched optimal quantization problem* defined as

$$\mathcal{E}^\alpha(\nu, N) \coloneqq \inf \left\{ \mathbf{d}^\alpha(\mu, \nu) : \#\operatorname{spt}\mu \leq N \right\},$$

where $\mathbf{d}^\alpha$ is the branched transport distance, given as the infimum of α-mass $\mathbb{M}^\alpha(\mathbf{P})$ amongst traffic plans $\mathbf{P}$ transporting μ to ν. When $\mathcal{E}^\alpha(\nu, N)$ is finite, an optimizer μ_N for this problem is called an *optimal $N-$point quantizer of ν.*

Our first main result is a branched transport version of a result by Zador [Zad82] valid for classical quantization.

Theorem 1.1. *Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a measure which is absolutely continuous.*

(i) If $(\mu_N)_{N\in\mathbb{N}^*}$ is a sequence of optimal N-point quantizers of ν,

$$\mu_N^\diamond \coloneqq \frac{1}{N} \sum_{\{x:\mu_N(\{x\})>0\}} \delta_x \xrightarrow[N\to+\infty]{\mathscr{C}_b'} M_{\alpha,d}(\nu)^{-1} \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}},$$

where $M_{\alpha,d}(\nu) \coloneqq \int_K \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}} \,\mathrm{d}x$.

(ii) The leading-order asymptotics of the optimal quantization error is given by

$$\lim_{N\to\infty} N^{\beta/d} \mathcal{E}^\alpha(\nu, N) = c_{\alpha,d} M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}$$

where $c_{\alpha,d} \in (0,+\infty)$ is the constant defined in (2.19) of Proposition 2.12.

The above theorem is a consequence of a more precise $\Gamma-$convergence result of $\mathbf{d}^\alpha$-distance to the continuum along sequences with fixed support density, given in Theorem 3.1. The latter is a branched transport analogue of the asymptotic result of [BJM02] for classical optimal quantization, and from which it is inspired.

Our second main result pertains to uniformity estimates on the optimal quantizer's support, adapting the general strategy of [Gru04], which is valid for classical quantization, to the branched quantization case. We provide bounds on the *covering radius* and on the *separation distance* for the support of optimal quantizers, both at the natural scale of $N^{-1/d}$, which is coherent with the principle that for an optimal quantizer, roughly speaking, a ball-like set of volume $\approx 1/N$ is assigned to each of the N points in the quantizer's support. The covering radius bound quantifies the property that the atoms of an optimal quantizer are never farther than $c_1 N^{-1/d}$ from the support of the quantized measure ν, and the separation bound indicates that the atoms of the quantizer are never closer than $c_2 N^{-1/d}$ from each other.

Theorem 1.2. *Assume $\alpha \in (1-1/d, 1)$. Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a compactly supported d-Ahlfors regular measure*[1] *with constants $c_A, C_A > 0$ on $\mathbb{R}^d$ and $\mu_N = \sum_{i\leq N} m_i \delta_{x_i}$ be an N-point optimal quantizer with atoms $\mathcal{X} = \{x_i\}_{1\leq i\leq N}$. Then the covering radius and separation distance respectively enjoy the following bounds:*

$$\omega(\operatorname{spt}\nu, \mathcal{X}) \coloneqq \sup_{x\in\mathcal{X}} \min_{\operatorname{spt}\nu} d(x, x') \leq c_2 N^{-1/d}, \tag{1.1}$$

[1] Meaning that $c_A r^d \leq \nu(B_r(x)) \leq C_A r^d$ for every $x \in \operatorname{spt}\nu, r \in (0, \operatorname{diam}(\operatorname{spt}\nu)])$ and some constants $c_A, C_A > 0$.

$$\delta(\mathcal{X}) \coloneqq \min_{x,x' \in \operatorname{spt}\nu : x \neq x'} d(x,x') \geq c_1 N^{-1/d}. \tag{1.2}$$

for some constants $c_1, c_2 > 0$ that depends on $(\alpha, d, c_A, C_A, \operatorname{diam}(\operatorname{spt}\nu))$ but not on N.

We follow the general strategy of [Gru04], which entails major extra difficulties. A crucial new technical result is established in Theorem 4.1, where we give a uniform Hölder control of the so-called *landscape function*, a substitute for classical Kantorovich potentials in branched transport theory. Recall that in classical optimal transport theory, so-called Kantorovich duality allows to transform the problem into a dual version based on Kantorovich potentials (see e.g. [San15, Chapter 1]). In turn, Kantorovich potentials can be used to show that for optimal quantization with cost $|x-y|^p$ interfaces of the quantization cells are straight (see e.g. in [MT21]).

In branched transport there is no useful analogue of Kantorovich duality, but optimal Kantorovich potentials have a partial analogue in the landscape function, which is in particular a (upper) first variation of the branched transport distance $\mathbf{d}^\alpha$. As a reference for the single-source landscape function $z_{\mathbf{P}}$ (corresponding to case $N=1$ in our notation) see e.g. [San07; BS11]. Its basic properties are recalled in Proposition 2.3. For general $N \geq 1$, in Theorem 4.1 we prove the following result, which holds for quantizers that are *mass-optimal*, a condition that is weaker than optimality and only requires the quantizer to be optimal among measures with a fixed support but with varying masses (see Definition 2.6).

Theorem 1.3 (Simplified statement of Theorem 4.1)**.** *Let $\nu \in \mathscr{M}_c(\mathbb{R}^d)$ be a compactly supported d-Ahlfors regular measure, and let $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$ be an optimal traffic plan where $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ is a N-point mass-optimal quantizer of ν with respect to $(x_i)_{1\leq i \leq N}$. There exists a unique function $z_{\mathbf{P}} : \operatorname{spt}\nu \to \mathbb{R}_+$ that we call* landscape function associated with $\mathbf{P}$ *which locally coincides with the single-source landscape functions $z_{\mathbf{P}^{x_i}}$ for each source x_i, and is β-Hölder continuous for $\beta = 1 + d\alpha - d \in (0,1)$, with a Hölder constant independent from N.*

This theorem provides at the same time a definition of landscape and its Hölder regularity in the case of a source measure that is a mass-optimal quantizer. By definition, this landscape function will inherit the same key properties as in the single-source case, as stated in Proposition 4.2. We emphasize that in the proof of Theorem 1.2 we make crucial use of the *uniform in N* Hölder control of $z_{\mathbf{P}}$, without which we do not expect the same results to hold.

Finally, we remark that the notion of mass-optimal quantizers allows us to define analogues of Voronoï cells in the context of branched transport, that we call *branched Voronoï basins*, which exhibit striking differences to Voronoï cells (see Remark 4.5).

1.3 Structure of the paper

- In Section 2 we complete the definitions underlying our main theorems, recall important foundational results in branched optimal transport and establish preliminary results on the optimal quantization and partition problems.
- In Section 3 we prove our main Γ-convergence and asymptotic results, Theorem 3.1 and Theorem 1.1.
- In Section 4 we prove the above Theorem 1.3 on the regularity of the landscape function.
- In Section 5 we prove the uniformity results of Theorem 1.2.

2 Background and preliminaries

2.1 Background in branched optimal transport

In this section we set up the static "Lagrangian" model of branched optimal transport based on *traffic plans* developed by [BCM05] and [MSM03]. The main reference on branched optimal transport is the book [BCM09]. The presentation, notation and definitions that we adopt in this paper have been slightly simplified following more recent works such as [Peg17b; CDM19; CDM21; Col+22].

Traffic plans

A *traffic plan* on $\mathbb{R}^d$ is a finite positive measure $\mathbf{P} \in \mathscr{M}^+(\Gamma^d)$ on the set of 1-Lipschitz curves $\Gamma^d := \mathrm{Lip}_1(\mathbb{R}^+, \mathbb{R}^d)$, endowed with the metrizable topology of uniform convergence on compact sets, which is concentrated on the set of curves with finite stopping time:

$$\mathbf{P}(\{\gamma \in \Gamma^d : T(\gamma) = +\infty\}) = 0, \tag{2.1}$$

where for every $\gamma \in \Gamma^d$,

$$T(\gamma) := \inf\{\tau \geq 0 : \gamma \text{ constant on } [\tau, +\infty)\} \in [0, +\infty].$$

We denote by $\mathbf{TP}^d$ the space of traffic plans over $\mathbb{R}^d$, and whenever $\mu^\pm \in \mathscr{M}^+(\mathbb{R}^d)$ have equal total mass we denote by $\mathbf{TP}(\mu^-, \mu^+)$ the set of traffic plans transporting μ^- to μ^+ i.e. such that $(e_0)_\sharp \mathbf{P} = \mu^-$ and $(e_\infty)_\sharp \mathbf{P} = \mu^+$ where $e_0(\gamma) := \gamma(0)$ and $e_\infty(\gamma) := \gamma(+\infty) := \gamma(T(\gamma))$ for every $\gamma \in \Gamma^d$. The measures μ^- and μ^+ are respectively called the *source and sink measures* of $\mathbf{P}$.

For every $x \in \mathbb{R}^d$, the *multiplicity*

$$\Theta_{\mathbf{P}}(x) := \int_{\Gamma^d} \#\gamma^{-1}(\{x\})\,\mathrm{d}\mathbf{P}(\gamma)$$

represents the amount of curves, measured by $\mathbf{P}$, which visit x (each curve being counted as many times as it visits x). The *network* of $\mathbf{P}$ is the (possibly empty) countably 1-rectifiable set[2]

$$\Sigma_{\mathbf{P}} := \{x \in \mathbb{R}^d : \Theta_{\mathbf{P}}(x) > 0\}.$$

The traffic plan $\mathbf{P}$ is said *rectifiable* if there exists a 1-rectifiable set Σ such that

$$\mathscr{H}^1(\gamma(\mathbb{R}^+) \setminus \Sigma) = 0 \text{ for } \mathbf{P}\text{-almost every } \gamma \in \Gamma^d, \tag{2.2}$$

in which case (2.2) holds with $\Sigma = \Sigma_{\mathbf{P}}$. It is said *simple* if it is concentrated on simple curves, i.e. curves $\gamma \in \Gamma^d$ such that γ is constant on $[s,t]$ whenever $\gamma(s) = \gamma(t)$ and $s < t$.

Finally, two traffic plans $\mathbf{P}_1, \mathbf{P}_2$ are said *disjoint* if there exists two disjoint sets $A_1, A_2 \subseteq \mathbb{R}^d$ such that for $i \in \{1,2\}$,

$$\mathscr{H}^1(\gamma(\mathbb{R}^+) \setminus A_i) = 0 \text{ for } \mathbf{P}_i\text{-almost every } \gamma \in \Gamma^d. \tag{2.3}$$

For rectifiable traffic plans $\mathbf{P}_1, \mathbf{P}_2$, it is equivalent to

$$\mathscr{H}^1(\Sigma_{\mathbf{P}_1} \cap \Sigma_{\mathbf{P}_2}) = 0 \quad \text{or equivalently} \quad \Theta_{\mathbf{P}_1}\mathscr{H}^1 \perp \Theta_{\mathbf{P}_2}\mathscr{H}^1. \tag{2.4}$$

[2]It is countably 1-rectifiable by [Peg17b, Section 2.1] or [BCM05, Lemma 6.3], meaning that it is included, up to a $\mathscr{H}^1$-null set, in a countable union of Lipschitz curves.

Concatenation of traffic plans

We follow the presentation of concatenations provided in [CDM19, §3.3]. If (γ_1, γ_2) belong to the set $\Lambda^d \subseteq \Gamma^d \times \Gamma^d$ of pairs of curves which have finite stopping time and satisfy $\gamma_1(+\infty) = \gamma_2(0)$, we set for every $t \in \mathbb{R}^+$

$$(\gamma_1 : \gamma_2)(t) := \begin{cases} \gamma_1(t) & \text{if } t \in [0, T(\gamma_1)), \\ \gamma_2(t - T(\gamma_1)) & \text{if } t \in [T(\gamma_1), +\infty). \end{cases}$$

We denote this map by $\mathrm{conc} : \Lambda^d \to \Gamma^d$.

If $\mathbf{P}_1, \mathbf{P}_2 \in \mathbf{TP}^d$ are such that $(e_\infty)_\sharp \mathbf{P}_1 = (e_0)_\sharp \mathbf{P}_2$, we say that $\mathbf{P}$ is a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ if there exists a measure $\bar{\mathbf{P}} \in \mathscr{M}^+(\Gamma^d \times \Gamma^d)$, which is concentrated on Λ^d and satisfies

$$\mathbf{P} = \mathrm{conc}_\# \bar{\mathbf{P}}$$
$$(p_i)_\sharp \bar{\mathbf{P}} = \mathbf{P}_i \quad \text{where } p_i : (\gamma_1, \gamma_2) \mapsto \gamma_i \text{ for } i \in \{1, 2\}.$$

We denote by $(\mathbf{P}_1 : \mathbf{P}_2)$ the set of concatenations of $\mathbf{P}_1$ and $\mathbf{P}_2$. We will need some properties of concatenations that are summarized in the following proposition, extracted from [CDM19, §3.3].

Proposition 2.1 ([CDM19, Lemma 3.6])**.** *If $\mathbf{P}_1, \mathbf{P}_2 \in \mathbf{TP}^d$ are such that $(e_\infty)_\sharp \mathbf{P}_1 = (e_0)_\sharp \mathbf{P}_2$, then:*

(i) $(\mathbf{P}_1 : \mathbf{P}_2)$ *is nonempty,*

(ii) $(e_0)_\sharp \mathbf{P} = (e_0)_\sharp \mathbf{P}_1$ *and* $(e_\infty)_\sharp \mathbf{P} = (e_\infty)_\sharp \mathbf{P}_2$ *for every* $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$*,*

(iii) for every $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$*,* $\Theta_{\mathbf{P}} = \Theta_{\mathbf{P}_1} + \Theta_{\mathbf{P}_2}$ *and thus* $\mathbb{M}^\alpha(\mathbf{P}) \le \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2)$*,*

(iv) if $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$ *and* $\mathbf{P}' \in (\mathbf{P'}_1 : \mathbf{P'}_2)$ *then* $\mathbf{P} + \mathbf{P}' \in (\mathbf{P}_1 + \mathbf{P'}_1 : \mathbf{P}_2 + \mathbf{P'}_2)$*.*

The α-mass functional and the irrigation problem

For $\alpha \in (0, 1)$, the α-mass[3] of a traffic plan is defined as

$$\mathbb{M}^\alpha(\mathbf{P}) = \begin{cases} \displaystyle\int_{\Sigma_{\mathbf{P}}} \Theta_{\mathbf{P}}(x)^\alpha \, \mathrm{d}\mathscr{H}^1(x) & \text{if } \mathbf{P} \text{ is rectifiable,} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

If $\mu^\pm$ are two positive measures on $\mathbb{R}^d$ of equal (finite) mass, the irrigation problem then reads as

$$\inf \quad \left\{ \mathbb{M}^\alpha(\mathbf{P}) \;\middle|\; \mathbf{P} \in \mathbf{TP}(\mu^-, \mu^+) \right\}, \tag{I$^\alpha$}$$

and we denote by $\mathbf{d}^\alpha(\mu^-, \mu^+)$ this infimum value.

Definition 2.2 (Optimal traffic plan)**.** We say that $\mathbf{P} \in \mathbf{TP}(\mu^-, \mu^+)$ is optimal if the infimum in (I$^\alpha$) is finite and attained by $\mathbf{P}$.

Let us state some known results that we shall use throughout the paper.

(1) **Subadditivity, additivity and disjointness.** The α-mass is subadditive in the sense that

$$\begin{gathered} \forall \mathbf{P}_1, \mathbf{P}_2 \in \mathbf{TP}^d, \quad \mathbb{M}^\alpha(\mathbf{P}_1 + \mathbf{P}_2) \le \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2) \\ \text{with equality when } \mathbf{P}_1 \text{ and } \mathbf{P}_2 \text{ are disjoint.} \end{gathered} \tag{2.6}$$

Besides, if $\mathbf{P}^1$ and $\mathbf{P}^2$ are optimal (for their own marginals) and are disjoint, then $\mathbf{P}^1 + \mathbf{P}^2$ is also optimal.

[3] It is the equivalent for traffic plans, of the α-mass of currents.

(2) **Irrigability and irrigation distance.** Contrary to the classical optimal transport problem, for some pair of compactly supported measures (μ^-, μ^+) and some exponent α it is possible that (2.5) admits no competitor of finite α-mass, typically when the measures spread on a set of large dimension while the exponent α is too small. However, when $\alpha > 1 - \frac{1}{d}$ any measure $\mu \in \mathscr{M}_c^+(\mathbb{R}^d)$ is *α-irrigable*, meaning that $\mathbf{d}^\alpha(\mu, \|\mu\|\delta_0) < +\infty$, and there exists a competitor of finite α-mass for any pair of measures $\mu^-, \mu^+ \in \mathscr{M}_c^+(\mathbb{R}^d)$ of equal total mass, as shown[4] in [Xia03]. In particular

$$(\alpha \in (1 - 1/d, 1) \quad \text{and} \quad K \subseteq \mathbb{R}^d \text{ compact})$$
$$\Longrightarrow$$
$$(\mathbf{d}^\alpha \text{ is a distance on } \mathscr{P}(K) \text{ which metrizes the weak-}\star \text{ convergence on } \mathscr{C}(K)').$$

(3) **Existence for the irrigation problem.** Problem (I^α) always admits a solution. It results for example from the existence of $\mathbb{E}^\alpha$ minimizers established in [BCM09, Section 3.4], where $\mathbb{E}^\alpha$ is a more complicated variant[5] of $\mathbb{M}^\alpha$, knowing that $\mathbb{E}^\alpha \geq \mathbb{M}^\alpha$ and that a minimizer of $\mathbb{E}^\alpha$ is also a minimizer of $\mathbb{M}^\alpha$ with the same cost.

(4) **Upper estimates on the α-mass.** If $\alpha \in (1 - \frac{1}{d}, 1)$, there exists a constant $C_{\mathsf{BOT}} = C_{\mathsf{BOT}}(\alpha, d) \in (0, +\infty)$ such that for any compactly supported measures $\mu^\pm$ of equal total mass,
$$\mathbf{d}^\alpha(\mu^-, \mu^+) \leq C_{\mathsf{BOT}} \operatorname{diam}(\operatorname{spt}(\mu^+ - \mu^-))\|\mu^+ - \mu^-\|^\alpha. \tag{2.7}$$
Indeed, it is proven in [Xia03] that $\mathbf{d}^\alpha(\delta_0, \mu) \leq C_{\mathsf{BOT}}(\alpha, d)/2$ for every $\mu \in \mathscr{P}(Q_1^d)$ and some best constant $C_{\mathsf{BOT}}(\alpha, d) \in (0, \infty)$, from which we deduce $\mathbf{d}^\alpha(\mu^-, \mu^+) \leq Crm^\alpha$ when $\operatorname{diam}(\operatorname{spt}(\mu^- + \mu^+)) \leq r$ and $\|\mu^-\| = \|\mu^+\| = m$ using the triangle inequality and the 1-homogeneity in space and α-homogeneity in mass of the α-mass. Applying this to the measures $\tilde{\mu}^\pm = \mu^\pm - \mu^- \wedge \mu^+$ yields (2.7) since $\|\tilde{\mu}^\pm\| = \|\mu^+ - \mu^-\|/2$, $\operatorname{spt}(\mu^+ - \mu^-) = \operatorname{spt}(\tilde{\mu}^+ + \tilde{\mu}^-)$, and $\mathbf{d}^\alpha(\mu^-, \mu^+) = \mathbf{d}^\alpha(\tilde{\mu}^-, \tilde{\mu}^+)$.

(5) **First variation of the α-mass.** If $\mathbf{P}, \tilde{\mathbf{P}}$ are traffic plans with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, then
$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha \int_{\Gamma^d} Z_{\mathbf{P}}(\gamma) \, \mathrm{d}(\tilde{\mathbf{P}} - \mathbf{P})(\gamma), \tag{2.8}$$
where
$$Z_{\mathbf{P}}(\gamma) \coloneqq \int_\gamma \Theta_{\mathbf{P}}^{\alpha-1} = \int_{\gamma(\mathbb{R})} \Theta_{\mathbf{P}}(x)^{\alpha-1} \# \gamma^{-1}(x) \, \mathrm{d}\mathscr{H}^1(x). \tag{2.9}$$
The proof of (2.8) relies on the concavity of $m \mapsto m^\alpha$ on $\mathbb{R}^+$ applied to $\Theta_{\tilde{\mathbf{P}}} = \Theta_{\mathbf{P}} + (\Theta_{\tilde{\mathbf{P}}} - \Theta_{\mathbf{P}})$ and on Fubini's theorem (we refer to [BCM09, Chapter 11], [San07, Theorem 3.1]). Notice that the integral $\int Z_{\mathbf{P}} \, \mathrm{d}(\tilde{\mathbf{P}} - \mathbf{P})$ is well-defined (possibly infinite) since by Fubini's theorem we may show that:
$$\infty > \mathbb{M}^\alpha(\mathbf{P}) = \int_{\Gamma^d} Z_{\mathbf{P}} \, \mathrm{d}\mathbf{P}$$
and $\int_{\Gamma^d} Z_{\mathbf{P}} \, \mathrm{d}\tilde{\mathbf{P}} \in [0, \infty]$.

[4] It is shown in the Eulerian model based on vector measures, but adapting the proof in our Lagrangian setting is straightforward, or one can also invoke the equivalence of the models [Peg17b].

[5] A reason for using this functional $\mathbb{E}^\alpha$ (denoted by $\mathcal{E}^\alpha$ in [BCM09]) was that one could establish its lower semicontinuity (on suitable subsets) via another expression (so-called energy formula). Nowadays, one may instead prove lower semicontinuity of $\mathbb{M}^\alpha$ and work with it directly.

(6) **Single-path property.** If $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$ is an optimal traffic plan, it is simple and satisfies the single-path property, which be stated in the single-source case where $\mu = m\delta_s$ as follows: for every $x \in \Sigma_{\mathbf{P}}$, there exists a (unique) injective curve parameterized by arc length $\gamma_{\mathbf{P},x} : [0,\ell] \to \mathbb{R}^d$ such that $\mathbf{P}$-a.e. curve γ passing by x follows the trajectory of $\gamma_{\mathbf{P},x}$, meaning: if $t_x(\gamma)$ denotes the greatest $t \in [0, T(\gamma)]$ such that $\gamma(t) = x$,

$$\text{for } \mathbf{P}\text{-a.e. } \gamma \text{ s.t. } x \in \gamma(\mathbb{R}_+), \quad \tilde{\gamma}_{[0,t_x(\tilde{\gamma})]} = \gamma_{\mathbf{P},x}, \tag{2.10}$$

where $\tilde{\gamma}$ denotes the unit-speed reparameterization of $\gamma \in \Gamma^d$. This fact is stated in [BCM09, Proposition 7.4].

Landscape function for a single source

Given an optimal irrigation plan $\mathbf{P} \in \mathbf{TP}(m\delta_s,\nu)$, following [San07] we say that a curve γ is $\mathbf{P}$-*good* if, recalling the notation (2.9),

- $Z_{\mathbf{P}}(\gamma) < +\infty$,
- for all $t < T(\gamma)$,
$$\Theta_{\mathbf{P}}(\gamma(t)) = \mathbf{P}(\{\tilde{\gamma} \in \Gamma(\mathbb{R}^d) : \gamma = \tilde{\gamma} \text{ on } [0,t]\}).$$

It is proven in [San07] that any optimal traffic plan $\mathbf{P}$ is concentrated on the set of $\mathbf{P}$-good curves, and that for all $\mathbf{P}$-good curve γ, the quantity $Z_{\mathbf{P}}(\gamma)$ depends only on the final point $\gamma(\infty)$ of the curve, thus we may define the landscape function $z_{\mathbf{P}}$ as follows:

$$z_{\mathbf{P}}(x) := \begin{cases} Z_{\mathbf{P}}(\gamma) & \text{if } \gamma \text{ is an } \mathbf{P}\text{-good curve s.t. } x = \gamma(\infty), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}$$

We summarize the properties on the landscape function, extracted from [San07], that we shall need.

Proposition 2.3. *If $\mathbf{P} \in \mathbf{TP}(m\delta_s,\nu)$ is optimal with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, $\alpha \in [0,1)$, and $z_{\mathbf{P}}$ is its landscape function, then $z_{\mathbf{P}}$ is lower semicontinuous and finite on $\Sigma_{\mathbf{P}} \cup \operatorname{spt}\nu$. Moreover:*

(A) $z_{\mathbf{P}}(x) \geq d(x,s)$ for every $x \in \mathbb{R}^d$;

(B) the α-mass may be expressed in terms of $z_{\mathbf{P}}$:
$$\mathbf{d}^\alpha(m\delta_s,\mu) = \mathbb{M}^\alpha(\mathbf{P}) = \int_{\Gamma^d} Z_{\mathbf{P}}(\gamma)\,\mathrm{d}\mathbf{P}(\gamma) = \int_{\mathbb{R}^d} z_{\mathbf{P}}(x)\,\mathrm{d}\nu(x);$$

(C) if $\tilde{\mathbf{P}} \in \mathbf{TP}(m\delta s,\tilde{\nu})$ is a traffic plan concentrated on $\mathbf{P}$-good curves, then:
$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}}\,\mathrm{d}(\tilde{\nu}-\nu),$$
and the inequality is strict if $\Theta_{\tilde{\mathbf{P}}} - \Theta_{\mathbf{P}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}}$;

(D) in particular, $z_{\mathbf{P}}$ is an upper first variation of the irrigation distance, *in the sense that for every $\tilde{\nu} \in \mathscr{M}^+(\mathbb{R}^d)$,*
$$\mathbf{d}^\alpha(\|\tilde{\nu}\|\delta_s,\tilde{\nu}) \leq \mathbf{d}^\alpha(\|\nu\|\delta_s,\nu) + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}}\,\mathrm{d}(\tilde{\nu}-\nu).$$

2.2 The optimal quantization problem and mass-optimal quantizers

For $\alpha \in (0,1)$, the optimal branched quantization problem is the following:

$$\mathcal{E}^\alpha(\nu, N) := \inf\left\{\mathbf{d}^\alpha(\mu,\nu) : \|\mu\| = \|\nu\| \text{ and } \#\operatorname{spt}\mu \le N\right\}. \tag{2.12}$$

An admissible candidate μ_N in this problem will be called a N-point quantizer of ν.

Definition 2.4 (Optimal quantizer)**.** When $\mathcal{E}^\alpha(\nu,N) < +\infty$ a solution of (2.12) will be called an *N-point optimal quantizer* of ν.

Theorem 2.5. *For any finite positive measure $\nu \in \mathscr{M}^+(\mathbb{R}^d)$ and any $N \in \mathbb{N}^*$, the optimal quantization problem (2.12) admits a solution.*

Proof. Take an integer $N \ge 1$ and ν a finite positive measure on $\mathbb{R}^d$, assuming without loss of generality that it has unit mass. Suppose that $\mathcal{E}^\alpha(\nu,N) < +\infty$ (otherwise there is nothing to prove) and take $(\mu^n)_{n\in\mathbb{N}}$ a minimizing sequence with $\sup_{n\in\mathbb{N}} \mathbf{d}^\alpha(\mu^n,\nu) =: \Lambda < +\infty$. Let us show that it is tight. Take $\varepsilon > 0$ and $R \ge 4\Lambda/\varepsilon$ large enough such that $\nu(\mathbb{R}^d \setminus B_{R/2}) \le \varepsilon/2$. If $\mathbf{P}^n \in \mathbf{TP}(\mu^n,\nu)$ is an optimal traffic plan, then

$$\begin{aligned}
\Lambda \ge \mathbb{M}^\alpha(\mathbf{P}^n) &\ge \int_{\Gamma^d} L(\gamma)\,\mathrm{d}\mathbf{P}^n(\gamma)\\
&\ge \int_{\Gamma^d} L(\gamma)\mathbf{1}_{\{\gamma:\gamma(0)\in\mathbb{R}^d\setminus B_R,\gamma(\infty)\in B_{R/2}\}}\,\mathrm{d}\mathbf{P}^n(\gamma)\\
&\ge \frac{R}{2}(\mu^n(\mathbb{R}^d\setminus B_R) - \nu(\mathbb{R}^d\setminus B_{R/2})),
\end{aligned}$$

which implies that

$$\mu^n(\mathbb{R}^d\setminus B_R) \le 2\Lambda/R + \nu(\mathbb{R}^d\setminus B_{R/2}) \le \varepsilon.$$

As a consequence, (μ^n) admits a subsequence converging narrowly to some $\mu \in \mathscr{P}(\mathbb{R}^d)$. Necessarily, μ has at most N atoms as well and by lower semicontinuity[6] of $\mathbf{d}^\alpha$ for the narrow convergence of probability measures μ is a N-point optimal quantizer. □

We will consider a class of quantizers which is broader than optimal quantizers and that we call *mass-optimal quantizers*. They will be used to establish the full Γ-convergence result in Section 3, and may also provide a notion of Voronoï cells, called *Voronoï basins*, in the setting of branched optimal transport (see Remark 4.5).

Definition 2.6 (Mass-optimal quantizer)**.** Let $\nu \in \mathscr{M}(\mathbb{R}^d)$ be a finite positive measure and $\mathcal{X} = \{x_i\}_{1\le i\le N}$ be a set of cardinal N. If μ is a measure supported on $\mathcal{X}$ such that the masses of its atoms are chosen in the best way to approximate ν in $\mathbf{d}^\alpha$ distance, i.e. $\mathbf{d}^\alpha(\mu,\nu) = \mathbf{d}^\alpha(\mathcal{X},\nu)$ where

$$\mathbf{d}^\alpha(\mathcal{X},\nu) := \inf\left\{\mathbf{d}^\alpha(\mu',\nu) : \operatorname{spt}\mu' \subseteq \mathcal{X}\right\},$$

we say that μ is an N-point *mass-optimal quantizer* with respect to $\{x_i\}_{1\le i\le N}$.

We will also need to decompose any traffic plan $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$, where μ is purely atomic, with respect to the atoms of μ, also called the *sources of* $\mathbf{P}$.

[6] In the compact case, lower semicontinuity of $\mathbf{d}^\alpha$ is obtained from [BCM09, Chapter 3], see for example [Peg17a, Section 1.2.3] for the case of $\mathbb{R}^d$.

Definition 2.7 (Restrictions and basins from a source)**.** If $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ where μ is purely atomic and x is an atom of μ, the *restriction of* $\mathbf{P}$ *from the source* x is defined by

$$\mathbf{P}^x \coloneqq \mathbf{P} \llcorner \{\gamma \in \Gamma^d : \gamma(0) = x\},$$

so that the following *source decomposition* of $\mathbf{P}$ holds:

$$\mathbf{P} = \sum_{x \in \operatorname{spt} \mu} \mathbf{P}^x.$$

The decomposition is said *disjoint* if all these restrictions $(\mathbf{P}^x)_{x \in \operatorname{spt} \mu}$ are pairwise disjoint (as defined in (2.3)). We also introduce the *basin from* x *with respect to* $\mathbf{P}$ as the support of the sink measure of $\mathbf{P}^x$:

$$\operatorname{Bas}(\mathbf{P}, x) \coloneqq \operatorname{spt}((e_\infty)_\sharp \mathbf{P}^x).$$

In the next lemma, we show that the source decomposition of an optimal traffic plan between a measure and a mass-optimal quantizer is disjoint in the above sense, and that the corresponding sink measures are mutually singular. This result plays a key role in the proof of equivalence between the optimal quantization and optimal partition, and will also be crucial in Section 4 to define and study the landscape function, and subsequently to show the disjointness of irrigation basins.

Lemma 2.8 (Disjointness properties of mass-optimal quantizers)**.** *Let $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ be an N-point mass-optimal quantizer of $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ with respect to $\mathcal{X} \coloneqq \{x_i\}_{1 \le i \le N}$ and $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan. Then:*

(i) the traffic plans $\mathbf{P}^{x_i}$ are disjoint. In particular they are optimal (for their own marginals), and

$$\mathbb{M}^\alpha(\mathbf{P}) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i});$$

(ii) the irrigated measures $\nu_i \coloneqq (e_\infty)_\sharp \mathbf{P}^{x_i}$ are mutually singular.

Proof. Let us start by proving (i). Since $\mathbf{P}$ is rectifiable, so are the $\mathbf{P}^{x_i}$'s, thus by (2.4) it suffices to show that the measures $\Theta_{\mathbf{P}^{x_i}} \mathscr{H}^1$ are mutually singular. By contradiction, assume that for some $i \neq j$, $\Theta_{\mathbf{P}^{x_i}} \mathscr{H}^1$ and $\Theta_{\mathbf{P}^{x_j}} \mathscr{H}^1$ are not mutually singular. We shall contradict the optimality of $\mathbf{P}$. There exists a Borel set $A \subseteq \Sigma_{\mathbf{P}}$ and a constant $m_0 > 0$ such that $\mathscr{H}^1(A) > 0$ and $\Theta_{\mathbf{P}^{x_i}}(x) \wedge \Theta_{\mathbf{P}^{x_j}}(x) \geq m_0$ for every $x \in A$. Pick a point $x \in A$ with $x \neq x_i$ and $x \neq x_j$ and consider for $k \in \{i, j\}$ a traffic plan

$$\mathbf{P}_k \leq \mathbf{P} \llcorner \{\gamma \in \Gamma^d : \gamma(0) = x_k, x \in \gamma(\mathbb{R}_+)\} \quad \text{such that} \quad \|\mathbf{P}_k\| = m_0.$$

For every $\varepsilon \in [0, 1]$, we will build a traffic plan $\mathbf{P}_\varepsilon$, obtained from $\mathbf{P}$ by taking a fraction ε of $\mathbf{P}_i$, replacing for each curve γ of $\mathbf{P}_i$ the curve segment between x_i and x by a segment of a curve $\tilde{\gamma}$ of $\mathbf{P}_j$ from x_j to x. To do this, consider the map $t_x : \gamma \mapsto \min \gamma^{-1}(\{x\})$ and the restriction maps $r_x^- : \gamma \mapsto \gamma_{|[0, t_x(\gamma)]}$, $r_x^+ : \gamma \mapsto \gamma_{|[t_x(\gamma), +\infty)}$. Then

$$(e_\infty)_\sharp (r_x^-)_\sharp \mathbf{P}_j = (e_0)_\sharp (r_x^+)_\sharp \mathbf{P}_i = m_0 \delta_x,$$

and by Proposition 2.1 (i) there exists a concatenation $\mathbf{Q} \in ((r_x^-)_\sharp \mathbf{P}_j : (r_x^+)_\sharp \mathbf{P}_i)$. For $\varepsilon \in [0, 1)$ set

$$\mathbf{P}_\varepsilon \coloneqq \mathbf{P} + \varepsilon(\mathbf{Q} - \mathbf{P}_i).$$

We shall do the converse operation for $\varepsilon \in (-1, 0)$, namely

$$\mathbf{P}_\varepsilon \coloneqq \mathbf{P} - \varepsilon(\mathbf{Q}' - \mathbf{P}_j), \quad \text{where} \quad \mathbf{Q}' \in ((r_x^-)_\sharp \mathbf{P}_i : (r_x^+)_\sharp \mathbf{P}_j).$$

Notice that for both possible signs of ε, $\mathbf{P}_\varepsilon$ is rectifiable, $\Sigma_{\mathbf{P}_\varepsilon} \subseteq \Sigma_{\mathbf{P}}$ and

$$\Theta_{\mathbf{P}_\varepsilon} = \Theta_{\mathbf{P}} + \varepsilon\Delta\Theta \quad \text{where} \quad \Delta\Theta := \Theta_{(r_x^-)_\sharp \mathbf{P}_j} - \Theta_{(r_x^-)_\sharp \mathbf{P}_i}. \tag{2.13}$$

Indeed if for example $\varepsilon \geq 0$ then by Proposition 2.1 (iii) we have $\Theta_{\mathbf{Q}} = \Theta_{(r_x^-)_\sharp \mathbf{P}_j} + \Theta_{(r_x^+)_\sharp \mathbf{P}_i}$, and (2.13) follows because $\Theta_{\mathbf{P}_i} = \Theta_{(r_x^-)_\sharp \mathbf{P}_i} + \Theta_{(r_x^+)_\sharp \mathbf{P}_i}$.

Now, the initial measure $\mu_\varepsilon := (e_0)_\sharp \mathbf{P}_\varepsilon$ of $\mathbf{P}_\varepsilon$ is still supported on $\{x_i : 1 \leq i \leq n\}$ and its final measure is still ν, thus by mass-optimality of μ,

$$\int_{\mathbb{R}^d} \Theta_{\mathbf{P}}^\alpha \, \mathrm{d}\mathscr{H}^1 = \mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}_\varepsilon) = \int_{\Sigma_{\mathbf{P}}} (\Theta_{\mathbf{P}} + \varepsilon\Delta\Theta)^\alpha \, \mathrm{d}\mathscr{H}^1.$$

For $k \in \{i, j\}$, by the single-path property recalled in (2.10), $\mathbf{P}$-a.e. curve starting at x_k and visiting x follows a trajectory given by a single (simple, parameterized by arc length) curve γ_k such that $\gamma_k(0) = x_k$, $\gamma_k(\infty) = x$ and $\gamma_k(\mathbb{R}_+) \subseteq \Sigma_{\mathbf{P}}$; in particular, $\Theta_{(r_x^-)_\sharp \mathbf{P}_k} = m_0 \mathbf{1}_{\gamma_k(\mathbb{R}_+)}$. Since x_i, x_j, x are distinct points, we get

$$\mathscr{H}^1(\{y \in \Sigma_{\mathbf{P}} : \Delta\Theta(y) \neq 0\}) = \mathscr{H}^1(\gamma_i(\mathbb{R}_+)\Delta\gamma_j(\mathbb{R}_+)) > 0,$$

and as $\alpha \in (0,1)$, the function $\varepsilon \mapsto \int_{\Sigma_{\mathbf{P}}} (\Theta_{\mathbf{P}} + \varepsilon\Delta\Theta)^\alpha \, \mathrm{d}\mathscr{H}^1$ is finite[7] and strictly concave on $(-1,1)$. But it is minimized at $\varepsilon = 0$: a contradiction. Consequently, $\Theta_{\mathbf{P}^{x_i}} \mathscr{H}^1$ and $\Theta_{\mathbf{P}^{x_j}} \mathscr{H}^1$ are mutually singular for every $i \neq j$.

Since $\mathbf{P} = \sum_{i=1}^N \mathbf{P}^{x_i}$ is a disjoint decomposition, by (2.6) we get that $\mathbb{M}^\alpha(\mathbf{P}) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i})$ and the optimality of each $\mathbf{P}^{x_i}$ for $i \in \{1, \ldots, N\}$ follows from that of $\mathbf{P}$.

Let us now prove (ii). For every $i \in \{1, \ldots, N\}$, the traffic plan $\mathbf{P}^{x_i}$ being optimal with a single source, we may consider its landscape functions $z_{\mathbf{P}^{x_i}}$ as in (2.11). By contradiction assume that $\nu_i \perp \nu_j$ does not hold for some $i \neq j$. Then we have:

$$m := \|\tilde{\nu}\| > 0 \quad \text{where} \quad \tilde{\nu} := \nu_i \wedge \nu_j.$$

For $k \in \{i, j\}$ consider a plan $\mathbf{P}_k \leq \mathbf{P}^{x_k}$ such that $\mathbf{P}_k \in \mathbf{TP}(m\delta_{x_k}, \tilde{\nu})$, and define for $\varepsilon \in (-1,1) \setminus \{0\}$ the competitor

$$\mathbf{P}_\varepsilon = \mathbf{P} + \varepsilon(\mathbf{P}_j - \mathbf{P}_i).$$

Its initial measure $(e_0)_\sharp \mathbf{P}_\varepsilon$ is still supported on $\{x_i : 1 \leq i \leq N\}$ and its final measure is still ν, thus by mass-optimality of μ we get

$$\begin{aligned}
&\mathbb{M}^\alpha(\mathbf{P}) = \mathbf{d}^\alpha(\mathcal{X}, \nu) \leq \mathbb{M}^\alpha(\mathbf{P}_\varepsilon) \\
&\leq \sum_{k \neq i,j} \mathbb{M}^\alpha(\mathbf{P}^{x_k}) + \mathbb{M}^\alpha(\mathbf{P}^{x_i} - \varepsilon\mathbf{P}_i) + \mathbb{M}^\alpha(\mathbf{P}^{x_j} + \varepsilon\mathbf{P}_j) \\
&< \sum_{k \neq i,j} \mathbb{M}^\alpha(\mathbf{P}^{x_k}) + \mathbb{M}^\alpha(\mathbf{P}^{x_i}) - \alpha\varepsilon \int_{\mathbb{R}^d} z_{\mathbf{P}^{x_i}} \, \mathrm{d}(e_\infty)_\sharp \mathbf{P}_i + \mathbb{M}^\alpha(\mathbf{P}^{x_j}) + \alpha\varepsilon \int_{\mathbb{R}^d} z_{\mathbf{P}^{x_j}} \, \mathrm{d}(e_\infty)_\sharp \mathbf{P}_j \\
&= \mathbb{M}^\alpha(\mathbf{P}) + \alpha\varepsilon \int_{\mathbb{R}^d} (z_{\mathbf{P}^{x_j}} - z_{\mathbf{P}^{x_i}}) \, \mathrm{d}\tilde{\nu},
\end{aligned}$$

where we have used the first variation inequality Proposition 2.3 (C) twice on the third line. The inequality in the third line is strict for $\epsilon \in (-1,1) \setminus \{0\}$, because for $k \in \{i, j\}$, $\Theta_{\mathbf{P}_k} \leq \Theta_{\mathbf{P}^{x_k}}$ thus for every $y \in \mathbb{R}^d$ such that $\Theta_{\mathbf{P}_k}(y) > 0$ we have $|\varepsilon|\Theta_{\mathbf{P}_k} < \Theta_{\mathbf{P}^{x_k}}$: this strict inequality holds on a $\mathscr{H}^1$-positive subset of $\Sigma_{\mathbf{P}^{x_k}}$. We can now choose ε such that $\varepsilon \int_{\mathbb{R}^d} (z_{\mathbf{P}^{x_j}} - z_{\mathbf{P}^{x_i}}) \, \mathrm{d}\tilde{\nu} \leq 0$, and we get the contradiction $\mathbb{M}^\alpha(\mathbf{P}) < \mathbb{M}^\alpha(\mathbf{P})$. □

[7] Since $|\Delta\Theta| \leq \Theta_{\mathbf{P}}$ and $\mathbb{M}^\alpha(\mathbf{P}) < +\infty$.

2.3 The optimal partition problem and equivalence with optimal quantization

For $\alpha \in (0,1)$ and any finite nonnegative measure ν, we define

$$\mathbf{C}^\alpha(\nu) := \inf_{x\in\mathbb{R}^d} \mathbf{d}^\alpha(\|\nu\|\delta_x, \nu).$$

Given a compactly supported measure ν on $\mathbb{R}^d$ and an integer $N > 1$, we define the optimal (branched) ν-partition problem as:

$$\inf\left\{\sum_{i=1}^N \mathbf{C}^\alpha(\nu \llcorner \Omega_i) : \ (\Omega_i)_{1\le i\le N} \subseteq \operatorname{spt}\nu, \nu(\mathbb{R}^d \setminus \bigcup_i \Omega_i) = 0 \text{ and } \nu(\Omega_i \cap \Omega_j) = 0\, (\forall i \neq j)\right\}. \tag{2.14}$$

It may be equivalently written in terms of measures as:

$$\inf\left\{\sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) : \ (\nu_i)_{1\le i\le N}, \nu = \sum_{1\le i\le N} \nu_i \text{ and } \nu_i \perp \nu_j\, (\forall i \neq j)\right\}. \tag{2.15}$$

For more classical transport costs, e.g. corresponding to Wasserstein distances W_p ($p \geq 1$), it is straightforward to see that optimal quantization is equivalent to an optimal partition problem, where the optimal partitions are given by Voronoï diagrams associated with a finite set of points. With our branched transportation cost $\mathbf{C}^\alpha$ (corresponding to the distance $\mathbf{d}^\alpha$), the situation is a priori much more difficult, since there is no clear decomposition of the target space into regions associated with the atoms of a quantizer: on the contrary, in branched transport it is expected that several atoms are first collected together along a graph, then irrigate some part of the target measure, so that we cannot associate these irrigated points with a single atom. However, we have seen in Lemma 2.8 that such situations do not occur for *(mass-)optimal* quantizers, resulting in the equivalence between the optimal quantization and optimal partition problems.

Note that the existence of minimizers for (2.15) is not direct from functional analysis results, since the condition $\nu_i \perp \nu_j$ does not pass to weak limits of measures. To prove existence of solutions, we introduce the following relaxed optimal partition problem

$$\inf\left\{\sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) : \ (\nu_i)_{1\le i\le N}, \nu = \sum_{1\le i\le N} \nu_i\right\}. \tag{2.16}$$

We shall see below that the optimal quantization problem and the original and relaxed partition problems are equivalent (Theorem 2.9), and obtain existence to (2.15) in Corollary 2.10.

Theorem 2.9 (Optimal quantization $\simeq$ optimal partition)**.** *Given a measure $\nu \in \mathscr{M}^+(\mathbb{R}^d)$, the minimal values of the optimal quantization problem (2.12) and the optimal partition problem (2.15), as well as its relaxation (2.16), are equal. Furthermore, when the optimal values are finite the minimizers of these problems are related as follows:*

(i) If $\mu_N = \sum_{i=1}^N m_i \delta_{x_i}$ is solution of (2.12) with optimal traffic plan $\mathbf{P} \in \mathbf{TP}(\mu_N, \nu)$ then the irrigated measures $\nu_i := (e_\infty)_\sharp \mathbf{P}^{x_i}$ for $1 \le i \le N$ form an optimizer of (2.15).

(ii) If $(\nu_i)_{1\le i\le N}$ is an optimizer of (2.15) and if for every i, $x_i \in \mathbb{R}^d$ and $\mathbf{P}_i \in \mathbf{TP}(\|\nu_i\|\delta_{x_i}, \nu_i)$ are optimal, i.e. $\mathbf{C}^\alpha(\nu_i) = \mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \mathbb{M}^\alpha(\mathbf{P}_i)$, then $\mu_N := \sum_{i=1}^N \|\nu_i\|\delta_{x_i}$ is an optimizer for (2.12) and $\mathbf{P} := \sum_{i=1}^N \mathbf{P}_i \in \mathbf{TP}(\mu_N, \nu)$ is an optimal traffic plan.

(iii) The optimal partition problem (2.15) and the relaxed problem (2.16) have the same minimizers and minimal value.

Proof of Theorem 2.9. Denote by $\mathcal{E}^\alpha_{\mathrm{p}}(\nu, N)$ and $\mathcal{E}^\alpha_{\mathrm{pr}}(\nu, N)$ the infima of (2.15) and (2.16) respectively. Assuming that $\mathcal{E}^\alpha(\nu, N) < +\infty$, take $\mu = \sum_{i=1}^N m_i\delta_{x_i}$ a minimizer of (2.12), which exists by Theorem 2.5, and an optimal traffic plan $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$. By Lemma 2.8 (ii), the irrigated measures $\nu_i = (e_\infty)_\sharp \mathbf{P}^{x_i}$ are mutually singular and $\nu = \sum_{i=1}^N \nu_i$. In particular, $(\nu_i)_{1\le i\le N}$ is a competitor for (2.15). Besides, by Lemma 2.8 (i) the traffic plans $\mathbf{P}^{x_i} \in \mathbf{TP}(m_i\delta_{x_i}, \nu_i)$ are disjoint and optimal, thus:

$$\mathcal{E}^\alpha_{\mathrm{p}}(\nu, N) \le \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) \le \sum_{i=1}^N \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu_i) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i}) = \mathbb{M}^\alpha(\mathbf{P}) = \mathcal{E}^\alpha(\nu, N). \tag{2.17}$$

Viceversa, assuming $\mathcal{E}^\alpha_{\mathrm{pr}}(\nu, N) < +\infty$, take a ε-minimizer $(\nu_i)_{1\le i\le N}$ of (2.16) for some fixed $\varepsilon > 0$. We can form a competitor for (2.12) by simply taking for each ν_i a point x_i that is optimal, i.e. such that $\mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \mathbf{C}^\alpha(\nu_i)$, and setting $\mu \coloneqq \sum_{i=1}^N \|\nu_i\|\delta_{x_i}$. Moreover, taking for every $i \in \{1, \dots, N\}$ an optimal traffic plan $\mathbf{P}_i \in \mathbf{TP}(\|\nu_i\|\delta_{x_i}, \nu_i)$, the traffic plan $\mathbf{P} \coloneqq \sum_{i=1}^N \mathbf{P}_i$ belongs to $\mathbf{TP}(\mu, \nu)$ where μ has at most N atoms, and therefore by subadditivity of the α-mass:

$$\begin{aligned}\mathcal{E}^\alpha(\nu, N) \le \mathbf{d}^\alpha(\mu, \nu) \le \mathbb{M}^\alpha(\mathbf{P}) &\le \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}_i) \\ &= \sum_{i=1}^N \mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) = \mathcal{E}^\alpha_{\mathrm{pr}}(\nu, N) + \varepsilon.\end{aligned} \tag{2.18}$$

Since ε is arbitrary, this shows $\mathcal{E}^\alpha(\nu, N) = \mathcal{E}^\alpha_{\mathrm{pr}}(\nu, N) = \mathcal{E}^\alpha_{\mathrm{p}}(\nu, N)$ and (i) holds because of (2.17) in the first part of the proof ; it implies in particular existence for the optimal partition problem and its relaxation. Besides, taking now $(\nu_i)_{1\le i\le N}$ a minimizer of the relaxed partition problem (2.16) instead of an ε-minimizer, and plugging it in the inequalities (2.18) (now $\varepsilon = 0$), shows that the quantizer μ built as above is optimal for ν, and the traffic plan $\mathbf{P}$ (also built as above) is optimal in $\mathbf{TP}(\mu, \nu)$. From (i) we deduce that the ν_i's are actually mutually singular and $(\nu_i)_{1\le i\le N}$ is a minimizer of (2.15), which in turn implies (iii). □

As direct corollary of Theorem 2.5 and Theorem 2.9, we obtain existence for the optimal partition problem.

Corollary 2.10. *For any finite positive measure $\nu \in \mathscr{M}^+(\mathbb{R}^d)$ and any $N \in \mathbb{N}^*$, the optimal partition problem (2.15) admits a solution.*

2.4 Asymptotic energy scaling and asymptotic constant

From now on and in all the following results of the paper, we shall assume

$$\alpha \in (1 - 1/d, 1)$$

where $d \in \mathbb{N}^*$ is the ambient dimension, and extensively use the exponent $\beta = \beta(\alpha, d)$ whose expression we recall:

$$\beta = 1 + d\alpha - d \in (0, 1).$$

We start by proving a general upper bound on the optimal quantization error by N points.

Lemma 2.11. *Let $\nu \in \mathscr{M}^+_c(\mathbb{R}^d)$ be a finite measure and $\alpha \in (1 - 1/d, 1)$. If ν is supported on a cube Q of edge length r, it holds:*

$$\mathcal{E}^\alpha(\nu, N) \le C_{\mathsf{BOT}}(\alpha, d) N^{-\beta/d} r \|\nu\|^\alpha.$$

Proof. Without loss of generality suppose that $Q = Q_1^d$. Take $n \in \mathbb{N}^*$ such that $n^d \le N < (n+1)^d$ and divide the cube Q into n^d subcubes $\{Q_i\}_{1\le i\le n^d}$ of edge length $\frac{1}{n}$. By concavity of $\mathbb{R}_+ \ni m \mapsto m^\alpha$ and (2.7) we have

$$\begin{aligned}\mathcal{E}^\alpha(\nu, N) &\le \mathcal{E}^\alpha(\nu, n^d) \le \sum_{i=1}^{n^d} \mathcal{E}^\alpha(\nu \llcorner Q_i, 1) \le \sum_{i=1}^{n^d} \frac{C_{\mathsf{BOT}}(\alpha, d)}{2} n^{-1} \nu(Q_i)^\alpha \\ &\le \frac{C_{\mathsf{BOT}}(\alpha, d)}{2} n^{-1} n^d \left(\frac{\|\nu\|}{n^d}\right)^\alpha = \frac{C_{\mathsf{BOT}}(\alpha, d)}{2} n^{-\beta} \|\nu\|^\alpha \le C_{\mathsf{BOT}}(\alpha, d) N^{-\beta/d} \|\nu\|^\alpha. \qquad \square\end{aligned}$$

We show that the optimal N-point quantization error of the unit cube behaves as some negative power of N times a nontrivial constant $c_{\alpha,d}$, when the Lebesgue measure is α-irrigable.

Proposition 2.12. *If $\alpha \in (1 - 1/d, 1)$, then there exists a constant $c_{\alpha,d} \in (0, +\infty)$ such that*

$$\lim_{N\to+\infty} N^{\beta/d} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) = c_{\alpha,d}. \tag{2.19}$$

The proof is based on a classical result on subadditive processes in ergodic theory (see e.g. [LM02]).

Proof. Define for every Borel set $A \subseteq \mathbb{R}^d$ and $N \in \mathbb{N}$,

$$\mathcal{S}^\alpha(A) = \mathcal{E}^\alpha(\mathscr{L}^d \llcorner A, \lfloor \mathscr{L}^d(A) \rfloor).$$

Notice that for every $N \in \mathbb{N}^*$, by 1-homogeneity in space and α-homogeneity in mass of the α-mass, we have

$$\begin{aligned}N^{\beta/d} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) &= \frac{1}{N} N^{1/d+\alpha} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) \\ &= \frac{1}{N} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0, N^{1/d}]^d, N) = \frac{\mathcal{S}^\alpha(Q_N)}{N}\end{aligned}$$

where $Q_N := [0, N^{1/d}]^d$ is a cube of volume N. By [LM02, Theorem 2.1], any nonnegative subadditive translation-invariant function $\mathcal{S}$ defined on bounded Borel subsets of $\mathbb{R}^d$ satisfies

$$\lim_{N\to+\infty} \frac{\mathcal{S}(Q_N)}{N} = \inf_{n\in\mathbb{N}^*} \frac{\mathcal{S}([0,n)^d)}{n^d},$$

hence it suffices to show that $\mathcal{S}^\alpha$ is subadditive, the translation invariance being trivial. Subadditivity is a direct consequence of the subadditivity of $\mathbb{M}^\alpha$ and the superadditivity of the integer part. Indeed, take A_1, A_2 two disjoint bounded Borel subsets of $\mathbb{R}^d$, then for any $i \in \{1, 2\}$ an optimal quantizer μ_i of $\mathscr{L}^d \llcorner A_i$ with at most $\lfloor \mathscr{L}^d(A_i) \rfloor$ atoms, and an optimal traffic plan $\mathbf{P}_i \in \mathbf{TP}(\mu_i, \mathscr{L}^d \llcorner A_i)$. Since $\mathbf{P}_1 + \mathbf{P}_2 \in \mathbf{TP}(\mu_1 + \mu_2, \mathscr{L}^d \llcorner (A_1 \sqcup A_2))$ and the number N of atoms of $\mu_1 + \mu_2$ satisfies

$$N \le \lfloor \mathscr{L}^d(A_1) \rfloor + \lfloor \mathscr{L}^d(A_1) \rfloor \le \lfloor \mathscr{L}^d(A_1 \sqcup A_2) \rfloor,$$

we obtain

$$\begin{aligned}\mathcal{S}^\alpha(A_1 \sqcup A_2) &= \mathcal{E}^\alpha(\mathscr{L}^d \llcorner (A_1 \sqcup A_2), \lfloor \mathscr{L}^d(A_1 \sqcup A_2) \rfloor) \\ &\le \mathbf{d}^\alpha(\mu_1 + \mu_2, \mathscr{L}^d \llcorner (A_1 \sqcup A_2)) \\ &\le \mathbb{M}^\alpha(\mathbf{P}_1 + \mathbf{P}_2) \\ &\le \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2) = \mathcal{S}^\alpha(A_1) + \mathcal{S}^\alpha(A_2).\end{aligned}$$

We have thus proven the existence of the constant $c_{\alpha,d} \in [0, +\infty]$ of the statement. It is finite because by Lemma 2.11,

$$c_{\alpha,d} = \inf_{n\in\mathbb{N}^*} \frac{\mathcal{S}^\alpha([0,n)^d)}{n^d} = \inf_{n\in\mathbb{N}^*} n^\beta \mathcal{E}^\alpha([0,1]^d, n^d) \le C_{\mathsf{BOT}}(\alpha, d) < +\infty.$$

We next show that $c_{\alpha,d}$ is strictly positive. By [PSX19, Theorem 2.1], the constant

$$e_{\alpha,d} \coloneqq \inf\left\{\mathbf{d}^\alpha(\delta_0,\rho) : \rho \in \mathscr{P}(\mathbb{R}^d), \rho \leq \mathscr{L}^d\right\} \tag{2.20}$$

is a strictly positive real number. Let $\mu_N = \sum_{i=1}^N m_i \delta_{x_i}$ be an N-point optimal quantizer of $\mathscr{L}^d \llcorner [0,1]^d$ and $\mathbf{P} \in \mathbf{TP}(\mu_N, \mathscr{L}^d \llcorner [0,1]^d)$ an optimal traffic plan. Using Lemma 2.8, noticing that for every $i \in \{1,\dots,N\}$, $\nu_i \coloneqq (e_\infty)_\sharp \mathbf{P}^{x_i} \leq \mathscr{L}^d$, and using again the homogeneity properties of the α-mass, we get:

$$\begin{aligned}
\mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) &= \sum_{i=1}^N \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu_i) \\
&\geq \sum_{i=1}^N m_i^{\alpha+\frac{1}{d}} e_{\alpha,d} \geq N(1/N)^{\alpha+\frac{1}{d}} e_{\alpha,d},
\end{aligned}$$

where the last inequality is due to the convexity of $m \mapsto m^{\alpha+\frac{1}{d}}$ (because $\alpha + \frac{1}{d} > 1$). This implies that $c_{\alpha,d} \geq e_{\alpha,d} > 0$ and concludes the proof.

□

3 Γ-convergence and Zador-type Theorem

We are now going to provide an equivalent for the optimal quantization error of a compactly supported finite measure $\nu \ll \mathscr{L}^d$ as the number of points goes to infinity, analogous to the classical Zador's Theorem (see [GL00, Theorem 6.2], or the original papers [Zad63; Zad82; BW82]), which states in particular that

$$\mathcal{E}^{W_2^2}(\nu) N^{-\frac{2}{d}} \xrightarrow{N\to+\infty} \mathcal{E}^{W_2^2}(\mathscr{L}^d \llcorner [0,1]^d) \|\nu\|_{\frac{d}{d+2}},$$

where $\mathcal{E}^{W_2^2}(\nu) \coloneqq \inf\{W_2(\mu_N,\nu)^2 : \mu_N \in \mathscr{M}_N\}$ and W_2 denotes the 2-Wasserstein distance over probability measures. We shall also be interested in the limit distribution of centers of N-point optimal quantizers μ_N, i.e. to the weak limit of

$$\mu_N^\diamond \coloneqq \frac{1}{\#\operatorname{spt}\mu_N} \sum_{\{x:\mu_N(\{x\})>0\}} \delta_x.$$

We tackle the two questions simultaneously by establishing a (stronger) Γ-convergence result, inspired from of [BJM02; BJM11].

3.1 A Γ-convergence result

We establish a Γ-convergence result in the spirit of [BJM02], in a form that is slighly more concise. We do not follow the extended approach of [BJM11], where the functionals depend on the quantizers μ_N and also on an extra variable that encodes the distributions of masses (as measures over $\mathbb{R}_+$), since the Γ-limit does not have a fully explicit expression in this case, and we are not able to derive useful informations from it. Instead, the functionals $\mathcal{F}_N$ that we consider will depend solely on sets Σ of N points, embedded in the space of probability measures through their empirical measures $\frac{1}{N}\sum_{s\in\Sigma}\delta_s$, leading to the definition

$$\mathscr{X}_N \coloneqq \left\{\frac{1}{N}\sum_{s\in\Sigma}\delta_s : \#S = N\right\} \qquad (\forall N \in \mathbb{N}^*).$$

We fix a compactly supported measure $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ such that $\nu \ll \mathscr{L}^d$. We consider the sequence of functionals $\mathcal{F}_N : \mathscr{P}(\mathbb{R}^d) \to [0,+\infty]$ defined for every $N \in \mathbb{N}^*$ by

$$\mathcal{F}_N(\rho) = \begin{cases} N^{\beta/d} \inf\{\mathbf{d}^\alpha(\mu,\nu) : \operatorname{spt}\mu \subseteq \operatorname{spt}\rho\} & \text{if } \rho \in \mathscr{X}_N, \\ +\infty & \text{otherwise.} \end{cases}$$

Determining the Γ-limit of the sequence $(\mathcal{F}_N)_{N\geq 1}$ amounts to seeking out the least (asymptotic) energy to approximate, in the sense of branched optimal transport, the measure ν by N-point quantizers μ_N while prescribing the limit density of the centers $(\mu_N^\diamond)_{N\geq 1}$, which will correspond to the ρ variable. We shall prove that the Γ-limit is the functional $\mathcal{F}_\infty : \mathscr{P}(\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\mathcal{F}_\infty(\rho) = c_{\alpha,d} \int_{\mathbb{R}^d} \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\frac{\beta}{d}}} \,\mathrm{d}x$$

where we recall $\beta = 1 + d\alpha - d$, $c_{\alpha,d}$ is the constant defined in (2.19) and $\rho_{\mathrm{ac}} = \frac{\mathrm{d}\rho}{\mathrm{d}\mathscr{L}^d}$.

Theorem 3.1. *Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ such that $\nu \ll \mathscr{L}^d$ and $\alpha \in (1-1/d,1)$. The sequence of functionals $(\mathcal{F}_N)_{N\geq 1}$ Γ-converges to $\mathcal{F}_\infty$ as $N\to\infty$ with respect to the narrow convergence of probability measures.*

We are going to use the following lemmas.

Lemma 3.2. *Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ and $\alpha \in (1-1/d,1)$. It holds:*

$$\lim_{\delta\to 0} \liminf_{N\to+\infty} N^{\beta/d} \inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} = \liminf_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu,N).$$

Proof. Suppose that ν is supported on a closed cube Q of edge length $r>0$. First of all, it is clear that

$$\inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} \leq \mathcal{E}^\alpha(\nu,N).$$

for every $\delta > 0$. Now, let us take a small $\lambda > 0$. For every N large enough, by Lemma 2.11 and subadditivity of the α-mass, we have for every $\nu' \leq \nu$:

$$\begin{aligned}\mathcal{E}^\alpha(\nu, N + \lceil\lambda N\rceil) &\leq \mathcal{E}^\alpha(\nu',N) + \mathcal{E}^\alpha(\nu-\nu', \lceil\lambda N\rceil) \\ &\leq \mathcal{E}^\alpha(\nu',N) + C(\alpha,d)N^{-\beta/d} r\|\nu-\nu'\|^\alpha \lambda^{-\beta/d},\end{aligned}$$

hence for every $\delta > 0$,

$$\begin{aligned}&\liminf_{N\to+\infty} N^{\beta/d} \inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} \\ \geq &\liminf_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu, N+\lceil\lambda N\rceil)) - C(\alpha,d) r\delta^\alpha\lambda^{-\beta/d} \\ \geq &(1+\lambda)^{-\beta/d} \liminf_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu,N) - C(\alpha,d)r\delta^\alpha\lambda^{-\beta/d}.\end{aligned}$$

Taking the liminf as $\delta\to 0$ and then $\lambda \to 0$ yields the result. □

Lemma 3.3. *Let $\nu \in \mathscr{M}^+(Q \subseteq \mathbb{R}^d)$ be a measure over a closed cube Q of edge length R such that $\nu \ll \mathscr{L}^d$, $\alpha \in (1-1/d,1)$ and $(A_i)_{1\leq i\leq I}$ be a $\mathscr{L}^d$-essential partition[8] of Q with $\operatorname{diam}(A_i) \leq r$ for $1 \leq i \leq I$. We set*

$$\nu' := \sum_{i=1}^I m_i \mathscr{L}^d \llcorner A_i$$

where $m_i := \nu(A_i)/\mathscr{L}^d(A_i)$ if $\nu(A_i) > 0$ and $m_i := 0$ otherwise. There is a constant $C'_{\mathsf{BOT}} = C'_{\mathsf{BOT}}(\alpha,d)$ depending only on α and d such that

$$\mathbf{d}^\alpha(\nu,\nu') \leq C'_{\mathsf{BOT}} R^{1-\beta} r^\beta \|\nu'-\nu\|^\alpha.$$

[8] Meaning $\mathscr{L}^d\left(Q\Delta \bigcup_{1\leq i\leq I} A_i = 0\right)$ and $\mathscr{L}^d(A_i \cap A_i) = 0$ for every $i \neq i$.

Proof. We know from [MS07, Proposition 0.1] (also [BCM09, Proposition 6.16]) that

$$\mathbf{d}^\alpha(\mu^-,\mu^+) \le C'_{\mathsf{BOT}} W_1(\mu^-,\mu^+)^\beta \le C'_{\mathsf{BOT}} W_\infty(\mu^-,\mu^+)^\beta$$

for every probability measures $\mu^\pm \in \mathscr{P}(Q_1^d)$ and some constant $C'_{\mathsf{BOT}} = C'_{\mathsf{BOT}}(\alpha,d)$. Applying it to $\mu^- = \nu - \nu \wedge \nu'$ and $\mu^+ = \nu' - \nu \wedge \nu'$ after appropriate rescalings in mass $m := \|\mu^-\|$ and distance R, we obtain:

$$\begin{aligned} &R^{-1}m^{-\alpha}\mathbf{d}^\alpha(\mu^-,\mu^+) \le C'_{\mathsf{BOT}} R^{-\beta} W_\infty(\mu^-,\mu^+)^\beta \\ \implies\ &\mathbf{d}^\alpha(\nu,\nu') \le \mathbf{d}^\alpha(\mu^-,\mu^+) \le C'_{\mathsf{BOT}} W_\infty(\mu^-,\mu^+)^\beta m^\alpha R^{1-\beta}. \end{aligned}$$

By construction, ν and ν' have equal mass on each A_i, thus the same goes for μ^- and μ^+, and since $\operatorname{diam}(A_i) \le r$ it implies that $W_\infty(\mu^-,\mu^+) \le r$. Since $\|\nu-\nu'\| = 2m$, we obtain the desired result. □

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. We are going to prove successively the $\Gamma-\liminf$ and $\Gamma-\limsup$ inequality, i.e.

$$\forall \rho \in \mathscr{P}(\mathbb{R}^d), \forall (\rho_N)_{N\in\mathbb{N}^*} \xrightarrow{\mathscr{C}_b'} \rho, \quad \liminf_N \mathcal{F}_N(\rho_N) \ge \mathcal{F}_\infty(\rho), \tag{3.1}$$

$$\forall \rho \in \mathscr{P}(\mathbb{R}^d), \exists (\rho_N)_{N\in\mathbb{N}^*} \xrightarrow{\mathscr{C}_b'} \rho, \quad \limsup_N \mathcal{F}_N(\rho_N) \le \mathcal{F}_\infty(\rho). \tag{3.2}$$

Proof of the Γ-liminf inequality (3.1). Let us take a sequence of probability measures $\rho_N \xrightarrow{\mathscr{C}_0'} \rho$, assuming without loss of generality that $\liminf_N \mathcal{F}_N(\rho_N) < +\infty$. Up to taking a subsequence, we may assume that $\mathcal{F}_N(\rho_N)$ converges to $\liminf_N \mathcal{F}_N(\rho_N)$ and

$$C := \sup_{N\in\mathbb{N}^*} \mathcal{F}_N(\rho_N) < +\infty.$$

In particular we know that for every $N \in \mathbb{N}^*$, $\rho_N = \frac{1}{N}\sum_{s\in\Sigma_N}\delta_s$ for some set Σ_N of cardinal N, and we take a mass-optimal quantizer μ_N of ν with respect to Σ_N, as well as an optimal traffic plan $\mathbf{P}_N \in \mathbf{TP}(\mu_N,\nu)$, so that

$$\mathcal{F}_N(\rho_N) = N^{\beta/d}\mathbf{d}^\alpha(\Sigma_N,\nu) = N^{\beta/d}\mathbf{d}^\alpha(\mu_N,\nu) = N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N) = N^{\beta/d}\int_{\Gamma^d} Z_{\mathbf{P}_N}\,\mathrm{d}\mathbf{P}_N(\gamma),$$

recalling $Z_{\mathbf{P}_N}$ is defined in (2.9). A standard strategy to show (3.1) is to express this energy as the total mass of some measure e_N, which converges up to subsequence to some measure e, then show a lower bound on e and use the lower semicontinuity of the norm on $\mathscr{M}(\mathbb{R}^d)$. In our branched optimal transport setting, in order to follow this strategy we will have to resort to outer measures rather than measures. More precisely, we shall bound from below the energy $\mathcal{F}_N(\rho_N)$ by the total mass $E'_N(\mathbb{R}^d)$ of some suitable outer measure E'_N, that in some sense becomes a measure asymptotically as $N \to +\infty$.

Notice that

$$CN^{-\beta/d} \ge \int_{\Gamma^d} Z_{\mathbf{P}_N}(\gamma)\,\mathrm{d}\mathbf{P}_N(\gamma) \ge \int_{\Gamma^d} L(\gamma)\,\mathrm{d}\mathbf{P}_N(\gamma),$$

so that by Markov's inequality for every $M > 0$:

$$\mathbf{P}_N(\{\gamma : L(\gamma) \ge MN^{-\beta/d}\}) \le \frac{C}{M}. \tag{3.3}$$

Consider an increasing sequence M_N tending to $+\infty$ and such that $M_N N^{-\beta/d} \to 0$, and set

$$\mathbf{P}'_N := \mathbf{P}_N \llcorner \Gamma_N \quad \text{where} \quad \Gamma_N := \{\gamma : \; L(\gamma) < M_N N^{-\beta/d}\}. \tag{3.4}$$

We define for every Borel set $A \subseteq \mathbb{R}^d$:

$$E_N(A) := N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N \llcorner e_\infty^{-1}(A)), \qquad E'_N(A) := N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A)).$$

We remark that

$$E_N(\mathbb{R}^d) = \mathcal{F}_N(\rho_N)$$

and E'_N (and also E_N) is an outer measure (being countably subadditive) and a priori is not a measure: it is possible that for two disjoint Borel sets A_1, A_2, the plans $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)$ and $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2)$ are not disjoint. However, E'_N becomes additive when $\operatorname{dist}(A_1, A_2) > 0$ and N becomes large enough. Indeed, if $M_N N^{-\beta/d} \leq \frac{1}{2}\operatorname{dist}(A_1, A_2)$, which is the case for N large enough, then for every curve $\gamma_i \in \Gamma_N \cap e_\infty^{-1}(A_i)$, $i \in \{1, 2\}$,

$$\gamma_1(\mathbb{R}) \cap \gamma_2(\mathbb{R}) = \emptyset,$$

which in turn implies that $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)$ and $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2)$ are disjoint, and thus by (2.6):

$$\mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1 \cup A_2)) = \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)) + \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2))$$

i.e.

$$E'_N(A_1 \cup A_2) = E'_N(A_1) + E'_N(A_2). \tag{3.5}$$

Notice that this additivity property does not hold a priori for E_N, which was the point for restricting it and use E'_N instead.

We know that ν-a.e. point $x \in \operatorname{spt} \nu$ satisfies

$$\fint_{Q_r(x)} |\nu - \nu(x)| \, \mathrm{d}x \to 0 \quad \text{and} \quad \nu(x) \in (0, +\infty). \tag{3.6}$$

Fix such a point x, take $\delta > 0$ such that $\rho(\partial Q_\delta(x)) = 0$ (this is true for all but countably many δ's), where $Q_\delta(x)$ denotes the closed cube $x + \delta[-1/2, 1/2]^d$, and consider the slightly smaller $\delta' = \tau\delta$ for $\tau \in (0, 1)$ (which we will send to 1 later). We denote for every $N \in \mathbb{N}^*$

$$n_{N,\delta} := \#(\Sigma_N \cap Q_\delta(x)), \qquad \nu_N := (e_\infty)_\# \mathbf{P}'_N,$$

and we define the δ'-rescalings around x

$$\begin{aligned}
\nu_{\delta'} &:= \frac{1}{\delta'^d}\left(y \mapsto \frac{y - x}{\delta'}\right)_\sharp \left(1 \wedge \frac{\nu}{\nu(x)} \mathscr{L}^d \llcorner Q_{\delta'(x)}\right), \\
\nu_{N,\delta'} &:= \frac{1}{\delta'^d}\left(y \mapsto \frac{y - x}{\delta'}\right)_\sharp \left(1 \wedge \frac{\nu_N}{\nu(x)} \mathscr{L}^d \llcorner Q_{\delta'(x)}\right).
\end{aligned}$$

For N large enough $Q_{\delta'}(x) + B(0, M_N N^{-\beta/d}) \subseteq Q_\delta(x)$ because $M_N N^{-\beta/d}$ converges to 0, hence we have the lower bounds:

$$\begin{aligned}
N^{-\beta/d} E'_N(Q_{\delta'}(x)) &= \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))) \\
&\geq \mathbf{d}^\alpha((e_0)_\#(\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))), \nu_N \llcorner Q_{\delta'}(x)) && (3.7) \\
&\geq \mathcal{E}^\alpha(\nu_N \llcorner Q_{\delta'}(x), n_N) && (3.8) \\
&\geq \mathcal{E}^\alpha(\nu_N \wedge \nu(x)\mathscr{L}^d \llcorner Q_{\delta'}(x), n_N) && (3.9) \\
&= \nu(x)^\alpha \delta'^{1+d\alpha} \mathcal{E}^\alpha(\nu_{N,\delta'}, n_N).
\end{aligned}$$

where (3.7) follows from the definition of $\mathbf{d}^\alpha$, (3.8) and (3.9) from the facts that the source measure of $\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))$ is a submeasure of $\mu_N \llcorner Q_\delta(x)$ (thus has at most n_N atoms) and that $\mathcal{E}^\alpha(\nu, n)$ is decreasing in n and increasing in ν.

For every $N \in \mathbb{N}^*$, $\nu_{N,\delta'}$ is a submeasure of $\nu_{\delta'}$ because $\nu_N \leq \nu$, and by (3.3) and (3.4) we know that $\|\nu_{\delta'} - \nu_{N,\delta'}\| \leq C/M_N \xrightarrow{N\to+\infty} 0$, thus multipliying (3.9) by $N^{\beta/d}$, passing to the liminf in N and using Lemma 3.2 yields:

$$\begin{aligned}\liminf_{N\to+\infty} E'_N(Q_\delta(x)) &\geq \nu(x)^\alpha(\delta\tau)^{1+d\alpha} \liminf_{N\to+\infty}\left(\frac{N}{n_N}\right)^{\beta/d} \mathcal{E}^\alpha(\nu_{N,\delta'}, n_N) n_N^{\beta/d} \\ &= \frac{(\delta\tau)^{1+d\alpha}\nu(x)^\alpha}{\rho(Q_\delta(x))^{\beta/d}} \liminf_{n\to+\infty} n^{\beta/d}\mathcal{E}^\alpha(\nu_{\delta'}, n)\end{aligned} \tag{3.10}$$

because

$$\frac{n_N}{N} = \rho_N(Q_\delta) \to \rho(Q_\delta(x)),$$

since $\rho_N \xrightarrow{\mathscr{C}'_0} \nu$ and $\rho(\partial Q_\delta(x)) = 0$. Notice that $\nu_{\delta'} \leq \mathscr{L}^d \llcorner Q_1$ and by (3.6) that $\|\nu_{\delta'}\| \to 1 = \mathscr{L}^d(Q_1)$ hence $\|\nu_{\delta'} - \mathscr{L}^d \llcorner Q_1\| \xrightarrow{\delta\to 0} 0$. We divide by δ^d, pass to the limsup as $\delta \to 0$, and then take $\tau \to 1$ recalling that $\delta' = \tau\delta$, and use Lemma 3.2 again, obtaining

$$\begin{aligned}\limsup_{\delta\to 0} \frac{\liminf_N E'_N(Q_\delta(x))}{\mathscr{L}^d(Q_\delta(x))} &\geq \nu(x)^\alpha \limsup_{\delta\to 0} \frac{\delta^\beta}{\rho(Q_\delta(x))^{\beta/d}} \liminf_{n\to+\infty} n^{\beta/d}\mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, n) \\ &= \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}} c_{\alpha,d},\end{aligned} \tag{3.11}$$

which holds for $\mathscr{L}^d$-a.e. (thus ν-a.e.) x by Radon-Nikodym Theorem.

Now, we conclude by applying a covering argument. For fixed $\varepsilon \in (0,1)$ we consider the collection $\mathcal{Q}_\varepsilon$ of cubes $Q_\delta(x)$, $\delta \in (0,1]$, $x \in \mathbb{R}^d$ such that

(i) $\frac{(\varepsilon^{-1}\wedge\rho(x))^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}}(x))^{\beta/d}} \geq ⨍_{Q_\delta(x)} \frac{(\varepsilon^{-1}\wedge\nu)^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}})^{\beta/d}} - \varepsilon$,

(ii) $\frac{\liminf_N E_N(Q_\delta(x))}{\mathscr{L}^d(Q_\delta(x))} \geq c_{\alpha,d}\frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}} - \varepsilon$.

For any fixed $R > 0$, the set of cubes $\mathcal{Q}_\epsilon$ form a fine cover of $\mathscr{L}^d$-a.e. $K_R := \{x \in \mathbb{R}^d : \nu(x) > 0\} \cap Q_R(0)$ because of (3.11) and the fact that for $\mathscr{L}^d$-a.e. $x \in K_R$ we have

$$\lim_{\delta\to 0} ⨍_{Q_\delta(x)} \frac{(\varepsilon^{-1}\wedge\nu)^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}})^{\beta/d}} = \frac{(\varepsilon^{-1}\wedge\nu(x))^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}}(x))^{\beta/d}}$$

thanks to Lebesgue differentiation theorem applied to $\frac{(\mu\wedge\varepsilon^{-1})^\alpha}{(\varepsilon\vee\nu_{\mathrm{ac}})^{\beta/d}} \in L^1(K_R)$. Then, using the Vitali-Besicovitch covering theorem, there exists a countable family of disjoint cubes $(Q_{\delta_i}(x_i))_{i<I} \subseteq \mathcal{Q}_\varepsilon$, $I \in \mathbb{N} \cup \{+\infty\}$, that cover K_R up to a $\mathscr{L}^d$-negligible set. Using above properties (i) and (ii) of the collection $\mathcal{Q}_\epsilon$, we get that for every $J < I$,

$$\begin{aligned}\liminf_{N\to+\infty} E'_N(K_R) &\geq \liminf_{N\to+\infty} E'_N\Big(\bigcup_{i\leq J} Q_{\delta_i}(x_i)\Big) \\ &\overset{(3.5)}{=} \liminf_{N\to+\infty} \sum_{i\leq J} E'_N(Q_{\delta_i}(x_i)) \\ &\overset{(ii)}{\geq} \sum_{i\leq J} c_{\alpha,d}\frac{\delta_i^d\nu(x_i)^\alpha}{\rho_{\mathrm{ac}}(x_i)^{\beta/d}} - \varepsilon\delta_i^d\end{aligned}$$

$$\overset{\text{(i)}}{\geq} c_{\alpha,d} \sum_{i\leq J} \int_{Q_{\delta_i}(x_i)} \frac{(\varepsilon^{-1} \wedge \nu)^\alpha}{(\varepsilon \vee \rho_{\rm ac})^{\beta/d}} - 2\varepsilon \mathscr{L}^d(K_R + Q_1).$$

Taking $J \to I$, then $\varepsilon \to 0$ and $R \to +\infty$, by the Monotone Convergence Theorem we get:

$$\liminf_{N\to+\infty} \mathcal{F}_N(\rho_N) \geq \lim_{R\to+\infty} \liminf_{N\to+\infty} E'_N(K_R) \geq \lim_{R\to+\infty} c_{\alpha,d} \int_{K_R} \frac{\nu(x)^\alpha}{\rho_{\rm ac}(x)^{\beta/d}} \,\mathrm{d}x = \mathcal{E}_\infty(\rho).$$

Proof of the Γ-limsup inequality (3.2). Let us remark that the subsets

$$\mathcal{A} := \mathscr{P}_c(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \quad \text{and} \quad \mathcal{A}' := \{\rho \in \mathcal{A} : \operatorname{spt}\rho \text{ is some cube } Q \text{ and } \operatorname{ess\,inf}_Q \rho > 0\}$$

are dense in the $\mathcal{F}_\infty$ energy for the narrow convergence of measures. First of all, it is clear[9] that $\mathscr{P}_c(\mathbb{R}^d)$ is dense in energy. Then let us approximate any $\rho \in \mathscr{P}_c(\mathbb{R}^d)$ by measures in $\mathcal{A}$. Assume that it decomposes as $\rho = \rho_{\rm ac}\mathscr{L}^d + \rho_{\rm s}$ where $\rho_{\rm s} \perp \mathscr{L}^d$, and $\rho_{\rm s} \neq 0$ (otherwise there is nothing to prove). We know that there exists $\rho_{\varepsilon,\rm s}$ for $\varepsilon \in (0,1)$ which are absolutely continuous with respect to $\mathscr{L}^d \llcorner \Omega_\varepsilon$, where Ω_ε are nondecreasing subsets of $\mathbb{R}^d$ such that $\mathscr{L}^d(\Omega_\varepsilon) \leq \varepsilon$, and such that $\rho_{\rm s,\varepsilon} \xrightarrow[\varepsilon\to 0]{\mathscr{C}'_b} \rho_{\rm s}$, and $\|\rho_{\varepsilon,\rm s}\| = \|\rho_{\rm s}\|$. We set

$$\rho_\varepsilon := \rho_{\rm ac}\mathscr{L}^d + \rho_{\rm s,\varepsilon}.$$

Notice that $(\rho_\varepsilon)_{\rm ac} \geq \rho_{\rm ac}$ so that

$$\mathcal{F}_\infty(\rho) \geq \mathcal{F}_\infty(\rho_\varepsilon) \geq c_{\alpha,d} \int_{\mathbb{R}^d\setminus\Omega_\varepsilon} \frac{\nu(x)^\alpha}{\rho_{\rm ac}(x)^{\beta/d}} \,\mathrm{d}x,$$

and by the Monotone Convergence Theorem we get $\mathcal{F}_\infty(\rho_\varepsilon) \xrightarrow{\varepsilon\to 0} \mathcal{F}_\infty(\rho)$. Now to approximate any $\rho \in \mathcal{A}$ by measures in $\mathcal{A}'$, set for every $\varepsilon > 0$

$$\rho_\varepsilon := \frac{\rho \vee \varepsilon}{\|\rho \vee \varepsilon\|}.$$

It is clear that $\|\rho_\varepsilon\| = 1$ and $\rho_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathscr{C}'_b} \rho$, and by the Monotone Convergence Theorem again we get $\mathcal{F}_\infty(\rho_\varepsilon) \xrightarrow[\varepsilon\to 0]{\mathscr{C}'_b} \mathcal{F}_\infty(\rho)$. As a consequence, to prove the $\Gamma - \limsup$ inequality it suffices to find a recovery sequence for any given $\rho \in \mathcal{A}'$. Several steps are standard and inspired from [BJM02], thus some of the constructions will be quickly done.

Step 1. *(Building approximation sequences.)* Take $\rho \in \mathcal{A}'$, whose support is by definition a cube Q and assume that $\mathcal{F}_\infty(\rho) < +\infty$ (otherwise there is nothing to prove), which implies that $\operatorname{spt}\nu \subseteq Q$. Consider the collection of subcubes of Q of edge length $\lambda N^{-1/d}$ given by:

$$\{Q_{N,i} : i \in I\} := \{\lambda N^{-1/d}(k + Q_1) \subseteq Q : k \in \mathbb{Z}^d\},$$

where $\lambda \geq 1$ is taken large (and will be sent to $+\infty$ later) and define piecewise constant approximations of ν:

$$\nu_N := \sum_{i\in I} \nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i} \quad \text{where} \quad \nu_{N,i} := \frac{\nu(Q_{N,i})}{\mathscr{L}^d(Q_{N,i})} \quad (\forall i \in I).$$

[9] Simply consider the family $\frac{\rho\llcorner Q_{1/\varepsilon}}{\|\rho\llcorner Q_{1/\varepsilon}\|} \xrightarrow[\varepsilon\to 0]{\mathscr{C}'_b} \rho$ for any given $\rho \in \mathscr{P}(\mathbb{R}^d)$.

Notice that $\nu_N \to \nu$ in $L^1(\mathbb{R}^d)$. Let us build suitable N-point approximations of ρ by putting the appropriate number of points $n_{N,i}$ in each cube $Q_{N,i}$. The number $n_{N,i}$ should be approximately given by

$$N\rho(Q_{N,i}) = N(\lambda N^{-1/d})^d \rho_{N,i} = \lambda^d \rho_{N,i} \quad \text{where} \quad \rho_{N,i} := \frac{\rho(Q_{N,i})}{\mathscr{L}^d(Q_{N,i})} \quad (\forall i \in I).$$

Since

$$\sum_{i\in I} \lfloor \lambda^d \rho_{N,i} \rfloor \leq \sum_{i\in I} \lambda^d \rho_{N,i} = N\rho\Big(Q \setminus \bigcup_{i\in I} Q_{N,i}\Big) = N + o_{N\to+\infty}(N)$$

and

$$\# I \sim_{N\to+\infty} N^d \geq N$$

then for N large enough, we may choose for every i an integer $n_{N,i}$ such that

$$\lfloor \lambda^d \rho_{N,i} \rfloor \leq n_{N,i} \leq \lfloor \lambda^d \rho_{N,i} \rfloor + 1 \quad \text{and} \quad \sum_{i\in I} n_{N,i} = N. \tag{3.12}$$

Notice that if we took λ large enough, $\lambda^d \rho_{N,i} \geq \lambda^d \kappa \geq 1$ where $\kappa := \operatorname{ess\,inf}_K \rho$, so that we may assume $n_{N,i} \in \mathbb{N}^*$ for every $i \in I$.

For every $i \in I$, we take $\Sigma_{N,i}$ included in the interior of $Q_{N,i}$ as the support of a $n_{N,i}$-point quantizer of $\mathscr{L}^d \llcorner Q_{N,i}$ which is δ_N-optimal, where $\delta_N := \left(\sum_{i\in I} \nu_{i,N}^\alpha\right)^{-1} N^{\beta/d-1}$, i.e.

$$\mathcal{E}^\alpha(\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \leq \mathbf{d}^\alpha(\Sigma_{N,i}, \mathscr{L}^d \llcorner Q_{N,i}) \leq \mathcal{E}^\alpha(\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) + \delta_N, \tag{3.13}$$

and we eventually define

$$\Sigma_N := \bigsqcup_{i\in I} \Sigma_{N,i} \quad \text{and} \quad \rho_N := \frac{1}{N} \sum_{s\in\Sigma_N} \delta_s.$$

We know that $\rho_N \xrightarrow[N\to+\infty]{\mathscr{C}_b'} \rho$ because by (3.12):

$$\sup_{i\in I} |\rho_N(Q_{N,i}) - \rho(Q_{N,i})| \leq \frac{1}{N} \sup_{i\in I} |n_{N,i} - \lambda^d \rho_{N,i}| \leq \frac{1}{N} \xrightarrow{N\to+\infty} 0.$$

By the triangle inequality and (3.13) we find the following:

$$\begin{aligned}
\mathcal{F}_N(\rho_N) = N^{\beta/d} \mathbf{d}^\alpha(\Sigma_N, \nu) &\leq N^{\beta/d} \mathbf{d}^\alpha(\Sigma_N, \nu_N) + N^{\beta/d} \mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d} \mathbf{d}^\alpha\Big(\bigcup_{i\in I} \Sigma_{N,i}, \sum_{i\in I} \nu_{N,i} \mathscr{L}^d \llcorner Q_{N,i}\Big) + N^{\beta/d} \mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d} \sum_{i\in I} \Big(\mathcal{E}^\alpha(\nu_{N,i} \mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) + \nu_{N,i}^\alpha \delta_N\Big) + N^{\beta/d} \mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d} \sum_{i\in I} \mathcal{E}^\alpha(\nu_{N,i} \mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) && (3.14) \\
&\qquad + N^{\beta/d} \mathbf{d}^\alpha(\nu_N, \nu) + \frac{1}{N} && (3.15)
\end{aligned}$$

Step 2. *(Bounding (3.14).)* We have for every $i \in I$:

$$\mathcal{E}^\alpha(\nu_{N,i} \mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) = \nu_{N,i}^\alpha (N^{-1/d} \lambda)^{1+d\alpha} \mathcal{E}^\alpha(Q_1, n_{N,i}),$$

and therefore, if we set $\tilde\rho_N \coloneqq \sum_{i\in I} \rho_{N,i}\mathbf{1}_{Q_{N,i}}$ and $X_N = \bigcup_{i\in I} Q_{N,i}$,

$$\begin{aligned}
&N^{\beta/d}\sum_{i\in I}\mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i})\\
&\le \sum_{i\in I} N^{-1}\nu_{N,i}^\alpha \lambda^{1+d\alpha}\mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \rho_{N,i}\rfloor)\\
&= \sum_{i\in I}\int_{Q_{N,i}} \nu_N(x)^\alpha \mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde\rho_N(x)\rfloor)\lambda^\beta \,\mathrm{d}x\\
&\le \int_{X_N} \frac{\nu_N(x)^\alpha}{\tilde\rho_N(x)^{\beta/d}}\mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde\rho_N(x)\rfloor)(\lambda^d\tilde\rho_N(x))^{\beta/d}\,\mathrm{d}x.
\end{aligned}$$

Now note that $\tilde\rho_N \ge \kappa > 0$ for a.e. $x \in X_N$, so that

$$\mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde\rho_N(x)\rfloor)(\lambda^d\tilde\rho_N(x))^{\beta/d} \le \sup_{n\ge\lfloor\lambda^d\kappa\rfloor}\mathcal{E}^\alpha(Q_1,n)(n+1)^{\beta/d} \le c_{\alpha,d}(1+\varepsilon(\lambda)),$$

where $\varepsilon(\lambda) \xrightarrow{\lambda\to+\infty} 0$ by Proposition 2.12. Besides, $(\tilde\rho_N)$ and (ν_N) converge to ν in $L^1(\mathbb{R}^d)$ thus[10] (ν_N^α) as well. Therefore by reverse Fatou's Lemma, taking the superior limit as $N\to+\infty$ then the limit $\lambda\to+\infty$ yields

$$\limsup_{N\to+\infty} N^{\beta/d}\sum_{i\in I}\mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \le c_{\alpha,d}\int_Q \frac{\nu(x)^\alpha}{\rho(x)^{\beta/d}}\,\mathrm{d}x.$$

Step 3. *(Bounding (3.15) and conclusion.)* We apply Lemma 3.3 to the measures ν and $\nu' = \nu_N$:

$$\mathbf{d}^\alpha(\nu,\nu_N) \le C'_{\mathsf{BOT}} R^{1-\beta}(\lambda N^{-1/d})^\beta \|\nu-\nu_N\|^\alpha,$$

so that

$$N^{\beta/d}\mathbf{d}^\alpha(\nu,\nu_N) \le C'_{\mathsf{BOT}} R^{1-\beta}\lambda^\beta \|\nu-\nu_N\|^\alpha.$$

Taking the limit $N\to+\infty$, since $\nu_N\to\nu$ in L^1, we get:

$$\limsup_{N\to+\infty} N^{\beta/d}\mathbf{d}^\alpha(\nu,\nu_N) = 0. \tag{3.16}$$

By Step 2. and (3.16) we thus have

$$\limsup_{N\to+\infty}\mathcal{F}_N(\rho_N) \le c_{\alpha,d}\int_\Omega \frac{\nu(x)^\alpha}{\rho(x)^{\beta/d}}\,\mathrm{d}x = \mathcal{F}_\infty(\rho),$$

as desired.

□

Remark 3.4. There are alternative approaches for the Γ-liminf part of the proof if we assume that the measure ν is d-Ahlfors regular (see (4.1)), since we may use the Hölder regularity of the landscape function and its consequences (in particular the bound on the diameter of basins in terms of their masses) that are established in Section 4. Indeed, we may use directly the outer measures E_N defined for every Borel set A by

$$E_N(A) \coloneqq N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N \llcorner e_\infty^{-1}(A)),$$

[10] Indeed, there are all supported on the compact set Q and (ν_N^α) converges to ν^α in $L^{1/\alpha}(Q)$.

rather than the restrictions E'_N, or even use the measures defined by

$$e'_N(A) := \int_A z_{\mathbf{P}_N} \, \mathrm{d}\mu.$$

The relevance of restricting the plans (and thus of passing from E_N to E'_N), is that we can then guarantee that E'_N satisfies additivity for sets at positive distance and N large enough. But one may check that this property holds directly for E_N thanks to Corollary 4.3 and Lemma 4.4. It is even easier with e_N which is by definition a measure, although in this case we need to adapt the series of inequalities (3.7)-(3.9) which give the lower bound.

Also note that similar considerations using Hölder regularity of the landscape function under Ahlfors regularity hypotheses may also apply to Proposition 3.5 to replace the outer measure E'_N by E_N or e_N in the statement on the equi-distribution of energy at the macroscopic scale.

3.2 Asymptotics of the quantization error and support of optimal quantizers

From the Γ-convergence established in the previous subsection, we may obtain the asymptotics of the optimal quantization error (a branched optimal transport variant of Zador's theorem) and the limit density of the centers of optimal quantizers, i.e. to establish Theorem 1.1.

Proof of Theorem 1.1. Take for every $N \in \mathbb{N}^*$ a N-point optimal quantizer μ_N of ν. Since ν is concentrated on some closed cube Q, it is straightforward to see that by optimality all the μ_N's must be concentrated on Q as well. Thus the sequence of probability measures $(\mu_N^\diamond)$ converges narrowly, up to subsequence, to a measure ρ. Since for every N,

$$\mathcal{F}_N(\mu_N^\diamond) = \inf \mathcal{F}_N,$$

by Theorem 3.1 the measure ρ minimizes the Γ-limit $\mathcal{F}_\infty$ and

$$\lim_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu, N) = \lim_{N\to+\infty} \mathcal{F}_N(\mu_N^\diamond) = \mathcal{F}_\infty(\rho) \doteq c_{\alpha,d} \int_K \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\alpha+\frac{1}{d}-1}} \, \mathrm{d}x.$$

As a consequence of minimality, ρ is absolutely continuous with respect to $\mathscr{L}^d$ and the Euler-Lagrange equation can be written as:

$$\nu(x)^\alpha = (M\rho)^{\alpha+\frac{1}{d}}$$

for $\mathscr{L}^d$-a.e. $x \in \Omega$, for a constant M which is given by

$$M = M_{\alpha,d}(\nu) := \int_K \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}} \, \mathrm{d}x.$$

In particular $\rho = M_{\alpha,d}(\nu)^{-1}\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$ and

$$\lim_N N^{\beta/d}\mathcal{E}^\alpha(\nu, N) = c_{\alpha,d} M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} = \left(\int_K \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}\right)^{\alpha+\frac{1}{d}}. \qquad \square$$

3.3 Equidistribution results at the macroscopic scale

From the Γ-convergence result and its proof, we obtain convergence of measures (or outer measures) of interest to understand uniformizing features at the macroscopic scale.

Proposition 3.5. *Let $(\mu_N)_{N\in\mathbb{N}^*}$ be a sequence of N-point optimal quantizers of ν.*

(A) The empirical measures converge as follows:

$$\mu_N^\diamond \xrightarrow{\mathscr{C}_0'} M_{\alpha,d}(\nu)^{-1}\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$$

In particular if $\nu = \mathscr{L}^d \llcorner X$ for some Borel set X satisfying $\mathscr{L}^d(X) = 1$, we obtain

$$\frac{1}{N}\#(\operatorname{spt}\mu_N \cap B) \to \mathscr{L}^d(B),$$

for every Borel set $B \subseteq X$ such that $\mathscr{L}^d(\partial B) = 0$.

(B) The energy outer measures (E_N') converge in the following sense:

$$\lim_{N\to+\infty} E_N'(B) = c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_B \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$$

for every Borel set B such that $\mathscr{L}^d(\partial B) = 0$. In particular if $\nu = \mathscr{L}^d \llcorner X$ for some Borel set X satisfying $\mathscr{L}^d(X) > 0$ then

$$\lim_{N\to+\infty} E_N'(B) = c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\mathscr{L}^d(B).$$

Proof. The first item (A) is a direct consequence of Theorem 1.1. For (B), we follow the proof of the $\Gamma - \liminf$ inequality in Theorem 3.1 and apply the covering argument to the subcollection $\mathcal{Q}_\varepsilon' \subseteq \mathcal{Q}_\varepsilon$ of cubes $Q_\delta(x)$ which are included in a given open subset $\Omega \subseteq \mathbb{R}^d$. We therefore get:

$$\liminf_{N\to+\infty} E_N'(\Omega) \geq c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_\Omega \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}.$$

If B is a Borel set with $\mathscr{L}^d(\partial B) = 0$, we may apply the above inequality to $B_{<\varepsilon} := \{x : d(x, B^c) > \varepsilon\}$ and $B_{>\varepsilon} = \{x : d(x, B) > \varepsilon\}$, and using the asymptotic additivity of E_N' to the sets B and $B_{>\varepsilon}$ which are at positive distance, we obtain

$$\begin{aligned}
c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_{B_{<\varepsilon}} \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}} &\leq \liminf_{N\to+\infty} E_N'(B_{<\varepsilon}) \leq \liminf_{N\to+\infty} E_N'(B) \leq \limsup_{N\to+\infty} E_N'(B) \\
&= \limsup_{N\to+\infty} E_N'(B \cup B_{>\varepsilon}) - E_N'(B_{>\varepsilon}) \\
&\leq \limsup_{N\to+\infty} E_N(\mathbb{R}^d) - \liminf_{N\to+\infty} E_N'(B_{>\varepsilon}) \\
&\leq c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_K \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}} - c_{\alpha,d}M_{\alpha,d}(\nu)^{-(\alpha+\frac{1}{d})}\int_{B_{>\varepsilon}} \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}} \\
&= c_{\alpha,d}M_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_{\mathbb{R}^d\setminus B_{>\varepsilon}} \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}.
\end{aligned}$$

Taking the limit $\varepsilon \to 0$, we get the desired result of (B). □

4 Landscape function for mass-optimal quantizers

This section is devoted to the landscape function, its definition and Hölder regularity. We stress that the classical definition of landscape function from [San07], recalled in Section 2.1, is only given in the case of a single source $\mu = m\delta_x$ and, as already said, an optimal traffic plan with several sources may in general not decompose disjointly according to its sources. This poses a serious issue to define and study the landscape function in such a case. An attempt at defining the landscape function for several sources (even in a more general setting) has been made in [Peg17a, Chapter 4], but the construction is quite technical and the Hölder constant computed there actually explodes when the number of sources tends to infinity. However, in the case of optimal quantizers or even mass-optimal quantizers, the disjointness result established in Lemma 2.8 allows us to give a simple ad hoc definition of landscape function, and, following the approach of [San07], we are able to show its Hölder regularity with a Hölder constant that is *uniform in the number of sources*, a crucial information to establish the uniform regularity properties in Section 5.

4.1 Uniform Hölder regularity

Our main result is the following:

Theorem 4.1 (extended version of Theorem 1.3)**.** *Let $\alpha \in (1-1/d, 1)$ and $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a measure which is d-Ahlfors regular with constants $0 < c_A \leq C_A$, i.e.*

$$c_A r^d \leq \nu(B_r(x)) \leq C_A r^d \quad (\forall x \in \operatorname{spt}\nu,\ \forall r \leq \operatorname{diam}(\operatorname{spt}\nu)), \tag{4.1}$$

and let $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan where $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ is a N-point mass-optimal quantizer of ν with respect to $\{x_i\}_{1\leq i\leq N}$. There exists a unique function $z_{\mathbf{P}} : \operatorname{spt}\nu \to \mathbb{R}_+$ that we call landscape function associated with $\mathbf{P}$ *satisfying:*

(i) for every $i \in \{1, \ldots, N\}$, $z_{\mathbf{P}^{x_i}} = z_{\mathbf{P}}$ everywhere on $\operatorname{Bas}(\mathbf{P}, x_i)$;

(ii) $z_{\mathbf{P}}$ is β-Hölder continuous where we recall $\beta = 1 + d\alpha - d \in (0, 1)$, with a Hölder constant smaller than a constant $C_H = C_H(c_A, C_A, \alpha, d)$.

Proof. Let us start by setting a candidate landscape function which is uniquely defined ν-almost everywhere on $\operatorname{spt}\nu$. The measures $\nu_i := (e_\infty)_\sharp \mathbf{P}^{x_i}$ are mutually singular and sum to ν thanks to Lemma 2.8, thus we may define a Borel function $z : \operatorname{spt}\nu \to \mathbb{R}_+$ such that for every $i \in \{1, \ldots, N\}$:

$$z = z_{\mathbf{P}^{x_i}} \quad \nu_i\text{-almost everywhere.}$$

Let us show that z admits a Hölder continuous representative through Campanato estimates, following the strategy of [San07]. Take a point $x \in \operatorname{spt}\nu$. For every $r \in (0, 2\operatorname{diam}(\operatorname{spt}\nu)]$ we denote by $z_r(x) := ⨍_{B_r(x)} z \,\mathrm{d}\nu$ the mean of z on $B_r(x)$, and by $\bar{z}_r(x)$ the central median of z on $B_r(x)$ with respect to ν, defined as the midpoint of the interval of values $\ell \in \mathbb{R}_+$ such that $B_r(x)$ may be partitioned into two subsets $A \sqcup B = B_r(x)$ with equal mass, i.e. $\nu(A) = \nu(B) = \nu(B_r(x))/2$, and such that $z \geq \ell$ on A and $z \leq \ell$ on B. Consider two such sets A, B for the central median $\ell = \bar{z}_r(x)$ and define the following variation of ν:

$$\tilde{\nu} := \nu - \nu \llcorner A + \nu \llcorner B = \sum_{i=1}^N \tilde{\nu}_i$$

where for every $i \in \{1, \ldots, N\}$,

$$\tilde{\nu}_i := \nu_i - \nu_i \llcorner A + \nu_i \llcorner B.$$

By Lemma 2.8 again, we know that the $\mathbf{P}^{x_i}$'s are disjoint and thus optimal traffic plans with single source x_i, thus we may use the first variation inequality (D) of Proposition 2.3 for every $i \in \{1, \ldots, N\}$ to obtain:

$$\begin{aligned}\mathbf{d}^\alpha(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i) &\leq \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu_i) + \alpha\left(\int_B z_{\mathbf{P}^{x_i}}\,\mathrm{d}\nu_i - \int_A z_{\mathbf{P}^{x_i}}\,\mathrm{d}\nu_i\right)\\ &= \mathbb{M}^\alpha(\mathbf{P}^{x_i}) + \alpha\left(\int_B z\,\mathrm{d}\nu_i - \int_A z\,\mathrm{d}\nu_i\right).\end{aligned} \tag{4.2}$$

We set $\tilde{\mathbf{P}} := \sum_{i=1}^N \tilde{\mathbf{P}}_i$ where $\tilde{\mathbf{P}}_i \in \mathbf{TP}(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i)$ is an optimal traffic plan for every $i \in \{1, \dots, N\}$. Summing (4.2) over i, using the subadditivity of the α-mass and the disjointness of the $\mathbf{P}^{x_i}$'s together with (2.6) yields:

$$\begin{aligned}\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \sum_{i=1}^N \mathbb{M}^\alpha(\tilde{\mathbf{P}}_i) &= \sum_{i=1}^N \mathbf{d}^\alpha(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i) \\ &\leq \sum_{i=1}^N \left(\mathbb{M}^\alpha(\mathbf{P}^{x_i}) + \alpha\left(\int_B z\,\mathrm{d}\nu_i - \int_A z\,\mathrm{d}\nu_i\right)\right) \\ &= \mathbb{M}^\alpha(\mathbf{P}) + \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right).\end{aligned} \tag{4.3}$$

Notice that $\tilde{\mathbf{P}} \in \mathbf{TP}(\tilde{\mu}, \tilde{\nu})$ where $\tilde{\mu} := \sum_{i=1}^N \|\tilde{\nu}_i\|\delta_{x_i}$. Take an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\tilde{\nu}, \nu)$ and consider a concatenation

$$\mathbf{P}' \in \tilde{\mathbf{P}} : \mathbf{Q} \subseteq \mathbf{TP}(\tilde{\mu}, \nu),$$

which is defined thanks to Proposition 2.1 (i). Since $\operatorname{spt}(\tilde{\nu} - \nu) \subseteq \bar{B}_r(x)$ and $\|\tilde{\nu} - \nu\| = \nu(B_r(x)) \leq C_A r^d$, by Proposition 2.1 (iii) and the branched transport upper estimate (2.7) we have:

$$\mathbb{M}^\alpha(\mathbf{P}') \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + \mathbb{M}^\alpha(\mathbf{Q}) \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + C_{\mathsf{BOT}}\ 2r\ (C_A r^d)^\alpha. \tag{4.4}$$

Now we remark that $\tilde{\mu}$ is still supported on $\{x_i : 1 \leq i \leq N\}$ and μ is a mass-optimal quantizer of ν with respect to the x_i's, so that $\mathbb{M}^\alpha(\mathbf{P}')$ is greater than $\mathbb{M}^\alpha(\mathbf{P})$, thus by combining (4.4) and (4.3):

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}') &\leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + 2C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha} \\ &\leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right) + 2C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha}.\end{aligned}$$

This implies that

$$0 \leq \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right) + 2C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha},$$

hence

$$\int_{B_r(x)} |z - \bar{z}_r(x)|\,\mathrm{d}\nu = \int_A z\,\mathrm{d}\nu - \int_B z\,\mathrm{d}\nu \leq 2\alpha^{-1} C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha}$$

and finally

$$\begin{aligned}\int_{B_r(x)} |z - z_r(x)|\,\mathrm{d}\nu &\leq \int_{B_r(x)} |z - \bar{z}_r(x)|\,\mathrm{d}\nu + \nu(B_r(x))|z_r(x) - \bar{z}_r(x)| \\ &\leq 2\int_{B_r(x)} |z - \bar{z}_r(x)|\,\mathrm{d}\nu \leq 4\alpha^{-1} C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha}.\end{aligned} \tag{4.5}$$

We now use Campanato estimates: for every $x \in \operatorname{spt}\nu$, $r \leq 2\operatorname{diam}(\operatorname{spt}\nu)$ and $r' \in [r/2, r]$,

$$\begin{aligned}|z_r(x) - z_{r'}(x)| &\leq ⨍_{B_{r'}(x)} |z - z_r(x)|\,\mathrm{d}\nu \\ &\leq \frac{1}{\nu(B_{r'}(x))}\int_{B_r(x)} |z - z_r(x)|\,\mathrm{d}\nu \leq \frac{4\alpha^{-1} C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha}}{c_A (r/2)^d} \leq C r^\beta,\end{aligned} \tag{4.6}$$

where we have set $C := \frac{2^{d+2} C_{\mathsf{BOT}} C_A^\alpha}{\alpha c_A}$, and as before $\beta = 1 + d\alpha - d \in (0, 1)$. Applying (4.6) to radii $r2^{-n}, r2^{-n-1}$ for $n \in \mathbb{N}$, we deduce that $(z_{r2^{-n}})_{n\in\mathbb{N}}$ is a Cauchy sequence, which in turn implies (using (4.6) again) that the following limit exists for every $x \in \operatorname{spt}\nu$:

$$z_{\mathbf{P}}(x) := \lim_{r\to 0} z_r(x) = \lim_{r\to 0} ⨍_{B_r(x)} z\,\mathrm{d}\nu.$$

By triangle inequality (4.6) yields

$$|z_r(x) - z_{\mathbf{P}}(x)| \leq \sum_{n=0}^{+\infty} |z_{r2^{-n}}(x) - z_{r2^{-(n+1)}}(x)| \leq \frac{Cr^\beta}{1-2^{-\beta}},$$

and combining with (4.5) we get

$$\fint_{B_r(x)} |z - z_{\mathbf{P}}(x)| \,\mathrm{d}\nu \leq \frac{2}{1-2^{-\beta}} Cr^\beta.$$

Finally, take $x, y \in \operatorname{spt}\nu$ such that $r := |y - x|$ and use the last two inequalities to get:

$$\begin{aligned}
|z_{\mathbf{P}}(y) - z_{\mathbf{P}}(x)| &\leq |z_{\mathbf{P}}(y) - z_r(y)| + |z_r(y) - z_{\mathbf{P}}(x)| \\
&\leq \frac{Cr^\beta}{1-2^{-\beta}} + \fint_{B_r(y)} |z - z_{\mathbf{P}}(x)| \\
&\leq \frac{Cr^\beta}{1-2^{-\beta}} + \frac{\nu(B_{2r}(x))}{\nu(B_r(y))} \fint_{B_{2r}(x)} |z - z_{\mathbf{P}}(x)| \,\mathrm{d}\nu \leq \left(\frac{1 + 2^{d+1}(C_A/c_A)}{1-2^{-\beta}}\right) Cr^\beta.
\end{aligned}$$

As a consequence, we get (ii) with

$$C_H := \frac{2^{2(d+2)} C_{\mathsf{BOT}} C_A^{1+\alpha}}{(1 - 2^{-(1+d\alpha-d)})\alpha c_A^2}.$$

Let us now prove (i). Since ν-a.e. point of $\operatorname{spt}\nu$ is a Lebesgue point of z (with respect to ν), we know that $z_{\mathbf{P}} = z$ ν-a.e. thus $z_{\mathbf{P}} = z_{\mathbf{P}^{x_i}}$ ν_i-a.e., but since $z_{\mathbf{P}^{x_i}}$ is lower semicontinuous and $z_{\mathbf{P}}$ is continuous on $\operatorname{spt}\nu_i$, we have $z_{\mathbf{P}^{x_i}} \leq z_{\mathbf{P}}$ everywhere on $\operatorname{Bas}(\mathbf{P}, x_i) = \operatorname{spt}\nu_i$. Let us show that we actually have equality. Given $x \in \operatorname{spt}\nu_i$ such that $z_{\mathbf{P}^{x_i}}(x) < \infty$ (otherwise there is nothing to prove), consider a $\mathbf{P}^{x_i}$-good curve γ_i from x_i to x. Fix $r \leq \operatorname{diam}(\operatorname{spt}\nu)$, take an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\nu(B_r(x))\delta_x, \nu \llcorner B_r(x))$ and by Proposition 2.1 (i) take a concatenation

$$\mathbf{P}' \in (\nu(B_r(x))\delta_{\gamma_i})) : \mathbf{Q} \in \mathbf{TP}(\nu(B_r(x))\delta_{x_i}, \nu \llcorner B_r(x)).$$

We build the competitor

$$\tilde{\mathbf{P}} := \mathbf{P} - \mathbf{P} \llcorner \{\gamma(\infty) \in B_r(x)\} + \mathbf{P}'$$

which belongs to $\mathbf{TP}(\tilde{\mu}, \nu)$ for some measure $\tilde{\mu}$ which is still supported on $\{x_i, 1 \leq i \leq N\}$. Using as above the first variation inequality (D) of Proposition 2.3 (applied to each $\mathbf{P}^{x_j}$, $j \in \{1, \ldots, N\}$), the subaddditivity of the α-mass and the mass-optimality of μ we must have:

$$\mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \mathbb{M}^\alpha(\mathbf{P}) - \alpha \int_{B_r(x)} z_{\mathbf{P}} \,\mathrm{d}\nu + \alpha\nu(B_r(x)) z_{\mathbf{P}^{x_i}}(x) + \mathbb{M}^\alpha(\mathbf{Q})$$

Since $\mathbb{M}^\alpha(\mathbf{Q}) \leq 2C_{\mathsf{BOT}} r^{1+d\alpha}$ it implies

$$\forall r \in (0, \operatorname{diam}(\operatorname{spt}\nu)), \quad \fint_{B_r(x)} z_{\mathbf{P}} \,\mathrm{d}\nu \leq z_{\mathbf{P}^{x_i}}(x) + \frac{2C_{\mathsf{BOT}}}{c_A} r^\beta \implies z_{\mathbf{P}}(x) \leq z_{\mathbf{P}_i}(x),$$

hence $z_{\mathbf{P}^{x_i}} = z_{\mathbf{P}}$ on $\operatorname{Bas}(\mathbf{P}, x_i)$ for every $i \in \{1, \ldots, N\}$, i.e. (i) holds true.

□

4.2 Applications of the landscape function

We now generalize the properties of the single-source landscape function of Proposition 2.3 to our setting. First, we extend the notion of $\mathbf{P}$-good curve when $\mathbf{P}$ is a traffic plan with N sources $\{x_1, \ldots, x_N\}$ such that the traffic plans $\mathbf{P}^{x_i}$'s are disjoint: we say that a curve γ is $\mathbf{P}$-*good* if it starts at some source x_i and it is $\mathbf{P}^{x_i}$-good.

Proposition 4.2. *Assume $\alpha \in (1-1/d, 1)$. Let ν be a compactly supported d-Ahlfors regular measure, μ be a N-point mass-optimal quantizer with respect to $\mathcal{X} := \{x_i\}_{1 \le i \le N}$ and $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, $\alpha \in [0,1)$. We consider a nonempty subset $\mathcal{X}' \subseteq \mathcal{X}$ and we set:*

$$\mu' := \mu \llcorner \mathcal{X}' \qquad \nu' := \nu \llcorner \bigcup_{s \in \mathcal{X}'} \operatorname{Bas}(\mathbf{P}, s) \qquad \mathbf{P}' := \sum_{s \in \mathcal{X}'} \mathbf{P}^s.$$

The landscape function $z_{\mathbf{P}} : \operatorname{spt} \nu \to \mathbb{R}_+$ given by Theorem 4.1 satisfies:

(A) $z_{\mathbf{P}}(x) \ge d(x, \mathcal{X}')$ for every $x \in \operatorname{spt} \nu'$;

(B) the α-distance writes as:

$$\mathbf{d}^\alpha(\mathcal{X}', \nu') = \mathbf{d}^\alpha(\mu', \nu') = \mathbb{M}^\alpha(\mathbf{P}') = \int_{\mathbb{R}^d} z_{\mathbf{P}}(x) \, \mathrm{d}\nu'(x);$$

(C) if $\tilde{\mathbf{P}} \in \mathbf{TP}(\tilde{\mu}_N, \tilde{\nu})$ is a traffic plan concentrated on $\mathbf{P}'$-good curves, then:

$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \le \mathbb{M}^\alpha(\mathbf{P}') + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu),$$

and the inequality is strict if for some $x_i \in \mathcal{X}'$, $\Theta_{\tilde{\mathbf{P}}^{x_i}} - \Theta_{\mathbf{P}^{x_i}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_i}}$;

(D) in particular, $z_{\mathbf{P}}$ is an upper first variation of the irrigation distance, *in the sense that for every $\tilde{\nu} \in \mathscr{M}^+(\mathbb{R}^d)$,*

$$\mathbf{d}^\alpha(\mathcal{X}', \tilde{\nu}) \le \mathbf{d}^\alpha(\mathcal{X}', \nu') + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu').$$

Sketch of proof. The results follow rather directly from Proposition 2.3, Theorem 4.1 and the optimality of the $\mathbf{P}^{x_i}$'s stated in Lemma 2.8. For (A) it suffices to note that for every $x \in \operatorname{spt} \nu$ there exists $x_i \in \mathcal{X}$ such that $z_{\mathbf{P}}(x) = z_{\mathbf{P}^{x_i}}(x)$, and thus using Proposition 2.3 (A) we find $z_{\mathbf{P}^{x_i}}(x) \ge d(x, x_i) \ge d(x, \mathcal{X})$. For the remaining points, we merely apply the corresponding points from Proposition 2.3 to the traffic plans $\mathbf{P}^{x_i}, x_i \in \mathcal{X}'$, and combine it with the disjointness properties from Lemma 2.8. $\square$

Corollary 4.3. *Under the assumptions of Theorem 4.1 and using the same notations, the basins $\operatorname{Bas}(\mathbf{P}, x_i)$ are closed subsets of $\operatorname{spt} \nu$ which form a partition of ν, in the sense that they are ν-essentially disjoint and that their reunion is equal to $\operatorname{spt} \nu$.*

Proof. The basins are closed since by definition $\operatorname{Bas}(\mathbf{P}, x_i) = \operatorname{spt} \nu_i$ where $\nu_i = (e_\infty)_\sharp \mathbf{P}^{x_i}$ for every $i \in \{1, \ldots, N\}$. Furthermore, $\nu = \sum_{i=1}^N \nu_i$ implies that $\operatorname{spt} \nu = \bigcup_{i=1}^N \operatorname{spt} \nu_i$.

Let us show that for every $i \ne j$, $\nu_j(\operatorname{Bas}(\mathbf{P}, x_i)) = 0$, which implies the result, since for $k \ne l$ we can then write

$$\begin{aligned} \nu(\operatorname{Bas}(\mathbf{P}, x_k) \cap \operatorname{Bas}(\mathbf{P}, x_l)) &= \sum_{j=1}^N \nu_j(\operatorname{Bas}(\mathbf{P}, x_k) \cap \operatorname{Bas}(\mathbf{P}, x_l)) \\ &\le \nu_k(\operatorname{Bas}(\mathbf{P}, x_l)) + \sum_{j \ne k} \nu_j(\operatorname{Bas}(\mathbf{P}, x_k)) = 0. \end{aligned}$$

Suppose by contradiction that $\nu_j(\mathrm{Bas}(\mathbf{P},x_i))>0$ for some $i\neq j$. Take as competitor

$$\mathbf{P}'=\mathbf{P}-\underbrace{\mathbf{P}^{x_j}\llcorner\{\gamma:\gamma(\infty)\in\mathrm{Bas}(\mathbf{P},x_i)\}}_{\mathbf{P}_{ij}}+\mathbf{Q},$$

where $\mathbf{Q}\in\mathbf{TP}(\nu_j(\mathrm{Bas}(\mathbf{P},x_i))\delta_{x_i},\nu_j\llcorner\mathrm{Bas}(\mathbf{P},x_i))$ is chosen so that $\mathbf{Q}$-a.e. curve γ is a $\mathbf{P}^{x_i}$-good curve, which is possible because $z_{\mathbf{P}^{x_i}}$ is finite everywhere on $\mathrm{Bas}(\mathbf{P},x_i)$ by Theorem 4.1. Since $\mathbf{P}'\in\mathbf{TP}(\mu',\nu)$ where μ' is concentrated on the x_i's, by mass-optimality of the quantizer μ, we have:

$$\mathbb{M}^\alpha(\mathbf{P})=\mathbf{d}^\alpha(\mu,\nu)\leq\mathbf{d}^\alpha(\mu',\nu)\leq\mathbb{M}^\alpha(\mathbf{P}').$$

Notice that $\mathbf{P}'$ is rectifiable, $\Sigma_{\mathbf{P}_{ij}}\subseteq\Sigma_{\mathbf{P}^{x_j}}$ and $\Sigma_{\mathbf{Q}}\subseteq\Sigma_{\mathbf{P}^{x_i}}$, since every $\mathbf{P}^{x_i}$-good curve is $\mathscr{H}^1$-a.e. included in $\Sigma_{\mathbf{P}^{x_i}}$. Besides, the $\mathbf{P}^{x_k}$'s are disjoint by Lemma 2.8, which implies thanks to (2.4) that the networks $\Sigma_{\mathbf{P}^{x_k}}$'s are $\mathscr{H}^1$-essentially disjoint. Thus the traffic plans $(\mathbf{P}')^{x_k}$'s are disjoint. We apply the upper first variation inequality Proposition 4.2(C) to the variation given by replacing $\mathbf{P}^{x_i}\mapsto(\mathbf{P}')^{x_i}=\mathbf{P}^{x_i}-\mathbf{P}_{ij}$ and $\mathbf{P}^{x_j}\mapsto(\mathbf{P}')^{x_j}=\mathbf{P}^{x_j}+\mathbf{Q}$, and we get:

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P})&=\mathbf{d}^\alpha(\mu,\nu)\leq\mathbf{d}^\alpha(\mu',\nu)\leq\mathbb{M}^\alpha(\mathbf{P}')\\&=\mathbb{M}^\alpha(\mathbf{P})+(\mathbb{M}^\alpha(\mathbf{P}^{x_i}-\mathbf{P}_{ij})-\mathbb{M}^\alpha(\mathbf{P}^{x_i}))+(\mathbb{M}^\alpha(\mathbf{P}^{x_j}+\mathbf{Q})-\mathbb{M}^\alpha(\mathbf{P}^{x_j}))\\&<\mathbb{M}^\alpha(\mathbf{P})-\alpha\int_{\mathrm{Bas}(\mathbf{P},x_i)\cap\mathrm{Bas}(\mathbf{P},x_j)}z_{\mathbf{P}^{x_i}}\,\mathrm{d}\nu_j+\alpha\int_{\mathrm{Bas}(\mathbf{P},x_i)\cap\mathrm{Bas}(\mathbf{P},x_j)}z_{\mathbf{P}^{x_j}}\,\mathrm{d}\nu_j\\&=\mathbb{M}^\alpha(\mathbf{P})-\alpha\int_{\mathrm{Bas}(\mathbf{P},x_i)\cap\mathrm{Bas}(\mathbf{P},x_j)}z_{\mathbf{P}}\,\mathrm{d}\nu_j+\alpha\int_{\mathrm{Bas}(\mathbf{P},x_i)\cap\mathrm{Bas}(\mathbf{P},x_j)}z_{\mathbf{P}}\,\mathrm{d}\nu_j=\mathbb{M}^\alpha(\mathbf{P}),\end{aligned}$$

where we used Theorem 4.1 in the last equality, and and Proposition 4.2 and noticed that the inequality is strict since $\Theta_{\mathbf{P}_{ij}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_i}}$ (or similarly that $\Theta_{\mathbf{Q}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_j}}$). This is a contradiction. □

The measure of basins can be controlled from above and below for optimal plans associated with mass-optimal quantizers.

Lemma 4.4. *Under the assumptions of Theorem 4.1 and with the same notations, for every source x_i of $\mathbf{P}$, we set*

$$\delta(\mathbf{P},x_i):=\max_{y\in\mathrm{Bas}(\mathbf{P},x_i)}\frac{|y-x_i|}{2}.$$

Then we have for every x_i:

$$c_{\mathrm{Bas}}\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i))^d\leq\nu(\mathrm{Bas}(\mathbf{P},x_i))\leq C_A\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i))^d,\tag{4.7}$$

$$\frac{1}{2}\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i))\leq\delta(\mathbf{P},x_i)\leq\left(\frac{C_A}{2c_{\mathrm{Bas}}}\right)^{\frac{1}{d}}\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i)),\tag{4.8}$$

$$c_H\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i))^\beta\leq\sup_{\mathrm{Bas}(\mathbf{P},x_i)}z_{\mathbf{P}_N}\leq C'_H\operatorname{diam}(\mathrm{Bas}(\mathbf{P},x_i))^\beta,\tag{4.9}$$

where $c_{\mathrm{Bas}}:=2^{-d}\left(C_H+\frac{2C_{\mathrm{BOT}}}{\alpha c_A^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$, $c_H:=2^{-\beta}C_A^{\alpha-1}$ *and* $C'_H:=2c_{\mathrm{Bas}}^{-\frac{1}{d}}C_A^{\frac{\beta}{d}}$.

Proof. The upper bound in (4.7) follows from the upper d-Ahlfors regularity of μ.

For the lower bound in (4.8), consider a point $y\in\mathrm{Bas}(\mathbf{P},x_i)$ such that $|y-x_i|=\max_{y\in\mathrm{Bas}(\mathbf{P},x_i)}|y-x_i|$, which exists because $\mathrm{Bas}(\mathbf{P},x_i)$ is compact. Take a $\mathbf{P}^{x_i}$-good curve γ_i from x_i to y and set $r:=|y-x_i|$. We build a competing traffic plan $\tilde{\mathbf{P}}$ by removing $\mathbf{P}\llcorner\{\gamma\in\Gamma^d:\gamma(\infty)\in B_r(y)\}$

then adding an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\nu(B_r(x_i))\delta_{x_i}, \nu \llcorner B_r(y))$. By optimality of $\mathbf{P}$ and mass-optimality of the source measure μ, using the first variation inequality Proposition 4.2 (C), and the branched transport upper estimate (2.7), we get

$$-\alpha \int_{B_r(y)} z_{\mathbf{P}} \,d\nu + C_{\mathsf{BOT}}(2r)\nu(B_r(y))^\alpha \geq 0 \implies \fint_{B_r(y)} z_{\mathbf{P}} \,d\nu \leq \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}} r^\beta,$$

which yields by Theorem 4.1,

$$z_{\mathbf{P}^{x_i}}(y) = z_{\mathbf{P}}(y) - \fint_{B_r(y)} z_{\mathbf{P}} \,d\nu + \fint_{B_r(y)} z_{\mathbf{P}} \,d\nu \leq C_H r^\beta + \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}} r^\beta =: C' r^\beta. \tag{4.10}$$

Now recall the definition of landscape function in the single-source case:

$$\begin{aligned} C' r^\beta \geq z_{\mathbf{P}^{x_i}}(y) &= \int_{\gamma_i} \Theta_{\mathbf{P}^{x_i}}(x)^{\alpha-1} \,dx \\ &\geq \mathscr{H}^1(\gamma_i(\mathbb{R}_+))\nu(\mathrm{Bas}(\mathbf{P}, x_i))^{\alpha-1} \geq r\nu(\mathrm{Bas}(\mathbf{P}, x_i))^{\alpha-1}. \end{aligned} \tag{4.11}$$

As a consequence

$$\nu(\mathrm{Bas}(\mathbf{P}, x_i)) \geq (C' r^{\beta-1})^{\frac{1}{\alpha-1}} = 2^{-d} C'^{\frac{1}{\alpha-1}} (2r)^d = c_{\mathrm{Bas}}(2r)^d$$

where $c_{\mathrm{Bas}} := 2^{-d}\left(C_H + \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$. Notice that $\mathrm{diam}(\mathrm{Bas}(\mathbf{P}, x_i)) \leq 2r$ by the triangle inequality, which yields (4.7) and (4.8). As for (4.9), the lower bound comes from (4.11) and the upper Ahlfors regularity, while the upper bound comes from (4.10), which implies by the triangle inequality that for every $y' \in \mathrm{Bas}(\mathbf{P}, x_i)$

$$\begin{aligned} z_{\mathbf{P}}(y') \leq C_H|y - y'|^\beta + C' r^\beta &\leq 2C' r^\beta \\ &= 2 2^{d(\alpha-1)} c_{\mathrm{Bas}}^{\alpha-1} r^\beta \\ &\leq 2^\beta c_{\mathrm{Bas}}^{\alpha-1} \left(\frac{C_A}{2c_{\mathrm{Bas}}}\right)^{\frac{\beta}{d}} \mathrm{diam}(\mathrm{Bas}(\mathbf{P}, x_i))^\beta. \\ &\leq C'_H \,\mathrm{diam}(\mathrm{Bas}(\mathbf{P}, x_i))^\beta \end{aligned}$$

where $C'_H := 2c_{\mathrm{Bas}}^{-\frac{1}{d}} C_A^{\frac{\beta}{d}}$. □

Remark 4.5 (Voronoi basins). In the case $\alpha = 1$, if ν has a compact convex support Ω and $\mathbf{P}$ is an optimal traffic plan between ν and a mass-optimal quantizer $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ associated with the points $\{x_i\}_{1 \leq i \leq N}$, then the basins will be exactly the *Voronoi cells* $(\Omega \cap V_i)_{1 \leq i \leq N}$ given by

$$\forall i \in \{1, \dots, N\}, \quad V_i := \left\{ x : |x - x_i| = \min_{1 \leq j \leq N} |x - x_j| \right\}. \tag{4.12}$$

When $\alpha \in (0, 1)$, the basins $(\mathrm{Bas}(\mathbf{P}, x_i))_{1 \leq i \leq n}$ thus extend the notion of Voronoi cells to the case of branched optimal transport, which we may call *(branched) Voronoi basins*. These Voronoi basins are also closed sets which form a partition of the given measure ν, as stated in Corollary 4.3, but they are much more complicated in several regards:

- They need not be convex polyhedra, but are rather thought to exhibit fractal pairwise boundaries.
- Classical Voronoi cells do not actually depend on the measure ν or its support but may be computed directly from the points $\{x_i\}_{1 \leq i \leq N}$ through (4.12), taking the intersection with the support afterwards. On the contrary, there is a priori no reason for Voronoi basins to behave in the same way, and it is well possible that the Voronoi basins for ν and $\nu' \geq \nu$ are not nested.
- Computing Voronoi basins is much more difficult, as the problem of optimizing the masses given the points does not admit an explicit solution in the form of (4.12).

5 Uniform properties of optimal quantizers and partitions

In this section we investigate the uniform properties of optimal quantizers at the microscopic scale, i.e. at the scale of $N^{-1/d}$, when the measure ν that is quantized is d-Ahlfors regular. Roughly speaking, we are going to show that the atoms of a N-point optimal quantizer are distributed somewhat uniformly at this scale, being well-separated and leaving no big hole in the support of ν, and that the basins are somewhat round, having inner and outer balls of comparable size. We also show uniformity bounds on the masses and energies associated with each atom.

5.1 Delone constants for optimal quantizers

In this section we prove Theorem 1.2. Given a set $X \subseteq \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^d$ a finite set of points, we define the *covering radius* (also called *mesh norm* or *fill radius*) of X by $\mathcal{X}$, as

$$\omega(X, \mathcal{X}) := \sup_{x \in \mathcal{X}} \min_{x' \in \mathcal{X}} d(x, x').$$

It is the smallest $r \geq 0$ such that the closed balls of radius r with centers in $\mathcal{X}$ cover X. The *separation distance* (corresponding to $1/2$ of the *packing radius*) of $\mathcal{X}$ is defined by

$$\delta(\mathcal{X}) := \min_{x, x' \in \mathcal{X} : x \neq x'} d(x, x').$$

A set with finite covering radius and nonzero separation distance is called a *Delone set* with respect to X, and (ω, δ) its *Delone constants*. Given a d-Ahlfors regular measure ν, the following theorem shows that the atoms of optimal N-point quantizers are Delone sets with respect to $\operatorname{spt} \nu$, providing bounds comparable to $N^{-1/d}$ on its Delone constants.

Our proof of Theorem 1.2 is inspired from ideas of [Gru04], dealing with classical optimal transport costs. We stress that the situation in the branched optimal transport case is much more involved, since the ground cost is not explicit (it depends on all the trajectories and is part of the optimization defining $\mathbf{d}^\alpha$), and the shapes of basins are not known at all (they are thought to have fractal boundaries). Thus, we shall need to estimate

- the cost for merging a "small" basin to a "neighbouring" basin ;
- the gain to remove part of a "large" basin.

The landscape function, its uniform Hölder regularity and its consequences established in Section 4 will play a crucial role.

Proof of Theorem 1.2. We proceed by proving successively the following.

(a) *At least one basin is not too large:* there is a constant $\tilde{c}_2$ (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \exists j \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)) \leq \tilde{c}_2 N^{-\frac{1}{d}}.$$

(b) *At least one basin is not too small:* there is a constant $\tilde{c}_1$ (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \exists j \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)) \geq \tilde{c}_1 N^{-\frac{1}{d}}.$$

(c) *All basins are small:* there is a constant c_2 (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \forall i \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i)) \leq c_2 N^{-\frac{1}{d}}.$$

(d) *All atoms are far from each other:* there exists a constant c_1 such that

$$\forall N \in \mathbb{N}^*, \forall (1 \leq j \neq k \leq N), \quad d(x_j, x_k) \geq c_1 N^{-\frac{1}{d}}.$$

Notice that (1.1) follows from (c), since the basins form a covering of $\operatorname{spt} \mu$ by Corollary 4.3, while (1.2) is merely a rephrasing of (d).

Proof of (a) and (b). First note that by Corollary 4.3, the basins form a partition of ν, thus

$$\sum_{j=1}^{N} \nu(\mathrm{Bas}(\mathbf{P}_N, x_j)) = \|\nu\|,$$

thus there exists an index $j \in \{1, \dots, N\}$ for which

$$\nu(\mathrm{Bas}(\mathbf{P}_N, x_j)) \leq \frac{\|\nu\|}{N} \leq \frac{C_A}{N} \operatorname{diam}(\operatorname{spt} \nu)^d,$$

and by Lemma 4.4, this implies that

$$\operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_j) \leq (\|\nu\|(c_{\mathrm{Bas}} N)^{-1})^{\frac{1}{d}} = \tilde{c}_2 N^{\frac{1}{d}} \quad \text{where} \quad \tilde{c}_2 := (C_A/c_{\mathrm{Bas}})^{-\frac{1}{d}} \operatorname{diam}(\operatorname{spt} \nu).$$

Similarly, there exists an index $i \in \{1, \dots, N\}$ such that

$$C_A \operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i))^d \geq \nu(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq \frac{\|\nu\|}{N} \geq \frac{c_A}{N} \operatorname{diam}(\operatorname{spt} \nu)^d,$$

which implies that

$$\operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i) \geq \tilde{c}_2 N^{-\frac{1}{d}} \quad \text{where} \quad \tilde{c}_2 := (c_A/C_A)^{\frac{1}{d}} \operatorname{diam}(\operatorname{spt} \nu).$$

Proof of (c). Applying (a) we take $j \leq N$ such that

$$\operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_j)) \leq \tilde{c}_2 N^{-\frac{1}{d}}. \tag{5.1}$$

Suppose that for some $t > 1$ there exists $i \leq N$ such that

$$\operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq t\tilde{c}_2 N^{-\frac{1}{d}}.$$

We are going to show a contradiction when t is too large (not depending on N). For this, let us build a better competitor than μ_N. We shall add an extra point q of $\mathrm{Bas}(\mathbf{P}_N, x_i)$ to irrigate a "costly" ball around q, then remove the point x_j and irrigate the former basin $\mathrm{Bas}(\mathbf{P}_N, x_j)$ from a neighbouring basin $\mathrm{Bas}(\mathbf{P}_N, x_k)$.

Let us find a "costly" ball. Consider

$$q \in \arg\max_{\mathrm{Bas}(\mathbf{P}_N, x_i)} z_{\mathbf{P}_N},$$

which exists because $z_{\mathbf{P}_N}$ is Hölder-continuous thanks to Theorem 4.1 and basins are compact sets thanks to Corollary 4.3. We consider the ball $B_{\varepsilon t\tilde{c}_2 N^{-\frac{1}{d}}}(q)$ for some small $\varepsilon \in (0, 1)$ to be fixed later.

Now, we want to remove the point x_j from the quantizer μ_N and to irrigate the basin $\mathrm{Bas}(\mathbf{P}_N, x_j)$ from another basin that is not too far, in order to control the extra cost. By (5.1) and Ahlfors-regularity of ν, for $s > 1$ we have

$$\nu(B_{s\tilde{c}_2 N^{-\frac{1}{d}}}(x_j) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j)) \geq c_A (s\tilde{c}_2)^d N^{-1} - C_A \tilde{c}_2^d N^{-1} = (c_A s^d - C_A)\tilde{c}_2^d N^{-1}.$$

This is strictly positive if we take for example $s := (2C_A/c_A)^{\frac{1}{d}}$, in which case there exists a point p such that

$$p \in \mathrm{Bas}(\mathbf{P}_N, x_k) \cap B_{s\tilde{c}_2 N^{-1/d}}(x_j) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j) \quad \text{for some} \quad k \neq j, \tag{5.2}$$

because the basins form of covering of $\operatorname{spt} \nu$.

We are now ready to build our competitor $\mathbf{P}_N^*$, modifying $\mathbf{P}_N$ according to the following sketch; the addition of curves (which increases the α-mass) are labeled by (A), while the removal of curves (which decreases the α-mass) are labeled by (R).

(R_1) Remove all curves starting at x_j.

(R_2) Remove all curves ending in $B_{\varepsilon\tilde{c}_2 N^{-1/d}}(q)$.

(A_1) Re-irrigate $\mathrm{Bas}(\mathbf{P}_N, x_j)$ by

- bringing a mass m_j from x_k to p following a $\mathbf{P}_N^{x_k}$-good curve γ,
- then concatenate an optimal traffic plan $\mathbf{Q}^1 \in \mathbf{TP}(m_j\delta_p, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_j))$.

(A_2) Add an optimal traffic plan $\mathbf{Q}^2$ from $m\delta_q$ to $\nu \llcorner (B_{\varepsilon t\tilde{c}_2 N^{-\frac{1}{d}}}(q) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j))$, where m is the mass of the latter and $\varepsilon > 0$ is a small number to be chosen later (independently from N and t).

We start by doing the modifications along the existing network, corresponding to (R_1), (R_2) and the first part of (A_1). Setting $\Gamma_{x_j} := \{\gamma : \gamma(0) = x_j\}$ and $\Gamma_q := \{\gamma : \gamma(+\infty) \in B_{\varepsilon\tilde{c}_2 N^{-\frac{1}{d}}}(q)\}$, we define

$$\mathbf{P}'_N := \mathbf{P}_N - \mathbf{P}_N \llcorner \Gamma_q - \mathbf{P}_N \llcorner (\Gamma_{x_j} \setminus \Gamma_q) + m_j\delta_\gamma.$$

Secondly, we add the new curves and pieces of curves corresponding to the second part of (A_1) and (A_2). We set $\nu' := (e_\infty)_\sharp \mathbf{P}'_N$, and $\tilde{\mathbf{Q}}^1 := \mathbf{Q}^1 + \iota_\sharp(\nu' - m_j\delta p)$ where $\iota : \mathbb{R}^d \to \Gamma^d$ denotes the canonical injection which sends a point x to the constant curve $\gamma_x \equiv x$. We define our competitor $\mathbf{P}^*_N$ by

$$\mathbf{P}^*_N := \mathbf{P}''_N + \mathbf{Q}^2 \quad \text{where} \quad \mathbf{P}''_N \in \mathbf{P}'_N : \tilde{\mathbf{Q}}^1.$$

We estimate the gain and cost of these operations. First of all, we have

$$\begin{aligned}
\mathbb{M}^\alpha(\mathbf{P}^*_N) &\le \mathbb{M}^\alpha(\mathbf{P}''_N) + \mathbb{M}^\alpha(\mathbf{Q}^2) \le \mathbb{M}^\alpha(\mathbf{P}'_N) + \mathbb{M}^\alpha(\mathbf{Q}^2) + \mathbb{M}^\alpha(\mathbf{Q}^1) \\
&\le \mathbb{M}^\alpha(\mathbf{P}'_N) + C_{\mathsf{BOT}} C_A^\alpha (\varepsilon t\tilde{c}_2 N^{-1/d})^{1+d\alpha} + C_{\mathsf{BOT}} C_A^\alpha (\tilde{c}_2(1+s)N^{-1/d})^{1+d\alpha} \\
&= \mathbb{M}^\alpha(\mathbf{P}'_N) + C_1 N^{-(\alpha+\frac{1}{d})}(1 + (\varepsilon t)^{1+d\alpha}),
\end{aligned} \tag{5.3}$$

for some constant C_1 which does dot depend on N. The α-mass of $\mathbf{P}'_N$ may then be estimated through the first variation formula Proposition 4.2 (C):

$$\begin{aligned}
\mathbb{M}^\alpha(\mathbf{P}'_N) &\le \mathbb{M}^\alpha(\mathbf{P}_N) - \int_{B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)} z_{\mathbf{P}_N}\, \mathrm{d}\nu - \int_{\mathbb{R}^d} z_{\mathbf{P}_N}\, \mathrm{d}(e_\infty)_\sharp(\mathbf{P}_N \llcorner (\Gamma_{x_j} \setminus \Gamma_q)) + m_j z_{\mathbf{P}_N}(p) \\
&\le \mathbb{M}^\alpha(\mathbf{P}_N) - \int_{B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)} z_{\mathbf{P}_N}\, \mathrm{d}\nu + m_j z_{\mathbf{P}_N}(p).
\end{aligned} \tag{5.4}$$

Let us estimate $z_{\mathbf{P}_N}(p)$ from above and $z_{\mathbf{P}_N}$ from below on $B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)$. By Lemma 4.4, we know that

$$z_{\mathbf{P}N}(q) \ge c_H \,\mathrm{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i))^\beta \ge c_H (t\tilde{c}_2)^\beta N^{-\beta/d},$$

thus for every $y \in B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)$

$$\begin{aligned}
z_{\mathbf{P}_N}(y) &\ge c_H (t\tilde{c}_2)^\beta N^{-\beta/d} - C_H |y - q|^\beta \\
&\ge (c_H - \varepsilon^\beta C_H)(t\tilde{c}_2)^\beta N^{-\beta/d} \ge (c_H/2)(t\tilde{c}_2)^\beta N^{-\beta/d},
\end{aligned} \tag{5.5}$$

provided we have chosen $\varepsilon^\beta \le (c_H/2C_H)$. Besides, by Lemma 4.4 again and (5.2),

$$\begin{aligned}
z_{\mathbf{P}_N}(p) &\le \sup_{y \in \mathrm{Bas}(\mathbf{P}_N, x_j)} |z_{\mathbf{P}_N}(y) - z_{\mathbf{P}_N}(p)| + \sup_{y \in \mathrm{Bas}(\mathbf{P}_N, x_j)} z_{\mathbf{P}_N}(y) \\
&\le C_H((s+1)\tilde{c}_2 N^{-1/d})^\beta + C'_H(\tilde{c}_2 N^{-1/d})^\beta \\
&= C N^{-\beta/d}
\end{aligned} \tag{5.6}$$

for some C which does not depend on N. Reporting (5.5) and (5.6) in (5.4) yields

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - (c_H/2)(t\tilde{c}_2)^\beta N^{-\beta/d} c_A(\varepsilon t\tilde{c}_2 N^{-1/d})^d + CN^{-\beta/d}C_A(\tilde{c}_2 N^{-1/d})^d\\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - N^{-(\alpha+\frac{1}{d})}(C_2\varepsilon^d t^{1+d\alpha} - C_3),\end{aligned}$$

for some constant $C_2, C_3 > 0$ which do not depend on N.

Injecting this into (5.3), we get

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \leq \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(C_1 + C_3 + C_1(\varepsilon t)^{1+d\alpha} - C_2\varepsilon^d t^{1+d\alpha}).$$

Notice that because $1 + d\alpha - d = \beta > 0$, it is possible to choose ε small enough, independently from N and t (e.g.[11] $\varepsilon^\beta = (C_2/2C_1) \wedge (c_H/2C_H)$) so that

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \leq \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(C_1 + C_3 - (C_2/2)\varepsilon^d t^{1+d\alpha}).$$

Now, if t is too large, depending on the constants $C_1, C_2, C_3, \varepsilon$ which we stress do not depend on N, it leads to $\mathbb{M}^\alpha(\mathbf{P}^*_N) < \mathbb{M}^\alpha(\mathbf{P}_N)$, which contradicts the optimality of μ_N, because by construction the target measure of $\mathbf{P}^*_N$ is ν, and its source measure is an N-point quantizer. As a conclusion, (c) holds true.

Proof of (d). Take $t > 0$ and suppose that there are two atoms x_j, x_k of μ_N, with $j \neq k$, such that $d(x_j, x_k) \leq tN^{-1/d}$. We are going to show a lower bound on $t > 0$ (not depending on N). Applying (b), choose an index $i \leq N$ such that

$$\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i)) \geq \tilde{c}_1 N^{-1/d}.$$

Up to interchanging j with k, we may assume that $i \neq j$ (k may be equal to i, but it will not matter). The strategy is very similar as what we did above: we shall add an extra point $q \in \operatorname{Bas}(\mathbf{P}_N, x_i)$ to irrigate a "costly" ball around q, while removing the point x_j from the quantizer and irrigating the basin $\operatorname{Bas}(\mathbf{P}_N, x_j)$ from the close point x_k.

Consider

$$q \in \operatorname{arg\,max}_{\operatorname{Bas}(\mathbf{P}_N, x_i)} z_{\mathbf{P}_N},$$

which is possibly because $z_{\mathbf{P}_N}$ is Hölder-continuous thanks to Theorem 4.1 and basins are compact sets thanks to Corollary 4.3. Take some small $\varepsilon \in (0, 1)$ that we shall fix later. We build a better competitor thanks to the following construction.

(R_1) Remove all curves ending in $B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$, of total mass m.

(A_1) Add an optimal traffic plan $\mathbf{Q}^1 \in \mathbf{TP}(m\delta_q, \nu \llcorner B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q))$ to irrigate $\nu \llcorner B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$ again.

(A_2) Irrigate $\operatorname{Bas}(\mathbf{P}_N, x_j)$ from the point x_k instead of x_j by concatenating a (unit-speed parameterization of) the segment $[x_k, x_j]$ to all curves starting at x_j.

The removal (R_1) produces the new traffic plan

$$\mathbf{P}'_N \coloneqq \mathbf{P}_N - \mathbf{P}_N \llcorner \Gamma_q \quad \text{where} \quad \Gamma_q \coloneqq \{\gamma : \gamma(\infty) \in B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)\}.$$

We show by Lemma 4.4, as in (5.5) above, that for every $y \in B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$

$$z_{\mathbf{P}_N}(y) \geq (c_H/2)(t\tilde{c}_1)^\beta N^{-\beta/d},$$

[11] Recall that we had the condition $\varepsilon^\beta \leq (c_H/2C_H)$ for (5.5).

provided we have chosen $\varepsilon^\beta \le (c_H/2C_H)$. Thus the cost gain can be estimated by using the first variation formula Proposition 4.2 (C):

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}'_N) &\le \mathbb{M}^\alpha(\mathbf{P}_N) - \alpha\int_{B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)} z_{\mathbf{P}_N}\,\mathrm{d}\nu \\ &\le \mathbb{M}^\alpha(\mathbf{P}_N) - C_A(\varepsilon\tilde{c}_1 N^{-1/d})^d(c_H/2)N^{-\beta/d} \\ &\le \mathbb{M}^\alpha(\mathbf{P}_N) - C_1\varepsilon^d N^{-(\alpha+\frac{1}{d})}.\end{aligned} \tag{5.7}$$

For the addition of the (pieces of) curves (A_1) and (A_2), we denote by $\gamma_{k,j}$ the unit-speed parameterized segment from x_k to x_j, $\mathbf{P}'_{N,j} := \mathbf{P}'_N \llcorner \{\gamma : \gamma(0) = x_j\}$, $m'_j := \|\mathbf{P}'_{N,j}\|$ and we set $\mathbf{Q}^2 := m'_j\delta_{\gamma_{k,j}} + (\iota\circ e_0)_\sharp(\mathbf{P}'_N - \mathbf{P}'_{N,j})$. We define our competitor $\mathbf{P}^*_N$ by

$$\mathbf{P}^*_N := \mathbf{P}''_N + \mathbf{Q}^1 \quad \text{where} \quad \mathbf{P}''_N \in \mathbf{Q}^2 : \mathbf{P}'_N.$$

From (c), we know that

$$m'_j \le m_j \le C_A(\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)))^d \le C_A c_2^d N^{-1}$$

and we compute, using (5.7):

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}^*_N) &\le \mathbb{M}^\alpha(\mathbf{P}''_N) + \mathbb{M}^\alpha(\mathbf{Q}^1) \le \mathbb{M}^\alpha(\mathbf{P}'_N) + \mathbb{M}^\alpha(\mathbf{Q}^1) + \mathbb{M}^\alpha(\mathbf{Q}^2) \\ &\le \mathbb{M}^\alpha(\mathbf{P}'_N) + C_{\mathsf{BOT}}C_A^\alpha(\varepsilon\tilde{c}_1 N^{-1/d})^{1+d\alpha} + (m'_j)^\alpha(tN^{-1/d}) \\ &\le \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(-C_1\varepsilon^d + C_2\varepsilon^{1+d\alpha} + t).\end{aligned}$$

Taking $\varepsilon > 0$ such that $\varepsilon^\beta \le c_H/2C_H$ and $\varepsilon^\beta \le C_1/2C_2$ (e.g. take ε^β to be the minimum of the two), we get $C_2\varepsilon^{1+d\alpha} \le (C_1)/2\varepsilon^d$ and thus

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \le \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(t - (C_1/2)\varepsilon^d),$$

which leads to a contradiction if t is too small (independently from N). Hence t is lower bounded by some constant $c_1 > 0$ and (d) holds true. Finally, we remark that the constants c_1, c_2 that we computed depend only on $(\alpha, d, c_A, C_A, \operatorname{diam}(\operatorname{spt}\nu))$. □

5.2 Inner and outer ball property of the basins

Proposition 5.1. *Assume $\alpha \in (1-1/d, 1)$. Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a d-Ahlfors regular measure with constants $0 < c_A \le C_A$. Let $\mu_N = \sum_{i=1}^N m_i\delta_{x_i}$ be a N-point optimal quantizer of ν and $\mathbf{P}_N$ an optimal traffic plan in $\mathbf{TP}(\mu_N, \nu)$. There are constants $c, C > 0$ depending only on $(\alpha, d, c_A, C_A, \operatorname{diam}(\operatorname{spt}\nu))$ such that for all $i \in \{1, \dots, N\}$,*

$$\operatorname{Bas}(\mathbf{P}_N, x_i) \subseteq B(x_i, CN^{-1/d}), \tag{5.8}$$

and

$$B(x_i, cN^{-1/d}) \subseteq \mathbb{R}^d \setminus \bigcup_{j\ne i} \operatorname{Bas}(\mathbf{P}_N, x_j). \tag{5.9}$$

Remark 5.2. In particular, if $\nu = \mathscr{L}^d \llcorner \Omega$ for some open bounded set Ω with Lipschitz boundary, then for every source x_i such that $d(x_i, \partial\Omega) > cN^{-1/d}$, (5.8) and (5.9) rewrite as

$$B(x_i, cN^{-1/d}) \subseteq \operatorname{Bas}(\mathbf{P}_N, x_i) \subseteq B(x_i, CN^{-1/d}).$$

Besides, this translates as uniform inner and outer ball properties of optimal partitions (solutions to (2.14)). Indeed, by the equivalence between the optimal quantization and optimal partition problems stated in Theorem 2.9, solutions $(\Omega_i)_{1\leq i\leq N}$ to (2.14) are actually $\mathscr{L}^d$-equivalent to the basins $(\mathrm{Bas}(\mathbf{P}_N, x_i))_{1\leq i\leq N}$ associated with some traffic plan $\mathbf{P}_N$ with sources $\{x_i\}_{1\leq i\leq N}$.

Finally, we remark that the number of points such that $d(x_i, \partial\Omega) > cN^{-1/d}$ is $N + O(N^{1-1/d})$ because the other points convey a mass $\approx N^{-1}$ by Proposition 5.3, and their basins are included in $\{d(\cdot, \partial\Omega) < C'N^{-1/d}\}$ for some C', a set of volume $\approx N^{-1/d}$. However, without extra assumptions on Ω, some points x_i may very well belong to $\mathbb{R}^d \setminus \bar{\Omega}$. This is ruled out for example when Ω is convex, but then it is not clear whether $x_i \in \Omega$ for all x_i's.

Proof of Proposition 5.1. The outer ball property (5.8) holds with $C := c_2(C_A/(2c_{\mathrm{Bas}}))^{1/d}$, by (4.8) in Lemma 4.4 and (c) in the proof of (1.1). For the inner ball property (5.9), assume that for some $i \neq j$, $d(x_i, \mathrm{Bas}(\mathbf{P}_N, x_j)) \leq \varepsilon N^{-1/d}$. We shall find a lower bound on ε that depends only on $(\alpha, d, c_A, C_A, \mathrm{diam}(\mathrm{spt}\,\nu))$. If $\varepsilon \leq c_1/2$ where c_1 is the separation constant in (1.2), then taking a point $x \in B(x_i, \varepsilon N^{-1/d}) \cap \mathrm{Bas}(\mathbf{P}_N, x_j)$, we have

$$d(x, x_j) \geq d(x_i, x_j) - d(x_i, x) \geq (c_1/2)N^{-1/d},$$

thus taking γ_j a $\mathbf{P}_N$-good curve from x_j to x, its length is greater than $(c_1/2)N^{-1/d}$ hence

$$z_{\mathbf{P}_N}(x) = \int_{\gamma_j} \Theta^{\alpha-1}_{\mathbf{P}^{x_j}_N} \geq (c_1/2)N^{-1/d}\nu(\mathrm{Bas}(\mathbf{P}_N, x_j))^{\alpha-1} \geq \frac{1}{2}c_1(C_A c_2^d)^{\alpha-1}N^{-\beta/d}.$$

As a consequence, setting $c' := \frac{1}{2}c_1(C_A c_2^d)^{\alpha-1}$ and $c'' := (c'/2C_H)^{1/\beta}$, for every $y \in B(x, c''N^{-1/d}) \cap \mathrm{spt}\,\nu$, by Theorem 4.1,

$$z_{\mathbf{P}_N}(y) \geq z_{\mathbf{P}_N}(x) - C_H|y-x|^\beta \geq c'/2N^{-\beta/d}.$$

If $\varepsilon \leq c''$ we build a competitor $\mathbf{P}'_N$ by removing from $\mathbf{P}_N$ all curves going to $B(x, \varepsilon N^{-1/d})$, and adding an optimal traffic plan $Q \in \mathbf{TP}(m_\varepsilon \delta_{x_i}, \nu \llcorner B(x, \varepsilon N^{-1/d})$ where $m_\varepsilon := \nu(B(x, \varepsilon N^{-1/d}))$. Using the first variation formula Proposition 4.2 (C) and the subadditivity of the α-mass, we get by optimality of μ_N

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}_N) \leq \mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - \alpha\int_{B(x,\varepsilon N^{-1/d})} z_{\mathbf{P}_N}\,\mathrm{d}\nu + C_{\mathsf{BOT}}(2\varepsilon N^{-1/d})(C_A\varepsilon^d N^{-1})^\alpha \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - (\alpha c_A c'/2)\varepsilon^d N^{-(\alpha+\frac{1}{d})} + 2C_{\mathsf{BOT}}C_A^\alpha \varepsilon^{1+d\alpha}N^{-(\alpha+\frac{1}{d})}.\end{aligned}$$

We get a contradiction when ε is smaller than some constant $c''' > 0$ (depending only on α, d, c_A, c', C_{BOT}, C_A) because $d < 1 + d\alpha$. During the reasoning we made, recall that we assumed $\varepsilon \leq c_1/2$ then $\varepsilon \leq c''$, thus we must have:

$$\varepsilon \geq c := \min\{c_1/2, c'', c'''\},$$

and (5.9) holds with this constant $c > 0$. □

5.3 Uniformity bounds for masses and energies

Proposition 5.3. *Assume $\alpha \in (1-1/d, 1)$. Let $\nu \in \mathscr{M}^+_c(\mathbb{R}^d)$ be a d-Ahlfors regular measure with constants $c_A, C_A > 0$. Let $\mu_N = \sum_{i=1}^N m_i\delta_{x_i}$ be an N-point optimal quantizer and $\mathbf{P}_N \in \mathbf{TP}(\mu_N, \nu)$ be an optimal traffic plan. There are constants $c, C > 0$ depending only on $(\alpha, d, c_A, C_A, \mathrm{diam}(\mathrm{spt}\,\nu))$ such that for all $i \in \{1, \ldots, N\}$,*

$$cN^{-1} \leq m_i \leq CN^{-1}$$

and

$$cN^{-(\alpha+1/d)} \leq \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \leq CN^{-(\alpha+1/d)}.$$

Proof. The upper bounds come from the fact that, by Theorem 1.2, for every $i \in \{1, \dots, N\}$, $\mathrm{Bas}(\mathbf{P}_N, x_i)$ has diameter less than $c_2 N^{-1/d}$, thus

$$m_i = \nu(\mathrm{Bas}(\mathbf{P}_N, x_i)) \leq C_A c_2^d N^{-1},$$

and thus from the usual estimate of branched transport cost we get

$$\mathbf{d}^\alpha(m_i \delta_{x_i}, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \leq C_{\mathsf{BOT}}(c_2 N^{-1/d})(C_A c_2^d N^{-1})^\alpha.$$

The lower bounds results from Proposition 5.1, which implies that (c, C being the inner and outer ball constants):

$$\nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i) \geq \nu \llcorner B(x_i, cN^{-1/d}),$$

and thus

$$\nu(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq c_A c^d N^{-1}.$$

Since ν is d-Ahlfors regular on $\mathbb{R}^d$, it may be written $\nu = f\mathscr{L}^d \llcorner X$ with $c_A/\omega_d \leq f \leq C_A/\omega_d$ on some Borel set $X \subseteq \operatorname{spt} \nu$. Therefore, $\omega_d C_A^{-1} \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)$ is a measure which is absolutely continuous with respect to Lebesgue, with density in $[0, 1]$, and total mass greater than $m := \omega_d c_A / C_A c^d N^{-1}$, thus

$$\mathbf{d}^\alpha(m_i \delta_{x_i}, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \geq e_{\alpha,d} m^{\alpha + \frac{1}{d}},$$

where $e_{\alpha,d} > 0$ is the constant from the optimal shape problem studied in [PSX19], whose definition is given in (2.20). ☐

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