Optimal Quantization with Branched Optimal Transport distances

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We consider the problem of optimal approximation of a target measure by an atomic measure with N atoms, in branched optimal transport distance. This is a new branched transport version of optimal quantization problems. New difficulties arise, as in previously known Wasserstein semi-discrete transport results the interfaces between cells associated with neighboring atoms had Voronoi structure and satisfied an explicit description. This description is missing for our problem, in which the cell interfaces are thought to have fractal boundary. We study the asymptotic behaviour of optimal quantizers for absolutely continuous measures as the number N of atoms grows to infinity. We compute the limit distribution of the corresponding point clouds and show in particular a branched transport version of Zador's theorem. Moreover, we establish uniformity bounds of optimal quantizers in terms of separation distance and covering radius of the atoms, when the measure is d-Ahlfors regular. A crucial technical tool is the uniform in N Hölder regularity of the landscape function, a branched transport analog to Kantorovich potentials in classical optimal transport.

Keywords. optimal transport, branched transport, quantization, clustering, optimal partitions, uniform point configurations, Gamma-convergence, convergence of measures.
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Contents

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Notation

Notation on traffic plans

$\mathbf{TP}^d$	space of traffic plans on $\mathbb{R}^d$
$\mathbf{TP}(\mu^-, \mu^+)$	set of traffic plans $\mathbf{P}$ on $\mathbb{R}^d$ such that $(e_0)_\sharp \mathbf{P} = \mu^-$, $(e_\infty)_\sharp \mathbf{P} = \mu^+$ where $\mu^\pm \in \mathscr{M}^+(\mathbb{R}^d)$
$\mathbf{P}_n \overset{\star}{\rightharpoonup} \mathbf{P}$	weak-$\star$ convergence in $\mathscr{M}^+(\Gamma^d)$ in the duality with $\mathscr{C}(\Gamma^d)$
$\Theta_{\mathbf{P}}(x)$	$:= \int_{\Gamma^d} \#\gamma^{-1}(\{x\})\,\mathrm{d}\mathbf{P}(\gamma)$, multiplicity at x w.r.t. $\mathbf{P}$
$\Sigma_{\mathbf{P}}$	$:= \{x : \Theta_{\mathbf{P}}(x) > 0\}$ network associated with $\mathbf{P}$.
$\mathbb{M}^\alpha(\mathbf{P})$	$:= \int_{\mathbb{R}^d} \Theta_{\mathbf{P}}(x)^\alpha \mathscr{H}^1(x)$, α-mass of $\mathbf{P}$;

We use the conventions $0^b = +\infty$ if $b < 0$, $0^0 = 1$, and $+\infty \times 0 = 0$.

1 Introduction

In this work we study for the first time the asymptotics and uniformity properties of Optimal Quantization with interactions given via Branched Optimal Transport distances, which we will also call, for brevity, *Branched Quantization*. The fields of Branched Optimal Transport and Optimal Quantization both have a large variety of applications but have not been connected before. We give a very short review and motivations of both, after which we point out why building a connection is interesting to explore.

1.1 Branched Optimal Transport and Optimal Quantization motivations

Branched Optimal Transport (or Branched Transport for short) is an umbrella term for a class of optimization problems, related to classical optimal transport, in which mass particles are assumed to interact (as opposed to traveling independently) while moving from a source to a target distribution. The interaction favours the transportation of particles in a grouped way by lowering the transportation cost, which is justified in many practical situations by an *economy of scale*. A consequence of this assumption is that the particles' paths form a one-dimensional network with a branched structure. The most common model assumes a cost of the form $\ell \times m^\alpha$ to move a group of particles of total mass m over a distance ℓ, where $\alpha \in [0, 1]$, so that the cost is a concave power of the mass. This problem was first introduced in [Gil67], in a discrete setting, to optimize communications network, and was extended to two different continuous settings in [Xia03b; MSM03] (both are actually equivalent [PS06; Peg17b]). For an introduction to the theory of branched transport we refer to the book [BCM09b]. In pure mathematics, mass minimization amongst 1-dimensional flat G-chains with fixed boundary provides versions of branched transport, see e.g. [Xia04; PS06; MM16] and the fundamental results in [Fle66;

Whi99a; Whi99b]. We also mention [PZ18] for a first result on the classification of groups G that produce branching. This last work, together with the classification of homotopy groups of spheres and the classification [HR08] indicates that branched transport costs must commonly appear in connections between vortices of nonlinear Sobolev maps. This has recently lead to important insights into weak density results such as [Bet20]. Branched transport is also connected to size-minimization problems [DH03] and the Steiner problem [MM16], network transport systems [Dur06; But+09], superconductivity [Con+18], traffic flow optimization [ILB21], models of tree roots and branches [BS18; BPS20], models of river systems [RR01], mailing problems [Col+17], urban planning problems [BW15], amongst others. Finding the optimal branched transport map is in general NP-hard (while for classical optimal transport the complexity is $O(n^3 \log n)$ for $n-$point masses), therefore computational approximations are an interesting direction of research, see [OS11], [Mon17].

Classical optimal quantization problems consider the question of how to discretize a given positive measure $\nu \in \mathcal{M}_+(X)$ (in which $X = \mathbb{R}^d$ or X is a more general metric space), in such a way that the discrete $N-$point approximant $\nu_N \in \mathcal{M}(\mathbb{R}^d)$ is at minimum distance according to a distance or cost $c : X \times X \to (0, \infty)$ (which usually is a power of the distance over $X = \mathbb{R}^d$, i.e. $c(x, y) = |x - y|^p, p \geq 1$, see [Mér11]). We refer to general rerference books [GL00], [GG12] for an overview of the problem. Historical first references are [Fej59] and [Ger79]. Applications of optimal quantization range from from clustering [MB02], [Boo+09], to signal processing [GG92], to numerical integration/quadrature [Pag98], to economy [BS72]. The quantization problem can be reformulated as a semi-discrete optimal transport problem, see e.g. [Mér11], and important applications appeared in material science [BPT14; BC21; BPR23]. Asymptotics and continuum limits of the problem as the number of discretization points tends to infinity have been studied for the classical optimal quantization problems by Zador [Zad82] and by Bouchitté et al. [BJM02][BJM11], and we also mention the important work of Gruber [Gru04].

Further problems that are not directly formulated as a quantization problem but can be seen as generalizations of the problem in the same way as the case of introducing a branched transport distance is, appear in minimization of energies of a large number of "charges" under Riesz-Coulomb interactions: see the book [BHS19], and the crystallization survey [BL15]. We mention the related problems of optimal unconstrained polarization [HPS20], jellium equidistribution [PR18], Voronoidal tessellations [DFG99], power diagrams [AHA98], clustering [Sax+17], amongst others.

In this work, we focus on the case in which the cost underlying an optimal quantization problem is replaced by a branched transport cost. The motivations for formulating this new problem come from both a mathematically interesting new difficulty, and for the potential of direct applications.

Mathematically, the most important difficulty with quantization problems under branched transport distances, is that the regularity of interfaces is not known, and the interfaces do not satisfy an explicit condition. This makes Branched Quantization much more challenging than classical optimal quantization, and required us to give replacements for the main steps in the proof. We expect that our approach will allow to study some classes of problems involving random interfaces as well, since we do not make direct use of properties of the interfaces in our estimates.

In terms of applicability, in computational problems many clustering problems use classical distances only due to their being "a simple first choice" and complex clustering tasks are better approximated via hierarchical tree-like clustering structures, such as those formed by branched transport networks. Many biological models such as the study of plant root competition (a natural extension to models such as [BS18]) would directly lead to branched quantization formulations. The same goes for supply chains modelling, in which several sources have to be optimized in order to supply a target density of users.

1.2 Main results

In this section we give simplified statements of our asymptotics and uniformity results for optimal branched quantizers, using a minimal amount of definitions. For further full definitions and background results, see Section 2.

We will define in Section 2.1 traffic plans $\mathbf{P}$ between probability measures μ, ν, as suitable measures over $1-$Lipschitz curves transporting μ to ν. For $\alpha \in [0,1]$ we define the α-mass $\mathbb{M}^\alpha(\mathbf{P})$ as the integral of the α-th power of the transported mass flux $\Theta_{\mathbf{P}}$ (called *multiplity*), over the network $\Sigma_{\mathbf{P}}$ induced by $\mathbf{P}$, when the latter is 1-rectifiable (see full definition in Section 2.1). It is indeed proportional to $m^\alpha \times \ell$ when moving a total mass m over a distance ℓ. Then for $N \geq 1$ we consider the *branched optimal quantization problem* defined as

$$\mathcal{E}^\alpha(\nu, N) \coloneqq \min\left\{\mathbf{d}^\alpha(\mu,\nu) : \#\operatorname{spt}\mu \leq N\right\}, \tag{1.1}$$

in which $\mathbf{d}^\alpha$ is the branched transport distance, given as the infimum of $\alpha-$mass $\mathbb{M}^\alpha(\mathbf{P})$ amongst traffic plans $\mathbf{P}$ transporting μ to ν. An optimimzer for this minimization problem is called an *optimal $N-$point quantizer of μ.*

Recall that a natural scaling power is $\beta := 1 + d\alpha - d$, which will be used below. Furthermore, Ahlfors regularity condition on a measure μ requires that the measure of balls centered in its support is comparable to Lebesgue measure (see (4.2) for the precise definition).

Then the first main result is a branched transport version of a result by Zador [Zad82] valid for classical quantization.

Theorem 1.1 (See Theorem 3.5)**.** *Let ν be a d-Ahlfors regular probability measure on $\Omega \subseteq \mathbb{R}^d$. Then:*

(A) if $(\mu_N)_{N\in\mathbb{N}}$ is a sequence of optimal N-point quantizers μ_N,

$$\mu_N^\diamond \doteq (\#\operatorname{spt}\mu_N)^{-1} \sum_{\{x:\mu_N(\{x\})>0\}} \delta_x \overset{\star}{\rightharpoonup} C\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}},$$

where $C = \left(\int_\Omega \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}}\,\mathrm{d}x\right)^{-1}$.

(B) assimilating ν with its density with respect to the Lebesgue measure $\mathscr{L}^d$,

$$\lim_{N\to\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu, N) = c_{\alpha,d}\left(\int_\Omega \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}}\,\mathrm{d}x\right)^{\alpha+\frac{1}{d}}; \tag{1.2}$$

The above theorem is a consequence of a more precise $\Gamma-$convergence result of $\mathbf{d}^\alpha$-distance to the continuum along sequences with fixed limit mass density and support density, given in Theorem 3.1. The result of Theorem 3.1 is the branched transport analogue of the asymptotic result of [BJM11] for classical optimal quantization.

Our uniformity results reinstate the for branched quantization, the general strategy of [Gru04], valid for classical quantization. This result describes bounds on the *covering radius* and on the *separation distance* for optimal quantizers, both at the natural scale of $cN^{-1/d}$, which is coherent with the principle that for an optimal quantizer, roughly speaking, an euclidean ball of volume c/N is assigned to each of N points in the quantizer support. The covering radius bound quantifies the property that the atoms of an optimal quantizer are never farther than $c_1N^{-1/d}$ from the support of the quantized measure μ, and the separation bound indicates that the atoms of the quantizer are never closer than $c_2N^{-1/d}$ from each other.

Theorem 1.2 (See Theorem 5.1)**.** *Let ν be a compactly supported d-Ahlfors regular measure on $\mathbb{R}^d$ and $\mu_N = \sum_{1\leq i\leq N} m_i\delta_{x_i}$ be an N-point optimal quantizer with atoms $\mathcal{X} = \{x_i\}_{1\leq i\leq N}$. Then the covering radius $\omega(\operatorname{spt}\nu, \mathcal{X})$ and separation distance $\delta(\mathcal{X})$ enjoy the following bounds:*

$$\omega(\operatorname{spt}\nu, \mathcal{X}) \coloneqq \sup_{x\in\operatorname{spt}\nu}\min_i d(x_i, x) \leq c_2N^{-1/d},$$
$$\delta(\mathcal{X}) \coloneqq \min_{j\neq k} d(x_j, x_k) \geq c_1N^{-1/d}.$$

for some constants $c_1, c_2 > 0$ that do not depend on N.

The new technical result that allows to prove our uniformity results is given in Theorem 4.1, and gives a uniform Hölder control of the so-called *landscape function*, a substitute for classical Kantorovich potentials in Branched Transport theory. Recall that classical Optimal Transport theory, so-called Kantorovich duality allow to transform the problem to a dual version based on Kantorovich potentials (see e.g. [San15]). In turn, Kantorovich potentials can be used to show that for optimal quantization with cost $|x-y|^p$ interfaces of the quantization cells are straight (see e.g. in [MT21]).

In Branched Transport there is no precise analogue of Kantorovich duality, but optimal Kantorovich potentials have a partial analogue in the landscape function. As a reference for the single-source landscape function $z_{\mathbf{P}^x}$ (corresponding to case $N = 1$ in our notation) see e.g. [PSX19] and references therein. The basic properties of $z_{\mathbf{P}^x}$ are recalled in Proposition 2.2. For general N, in Theorem 4.1 we prove the following result, for which need a weaker property of a quantizer to be *mass-optimal*, i.e. we require it to be a minimizer restricted to the class of measures for which the support of the quantizer competitors is kept fixed but the masses are allowed to vary.

Theorem 1.3 (see Theorem 4.1)**.** *Let $\nu \in \mathscr{M}_c(\mathbb{R}^d)$ be a compactly supported d-Ahlfors regular measure, and let $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan where $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ is a N-point mass-optimal quantizer of ν with respect to $(x_i)_{1\le i\le N}$. There exists a unique function $z_{\mathbf{P}} : \operatorname{spt}\nu \to \mathbb{R}_+$ that we call* landscape function associated with $\mathbf{P}$ *which locally coincides with the single-source landscape functions $z_{\mathbf{P}^{x_i}}$ and is β-Hölder continuous for $\beta := 1 + d\alpha - d \in (0, 1]$, with a Hölder constant independent on N.*

We emphasize that in the proof of Theorem 1.2 we make crucial use of the *uniform in N* Hölder control of $z_{\mathbf{P}}$, without which we do not expect the same results to hold.

1.3 Structure of the paper

- In Section 2 we complete the definitions underlying our main theorems, recall important foundational results in Branched Optimal Transport and establish preliminary results on the optimal quantization and partition problems.
- In Section 3 we prove our main Γ-convergence and asymptotic results, Theorem 3.1 and Theorem 1.2.
- In Section 4 we prove the above Theorem 1.3 on the regularity of the landscape function.
- In Section 5 we prove the uniformity result of Theorem 1.2.

2 Background and preliminaries

2.1 Background in branched optimal transport

In this section we set up the static "Lagrangian" model of branched optimal transport based on *traffic plans* developed by [BCM05] and [MSM03]. The main reference on branched optimal transport is the book [BCM09b]. The presentation, notation and definitions that we adopt in this paper have been slightly simplified following more recent works.

Traffic plans

A *traffic plan* $\mathbf{P}$ on $\mathbb{R}^d$ is a finite positive measure on the set of 1-Lipschitz curves $\Gamma^d := \mathrm{Lip}_1(\mathbb{R}^+, \mathbb{R}^d)$, endowed with the metrizable topology of uniform convergence on compact sets, which is concentrated on the set of curves with finite stopping time:

$$\mathbf{P}(\{\gamma \in \Gamma^d : T(\gamma) = +\infty\}) = 0, \tag{2.1}$$

where for every $\gamma \in \Gamma^d$,

$$T(\gamma) := \inf\{\tau \ge 0 : \gamma \text{ constant on } [\tau, +\infty)\} \in [0, +\infty]. \tag{2.2}$$

Given two probability measures $\mu^\pm \in \mathscr{M}^+(\mathbb{R}^d)$, we say that a traffic plan $\mathbf{P}$ transports μ^- to μ^+, and we write $\mathbf{P} \in \mathbf{TP}(\mu^-, \mu^+)$ if one recovers μ^- and μ^+ by assigning the mass of each curve to its initial or final point, namely if $(e_0)_\sharp \mathbf{P} = \mu^-$ and $(e_\infty)_\sharp \mathbf{P} = \mu^+$ where $e_0(\gamma) = \gamma(0)$ and $e_\infty(\gamma) = \lim_{t\to\infty} \gamma(t)$ for every $\gamma \in \Gamma^d$. The measures μ^- and μ^+ are respectively called the source and sink measures of $\mathbf{P}$.

For every $x \in \mathbb{R}^d$, the *multiplicity*

$$\Theta_{\mathbf{P}}(x) \coloneqq \int_{\Gamma^d} \#\gamma^{-1}(\{x\}) \, \mathrm{d}\mathbf{P}(\gamma) \tag{2.3}$$

represents the amount of curves, measured by $\mathbf{P}$, which visit x (each curve being counted as many times as it visits x). The *network* of $\mathbf{P}$ is the (possibly empty) 1-rectifiable set[1]

$$\Sigma_{\mathbf{P}} \coloneqq \{x \in \mathbb{R}^d : \Theta_{\mathbf{P}}(x) > 0\}. \tag{2.4}$$

The traffic plan $\mathbf{P}$ is said *rectifiable* if there exists a 1-rectifiable set Σ such that

$$\mathscr{H}^1(\gamma(\mathbb{R}^+) \setminus \Sigma) = 0 \text{ for } \mathbf{P}\text{-almost every } \gamma \in \Gamma^d, \tag{2.5}$$

in which case (2.5) holds with $\Sigma = \Sigma_{\mathbf{P}}$. It is said *simple* if it is concentrated on simple curves, i.e. curves $\gamma \in \Gamma^d$ such that γ is constant on $[s, t]$ whenever $\gamma(s) = \gamma(t)$ and $s < t$.

Finally, two traffic plans, $\mathbf{P}_1, \mathbf{P}_2$ are said *disjoint* if there exists two disjoint sets $A_1, A_2 \subseteq \mathbb{R}^d$ such that for $i \in \{1, 2\}$,

$$\mathscr{H}^1(\gamma(\mathbb{R}^+) \setminus A_i) = 0 \text{ for } \mathbf{P}_i\text{-almost every } \gamma \in \Gamma^d. \tag{2.6}$$

For rectifiable traffic plans $\mathbf{P}_1, \mathbf{P}_2$, it is equivalent to

$$\mathscr{H}^1(\Sigma_{\mathbf{P}_1} \cap \Sigma_{\mathbf{P}_2}) = 0. \tag{2.7}$$

Concatenation of traffic plans

We follow the presentation of concatenations provided in [CDM19, §3.3]. If $\gamma_1, \gamma_2 \in \Gamma^d$ have finite stopping time and $\gamma_1(+\infty) = \gamma_2(0)$ we set for every $t \in \mathbb{R}^+$

$$(\gamma_1 : \gamma_2)(t) \coloneqq \begin{cases} \gamma_1(t) & \text{if } t \in [0, T(\gamma_1)), \\ \gamma_2(t - T(\gamma_1)) & \text{if } t \in [T(\gamma_1), +\infty). \end{cases}$$

We denote this map by $\mathrm{conc} : \Lambda^d \subseteq \Gamma^d \times \Gamma^d \to \Gamma^d$.

If $\mathbf{P}_1, \mathbf{P}_2 \in \mathbf{TP}^d$ are such that $(e_\infty)_\sharp \mathbf{P}_1 = (e_0)_\sharp \mathbf{P}_2$, we say that $\mathbf{P}$ is a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ if there exists a measure $\bar{\mathbf{P}} \in \mathscr{M}^+(\Gamma^d \times \Gamma^d)$, called a *recovery plan*, which is concentrated on Λ^d and satisfies

$$\mathbf{P} = \mathrm{conc}_\# \bar{\mathbf{P}} \tag{2.8}$$

$$(p_i)_\sharp \bar{\mathbf{P}} = \mathbf{P}_i \quad \text{where } p_i : (\gamma_1, \gamma_2) \mapsto \gamma_i \text{ for } i \in \{1, 2\}. \tag{2.9}$$

We denote by $(\mathbf{P}_1 : \mathbf{P}_2)$ the set of concatenations of $\mathbf{P}_1$ and $\mathbf{P}_2$. We will need some properties of concatenations that are summarized in the following proposition, extracted from [CDM19, §3.3].

Proposition 2.1 ([CDM19, Lemma 3.6])**.** *If $\mathbf{P}_1, \mathbf{P}_2 \in \mathbf{TP}^d$ are such that $(e_\infty)_\sharp \mathbf{P}_1 = (e_0)_\sharp \mathbf{P}_2$, then:*

(i) $(\mathbf{P}_1 : \mathbf{P}_2)$ *is nonempty,*

(ii) $(e_0)_\sharp \mathbf{P} = (e_0)_\sharp \mathbf{P}_1$ *and* $(e_\infty)_\sharp \mathbf{P} = (e_\infty)_\sharp \mathbf{P}_2$ *for every* $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$,

(iii) for every $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$, $\Theta_{\mathbf{P}} = \Theta_{\mathbf{P}_1} + \Theta_{\mathbf{P}_2}$ *and thus* $\mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2)$,

(iv) if $\mathbf{P} \in (\mathbf{P}_1 : \mathbf{P}_2)$ *and* $\mathbf{P}' \in (\mathbf{P}'_1 : \mathbf{P}'_2)$ *then* $\mathbf{P} + \mathbf{P}' \in (\mathbf{P}_1 + \mathbf{P}'_1 : \mathbf{P}_2 + \mathbf{P}'_2)$.

[1] It is 1-rectifiable by [Peg17b, Section 2.1] or [BCM05, Lemma 6.3].

The α-mass functional and the irrigation problem

For $\alpha \in [0,1]$, the α-mass[2] of a traffic plan is defined as

$$\mathbb{M}^\alpha(\mathbf{P}) = \begin{cases} \displaystyle\int_{\Sigma_\mathbf{P}} \Theta_\mathbf{P}(x)^\alpha \,\mathrm{d}\mathscr{H}^1(x) & \text{if } \mathbf{P} \text{ is rectifiable,} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

If $\mu^\pm$ are two positive measures on $\mathbb{R}^d$ of equal (finite) mass, the irrigation problem then reads as

$$\inf \quad \left\{ \mathbb{M}^\alpha(\mathbf{P}) \;\middle|\; \mathbf{P} \in \mathbf{TP}(\mu^-,\mu^+) \right\}, \tag{I$^\alpha$}$$

and we denote by $\mathbf{d}^\alpha(\mu^-,\mu^+)$ this infimum value. We say that $\in\mathbf{OTP}(\mu^-,\mu^+)$ if it realizes the infimum in (I$^\alpha$).

Let us state or recall some results that we shall use thoughout the paper.

(1) **Irrigability and irrigation distance.** Contrary to the classical optimal transport problem, for some pair of compactly supported measures (μ^-,μ^+) and some exponent α it is possible that (2.10) admits no competitor of finite α-mass, typically when the measures spread on a set of large dimension while the exponent α is too small. However, when $\alpha > 1-\frac{1}{d}$, there exists a competitor of finite α-mass for any pair of measures (μ^-,μ^+) of same total mass which are compactly supported on $\mathbb{R}^d$, as shown[3] in [Xia03a]. In particular

$$\begin{aligned} &\alpha > 1 - \frac{1}{d} \quad \text{ and } K \subseteq \mathbb{R}^d \text{ compact} \\ \implies &\mathbf{d}^\alpha \text{ is a distance on } \mathscr{P}(K) \text{ which metrizes the weak-}* \text{ convergence of } \mathscr{C}(K)'. \end{aligned} \tag{2.11}$$

(2) **Existence for the irrigation problem.** Whenever (I$^\alpha$) admits a competitor of finite cost, it admits an minimizer. It results for example from the existence of $\mathcal{E}^\alpha$ minimizers established in [BCM09b], where $\mathcal{E}^\alpha$ is a more complicated variant[4] of $\mathbb{M}^\alpha$, knowing that $\mathcal{E}^\alpha \geq \mathbb{M}^\alpha$, with equality for $\mathcal{E}^\alpha$ minimizers.

(3) **Upper estimates on the α-mass.** If $\alpha > 1 - \frac{1}{d}$, there exists a constant $C_{\mathsf{BOT}} = C_{\mathsf{BOT}}(\alpha,d) \in (0,+\infty)$ such that for any compactly supported measures $\mu^\pm$ of same total mass,

$$\mathbf{d}^\alpha(\mu^-,\mu^+) \leq C_{\mathsf{BOT}} \operatorname{diam}(\operatorname{spt}(\mu^+ - \mu^-)) \|\mu^+ - \mu^-\|^\alpha. \tag{2.12}$$

Indeed, it is proven in [Xia03a] that $\mathbf{d}^\alpha(\delta_0,\mu) \leq C(\alpha,d)/2$ for every $\mu \in \mathscr{P}([-1/2,1/2]^d)$ and some constant $C(\alpha,d) \in (0,\infty)$, from which we deduce $\mathbf{d}^\alpha(\mu^-,\mu^+) \leq Crm^\alpha$ when $\operatorname{diam}(\operatorname{spt}(\mu^-+\mu^+)) \leq r$ and $\|\mu^-\| = \|\mu^+\| = m$ using the triangle inequality and the 1-homogeneity in space and α-homogeneity in mass of the α-mass. Applying this to the measures $\tilde\mu^\pm = \mu^\pm - \mu^- \wedge \mu^+$ yields (2.12) since $\|\tilde\mu^\pm\| = \|\mu^+ - \mu^-\|/2$, $\operatorname{spt}(\mu^+ - \mu^-) = \operatorname{spt}(\tilde\mu^+ + \tilde\mu^-)$, and $\mathbf{d}^\alpha(\mu^-,\mu^+) = \mathbf{d}^\alpha(\tilde\mu^-,\tilde\mu^+)$.

(4) **First variation of the α-mass.** If $\mathbf{P}, \tilde{\mathbf{P}}$ are traffic plans with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, then

$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha \int_{\Gamma^d} Z_\mathbf{P}(\gamma)\,\mathrm{d}(\tilde{\mathbf{P}} - \mathbf{P})(\gamma), \tag{2.13}$$

where

$$Z_\mathbf{P}(\gamma) \coloneqq \int_\gamma \Theta_\mathbf{P}^{\alpha-1} = \int_{\gamma(\mathbb{R})} \Theta_\mathbf{P}(x)^{\alpha-1} \# \gamma^{-1}(x)\,\mathrm{d}\mathscr{H}^1(x). \tag{2.14}$$

[2] It is the equivalent for traffic plans, of the α-mass of currents.

[3] It is shown in the Eulerian model based on vector measures, but adapting the proof in our Lagrangian setting is straightforward, or one can also invoke the equivalence of the models [Peg17b].

[4] The reason for using this functional $\mathcal{E}^\alpha$ instead of the α-mass was that it was only possible at that time to establish the lower semicontinuity of $\mathcal{E}^\alpha$ (on suitable subsets).

The proof of (2.13) relies on the concavity of $m \mapsto m^\alpha$ applied to $\Theta_{\tilde{\mathbf{P}}} = \Theta_{\mathbf{P}} + (\Theta_{\tilde{\mathbf{P}}} - \Theta_{\mathbf{P}})$ and on Fubini's theorem (we refer to [BCM09b] or [Peg17b]). Notice that the integral $\int Z_{\mathbf{P}} \, \mathrm{d}(\tilde{\mathbf{P}} - \mathbf{P})$ is well-defined (possibly infinite) since by Fubini's theorem

$$\infty > \mathbb{M}^\alpha(\mathbf{P}) = \int_{\Gamma^d} Z_{\mathbf{P}} \, \mathrm{d}\mathbf{P}$$

and $\int_{\Gamma^d} Z_{\mathbf{P}} \, \mathrm{d}\tilde{\mathbf{P}} \in [0, \infty]$.

(5) **Single-path property.** If $\mathbf{P} \in \mathbf{TP}(m\delta_s, \nu)$ is an optimal traffic plan, it is simple and satisfies the single-path property, which be stated in the single-source case as follows: for every $x \in \Sigma_{\mathbf{P}}$, there exists a (unique) curve parameterized by arc length $\gamma_{\mathbf{P},x} : [0, \ell] \to \mathbb{R}^d$ such that $\mathbf{P}$-a.e. curve γ passing by x follows the trajectory of $\gamma_{\mathbf{P},x}$, i.e. if $t_x(\gamma)$ ednotes the unique $t \in [0, T(\gamma)]$ such that $\gamma(t) = x$,

$$\text{for } \mathbf{P}\text{-a.e. } \gamma \text{ s.t. } x \in \gamma(\mathbb{R}_+), \quad \tilde{\gamma}_{[0, t_x(\tilde{\gamma})]} = \gamma_{\mathbf{P},x}, \tag{2.15}$$

where $\tilde{\gamma}$ denotes the unit-speed reparameterization of $\gamma \in \Gamma^d$. This fact is stated in [BCM09b, Proposition 7.4].

Landscape function for a single source

Given an optimal irrigation plan $\mathbf{P} \in \mathbf{TP}(m\delta_s, \nu)$, following [San07] we say that a curve γ is $\mathbf{P}$-*good* if, recalling the notation (2.14),

- $Z_{\mathbf{P}}(\gamma) = \int_\gamma \Theta_{\mathbf{P}}(x)^{\alpha-1} \, \mathrm{d}x < +\infty$,
- for all $t < T(\gamma)$,
$$\Theta_{\mathbf{P}}(\gamma(t)) = \mathbf{P}(\{\tilde{\gamma} \in \Gamma(\mathbb{R}^d) : \gamma = \tilde{\gamma} \text{ on } [0, t]\}).$$

It is proven in [San07] that any traffic plan $\mathbf{P}$ which is optimal (with finite cost) is concentrated on the set of $\mathbf{P}$-good curves, and that for all $\mathbf{P}$-good curve γ, the quantity $Z_{\mathbf{P}}(\gamma)$ depends only on the final point $\gamma(\infty)$ of the curve, thus we may define the landscape function $z_{\mathbf{P}}$ as follows:

$$z_{\mathbf{P}}(x) := \begin{cases} Z_{\mathbf{P}}(\gamma) & \text{if } \gamma \text{ is an } \mathbf{P}\text{-good curve s.t. } x = \gamma(\infty), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.16}$$

We summarize the properties on the landscape function, extracted from [San07], that we shall need.

Proposition 2.2. *If $\mathbf{P} \in \mathbf{TP}(m\delta_s, \nu)$ is optimal with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, $\alpha \in [0, 1)$, and $z_{\mathbf{P}}$ is its landscape function, then $z_{\mathbf{P}}$ is lower semicontinuous and finite on $\Sigma_{\mathbf{P}} \cup \operatorname{spt} \nu$. Moreover:*

(A) $z_{\mathbf{P}}(x) \geq d(x, s)$ for every $x \in \mathbb{R}^d$;

(B) the α-mass may be expressed in terms of $z_{\mathbf{P}}$:

$$\mathbf{d}^\alpha(m\delta_s, \mu) = \mathbb{M}^\alpha(\mathbf{P}) = \int_{\Gamma^d} Z_{\mathbf{P}}(\gamma) \, \mathrm{d}\mathbf{P}(\gamma) = \int_{\mathbb{R}^d} z_{\mathbf{P}}(x) \, \mathrm{d}\nu(x);$$

(C) if $\tilde{\mathbf{P}} \in \mathbf{TP}(m\delta s, \tilde{\nu})$ is a traffic plan concentrated on $\mathbf{P}$-good curves, then:

$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu),$$

and the inequality is strict if $\Theta_{\tilde{\mathbf{P}}} - \Theta_{\mathbf{P}}$ is not zero $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}}$;

(D) in particular, $z_{\mathbf{P}}$ is an upper first variation of the irrigation distance*, in the sense that for every $\tilde{\nu} \in \mathscr{M}^+(\mathbb{R}^d)$,*

$$\mathbf{d}^\alpha(\|\tilde{\nu}\|\delta_p, \tilde{\nu}) \leq \mathbf{d}^\alpha(\|\nu\|\delta_s, \nu) + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu).$$

2.2 The optimal quantization problem and mass-optimal quantizers

The optimal branched quantization problem is the following:

$$\mathcal{E}^\alpha(\nu, N) \coloneqq \min\left\{\mathbf{d}^\alpha(\mu,\nu) : \|\mu\| = \|\nu\| \text{ and } \#\operatorname{spt}\mu \le N\right\}. \tag{2.17}$$

An admissible candidate μ_N in this problem will be called a quantizer of ν, and a solution will be called an *N-point optimal quantizer* of ν. Existence of optimal quantizers is shown next.

Theorem 2.3. *For any finite positive measure $\nu \in \mathscr{M}(\mathbb{R}^d)$ and any $N \in \mathbb{N}^*$, the optimal quantization problem (2.17) admits a solution.*

Proof. Take an integer $N \ge 1$ and ν a finite positive measure on $\mathbb{R}^d$, assuming without loss of generality that it has unit mass. Suppose that $\mathcal{E}^\alpha(\nu, N) < +\infty$ (otherwise there is nothing to prove) and take $(\mu^n)_{n\in\mathbb{N}}$ a minimizing sequence with $\sup_{n\in\mathbb{N}} \mathbf{d}^\alpha(\mu^n,\nu) =: \Lambda < +\infty$. Let us show that it is tight. Take $\varepsilon > 0$ and $R \ge \Lambda/\varepsilon$ large enough such that $\nu(\mathbb{R}^d \setminus B_{R/2}) \le \varepsilon/2$. If $\mathbf{P}^n \in \mathbf{TP}(\mu^N, \nu)$ is an optimal traffic plan, then

$$\begin{aligned}
\Lambda \ge \mathbb{M}^\alpha(\mathbf{P}^n) &\ge \int_{\Gamma^d} L(\gamma)\,\mathrm{d}\mathbf{P}^n(\gamma)\\
&\ge \int_{\Gamma^d} L(\gamma)\mathbf{1}_{\{\gamma:\gamma(0)\in\mathbb{R}^d\setminus B_R,\gamma(\infty)\in B_{R/2}\}}\,\mathrm{d}\mathbf{P}^n(\gamma)\\
&\ge R(\mu_n(\mathbb{R}^d\setminus B_R) - \nu(\mathbb{R}^d\setminus B_{R/2})),
\end{aligned}$$

which implies that

$$\mu^n(\mathbb{R}^d\setminus B_R) \le \Lambda/R + \nu(\mathbb{R}^d\setminus B_{R/2}) \le \varepsilon.$$

As a consequence, (μ^n) admits a subsequence converging narrowly to some $\mu \in \mathscr{P}(\mathbb{R}^d)$. Necessarily, μ has at most N atoms as well and by lower semicontinuity of $\mathbf{d}^\alpha$ for the weak convergence of probability measures μ is a N-point optimal quantizer. □

We will consider a class of quantizers which include optimal quantizers and that we call *mass-optimal quantizers*. They will be used to establish the full Γ-convergence result in Section 3, and may also provide a notion of Voronoï cells, called *Voronoï basins*, in the setting of branched optimal transport (see Remark 4.5).

Definition 2.4 (Mass-optimal quantizers)**.** Let $\nu \in \mathscr{M}(\mathbb{R}^d)$ be a finite positive measure and $\mathcal{X} = \{x_i\}_{1\le i\le N}$ be a set of cardinal N. If μ is a measure supported on $\mathcal{X}$ such that the masses of its atoms are chosen in the best way to approximate ν in $\mathbf{d}^\alpha$ distance, i.e. $\mathbf{d}^\alpha(\mu,\nu) = \mathbf{d}^\alpha(\mathcal{X},\nu)$ where

$$\mathbf{d}^\alpha(\mathcal{X},\nu) \coloneqq \inf\left\{\mathbf{d}^\alpha(\mu',\nu) : \operatorname{spt}\mu' \subseteq \mathcal{X}\right\}, \tag{2.18}$$

we say that μ is an N-point *mass-optimal quantizer* with respect to $\{x_i\}_{1\le i\le N}$.

We will also need to decompose any traffic plan $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$, where μ is purely atomic, with respect to the atoms of μ, also called the *sources of* $\mathbf{P}$.

Definition 2.5 (Restrictions and basins from a source)**.** If $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$ where μ is purely atomic and x is an atom of μ, the *restriction of* $\mathbf{P}$ *from the source* x is defined by

$$\mathbf{P}^x \coloneqq \mathbf{P} \llcorner \{\gamma \in \Gamma^d : \gamma(0) = x\}, \tag{2.19}$$

so that the following *source decomposition* of $\mathbf{P}$ holds:

$$\mathbf{P} = \sum_{x\in\operatorname{spt}\mu} \mathbf{P}^x. \tag{2.20}$$

The decomposition is said *disjoint* if all these restrictions $(\mathbf{P}^x)_{x\in\operatorname{spt}\mu}$ are pairwise disjoint. We also introduce the support of the sink measure of $\mathbf{P}^x$, also called the *basin from x with respect to* $\mathbf{P}$:

$$\operatorname{Bas}(\mathbf{P}, x) \coloneqq \operatorname{spt}((e_\infty)_\sharp \mathbf{P}^x). \tag{2.21}$$

In the next lemma, we show that the source decomposition of an optimal plan between a measure and a mass-optimal quantizer is disjoint in the above sense, and that the corresponding sink measures are mutually singular. This result plays a key role in the proof of equivalence between optimal quantization and optimal partition, and will also be crucial in Section 4 to define and study the landscape function, and to show the disjointness of irrigation basins.

Lemma 2.6 (Disjointness properties of mass-optimal quantizers)**.** *Let* $\mu = \sum_{i=1}^N m_i\delta_{x_i}$ *be an* N*-point mass-optimal quantizer of* $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ *with respect to* $\mathcal{X} := \{x_i\}_{1\le i\le N}$ *and* $\mathbf{P} \in \mathbf{TP}(\mu,\nu)$ *be an optimal traffic plan. Then:*

(i) the traffic plans $\mathbf{P}^{x_i}$ *are* disjoint*, in the sense that the measures* $\Theta_{\mathbf{P}^{x_i}}\mathscr{H}^1$ *are mutually singular. In particular they realize the minimum–cost branched transport distance between their own marginals, and*

$$\mathbb{M}^\alpha(\mathbf{P}) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i});$$

(ii) the irrigated measures $\nu_i := (e_\infty)_\sharp \mathbf{P}^{x_i}$ *are mutually singular.*

Proof. Let us start by proving (i). By contradiction, assume that for some $i \neq j$, $\Theta_{\mathbf{P}^{x_i}}\mathscr{H}^1$ and $\Theta_{\mathbf{P}^{x_j}}\mathscr{H}^1$ are not mutually singular, and we contradict the optimality of $\mathbf{P}$. Then there exists a Borel set $A \subseteq \Sigma_\mathbf{P}$ and a constant $m_0 > 0$ such that $\mathscr{H}^1(A) > 0$ and $\Theta_{\mathbf{P}^{x_i}}(x) \wedge \Theta_{\mathbf{P}^{x_j}}(x) \geq m_0$ for every $x \in A$. Pick a point $x \in A$ with $x \neq x_i$ and $x \neq x_j$ and consider for $k \in \{i,j\}$ a traffic plan

$$\mathbf{P}_k \leq \mathbf{P} \llcorner \{\gamma \in \Gamma^d : \gamma(0) = x_k, x \in \gamma(\mathbb{R}_+)\} \quad \text{such that} \quad \|\mathbf{P}_k\| = m_0.$$

For every $\varepsilon \in [0,1]$, we will build a traffic plan $\mathbf{P}_\varepsilon$, obtained from $\mathbf{P}$ by taking a fraction ε of $\mathbf{P}_i$, replacing for each curve γ of $\mathbf{P}_i$ the curve segment between x_i and x by a segment of a curve $\tilde\gamma$ of $\mathbf{P}_j$ from x_j to x. To do this, consider the map $t_x : \gamma \mapsto \min \gamma^{-1}(\{x\})$ and the restriction maps $r_x^- : \gamma \mapsto \gamma_{|[0,t_x(\gamma)]}$, $r_x^+ : \gamma \mapsto \gamma_{|[t_x(\gamma),+\infty)}$. Then

$$(e_\infty)_\sharp (r_x^-)_\sharp \mathbf{P}_j = (e_0)_\sharp (r_x^+)_\sharp \mathbf{P}_i = m_0\delta_x,$$

and by Proposition 2.1 (i) the concatenation $\mathbf{Q} \in ((r_x^-)_\sharp\mathbf{P}_j : (r_x^+)_\sharp\mathbf{P}_i)$ is defined. For $\varepsilon \in [0,1)$ set

$$\mathbf{P}_\varepsilon := \mathbf{P} + \varepsilon(\mathbf{Q} - \mathbf{P}_i).$$

We shall do the converse operation for $\varepsilon \in (-1,0)$, namely

$$\mathbf{P}_\varepsilon := \mathbf{P} - \varepsilon(\mathbf{Q}' - \mathbf{P}_j), \quad \text{where} \quad \mathbf{Q}' \in ((r_x^-)_\sharp\mathbf{P}_i : (r_x^+)_\sharp\mathbf{P}_j).$$

Notice that for both possible signs of ε we have $\Sigma_{\mathbf{P}_\varepsilon} \subseteq \Sigma_\mathbf{P}$ and

$$\Theta_{\mathbf{P}_\varepsilon} = \Theta_\mathbf{P} + \varepsilon\Delta\Theta \quad \text{where} \quad \Delta\Theta := \Theta_{(r_x^-)_\sharp\mathbf{P}_j} - \Theta_{(r_x^-)_\sharp\mathbf{P}_i}. \tag{2.22}$$

Indeed if for example $\varepsilon \geq 0$ then by Proposition 2.1 (iii) we have $\Theta_\mathbf{Q} = \Theta_{(r_x^-)_\sharp\mathbf{P}_j} + \Theta_{(r_x^+)_\sharp\mathbf{P}_i}$, and (2.22) follows because $\Theta_{\mathbf{P}_i} = \Theta_{(r_x^-)_\sharp\mathbf{P}_i} + \Theta_{(r_x^+)_\sharp\mathbf{P}_i}$.

Now, the initial measure $\mu_\varepsilon := (e_0)_\sharp\mathbf{P}_\varepsilon$ of $\mathbf{P}_\varepsilon$ is still supported in $\{x_i : 1 \leq i \leq n\}$ and its final measure is still ν, thus by mass-optimality of μ,

$$\int_{\mathbb{R}^d} \Theta_\mathbf{P}^\alpha \, \mathrm{d}\mathscr{H}^1 = \mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}_\varepsilon) = \int_{\Sigma_\mathbf{P}} (\Theta_\mathbf{P} + \varepsilon\Delta\Theta)^\alpha \, \mathrm{d}\mathscr{H}^1.$$

For $k \in \{i,j\}$, by the single-path property recalled in (2.15), $\mathbf{P}$-a.e. curve starting at x_k and visiting x follows a trajectory given by a single (simple, parameterized by arc length) curve γ_k such that $\gamma_k(0) = x_k$, $\gamma_k(\infty) = x$ and $\gamma_k(\mathbb{R}_+) \subseteq \Sigma_\mathbf{P}$; in particular, $\Theta_{(r_x^-)_\sharp\mathbf{P}_k} = m_0\mathbf{1}_{\gamma_k(\mathbb{R}_+)}$. Since x_i, x_j, x are distinct points, we get

$$\mathscr{H}^1(\{y \in \Sigma_\mathbf{P} : \Delta\Theta(y) \neq 0\}) = \mathscr{H}^1(\gamma_i(\mathbb{R}_+)\Delta\gamma_j(\mathbb{R}_+)) > 0,$$

and as $\alpha \in (0,1)$, the function $\varepsilon \mapsto \int_{\Sigma_{\mathbf{P}}} (\Theta + \varepsilon\Delta\Theta)^\alpha \, d\mathscr{H}^1$ is strictly concave on $(-1,1)$. But it is minimized at $\varepsilon = 0$: a contradiction. Consequently, $\Theta_{\mathbf{P}^{x_i}}\mathscr{H}^1$ and $\Theta_{\mathbf{P}^{x_j}}\mathscr{H}^1$ are mutually singular for every $i \neq j$.

Since $\mathbf{P} = \sum_{i=1}^N \mathbf{P}^{x_i}$, we get that $\mathbb{M}^\alpha(\mathbf{P}) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i})$, and the optimality of each $\mathbf{P}^{x_i}$ for $i \in \{1,\dots,N\}$ follows from that of $\mathbf{P}$.

Let us now prove (ii). For every $i \in \{1,\dots,N\}$, the traffic plan $\mathbf{P}^{x_i}$ being optimal with a single source, we may consider its landscape functions $z_{\mathbf{P}_i}$ as in (2.16). By contradiction assume that $\nu_i \perp \nu_j$ does not hold for some $i \neq j$. Then we have:

$$m := \|\tilde{\nu}\| > 0 \quad \text{where} \quad \tilde{\nu} := \nu_i \wedge \nu_j.$$

For $k \in \{i,j\}$ consider a plan $\mathbf{P}_k \leq \mathbf{P}^{x_k}$ such that $\mathbf{P}_k \in \mathbf{TP}(m\delta_{x_k}, \tilde{\nu})$, and define for $\varepsilon \in (-1,1) \setminus \{0\}$ the competitor

$$\mathbf{P}_\varepsilon = \mathbf{P} + \varepsilon(\mathbf{P}_j - \mathbf{P}_i).$$

Its initial measure $(e_0)_\sharp \mathbf{P}_\varepsilon$ is still supported on $x_i : 1 \leq i \leq N$ and its final measure is still ν, thus by mass-optimality of μ we get

$$\begin{aligned}
&\mathbb{M}^\alpha(\mathbf{P}) = \mathbf{d}^\alpha(\mathcal{X},\nu) \leq \mathbb{M}^\alpha(\mathbf{P}_\varepsilon) \\
&\leq \sum_{k\neq i,j} \mathbb{M}^\alpha(\mathbf{P}^{x_k}) + \mathbb{M}^\alpha(\mathbf{P}^{x_i} - \varepsilon\mathbf{P}_i) + \mathbb{M}^\alpha(\mathbf{P}^{x_j} + \varepsilon\mathbf{P}_j) \\
&< \sum_{k\neq i,j} \mathbb{M}^\alpha(\mathbf{P}^{x_k}) + \mathbb{M}^\alpha(\mathbf{P}^{x_i}) - \alpha\varepsilon \int_{\mathbb{R}^d} z_{\mathbf{P}^{x_i}} \, d(e_\infty)_\sharp \mathbf{P}_i + \mathbb{M}^\alpha(\mathbf{P}^{x_j}) + \alpha\varepsilon \int_{\mathbb{R}^d} z_{\mathbf{P}^{x_j}} \, d(e_\infty)_\sharp \mathbf{P}_j \\
&= \mathbb{M}^\alpha(\mathbf{P}) + \alpha\varepsilon \int_{\mathbb{R}^d} (z_{\mathbf{P}^{x_j}} - z_{\mathbf{P}^{x_i}}) \, d\tilde{\nu},
\end{aligned}$$

where we have used the first variation inequality Proposition 2.2 (C) twice on the third line. The inequality in the third line is strict for $\epsilon \in (-1,1) \setminus \{0\}$, because for $k \in \{i,j\}$, $\Theta_{\mathbf{P}_k} \leq \Theta_{\mathbf{P}^{x_k}}$ thus for every $y \in \mathbb{R}^d$ such that $\Theta_{\mathbf{P}_k}(y) > 0$ we have $|\varepsilon|\Theta_{\mathbf{P}_k} < \Theta_{\mathbf{P}^{x_k}}$: this strict inequality holds on a $\mathscr{H}^1$-positive subset of $\Sigma_{\mathbf{P}^{x_k}}$. We can now choose ε such that $\varepsilon \int_{\mathbb{R}^d} (z_{\mathbf{P}^{x_j}} - z_{\mathbf{P}^{x_i}}) \, d\tilde{\nu} \leq 0$, and we get the contradiction $\mathbb{M}^\alpha(\mathbf{P}) < \mathbb{M}^\alpha(\mathbf{P})$. □

2.3 The optimal partition problem and equivalence with optimal quantization

For any finite nonnegative measure ν, we define

$$\mathbf{C}^\alpha(\nu) := \inf_{x\in\mathbb{R}^d} \mathbf{d}^\alpha(\|\nu\|\delta_x, \nu). \tag{2.23}$$

Given a compactly supported measure ν on $\mathbb{R}^d$ and an integer $N > 1$, we define the optimal (branched) ν-partition problem as:

$$\inf \left\{ \sum_{i=1}^N \mathbf{C}^\alpha(\nu \llcorner \Omega_i) : \ (\Omega_i)_{1\leq i\leq N} \subseteq \operatorname{spt}\nu, \nu(\mathbb{R}^d \setminus \bigcup_i \Omega_i) = 0 \text{ and } \mu(\Omega_i \cap \Omega_j) = 0 \, (\forall i \neq j) \right\}. \tag{2.24}$$

It may be equivalently written in terms of measures as:

$$\inf \left\{ \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) : \ (\nu_i)_{1\leq i\leq N}, \nu = \sum_{1\leq i\leq N} \nu_i \text{ and } \nu_i \perp \nu_j \, (\forall i \neq j) \right\}. \tag{2.25}$$

For more classical costs, e.g. corresponding to Wasserstein distance W_p, it is straightforward to see that optimal quantization is equivalent to an optimal partition problem, where the optimal partitions are given by Voronoi diagrams associated with a finite set of points. With our branched transportation cost $\mathbf{C}^\alpha$ (corresponding to distance $\mathbf{d}^\alpha$), the situation is a priori much more difficult, since there is no clear decomposition of the target space into regions associated with the atoms of a quantizer: on the contrary, in branched transport it is expected that several atoms are first collected together along a graph, then irrigate some part of the target measure, so that we cannot associate these irrigated points with a single atom. However, we shall see that such situations do not occur for *optimal* quantizers, resulting in the equivalence between the optimal quantization and optimal partition problems.

Note that the existence of minimizers for (2.25) is not direct from functional analysis results, since the condition $\nu_i \perp \nu_j$ does not pass to weak limits of measures. To prove existence of solutions, we introduce the following relaxed optimal partition problem

$$\inf \left\{ \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) : \ (\nu_i)_{1\le i\le N}, \nu = \sum_{1\le i\le N} \nu_i \right\}. \tag{2.26}$$

We shall see below that the optimal quantization problem and the original and relaxed partition problems are equivalent (Theorem 2.7), and obtain existence to (2.25) in Corollary 2.8.

Theorem 2.7 (Optimal Quantization $\simeq$ Optimal Partition)**.** *Given a measure $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$, the minimal values of the optimal quantization problem (2.17) and the optimal partition problem (2.25), as well as its relaxation (2.26), are equal. Furthermore the minimizers of these problems are related as follows:*

(i) If $\mu_N = \sum_{i=1}^N m_i\delta_{x_i}$ is solution of (2.17) with optimal plan $\mathbf{P} \in \mathbf{TP}(\mu_N, \nu)$ then the irrigated measures $\nu_i \coloneqq (e_\infty)_\sharp \mathbf{P}^{x_i}$ for $1 \le i \le N$ form an optimizer of (2.25).

(ii) If $(\nu_i)_{1\le i\le N}$ is an optimizer of (2.25) and if for every i, $x_i \in \mathbb{R}^d$ and $\mathbf{P}_i \in \mathbf{TP}(\|\nu_i\|\delta_{x_i}, \nu_i)$ are optimal, i.e. $\mathbf{C}^\alpha(\nu_i) = \mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \mathbb{M}^\alpha(\mathbf{P}_i)$, then $\mu_N \coloneqq \sum_{i=1}^N \|\nu_i\|\delta_{x_i}$ is an optimizer for (2.17) and $\mathbf{P} \coloneqq \sum_{i=1}^N \mathbf{P}_i \in \mathbf{TP}(\mu_N, \nu)$ is an optimal traffic plan.

(iii) The optimal partition problem (2.25) and the relaxed problem (2.26) have the same minimizers and minimal value.

Proof of Theorem 2.7. Denote by $\mathcal{E}_{\mathrm{p}}^\alpha(\nu, N)$ and $\mathcal{E}_{\mathrm{pr}}^\alpha(\nu, N)$ the infima of (2.25) and (2.26) respectively and take $\mu = \sum_{i=1}^N m_i\delta_{x_i}$ a minimizer of (2.17), which exists by Theorem 2.3, and an optimal traffic plan $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$. By Lemma 2.6 (ii), the irrigated measures $\nu_i = (e_\infty)_\sharp \mathbf{P}^{x_i}$ are mutually singular and $\nu = \sum_{i=1}^N \nu_i$. In particular, $(\nu_i)_{1\le i\le N}$ is a competitor for (2.25). Besides, by Lemma 2.6 (i) the traffic plans $\mathbf{P}^{x_i} \in \mathbf{TP}(m_i\delta_{x_i}, \nu_i)$ are disjoint and optimal, thus:

$$\mathcal{E}_{\mathrm{p}}^\alpha(\nu, N) \le \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) \le \sum_{i=1}^N \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu_i) = \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}^{x_i}) = \mathbb{M}^\alpha(\mathbf{P}) = \mathcal{E}^\alpha(\nu, N). \tag{2.27}$$

Viceversa, take a ε-minimizer $(\nu_i)_{1\le i\le N}$ of (2.26) for some fixed $\varepsilon > 0$. We can form a competitor for (2.17) by simply taking for each ν_i a point x_i that is optimal, i.e. such that $\mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \mathbf{C}^\alpha(\nu_i)$, and setting $\mu \coloneqq \sum_{i=1}^N \|\nu_i\|\delta_{x_i}$. Moreover, taking for every $i \in \{1, \dots, N\}$ an optimal traffic plan $\mathbf{P}_i \in \mathbf{TP}(\|\nu_i\|\delta_{x_i}, \nu_i)$, the traffic plan $\mathbf{P} \coloneqq \sum_{i=1}^N \mathbf{P}_i$ belongs to $\mathbf{TP}(\mu, \nu)$ where μ has at most N atoms, and therefore by subadditivity of the α-mass:

$$\begin{aligned} \mathcal{E}^\alpha(\nu, N) \le \mathbf{d}^\alpha(\mu, \nu) \le \mathbb{M}^\alpha(\mathbf{P}) &\le \sum_{i=1}^N \mathbb{M}^\alpha(\mathbf{P}_i) \\ &= \sum_{i=1}^N \mathbf{d}^\alpha(\|\nu_i\|\delta_{x_i}, \nu_i) = \sum_{i=1}^N \mathbf{C}^\alpha(\nu_i) = \mathcal{E}_{\mathrm{pr}}^\alpha(\nu, N) + \varepsilon. \end{aligned} \tag{2.28}$$

Since ε is arbitrary, this shows $\mathcal{E}^\alpha(\nu, N) = \mathcal{E}_{\mathrm{pr}}^\alpha(\nu, N) = \mathcal{E}_{\mathrm{p}}^\alpha(\nu, N)$ and (i) holds because of (2.27) in the first part of the proof ; it implies in particular existence for the optimal partition problem and its relaxation. Besides, taking now $(\nu_i)_{1\le i\le N}$ a minimizer of the relaxed partition problem (2.26) instead of an ε-minimizer, and plugging it in the inequalities (2.28) (now $\varepsilon = 0$), shows that the quantizer μ built as above is optimal for ν, and the traffic plan $\mathbf{P}$ (also built as above) is optimal in $\mathbf{TP}(\mu, \nu)$. From (i) we deduce that the ν_i's are actually mutually singular and $(\nu_i)_{1\le i\le N}$ is an minimizer of (2.25), which in turn implies (iii). ☐

As direct corollary from Theorem 2.3 and Theorem 2.7, we obtain existence for the optimal partition problem.

Corollary 2.8. *For any finite positive measure $\nu \in \mathscr{M}(\mathbb{R}^d)$ and any $N \in \mathbb{N}^*$, the optimal partition problem (2.25) admits a solution.*

2.4 Asymptotic energy scaling and asymptotic constant

We start by proving a general upper bound on the optimal quantization error by N points.

Lemma 2.9. *Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a finite measure and $\alpha \in \left(1 - \frac{1}{d}, 1\right]$. If ν is supported on a cube Q of edge length r, it holds:*

$$\mathcal{E}^\alpha(\nu, N) \leq C(\alpha, d) N^{-\beta/d} r \|\nu\|^\alpha,$$

recalling that $C(\alpha, d) = 2\mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, 1)$

Proof. Suppose that $Q := [-1/2, 1/2]^d$ and take $n \in \mathbb{N}^*$ such that $n^d \leq N < (n+1)^d$. Divide the cube Q into n^d subcubes $\{Q_i\}_{1 \leq i \leq n^d}$ of edge length $\frac{1}{n}$. We have

$$\begin{aligned}
\mathcal{E}^\alpha(\nu, N) \leq \mathcal{E}^\alpha(\nu, n^d) \leq \sum_{i=1}^{n^d} \mathcal{E}^\alpha(\nu \llcorner Q_i, 1) &\leq \sum_{i=1}^{n^d} \frac{C(\alpha, d)}{2} n^{-1} \nu(Q_i)^\alpha \\
&\leq \frac{C(\alpha, d)}{2} n^{-1} n^d \left(\frac{\|\nu\|}{n^d}\right)^\alpha = \frac{C(\alpha, d)}{2} n^{-\beta} \|\nu\|^\alpha \leq C(\alpha, d) N^{-\beta/d} \|\nu\|^\alpha.
\end{aligned}$$

□

We show that the optimal N-point quantization error of the unit cube behaves as some negative power of N times a nontrivial constant $c_{\alpha,d}$, when the Lebesgue measure is α-irrigable.

Proposition 2.10. *If $\alpha \in (1 - \frac{1}{d}, 1]$, then there exists a constant $c_{\alpha,d} \in (0, +\infty)$ such that*

$$\lim_{N \to +\infty} N^{\beta/d} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) = c_{\alpha,d} \quad \textit{where} \quad \beta := 1 + d\alpha - d,$$

and $\mathcal{E}^\alpha$ is the optimal branched quantization error defined in (2.17).

The proof is based on a classical result of subadditive processes in ergodic theory (see e.g. [LM02]).

Proof. Define for every borel set $A \subseteq \mathbb{R}^d$ and $N \in \mathbb{N}$,

$$\mathcal{S}^\alpha(A) = \mathcal{E}^\alpha(\mathscr{L}^d \llcorner A, \lfloor \mathscr{L}^d(A) \rfloor). \tag{2.29}$$

Notice that for every $N \in \mathbb{N}^*$, by 1-homogeneity in space and α-homogeneity in mass of the α-mass, we have

$$\begin{aligned}
N^{\beta/d} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) &= \frac{1}{N} N^{1/d + \alpha} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) \\
&= \frac{1}{N} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0, N^{1/d}]^d, N) = \frac{\mathcal{S}^\alpha(Q_N)}{N}
\end{aligned}$$

where $Q_N := [0, N^{1/d}]^d$ is a cube of volume N. By [LM02, Theorem 2.1], any nonnegative subadditive translation-invariant function $\mathcal{S}$ defined on bounded Borel subsets of $\mathbb{R}^d$ satisfies

$$\lim_{N \to +\infty} \frac{\mathcal{S}(Q_N)}{N} = \inf_{n \in \mathbb{N}^*} \frac{\mathcal{S}([0,n)^d)}{n^d},$$

hence it suffices to show that $\mathcal{S}^\alpha$ is subadditive, the invariance by translation being trivial. Subadditivity is a direct consequence of the subadditivity of $\mathbb{M}^\alpha$ and the superadditivity of the integer part. Indeed, take A_1, A_2 two disjoint bounded Borel subsets of $\mathbb{R}^d$, then for any $i \in \{1,2\}$ an optimal quantizer μ_i of $\mathscr{L}^d \llcorner A_i$ with at most $\lfloor \mathscr{L}^d(A_i) \rfloor$ atoms, and an optimal traffic plan $\mathbf{P}_i \in \mathbf{TP}(\mu_i, \mathscr{L}^d \llcorner A_i)$. Since $\mathbf{P}_1 + \mathbf{P}_2 \in \mathbf{TP}(\mu_1 + \mu_2, \mathscr{L}^d \llcorner (A_1 \sqcup A_2))$ and the number N of atoms of $\mu_1 + \mu_2$ satisfies

$$N \leq \lfloor \mathscr{L}^d(A_1) \rfloor + \lfloor \mathscr{L}^d(A_1) \rfloor \leq \lfloor \mathscr{L}^d(A_1 \sqcup A_2) \rfloor,$$

we obtain

$$\begin{aligned} \mathcal{S}^\alpha(A_1 \sqcup A_2) &= \mathcal{E}^\alpha(\mathscr{L}^d \llcorner (A_1 \sqcup A_2), \lfloor \mathscr{L}^d(A_1 \sqcup A_2) \rfloor) \\ &\leq \mathbf{d}^\alpha(\mu_1 + \mu_2, \mathscr{L}^d \llcorner (A_1 \sqcup A_2)) \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_1 + \mathbf{P}_2) \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2) = \mathcal{S}^\alpha(A_1) + \mathcal{S}^\alpha(A_2). \end{aligned}$$

We have thus proven the existence of the constant $c_{\alpha,d} \in [0, +\infty]$ of the statement. It is finite because by Lemma 2.9, with constant $C(\alpha, d)$ as in that lemma,

$$c_{\alpha,d} = \inf_{n \in \mathbb{N}^*} \frac{\mathcal{S}^\alpha([0,n)^d)}{n^d} \leq C(\alpha, d) < +\infty.$$

We next show that $c_{\alpha,d}$ is strictly positive. By [PSX19, Theorem 2.1], the constant

$$e_{\alpha,d} \coloneqq \inf \left\{ \mathbf{d}^\alpha(\delta_0, \rho) : \rho \in \mathscr{P}(\mathbb{R}^d), \rho \leq \mathscr{L}^d \right\} \tag{2.30}$$

is a strictly positive real number. Let $\mu_N = \sum_{i=1}^N m_i \delta_{x_i}$ be an N-point optimal quantizer of $\mathscr{L}^d \llcorner [0,1]^d$ and $\mathbf{P} \in \mathbf{TP}(\mu_N, \mathscr{L}^d \llcorner [0,1]^d)$ an optimal traffic plan. Using Lemma 2.6, noticing that for every $i \in \{1, \ldots, N\}$, $\nu_i \coloneqq (e_\infty)_\sharp \mathbf{P}^{x_i} \leq \mathscr{L}^d$, and using again the homogeneity properties of the α-mass, we get:

$$\begin{aligned} \mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, N) &= \sum_{i=1}^N \mathbf{d}^\alpha(m_i \delta_{x_i}, \nu_i) \\ &\geq \sum_{i=1}^N m_i^{\alpha + \frac{1}{d}} e_{\alpha,d} \geq N(1/N)^{\alpha + \frac{1}{d}} e_{\alpha,d}, \end{aligned}$$

where the last inequality is due to the convexity of $m \mapsto m^{\alpha + \frac{1}{d}}$ (because $\alpha + \frac{1}{d} > 1$). This implies that $c_{\alpha,d} \geq e_{\alpha,d} > 0$ and concludes the proof.

□

3 Γ-convergence and Zador-type Theorem

We are now going to provide an equivalent for the optimal quantization error of a compactly supported finite measure $\nu \ll \mathscr{L}^d$ as the number of points goes to infinity, analogous to the classical Zador's Theorem (see [GL00, Theorem 6.2], or the original papers [Zad64; Zad82; BW82]), which states in particular that

$$\mathcal{E}^{W_2^2}(\nu) N^{-\frac{2}{d}} \xrightarrow{N \to +\infty} \mathcal{E}^{W_2^2}(\mathscr{L}^d \llcorner [0,1]^d) \|\nu\|_{\frac{d}{d+2}},$$

where $\mathcal{E}^{W_2^2}(\nu) \coloneqq \inf\{W_2(\mu_N, \nu)^2 : \mu_N \in \mathscr{M}_N\}$. We shall also be interested in the limit distribution of centers of N-point optimal quantizers μ_N, i.e. to the weak limit of

$$\mu_N^\diamond \coloneqq \frac{1}{\# \operatorname{spt} \mu} \sum_{\{x : \mu_N(\{x\}) > 0\}} \delta_x.$$

We tackle the two questions simultaneously by establishing a (stronger) Γ-convergence result, inspired from of [BJM02; BJM11].

3.1 A Γ-convergence result

We establish a Γ-convergence result in the spirit of [BJM02], in a form that is slighly more concise. We do not follow the extended approach of [BJM11], where the functionals depend on the quantizers μ_N and also on an extra variable that encodes the distributions of masses (as measures over $\mathbb{R}_+$), since the Γ-limit does not have a fully explicit expression in this case, and we are not able to derive useful informations from it. Instead, the functionals $\mathcal{F}_N$ that we consider will depend solely on sets Σ of N points, embedded in the space of probability measures through their empirical measures $\frac{1}{N}\sum_{s\in\Sigma}\delta_s$, leading to the definition

$$\mathscr{X}_N \coloneqq \left\{\frac{1}{N}\sum_{s\in\Sigma}\delta_s : \#S = N\right\} \qquad (\forall N\in\mathbb{N}^*).$$

We fix a nontrivial closed cube $K \subseteq \mathbb{R}^d$ and a measure $\nu \in \mathscr{M}^+(K)$ such that $\nu \ll \mathscr{L}^d$. We consider the sequence of functionals $\mathcal{F}_N : \mathscr{P}(K) \to [0,+\infty]$ defined for every $N \in \mathbb{N}^*$ by

$$\mathcal{F}_{N,\nu}(\rho) = \begin{cases} N^{\beta/d}\inf\{\mathbf{d}^\alpha(\mu,\nu) : \operatorname{spt}\mu \subseteq \operatorname{spt}\rho\} & \text{if } \rho \in \mathscr{X}_N, \\ +\infty & \text{otherwise.}\end{cases} \tag{3.1}$$

Determining the Γ-limit of the sequence $(\mathcal{F}_N)_{N\geq 1}$ amounts to seeking out the least (asymptotic) energy to approximate, in the sense of branched optimal transport, the measure ν by N-point quantizers μ_N while prescribing the limit density of the centers $(\mu_N^\diamond)_{N\geq 1}$, which will correspond to the ρ variable. We shall prove that the Γ-limit is the functional $\mathcal{F}_\infty : \mathscr{P}(K) \to [0,+\infty]$ defined by

$$\mathcal{F}_\infty(\rho) = c_{\alpha,d}\int_{\mathbb{R}^d} \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\frac{\beta}{d}}}\,\mathrm{d}x, \tag{3.2}$$

where we recall $\beta = 1 + d\alpha - d$

Theorem 3.1. *Let $\nu \in \mathscr{M}^+(Q)$ where $Q \subseteq \mathbb{R}^d$ is a nontrivial closed cube. If $\nu \ll \mathscr{L}^d$ and $\alpha > 1 - \frac{1}{d}$, the sequence of functionals $(\mathcal{F}_N)_{N\geq 1}$ Γ-converges to $\mathcal{F}_\infty$ as $N \to \infty$ with respect to weak convergence of measures.*

We are going to use the following lemmas.

Lemma 3.2. *Let $\nu \in \mathscr{M}_c^+(\mathbb{R}^d)$ be a measure supported on a cube Q and $\alpha \in (1-\frac{1}{d}, 1)$. It holds*

$$\lim_{\delta\to 0}\liminf_{N\to+\infty} N^{\beta/d}\inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} = \liminf_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu,N).$$

Proof. Suppose that Q is a cube of edge length $r > 0$. First of all, it is clear that

$$\inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} \leq \mathcal{E}^\alpha(\nu,N).$$

for every $\delta \in (0,\delta_0)$. Now, let us take a small $\lambda > 0$. For every N large enough, by Lemma 2.9 and subadditivity of the α-mass, we have for every $\nu' \leq \nu$

$$\begin{aligned}\mathcal{E}^\alpha(\nu, N + \lceil\lambda N\rceil) &\leq \mathcal{E}^\alpha(\nu',N) + \mathcal{E}^\alpha(\nu-\nu',\lceil\lambda N\rceil) \\ &\leq \mathcal{E}^\alpha(\nu',N) + C(\alpha,d)N^{-\beta/d}r\|\nu-\nu'\|^\alpha\lambda^{-\beta/d},\end{aligned}$$

hence for every $\delta > 0$

$$\begin{aligned}&\liminf_{N\to+\infty} N^{\beta/d}\inf\{\mathcal{E}^\alpha(\nu',N) : \nu' \leq \nu, \|\nu-\nu'\| \leq \delta\} \\ \geq &(1+\lambda)^{-\beta/d}\liminf_{N\to+\infty} N^{\beta/d}\mathcal{E}^\alpha(\nu,N) - C(\alpha,d)r\delta^\alpha\lambda^{-\beta/d}.\end{aligned}$$

Taking the liminf as $\delta \to 0$ and then $\lambda \to 0$ yields the result. ☐

Lemma 3.3. *Let $\nu \in \mathscr{M}^+(K)$ be an absolutely continuous measure over a cube K of edge length R and $(A_i)_{1\le i\le I}$ be a $\mathscr{L}^d$-essential partition of K with $\operatorname{diam}(A_i) \le r$ for $1 \le i \le I$. We set*

$$\nu' := \sum_{i=1}^{I} \nu_i \mathscr{L}^d \llcorner A_i$$

where $\nu_i := \nu(A_i)/\mathscr{L}^d(A_i)$ if $\nu(A_i) > 0$ and $\nu_i := 0$ otherwise. Then there is a constant c depending only on α and d such that

$$\mathbf{d}^\alpha(\nu,\nu') \le c R^{1-\beta} r^\beta \|\nu' - \nu\|^\alpha.$$

Proof. We know from [MS07, Proposition 0.1] (also [BCM09a, Proposition 6.16]) that

$$\mathbf{d}^\alpha(\mu^-,\mu^+) \le c W_1(\mu^-,\mu^+)^\beta$$

for every probability measures $\mu^\pm \in \mathscr{P}(Q_1)$ and some constant $c = c(\alpha,d)$. Applying it to $\mu^- = \nu - \nu\wedge\nu'$ and $\mu^+ = \nu' - \nu\wedge\nu'$ with appropriate rescalings in mass and distance, we obtain

$$\mathbf{d}^\alpha(\nu,\nu') \le \mathbf{d}^\alpha(\mu^-,\mu^+) \le c W_1(\mu^-,\mu^+)^\beta \|\mu^-\|^{\alpha-\beta} R^{1-\beta} \le c W_\infty(\mu^-,\mu^+)^\beta \|\mu^-\|^\alpha R^{1-\beta}.$$

By construction, ν and ν' have same mass on each A_i, thus the same goes for μ^- and μ^+, and since $\operatorname{diam}(A_i) \le r$ it implies that $W_\infty(\mu^-,\mu^+) \le r$. Since $\|\nu - \nu'\| = 2\|\mu^-\|$, we obtain the desired result. □

Proof of Theorem 3.1. We are going to prove successively the $\Gamma - \liminf$ and $\Gamma - \limsup$ inequality, i.e.

$$\forall \rho \in \mathscr{P}(K), \forall (\rho_N)_{N\in\mathbb{N}^*} \rightharpoonup \rho, \quad \liminf_N \mathcal{F}_N(\rho_N) \ge \mathcal{F}_\infty(\rho), \tag{3.3}$$

$$\forall \rho \in \mathscr{P}(K), \exists (\rho_N)_{N\in\mathbb{N}^*} \rightharpoonup \rho, \quad \limsup_N \mathcal{F}_N(\rho_N) \le \mathcal{F}_\infty(\rho). \tag{3.4}$$

Proof of the Γ-liminf inequality (3.3). Let us take a sequence of probability measures $\rho_N \rightharpoonup \rho$, assuming without loss of generality that $\liminf_N \mathcal{F}_N(\rho_N) < +\infty$. Up to taking a subsequence, we may assume that $\mathcal{F}_N(\rho_N)$ converges to $\liminf_N \mathcal{F}_N(\rho_N)$ and

$$C := \sup_N \mathcal{F}_N(\rho_N) < +\infty.$$

In particular we know that for every $N \in \mathbb{N}^*$, $\rho_N = \frac{1}{N}\sum_{s\in\Sigma_N}\delta_s$ for some set Σ_N of cardinal N, and we take a mass-optimal quantizer μ_N of ν with respect to Σ_N, as well as an optimal traffic plan $\mathbf{P}_N \in \mathbf{TP}(\mu_N,\nu)$, so that

$$\mathcal{F}_N(\rho_N) = N^{\beta/d}\mathbf{d}^\alpha(\Sigma_N,\nu) = N^{\beta/d}\mathbf{d}^\alpha(\mu_N,\nu) = N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N) = N^{\beta/d}\int_{\Gamma^d} Z_{\mathbf{P}_N}\,\mathrm{d}\mathbf{P}_N(\gamma).$$

A standard strategy to show (3.3) is to express this energy as the total mass of some measure e_N, which converges up to subsequence to some measure e, then show a lower bound on e and use the lower semicontinuity of the total mass. In our branched optimal transport setting, in order to follow this strategy we will have to resort to outer measures rather than measures. More precisely, we shall bound from below the energy $\mathcal{F}_N(\rho_N)$ by the total mass $E'_N(K)$ of some suitable outer measure E'_N, that in some sense becomes becomes a measure asymptotically as $N \to +\infty$, in some sense.

Notice that

$$CN^{-\beta/d} \ge \int_\Gamma Z_{\mathbf{P}_N}(\gamma)\,\mathrm{d}\mathbf{P}_N(\gamma) \ge \int_\Gamma L(\gamma)\,\mathrm{d}\mathbf{P}_N(\gamma),$$

so that by Markov's inequality

$$\mathbf{P}_N(\{\gamma : L(\gamma) \ge MN^{-\beta/d}\}) \le \frac{C}{M}. \tag{3.5}$$

Consider an increasing sequence M_N tending to $+\infty$ and such that $M_N N^{-\beta/d} \to 0$, and set

$$\mathbf{P}'_N := \mathbf{P}_N \llcorner \Gamma_N \quad \text{where} \quad \Gamma_N := \{\gamma : \ L(\gamma) < M_N N^{-\beta/d}\}. \tag{3.6}$$

We define for every Borel set $A \subseteq \mathbb{R}^d$:

$$E_N(A) := N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N \llcorner e_\infty^{-1}(A)), \qquad E'_N(A) := N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A)). \tag{3.7}$$

We remark that

$$E_N(K) = \mathcal{F}_N(\rho_N)$$

and E'_N (and also E_N) is an outer measure (being countably subadditive) and not a priori a measure: it is posssible that for two disjoint Borel sets A_1, A_2, the plans $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)$ and $\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2)$ are not disjoint. However, E'_N becomes additive when $\operatorname{dist}(A_1, A_2) > 0$ and N becomes large enough. Indeed, if $M_N N^{-\beta/d} \leq \frac{1}{2}\operatorname{dist}(A_1, A_2)$, which is the case for N large enough, then for every curve $\gamma_i \in \Gamma_N \cap e_\infty^{-1}(A_i)$, $i \in \{1, 2\}$,

$$\gamma_1(\mathbb{R}) \cap \gamma_2(\mathbb{R}) = \emptyset,$$

which in turn implies that $\Theta_{\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)}\mathscr{H}^1$ and $\Theta_{\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2)}\mathscr{H}^1$ are mutually singular, and thus

$$\mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1 \cup A_2)) = \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_1)) + \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(A_2))$$

i.e.

$$E'_N(A_1 \cup A_2) = E'_N(A_1) + E'_N(A_2). \tag{3.8}$$

Notice that this additivity property does not hold a priori for E_N, which was the point for restricting the measure and use E'_N instead.

We know that ν-a.e. point $x \in \operatorname{spt}\nu$ satisfies

$$\fint_{Q_r(x)} |\nu - \nu(x)| \,\mathrm{d}x \to 0 \quad \text{and} \quad \nu(x) \in (0, +\infty). \tag{3.9}$$

Fix such a point x, take $\delta > 0$ such that $\rho(\partial Q_\delta(x)) = 0$ (this is true for all but countably many δ's), where $Q_\delta(x)$ denotes the closed cube $x + \delta[-1/2, 1/2]^d$, and consider the slightly smaller $\delta' = \tau\delta$ for $\tau \in (0, 1)$ (which we will send to 1 later). We denote for every $N \in \mathbb{N}^*$

$$n_{N,\delta} := \#(\Sigma_N \cap Q_\delta(x)), \qquad \nu_N := (e_\infty)_\#\mathbf{P}'_N,$$

and we define the δ'-rescalings around x

$$\nu_{\delta'} := \frac{1}{\delta'^d}\left(y \mapsto \frac{y - x}{\delta'}\right)_\sharp \left(1 \wedge \frac{\nu'}{\nu(x)}\mathscr{L}^d \llcorner Q_{\delta'(x)}\right),$$
$$\nu_{N,\delta'} := \frac{1}{\delta'^d}\left(y \mapsto \frac{y - x}{\delta'}\right)_\sharp \left(1 \wedge \frac{\nu_N}{\nu(x)}\mathscr{L}^d \llcorner Q_{\delta'(x)}\right).$$

For N large enough, $Q_{\delta'}(x) + B(0, M_N N^{-\beta/d}) \subseteq Q_\delta(x)$ hence we have the lower bounds:

$$\begin{aligned}
N^{-\beta/d}E'_N(Q_{\delta'}(x)) &= \mathbb{M}^\alpha(\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))) \\
&\geq \mathbf{d}^\alpha((e_0)_\#(\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))), \nu_N \llcorner Q_{\delta'}(x)) && (3.10)\\
&\geq \mathcal{E}^\alpha(\nu_N \llcorner Q_{\delta'}(x), n_N) && (3.11)\\
&\geq \mathcal{E}^\alpha(\nu_N \wedge \nu(x)\mathscr{L}^d \llcorner Q_{\delta'}(x), n_N) && (3.12)\\
&= \nu(x)^\alpha \delta'^{1+d\alpha}\mathcal{E}^\alpha(\nu_{N,\delta'}, n_N).
\end{aligned}$$

where (3.10) follows from the definition of $\mathbf{d}^\alpha$, (3.11) and (3.12) from the facts that the source measure of $\mathbf{P}'_N \llcorner e_\infty^{-1}(Q_{\delta'}(x))$ is a submeasure of $\mu_N \llcorner Q_\delta(x)$ (thus has at most n_N atoms) and that $\mathcal{E}^\alpha(\nu, n)$ is decreasing in n and increasing in ν.

By (3.5) and (3.6), $(\nu_{N,\delta'})_{N\in\mathbb{N}^*}$ is a submeasure of $\nu_{\delta'}$ such that $\|\nu_{\delta'} - \nu_{N,\delta'}\| \le C/M_N \xrightarrow{N\to+\infty} 0$, thus multipliying (3.12) by $N^{\beta/d}$, passing to the liminf in N and using Lemma 3.2 yields:

$$\liminf_N E'_N(Q_\delta(x)) \ge \nu(x)^\alpha(\delta\tau)^{1+d\alpha} \liminf_{N\to+\infty} \left(\frac{N}{n_N}\right)^{\beta/d} \mathcal{E}^\alpha(\nu_{N,\delta'}, n_N) n_N^{\beta/d} \tag{3.13}$$

$$= \frac{(\delta\tau)^{1+d\alpha}\nu(x)^\alpha}{\rho(Q_\delta(x))^{\beta/d}} \liminf_{n\to+\infty} n^{\beta/d}\mathcal{E}^\alpha(\nu_{\delta'}, n) \tag{3.14}$$

because

$$\frac{n_N}{N} = \rho_N(Q_\delta) \to \rho(Q_\delta(x)),$$

since $\rho_N \rightharpoonup \nu$ and $\rho(\partial Q_\delta(x)) = 0$. Notice that $\nu_{\delta'} \le \mathscr{L}^d \llcorner Q_1$ and by (3.9) that $\|\nu_{\delta'}\| \to 1$ hence $\|\nu_{\delta'} - \mathscr{L}^d \llcorner Q_1\| \to 0$. We divide by δ^d, pass to the limsup as $\delta \to 0$, and then take $\tau \to 1$ recalling that $\delta' = \tau\delta$, and use Lemma 3.2 again, obtaining

$$\begin{aligned}\limsup_{\delta\to 0} \frac{\liminf_N E'_N(Q_\delta(x))}{\mathscr{L}^d(Q_\delta(x))} &\ge \nu(x)^\alpha \limsup_{\delta\to 0} \frac{\delta^\beta}{\rho(Q_\delta(x))^{\beta/d}} \liminf_n n^{\beta/d}\mathcal{E}^\alpha(\mathscr{L}^d \llcorner [0,1]^d, n) \\ &= \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}} c_{\alpha,d}\end{aligned} \tag{3.15}$$

for ν-a.e. x by Radon-Nikodym Theorem.

Now, we conclude by applying a covering argument. For fixed $\varepsilon \in (0,1)$ we consider the collection $\mathcal{Q}_\varepsilon$ of cubes $Q_\delta(x)$, $\delta > 0$, $x \in \mathbb{R}^d$ such that

(i) $Q_\delta(x) \in K + Q_1$,

(ii) $\frac{(\varepsilon^{-1}\wedge\rho(x))^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}}(x))^{\beta/d}} \ge \fint_{Q_\delta(x)} \frac{(\varepsilon^{-1}\wedge\nu)^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}})^{\beta/d}} - \varepsilon$,

(iii) $\frac{\liminf_N E_N(Q_\delta(x))}{\mathscr{L}^d(Q_\delta(x))} \ge c_{\alpha,d} \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}} - \varepsilon$.

The set of cubes $\mathcal{Q}_\epsilon$ form a fine cover of $\mathscr{L}^d$-a.e. $\tilde{K} := \{x \in K : \nu(x) > 0\}$ because of (3.15) and the fact that for $\mathscr{L}^d$-a.e. x, since $\frac{(\mu\wedge\varepsilon^{-1})^\alpha}{(\varepsilon\vee\nu_{\mathrm{ac}})^{\beta/d}}$, by Lebesgue theorem we have

$$\lim_{\delta\to 0} \fint_{Q_\delta(x)} \frac{(\varepsilon^{-1}\wedge\nu)^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}})^{\beta/d}} = \frac{(\varepsilon^{-1}\wedge\nu(x))^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}}(x))^{\beta/d}}.$$

Then, by Vitali-Besicovitch covering theorem, there exists a countable family of disjoint cubes $(Q_{\delta_i}(x_i))_{i<I} \subseteq \mathcal{Q}_\varepsilon$, $I \in \mathbb{N}\cup\{+\infty\}$, that cover $\tilde{K}$ up to a $\mathscr{L}^d$-negligible set. Using above properties (i)-(iii) of the collection $\mathcal{Q}_\epsilon$, we get that for every $J < I$,

$$\liminf_N E'_N(K) \ge \liminf_N E'_N\Big(\bigcup_{i\le J} Q_{\delta_i}(x_i)\Big) \tag{3.16}$$

$$\overset{(3.8)}{=} \liminf_N \sum_{i\le J} E'_N(Q_{\delta_i}(x_i)) \tag{3.17}$$

$$\overset{\text{(iii)}}{\ge} \sum_{i\le J} c_{\alpha,d}\frac{\delta_i^d\nu(x_i)^\alpha}{\rho_{\mathrm{ac}}(x_i)^{\beta/d}} - \varepsilon\delta_i^d \tag{3.18}$$

$$\overset{\text{(i) and (ii)}}{\ge} c_{\alpha,d}\sum_{i\le J}\int_{Q_{\delta_i}(x_i)} \frac{(\varepsilon^{-1}\wedge\nu)^\alpha}{(\varepsilon\vee\rho_{\mathrm{ac}})^{\beta/d}} - 2\varepsilon\mathscr{L}^d(K+Q_1). \tag{3.19}$$

Taking $J \to I$, then $\varepsilon \to 0$, by the monotone convergence theorem we get:

$$\liminf_N \mathcal{F}_N(\rho_N) \ge \liminf_N E'_N(K) \ge c_{\alpha,d}\int_K \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}}\,\mathrm{d}x = \mathcal{E}_\infty(\rho).$$

Proof of the Γ-limsup inequality (3.4). Let us remark that the subsets

$$\mathcal{A} := \mathscr{P}(K) \cap L^1(K), \quad \text{and} \quad \mathcal{A}' := \{\rho \in \mathcal{A} : \operatorname{ess\,inf}_K \rho > 0\}$$

are dense in the $\mathcal{F}_\infty$ for the weak convergence of measures. Let us first show it for $\mathcal{A}$. Let $\rho \in \mathscr{P}(K)$ and assume that it decomposes as $\rho = \rho_{\mathrm{ac}}\mathscr{L}^d + \rho_{\mathrm{s}}$ where $\rho_{\mathrm{s}} \perp \mathscr{L}^d$, and $\rho_{\mathrm{s}} \neq 0$ (otherwise there is nothing to prove). We know that there exists $\rho_{\varepsilon,\mathrm{s}}$ for $\varepsilon \in (0,1)$ which are absolutely continuous with respect to $\mathscr{L}^d \llcorner (\Omega_\varepsilon \cap K)$, where Ω_ε are nondecreasing subsets of $\mathbb{R}^d$ such that $\mathscr{L}^d(\Omega_\varepsilon) \leq \varepsilon$, and such that $\rho_{\varepsilon,\mathrm{s}} \xrightarrow[\varepsilon\to 0]{} \rho_{\mathrm{s}}$, and $\|\rho_{\varepsilon,\mathrm{s}}\| = \|\rho_{\mathrm{s}}\|$. We set

$$\rho_\varepsilon := \rho_{\mathrm{ac}} + \rho_{\varepsilon,\mathrm{s}},$$

notice that $\rho_\varepsilon \geq \rho_{\mathrm{ac}}$ so that

$$\mathcal{F}_\infty(\rho) \geq \mathcal{F}_\infty(\rho_\varepsilon) \geq c_{\alpha,d} \int_{K\setminus\Omega_\varepsilon} \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\beta/d}}\,\mathrm{d}x,$$

and by the Monotone Convergence theorem we get $\mathcal{F}_\infty(\rho_\varepsilon) \xrightarrow{\varepsilon\to 0} \mathcal{F}_\infty(\rho)$. Now let us take $\rho \in \mathcal{A}$, and set for every $\varepsilon > 0$

$$\rho_\varepsilon := \frac{\rho \vee \varepsilon}{\|\rho \vee \varepsilon\|}.$$

It is clear that $\|\rho_\varepsilon\| = 1$ and $\rho_\varepsilon \rightharpoonup \rho$. Besides by monotone convergence,

$$\mathcal{F}_\infty(\rho_\varepsilon) = c_{\alpha,d}\|\rho_\varepsilon\|^{\beta/d} \int_K \frac{\nu(x)^\alpha}{(\rho\vee\varepsilon)^{\beta/d}}\,\mathrm{d}x \to c_{\alpha,d}\int_K \frac{\nu(x)^\alpha}{\rho^{\beta/d}}\,\mathrm{d}x = \mathcal{F}_\infty(\rho).$$

As a consequence, it suffices to find a recovery sequence for any given $\rho \in \mathcal{A}'$. Several steps are standard, thus some of the constructions will be only sketched.

Step 1. *(Building approximation sequences.)* Partition $\mathbb{R}^d$ in cubes $Q_{N,i}$ of edge $\lambda N^{-1/d}$ where $\lambda \geq 1$ is taken large (and will be sent to $+\infty$ later), and define piecewise constant approximations of ν:

$$\nu_N := \sum_i \nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i} \quad \text{where} \quad \nu_{N,i} := \frac{\nu(Q_{N,i})}{\mathscr{L}^d(Q_{N,i})} \quad (\forall i).$$

Notice that $\nu_N \to \nu$ in $L^1(K)$. Let us build suitable N-point approximations of ρ by putting the appropriate number of points $n_{N,i}$ in each cube $Q_{N,i}$. The number $n_{N,i}$ should be approximately given by

$$N\rho(Q_{N,i}) = N(\lambda N^{-1/d})^d \rho_{N,i} = \lambda^d \rho_{N,i} \quad \text{where} \quad \rho_{N,i} := \frac{\rho(Q_{N,i})}{\mathscr{L}^d(Q_{N,i})} \quad (\forall i).$$

Notice that if we take λ large enough, $\lambda^d \rho_{N,i} \geq \lambda^d \kappa \geq 1$ where $\kappa := \operatorname{ess\,inf}_K \rho$, and

$$\sum_i \lfloor \lambda^d \rho_{N,i} \rfloor \leq N = \sum_i \lambda^d \rho_{N,i} \leq \sum_i \lceil \lambda^d \rho_{N,i} \rceil,$$

thus we may choose for every i an integer $n_{N,i} \geq 1$ such that

$$n_{N,i} \in \{\lfloor \lambda^d \rho_{N,i} \rfloor, \lceil \lambda^d \rho_{N,i} \rceil\}$$

and

$$\sum_i n_{N,i} = N.$$

We take $\Sigma_{N,i}$ included in the interior of $Q_{N,i}$ as the support of a $n_{N,i}$-point quantizer of $\mathscr{L}^d \llcorner Q_{N,i}$ such that

$$\mathcal{E}^\alpha(\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \leq \mathbf{d}^\alpha(\Sigma_{N,i}, \mathscr{L}^d \llcorner Q_{N,i}) \leq \mathcal{E}^\alpha(\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) + \frac{1}{N^{1+1/d}}, \tag{3.20}$$

and we eventually define

$$\Sigma_N \coloneqq \bigsqcup_i \Sigma_{N,i} \quad \text{and} \quad \rho_N \coloneqq \frac{1}{N} \sum_{s \in \Sigma_N} \delta_s.$$

We know that $\rho_N \rightharpoonup \rho$ because

$$\sup_i |\rho_N(Q_{N,i}) - \rho(Q_{N_i})| \leq \frac{1}{N} \sup_i |n_{N,i} - \lambda^d \rho_{N,i}| \leq \frac{1}{N} \to 0.$$

By the triangle inequality and (3.20) we find the following:

$$\begin{aligned}
\mathcal{F}_N(\rho_N) = N^{\beta/d}\mathbf{d}^\alpha(\Sigma_N, \nu) &\leq N^{\beta/d}\mathbf{d}^\alpha(\Sigma_N, \nu_N) + N^{\beta/d}\mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d}\mathbf{d}^\alpha\Big(\bigcup_i \Sigma_{N,i}, \sum_i \nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}\Big) + N^{\beta/d}\mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d}\sum_i \Big(\mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) + \nu_{N,i}^\alpha N^{-(1+1/d)}\Big) \\
&\qquad\qquad\qquad\qquad\qquad + N^{\beta/d}\mathbf{d}^\alpha(\nu_N, \nu) \\
&\leq N^{\beta/d}\sum_i \mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \qquad (3.21) \\
&\quad + N^{\beta/d}\mathbf{d}^\alpha(\nu_N, \nu) + \frac{1}{N}\|\nu\|^\alpha \qquad (3.22)
\end{aligned}$$

(3.23)

Step 2. *(Bounding (3.21).)* We have

$$\mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) = \nu_{N,i}^\alpha (N^{-1/d}\lambda)^{1+d\alpha}\mathcal{E}^\alpha(Q_1, n_{N,i}),$$

and therefore, if we set $\tilde{\rho}_N \coloneqq \sum_i \rho_{N,i}\mathbf{1}_{Q_{N,i}}$,

$$\begin{aligned}
&N^{\beta/d}\sum_i \mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \\
&\leq \sum_i N^{-1}\rho_{N,i}^\alpha \lambda^{1+d\alpha}\mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \rho_{N,i} \rfloor) \\
&= \sum_i \int_{Q_{N,i}} \nu_N(x)^\alpha \mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde{\rho}_N(x) \rfloor)\lambda^\beta \, \mathrm{d}x \\
&\leq \int_K \frac{\nu_N(x)^\alpha}{\tilde{\rho}_N(x)^{\beta/d}} \mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde{\rho}_N(x) \rfloor)(\lambda^d \tilde{\rho}_N(x))^{\beta/d} \, \mathrm{d}x.
\end{aligned}$$

Now note that $\tilde{\rho}_N \geq \kappa > 0$ a.e. in K, so that

$$\mathcal{E}^\alpha(Q_1, \lfloor \lambda^d \tilde{\rho}_N(x) \rfloor)(\lambda^d \tilde{\rho}_N(x))^{\beta/d} \leq \sup_{n \geq \lfloor \lambda^d \kappa \rfloor} \mathcal{E}^\alpha(Q_1, n)(n+1)^{\beta/d} \leq c_{\alpha,d}(1 + \varepsilon(\lambda)),$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to +\infty$ by Proposition 2.10. Besides, (ν_N) converges to ν in L^1 thus (ν_N^α) converges to ν^α in $L^{1/\alpha}$ thus in L^1, and $(\tilde{\rho}_N)$ converges a.e. to ρ. Therefore by the Dominated Convergence Theorem, taking the limit $N \to +\infty$ then $\lambda \to +\infty$ yields

$$\limsup_{N \to +\infty} N^{\beta/d}\sum_i \mathcal{E}^\alpha(\nu_{N,i}\mathscr{L}^d \llcorner Q_{N,i}, n_{N,i}) \leq c_{\alpha,d}\int_K \frac{\nu(x)^\alpha}{\rho(x)^{\beta/d}} \, \mathrm{d}x.$$

Step 3. *(Bounding (3.22) and conclusion.)* We apply Lemma 3.3 to the measures ν and $\nu' = \nu_N$:

$$\mathbf{d}^\alpha(\nu, \nu_N) \leq cR^{1-\beta}(\lambda N^{-1/d})^\beta \|\nu - \nu_N\|^\alpha,$$

so that

$$N^{\beta/d}\mathbf{d}^\alpha(\nu, \nu_N) \leq cR^{1-\beta}\lambda^\beta \|\nu - \nu_N\|^\alpha.$$

Taking the limit $N \to +\infty$, since $\nu_N \to \nu$ in L^1, we get:

$$\limsup_{N\to+\infty} N^{\beta/d}\mathbf{d}^\alpha(\nu, \nu_N) = 0. \tag{3.24}$$

By Step 2. and (3.24) we thus have

$$\limsup_{N\to+\infty} \mathcal{F}_N(\rho_N) \leq c_{\alpha,d} \int_\Omega \frac{\nu(x)^\alpha}{\rho(x)^{\beta/d}} \,\mathrm{d}x = \mathcal{F}_\infty(\rho),$$

as desired.

□

Remark 3.4. There are alternative approaches for the Γ-liminf part of the proof if we assume that the measure ν is d-Ahflors regular, since we may use the Hölder regularity of the landscape function and its consequences (in particular the bound on the diameter of basins in terms of their masses) that are established in Section 4. Indeed, we may use directly the outer measures E_N defined for every Borel set A by

$$E_N(A) \coloneqq N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N \llcorner e_\infty^{-1}(A)),$$

rather than the restrictions

$$e_N : A \mapsto N^{\beta/d}\mathbb{M}^\alpha(\mathbf{P}_N \llcorner e_\infty^{-1}(A) \cap \{L < M_N N^{-\beta/d}\}),$$

or even use the measures defined by

$$e'_N(A) \coloneqq \int_A z_{\mathbf{P}_N} \,\mathrm{d}\mu.$$

The relevance of restricting the plans (and thus of passing from E_N to e_N), is that we can then guarantee that e_N satisfy additivity for sets at positive distance and N large enough. But one may check that this property holds directly for E_N thanks to Corollary 4.3 and Lemma 4.4. It is even easier with e'_N which is by definition a measure, although in this case we need to adapt the series of inequalities (3.10)-(3.12) which give the lower bound.

Also note that similar considerations using Hölder regularity of the landscape function under Ahlfors-regularity hypotheses may also apply to Proposition 3.6 to replace the outer measure E'_N by E_N or e_N in the statement on the equi-distribution of energy at the macroscopic scale.

3.2 Asymptotics of the quantization error and support of optimal quantizers

From the Γ-convergence established in the previous subsection, we may obtain the asymptotics of the optimal quantization error (a branched optimal transport variant of Zador's theorem) and the limit density of the centers of optimal quantizers.

Theorem 3.5 (Extended version of Theorem 1.1)**.** *Let $\nu \in \mathscr{M}^+(K)$ be an absolutely continuous measure on a cube $K \subseteq \mathbb{R}^d$. Then:*

(A) if $(\mu_N)_{N\in\mathbb{N}}$ is a sequence of optimal N-point quantizers of ν,

$$\mu_N^\diamond \doteq \frac{1}{N} \sum_{\{x:\mu_N(\{x\})>0\}} \delta_x \overset{\star}{\rightharpoonup} C_{\alpha,d}(\nu)^{-1}\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}},$$

where $C_{\alpha,d}(\nu) \coloneqq \int_K \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}} \,\mathrm{d}x$;

(B) the leading-order asymptotics of the optimal quantization error is given by

$$\lim_{N\to\infty} N^{\beta/d}\mathcal{E}^{\alpha}(\nu,N) = c_{\alpha,d}C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}. \tag{3.25}$$

Proof of Theorem 3.5. Take for every $N\in\mathbb{N}^*$ an N-point optimal quantizer μ_N of ν. The sequence of probability measures $(\mu_N^\diamond)$ converges up to subsequence to a measure ρ. Since for every N,

$$\mathcal{F}_N(\mu_N^\diamond) = \inf \mathcal{F}_N,$$

by Theorem 3.1 the measure ρ minimizes the Γ-limit $\mathcal{F}_\infty$ and

$$\lim_{N\to+\infty} N^{\beta/d}\mathcal{E}^{\alpha}(\nu,N) = \lim_{N\to+\infty}\mathcal{F}_N(\mu_N^\diamond) = \mathcal{F}_\infty(\rho) \doteq c_{\alpha,d}\int_K \frac{\nu(x)^\alpha}{\rho_{\mathrm{ac}}(x)^{\alpha+\frac{1}{d}-1}}\,\mathrm{d}x.$$

As a consequence of minimality, ρ is absolutely continuous with respect to $\mathscr{L}^d$ and the Euler-Lagrange equation can be written as:

$$\nu(x)^\alpha = (C\rho)^{\alpha+\frac{1}{d}}$$

for $\mathscr{L}^d$-a.e. $x\in\Omega$, for a constant C which is given by

$$C = C_{\alpha,d}(\nu) := \int_K \nu(x)^{\frac{\alpha}{\alpha+\frac{1}{d}}}\,\mathrm{d}x.$$

In particular $\rho = C_{\alpha,d}(\nu)^{-1}\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$ and

$$\lim_N N^{\beta/d}\mathcal{E}^{\alpha}(\nu,N) = c_{\alpha,d}C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} = \left(\int_K \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}\right)^{\alpha+\frac{1}{d}}. \qquad \square$$

3.3 Equidistribution results at the macroscopic scale

From the Γ-convergence result and its proof, we obtain convergence of measures (or outer measures) of interest to understand uniformizing features at the macroscopic scale.

Proposition 3.6. *Let (μ_N) be a sequence of N-point optimal quantizers of ν (i.e. solutions to (2.17)).*

(A) The empirical measures converge as follows:

$$\mu_N^\diamond \rightharpoonup C_{\alpha,d}(\nu)^{-1}\nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$$

In particular if $\nu = \mathscr{L}^d \llcorner K$ with $\mathscr{L}^d(K)=1$, we obtain

$$\frac{1}{N}\#(\operatorname{spt}\mu_N\cap B) \to \mathscr{L}^d(B),$$

for every Borel set $B\subseteq\Omega$ such that $\mathscr{L}^d(\partial B)=0$.

(B) The energy outer measures (E'_N) converge in the following sense:

$$\lim_{N\to+\infty} E'_N(B) = c_{\alpha,d}C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\int_B \nu^{\frac{\alpha}{\alpha+\frac{1}{d}}}$$

for every Borel set B such that $\mathscr{L}^d(\partial B)=0$. In particular if $\nu=\mathscr{L}^d\llcorner K$ then

$$\lim_N E'_N(B) = c_{\alpha,d}C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}}\mathscr{L}^d(B).$$

Proof. The first item (A) is a direct consequence of Theorem 3.5. For (B), we follow the proof of the $\Gamma - \liminf$ inequality in Theorem 3.1 and apply the covering argument to the subcollection $\mathcal{Q}'_\varepsilon \subseteq \mathcal{Q}_\varepsilon$ of cubes $Q_\delta(x)$ which are included in a given open subset Ω of K, thus building a disjoint collection of subcubes which cover a.e. $\Omega \cap \{\nu > 0\}$. We therefore get:

$$\liminf_N e_N(\Omega) \geq c_{\alpha,d} C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} \int_\Omega \mu^{\frac{\alpha}{\alpha+\frac{1}{d}}}.$$

If B is a Borel set with $\mathscr{L}^d(\partial B) = 0$, we may apply the above inequality to $B_{<\varepsilon} \coloneqq \{x : d(x, B^c) > \varepsilon\}$ and $B_{>\varepsilon} = \{x : d(x, B) > \varepsilon\}$, and using the asymptotic additivity of E_N to the sets B and $B_{>\varepsilon}$ which are at positive distance, we obtain

$$\begin{aligned}
c_{\alpha,d} C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} \int_{B_{<\varepsilon}} \mu^{\frac{\alpha}{\alpha+\frac{1}{d}}} &\leq \liminf_{N\to+\infty} E'_N(B_{<\varepsilon}) \leq \liminf_{N\to+\infty} E'_N(B) \leq \limsup_{N\to+\infty} E'_N(B) \\
&= \limsup_{N\to+\infty} E'_N(B \cup B_{>\varepsilon}) - E'_N(B_{>\varepsilon}) \\
&\leq \limsup_{N\to+\infty} E_N(K) - \liminf_{N\to+\infty} E'_N(B_{>\varepsilon}) \\
&\leq c_{\alpha,d} C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} \int_K \mu^{\frac{\alpha}{\alpha+\frac{1}{d}}} - c_{\alpha,d} C_\mu^{-(\alpha+\frac{1}{d})} \int_{B_{>\varepsilon}} \mu^{\frac{\alpha}{\alpha+\frac{1}{d}}} \\
&= c_{\alpha,d} C_{\alpha,d}(\nu)^{\alpha+\frac{1}{d}} \int_{K\setminus B_{>\varepsilon}} \mu^{\frac{\alpha}{\alpha+\frac{1}{d}}}.
\end{aligned}$$

Taking the limit $\varepsilon \to 0$, we get the desired result of (B). □

4 Landscape function for mass-optimal quantizers

This section is devoted to the landscape function, its definition and Hölder regularity. We stress that the classical definition of landscape function from [San07], recalled in Section 2.1, is only given in the case of a single source $\mu = m\delta_x$ and, as already said, an optimal traffic plan with several sources may in general not decompose disjointly according to its sources. This poses a serious issue to define and study the landscape function in such a case. An attempt at defining the landscape function for several sources (even in a more general setting) has been made in [Peg17a, Chapter 4], but the construction is quite technical and the Hölder constant computed there actually explodes when the number of sources tends to infinity. However, in the case of optimal quantizers or even mass-optimal quantizers, the disjointness result established in Lemma 2.6 allows us to give a simple ad hoc definition of landscape function, and we are able to show its Hölder regularity with a Hölder constant that is *uniform in the number of sources*, a crucial information to establish the uniform regularity properties in Section 5.

4.1 Uniform Hölder regularity

From now on and for the rest of the article, we assume that the exponent α is above the critical threshold in $\mathbb{R}^d$:

$$1 - \frac{1}{d} < \alpha \leq 1. \tag{4.1}$$

Theorem 4.1 (extended version of Theorem 1.3)**.** *Let $\nu \in \mathscr{M}_c(\mathbb{R}^d)$ be a compactly supported measure which is d-Ahlfors regular with constants $0 < c_A \leq C_A$, i.e.*

$$c_A r^d \leq \nu(B_r(x)) \leq C_A r^d \quad (\forall x \in \operatorname{spt}\nu, \forall r \leq \operatorname{diam}(\operatorname{spt}\nu)), \tag{4.2}$$

and let $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan where $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ is a N-point mass-optimal quantizer of ν with respect to $(x_i)_{1\leq i\leq N}$. There exists a unique function $z_{\mathbf{P}} : \operatorname{spt}\nu \to \mathbb{R}_+$ that we call landscape function associated with $\mathbf{P}$ *satisfying:*

(i) for every $i \in \{1,\dots,N\}$*,* $z_{\mathbf{P}^{x_i}} = z_{\mathbf{P}}$ *everywhere on* $\mathrm{Bas}(\mathbf{P}, x_i)$*;*

(ii) $z_{\mathbf{P}}$ *is* β*-Hölder continuous where* $\beta := 1 + d\alpha - d \in (0,1]$*, with a Hölder constant smaller than a constant* $C_H = C_H(c_A, C_A, \alpha, d)$*.*

Proof. Let us start by setting a candidate landscape function which is uniquely defined ν-almost everywhere on $\operatorname{spt}\nu$. The measures $\nu_i := (e_\infty)_\sharp \mathbf{P}^{x_i}$ are mutually singular and sum to ν thanks to Lemma 2.6, thus we may define a Borel function $z : \operatorname{spt}\nu \to \mathbb{R}_+$ such that for every $i \in \{1,\dots,N\}$:

$$z = z_{\mathbf{P}^{x_i}} \quad \nu_i\text{-almost everywhere.}$$

Let us show that z admits a Hölder continuous representative through Campanato estimates, following the strategy of [San07]. Take a point $x \in \operatorname{spt}\nu$. For every $r \in (0, 2\operatorname{diam}(\operatorname{spt}\nu)]$ denote by $z_r(x)$ the central median of z on $B_r(x)$ with respect to ν, defined as the midpoint of the interval of values $\ell \in \mathbb{R}_+$ such that $B_r(x)$ may be partitioned into two subsets $A \sqcup B = B_r(x)$ with equal mass, i.e. $\nu(A) = \nu(B) = \nu(B_r(x))/2$, and such that $z \geq \ell$ on A and $z \leq \ell$ on B. Consider two such sets A, B for the central median $\ell = z_r(x)$ and define the following variation of ν:

$$\tilde{\nu} := \nu - \nu \llcorner A + \nu \llcorner B = \sum_{i=1}^N \tilde{\nu}_i$$

where for every $i \in \{1,\dots,N\}$,

$$\tilde{\nu}_i := \nu_i - \nu_i \llcorner A + \nu_i \llcorner B.$$

By Lemma 2.6 again, we know that the $\mathbf{P}^{x_i}$'s are optimal traffic plans with single source x_i, thus we may use the first variation inequality (D) of Proposition 2.2 for every $i \in \{1,\dots,N\}$ to obtain:

$$\begin{aligned}
\mathbf{d}^\alpha(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i) &\leq \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu_i) + \alpha\left(\int_B z_{\mathbf{P}^{x_i}}\,\mathrm{d}\nu_i - \int_A z_{\mathbf{P}^{x_i}}\,\mathrm{d}\nu_i\right) \\
&= \mathbb{M}^\alpha(\mathbf{P}^{x_i}) + \alpha\left(\int_B z\,\mathrm{d}\nu_i - \int_A z\,\mathrm{d}\nu_i\right).
\end{aligned} \tag{4.3}$$

We set $\tilde{\mathbf{P}} := \sum_{i=1}^N \tilde{\mathbf{P}}_i$ where $\tilde{\mathbf{P}}_i \in \mathbf{TP}(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i)$ is an optimal traffic plan for every $i \in \{1,\dots,N\}$. Summing (4.3) over i, using the subadditivity of the α-mass and the disjointness of the $\mathbf{P}^{x_i}$'s, which results from Lemma 2.6, yields:

$$\begin{aligned}
\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \leq \sum_{i=1}^N \mathbb{M}^\alpha(\tilde{\mathbf{P}}_i) &= \sum_{i=1}^N \mathbf{d}^\alpha(\|\tilde{\nu}_i\|\delta_{x_i}, \tilde{\nu}_i) \\
&\leq \sum_{i=1}^N \left(\mathbb{M}^\alpha(\mathbf{P}^{x_i}) + \alpha\left(\int_B z\,\mathrm{d}\nu_i - \int_A z\,\mathrm{d}\nu_i\right)\right) \\
&= \mathbb{M}^\alpha(\mathbf{P}) + \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right).
\end{aligned} \tag{4.4}$$

Notice that $\tilde{\mathbf{P}} \in \mathbf{TP}(\tilde{\mu}, \tilde{\nu})$ where $\tilde{\mu} := \sum_{i=1}^N \|\tilde{\nu}_i\|\delta_{x_i}$. Take an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\tilde{\nu}, \nu)$ and consider a concatenation

$$\mathbf{P}' \in \tilde{\mathbf{P}} : \mathbf{Q} \subseteq \mathbf{TP}(\tilde{\mu}, \nu),$$

which is defined thanks to Proposition 2.1 (i). Since $\operatorname{spt}(\tilde{\nu} - \nu) \subseteq \bar{B}_r(x)$ and $\|\tilde{\nu} - \nu\| = \nu(B_r(x)) \leq C_A r^d$, by Proposition 2.1 (iii) and the branched transport upper estimate (2.12) we have:

$$\mathbb{M}^\alpha(\mathbf{P}') \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + \mathbb{M}^\alpha(\mathbf{Q}) \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + C_{\mathsf{BOT}}\, 2r\, (C_A r^d)^\alpha. \tag{4.5}$$

Now we remark that $\tilde{\mu}$ is still supported on $\{x_i : 1 \leq i \leq N\}$ and μ is a mass-optimal quantizer of ν with respect to the x_i's, so that $\mathbb{M}^\alpha(\mathbf{P}')$ is greater than $\mathbb{M}^\alpha(\mathbf{P})$, thus by combining (4.5) and (4.4):

$$\mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}') \leq \mathbb{M}^\alpha(\tilde{\mathbf{P}}) + 2C_{\mathsf{BOT}} C_A^\alpha r^{1+d\alpha}$$

$$\leq \mathbb{M}^\alpha(\mathbf{P}) + \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right) + 2C_{\mathsf{BOT}}C_A^\alpha r^{1+d\alpha}.$$

This implies that

$$0 \leq \alpha\left(\int_B z\,\mathrm{d}\nu - \int_A z\,\mathrm{d}\nu\right) + 2C_{\mathsf{BOT}}C_A^\alpha r^{1+d\alpha},$$

hence

$$\int_{B_r(x)} |z - z_r(x)|\,\mathrm{d}\nu = \int_A z\,\mathrm{d}\nu - \int_B z\,\mathrm{d}\nu \leq 2\alpha^{-1}C_{\mathsf{BOT}}C_A^\alpha r^{1+d\alpha}. \tag{4.6}$$

We now use Campanato estimates: for every $x \in \operatorname{spt}\nu$, $r \leq 2\operatorname{diam}(\operatorname{spt}\nu)$ and $r' \in [r/2, r]$,

$$\begin{aligned} |z_r(x) - z_{r'}(x)| &\leq \fint_{B_{r'}(x)} |z - z_r(x)|\,\mathrm{d}\nu \\ &\leq \frac{1}{\nu(B_{r'}(x))}\int_{B_r(x)} |z - z_r(x)|\,\mathrm{d}\nu \leq \frac{2\alpha^{-1}C_{\mathsf{BOT}}C_A^\alpha r^{1+d\alpha}}{c_A(r/2)^d} \leq Cr^\beta, \end{aligned} \tag{4.7}$$

where we have set $C := \frac{2^{d+1}C_{\mathsf{BOT}}C_A^\alpha}{\alpha c_A}$, and as before $\beta = 1 + d\alpha - d \in (0,1]$. Applying (4.7) to radii $r2^{-n}, r2^{-n-1}$, we deduce that $(z_{r2^{-n}})_{n\in\mathbb{N}}$ is a Cauchy sequence, which in turn implies (using (4.7) again) that the following limit exists for every $x \in \operatorname{spt}\nu$:

$$z_{\mathbf{P}}(x) := \lim_{r\to 0} z_r(x) = \lim_{r\to 0}\fint_{B_r(x)} z\,\mathrm{d}\nu.$$

By triangle inequality (4.7) yields

$$|z_r(x) - z_{\mathbf{P}}(x)| \leq \sum_{n=0}^{+\infty}|z_{r2^{-n}}(x) - z_{r2^{-(n+1)}}(x)| \leq \frac{Cr^\beta}{1-2^{-\beta}},$$

and combining with (4.6) we get

$$\fint_{B_r(x)} |z - z_{\mathbf{P}}(x)|\,\mathrm{d}\nu \leq \frac{2}{1-2^{-\beta}}Cr^\beta.$$

Finally, take $x, y \in \operatorname{spt}\nu$ such that $r := |y - x|$ and use the last two inequalities to get:

$$\begin{aligned} |z_{\mathbf{P}}(y) - z_{\mathbf{P}}(x)| &\leq |z_{\mathbf{P}}(y) - z_r(y)| + |z_r(y) - z_{\mathbf{P}}(x)| \\ &\leq \frac{Cr^\beta}{1-2^{-\beta}} + \fint_{B_r(y)} |z - z_{\mathbf{P}}(x)| \\ &\leq \frac{Cr^\beta}{1-2^{-\beta}} + \frac{\nu(B_{2r}(x))}{\nu(B_r(y))}\fint_{B_{2r}(x)} |z - z_{\mathbf{P}}(x)|\,\mathrm{d}\nu \leq \left(\frac{1 + 2^{d+1}(C_A/c_A)}{1-2^{-\beta}}\right)Cr^\beta. \end{aligned}$$

As a consequence, we get (ii) with

$$C_H := \frac{2^{(d+2)^2}C_{\mathsf{BOT}}C_A^{1+\alpha}}{(1-2^{-(1+d\alpha-d)})\alpha c_A^2}.$$

Let us now prove (i). Since ν-a.e. point of $\operatorname{spt}\nu$ is a Lebesgue point of z (with respect to ν), we know that $z_{\mathbf{P}} = z$ ν-a.e. thus $z_{\mathbf{P}} = z_{\mathbf{P}^{x_i}}$ ν_i-a.e., but since $z_{\mathbf{P}^{x_i}}$ is lower semicontinuous and $z_{\mathbf{P}}$ is continuous on $\operatorname{spt}\nu_i$, we have $z_{\mathbf{P}^{x_i}} \leq z_{\mathbf{P}}$ everywhere on $\operatorname{Bas}(\mathbf{P}, x_i) = \operatorname{spt}\nu_i$. Let us show that we actually have equality. Given $x \in \operatorname{spt}\nu_i$ such that $z_{\mathbf{P}^{x_i}}(x) < \infty$ (otherwise there is nothing to prove), consider a $\mathbf{P}^{x_i}$-good curve γ_i from x_i to x. Fix $r \leq \operatorname{diam}(\operatorname{spt}\nu)$, take an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\nu(B_r(x))\delta_x, \nu \llcorner B_r(x))$ and by Proposition 2.1 (i) take a concatenation

$$\mathbf{P}' \in (\nu(B_r(x))\delta_{\gamma_i})) : \mathbf{Q} \in \mathbf{TP}(\nu(B_r(x))\delta_{x_i}, \nu \llcorner B_r(x)).$$

We build the competitor

$$\tilde{\mathbf{P}} := \mathbf{P} - \mathbf{P} \llcorner \{\gamma(\infty) \in B_r(x)\} + \mathbf{P}'$$

which belongs to $\mathbf{TP}(\tilde{\mu}, \nu)$ for some measure $\tilde{\mu}$ which is still supported on $\{x_i, 1 \le i \le N\}$. Using as above the first variation inequality (D) of Proposition 2.2 (applied to each $\mathbf{P}^{x_j}$, $j \in \{1, \dots, N\}$), the subaddditivity of the α-mass and the mass-optimality of μ we must have:

$$\mathbb{M}^\alpha(\mathbf{P}) \le \mathbb{M}^\alpha(\tilde{\mathbf{P}}) \le \mathbb{M}^\alpha(\mathbf{P}) - \alpha \int_{B_r(x)} z_{\mathbf{P}} \, \mathrm{d}\nu + \alpha \nu(B_r(x)) z_{\mathbf{P}^{x_i}}(x) + \mathbb{M}^\alpha(\mathbf{Q})$$

Since $\mathbb{M}^\alpha(\mathbf{Q}) \le 2 C_{\mathsf{BOT}} r^{1+d\alpha}$ it implies

$$\forall r \in (0, \operatorname{diam}(\operatorname{spt} \nu)), \quad \fint_{B_r(x)} z_{\mathbf{P}} \, \mathrm{d}\nu \le z_{\mathbf{P}_i}(x) + \frac{2 C_{\mathsf{BOT}}}{c_A} r^\beta \implies z_{\mathbf{P}}(x) \le z_{\mathbf{P}_i}(x),$$

hence $z_{\mathbf{P}^{x_i}} = z_{\mathbf{P}}$ on $\operatorname{Bas}(\mathbf{P}, x_i)$ for every $i \in \{1, \dots, N\}$, i.e. (i) holds true.

□

4.2 Applications of the landscape function

We now generalize the properties of the single-source landscape function of Proposition 2.2 to our setting. First, we extend the notion of $\mathbf{P}$-good curve when $\mathbf{P}$ is a traffic plan with N sources $\{x_1, \dots, x_N\}$ such that the traffic plans $\mathbf{P}^{x_i}$'s are disjoint: we say that a curve γ is $\mathbf{P}$-*good* if it starts at some source x_i and it is $\mathbf{P}^{x_i}$-good.

Proposition 4.2. *Let ν be a compactly supported d-Ahlfors regular measure, μ be a N-point mass-optimal quantizer with respect to $\mathcal{X} := \{x_i\}_{1 \le i \le N}$ and $\mathbf{P} \in \mathbf{TP}(\mu, \nu)$ be an optimal traffic plan with $\mathbb{M}^\alpha(\mathbf{P}) < \infty$, $\alpha \in [0, 1)$. We consider a nonempty subset $\mathcal{X}' \subseteq \mathcal{X}$ and we set:*

$$\mu' := \mu \llcorner \mathcal{X}' \qquad \nu' := \nu \llcorner \bigcup_{s \in \mathcal{X}'} \operatorname{Bas}(\mathbf{P}, s) \qquad \mathbf{P}' := \sum_{s \in \mathcal{X}'} \mathbf{P}^s.$$

The landscape function $z_{\mathbf{P}} : \operatorname{spt} \nu \to \mathbb{R}_+$ given by Theorem 4.1 satisfies:

(A) $z_{\mathbf{P}}(x) \ge d(x, \mathcal{X}')$ for every $x \in \operatorname{spt} \nu'$;

(B) for every $\mathcal{X}' \subseteq \Sigma$, the α-distance writes as:

$$\mathbf{d}^\alpha(\mathcal{X}', \nu') = \mathbf{d}^\alpha(\mu', \nu') = \mathbb{M}^\alpha(\mathbf{P}') = \int_{\mathbb{R}^d} z_{\mathbf{P}}(x) \, \mathrm{d}\nu'(x),$$

(C) if $\tilde{\mathbf{P}} \in \mathbf{TP}(\tilde{\mu}_N, \tilde{\nu})$ is a traffic plan concentrated on $\mathbf{P}'$-good curves, then:

$$\mathbb{M}^\alpha(\tilde{\mathbf{P}}) \le \mathbb{M}^\alpha(\mathbf{P}') + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu),$$

and the inequality is strict if for some $x_i \in \mathcal{X}'$, $\Theta_{\tilde{\mathbf{P}}^{x_i}} - \Theta_{\mathbf{P}^{x_i}}$ is not zero $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_i}}$;

(D) in particular, $z_{\mathbf{P}}$ is an upper first variation of the irrigation distance, *in the sense that for every $\tilde{\nu} \in \mathscr{M}^+(\mathbb{R}^d)$,*

$$\mathbf{d}^\alpha(\mathcal{X}', \tilde{\nu}) \le \mathbf{d}^\alpha(\mathcal{X}', \nu') + \alpha \int_{\mathbb{R}^d} z_{\mathbf{P}} \, \mathrm{d}(\tilde{\nu} - \nu').$$

Sketch of proof: The results follow from Proposition 2.2 rather directly, using the result of Theorem 4.1. For (A) it suffices to note that for every $x \in \operatorname{spt} \nu$ there exists $x_i \in \mathcal{X}$ such that $z_{\mathbf{P}}(x) = z_{\mathbf{P}^{x_i}}(x)$, and thus using Proposition 2.2 (A) we find $z_{\mathbf{P}^{x_i}}(x) \ge d(x, x_i) \ge d(x, \mathcal{X})$. For the remaining points, the application of corresponding points from Proposition 2.2 applied separately to the basins $\operatorname{Bas}(\mathbf{P}, s)$, $s \in \mathcal{X}'$, together with the disjointness properties from Theorem 4.1. □

Corollary 4.3. *Under the assumptions of Theorem 4.1 and using the same notations, the basins* $\mathrm{Bas}(\mathbf{P}, x_i)$ *are closed subsets of* $\operatorname{spt}\nu$ *which form a partition of* ν*, in the sense that they are* ν*-essentially disjoint and that their reunion is equal to* $\operatorname{spt}\nu$*.*

Proof. The basins are closed since by definition $\mathrm{Bas}(\mathbf{P}, x_i) = \operatorname{spt}\nu_i$ where $\nu_i = (e_\infty)_\sharp \mathbf{P}^{x_i}$ for every $i \in \{1, \dots, N\}$. Furthermore, $\nu = \sum_{i=1}^N \nu_i$ implies that $\operatorname{spt}\nu = \bigcup_{i=1}^N \operatorname{spt}\nu_i$.

Let us show that for every $i \neq j$, $\nu_j(\mathrm{Bas}(\mathbf{P}, x_i)) = 0$, which implies the result, since for $k \neq l$ we can write

$$\begin{aligned}\nu(\mathrm{Bas}(\mathbf{P}, x_k) \cap \mathrm{Bas}(\mathbf{P}, x_l)) &= \sum_{j=1}^N \nu_j(\mathrm{Bas}(\mathbf{P}, x_k) \cap \mathrm{Bas}(\mathbf{P}, x_l)) \\ &\leq \nu_k(\mathrm{Bas}(\mathbf{P}, x_l)) + \sum_{j \neq k} \nu_j(\mathrm{Bas}(\mathbf{P}, x_k)) = 0.\end{aligned}$$

Suppose by contradiction that $\nu_j(\mathrm{Bas}(\mathbf{P}, x_i)) > 0$ for some $i \neq j$. Take as competitor

$$\mathbf{P}' = \mathbf{P} - \underbrace{\mathbf{P}^{x_j} \llcorner \{\gamma : \gamma(\infty) \in \mathrm{Bas}(\mathbf{P}, x_i)\}}_{\mathbf{P}_{ij}} + \mathbf{Q},$$

where $\mathbf{Q} \in \mathbf{TP}(\nu_j(\mathrm{Bas}(\mathbf{P}, x_i))\delta_{x_i}, \nu_j \llcorner \mathrm{Bas}(\mathbf{P}, x_i))$ is chosen so that $\mathbf{Q}$-a.e. curve γ is a $\mathbf{P}^{x_i}$-good curve, which is possible because $z_{\mathbf{P}^{x_i}}$ is finite everywhere on $\mathrm{Bas}(\mathbf{P}, x_i)$ by Theorem 4.1. Since $\mathbf{P}' \in \mathbf{TP}(\mu', \nu)$ where μ' is concentrated on the x_i's, by mass-optimality of the quantizer μ, we have:

$$\mathbb{M}^\alpha(\mathbf{P}) = \mathbf{d}^\alpha(\mu, \nu) \leq \mathbf{d}^\alpha(\mu', \nu) \leq \mathbb{M}^\alpha(\mathbf{P}').$$

Notice that $\mathbf{P}'$ is rectifiable and $\Sigma_{\mathbf{P}_{ij}} \subseteq \Sigma_{\mathbf{P}^{x_j}}$ and $\Sigma_{\mathbf{Q}} \subseteq \Sigma_{\mathbf{P}^{x_i}}$, since every $\mathbf{P}^{x_i}$-good curve is $\mathscr{H}^1$-a.e. included in $\Sigma_{\mathbf{P}^{x_i}}$. Besides, the $\mathbf{P}^{x_k}$'s are disjoint by Lemma 2.6, which implies thanks to (2.7) that the networks $\Sigma_{\mathbf{P}^{x_k}}$'s are $\mathscr{H}^1$-essentially disjoint. Thus the traffic plans $(\mathbf{P}')^{x_k}$'s are disjoint. We apply the upper first variation inequality Proposition 4.2(C) to the variation given by replacing $\mathbf{P}^{x_i} \mapsto (\mathbf{P}')^{x_i} = \mathbf{P}^{x_i} - \mathbf{P}_{ij}$ and $\mathbf{P}^{x_j} \mapsto (\mathbf{P}')^{x_j} = \mathbf{P}^{x_j} + \mathbf{Q}$, and we get:

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}) = \mathbf{d}^\alpha(\mu, \nu) &\leq \mathbf{d}^\alpha(\mu', \nu) \leq \mathbb{M}^\alpha(\mathbf{P}') \\ &= \mathbb{M}^\alpha(\mathbf{P}) + (\mathbb{M}^\alpha(\mathbf{P}^{x_i} - \mathbf{P}_{ij}) - \mathbb{M}^\alpha(\mathbf{P}^{x_i})) + (\mathbb{M}^\alpha(\mathbf{P}^{x_j} + \mathbf{Q}) - \mathbb{M}^\alpha(\mathbf{P}^{x_j})) \\ &< \mathbb{M}^\alpha(\mathbf{P}) - \alpha \int_{\mathrm{Bas}(\mathbf{P}, x_i) \cap \mathrm{Bas}(\mathbf{P}, x_j)} z_{\mathbf{P}^{x_i}} \, \mathrm{d}\nu_j + \alpha \int_{\mathrm{Bas}(\mathbf{P}, x_i) \cap \mathrm{Bas}(\mathbf{P}, x_j)} z_{\mathbf{P}^{x_j}} \, \mathrm{d}\nu_j \\ &= \mathbb{M}^\alpha(\mathbf{P}) - \alpha \int_{\mathrm{Bas}(\mathbf{P}, x_i) \cap \mathrm{Bas}(\mathbf{P}, x_j)} z_{\mathbf{P}} \, \mathrm{d}\nu_j + \alpha \int_{\mathrm{Bas}(\mathbf{P}, x_i) \cap \mathrm{Bas}(\mathbf{P}, x_j)} z_{\mathbf{P}} \, \mathrm{d}\nu_j = \mathbb{M}^\alpha(\mathbf{P}),\end{aligned}$$

in which we have used Theorem 4.1 and Proposition 4.2 (C), knowing that $\Theta_{\mathbf{P}_{ij}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_i}}$ (or similarly that $\Theta_{\mathbf{Q}}$ does not vanish $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}^{x_j}}$). This is a contradiction. □

The measure of basins can be controlled from above and below for optimal plans associated with mass-optimal quantizers.

Lemma 4.4. *Under the assumptions of Theorem 4.1 and with the same notations, for every source* x_i *of* $\mathbf{P}$*, we set*

$$\delta(\mathbf{P}, x_i) \coloneqq \max_{y \in \mathrm{Bas}(\mathbf{P}, x_i)} \frac{|y - x_i|}{2}.$$

Then we have

$$c_{\mathrm{Bas}} \operatorname{diam}(\mathrm{Bas}(\mathbf{P}, x_i))^d \leq \nu(\mathrm{Bas}(\mathbf{P}, x_i)) \leq C_A \operatorname{diam}(\mathrm{Bas}(\mathbf{P}, x_i))^d, \tag{4.8}$$

$$\frac{1}{2} \operatorname{diam}(\mathrm{Bas}(\mathbf{P}, x_i)) \leq \delta(\mathbf{P}, x_i) \leq \left(\frac{C_A}{2c_{\mathrm{Bas}}}\right)^{\frac{1}{d}} \operatorname{diam}(\mathrm{Bas}(\mathbf{P}, x_i)), \tag{4.9}$$

$$c_H \operatorname{diam}(\operatorname{Bas}(\mathbf{P}, x_i))^\beta \leq \sup_{\operatorname{Bas}(\mathbf{P},x_i)} z_{\mathbf{P}_N} \leq C'_H \operatorname{diam}(\operatorname{Bas}(\mathbf{P}, x_i))^\beta, \tag{4.10}$$

where $c_{\mathrm{Bas}} := 2^{-d}\left(C_H + \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$, $c_H := 2^{-\beta} C_A^{\alpha-1}$ *and* $C'_H := 2^\beta c_{\mathrm{Bas}}^{\frac{1}{d}} C_A^{\frac{\beta}{d}}$.

Proof. The upper bound (first inequality) follows from the upper d-Ahlfors regularity of μ.

For the lower bound (second inequality), consider a point $y \in \operatorname{Bas}(\mathbf{P}, x_i)$ such that $|y - x_i| = \max_{y \in \operatorname{Bas}(\mathbf{P},x_i)} |y - x_i|$, which exists because $\operatorname{Bas}(\mathbf{P}, x_i)$ is compact. Take a $\mathbf{P}^{x_i}$-good curve γ_i from x_i to y and set $r := |y - x_i|$. We build a competing traffic plan $\tilde{\mathbf{P}}$ by removing $\mathbf{P} \llcorner \{\gamma \in \Gamma^d : \gamma(\infty) \in B_r(y)\}$ then adding an optimal traffic plan $\mathbf{Q} \in \mathbf{TP}(\nu(B_r(x_i))\delta_{x_i}, \nu \llcorner B_r(y))$. By optimality of $\mathbf{P}$ and mass-optimality of the source measure μ, using the first variation inequality Proposition 4.2 (C), and the branched transport upper estimate (2.12), we get

$$-\alpha \int_{B_r(y)} z_{\mathbf{P}} \,\mathrm{d}\nu + C_{\mathsf{BOT}}(2r)\nu(B_r(y))^\alpha \geq 0 \implies \fint_{B_r(y)} z_{\mathbf{P}} \,\mathrm{d}\nu \leq \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}} r^\beta, \tag{4.11}$$

which yields by Theorem 4.1,

$$z_{\mathbf{P}^{x_i}}(y) = z_{\mathbf{P}}(y) - \fint_{B_r(y)} z_{\mathbf{P}} \,\mathrm{d}\nu + \fint_{B_r(y)} z_{\mathbf{P}} \,\mathrm{d}\nu \leq C_H r^\beta + \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}} r^\beta =: C' r^\beta. \tag{4.12}$$

Now recall the definition of landscape function in the single-source case:

$$\begin{aligned} C' r^\beta \geq z_{\mathbf{P}^{x_i}}(y) &= \int_{\gamma_i} \Theta_{\mathbf{P}^{x_i}}(x)^{\alpha-1} \,\mathrm{d}x \\ &\geq \mathscr{H}^1(\gamma_i(\mathbb{R}_+))\nu(\operatorname{Bas}(\mathbf{P}, x_i))^{\alpha-1} \geq r\nu(\operatorname{Bas}(\mathbf{P}, x_i))^{\alpha-1}. \end{aligned} \tag{4.13}$$

As a consequence

$$\nu(\operatorname{Bas}(\mathbf{P}, x_i)) \geq (C' r^{\beta-1})^{\frac{1}{\alpha-1}} = 2^{-d} C'^{\frac{1}{\alpha-1}} (2r)^d = c_{\mathrm{Bas}}(2r)^d$$

where $c_{\mathrm{Bas}} := 2^{-d}\left(C_H + \frac{2C_{\mathsf{BOT}}}{\alpha c_A^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$. Notice that $\operatorname{diam}(\operatorname{Bas}(\mathbf{P}, x_i)) \leq 2r$ by the triangle inequality, which yields (4.8) and (4.9). As for (4.10), the lower bound comes from (4.13) and the upper Ahlfors regularity, while the upper bound comes from (4.12), which implies by the triangle inequality that

$$\begin{aligned} z_{\mathbf{P}}(x) \leq C_H |y - x|^\beta + C' r^\beta &\leq 2C' r^\beta \\ &= 2 2^{d(\alpha-1)} c_{\mathrm{Bas}}^{\alpha-1} r^\beta \\ &\leq 2^\beta c_{\mathrm{Bas}}^{\alpha-1} \left(\frac{C_A}{2c_{\mathrm{Bas}}}\right)^{\frac{\beta}{d}} \operatorname{diam}(\operatorname{Bas}(\mathbf{P}, x_i))^\beta. \\ &= C'_H \operatorname{diam}(\operatorname{Bas}(\mathbf{P}, x_i))^\beta \end{aligned}$$

where $C'_H := 2^\beta c_{\mathrm{Bas}}^{\frac{1}{d}} C_A^{\frac{\beta}{d}}$. □

Remark 4.5 (Voronoi basins). In the case $\alpha = 1$, if ν has a compact convex support Ω and $\mathbf{P}$ is an optimal traffic plan between ν and a mass-optimal quantizer $\mu = \sum_{i=1}^N m_i \delta_{x_i}$ associated with the points $(x_i)_{1 \leq i \leq N}$, then the basins will be exactly the *Voronoi cells* $(\Omega \cap V_i)_{1 \leq i \leq N}$ given by

$$\forall i \in \{1, \dots, N\}, \quad V_i := \left\{x : |x - x_i| = \min_{1 \leq j \leq N} |x - x_j|\right\}. \tag{4.14}$$

When $\alpha \in (0, 1)$, the basins $(\operatorname{Bas}(\mathbf{P}, x_i))_{1 \leq i \leq n}$ thus extend the notion of Voronoi cells to the case of branched optimal transport, which we may call *(branched) Voronoi basins*. These Voronoi basins are also closed sets which form a partition of the given measure ν, as stated in Corollary 4.3, but they are much more complicated in several regards:

- They need not be convex polyhedra, but are rather thought to exhibit fractal pairwise boundaries.
- Classical Voronoi cells do not actually depend on the measure ν or its support but may be computed directly from the points $(x_i)_{1\leq i\leq N}$ through (4.14), taking the intersection with the support afterwards. On the contrary, there is a priori no reason for Voronoi basins to behave in the same way, and it is well possible that the Voronoi basins for ν and $\nu' \geq \nu$ are not nested.
- Computing Voronoi basins is much more difficult, as the problem of optimizing the masses given the points does not admit an explicit solution in the form of (4.14).

5 Uniform properties of optimal quantizers and partitions

In this section we investigate the uniform properties of optimal quantizers at the microscopic scale, i.e. at a the scale of $N^{-1/d}$, when the measure ν that is quantized is d-Ahlfors regular. Roughly speaking, we are going to show that the atoms of a N-point optimal quantizer are distributed somewhat uniformly at this scale, being well-separated and leaving no big hole in the support of ν, and that the basins are somewhat round, having inner and outer balls of comparable size. We also show uniformity bounds on the masses and energies associated with each atom.

5.1 Delone constants for optimal quantizers

Given a set $X \subseteq \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^d$ a finite set of points, we define the *covering radius* (also called *mesh norm* or *fill radius*) of X by $\mathcal{X}$, as

$$\omega(X, \mathcal{X}) := \sup_{x\in X} \min_{x'\in\mathcal{X}} d(x, x').$$

It is the smallest $r \geq 0$ such that the closed balls of radius r with centers in $\mathcal{X}$ cover X. The *separation distance* (corresponding to $1/2$ of the *packing radius*) of $\mathcal{X}$ is defined by

$$\delta(\mathcal{X}) := \min_{x\neq x', x,x'\in\mathcal{X}} d(x, x').$$

A set with finite covering radius and nonzero separation distance is called a *Delone set* with respect to X, and (ω, δ) its *Delone constants*. Given a d-Ahlfors regular measure ν, the following theorem provides shows that the atoms of optimal N-point quantizers are Delone sets with respect to $\operatorname{spt}\nu$, providing bounds comparable to $N^{-1/d}$ on its Delone constants.

Theorem 5.1 (Extended version of Theorem 1.2)**.** *Let ν be a compactly supported d-Ahlfors regular measure on $\mathbb{R}^d$ and $\mu_N = \sum_{i\leq N} m_i\delta_{x_i}$ be an N-point optimal quantizer (i.e. a solution to (2.17)) with atoms $\mathcal{X} = \{x_i\}_{1\leq i\leq N}$. Then the covering radius $\omega(\operatorname{spt}\nu, \mathcal{X})$ and separation distance $\delta(\mathcal{X})$ enjoy the following bounds:*

$$\omega(\operatorname{spt}\nu, \mathcal{X}) \leq c_2 N^{-1/d}, \tag{5.1}$$

$$\delta(\mathcal{X}) \geq c_1 N^{-1/d}. \tag{5.2}$$

for some constants $c_1, c_2 > 0$ that do not depend on N.

Our proof is inspired from ideas of [Gru04], dealing with classical optimal transport costs. We stress that the situation in the branched optimal transport case is much more involved, since the ground cost is not explicit (it depends on all the trajectories and is part of the optimization defining $\mathbf{d}^\alpha$), and the shapes of basins are not known at all (they are thought to have fractal boundaries). Thus, we shall need to estimate

- the cost for merging a “small” basin to a “neighbouring” basin ;
- the gain to remove part of a “large” basin.

The landscape function, its uniform Hölder regularity and its consequences established in Section 4 will play a crucial role.

Proof of (5.1). We proceed by proving successively the following.

(a) *At least one basin is not too large:* there is a constant $\tilde{c}_2$ (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \exists j \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)) \leq \tilde{c}_2 N^{-\frac{1}{d}}.$$

(b) *At least one basin is not too small:* there is a constant $\tilde{c}_1$ (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \exists j \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)) \leq \tilde{c}_1 N^{-\frac{1}{d}}.$$

(c) *All basins are small:* there is a constant c_2 (not depending on N) such that

$$\forall N \in \mathbb{N}^*, \forall i \leq N, \quad \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i)) \leq c_2 N^{-\frac{1}{d}}.$$

(d) *All atoms are far from each other:* there exists a constant c_2 such that

$$\forall N \in \mathbb{N}^*, \forall (1 \leq j \neq k \leq N), \quad d(x_j, x_k) \geq c_1 N^{-\frac{1}{d}}.$$

Notice that (5.1) follows from (c), since the basins form a covering of $\operatorname{spt}\mu$ by Corollary 4.3, while (5.2) is merely a rephrasing of (d).

Proof of (a) and (b). First note that by Corollary 4.3, the basins form a partition of ν, thus

$$\sum_{j=1}^{N} \nu(\operatorname{Bas}(\mathbf{P}_N, x_j)) = \|\nu\|, \tag{5.3}$$

thus there exists an index $j \in \{1, \dots, N\}$ for which

$$\nu(\operatorname{Bas}(\mathbf{P}_N, x_j)) \leq \frac{\|\nu\|}{N},$$

and by Lemma 4.4, this implies that

$$\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j) \leq (\|\nu\|(c_{\mathrm{Bas}} N)^{-1})^{\frac{1}{d}} = \tilde{c}_2 N^{-\frac{1}{d}} \quad \text{where} \quad \tilde{c}_2 := (\|\nu\| c_{\mathrm{Bas}}^{-1})^{-\frac{1}{d}}.$$

Similarly, there exists an index $i \in \{1, \dots, N\}$ such that

$$C_A \operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i) \geq \nu(\operatorname{Bas}(\mathbf{P}_N, x_i)) \geq \frac{\|\nu\|}{N},$$

which implies that

$$\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i) \geq \tilde{c}_2 N^{-\frac{1}{d}} \quad \text{where} \quad \tilde{c}_2 := (\|\nu\| C_A^{-1})^{-\frac{1}{d}}.$$

Proof of (c). Applying (a) we find that there exists $j \leq N$ such that

$$\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_j)) \leq \tilde{c}_2 N^{-\frac{1}{d}}. \tag{5.4}$$

Suppose that for some $t > 1$ there exists $i \leq N$ such that

$$\operatorname{diam}(\operatorname{Bas}(\mathbf{P}_N, x_i)) \geq t \tilde{c}_2 N^{-\frac{1}{d}}.$$

We are going to show a contradiction when t is too large (not depending on N). For this, let us build a better competitor than μ_N. We shall add an extra point q of $\operatorname{Bas}(\mathbf{P}_N, x_i)$ to irrigate a "costly" ball around q, then remove the point x_j and irrigate the former basin $\mathrm{Bas}(\mathbf{P}_N, x_j)$ from a neighbouring basin $\mathrm{Bas}(\mathbf{P}_N, x_k)$.

Let us find a "costly" ball. Consider

$$q \in \arg\max_{\mathrm{Bas}(\mathbf{P}_N, x_i)} z_{\mathbf{P}_N},$$

which exists because $z_{\mathbf{P}_N}$ is Hölder-continuous thanks to Theorem 4.1 and basins are compact sets thanks to Corollary 4.3. We consider the ball $B_{\varepsilon t\tilde{c}_2 N^{-\frac{1}{d}}}(q)$ for some small $\varepsilon \in (0,1)$ to be fixed later.

Now, we want to remove the point x_i from the quantizer μ_N and to irrigate the basin $\mathrm{Bas}(\mathbf{P}_N, x_i)$ from another basin that is not too far, in order to control the extra cost. By (5.4) and Ahlfors-regularity of ν, for $s > 1$ we have

$$\nu(B_{s\tilde{c}_2 N^{-\frac{1}{d}}}(x_j) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j)) \geq c_A (s\tilde{c}_2)^d N^{-1} - C_A \tilde{c}_2^d N^{-1} = (c_A s^d - C_A)\tilde{c}_2^d N^{-1}.$$

This is strictly positive if we take for example $s \coloneqq (2C_A/c_A)^{\frac{1}{d}}$, in which case there exists a point p such that

$$p \in \mathrm{Bas}(\mathbf{P}_N, x_k) \cap B_{s\tilde{c}_2 N^{-1/d}}(x_j) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j) \quad \text{for some} \quad k \neq j,$$

because the basins form of covering of $\operatorname{spt}\nu$.

We are now ready to build our competitor $\mathbf{P}_N^*$, modifying $\mathbf{P}_N$ according to the following sketch; the addition of curves (which increase the α-mass) are labeled by (A), while the removal of curves (which decreases the α-mass) are labeled by (R).

(R_1) Remove all curves starting at x_j.

(R_2) Remove all curves ending in $B_{\varepsilon\tilde{c}_2 N^{-1/d}}(q)$.

(A_1) Re-irrigate $\mathrm{Bas}(\mathbf{P}_N, x_j)$ by

- bringing a mass m_j from x_k to p following a $\mathbf{P}_N^{x_k}$-good curve γ,
- then concatenate an optimal traffic plan $\mathbf{Q}^1 \in \mathbf{TP}(m_j\delta_p, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_j))$.

(A_2) Add an optimal traffic plan $\mathbf{Q}^2$ from $m\delta_q$ to $\nu \llcorner (B_{\varepsilon t\tilde{c}_2 N^{-\frac{1}{d}}}(q) \setminus \mathrm{Bas}(\mathbf{P}_N, x_j))$, where m is the mass of the latter and $\varepsilon > 0$ is a small number to be chosen later (independently from N and t).

We start by doing the modifications along the existing network, corresponding to (R_1), (R_2) and the first part of (A_1). Setting $\Gamma_{x_j} \coloneqq \{\gamma : \gamma(0) = x_j\}$ and $\Gamma_q \coloneqq \{\gamma : \gamma(+\infty) \in B_{\varepsilon\tilde{c}_2 N^{-\frac{1}{d}}}(q)\}$, we define

$$\mathbf{P}'_N \coloneqq \mathbf{P}_N - \mathbf{P}_N \llcorner \Gamma_q - \mathbf{P}_N \llcorner (\Gamma_{x_j} \setminus \Gamma_q) + m_j \delta_\gamma.$$

Secondly, we add the new curves and pieces of curves corresponding to the second part of (A_1) and (A_2). We set $\nu' \coloneqq (e_\infty)_\sharp \mathbf{P}'_N$, and $\tilde{\mathbf{Q}}_1 \coloneqq \mathbf{Q}_1 + \iota_\sharp(\nu' - m_j\delta p)$ where $\iota : \mathbb{R}^d \to \Gamma^d$ denotes the canonical injection which sends a point x to the constant curve $\gamma_x \equiv x$. We define our competitor $\mathbf{P}_N^*$ by

$$\mathbf{P}_N^* \coloneqq \mathbf{P}''_N + \mathbf{Q}_2 \quad \text{where} \quad \mathbf{P}''_N \in \mathbf{P}'_N : \tilde{\mathbf{Q}}_1.$$

We estimate the gain and cost of these operations. First of all, we have

$$\begin{aligned}
\mathbb{M}^\alpha(\mathbf{P}_N^*) &\leq \mathbb{M}^\alpha(\mathbf{P}''_N) + \mathbb{M}^\alpha(\mathbf{Q}_2) \leq \mathbb{M}^\alpha(\mathbf{P}'_N) + \mathbb{M}^\alpha(\mathbf{Q}_2) + \mathbb{M}^\alpha(\mathbf{Q}_1) \\
&\leq \mathbb{M}^\alpha(\mathbf{P}'_N) + C_{\mathsf{BOT}} C_A^\alpha (\varepsilon t\tilde{c}_2 N^{-1/d})^{1+d\alpha} + C_{\mathsf{BOT}} C_A^\alpha (\tilde{c}_2 N^{-1/d})^{1+d\alpha} \\
&= \mathbb{M}^\alpha(\mathbf{P}'_N) + C_1 N^{-(\alpha+\frac{1}{d})}(1 + (\varepsilon t)^{1+d\alpha}),
\end{aligned} \tag{5.5}$$

for some constant C_1 which does dot depend on N. The α-mass of $\mathbf{P}'_N$ may then be estimated through the first variation formula Proposition 4.2 (C):

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - \int_{B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)} z_{\mathbf{P}_N}\,\mathrm{d}\nu - \int_{\mathbb{R}^d} z_{\mathbf{P}_N}\,\mathrm{d}(e_\infty)_\sharp(\mathbf{P}_N \llcorner (\Gamma_{x_j}\setminus\Gamma_q)) + m_j z_{\mathbf{P}_N}(p)\\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - \int_{B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)} z_{\mathbf{P}_N}\,\mathrm{d}\nu + m_j z_{\mathbf{P}_N}(p).\end{aligned} \tag{5.6}$$

Let us estimate $z_{\mathbf{P}_N}(p)$ from above and $z_{\mathbf{P}_N}$ from below on $B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)$. By Lemma 4.4, we know that

$$z_{\mathbf{P}N}(q) \geq c_H \operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i))^\beta \geq c_H (t\tilde{c}_2)^\beta N^{-\beta/d},$$

thus for every $y \in B_{\varepsilon t\tilde{c}_2 N^{-1/d}}(q)$

$$\begin{aligned}z_{\mathbf{P}_N}(y) &\geq c_H (t\tilde{c}_2)^\beta N^{-\beta/d} - C_H|y-q|^\beta\\ &\geq (c_H - \varepsilon^\beta C_H)(t\tilde{c}_2)^\beta N^{-\beta/d} \geq (c_H/2)(t\tilde{c}_2)^\beta N^{-\beta/d},\end{aligned} \tag{5.7}$$

provided we have chosen $\varepsilon^\beta \leq (c_H/2C_H)$. Besides, by Lemma 4.4 again

$$\begin{aligned}z_{\mathbf{P}_N}(p) &\leq \sup_{y\in\mathrm{Bas}(\mathbf{P}_N,x_j)} |z_{\mathbf{P}_N}(y) - z_{\mathbf{P}_N}(p)| + \sup_{y\in\mathrm{Bas}(\mathbf{P}_N,x_j)} z_{\mathbf{P}_N}(y)\\ &\leq C_H((s+1)\tilde{c}_2 N^{-1/d})^\beta + C'_H(\tilde{c}_2 N^{-1/d})^\beta\\ &= CN^{-\beta/d}\end{aligned} \tag{5.8}$$

for some C which does not depend on N. Reporting (5.7) and (5.8) in (5.6) yields

$$\begin{aligned}\mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - (c_H/2)t^\beta N^{-\beta/d} c_A(\varepsilon t\tilde{c}_2 N^{-1/d})^d + CN^{-\beta/d} C_A(s\tilde{c}_2 N^{-1/d})^d\\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - N^{-(\alpha+\frac{1}{d})}(C_2\varepsilon^d t^{1+d\alpha} - C_3),\end{aligned}$$

for some constant $C_2, C_3 > 0$ which do not depend on N.

Injecting this into (5.5), we get

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \leq \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(C_1 + C_3 + C_1(\varepsilon t)^{1+d\alpha} - C_2\varepsilon^d t^{1+d\alpha}).$$

Notice that because $1 + d\alpha - d = \beta > 0$, it is possible to choose ε small enough, independently from N and t (e.g.[5] $\varepsilon^\beta = (C_2/2C_1) \wedge (c_H/2C_H)$) so that

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \leq \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(C_1 + C_3 - (C_2/2)\varepsilon^d t^{1+d\alpha}).$$

Now, if t is too large, depending on the constants $C_1, C_2, C_3, \varepsilon$ which we stress do not depend on N, it leads to $\mathbb{M}^\alpha(\mathbf{P}^*_N) < \mathbb{M}^\alpha(\mathbf{P}_N)$, which contradicts the optimality of μ_N, because by construction the target measure of $\mathbf{P}^*_N$ is ν, and its source measure is an N-point quantizer. As a conclusion, (c) holds true.

Proof of (d). Take $t > 0$ and suppose that there are two atoms x_j, x_k of μ_N, with $j \neq k$, of μ_N such that $d(x_j, x_k) \leq tN^{-1/d}$. We are going to show a lower bound on $t > 0$ (not depending on N). Choose a $i \leq N$ such that

$$\mathscr{L}^d(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq \frac{\|\nu\|}{N} \quad \text{and} \quad \operatorname{diam}(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq \tilde{c}_1 N^{-1/d}$$

which is possible thanks to (b). Up to interchanging j with k, we may assume that $i \neq j$ (k may be equal to i, but it will not matter). The strategy is very similar as what we did above: we shall add an

[5] Recall that we had the condition $\varepsilon^\beta \leq (c_H/2C_H)$ for (5.7).

extra point $q \in \mathrm{Bas}(\mathbf{P}_N, x_i)$ to irrigate a "costly" ball around q, while removing the point x_j from the quantizer and irrigating the basin $\mathrm{Bas}(\mathbf{P}_N, x_j)$ from the close point x_k.

Consider

$$q \in \arg\max_{\mathrm{Bas}(\mathbf{P}_N, x_i)} z_{\mathbf{P}_N},$$

which is possibly because $z_{\mathbf{P}_N}$ is Hölder-continuous thanks to Theorem 4.1 and basins are compact sets thanks to Corollary 4.3. Take some small $\varepsilon \in (0,1)$ that we shall fix later. We build a better competitor thanks to the following construction.

(R_1) Remove all curves ending in $B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$, of total mass m.

(A_1) Add an optimal traffic plan $\mathbf{Q}^1 \in \mathbf{TP}(m\delta_q, \nu \llcorner B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q))$ to irrigate $\nu \llcorner B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$ again.

(A_2) Irrigate $\mathrm{Bas}(\mathbf{P}_N, x_j)$ from the point x_k instead of x_j by concatenating a (unit-speed parameterization of) the segment $[x_k, x_j]$ to all curves starting at x_j.

The removal (R_1) produces the new traffic plan

$$\mathbf{P}'_N \coloneqq \mathbf{P}_N - \mathbf{P}_N \llcorner \Gamma_q \quad \text{where} \quad \Gamma_q \coloneqq \{\gamma : \gamma(\infty) \in B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)\}.$$

We show as above by Lemma 4.4 that for every $y \in B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)$

$$z_{\mathbf{P}_N}(y) \geq (c_H/2)(t\tilde{c}_1)^\beta N^{-\beta/d}, \tag{5.9}$$

provided we have chosen $\varepsilon^\beta \leq (c_H/2C_H)$. Thus the cost gain can be estimated by using the first variation formula Proposition 4.2 (C):

$$\begin{aligned} \mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - \alpha \int_{B_{\varepsilon\tilde{c}_1 N^{-1/d}}(q)} z_{\mathbf{P}_N} \,\mathrm{d}\nu \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - C_A(\varepsilon\tilde{c}_1 N^{-1/d})^d (c_H/2) N^{-\beta/d} \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - C_1\varepsilon^d N^{-(\alpha+\frac{1}{d})}. \end{aligned} \tag{5.10}$$

For the addition of the (pieces of) curves (A_1) and (A_2), we denote by $\gamma_{k,j}$ the unit-speed parameterized segment from x_k to x_j, $\mathbf{P}'_{N,j} \coloneqq \mathbf{P}'_N \llcorner \{\gamma : \gamma(0) = x_j\}$, $m'_j \coloneqq \|\mathbf{P}'_{N,j}\|$ and we set $\mathbf{Q}_2 \coloneqq m'_j \delta_{\gamma_{k,j}} + (\iota \circ e_0)_\sharp(\mathbf{P}'_N - \mathbf{P}'_{N,j})$. We define our competitor $\mathbf{P}^*_N$ by

$$\mathbf{P}^*_N \coloneqq \mathbf{P}''_N + \mathbf{Q}_1 \quad \text{where} \quad \mathbf{P}''_N \in \mathbf{Q}_2 : \mathbf{P}'_N.$$

From (c), we know that

$$m'_j \leq m_j \leq C_A(\mathrm{diam}(\mathrm{Bas}(\mathbf{P}_N, x_j)))^d \leq C_A c_2^d N^{-1}$$

and we compute, using (5.10):

$$\begin{aligned} \mathbb{M}^\alpha(\mathbf{P}^*_N) &\leq \mathbb{M}^\alpha(\mathbf{P}''_N) + \mathbb{M}^\alpha(\mathbf{Q}_1) \leq \mathbb{M}^\alpha(\mathbf{P}'_N) + \mathbb{M}^\alpha(\mathbf{Q}_1) + \mathbb{M}^\alpha(\mathbf{Q}_2) \\ &\leq \mathbb{M}^\alpha(\mathbf{P}'_N) + C_{\mathsf{BOT}} C_A^\alpha (\varepsilon\tilde{c}_1 N^{-1/d})^{1+d\alpha} + (m'_j)^\alpha (tN^{-1/d}) \\ &= \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(-C_1\varepsilon^d + C_2\varepsilon^{1+d\alpha} + t). \end{aligned} \tag{5.11}$$

Taking $\varepsilon > 0$ such that $\varepsilon^\beta \leq c_H/2C_H$ and $\varepsilon^\beta \leq C_1/2C_2$ (e.g. take ε^β to be the minimum of the twi), we get $C_2\varepsilon^{1+d\alpha} \leq (C_1)/2\varepsilon^d$ and thus

$$\mathbb{M}^\alpha(\mathbf{P}^*_N) \leq \mathbb{M}^\alpha(\mathbf{P}_N) + N^{-(\alpha+\frac{1}{d})}(t - (C_1/2)\varepsilon^d),$$

which leads to a contradiction if t is too small (independently from N). Hence t is lower bounded by some constant $c_1 > 0$ and (d) holds true. □

5.2 Inner and outer ball property of the basins

Proposition 5.2. *Let $\nu \in \mathscr{M}^+(K)$ be a d-Ahlfors regular measure with constants $0 < c_A \leq C_A$. Let $\mu_N = \sum_{i=1}^N m_i \delta_{x_i}$ be a N-point optimal quantizer of ν and $\mathbf{P}_N$ an optimal traffic plan in $\mathbf{TP}(\mu_N, \nu)$. There are constants $c, C > 0$ depending only on α, d, c_A, C_A such that for all $i \in \{1, \dots, N\}$,*

$$\mathrm{Bas}(\mathbf{P}_N, x_i) \subseteq B(x_i, CN^{-1/d}), \tag{5.12}$$

and

$$B(x_i, cN^{-1/d}) \subseteq \mathbb{R}^d \setminus \bigcup_{j \neq i} \mathrm{Bas}(\mathbf{P}_N, x_j). \tag{5.13}$$

Remark 5.3. In particular, if $\nu = \mathscr{L}^d \llcorner \Omega$ for some open bounded set Ω with Lipschitz boundary, then for every source x_i such that $d(x_i, \partial\Omega) > cN^{-1/d}$, (5.12) and (5.13) rewrite as

$$B(x_i, cN^{-1/d}) \subseteq \mathrm{Bas}(\mathbf{P}_N, x_i) \subseteq B(x_i, CN^{-1/d}).$$

Besides, this translates as uniform inner and outer ball properties of optimal partitions (solutions to (2.24)). Indeed, by the equivalence between the optimal quantization and optimal partition problems stated in Theorem 2.7, solutions $(\Omega_i)_{1 \leq i \leq N}$ to (2.24) are actually $\mathscr{L}^d$-equivalent to basins $(\mathrm{Bas}(\mathbf{P}_N, x_i))_{1 \leq i \leq N}$ for some traffic plan $\mathbf{P}_N$ with sources $(x_i)_{1 \leq i \leq N}$.

Finally, we remark that the number of points such that $d(x_i, \partial\Omega) > cN^{-1/d}$ is $N + O(N^{1-1/d})$ because the other points convey a mass $\approx N^{-1}$ by Proposition 5.4, and their basins are included in $\{d(\cdot, \partial\Omega) < C'N^{-1/d}\}$ for some C', a set of volume $\approx N^{-1/d}$. However, without extra assumptions on Ω, some points x_i may very well belong to $\mathbb{R}^d \setminus \bar{\Omega}$. Thus is ruled out for example when Ω is convex, but then it is not guaranteed that $x_i \in \Omega$ for all x_i's for general Ω.

Proof of Proposition 5.2. The outer ball property (5.12) holds with $C := c_2$, the constant in (c) in the proof of (5.1). For the inner ball property (5.13), assume that $d(x_i, \mathrm{Bas}(\mathbf{P}_N, x_j)) \leq \varepsilon N^{-1/d}$ for some $i \neq j$. We shall find a lower bound on ε that depends only on $(\alpha, d, c_A, C_A, \mathrm{diam}(\nu))$. If $\varepsilon \leq c_1/2$ where c_1 is the separation constant in (5.2), then taking a point $x \in B(x_i, \varepsilon N^{-1/d}) \cap \mathrm{Bas}(\mathbf{P}_N, x_j)$, we have

$$d(x, x_j) \geq d(x_i, x_j) - d(x_i, x) \geq (c_1/2)N^{-1/d},$$

thus taking γ_j a $\mathbf{P}_N$-good curve from x_j to x, its length is greater than $(c_1/2)N^{-1/d}$ hence

$$z_{\mathbf{P}_N}(x) = \int_{\gamma_j} \Theta_{\mathbf{P}_N^{x_j}}^{\alpha-1} \geq (c_1/2)N^{-1/d} \nu(\mathrm{Bas}(\mathbf{P}_N, x_j))^{\alpha-1} \geq \frac{1}{2} c_1 (C_A c_2)^{\alpha-1} N^{-\beta/d}.$$

As a consequence, setting $c' := \frac{1}{2} c_1 (C_A c_2)^{\alpha-1}$ and $c'' := (c'/2C_H)^{1/\beta}$, for every $y \in B(x, c''N^{-1/d}) \cap \mathrm{spt}\, \nu$, by Theorem 4.1

$$z_{\mathbf{P}_N}(y) \geq z_{\mathbf{P}_N}(x) - C_H |y - x|^\beta \geq c'/2N^{-\beta/d}.$$

If $\varepsilon \leq c''$ we build a competitor $\mathbf{P}'_N$ by removing from $\mathbf{P}_N$ all curves going to $B(x, \varepsilon N^{-1/d})$, and adding an optimal traffic plan $Q \in \mathbf{TP}(m_\varepsilon \delta_{x_i}, \nu \llcorner B(x, \varepsilon N^{-1/d})$ where $m_\varepsilon := \nu(B(x, \varepsilon N^{-1/d}))$. Using the first variation formula Proposition 4.2 (C) and the subadditivity of the α-mass, we get by optimality of μ_N

$$\begin{aligned} \mathbb{M}^\alpha(\mathbf{P}_N) \leq \mathbb{M}^\alpha(\mathbf{P}'_N) &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - \alpha \int_{B(x, \varepsilon N^{-1/d})} z_{\mathbf{P}_N} \, \mathrm{d}\nu + C_{\mathsf{BOT}} (2\varepsilon N^{-1/d})(C_A \varepsilon^d N^{-1})^\alpha \\ &\leq \mathbb{M}^\alpha(\mathbf{P}_N) - (\alpha c_A c'/2) \varepsilon^d N^{-(\alpha + \frac{1}{d})} + 2 C_{\mathsf{BOT}} C_A^\alpha \varepsilon^{1+d\alpha} N^{-(\alpha + \frac{1}{d})}. \end{aligned}$$

We get a contradiction when ε is smaller than some constant $c''' > 0$ (depending only on $(\alpha, c_a, c', C_{\mathsf{BOT}}, C_A)$) because $d < 1 + d\alpha$. During the reasoning we made, recall that we assumed $\varepsilon \leq c_1/2$ then $\varepsilon \leq c''$, thus we must have:

$$\varepsilon \geq c := \min\{c_1/2, c'', c'''\},$$

and (5.13) holds with this constant $c > 0$. ∎

5.3 Uniformity bounds for masses and energies

Proposition 5.4. *Let $\nu \in \mathscr{M}^+(K)$ be a d-Ahlfors regular measure with constants c_A, C_A. Let $\mu_N = \sum_{i=1}^N m_i \delta_{x_i}$ be an N-point optimal quantizer (i.e. a solution to (2.17)) and $\mathbf{P}_N \in \mathbf{OTP}^\alpha(\mu_N, \nu)$. There are constants $c, C > 0$ depending only on (α, d, c_A, C_A) such that for all $i \in \{1, \ldots, N\}$,*

$$cN^{-1} \leq m_i \leq CN^{-1}$$

and

$$cN^{-(\alpha+1/d)} \leq \mathbf{d}^\alpha(m_i\delta_{x_i}, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \leq CN^{-(\alpha+1/d)}.$$

Proof. The upper bounds come from the fact that, by Theorem 5.1, for every $i \in \{1, \ldots, N\}$, $\mathrm{Bas}(\mathbf{P}_N, x_i)$ has diameter less than $c_2 N^{-1/d}$, thus

$$m_i = \nu(\mathrm{Bas}(\mathbf{P}_N, x_i)) \leq C_A c_2^d N^{-1},$$

and thus from the usual estimate of branched transport cost we get

$$\mathbf{d}^\alpha(m_i\delta_{x_i}, \nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \leq C_{\mathsf{BOT}}(c_2 N^{-1/d})(C_A c_2^d N^{-1})^\alpha.$$

The lower bounds results from Proposition 5.2, which implies that (c, C being the inner and outer ball constants):

$$\mu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i) \geq \mu \llcorner B(x_i, cN^{-1/d}),$$

and thus

$$\mu(\mathrm{Bas}(\mathbf{P}_N, x_i)) \geq c_A c^d N^{-1}.$$

Since μ is d-Ahlfors regular, it may be written $\mu = f\mathscr{L}^d \llcorner X$ with $c_A/\omega_d \leq f \leq C_A/\omega_d$ on some Borel set $X \subseteq \operatorname{spt}\nu$. Therefore, $C_A^{-1}\nu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)$ is a measure which is absolutely continuous with respect to Lebesgue, with density in $[0, 1]$, and total mass greater than $m := c_A/C_A c^d N^{-1}$, thus

$$\mathbf{d}^\alpha(m_i \delta x_i, \mu \llcorner \mathrm{Bas}(\mathbf{P}_N, x_i)) \geq C_A^\alpha e_{\alpha,d} m^{\alpha+\frac{1}{d}},$$

where $e_{\alpha,d} > 0$ is the constant from the optimal shape problem studied in [PSX19], whose definition is given in (2.30). □

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