INTRINSIC REGULAR SUBMANIFOLDS IN HEISENBERG GROUPS ARE DIFFERENTIABLE GRAPHS

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ABSTRACT. We characterize intrinsic regular submanifolds in the Heisenberg group as intrinsic differentiable graphs.

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1. INTRODUCTION

The notion of rectifiable set is a key one in calculus of variations and in geometric measure theory. To develop a satisfactory theory of rectifiable sets inside Carnot groups has been the object of much research in the last ten years (see e.g. [2], [3], [8], [9], [14], [16], [17], [26], [27], [28], [33]).

Rectifiable sets, in Euclidean spaces, are natural generalizations of C^1 submanifolds; moreover they are often defined, (but for a negligeable set), as a countable union of compact subsets contained in C^1 submanifolds.

Hence, understanding the objects that, inside Carnot groups, naturally take the role of C^1 submanifolds is a preliminary task in developing a satisfactory theory of rectifiable sets inside Carnot groups.
In this paper we consider functions acting between complementary subgroups of a given Carnot group $\mathbb{G}$ and, for them, we introduce the notions

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of intrinsic Lipschitz continuity and intrinsic differentiability. After we use these notions to characterize, inside Heisenberg groups $\mathbb{H}^n$, *intrinsic* C^1 submanifolds as, locally, *intrinsic differentiable graphs.*

Intrinsic graphs came out naturally in [14], (see also [8]), while studying level sets of Pansu differentiable functions from $\mathbb{H}^n$ to $\mathbb{R}$. They gave the possibility of proving an implicit function theorem for these level sets, that indeed are, locally, intrinsic graphs, (see Theorem 2.9 and also [17], [19], [20]). The simple idea of intrinsic graph is the following one: let $\mathbb{G}_1$ and $\mathbb{G}_2$ be *complementary subgroups* of a Carnot group $\mathbb{G}$, that is homogeneous subgroups, such that $\mathbb{G}_1 \cap \mathbb{G}_2 = e$ and $\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2$ (here $\cdot$ indicates the group operation in $\mathbb{G}$ and $e = (0, \dots, 0)$ is the unit element), then the intrinsic (left) graph of $f : \mathbb{G}_1 \to \mathbb{G}_2$ is the set

$$\operatorname{graph}(f) = \{g \cdot f(g) : g \in \mathbb{G}_1\}.$$

In this case we say that $\operatorname{graph}(f)$ is a graph over $\mathbb{G}_1$ in direction $\mathbb{G}_2$. More generally, we say (Definition 3.6) that a subset S of a Carnot group $\mathbb{G}$, is a *(left) intrinsic graph*, in direction of a homogeneous subgroup $\mathbb{H}$, if S intersects each left coset of $\mathbb{H}$ in at most a single point.

The notions of intrinsic Lipschitz continuity and of intrinsic differentiability - for functions acting between complementary subgroups of a Carnot group $\mathbb{G}$ - are given, as follows, trying to respect the geometric structure of the ambient space $\mathbb{G}$.

A function $f : \mathbb{G}_1 \to \mathbb{G}_2$ is said to be *intrinsic Lipschitz* (Definition 3.12) if it is possible to put, at each point $p \in \operatorname{graph}(f)$, an intrinsic cone (Definition 3.10), with vertex p, axis $\mathbb{G}_2$ and fixed opening, intersecting $\operatorname{graph}(f)$ only at p.

A function $f : \mathbb{G}_1 \to \mathbb{G}_2$ is *intrinsic differentiable at* $g \in \mathbb{G}_1$ if there is a homogeneous subgroup $\mathbb{H}$ of $\mathbb{G}$ such that, in the point $p = g \cdot f(g) \in \operatorname{graph}(f)$, the left coset $p \cdot \mathbb{H}$ is the limit of group dilations of $\operatorname{graph}(f)$ centered in p, or, in other words, if $p \cdot \mathbb{H}$ is the tangent plane to $\operatorname{graph}(f)$ in p (Definition 3.17). A uniform version of intrinsic differentiability is introduced in Definition 3.20.

Let us come now to intrinsic C^1 surfaces. In $\mathbb{H}^n$, or sometimes in more general Carnot groups, a class of surfaces that proved themselves to be a good generalization, to the group setting, of C^1 submanifolds are the so called H-regular submanifolds, (see Definition 4.1 and the references [17], [2], [8], [27], [33] and [37]).

In $\mathbb{H}^n$, H-regular submanifolds are defined in different ways according to their topological dimension k. Precisely, if $k \leq n$, a k-dimensional or *low dimensional* H-regular submanifold is, locally, the image in $\mathbb{H}^n$ of an open set of $\mathbb{R}^k$, through an injective, Pansu differentiable function; while a k-codimensional, or *low codimensional*, H-regular submanifold is, locally, the non critical level set of a Pansu differentiable function $\mathbb{H}^n \to \mathbb{R}^k$.

The surfaces contained in these two classes are very different from each other; indeed low dimensional H-regular surfaces are Legendrian, euclidean C^1 submanifolds, (see [17], Theorem 3.5), while the low codimensional ones can be very irregular, even fractals, from an euclidean point of view (see [17] and [23]). Nevertheless H-regular submanifolds can, very reasonably, be considered as C^1 submanifolds because (i) they have a tangent plane at each point, the tangent plane being the coset of a homogeneous subgroup of the ambient space $\mathbb{H}^n$, (ii) the tangent planes depend continuously on the point, (iii) they have locally finite Hausdorff measures, that can also be obtained by integration with appropriate area type formulas (see [17]).

In this paper we show, in our main result (see Theorem 4.3), a common characterization of H-regular surfaces, both low dimensional and low codimensional, proving that they are *uniformly, intrinsic differentiable graphs* of functions acting between *complementary homogeneous subgroups* of $\mathbb{H}^n$.

Our proof of equivalence of uniformly intrinsic differentiable graphs and of intrinsic C^1 submanifolds in $\mathbb{H}^n$, suggests also that *uniformly intrinsic differentiable* functions, acting between complementary subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$, are, inside $\mathbb{G}$, the group version of euclidean C^1 functions from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$ inside $\mathbb{R}^n$.

Finally, it seems to us that describing regular submanifolds as (intrinsic differentiable) graphs is more general and flexible than using parametrizations or level sets. Indeed, differently from $\mathbb{R}^n$ - where d-dimensional C^1 embedded submanifolds are *equivalently* defined as non-critical level sets of differentiable functions $\mathbb{R}^n \to \mathbb{R}^{n-d}$ or as images of injective differentiable maps $\mathbb{R}^d \to \mathbb{R}^n$ (or as *graphs of C^1 functions* $\mathbb{R}^d \to \mathbb{R}^{n-d}$) - in $\mathbb{H}^n$, low dimensional H-regular surfaces cannot be seen as non critical level sets and low codimensional ones cannot be seen as (bilipschitz) images of open sets. The reasons for this are rooted in the algebraic structure of $\mathbb{H}^n$; indeed, low dimensional horizontal subgroups of $\mathbb{H}^n$ are not normal subgroups, hence they cannot appear as kernels of homogeneous homomorphisms $\mathbb{H}^n \to \mathbb{R}^{n-d}$; on the other side, injective homogeneous homomorphism $\mathbb{R}^d \to \mathbb{H}^n$ do not exist, if $d \geq n+1$ (see [2] and [26]).

We recall that the class of uniformly differentiable functions has been studied, with different names and approaches, for 1-codimensional graphs, in [1], [8] and in [37]. In one section of this paper we tried to explain some of the connections between the two classes of functions.

We hope that the here taken approach to the study of H-regular submanifolds in $\mathbb{H}^n$, might prove itself to be useful also for defining intrinsic C^1 submanifolds in more general Carnot groups. With this aim, we made the effort of writing all statements and proofs in a coordinate free fashion. We hope that this will show, more clearly, how some of the concepts here discussed can find their natural setting in Carnot groups more general than Heisenberg groups.

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2. Notations and Preliminaries

For a general review on Carnot and Heisenberg groups see [4], [12], [13], [21], [22], [32], [34], [35], [36] and the recent ones [6] and [7]. Here we limit ourselves to fix some notations.

2.0.1. *Carnot groups.* A *graded group* of step k (see [21] or [32]) is a connected, simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$, of dimension n, is the direct sum of k subspaces $\mathfrak{g}_i$, of dimension m_i, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, such that

$$(1) \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for $1 \le i, j \le k$ and $\mathfrak{g}_i = 0$ for $i > k$.

A *Carnot group* $\mathbb{G}$ of step k is a graded group of step k, whose Lie algebra satisfies also

$$(2) \qquad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}.$$

for $i = 1, \dots, k$. That is, $\mathfrak{g}_1$ generates all the algebra.

The exponential map is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Let $X_1, \dots, X_n$ be a basis for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a basis for $\mathfrak{g}_1$ and, for $1 < j \le k$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a basis for $\mathfrak{g}_j$. Then any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1 X_1 + \dots + p_n X_n)$ and we can identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$. The explicit expression of the group operation $\cdot$, determined by the Campbell-Hausdorff formula (see [6] or [12]), has the form

$$(3) \qquad x \cdot y = x + y + \mathcal{Q}(x, y), \qquad \forall x, y \in \mathbb{R}^n$$

where $\mathcal{Q}(x,y) = (\mathcal{Q}_1(x,y), \dots, \mathcal{Q}_n(x,y)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Here, $Q_i(x,y) = 0$, for each $i = m_0, \dots, m_1$ $(m_0 = 1)$ and, for each $1 < j \le k$ and $m_{j-1} + 1 \le i \le m_j$, we have, $Q_i(x,y) = Q_i\big(x_1, \dots, x_{m_{j-1}}, y_1, \dots, y_{m_{j-1}}\big)$. Moreover, each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$.
If $p \in \mathbb{G}$, $p^{-1} = (-p_1, \dots, -p_n)$ is the inverse of p and $e = (0, \dots, 0)$ is the identity of $\mathbb{G}$.

If $\mathbb{G}$ is a graded group, for any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is an automorphism of $\mathbb{G}$ defined as

$$(4) \qquad \delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, ..., \lambda^{\alpha_n} x_n),$$

where the *homogeneity* of x_i is $\alpha_i \in \mathbb{N}$ and $\alpha_i = j$ if $m_{j-1} < i \le m_j$ (see [13] Chapter 1).

We shall use the following homogeneous norm and distance on $\mathbb{G}$

$$(5) \qquad d(x,y) = d(y^{-1} \cdot x, 0) = \left\| y^{-1} \cdot x \right\|,$$

where, if $p = (p_1, \dots, p_k) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k} = \mathbb{R}^n = \mathbb{G}$, then

$$(6) \qquad \|p\| = \max\{\varepsilon_j ||p_j||_{\mathbb{R}^{m_j}}^{1/\alpha_j}, j = 1, \dots, k\}.$$

Here $\varepsilon_1 = 1$, and $\varepsilon_2, \dots \varepsilon_k \in (0, 1]$ are suitable positive constants depending on $\mathbb{G}$ (see Theorem 5.1 of [16]).

The distance d is comparable with the Carnot Carathèodory distance of $\mathbb{G}$ and is well behaved with respect to left translations and dilations, that is

$$(7) \qquad d(z \cdot x, z \cdot y) = d(x, y) \quad , \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$. For $r > 0$ and $p \in \mathbb{G}$, we denote by $B(p, r)$ the open ball associated with d.

A *homogeneous subgroup* of a Carnot group $\mathbb{G}$ (see [34] 5.2.4) is a subgroup $\mathbb{H}$ such that, for all $g \in \mathbb{H}$ and $\lambda > 0$

$$\delta_\lambda g \in \mathbb{H}.$$

The *(linear) dimension* of a (sub)group is the dimension of its Lie algebra. The *metric dimension* of a subgroup, or of a subset, of $\mathbb{G}$ is the Hausdorff dimension, where the Hausdorff measures are constructed from the distance given in (5).

Any homogeneous subgroup of a Carnot group $\mathbb{G}$ of step k is, necessarily, a graded group, of step at most k, but in general it is not a Carnot group.

We will consider each homogeneous subgroup $\mathbb{H}$ of a group $\mathbb{G}$ as a metric space by restricting to $\mathbb{H}$ the distance defined in $\mathbb{G}$.

2.0.2. *Heisenberg groups.* The n-dimensional Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted as $p = (p_1, \dots, p_{2n}, p_{2n+1}) \in \mathbb{R}^{2n+1}$.
For $p, q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = \big(p_1 + q_1, \dots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{i=1}^{n}(p_i q_{i+n} - p_{i+n} q_i)\big).$$

For $\lambda > 0$, non isotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ are the automorphisms of the group defined as

$$\delta_\lambda p := (\lambda p_1, \dots, \lambda p_{2n}, \lambda^2 p_{2n+1}).$$

The Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ is spanned by the left invariant vector fields $X_1, \cdots, X_n, Y_1, \cdots, Y_n, T$, where

$$X_i(p) := \partial_i - \frac{1}{2} p_{i+n}\partial_{2n+1}, \quad Y_i(p) := \partial_{i+n} + \frac{1}{2} p_i \partial_{2n+1}, \quad T(p) := \partial_{2n+1},$$

for $i = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n, Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{8}$$

and we have

$$[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2. \tag{9}$$

The Lie algebra $\mathfrak{h}$ is also endowed with a scalar product $\langle \cdot, \cdot \rangle$ making the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$, T orthonormal.

The centre of $\mathbb{H}^n$ is the subgroup $\mathbb{T} := \exp(\mathfrak{h}_2) = \{(0, \dots, 0, p_{2n+1})\}$.

The *horizontal bundle* $H\mathbb{H}^n$ is the subbundle of the tangent bundle $T\mathbb{H}^n$ whose fibers $H\mathbb{H}^n_p$ are spanned by the horizontal vectors $X_1(p), \cdots, Y_n(p)$. The scalar product $\langle \cdot, \cdot \rangle$ induces naturally on each fiber $H\mathbb{H}^n_p$ a scalar product here denoted as $\langle \cdot, \cdot \rangle_p$.

If $p \in \mathbb{H}^n$, the homogeneous norm in (6) becomes

$$\|p\| := \max\{\|(p_1, \cdots, p_{2n})\|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$$

while the distance d is defined as in (5).

We define also the map $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ as

$$\pi(p) = \pi(p_1, \cdots, p_{2n}, p_{2n+1}) := (p_1, \cdots, p_{2n}).$$

Notice that any $p \in \mathbb{H}^n$ can be uniquely written as

$$p = (\pi(p), p_{2n+1}) = (\pi(p), 0) \cdot p_{\mathbb{T}}$$

where $p_{\mathbb{T}} = (0, \cdots, 0, p_{2n+1}) \in \mathbb{T}$ and $(\pi(p), 0) \in H\mathbb{H}^n_e$.

Proposition 2.1. *All homogeneous subgroups of $\mathbb{H}^n$ are either horizontal, that is contained in the horizontal fiber $H\mathbb{H}^n_e$, or vertical, that is containing the subgroup $\mathbb{T}$. A horizontal subgroup $\mathbb{V}$ has linear dimension and metric dimension k, with $1 \leq k \leq n$; moreover $\mathbb{V}$ is algebraically isomorphic and isometric to $\mathbb{R}^k$. A vertical subgroup $\mathbb{W}$ can have any dimension d, with $1 \leq d \leq 2n+1$, and its metric dimension is $d+1$.*

Proof. Observe that, $\mathbb{V} \subset \mathbb{H}^n$ is a homogeneous subgroup of $\mathbb{H}^n$, if and only if, $\mathbb{V} = \exp \mathfrak{v}$, where $\mathfrak{v}$ is a homogeneous subalgebra of $\mathfrak{h}$. Then, there exist linearly independent $v_1, \dots, v_k \in \mathfrak{h}$, with $1 \leq k \leq 2n+1$, such that $\mathfrak{v} := \mathrm{span}\,(v_1, \dots, v_k)$ and it must be $[v_i, v_j] \in \mathfrak{v}$, for each $i, j = 1, \dots, k$. It follows that, if $\mathbb{V}$ is horizontal, that is, if $v_i \in \mathfrak{h}_1$, for each $i = 1, \dots, k$, then necessarily we have $[v_i, v_j] = 0$ for each $i, j = 1, \dots, k$ and it must be $k \leq n$. Otherwise, suppose there exists $v \in \mathfrak{h}_1$, such that $v + T \in \mathfrak{v}$. Then both $\lambda v + \lambda T \in \mathfrak{v}$ and $\lambda v + \lambda^2 T \in \mathfrak{v}$, yielding that $T \in \mathfrak{v}$. Finally, observe that, if $\mathbb{V}$ is a horizontal subgroup with $\dim \mathfrak{v} = k$, then it is isomorphic and also isometric to $\mathbb{R}^k$, for, in this case, if $x, y \in \mathbb{V}$, the points $x \cdot \delta_\lambda(x^{-1} \cdot y) \in \mathbb{V}$ for each $0 \leq \lambda \leq 1$, form an horizontal segment connecting them. On the contrary, if $\mathbb{W}$ is a vertical subgroup with $\dim \mathfrak{w} = k$, then, in general, $\mathbb{W}$ is not isomorphic to $\mathbb{R}^k$ and is never isometric to $\mathbb{R}^k$, having metric dimension equal to $k+1$ (see [29], Theorem 2). $\square$

2.0.3. *Calculus.* The following notion of differentiability, for functions acting between graded groups, was introduced by Pansu in [32].

Definition 2.2. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graded groups endowed with homogeneous norms $\|\cdot\|_1$ and $\|\cdot\|_2$. A function $L : \mathbb{G}_1 \to \mathbb{G}_2$ is said to be *H-linear* or *horizontal linear* (the name was introduced in [26]), if L is a *homogeneous homomorphism*, that is if L is a group homomorphism and if, for all $g \in \mathbb{G}_1$ and for all $\lambda > 0$,

$$L(\delta^1_\lambda g) = \delta^2_\lambda L(g).$$

Here δ^i_λ is the group dilation in $\mathbb{G}_i$. The set of all H-linear function between $\mathbb{G}_1$ and $\mathbb{G}_2$ can be endowed with the norm

$$\|L\| = \sup\{\|L(g)\|_2 : \|g\|_1 \leq 1\}.$$

Definition 2.3. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graded groups endowed with homogeneous norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is said to be *P-differentiable* or *Pansu differentiable* in $g_0 \in \mathcal{A}$ if there is a H-linear function $df_{g_0} : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\lim_{g \to g_0} \frac{\| \left(df_{g_0}(g_0^{-1} \cdot g) \right)^{-1} \cdot f(g_0)^{-1} \cdot f(g) \|_2}{\|g_0^{-1} \cdot g\|_1} = 0.$$

The H-linear function df_{g_0} is called the P-differential of f. Analogously, $f : \mathcal{A} \to \mathbb{G}_2$ is said to be *continuosly P-differentiable* in $\mathcal{A}$ and we write

$$f \in C^1_H(\mathcal{A}, \mathbb{G}_2)$$

if it is P-differentiable in every $g \in \mathcal{A}$ and if the P-differential df_g depends continuously on x.

We have a characterization of real valued, P-differentiable functions due to Pansu (see [32]) and an explicit representation of P-differentials in terms of horizontal derivatives.

Proposition 2.4. *Let $\mathcal{U}$ be open in $\mathbb{H}^n$ and $f : \mathcal{U} \to \mathbb{R}$ be continuous. Then*

$$f \in C^1_H(\mathcal{U}) := C^1_H(\mathcal{U}, \mathbb{R})$$

if and only if, for $i = 1, \dots, n$, the distributional derivatives $X_i f$, $Y_i f$ are continuous in $\mathcal{U}$. Moreover

$$g = \left(g^1, \cdots, g^k\right) \in C^1_H(\mathcal{U}, \mathbb{R}^k)$$

if and only if $g^j \in C^1_H(\mathcal{U})$, for $j = 1, \cdots, k$.

Remember that $C^1(\mathcal{U}) \subset C^1_H(\mathcal{U})$ and that the inclusion is strict (see [14], Remark 5.9).

Proposition 2.5. *Let $\mathcal{U}$ be open in $\mathbb{H}^n$ and $f \in C^1_H(\mathcal{U})$. We define the* horizontal gradient $\nabla_H f : \mathcal{U} \to \mathbb{R}^{2n}$ *as the continuous function*

$$\nabla_H f := (X_1 f, \cdots, Y_n f)$$

or, equivalently, as the continuous function $\nabla_H f : \mathcal{U} \to H\mathbb{H}^n$

$$\nabla_H f := \sum_{i=1}^n (X_i f) X_i + (Y_i f) Y_i.$$

Then for all $p, p_0 \in \mathcal{U} \subset \mathbb{H}^n$,

$$\begin{aligned} f(p) &= f(p_0) + \langle \nabla_H f(p_0), \pi(p_0^{-1} \cdot p) \rangle_{\mathbb{R}^{2n}} + o(d(p, p_0)) \\ &= f(p_0) + df_{p_0}(p_0^{-1} \cdot p) + o(d(p, p_0)), \text{ as } d(p, p_0) \to 0. \end{aligned} \tag{10}$$

Proposition 2.6. *If $f \in C^1_H(\mathcal{U}, \mathbb{R}^k)$, then f is P-differentiable in $\mathcal{U}$ and*

$$df_{p_0}(p) = (J_H f)_{p_0}(\pi(p)),$$

where $J_H f$ is the horizontal Jacobian *of f. That is $J_H f$ is the $k \times 2n$ matrix valued function whose rows are the horizontal gradients of the components f^i of f.*

The following characterizations of H-linear functions are proved in [26] (see also [17] and [14]).

Proposition 2.7. *Let $1 \le k \le n$, and let $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ be the $2n \times 2n$ symplectic matrix. Then*

(i) *$L : \mathbb{R}^k \to \mathbb{H}^n$ is H-linear if and only if there is a $2n \times k$ matrix A with $A^T J A = 0$ such that, for all $x \in \mathbb{R}^k$,*

$$L(x) = (Ax, 0).$$

(ii) *$L : \mathbb{H}^n \to \mathbb{R}^k$ is H-linear, if and only if there is a $k \times 2n$ matrix A, such that for all $p \in \mathbb{H}^n$*

$$L(p) = A\,\pi(p)^t.$$

The following theorems, a Taylor's inequality, an implicit function theorem and a Whitney's extension theorem, are proved respectively in [13], in [17] and in [14] (see also [15]).

Theorem 2.8. *Let $\mathcal{U}$ be open in $\mathbb{H}^n$. Then there are $c = c(\mathbb{H}^n) > 1$ and $C = C(\mathbb{H}^n, k) > 0$ such that, if $B(p_0, cr) \subset \mathcal{U}$,*

$$\left\| f(q) - f(p) - df_p(p^{-1} \cdot q) \right\|_{\mathbb{R}^k} \leq C \sup_{x \in B(p_0, cr)} \|df_x - df_p\| \cdot d(p, q),$$

for $f \in C^1_H(\mathcal{U}, \mathbb{R}^k)$ and for all p, $q \in B(p_0, r)$.

Theorem 2.9. (Implicit function Theorem) *Let $1 \leq k \leq n$ and S be a k-codimensional H-regular surface. That is we assume that, for each $p_0 \in S$, there is an open neighborhood $\mathcal{U}$ of p_0 and $f : \mathcal{U} \to R^k$ with continuous and surjiective Pansu differential $df : \mathbb{H}^n \to \mathbb{R}^k$ such that*

$$S = \{p \in \mathcal{U} \subset \mathbb{H}^n : f(p) = 0\}.$$

Then, possibly choosing a smaller $\mathcal{U} \ni p_0$, there are homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$, with $\mathbb{V}$ k-dimensional, horizontal and $\mathbb{W}$ normal, such that

$$\begin{aligned} &\mathbb{W} \cap \mathbb{V} = \{e\} \\ &p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}, \end{aligned} \tag{11}$$

for all $p \in \mathbb{H}^n$, with $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{V}} \in \mathbb{V}$, there is

$$(df_p)_{|\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k \text{ one to one for all } p \in \mathcal{U}; \tag{12}$$

moreover, there are a relative open set $\mathcal{A} \subset \mathbb{W}$ and a continuous $\varphi : \mathcal{A} \to \mathbb{V}$ such that

$$S \cap \mathcal{U} = \{w \cdot \varphi(w),\ w \in \mathcal{A}\}.$$

Finally, there is a constant $L > 0$ such that for all $\bar{w}, w \in \mathcal{A}$ we have

$$\left\| \varphi(\bar{w})^{-1} \cdot \varphi(w) \right\| \leq L \left\| \varphi(\bar{w})^{-1} \cdot (\bar{w}^{-1} \cdot w) \cdot \varphi(\bar{w}) \right\|. \tag{13}$$

Theorem 2.10. *Let $\mathcal{F} \subset \mathbb{H}^n$ be closed. Given $f : \mathcal{F} \to \mathbb{R}^k$ continuous and $Q : \mathcal{F} \to \mathbb{R}^{k,2n}$ continuous and matrix valued, we define*

$$R(p, q) := \frac{f(q) - f(p) - Q_p \pi(p^{-1} \cdot q)^t}{d(p, q)},$$

for all $p, q \in \mathcal{F}$. Given $\mathcal{K} \subset \mathcal{F}$, compact and $\delta > 0$, we define

$$\rho_{\mathcal{K}}(\delta) := \sup\{\|R(p, q)\|_{\mathbb{R}^k} : p, q \in \mathcal{K}, 0 < d(p, q) < \delta\}.$$

Then, if for all compact $\mathcal{K} \subset \mathcal{F}$,

$$\rho_{\mathcal{K}}(\delta) \to 0 \text{ as } \delta \to 0,$$

there exists a function $\bar{f} \in C^1_H(\mathbb{H}^n, \mathbb{R}^k)$ such that, denoted by $J_H \bar{f}$ the horizontal Jacobian matrix of $\bar{f}$,

$$\bar{f}(p) = f(p), \qquad J_H \bar{f}(p) = Q_p,$$

for all $p \in \mathcal{F}$.

3. Intrinsic graphs

3.1. **Complementary subgroups and graphs.**

Definition 3.1. We say that two homogeneous subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of a Carnot group $\mathbb{G}$ are *complementary subgroups* in $\mathbb{G}$, and we write

$$\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2,$$

if $\mathbb{G}_1 \cap \mathbb{G}_2 = \{e\}$ and if, for all $g \in \mathbb{G}$, there are $g_{\mathbb{G}_1} \in \mathbb{G}_1$ and $g_{\mathbb{G}_2} \in \mathbb{G}_2$ such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}$.

If $\mathbb{G}_1$, $\mathbb{G}_2$ are complementary subgroups in $\mathbb{G}$ and one of them is a normal subgroup we say that $\mathbb{G}$ is a *semidirect product* of $\mathbb{G}_1$ and $\mathbb{G}_2$.

Example 3.2. Let $\mathbb{G} = \mathbb{H}^n$ and

$$\mathbb{V} = \{(p_1, 0, \dots, 0))\}, \quad \mathbb{W} = \{(0, p_2, \dots, p_{2n+1})\}.$$

Then $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ and the product is semidirect.

Example 3.3. $\mathbb{E} = (\mathbb{R}^4, \cdot)$ with the group law defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + (x_1 y_2 + x_2 y_1)/2 \\ x_4 + y_4 + [(x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2)]/2 \\ +(x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1)/12 \end{pmatrix} \tag{14}$$

and the family of dilation

$$\delta_\lambda(p_1, p_2, p_3, p_4) = (\lambda p_1, \lambda p_2, \lambda^2 p_3, \lambda^3 p_4). \tag{15}$$

The subgroups

$$\begin{aligned} \mathbb{G}_1 &= \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\} \\ \mathbb{G}_2 &= \{(0, x_2, x_3, x_4) : x_2, x_3, x_4 \in \mathbb{R}\}, \end{aligned} \tag{16}$$

are complementary subgroups in $\mathbb{E}$, $\mathbb{G}_2$ is a normal subgroup, hence $\mathbb{E}$ is the semidirect product of $\mathbb{G}_1$ and $\mathbb{G}_2$.

Another couple of complementary subgroups is given by

$$\begin{aligned} \mathbb{E}_1 &= \{(x_1, -x_1, x_3, 0) : x_1, x_3 \in \mathbb{R}\} \\ \mathbb{E}_2 &= \{(x_1, 0, 0, x_4) : x_1, x_4 \in \mathbb{R}\}. \end{aligned} \tag{17}$$

Neither of them is a normal subgroup, indeed a direct computation shows

$$p^{-1} \cdot x \cdot p = \big(x_1, -x_1, x_3 + x_1(p_1 + p_2), -(p_1 + p_2)(2x_3 + x_1(p_1 + p_2)/2)\big)$$

for $x \in \mathbb{E}_1$ and $p \in \mathbb{E}$; hence $p^{-1} \cdot x \cdot p \notin \mathbb{E}_1$; analogously

$$p^{-1} \cdot x \cdot p = \big(x_1, 0, x_1 p_2, x_4 - x_1(p_1 p_2 + p_2^2 - 2p_3)/2)\big)$$

for $x \in \mathbb{E}_2$ and $p \in \mathbb{E}$; hence $p^{-1} \cdot x \cdot p \notin \mathbb{E}_2$. Hence $\mathbb{E} = \mathbb{E}_1 \cdot \mathbb{E}_2$, but the product is not semidirect.

Proposition 3.4. *If $\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2$ as in Definition 3.1, each $g \in \mathbb{G}$ has unique* components *$g_{\mathbb{G}_1} \in \mathbb{G}_1$, $g_{\mathbb{G}_2} \in \mathbb{G}_2$, such that*

$$g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}.$$

The maps

$$g \to g_{\mathbb{G}_1} \text{ and } g \to g_{\mathbb{G}_2}$$

are continuous and there is a constant $c = c(\mathbb{G}_1, \mathbb{G}_2) > 0$ *such that*

$$c\left(\|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|\right) \le \|g\| \le \|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|\,. \tag{18}$$

Proof. Assume there are $g_{\mathbb{G}_1}, g'_{\mathbb{G}_1}, g_{\mathbb{G}_2}, g'_{\mathbb{G}_2}$, such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2} = g'_{\mathbb{G}_1} \cdot g'_{\mathbb{G}_2}$. Then $(g'_{\mathbb{G}_1})^{-1} \cdot g_{\mathbb{G}_1} = g'_{\mathbb{G}_2} \cdot (g_2)^{-1} = e$. Hence $g'_{\mathbb{G}_1} = g_{\mathbb{G}_1}$ and $g'_{\mathbb{G}_2} = g_{\mathbb{G}_2}$.

The continuity of the maps $g \to g_{\mathbb{G}_1}$ and $g \to g_{\mathbb{G}_2}$ follows by a direct consideration of the form of the product in $\mathbb{G}$ (see (3)). Indeed, the m_1 components in the first layer of $g_{\mathbb{G}_1}$ and $g_{\mathbb{G}_2}$ are the components of the euclidean projections of the first m_1 components of g, hence depend continuosly on g. This given, the values of the polynomials $Q_{m_1+1}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2}), \cdots, Q_{m_2}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2})$ are determined and are continuosly dependent on g. Now, the components of the second layer are given by the projections of $(g_{m_1+1} - Q_{m_1+1}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2}), \cdots, g_{m_2} - Q_{m_2}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2}))$, and so on.

By homogeneity, it is enough to prove the left hand side of (18) when $\|g\| = 1$, hence it follows by a compactness argument. The right hand side of (18) is just triangular inequality. □

Remark 3.5. Notice that if $\mathbb{G}_1$ and $\mathbb{G}_2$ are complementary subgroups in $\mathbb{G}$, we can write equivalently $\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2$ or $\mathbb{G} = \mathbb{G}_2 \cdot \mathbb{G}_1$. But the components $g_{\mathbb{G}_1}$ and $g_{\mathbb{G}_2}$, of a given element g, depend on the order in which we are considering the two subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$.

Definition 3.6. If $\mathbb{G}_1$ and $\mathbb{G}_2$ are complementary subgroups of $\mathbb{G}$ we denote $\xi \cdot \mathbb{G}_2$, $\xi \in \mathbb{G}_1$, a left coset of $\mathbb{G}_2$. We say that $S \subset \mathbb{G}$ is a *(left) graph over* $\mathbb{G}_1$ *along* $\mathbb{G}_2$ (or from $\mathbb{G}_1$ to $\mathbb{G}_2$) if

$$S \cap (\xi \cdot \mathbb{G}_2) \text{ contains at most one point,}$$

for all $\xi \in \mathbb{G}_1$. Equivalently, S is a graph if there is a function

$$\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$$

such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}$$

and we say that S *is the graph of* φ, $S = \text{graph}(\varphi)$.

A strong motivation supporting this definition of intrinsic graphs was provided by the implicit function Theorem 2.9. Indeed, in that theorem, it was proved that k-codimensional non critical level sets of C^1_H functions are, locally, intrinsic graphs.

Notice that, from Proposition 3.4, given the complementary subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$, if S is a graph from $\mathbb{G}_1$ to $\mathbb{G}_2$, then the function $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ is uniquely determined.

Remark 3.7. A more general definition of graph inside $\mathbb{G}$ can be considered. Assume that $\mathbb{H}$ is a homogeneous subgroup of $\mathbb{G}$. Even if there exist no complementary subgroup of $\mathbb{H}$ in $\mathbb{G}$, we can say that a set $S \subset \mathbb{G}$ is a *graph along* $\mathbb{H}$ if S intersect each coset of $\mathbb{H}$ in at most one point. Such a notion has been used many times in the literature, particularly inside $\mathbb{H}^n$. Many authors indeed considered sets as $S = \{(x_1, \cdots, y_n, \varphi(x_1, \cdots, y_n))\} \subset \mathbb{H}^n$, with φ real valued, that are graphs along $\mathbb{T}$. We recall that $\mathbb{T}$ has no complementary subgroup in $\mathbb{H}^n$ (see Proposition 3.21).

A trivial but key feature of (left) graphs is their keeping being graphs after dilations and (left) translations. Precisely, if $S = \mathrm{graph}\,(\varphi)$ with $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ then both $\delta_\lambda S$ and $q \cdot S$ are graphs from $\mathbb{G}_1$ to $\mathbb{G}_2$; if $\mathbb{G}$ is also the semidirect product of $\mathbb{G}_1$ and $\mathbb{G}_2$ then the algebraic form of the translated function can be explicitly given (see also [27]).

Proposition 3.8. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$, $\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ and $S = \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\} = \mathrm{graph}\,(\varphi)$. Then, for all $\lambda > 0$, the dilated set $\delta_\lambda S$ is a graph, precisely*

$$\delta_\lambda S = graph(\varphi_\lambda),$$

with $\varphi_\lambda := \delta_\lambda \circ \varphi \circ \delta_{1/\lambda} : \delta_\lambda \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$.

Proof. Just observe that $\delta_\lambda S = \delta_\lambda(\xi \cdot \varphi(\xi)) = \delta_\lambda \xi \cdot \delta_\lambda(\varphi(\xi))$. □

Proposition 3.9. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$, $\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ and $S = \mathrm{graph}\,(\varphi)$, then, for any $q \in \mathbb{G}$, there is $\varphi_q : \mathcal{E}_q \subset \mathbb{G}_1 \to \mathbb{G}_2$, such that*

$$\mathrm{graph}\,(\varphi_q) := q \cdot S = \{\eta \cdot \varphi_q(\eta) : \; \eta \in \mathcal{E}_q\}.$$

where φ_q is as in (19). *The statement can be made more explicit if we assume that $\mathbb{G}$ is the semidirect product of $\mathbb{G}_1$ and $\mathbb{G}_2$. In this case we have,*

(i): *If $\mathbb{G}_1$ is normal in $\mathbb{G}$ then*

$$\mathcal{E}_q := q \cdot \mathcal{E} \cdot (q_{\mathbb{G}_2})^{-1} \subset \mathbb{G}_1,$$

and, for $y \in \mathcal{E}_q$,

$$\varphi_q(y) = q_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_2}^{-1} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot q_{\mathbb{G}_2}).$$

(ii): *If $\mathbb{G}_2$ is normal in $\mathbb{G}$ then*

$$\mathcal{E}_q := q_{\mathbb{G}_1} \cdot \mathcal{E} \subset \mathbb{G}_1$$

and, for $y \in \mathcal{E}_q$,

$$\varphi_q(y) = y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y).$$

Proof. Observe that the map $\tau_q : \mathbb{G}_1 \to \mathbb{G}_1$ defined as $\tau_q(x) := (q \cdot x)_{\mathbb{G}_1}$ is injective. Indeed, from

$$q \cdot x = (q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2}, \quad q \cdot x' = (q \cdot x')_{\mathbb{G}_1} \cdot (q \cdot x')_{\mathbb{G}_2}, \quad (q \cdot x)_{\mathbb{G}_1} = (q \cdot x')_{\mathbb{G}_1}$$

we get $q \cdot x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = q \cdot x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$. Hence $x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$ and finally $x = x'$ because of the uniqueness of the components (see Proposition 3.4). Hence,

$$\begin{aligned} q \cdot S &= \{q \cdot x \cdot \varphi(x) : \; x \in \mathcal{E}\} \\ &= \{(q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2} \cdot \varphi(x) : \; x \in \mathcal{E}\} \\ &= \{y \cdot \varphi_q(y) : \; y \in \mathcal{E}_q\}, \end{aligned}$$

where, $\mathcal{E}_q = \{(q \cdot x)_{\mathbb{G}_1} : x \in \mathcal{E}\}$ and

$$\varphi_q(y) = (q \cdot \tau_q(y)^{-1})_{\mathbb{G}_2} \cdot \varphi(\tau_q(y)^{-1}) \tag{19}$$

for $y = (q \cdot x)_{\mathbb{G}_1} \in \mathcal{E}_q$. This concludes the proof of the first part.

Case (i): Assume $\mathbb{G}_1$ is a normal subgroup. Because

$$q \cdot x = q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot x = q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot x \cdot q_{\mathbb{G}_2}^{-1} \cdot q_{\mathbb{G}_2}$$

then $(q \cdot x)_{\mathbb{G}_1} = q \cdot x \cdot q_{\mathbb{G}_2}^{-1}$. It follows that

$$\mathcal{E}_q = \{q \cdot x \cdot q_{\mathbb{G}_2}^{-1} : x \in \mathcal{E}\},$$

and that $\tau_q(y)^{-1} = q^{-1} \cdot y \cdot q_{\mathbb{G}_2}$ for $y \in \mathcal{E}_q$. Hence

$$\begin{aligned}\varphi_q(y) &= (q_{\mathbb{G}_2} \cdot q^{-1} \cdot y \cdot q_{\mathbb{G}_2})_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2}) \\ &= (q_{\mathbb{G}_1}^{-1} \cdot y \cdot q_{\mathbb{G}_2})_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2}) \\ &= q_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2}),\end{aligned}$$

for $y \in \mathcal{E}_q$.

Case (ii): Assume $\mathbb{G}_2$ is a normal subgroup.
Then $(q \cdot x)_{\mathbb{G}_1} = (q_{\mathbb{G}_1} \cdot x \cdot x^{-1} \cdot q_{\mathbb{G}_2} \cdot x)_{\mathbb{G}_1} = q_{\mathbb{G}_1} \cdot x$. It follows that

$$\mathcal{E}_q = \{q_{\mathbb{G}_1} \cdot x : \; x \in \mathcal{E}\} = q_{\mathbb{G}_1} \cdot \mathcal{E}$$

and that $\tau_q(y)^{-1} = q_{\mathbb{G}_1}^{-1} \cdot y$ for $y \in \mathcal{E}_q$. Hence, for $y \in \mathcal{E}_q$,

$$\begin{aligned}\varphi_q(y) &= (q \cdot q_{\mathbb{G}_1}^{-1} \cdot y)_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y) \\ &= (y \cdot y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y)_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y) \\ &= y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y).\end{aligned}$$

□

3.2. **Intrinsic Lipschitz graphs.** The notion of *intrinsic Lipschitz continuity*, for function acting between complementary subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of $\mathbb{G}$, was originally suggested by (13) (see the definitions given in [19] and in [20]). We propose here an equivalent, more geometric, definition. We say that $f : \mathbb{G}_1 \to \mathbb{G}_2$, is intrinsic Lipschitz continuous if, at each $p \in \text{graph}\,(f)$, there is an (intrinsic) closed cone with vertex p, axis $\mathbb{G}_2$ and fixed opening, intersecting graph (f) only in p. The equivalence of this definition and other ones, more algebraic, is the content of Propositions 3.14 and 3.23.

Notice also that $\mathbb{G}_1$ and $\mathbb{G}_2$ are metric spaces, being subsets of $\mathbb{G}$, hence it makes sense to speak also of *metric Lipschitz continuous* functions $\mathbb{G}_1 \to \mathbb{G}_2$. As usual, we say that $f : \mathbb{G}_1 \to \mathbb{G}_2$ is metric Lipschitz if there is a constant $L > 0$ such that

$$\left\|f(g)^{-1} \cdot f(g')\right\| = d\big(f(g), f(g')\big) \leq L d(g, g') = L \left\|g^{-1} \cdot g'\right\|, \tag{20}$$

for all $g, g' \in \mathbb{G}_1$. The notions of intrinsic Lipschitz continuity and of metric Lipschitz continuity are different ones (see Example 3.24) and we stress here, and will try to convince the reader, that intrinsic Lipschitz continuity seems a more useful notion in the context of functions acting between subgroups of a given Carnot group.

Let us come to the basic definitions. By *intrinsic (closed) cone* we mean

Definition 3.10. Let $\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2$. The closed *cone* $C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha)$ with *base* $\mathbb{G}_1$*, axis* $\mathbb{G}_2$*, vertex* $q \in \mathbb{G}$*, opening* $\alpha > 0$ is

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) := q \cdot C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)$$

where

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha) := \{p \in \mathbb{G} : \|p_{\mathbb{G}_1}\| \leq \alpha \|p_{\mathbb{G}_2}\|\}.$$

Clearly,

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e,0) = \mathbb{G}_2,$$
$$C_{\mathbb{G}_1,\mathbb{G}_2}(q,\alpha) \subset C_{\mathbb{G}_1,\mathbb{G}_2}(q,\beta), \text{ if } 0<\alpha<\beta.$$

Moreover $\cup_{\alpha>0} C_{\mathbb{G}_1,\mathbb{G}_2}(e,\alpha) = (\mathbb{G}\setminus\mathbb{G}_1)\cup\{e\}$

Intrinsic cones are invariant under group dilations. Indeed,

Proposition 3.11. *For all $\alpha, t > 0$,*

$$\delta_t\big(C_{\mathbb{G}_1,\mathbb{G}_2}(e,\alpha)\big) = C_{\mathbb{G}_1,\mathbb{G}_2}(e,\alpha)$$

Proof. By the uniqueness of the components $p_{\mathbb{G}_1}$ and $p_{\mathbb{G}_2}$ of $p\in\mathbb{G}$, we have $(\delta_t p)_{\mathbb{G}_2} = \delta_t(p_{\mathbb{G}_2})$ and $(\delta_t p)_{\mathbb{G}_1} = \delta_t(p_{\mathbb{G}_1})$. Then the assertion follows since $\|\delta_t(p_{\mathbb{G}_2})\| = t\|p_{\mathbb{G}_2}\|$ and $\|\delta_t(p_{\mathbb{G}_1})\| = t\|p_{\mathbb{G}_1}\|$. □

Definition 3.12. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. We say that $f:\mathcal{A}\subset\mathbb{G}_1\to\mathbb{G}_2$ is *intrinsic Lipschitz continuous* in $\mathcal{A}$, if there is $M>0$ such that

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q,1/M)\cap \mathrm{graph}\,(f) = \{q\}, \tag{21}$$

for all $q\in\mathrm{graph}\,(f)$. If f is intrinsic Lipschitz continuous in $\mathcal{A}$, the Lipschitz constant of f in $\mathcal{A}$ is the infimum of the numbers M such that (21) holds. An intrinsic Lipschitz continuous function, with Lipschitz constant not exceeding $L>0$, will be called *L Lipschitz function.*

Remark 3.13. Notice that being intrinsic L Lipschitz is invariant under left translations of the graph. That is

$$f:\mathbb{G}_1\to\mathbb{G}_2 \text{ is } L \text{ Lipschitz, if and only if } f_q:\mathbb{G}_1\to\mathbb{G}_2 \text{ is } L \text{ Lipschitz,}$$

for all $q\in\mathbb{G}$.

We give now algebraic characterizations of intrinsic Lipschitz continuous functions.

Proposition 3.14. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. Then $f:\mathcal{E}\subset\mathbb{G}_1\to\mathbb{G}_2$ is intrinsic Lipschitz continuous in $\mathcal{E}$, if and only if there is $L>0$ such that,*

$$\left\|f_{q^{-1}}(x)\right\| \le L\,\|x\|\,, \tag{22}$$

for all $q\in\mathrm{graph}\,(f)$ and for all $x\in\mathcal{A}_{q^{-1}}$.

Proof. The equivalence of (22) and (21) follows from Definition 3.10 and from (ii) of Proposition 3.9 observing that, if $q\in\mathrm{graph}\,(f)$, then

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q,1/L)\cap\mathrm{graph}\,(f) = \{q\}$$

is equivalent with

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e,1/L)\cap\mathrm{graph}\,(f_{q^{-1}}) = \{e\}.$$

□

Remark 3.15. If $f:\mathcal{E}:\mathbb{G}_1\to\mathbb{G}_2$ is intrinsic Lipschitz continuous, then it is continuous. Indeed, if $f(e)=e$ then, by (22), f is continuous in e. To prove the continuity in a generic $x\in\mathcal{E}$, simply observe that $f_{q^{-1}}$ is continuous in e, where $q = x\cdot f(x)$.

3.3. **Intrinsic differentiable graphs.** We come now to the definition of differentiability - *intrinsic differentiability* - for functions acting between complementary subgroups of $\mathbb{G}$. As usual differentiability amounts to the existence of approximating linear functions. Hence we begin defining *intrinsic linear* functions - acting between complementary subgroups - as functions whose graphs are homogeneous subgroups. After we will give simple algebraic characterizations of this notion.

Definition 3.16. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. We say that $L : \mathbb{G}_1 \to \mathbb{G}_2$ is an *intrinsic linear function* if $\text{graph}\,(L) := \{g \cdot L(g) : g \in \mathbb{G}_1\}$ is an homogeneous subgroup of $\mathbb{G}$.

Notice that $\text{graph}\,(L)$ is a closed set and that intrinsic linear functions are continuous functions from $\mathbb{G}_1$ to $\mathbb{G}_2$.

Given the notion of intrinsic linear function, we say - as usual - that a function $f : \mathbb{G}_1 \to \mathbb{G}_2$, such that $f(e) = e$, is *intrinsic differentiable in* e if there is an intrinsic linear map $L : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\left\|L(g)^{-1} \cdot f(g)\right\| = o(\|g\|), \tag{23}$$

for $g \in \mathbb{G}_1$, as $\|g\| \to 0$, where, with a standard notation, $o(t)/t \to 0$ as $t \to 0^+$.

Up to this point the definition of intrinsic differentiability is the same as the definition of P-differentiability. The differences appear (see Definition 2.3), when we extend the previous notion to any point $\bar{g}$ of $\mathbb{G}_1$ using (23) in a *translation invariant* way. That is, given $\bar{g} \in \mathbb{G}_1$ we consider $\bar{p} = \bar{g} \cdot f(\bar{g})$ and the translated function $f_{\bar{p}^{-1}}$ that, by definition, satisfies $f_{\bar{p}^{-1}}(e) = e$. Now we say that f is intrinsic differentiable in $\bar{g}$ if and only if $f_{\bar{p}^{-1}}$ satisfies (23) (see Definition 3.17). We give also a uniform version of Definition 3.17 in Definition 3.20 and algebraic characterizations of both, when $\mathbb{G} = \mathbb{H}^n$, in Propositions 3.28 and 3.29.

Definition 3.17. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ with $\mathcal{A}$ relatively open in $\mathbb{G}_1$. For $\bar{p} := \bar{g} \cdot f(\bar{g}) \in \text{graph}\,(f)$ we consider the translated function $f_{\bar{p}^{-1}}$ defined in the neighborhood $\mathcal{A}_{\bar{p}^{-1}}$ of e in $\mathbb{G}_1$, (see Proposition 3.8). We say that f is *intrinsic differentiable in* $\bar{g} \in \mathcal{A}$ if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\left\|df_{\bar{g}}(g)^{-1} \cdot f_{\bar{p}^{-1}}(g)\right\| = o(\|g\|), \tag{24}$$

for $g \in \mathcal{A}_{\bar{p}^{-1}}$ and $\|g\| \to 0$. The map $df_{\bar{g}}$ is called the *intrinsic differential* of f.

Remark 3.18. It is natural to ask about the relations between the notions of P-differentiability and of intrinsic differentiability.

The two notions in general are different. Indeed, if we assume, with the notations of Example 3.2, that $\mathbb{G}_1 = \mathbb{W}$ and $\mathbb{G}_2 = \mathbb{V} \equiv \mathbb{R}$, then the characterization of intrinsic differentiability in $\mathbb{H}^n$ given in (ii) of Proposition 3.28 states that $f : \mathbb{W} \to \mathbb{V}$ is intrinsic differentiable in $w \in \mathbb{W}$ if

$$\left\|df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w')\right\| = o\big(\left\|f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w)\right\|\big),$$

for all $w' \in \mathbb{W}$; while f is P-differentiable in $w \in \mathbb{W}$ if it satisfies the different equation

$$\left\|df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w')\right\| = o\big(\left\|w^{-1} \cdot w'\right\|\big),$$

for all $w' \in \mathbb{W}$.

We notice that if $\mathbb{G} := \mathbb{G}_1 \times \mathbb{G}_2$, it is easy to convince oneself that

$$f : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is P-differentiable}$$

if and only if

$$f : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is intrinsic differentiable.}$$

This way intrinsic differentiability can be seen as a generalization of the notion of P-differentiability.

Remark 3.19. Notice that the notion of intrinsic differentiability, as the one of intrinsic Lipschitz continuity, are - as they should be - invariant by translations. Indeed, let $q_1 = g_1 \cdot f(g_1)$ and $q_2 = g_2 \cdot f(g_2) \in \text{graph}\,(f)$; then f is intrinsic differentiable in $g_1 \in \mathbb{G}_1$ if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e. Consequently, f is intrinsic differentiable in g_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv (f_{q_1^{-1}})_{q_2}$ is intrinsic differentiable in g_2.

Finally we give the following notion of *uniform intrinsic differentiability.*

Definition 3.20. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$. We say that f is *uniformly intrinsic differentiable* in $\mathcal{A}$ if

(i): f is intrinsic differentiable at each $\bar{g} \in \mathcal{A}$;

(ii): $df_{\bar{g}} : \mathbb{G}_1 \to \mathbb{G}_2$ depends continuously on $\bar{g}$, that is, for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\eta = \eta_{\mathcal{K}} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\eta(t) \to 0$ as $t \downarrow 0$ such that

$$\sup_{g \in \mathcal{K}} \left\| df_{\bar{g}_1}(g)^{-1} \cdot df_{\bar{g}_2}(g) \right\| \leq \eta\big(\left\| g_1^{-1} \cdot g_2 \right\| \big); \tag{25}$$

(iii): for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\varepsilon = \varepsilon_{\mathcal{A},\mathcal{K}} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varepsilon(t) \to 0$ as $t \downarrow 0$, and such that

$$\left\| df_{\bar{g}}(g)^{-1} \cdot f_{\bar{p}^{-1}}(g) \right\| \leq \varepsilon(\|g\|)\, \|g\|\,, \tag{26}$$

for all $g \in \mathcal{K}_{\bar{p}^{-1}}$ and for all $\bar{g} \in \mathcal{K}$.

3.4. **Graphs in Heisenberg groups.** When the Carnot group $\mathbb{G}$ is one of the Heisenberg groups $\mathbb{H}^n$, all the notions of the preceding sections can be made more explicit. One of the key points is that if $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ then the product is semidirect. More precisely, one of the two complementary subgroups is a normal subgroup, containing the centre $\mathbb{T}$ of $\mathbb{H}^n$, while the other one is a commutative subgroup contained in $H\mathbb{H}^n_e$. This is the content of the following Proposition, that is an immediate consequence of Proposition 2.1.

Proposition 3.21. *In all the possible couples of complementary subgroups of $\mathbb{H}^n$ there are*

(i) *a horizontal subgroup, from now on called $\mathbb{V}$, of dimension $k \leq n$, isomorphic and isometric to $\mathbb{R}^k$,*

(ii) *a normal subgroup, from now on called $\mathbb{W}$, of dimension $2n+1-k$, containing the subgroup $\mathbb{T}$.*

Proof. It follows readily from Proposition 2.1. Indeed if $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$, clearly $\mathbb{G}_1$, $\mathbb{G}_2$ cannot be both vertical subgroups or horizontal subgroups. $\square$

Remark 3.22. Keeping the same notations of the previous Proposition, notice that, if $f : \mathbb{V} \to \mathbb{W}$ then the intrinsic graph of f is also an Euclidean graph over $\mathbb{V}$ identified with a k-dimensional vector subspace of $\mathbb{R}^{2n+1}$. On the contrary, if $f : \mathbb{W} \to \mathbb{V}$, then in general graph$\,(f)$ is not an Euclidean graph. See Example 3.8 in ([17]).

The characterization (22) of intrinsic Lipschitz continuous functions, when given in $\mathbb{H}^n$, can be made more explicit using the characterizations of translated functions given in Proposition 3.9. The result is different depending if f is defined on a horizontal subgroup $\mathbb{V}$ or on a vertical normal subgroup $\mathbb{W}$.

Proposition 3.23. *Assume $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. Then*

(i) $f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ *is* intrinsic Lipschitz continuous*, if and only if the parametrization map $\Phi_f : \mathcal{A} \to \mathbb{H}^n$, defined as $\Phi_f(v) := v \cdot f(v)$, is metric Lipschitz continuous in $\mathcal{A}$, that is if and only if there is $\tilde{L} > 0$ such that*

$$\left\| \Phi_f(\bar{v})^{-1} \cdot \Phi_f(v) \right\| \le \tilde{L} \left\| \bar{v}^{-1} \cdot v \right\|, \tag{27}$$

for all $v, \bar{v} \in \mathcal{A}$.

(ii) $f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ *is* intrinsic Lipschitz continuous in $\mathcal{A}$*, if and only if there is $L > 0$ such that*

$$\left\| f(w)^{-1} \cdot f(w') \right\| \le L \left\| f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w) \right\|, \tag{28}$$

for all $w, w' \in \mathcal{A}$.

Proof. To prove (i) recall (ii) of Proposition 3.9. If $q = x \cdot f(x) \in \text{graph}\,(f)$ then, for all $\eta \in \mathcal{A}_{q^{-1}}$,

$$f_{q^{-1}}(\eta) = \eta^{-1} \cdot f(x)^{-1} \cdot \eta \cdot f(x \cdot \eta).$$

Hence, from (27), setting $\eta = x^{-1} \cdot v$, we have

$$\begin{aligned} \left\| f_{q^{-1}}(\eta) \right\| &= \left\| v^{-1} \cdot x \cdot f(x)^{-1} \cdot x^{-1} \cdot v \cdot f(v) \right\| \\ &\le \left\| v^{-1} \cdot x \right\| + \left\| f(x)^{-1} \cdot x^{-1} \cdot v \cdot f(v) \right\| \\ &= \left\| v^{-1} \cdot x \right\| + \left\| \Phi_f(x)^{-1} \cdot \Phi_f(v) \right\| \le (1 + \tilde{L}) \left\| \eta \right\|. \end{aligned}$$

On the other side,

$$\begin{aligned} &\Phi_f(v)^{-1} \cdot \Phi_f(\bar{v}) = f(v)^{-1} \cdot v^{-1} \cdot \bar{v} \cdot f(\bar{v}) \\ &= f(v)^{-1} \cdot v^{-1} \cdot x \cdot v \cdot f(x \cdot v) \\ &= x \cdot x^{-1} \cdot f(v)^{-1} \cdot x \cdot f(x \cdot v), \end{aligned}$$

where $x = v^{-1} \cdot \bar{v}$. Now from (22) we get (27).

To prove (ii) observe that, from (22) and (i) of Proposition 3.9, for any $\bar{x} \in \mathcal{A}$, and for any y in the domain of $f_{q^{-1}}$,

$$\left\| f_{q^{-1}}(y) \right\| \equiv \left\| f(\bar{x})^{-1} \cdot f\left(\bar{x} \cdot f(\bar{x}) \cdot y \cdot f(\bar{x})^{-1}\right) \right\| \le L \left\| y \right\|.$$

Changing variables, setting $x = \bar{x} \cdot f(\bar{x}) \cdot y \cdot f(\bar{x})^{-1}$, that is $y = f(\bar{x})^{-1} \cdot \bar{x}^{-1} \cdot x \cdot f(\bar{x})$, it follows that, $\forall x, \bar{x} \in \mathcal{A}$,

$$\left\| f(\bar{x})^{-1} \cdot f(x) \right\| \le L \left\| f(\bar{x})^{-1} \cdot (\bar{x}^{-1} \cdot x) \cdot f(\bar{x}) \right\|.$$

This completes the proof of (ii). ∎

The following examples show both that condition (20) is not invariant under left translation of the graph and that neither intrinsic Lipschitz continuity implies metric Lipschitz continuity nor the opposite.

Example 3.24. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$ where

$$\mathbb{V} = \{v = (v_1, 0, 0)\}, \quad \mathbb{W} = \{w = (0, w_2, w_3)\}.$$

Then,

$$\|w\| = \max\{|w_2|, |w_3|^{1/2}\}, \quad \|v\| = |v_1|,$$

for all $w \in \mathbb{W}$ and $v \in \mathbb{V}$.

(1) Let $\varphi : \mathbb{W} \to \mathbb{V}$, defined as

$$\varphi(w) := \left(1 + |w_3|^{1/2}, 0, 0\right).$$

It is easy to check that φ satisfies (20) with $L = 1$, hence φ is metric Lipschitz. On the contrary φ is not intrinsic Lipschitz. Indeed, let $p := (1, 0, 0) \in \mathrm{graph}(\varphi)$, from Proposition 3.9 we have $\varphi_{p^{-1}}(w) = \left(|w_2 + w_3|^{1/2}, 0, 0\right)$. For $\varphi_{p^{-1}}$, (22) does not hold. This shows also that condition (20) is not invariant under graph translations.

(2) Let $\psi : \mathbb{W} \to \mathbb{V}$ defined as

$$\psi(w) := \left(1 + |w_3 - w_2|^{1/2}, 0, 0\right).$$

ψ is intrinsic Lipschitz, indeed, with $p = (1, 0, 0)$ and $\varphi(w) := \left(|w_3|^{1/2}, 0, 0\right)$ we have $\psi(w) = \varphi_p(w)$, so that ψ is intrinsic Lipschitz because φ is intrinsic Lipschitz. On the contrary ψ is not metric Lipschitz, in the sense of (20), as can be easily observed.

Analogously, it can be checked that

(1) the constant function $\varphi : \mathbb{V} \to \mathbb{W}$ defined as

$$\varphi(v) := (0, 1, 0),$$

for all $v \in \mathbb{V}$, is metric Lipschitz continuous, but it is not intrinsic Lipschitz;

(2) $\psi : \mathbb{V} \to \mathbb{W}$ defined as

$$\psi(v) := (0, 1, -v_1)$$

for all $v \in \mathbb{V}$, is intrinsic Lipschitz continuous but it is not metric Lipschitz continuous.

The following technical result is related with Proposition 3.1 of [1]. It states that, for each single intrinsic Lipschitz function $f : \mathbb{W} \to \mathbb{V}$, it is possible to define a distance d_f on the domain $\mathbb{W}$ - the d_f distance of two points of $\mathbb{W}$ being the distance in $\mathbb{H}^n$ of the corresponding points on graph (f) - such that f is metric Lipschitz from $(\mathbb{W}, d_f) \to \mathbb{V}$.

Proposition 3.25. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. Let $f : \mathbb{W} \to \mathbb{V}$, $\Phi_f : \mathbb{W} \to \mathbb{H}^n$ defined as $\Phi_f(w) := w \cdot f(w)$. Define,*

$$\tau_f(w, w') := \|f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w)\|,$$

for all $w, w' \in \mathbb{W}$.

If f is intrinsic L Lipschitz continuous, then

$$c\,\tau_f(w,w') \le \|\Phi_f(w)^{-1}\cdot\Phi_f(w')\| \le (1+L)\tau_f(w,w'), \tag{29}$$

where $c = c(\mathbb{W},\mathbb{V}) \in (0,1)$ is the constant in (18). Finally, defining

$$\sigma_f(w,w') := \left(\tau_f(w,w') + \tau_f(w',w)\right)/2,$$

for all $w, w' \in \mathbb{W}$, σ_f is a quasi metric in $\mathbb{W}$.

Proof. From (ii) of Proposition 3.14, it follows

$$\begin{aligned}\left\|\Phi_f(w)^{-1}\cdot\Phi_f(w')\right\| &= \left\|f(w)^{-1}\cdot w^{-1}\cdot w'\cdot f(w)\cdot f(w)^{-1}\cdot f(w')\right\| \\ &\le \tau_f(w,w') + \left\|f(w)^{-1}\cdot f(w')\right\| \le (1+L)\tau_f(w,w').\end{aligned}$$

Moreover, notice that

$$f(w)^{-1}\cdot w^{-1}\cdot w'\cdot f(w) = \left(\Phi_f(w)^{-1}\cdot\Phi_f(w')\right)_{\mathbb{W}}.$$

Hence, by (18),

$$c\,\tau_f(w,w') = c\left\|\left(\Phi_f(w)^{-1}\cdot\Phi_f(w')\right)_{\mathbb{W}}\right\| \le \left\|\Phi_f(w)^{-1}\cdot\Phi_f(w')\right\|.$$

□

The following Proposition gives Heisenberg characterizations of intrinsic linear functions. Once more, the characterizations are different depending if the intrinsic linear map L is defined on a horizontal k-dimensional subgroup $\mathbb{V}$ ($1 \le k \le n$) or on a normal $(2n+1-k)$-dimensional subgroup $\mathbb{W}$.

Proposition 3.26. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ as in Proposition 3.21. Then*

(i) *$L : \mathbb{V} \to \mathbb{W}$ is intrinsic linear if and only if the parametric map $\Phi_L : \mathbb{V} \to \mathbb{H}^n$, defined as $\Phi_L(v) := v\cdot L(v)$, is H-linear.*

(ii) *$L : \mathbb{W} \to \mathbb{V}$ is intrinsic linear if and only if it is H-linear.*

Proof. Part (i): Assume that Φ_L is H-linear, then,

$$v\cdot L(v)\cdot v'\cdot L(v') = \Phi_L(v)\cdot\Phi_L(v') = \Phi_L(v\cdot v') = v\cdot v'\cdot L(v\cdot v'),$$

and

$$\delta_\lambda(v\cdot L(v)) = \delta_\lambda(\Phi_L(g)) = \Phi_L(\delta_\lambda g) = \delta_\lambda g\cdot L(\delta_\lambda g),$$

for all $v, v' \in \mathbb{V}$ and $\lambda \in \mathbb{R}$. These two together, imply that graph (L) is a homogeneous subgroup.

Inversely, if graph (L) is a homogeneous subgroup, by homogeneity, for each $v \in \mathbb{V}$ and $\lambda > 0$ there is $\bar{v} \in \mathbb{V}$ such that $\delta_\lambda(v\cdot L(v)) = \bar{v}\cdot L(\bar{v})$. Hence $\delta_\lambda v\cdot\delta_\lambda(L(v)) = \bar{v}\cdot L(\bar{v})$. By uniqueness of the components on $\mathbb{V}$ and $\mathbb{W}$ (Proposition 3.4) it follows

$$\delta_\lambda v = \bar{v}, \text{ so that } L(\delta_\lambda v) = L(\bar{v}) = \delta_\lambda(L(v)),$$

this proves, in particular, that L is homogeneous and also that

$$\Phi_L(\delta_\lambda v) = \delta_\lambda v\cdot L(\delta_\lambda v) = \delta_\lambda v\cdot\delta_\lambda(L(v)) = \delta_\lambda(\Phi_L(v)),$$

that is Φ_L is homogeneous.

Moreover, for $v, v' \in \mathbb{V}$ there is $\bar{v} \in \mathbb{V}$ such that

$$v\cdot L(v)\cdot v'\cdot L(v') = \bar{v}\cdot L(\bar{v})$$

hence

$$v\cdot v'\cdot v'^{-1}\cdot L(v)\cdot v'\cdot L(v') = \bar{v}\cdot L(\bar{v}).$$

Use once more the uniqueness of the components and the assumption that $\mathbb{W}$ is a normal subgroup to get $v \cdot v' = \bar{v}$ and, consequently,

$$v'^{-1} \cdot L(v) \cdot v' \cdot L(v') = L(\bar{v}) = L(v \cdot v'); \tag{30}$$

From (30) the additivity of Φ_L follows, indeed

$$\begin{aligned}\Phi_L(v \cdot v') &= v \cdot v' \cdot L(v \cdot v') \\ &= v \cdot v' \cdot v'^{-1} \cdot L(v) \cdot v' \cdot L(v') = \Phi_L(v) \cdot \Phi_L(v'),\end{aligned}$$

for all $v, v' \in \mathbb{V}$.

Part (ii): Assume that $L : \mathbb{W} \to \mathbb{V}$ is H-linear. Then, as before, for all $w \in \mathbb{W}$ and $\lambda > 0$, we have $\delta_\lambda(w \cdot L(w)) = \delta_\lambda w \cdot \delta_\lambda(L(w)) = \delta_\lambda w \cdot L(\delta_\lambda w)$, showing that graph (L) is homogeneous.
Now observe that, because L is H-linear, $\mathbb{W}$ is normal and $\mathbb{V} \equiv \mathbb{R}^k$ is commutative, we have

$$L(g^{-1} \cdot w \cdot g) = L(w), \tag{31}$$

for all $g \in \mathbb{H}^n$ and $w \in \mathbb{W}$. Indeed, as proved in [26], in our assumptions $L(w)$ does not depend on the $(2n+1)^{th}$ component of w. On the other side, the first $2n$ components of $g^{-1} \cdot w \cdot g$ and of w coincide.
From (31), for all $w, w' \in \mathbb{W}$ we have

$$\begin{aligned} w \cdot L(w) \cdot w' \cdot L(w') &= \underbrace{w \cdot L(w) \cdot w' \cdot (L(w))^{-1}}_{=\bar{w} \in \mathbb{W}} \cdot \underbrace{L(w) \cdot L(w')}_{\in \mathbb{V}} \\ &= \bar{w} \cdot L(w) \cdot L\big(L(w) \cdot w' \cdot L(w)^{-1}\big) \\ &= \bar{w} \cdot L(\bar{w}).\end{aligned}$$

This proves that graph (L) is a homogeneous group.

Conversely, assume that graph (L) is a homogeneous group.
Working as above, we have $\delta_\lambda(w \cdot L(w)) = \delta_\lambda w \cdot \delta_\lambda(L(w)) = \bar{w} \cdot L(\bar{w})$ and by the uniqueness of the components we get that $\bar{w} = \delta_\lambda w$ and $\delta_\lambda(L(w)) = L(\delta_\lambda w)$ showing that L is homogeneous.

Then, because graph (L) is a group, for all $w, w' \in \mathbb{W}$, there is $\bar{w} \in \mathbb{W}$ such that $w \cdot L(w) \cdot w' \cdot L(w') = \bar{w} \cdot L(\bar{w})$. As before, this implies that

$$\bar{w} = w \cdot L(w) \cdot w' \cdot L(w)^{-1} \text{ and } L(\bar{w}) = L(w) \cdot L(w'),$$

now letting $\tilde{w} := L(w) \cdot w' \cdot L(w)^{-1}$ we get that,

$$L(w \cdot \tilde{w}) = L(w) \cdot L\big(L(w)^{-1} \cdot \tilde{w} \cdot L(w)\big), \tag{32}$$

for all $w, \tilde{w} \in \mathbb{W}$.
Now observe that for all $w_\mathbb{T} \in \mathbb{W} \cap \mathbb{T}$ we have $w_\mathbb{T} \cdot w_\mathbb{T} = \delta_{\sqrt{2}}(w_\mathbb{T})$. Moreover, because $w_\mathbb{T}$ is in the centre of $\mathbb{H}^n$, (32) gives that $L(w_\mathbb{T} \cdot w_\mathbb{T}) = L(w_\mathbb{T}) \cdot L(w_\mathbb{T})$ and, in turn $L(w_\mathbb{T}) \cdot L(w_\mathbb{T}) = \delta_2(L(w_\mathbb{T}))$ because $L(w_\mathbb{T})$ belongs to the horizontal subgroup $\mathbb{V}$. Hence, by the homogeneity of L we get that

$$\delta_{\sqrt{2}}(L(w_\mathbb{T})) = L(w_\mathbb{T} \cdot w_\mathbb{T}) = L(w_\mathbb{T}) \cdot L(w_\mathbb{T}) = \delta_2(L(w_\mathbb{T}))$$

that eventually gives

$$L(w_\mathbb{T}) = e. \tag{33}$$

Recall that any $w \in \mathbb{W}$ can be written in a unique way as $w = \pi(g) \cdot w_{\mathbb{T}}$, with $w_{\mathbb{T}} \in \mathbb{T}$ and $\pi(w) \in \mathbb{W} \cap H\mathbb{H}^n_e$. Hence, from (32) and (33),

$$L(w) = L(\pi(w) \cdot w_{\mathbb{T}}) = L(\pi(w)) \cdot L(w_{\mathbb{T}}) = L(\pi(w)) \tag{34}$$

for all $w \in \mathbb{W}$. So that, because $\pi(g^{-1} \cdot w \cdot g) = \pi(w)$, for all $w \in \mathbb{W}$ and $g \in \mathbb{H}^n$, from (32) and (34), we get

$$\begin{aligned} L(w \cdot \tilde{w}) &= L(w) \cdot L\big(\pi((L(w)^{-1} \cdot \tilde{w} \cdot L(w))\big) \\ &= L(w) \cdot L(\pi(\tilde{w})) = L(w) \cdot L(\tilde{w}), \end{aligned}$$

for all $w, \tilde{w} \in \mathbb{W}$. This proves the additivity of L and concludes the Proposition. $\square$

The following examples show that each different characterization, given in Proposition 3.26, cannot be extended to be a characterization of all intrinsic linear functions.

Example 3.27. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, with $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$.

(1) For any fixed $a \in \mathbb{R}$, the function $L : \mathbb{V} \to \mathbb{W}$ defined as

$$L(v) = (0, av_1, -av_1^2/2)$$

is intrinsic linear because graph $(L) = \{(t, at, 0) : t \in \mathbb{R}\}$ is a horizontal 1-dimensional subgroup of $\mathbb{H}^1$. But L is not a group homomorphism from $\mathbb{V}$ to $\mathbb{W}$.

(2) For any fixed $a \in \mathbb{R}$, the function $L : \mathbb{W} \to \mathbb{V}$ defined as

$$L(w) = (aw_2, 0, 0)$$

is intrinsic linear because graph $(L) = \{(at, t, s) : t, s \in \mathbb{R}\}$ is a vertical 2-dimensional subgroup of $\mathbb{H}^1$. The parametric function $\Phi_L : \mathbb{W} \to \mathbb{H}^1$ acts as $\Phi_L(w) = (aw_2, w_2, w_3 - aw_2^2/2)$ and, consequently, Φ_L is not a group homomorphism from $\mathbb{V}$ to $\mathbb{H}^n$.

The following Propositions give algebraic characterizations of intrinsic differentiable functions. Notice that, once more, the characterizations are different if the subgroup, where f is defined, is a horizontal k-dimensional subgroup $\mathbb{V}$ $(1 \leq k \leq n)$ or a normal $(2n+1-k)$-dimensional subgroup $\mathbb{W}$.

Proposition 3.28. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. Then,*

(i) *$f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if the parameterization map $\Phi_f : \mathcal{A} \to \mathbb{H}^n$, $\Phi_f(g) := g \cdot f(g)$, is P-differentiable in $\bar{g}$ and, for all $g \in \mathbb{V}$,*

$$d\Phi_{f,\bar{g}}(g) = g \cdot df_{\bar{g}}(g). \tag{35}$$

(ii) *$f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{W} \to \mathbb{V}$, such that*

$$\big\|df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g)\big\| = o\left(\big\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\big\|\right)$$

for $g \in \mathcal{A}$ and $\big\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\big\| \to 0$.

Proof. Case (i): If f is intrinsic differentiable in $\bar{g}$ with intrinsic differential $df_{\bar{g}}$, by Proposition 3.26 the map $g \mapsto g \cdot df_{\bar{g}}(g)$ is a homogeneous homomorphism $\mathbb{V} \to \mathbb{H}^n$. We define

$$d\Phi_{f,\bar{g}} : \mathbb{V} \to \mathbb{H}^n \text{ as } d\Phi_{f,\bar{g}}(g) := g \cdot df_{\bar{g}}(g).$$

Observe that, from Proposition 3.9, if $\bar{p} = \bar{g} \cdot f(\bar{g})$ then

$$df_{\bar{g}}(\eta)^{-1} \cdot f_{\bar{p}^{-1}}(\eta) = df_{\bar{g}}(\eta)^{-1} \cdot \eta^{-1} \cdot f(\bar{g})^{-1} \cdot \eta \cdot f(\bar{g} \cdot \eta)$$

and, defining $g := \bar{g} \cdot \eta$,

$$\begin{aligned} & df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f_{\bar{p}^{-1}}(\bar{g}^{-1} \cdot g) \\ & = df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot (\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(g) \\ & = d\Phi_{f,\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot \Phi_f(\bar{g})^{-1} \cdot \Phi_f(g). \end{aligned} \tag{36}$$

Hence (24) yields

$$\left\| d\Phi_{f,\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot \Phi_f(\bar{g})^{-1} \cdot \Phi_f(g) \right\| = o\left(\left\| \bar{g}^{-1} \cdot g \right\| \right) \tag{37}$$

as $\left\| \bar{g}^{-1} \cdot g \right\| \to 0$, that is Φ_f is P-differentiable in $\bar{g}$.

Conversely, if Φ_f is P-differentiable in $\bar{g}$ then, by definition, its P-differential $d\Phi_{f,\bar{g}}$ is a homogeneous homomorphism $\mathbb{V} \to \mathbb{H}^n$. We prove that

$$d\Phi_{f,\bar{g}}(g) = g \cdot L_{f,\bar{g}}(g), \tag{38}$$

with $L_{f,\bar{g}} : \mathbb{V} \to \mathbb{W}$.

Indeed, by definition of P-differentiability and from (18) we have that both the $\mathbb{W}$ component and the $\mathbb{V}$ component of the left hand side of (37) have to be $o\left(\left\| \bar{g}^{-1} \cdot g \right\| \right)$. Looking at the $\mathbb{V}$ component we get

$$\left\| (d\Phi_{f,\bar{g}}(\bar{g}^{-1} \cdot g))_{\mathbb{V}}^{-1} \cdot \bar{g}^{-1} \cdot g \right\| = o\left(\left\| \bar{g}^{-1} \cdot g \right\| \right). \tag{39}$$

Notice that, by (i) of Proposition 2.7, $(d\Phi_{f,\bar{g}})_{\mathbb{V}}$ is a linear map from $\mathbb{V} \equiv \mathbb{R}^k$ to itself. Hence from (39) we get that

$$(d\Phi_{f,\bar{g}})_{\mathbb{V}} = I_{\mathbb{V}}.$$

This proves (38).

From (38) and from Proposition 3.26 we have that $L_{f,\bar{g}}$ is an intrinsic linear map $\mathbb{V} \to \mathbb{W}$.

We define $L_{f,\bar{g}}$ as the intrinsic differential of f at $\bar{g}$, that is

$$L_{f,\bar{g}} := df_{\bar{g}}.$$

Now, (37) and (36) yield the intrinsic differentiability of f in $\bar{g}$.

Case (ii): from (i) of Proposition 3.9, for all $\eta \in \mathbb{W}$, we have

$$df_{\bar{g}}(\eta)^{-1} \cdot f_{\bar{p}^{-1}}(\eta) = df_{\bar{g}}(\eta)^{-1} \cdot f(\bar{g})^{-1} \cdot f\left(\bar{g} \cdot f(\bar{g}) \cdot \eta \cdot f(\bar{g})^{-1} \right)$$

and defining g such that $\eta = f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})$

$$\begin{aligned} & = df_{\bar{g}}\left(f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g}) \right)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g) \\ & = df_{\bar{g}}\left(\bar{g}^{-1} \cdot g \right)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g), \end{aligned}$$

where in the last equality we have used that $df_{\bar{g}}$ is an homogeneous homomorphism and that $\mathbb{V}$ is commutative. Now the equivalence of Definition 3.17 and (2) of this Proposition is clear. $\square$

An analogous characterization holds for uniformly intrinsic differentiability.

Proposition 3.29. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, as in Proposition 3.21. Then*

(i) *$f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ is uniformly intrinsic differentiable in $\mathcal{A}$ if and only if the parameterization map $\Phi_f : \mathcal{A} \to \mathbb{H}^n$, is continuosly P-differentiable in $\mathcal{A}$.*

(ii) *$f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ is uniformly intrinsic differentiable in $\mathcal{A}$ if and only if it is intrinsic differentiable at each $g \in \mathcal{A}$ with differential df_g continuously dependent on g and if, for each compact $\mathcal{K} \subset \mathcal{A}$,*

$$\sup_{\substack{\bar{g}, g \in \mathcal{K} \\ 0 < \|\bar{g}^{-1} \cdot g\| < \delta}} \frac{\left\| df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g) \right\|}{\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\|} \to 0 \text{ as } \delta \to 0. \tag{40}$$

Proof. Case (i): the equivalence between uniformly intrinsic differentiability of f and continuous P-differentiability of Φ_f follows immediately from (35) and applying Theorem 4.6 in [27].

Case (ii): because f is continuous in $\mathcal{A}$, then f is bounded in each compact $\mathcal{K} \subset \mathcal{A}$; hence, $\left\| f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g}) \right\|$ is comparable with $\left\| \bar{g}^{-1} \cdot g \right\|$ in $\mathcal{K}$. Now the equivalence between (40) and the two conditions (ii) and (iii) of Definition 3.20, follows from the same steps used in the proof of Case (ii) of Proposition 3.28. □

The following Proposition states precisely a natural relation between intrinsic differentiability and intrinsic Lipschitz continuity.

Proposition 3.30. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. If $f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ is uniformly intrinsic differentiable in $\mathcal{A}$, then it is locally intrinsic Lipschitz continuos in $\mathcal{A}$.*

Proof. First we observe that an intrinsic linear function $L : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz continuous. Indeed for all $w \in \mathbb{W}$ and for all $t \geq 0$

$$\|L(w)\| = \|\delta_t L(\delta_{1/t} w)\| = t \|L(\delta_{1/t} w)\|;$$

choosing $t = \|w\|$ and defining $k := \sup_{\|\xi\|=1} \|L(\xi)\|$ it follows $\|L(w)\| \leq k \|w\|$, for all $w \in \mathbb{W}$. Finally, for each $p = \xi \cdot L(\xi) \in \operatorname{graph}(L)$, from Proposition 3.9 we see that $L_{p^{-1}}$ coincides with L, hence

$$\|L_{p^{-1}}(w)\| = \|L(w)\| \leq k \|w\|, \quad \forall w \in \mathbb{W}, \tag{41}$$

showing that L is intrinsic Lipschitz continuous.

For a fixed $w_0 \in \mathcal{A}$, let $\bar{r} > 0$ be such that for all $w, \eta \in \mathbb{W} \cap B(w_0, \bar{r})$

$$\sup_{w, \eta \in B(w_0, \bar{r})} \frac{\|df_w(w^{-1} \cdot \eta)^{-1} \cdot f(w)^{-1} \cdot f(\eta)\|}{\|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\|} \leq 1$$

and

$$\sup_{w \in B(w_0, \bar{r})} \|df_w\| = K < +\infty.$$

Then, for all $w, \eta \in \mathbb{W} \cap B(w_0, \bar{r})$, recalling also (31),

$$\begin{aligned} \left\| f(w)^{-1} \cdot f(\eta) \right\| &\leq \left\| df_w(w^{-1} \cdot \eta) \right\| + \left\| df_w(w^{-1} \cdot \eta)^{-1} \cdot f(w)^{-1} \cdot f(\eta) \right\| \\ &\leq (K+1) \left\| f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w) \right\|. \end{aligned}$$

from which the thesis. □

4. H-regular submanifolds are intrinsic differentiable graphs

This section contains our main theorem. We prove that $S \subset \mathbb{H}^n$ is a H-regular submanifold, as given in Definition 4.1, if and only if S is, locally, a uniformly intrinsic differentiable graph.

We begin recalling the definitions of H-regular submanifolds, of dimension k or of codimension k (see [17] and also [14] or [37]).

Definition 4.1. Let k be an integer, $1 \leq k \leq n$.

(i) A subset $S \subset \mathbb{H}^n$ is a *k-dimensional* H-regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$ with $p \in \mathcal{U}$, an open $\mathcal{A} \subset \mathbb{R}^k$ and an injective, continuosly Pansu differentiable $f : \mathcal{A} \to \mathcal{U}$, with injective Pansu differential, such that
$$S \cap \mathcal{U} = f(\mathcal{A}).$$

(ii) A subset $S \subset \mathbb{H}^n$ is a *k-codimensional* H-regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, and $f : \mathcal{U} \to \mathbb{R}^k$, $f \in C^1_H(\mathcal{U}; \mathbb{R}^k)$ with surjective Pansu differential, such that
$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\}.$$

Remark 4.2. These notions of H-regular submanifolds are different from the corresponding Euclidean ones and are also very different from each other. Indeed k-dimensional H-regular submanifolds of $\mathbb{H}^n$ are a subclass of k-dimensional Euclidean C^1 submanifolds of $\mathbb{R}^{2n+1}$ (see [17] and Theorem 6.1). On the contrary, k-codimensional H-regular submanifolds can be very irregular objects from an Euclidean point of view. A striking example of a 1-codimensional H-regular surface in $\mathbb{H}^1 \equiv \mathbb{R}^3$, with fractional Euclidean dimension equal to 2.5, is provided in [23]. An easier example is the surface
$$S = \{(x, y, t) : x = \sqrt{x^4 + y^4 + t^2}\} \subset \mathbb{H}^1.$$
S is a H-regular hypersurface (i.e. 1-codimensional H-regular) but S is not euclidean regular at the origin. On the other side, the horizontal plane $\{t = 0\}$ is Euclidean regular but not intrinsic regular at the origin.

Theorem 4.3. *The following statements are equivalent*

(1) *$S \subset \mathbb{H}^n$ is a H-regular submanifold.*

(2) *$\forall p \in S$ there is an open $\mathcal{U} \ni p$ such that $S \cap \mathcal{U}$ is the graph of a uniformly intrinsic differentiable function φ acting between complementary homogeneous subgroups of $\mathbb{H}^n$.*

More precisely, with $1 \leq k \leq n$, if S is k-dimensional H-regular then φ is defined on a k-dimensional horizontal subgroup and if S is k-codimensional H-regular then φ is defined on a $(2n+1-k)$-dimensional normal subgroup.

Proof. We divide the proof in two parts; in the first part we deal with the case of a k-dimensional S and in the second part S is a k-codimensional submanifold.

First part:

(1) $\Longrightarrow$ (2). By Definition 4.1, if S is a k dimensional H-regular submanifold then for each $p \in S$ there are an open neighborhood $\mathcal{U}$ of p and an open $\mathcal{A} \subset \mathbb{R}^k$ such that $S \cap \mathcal{U} = f(\mathcal{A})$ with f injective and uniformly P-differentiable in $\mathcal{A}$.

In Theorem 3.5 of [17] it is proved that any k-dimensional H-regular$S$ is also an Euclidean C^1 k-dimensional submanifold of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and that, at each $p \in S$, there is a k-dimensional horizontal subgroup $\mathbb{V}$ such that its coset $p \cdot \mathbb{V}$ is equal to the Euclidean tangent k-plane T_pS. (Notice that T_pS is also the limit of the Heisenberg dilations of S centered in p).

Let us fix $p \in S$ and let $\mathbb{V} := p^{-1} \cdot T_pS$ and $\mathbb{W} := \mathbb{V}^\perp$. The orthogonality is meant here with respect to the scalar product $\langle \cdot, \cdot \rangle_e$ that is the same as the Euclidean scalar product.

Choosing a small enough open neighborhood of the origin $\mathcal{V}$, we have that $(p^{-1} \cdot S) \cap \mathcal{V}$ is an Euclidean C^1 graph over the subgroup (or k-dimensional vector subspace of $\mathbb{R}^{2n+1}$) $\mathbb{V}$ in direction of $\mathbb{W}$. Precisely, there are an open $\mathcal{O} \subset \mathbb{V}$ and a function $\varphi : \mathcal{O} \to \mathbb{W}$, continuously differentiable in $\mathcal{O}$, such that

$$(p^{-1} \cdot S) \cap \mathcal{V} = \{v + \varphi(v) : v \in \mathcal{O}\}$$

The map $\Phi : \mathcal{O} \to \mathbb{H}^n$ defined as

$$\Phi(v) := v + \varphi(v)$$

is Euclidean C^1; once more by Theorem 3.5 of [17], the image of the Euclidean differential $d_{euc}\Phi_v$ is an horizontal k-dimensional subgroup of $\mathbb{H}^n$, for all $v \in \mathcal{O}$, hence, by Theorem 1.1 of [27], Φ is continuously P-differentiable in $\mathcal{O}$.

Finally,

$$(p^{-1} \cdot S) \cap \mathcal{V} = \{v + \varphi(v) : v \in \mathcal{O}\} = \{v \cdot \psi(v) : v \in \mathcal{O}\}$$

where $\psi : \mathcal{O} \to \mathbb{W}$ is given by

$$\psi(v) = \left(\varphi_1(v), \cdots, \varphi_{2n}(v), \varphi_{2n+1}(v) - \frac{1}{2} \sum_{i=1}^{n} \left(v_i \varphi_{n+i}(v) - v_{n+i} \varphi_i(v) \right) \right).$$

By Proposition 3.29, the function ψ is uniformly intrinsic differentiable in $\mathcal{O}$ because, being $\Phi_\psi \equiv \Phi$, the associated parametric map Φ_ψ is continuously P-differentiable.

So that we have proved that $(p^{-1} \cdot S) \cap \mathcal{V} = \text{graph}\,(\psi)$. Hence $S \cap (p \cdot \mathcal{V})$ is the graph of the uniformly intrinsic differentiable translated function ψ_p.

(2) $\implies$ (1). Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, as in Definition 3.1, and let $f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ be uniformly intrinsic differentiable in $\mathcal{A}$.

By Proposition 3.29, $\Phi_f : \mathcal{A} \to \mathbb{H}^n$ is uniformly P-differentiable in $\mathcal{A}$.

Hence, $\text{graph}\,(f) = \Phi_f(\mathcal{A})$ is a k-dimensional H-regular submanifold.

Second part:

(1) $\implies$ (2). Let $S \subset \mathbb{H}^n$ be a H-regular surface of codimension k. Then, (see Theorem 2.9), for each $p \in S$ there are an open neighborhood $\mathcal{U}$ of p and a function $f \in C^1_H(\mathcal{U}, \mathbb{R}^k)$ such that

$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\}. \tag{42}$$

Moreover there are homogeneous subgroups $\mathbb{V}$ and $\mathbb{W}$, such that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Definition 3.1, and, for all $x \in \mathcal{U}$,

$$df_{x|\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k \text{ is one to one.} \tag{43}$$

Here $df_x : \mathbb{H}^n \to \mathbb{R}^k$ is the P-differential of f in $x \in \mathcal{U}$. Finally, there are a relatively open $\mathcal{A} \subset \mathbb{W}$, with $p_{\mathbb{W}} \in \mathcal{A}$ and a continuos function $\varphi : \mathcal{A} \to \mathbb{V}$ such that

$$S \cap \mathcal{U} = \operatorname{graph}(\varphi) = \{w \cdot \varphi(w) : w \in \mathcal{A}\}. \tag{44}$$

We have to prove that φ is uniformly intrinsic differentiable in $\mathcal{A}$.

For each $w \in \mathcal{A}$, let $x = w \cdot \varphi(w)$. Define the H-linear function $d\varphi_w : \mathbb{W} \to \mathbb{V}$ as

$$d\varphi_w := -\left(df_{x|\mathbb{V}}\right)^{-1} \circ df_{x|\mathbb{W}} \tag{45}$$

By (ii) of Proposition 3.26, $d\varphi_w : \mathbb{W} \to \mathbb{V}$ is an intrinsic linear function. We prove now that, for any compact $\mathcal{K} \subset \mathcal{A}$,

$$\sup_{\substack{w,\eta \in \mathcal{K} \\ 0<\|w^{-1}\cdot\eta\|<\delta}} \frac{\|d\varphi_w(w^{-1}\cdot\eta)^{-1}\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\|}{\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\|} \to 0 \tag{46}$$

as $\delta \to 0$. This will complete the proof, because, by the characterization of uniform intrinsic differentiability given in Proposition 3.29 and observing that $d\varphi_w$ depends continuosly on w, (46) shows both that $d\varphi_w$ is the intrinsic differential of φ at w and that φ is uniformly intrinsic differentiable in $\mathcal{A}$. Notice that, for all $\eta \in \mathbb{W}$ and $v \in \mathbb{V}$,

$$\begin{aligned} &\left(\left(df_{x|\mathbb{V}}\right)^{-1} \circ df_x\right)(\eta\cdot v) = \left(df_{x|\mathbb{V}}\right)^{-1}\left(df_{x|\mathbb{W}}(\eta)\cdot df_{x|\mathbb{V}}(v)\right) \\ &= \left(\left(df_{x|\mathbb{V}}\right)^{-1} \circ df_{x|\mathbb{W}}\right)(\eta)\cdot v \end{aligned} \tag{47}$$

By (47), recalling that $L(w\cdot v) = L(v\cdot w)$ for any H-linear function L with values in a commutative subgroup of $\mathbb{H}^n$, we have

$$\begin{aligned} &\left\|(d\varphi_w(w^{-1}\cdot\eta))^{-1}\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\right\| \\ &= \left\|\left(\left(df_{x|\mathbb{V}}\right)^{-1}\circ df_{x|\mathbb{W}}\right)(w^{-1}\cdot\eta)\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\right\| \\ &= \left\|\left(\left(df_{x|\mathbb{V}}\right)^{-1}\circ df_x\right)(w^{-1}\cdot\eta\cdot\varphi(w)^{-1}\cdot\varphi(\eta))\right\| \\ &= \left\|\left(df_{x|\mathbb{V}}\right)^{-1}\left(df_x(\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta))\right)\right\| \end{aligned}$$

where $\Phi_\varphi(w) := w\cdot\varphi(w)$; then, by Taylor's inequality, (see Theorem 2.8), there exists $\delta > 0$ such that

$$\begin{aligned} &\le \left\|\left(df_{x|\mathbb{V}}\right)^{-1}\right\| \left\|f(\Phi_\varphi(\eta)) - f(\Phi_\varphi(w)) - df_x(\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta))\right\|_{\mathbb{R}^k} \\ &\le \sup_{w\in\mathcal{K}}\left\|\left(df_{x|\mathbb{V}}\right)^{-1}\right\| C \sup_{\substack{w,\eta\in\mathcal{K} \\ \|w^{-1}\cdot\eta\|<\delta}} \left\|df_{w\cdot\varphi(w)} - df_{\eta\cdot\varphi(\eta)}\right\| \ \|\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta)\|. \end{aligned}$$

Now observe that, the function φ is intrinsic Lipschitz continuous in the compact set $\mathcal{K}$, then, by (29), we have

$$\|\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta)\| \le (1+L)\tau_\varphi(w,\eta) = (1+L)\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\|$$

where L is the Lipschitz constant of φ. So that, the required equation (46) follows, because the function df is uniformly continuous in the compact set $\tilde{\mathcal{K}} := \{w' \cdot \varphi(w') : w' \in \mathcal{K}\}$ and observing that, if $\|w^{-1} \cdot \eta\| < \delta$, it is also $\|\Phi_\varphi(w)^{-1} \cdot \Phi_\varphi(\eta)\| < c(\mathcal{K}, \delta)$, where $c(\mathcal{K}, \delta) \to 0^+$ as $\delta \to 0$.

(2) $\Longrightarrow$ (1). Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21 and $\mathcal{A}$ be open in $\mathbb{W}$. We have to prove that if $\varphi : \mathcal{A} \to \mathbb{V}$ is uniformly intrinsic differentiable in $\mathcal{A}$, then $S = \{w \cdot \varphi(w),\ w \in \mathcal{A}\}$ is a k-codimensional H-regular submanifold. That is, we have to prove that, given $\bar{p} = \bar{w} \cdot \varphi(\bar{w}) \in S$, there are an open neighborhood $\mathcal{U}$ of $\bar{p}$ and a function $f \in C^1_H(\mathcal{U}, \mathbb{R}^k)$, such that

$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\} \tag{48}$$

and

$$df_x : \mathbb{H}^n \to \mathbb{R}^k \text{ is surjective} \tag{49}$$

for all $x \in \mathcal{U}$.

Let $\mathcal{I} \Subset \mathcal{A}$, be an open in $\mathbb{W}$ neighborhood of $\bar{w}$, then

$$\mathcal{F} := \{w \cdot \varphi(w) :\ w \in \overline{\mathcal{I}}\}$$

is a compact set in $\mathbb{H}^n$. We want to determine the desired function f by appropriately extending, using Whitney's extension theorem (see Theorem 2.10), the function identically zero on $\mathcal{F}$.

Let us verify the assumptions of Whitney's theorem.

For every $x = w \cdot \varphi(w) \in \mathcal{F}$, let $h_x : \mathbb{H}^n \to \mathbb{R}^k$ be the H-linear map

$$h_x(p) := (d\varphi_w(p_{\mathbb{W}}))^{-1} \cdot p_{\mathbb{V}}, \tag{50}$$

for all $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \in \mathbb{H}^n$.

Notice that for each $x = w \cdot \varphi(w) \in \mathcal{F}$, h_x is a H-linear function. The homogeneity of h_x is obvious. Moreover, observing that $(p \cdot q)_{\mathbb{V}} = p_{\mathbb{V}} \cdot q_{\mathbb{V}}$ and $(p \cdot q)_{\mathbb{W}} = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1}$, we have

$$\begin{aligned} h_x(p \cdot q) &= (d\varphi_w[(p \cdot q)_{\mathbb{W}}])^{-1} \cdot (p \cdot q)_{\mathbb{V}} \\ &= \left(d\varphi_w(p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1})\right)^{-1} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{V}} \\ &= (d\varphi_w(p_{\mathbb{W}}))^{-1} \cdot (d\varphi_w(q_{\mathbb{W}}))^{-1} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{V}} \\ &= h_x(p) \cdot h_x(q), \end{aligned}$$

for all $p, q \in \mathbb{H}^n$. This completes the proof of the H-linearity of h_x.

The map, from $\mathcal{F}$ to the set of H-linear functions from $\mathbb{H}^n$ to $\mathbb{R}^k$, defined as $x \mapsto h_x$ is continuous. This fact follows from (50) and from the continuity of $d\varphi_w$, as a map from $\mathcal{A}$ to the set of H-linear functions from $\mathbb{W}$ to $\mathbb{V}$; this, in turn, is contained in the assumption of intrinsic uniform differentiability of φ in $\mathcal{A}$.

Hence, if we associate, as in Proposition 2.7, to each H-linear function h_x a matrix $Q_x \in \mathbb{R}^{k,2n}$, then the map $Q : \mathcal{F} \to \mathbb{R}^{k,2n}$, sending $x \in \mathcal{F}$ to Q_x, is continuous.

Now define

$$R(x, y) := -\frac{h_x(x^{-1} \cdot y)}{d(x, y)}$$

for $x, y \in \mathcal{F}$, $x \neq y$. If $\mathcal{K}$ is a compact subset of $\mathcal{F}$, then

$$\sup \{\|R(x,y)\| : x, y \in \mathcal{K},\ 0 < d(x,y) < \delta\} \to 0 \text{ as } \delta \to 0. \tag{51}$$

Indeed, from (29), there exists $c = c(\mathbb{W}, \mathbb{V}) > 0$ such that, for all $x = w \cdot \varphi(w)$ and $y = \eta \cdot \varphi(\eta)$ in $\mathcal{K}$, we have:

$$c \left\|\varphi(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \varphi(w)\right\| \equiv c\tau_\varphi(w,\eta) \leq d(x,y).$$

Hence,

$$\begin{aligned}\|R(x,y)\| &= \frac{\|h_x(x^{-1} \cdot y)\|}{d(x,y)} \\ &\leq \frac{1}{c} \frac{\|(d\varphi_w(w^{-1} \cdot \eta))^{-1} \cdot \varphi(w)^{-1} \cdot \varphi(\eta)\|}{\|\varphi(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \varphi(w)\|} \to 0\end{aligned}$$

Now, (51) follows from the assumption of uniform intrinsic differentiability of φ and from (40).

We can now apply Whitney's theorem to the couple of functions

$$g : \mathcal{F} \to \mathbb{R}, \qquad Q : \mathcal{F} \to \mathbb{R}^{k,2n},$$

where $g(x) = 0$ for all $x \in \mathcal{F}$, to get a function $f \in C^1_H(\mathbb{H}^n, \mathbb{R}^k)$, vanishing on $\mathcal{F}$ and with a surjective differential at all points of $\mathcal{F}$.

To check this last point, observe that, from definition (50), for all $x \in \mathcal{F}$, $h_{x|\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$ is one to one.

To conclude our proof we have to provide an open neighborhood $\mathcal{U}$ of $\bar{p}$ satisfying (48) and (49).

Fix $r > 0$ and define $\mathcal{U}$ as

$$\mathcal{U} = \{w \cdot v \in \mathbb{H}^n :\ w \in \mathcal{I}_r \subset \mathbb{W},\ v \in \mathbb{V} \cap B(\varphi(\bar{w}), r)\} \tag{52}$$

where $\mathcal{I}_r \subset \mathcal{I}$ is a neighborhood of $\bar{w}$, $\mathcal{I}_r$ is open in $\mathbb{W}$ and such that, $\varphi(\mathcal{I}_r) \subset \mathbb{V} \cap B(\varphi(\bar{w}), r)$.

By definition $\bar{p} \in \mathcal{U}$ and, if we choose r small enough, by continuity of df on $\mathbb{H}^n$, df_x is surjective for all $x \in \mathcal{U}$, hence (49) holds.

Moreover, by continuity, for r small, $df_x : \mathbb{V} \to \mathbb{R}^k$ is one to one, for all $x \in \mathcal{U}$. Hence, for each $\tilde{w} \in \mathcal{I}_r$, the map

$$v \mapsto f(\tilde{w} \cdot v)$$

is one to one in $\mathbb{V} \cap B(\varphi(\bar{w}), r)$. It follows that, if $x = w \cdot v \in \mathcal{U}$ is such that $f(x) = 0$, then $x = w \cdot \varphi(w) \in \mathcal{F}$.

So also (48) holds and the proof is completed. □

5. Directional derivatives

We give here a few hints about relations among the notions introduced here and the ones in [1], [37] and [8]. We begin defining *directional derivatives* of a function $f : \mathbb{W} \to \mathbb{V}$ and we show that, using this notion, it is possible to get an explicit representation of the intrinsic differential df.

Definition 5.1. Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21 and $f : \mathbb{W} \to \mathbb{V}$.

(i) The *difference quotient* of f, in w along $Y \in \mathfrak{w}$, is

$$R_Y f(w;t) := \delta_{1/t}\left(f(w)^{-1} \cdot f(w \cdot f(w) \cdot \exp tY \cdot f(w)^{-1})\right),$$

for all $w \in \mathbb{W}$ and $t \in \mathbb{R}$. We use here the convention $\delta_\lambda p = \left(\delta_{|\lambda|} p\right)^{-1}$, if $\lambda < 0$.

(ii) The *intrinsic directional derivative* $D_Y f(w)$ is

$$D_Y f(w) := \lim_{t \to 0} R_Y f(w; t)$$

provided that the limit exists finite.

Notice that, if $f(\bar{w}) = e$, then difference quotients and derivatives in $\bar{w}$ have the following, more familiar, aspect

$$R_Y f(\bar{w}; t) = \delta_{1/t} f(\bar{w} \cdot \exp tY),$$

and

$$D_Y f(\bar{w}) = Y f(\bar{w}).$$

Intrinsic difference quotients and intrinsic directional derivatives are translation invariant. That is, if $p = w \cdot f(w)$, then

$$R_Y f(w; t) = R_Y f_{p^{-1}}(e; t) = \delta_{1/t} \left(f_{p^{-1}}(\exp tY)\right).$$

and

$$D_Y f(w) = D_Y f_{p^{-1}}(e) = Y f_{p^{-1}}(e). \tag{53}$$

Remark 5.2. It is not difficult to prove (see [18]) that $f : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz continuous with Lipschitz constant L if and only if

$$\|R_V f(w; t)\| \le L,$$

for all $V \in \mathfrak{w}$, with $\|V\| \le 1$ and for all $t > 0$, $w \in \mathbb{W}$.

Indeed, the following stronger result holds, (see [18]),

Proposition 5.3. *Assume that $V_1, \cdots, V_{2n-k}$ is an orthonormal basis of $\mathfrak{w} \cap \mathfrak{h}_1$. If $f : \mathbb{W} \to \mathbb{V}$ is such that*

$$R_{V_i} f(w; t) \le L$$

for $i = 1, \cdots, 2n-k$ and $t > 0$ then f is locally intrinsic Lipschitz continuous in $\mathbb{W}$.

Uniformly intrinsic differentiable functions $f : \mathbb{W} \to \mathbb{V}$ have continuous directional derivatives.

Proposition 5.4. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. If $\mathcal{O}$ is open in $\mathbb{W}$ and $f : \mathcal{O} \to \mathbb{V}$ is intrinsic differentiable in $w \in \mathcal{O}$, then its directional derivatives exist in w and*

$$D_Y f(w) = df_w(\exp Y).$$

Proof. By definition of intrinsic differentiability, if $w \in \mathcal{O}$ and $p = w \cdot f(w)$,

$$\|df_w(\exp tY)^{-1} \cdot f_{p^{-1}}(\exp tY)\| = o(\|\exp tY\|) = o(t)$$

as $t \to 0$. Hence, for $t > 0$,

$$f_{p^{-1}}(\exp tY) = df_w(\exp tY) \cdot \varepsilon(t),$$

where $\|\varepsilon(t)\| / t \to 0$ as $t \to 0$. Since,

$$\begin{aligned} R_Y f(w; t) &= \delta_{1/t} f_{p^{-1}}(\exp tY) = \delta_{1/t} \left(df_w(\exp tY) \cdot \varepsilon(t)\right) \\ &= df_w(\exp Y) \cdot \delta_{1/t} \varepsilon(t) \end{aligned}$$

we get,

$$\left\| df_w(\exp Y)^{-1} \cdot R_Y f(w;t) \right\| = \|\varepsilon(t)\| / t$$

and the thesis follows. $\square$

We compute explicitly directional derivatives for 1-codimensional and 2-codimensional graphs. We particularly want to show that directional derivatives can be, in some cases, first order non linear differential operators

Example 5.5. *(Codimension 1 graphs in $\mathbb{H}^n$)* Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, with

$$\mathbb{W} = \{(0, x_2, \cdots, x_{2n+1})\} \text{ and } \mathbb{V} = \{(x_1, 0, \cdots, 0)\}$$

and $f = (\varphi, 0, \cdots, 0) : \mathbb{W} \to \mathbb{V}$. Then (53) gives,

$$D_{X_j} f(x) = (X_j \varphi(x), 0, \cdots, 0), \quad D_{Y_j} f(x) = (Y_j \varphi(x), 0, \cdots, 0),$$

for $2 \le j \le n$, and

$$\begin{aligned} & D_{Y_1} f(x) \\ & = \left(\lim_{t \to 0} \frac{1}{t} \left(\varphi(0, \cdots, x_{n+1} + t, \cdots, x_{2n+1} + \varphi(x)t) - \varphi(x) \right), 0, \ldots, 0 \right) \\ & = \left(\partial_{n+1}\varphi(x) + \varphi(x) \partial_{2n+1} \varphi(x), 0, \ldots, 0 \right). \end{aligned}$$

Setting $D_{Y_1}\varphi = \partial_{n+1}\varphi + \varphi\partial_{2n+1}\varphi$ and using the notation introduced in [1], we have

$$W^{\varphi}\varphi = \big(X_2\varphi, \ldots, X_n\varphi, D_{Y_1}\varphi, Y_2\varphi, \ldots, Y_n\varphi \big).$$

We recall that the non linear operator $W^{\varphi}\varphi$ has been used in [1] and [5] - and with a different but equivalent notation in [8] - to give the first characterization of the class of functions $\varphi : \mathbb{W} \to \mathbb{V}$, (when $\mathbb{V}$ is 1-dimensional), that we call here uniformly intrinsic differentiable. Precisely, they prove that 1-codimensional H-regular surfaces in $\mathbb{H}^n$ are intrinsic graphs of functions $\varphi : \mathbb{W} \to \mathbb{V} \equiv \mathbb{R}$ such that $W^{\varphi}\varphi$ - appropriately interpreted - is continuous. We have also to mention that in [8] the authors work in the more general setting of sub-riemannian manifolds.

This final example, simply indicates how it looks the non linear system, analogous to the $W^{\varphi}\varphi$ operator, that we get when we deal with 2-codimensional surfaces.

Example 5.6. *(Codimension 2 submanifolds in $\mathbb{H}^2$)* Let $\mathbb{H}^2 = \mathbb{W} \cdot \mathbb{V}$, with

$$\mathbb{W} = \{(0, 0, w_3, w_4, w_5)\}, \quad \mathbb{V} = \{(v_1, v_2, 0, 0, 0)\}.$$

Let $f : \mathbb{W} \to \mathbb{V}$ be a sufficiently regular function defined as

$$f(w) = (f_1(w), f_2(w), 0, 0, 0).$$

If $f(e) = e$, then

$$\begin{aligned} D_{Y_1} f(e) &= \left(\partial_3 f_{1|_e}, \partial_3 f_{2|_e}, 0, \ldots, 0 \right) \\ D_{Y_2} f(e) &= \left(\partial_4 f_{1|_e}, \partial_4 f_{2|_e}, 0, \ldots, 0 \right). \end{aligned}$$

For the general case, let $p = w \cdot f(w)$ then, from Proposition 3.9, for all $x = (0, 0, x_3, x_4, x_5) \in \mathbb{W}$,

$$
\begin{aligned}
&f_{p^{-1}}(x)\\
&= f(w)^{-1} \cdot f\left(w \cdot f(w) \cdot x \cdot f(w)^{-1}\right)\\
&= \big(f_1(0, 0, w_3 + x_3, w_4 + x_4, w_5 + x_5 + f_1(w)x_3 + f_2(w)x_4) - f_1(w),\\
&\qquad f_2(0, 0, w_3 + x_3, \text{etc.}) - f_2(w), 0, \dots, 0\big).
\end{aligned}
$$

Choosing $x = \exp tY_1$ or $x = \exp tY_2$ and recalling (53), we get

$$
\begin{aligned}
D_{Y_1} f(w) &= D_{Y_1} f_{p^{-1}}(e) = \big((\partial_3 f_1 + f_1 \partial_5 f_1)_{|w}, (\partial_3 f_2 + f_1 \partial_5 f_2)_{|w}, 0, \dots, 0\big)\\
D_{Y_2} f(w) &= D_{Y_2} f_{p^{-1}}(e) = \big((\partial_4 f_1 + f_2 \partial_5 f_1)_{|w}, (\partial_4 f_2 + f_2 \partial_5 f_2)_{|w}, 0, \dots, 0\big).
\end{aligned}
$$

Setting $D_{Y_j} f_i(w) := \partial_3 f_i(w) + f_j(w)\partial_5 f_i(w)$, for $i, j = 1, 2$ then, from Proposition 5.4, we get the following representation of df_w

$$
df_w(x) = \big(x_3 D_{Y_1} f_1(w) + x_4 D_{Y_2} f_1(w), x_3 D_{Y_1} f_2(w) + x_4 D_{Y_2} f_2(w), 0, \dots, 0\big).
$$

6. Euclidean and H-regular submanifolds

We gather here a couple of results showing a few relations between euclidean C^1 submanifolds and H-regular submanifolds.

Theorem 6.1 deals with low dimensional H-regular surfaces and we show that they coincide with the subset of euclidean C^1 submanifolds of $\mathbb{R}^{2n+1}$ whose (euclidean) tangent planes are cosets of horizontal subgroups of $\mathbb{H}^n$. For a general approach to this topic see [38] and [27].

Theorem 6.2 deals with low codimensional H-regular submanifolds. We show that intrinsic graphs of euclidean C^1 functions, defined over a vertical k-codimensional subgroup, are H-regular submanifolds. Hence, by Theorem 4.3, these euclidean C^1 functions are uniformly intrinsic differentiable. This result should be compared with Theorem 3.8 of [1], although the proof given here follows a different procedure.

Theorem 6.1. *Let $1 \le k \le n$. The following statements are equivalent*

(1) *$S \subset \mathbb{H}^n$ is a H-regular submanifold of dimension k.*
(2) *S is a k-dimensional euclidean C^1 submanifold of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and the euclidean tangent k-planes to S are cosets of k-dimensional horizontal subgroups of $\mathbb{H}^n$.*

Proof. **(1) $\Longrightarrow$ (2)** is proved in Theorem 3.5 of [17].

(2) $\Longrightarrow$ (1) Fixed a point $p \in S$, let $\mathcal{U} \subset \mathbb{H}^n$ and $\mathcal{A} \subset \mathbb{R}^k$ be respectively an open neighborhood of p and an open set in $\mathbb{R}^k$ and let $f : \mathcal{A} \to \mathcal{U}$ be an injective function, $f \in C^1(\mathcal{A}; \mathbb{H}^n)$ with injective euclidean differential, such that $S \cap \mathcal{U} = f(\mathcal{A})$. By assumption, fixed $\eta_0 \in \mathcal{A}$, for each $k-$tuple $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$, the euclidean differential of f in η_0 can be expressed as

$$
(d_e f)_{\eta_0}(\eta) = \langle \nabla f_1(\eta_0), \eta \rangle X_1(f(x_0)) + \dots + \langle \nabla f_{2n}(\eta_0), \eta \rangle X_{2n}(f(\eta_0));
$$

thus, recalling that, for $i = 1, \ldots, n$,

$$X_i(f(\eta_0)) = (0, \ldots, 1, \ldots\ldots, 0, -\frac{1}{2} f_{n+i}(\eta_0)),\ (1 \text{ at place } i)$$
$$X_{n+i}(f(\eta_0)) = (0, \ldots\ldots, 1, \ldots, 0, \frac{1}{2} f_i(\eta_0)),\ (1 \text{ at place } n+i),$$

it takes the form

$$\begin{aligned}(d_e f)_{\eta_0}(\eta) =& \big(\langle \nabla f_1(\eta_0), \eta\rangle, \ldots, \langle \nabla f_{2n}(\eta_0), \eta\rangle, \\ & - \frac{1}{2}(\sum_{i=1}^n f_{n+i}(\eta_0)\langle \nabla f_i(\eta_0), \eta\rangle - f_i(\eta_0)\langle \nabla f_{n+i}(x_0), \eta\rangle)\big) \\ =& (\langle \nabla f_1(\eta_0), \eta\rangle, \ldots, \langle \nabla f_{2n}(\eta_0), \eta\rangle).\end{aligned}$$

To obtain the H-regularity of S, it suffices to prove that the homogeneous homomorphism defined as

$$(d_H f)_{\eta_0}(\eta) := (\langle \nabla f_1(\eta_0), \eta\rangle, \ldots, \langle \nabla f_{2n}(\eta_0), \eta\rangle, 0),$$

for all $\eta \in \mathbb{R}^k$, is the Pansu differential of f in η_0.

Indeed, supposing, without loss of generality, $\eta_0 = 0$ and $f(\eta_0) = 0$ because, otherwise, it suffices to apply the following result to the translated surface $\tau_{p^{-1}} S$ passing through the origin, we have to prove that

$$\|(-\langle \nabla f_1(0), \eta\rangle, \ldots, -\langle \nabla f_{2n}(0), \eta\rangle, 0) \cdot f(\eta)\| = o(\|\eta\|_{\mathbb{R}^k}) \tag{54}$$

that is, for our assumption,

$$|f_{2n+1}(\eta) + \frac{1}{2}\sum_{i=1}^n (f_i(\eta)\langle \nabla f_{n+i}(0), \eta\rangle - f_{n+i}(\eta)\langle \nabla f_i(0), \eta\rangle)| = o(\|\eta\|^2_{\mathbb{R}^k}). \tag{55}$$

Indeed, there exists $\theta = \theta(\eta) \in \mathbb{R}^k$, θ in the line segment joining 0 and η, such that the left side of (55) becomes

$$\begin{aligned}&\Big| -\frac{1}{2}\sum_{i=1}^n (f_{n+i}(\theta)\langle \nabla f_i(\theta), \eta\rangle - f_i(\theta)\langle \nabla f_{n+i}(\theta), \eta\rangle) \\ &\qquad + \frac{1}{2}\sum_{i=1}^n (f_i(\eta)\langle \nabla f_{n+i}(0), \eta\rangle - f_{n+i}(\eta)\langle \nabla f_i(0), \eta\rangle)\Big| \\ &\leq \frac{1}{2}|\eta| \sum_{i=1}^n \Big(|f_{n+i}(\theta)\nabla f_i(\theta) - f_i(\theta)\nabla f_{n+i}(\theta)| - \\ &\qquad - |f_i(\eta)\nabla f_{n+i}(0) - f_{n+i}(\eta)\nabla f_i(0)|\Big) = o(\|\eta\|^2_{\mathbb{R}^k}),\end{aligned}$$

where the last equality follows from the continuity of ∇f in $\mathcal{A}$. □

Theorem 6.2. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.21. If $\varphi : \mathcal{A} \subset \mathbb{W} \equiv \mathbb{R}^{2n+1-k} \to \mathbb{V} \equiv \mathbb{R}^k$ is an Euclidean C^1 function, then* graph (φ) *is a H-regular submanifold of codimension k and φ is uniformly intrinsic differentiable in $\mathcal{A}$.*

Proof. Choosing exponential coordinate in $\mathbb{H}^n$ related to $\mathbb{W}$ and $\mathbb{V}$, we can identify $\mathbb{V}$ with $\mathbb{R}^k$ and $\mathbb{W}$ with $\mathbb{R}^{2n+1-k}$. Let $\mathcal{B} = \{p \in \mathbb{H}^n : p_{\mathbb{W}} \in \mathcal{A}\}$. Then $\mathcal{B}$ is open because $\mathcal{A}$ is relatively open in $\mathbb{W}$.

Define $f : \mathcal{B} \subset \mathbb{H}^n \to \mathbb{V} \equiv \mathbb{R}^k$ as

$$f(p) := \varphi(p_{\mathbb{W}})^{-1} \cdot p_{\mathbb{V}}.$$

Notice that f is Euclidean C^1 hence it is also in $C^1_H(\mathcal{B}, \mathbb{R}^k)$. Moreover $df_{|\mathbb{V}} : \mathbb{V} \to \mathbb{V} \equiv \mathbb{R}^k$ is the identity.

Hence, by Theorem 2.9, graph$\,(\varphi) = \{p : f(p) = 0\}$ is a k-codimensional H-regular submanifold and by Theorem 4.3 φ is uniformly intrinsic differentiable in $\mathcal{A}$.

□

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