HOMOGENISATION OF NONLINEAR DIRICHLET PROBLEMS IN RANDOMLY PERFORATED DOMAINS UNDER MINIMAL ASSUMPTIONS ON THE SIZE OF PERFORATIONS

LUCIA SCARDIA, KONSTANTINOS ZEMAS, AND CATERINA IDA ZEPPIERI

Abstract. In this paper we study the convergence of nonlinear Dirichlet problems for systems of variational elliptic PDEs defined on randomly perforated domains of $\mathbb{R}^n$. Under the assumption that the perforations are small balls whose centres and radii are generated by a *stationary short-range marked point process*, we obtain in the critical-scaling limit an averaged nonlinear analogue of the extra term obtained in the classical work of Cioranescu and Murat [12]. In analogy to the random setting recently introduced by Giunti, Höfer, and Velázquez [21] to study the Poisson equation, we only require that the random radii have finite $(n-q)$-moment, where $1<q<n$ is the growth-exponent of the associated energy functionals. This assumption on the one hand ensures that the expectation of the nonlinear q-capacity of the spherical holes is finite, and hence that the limit problem is well defined. On the other hand, it does not exclude the presence of balls with large radii, that can cluster up. We show however that the critical rescaling of the perforations is sufficient to ensure that no percolating-like structures appear in the limit.

1. Introduction

In this paper we study the limit behaviour as $\varepsilon \to 0^+$ of sequences of *nonlinear* Dirichlet boundary value problems of the type

$$\begin{cases} \mathcal{A}[u]=0 & \text{in} \quad D_\varepsilon \\ u=0 & \text{on} \quad \partial D_\varepsilon\,, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is an elliptic differential operator and $D_\varepsilon := D\setminus H_\varepsilon$ is obtained by removing from an open, bounded, Lipschitz set $D\subset\mathbb{R}^n$ a collection H_ε of small, spherical inclusions. Here we assume that $H_\varepsilon = H_\varepsilon^\omega$ is a *random* set, namely, that the centres of the spherical holes are generated according to a stationary point process in $\mathbb{R}^n$ and that the associated radii are (suitably scaled) unbounded random variables with short-range correlations. Under minimal assumptions both on the nonlinearity and on the set of perforations, we prove that, *almost surely*, solutions to (1.1) converge weakly in a suitable Sobolev space to the solution of a limit problem

$$\begin{cases} \mathcal{A}_0[u]=0 & \text{in} \quad D \\ u=0 & \text{on} \quad \partial D\,. \end{cases} \tag{1.2}$$

In (1.2), the nonlinear homogenised operator $\mathcal{A}_0$ depends both on $\mathcal{A}$ and on the geometry of the perforated domain, through some sort of limit averaged nonlinear capacity density of H_ε^ω. In particular our result extends the stochastic homogenisation result of Giunti, Höfer, and Velázquez [21] for the Poisson equation to the nonlinear setting.

The study of homogenisation problems in perforated domains has a long history with seminal contributions of Marchenko and Khruslov [22, 23], Cioranescu and Murat [12], and Papanicolaou and Varadhan [24] (see also [10, 15, 16]). In a *periodic* setting a typical H_ε is chosen as

$$H_\varepsilon = \bigcup_{i\in\mathbb{Z}^n} \overline{B}_{\varepsilon^\alpha\rho}(\varepsilon i)\,, \tag{1.3}$$

for some $\rho>0$ and $\alpha>1$. In (1.3) the parameter ε represents the characteristic distance between the centres of the spherical holes, while $\varepsilon^\alpha \ll \varepsilon$ is proportional to the size of their (common) radius. If, moreover, the *linear* case of the Poisson equation is considered, the boundary value problem (1.1) becomes

$$\begin{cases} -\Delta u = \psi & \text{in} \quad D_\varepsilon \\ u = 0 & \text{on} \quad \partial D_\varepsilon \,, \end{cases} \tag{1.4}$$

where $\psi \in W^{-1,2}(D)$. In this linear, periodic framework, Cioranescu and Murat [12] showed the existence of a critical scaling for the perforation radius such that the sequence of solutions (u_ε) to (1.4) converges weakly to the solution of a limit Dirichlet problem. Namely, assuming that $n > 2$, for $\alpha = n/(n-2)$ and $\varepsilon > 0$, the unique solution $u_\varepsilon \in W_0^{1,2}(D_\varepsilon)$ to (1.4) converges weakly in $W^{1,2}(D)$, as $\varepsilon \to 0^+$, to the unique solution $u_0 \in W_0^{1,2}(D)$ of

$$\begin{cases} -\Delta u + \mu_0 u = \psi & \text{in} \quad D \\ u = 0 & \text{on} \quad \partial D. \end{cases} \tag{1.5}$$

In (1.5) the zero-order term $\mu_0 u$ is reminiscent of the homogeneous Dirichlet boundary conditions prescribed on the boundary of the spherical holes in (1.4), and μ_0 is a positive constant of geometric nature which represents the limit *capacity density* generated by the set H_ε. Namely, we have

$$\mu_0 = \lim_{\varepsilon \to 0^+} \operatorname{Cap}(H_\varepsilon \cap Q, \mathbb{R}^n)\,,$$

where Q is a unit cube in $\mathbb{R}^n$ and $\operatorname{Cap}(H_\varepsilon \cap Q, \mathbb{R}^n)$ denotes the elliptic, or 2-capacity of $H_\varepsilon \cap Q$ in $\mathbb{R}^n$; *i.e.*,

$$\operatorname{Cap}(H_\varepsilon \cap Q, \mathbb{R}^n) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}x \colon v \in W_0^{1,2}(\mathbb{R}^n), \ v \equiv 1 \text{ on } H_\varepsilon \cap Q \right\}.$$

In view of (1.3), by the subadditivity of the capacity and the fact that $\varepsilon^\alpha \ll \varepsilon$ it is easy to see that

$$\mu_0 = \lim_{\varepsilon \to 0^+} \operatorname{Cap}\left(\bigcup_{i \in \mathbb{Z}^n} B_{\varepsilon^\alpha \rho}(\varepsilon i) \cap Q, \mathbb{R}^n \right) = \operatorname{Cap}(B_\rho(0), \mathbb{R}^n)\,, \tag{1.6}$$

where the last equality follows from the fact that the number of holes in Q is of order ε^{-n}, and $\operatorname{Cap}(B_{\varepsilon^\alpha \rho}(\varepsilon i), \mathbb{R}^n) = \varepsilon^n \operatorname{Cap}(B_\rho(0), \mathbb{R}^n)$ for every $i \in \mathbb{Z}^n$, if $\alpha = n/(n-2)$. Moreover, in this case the constant μ_0 is explicit and given by

$$\mu_0 = (n-2)\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\rho^{n-2}\,.$$

Choices for the scaling of the perforation radius different from the critical value $\alpha = n/(n-2)$ give trivial convergence results. More precisely, in the case of *tiny holes*, corresponding to the choice $\alpha > n/(n-2)$, it is immediate to see that $\mu_0 = 0$, so that the limit problem (1.5) reduces to

$$\begin{cases} -\Delta u = \psi & \text{in} \quad D \\ u = 0 & \text{on} \quad \partial D\,. \end{cases}$$

For *large holes*, corresponding to choosing $\alpha < n/(n-2)$, the sequence of solutions to (1.4) converges strongly to zero in $W^{1,2}(D)$ (see [1, Lemma 3.4.1]).

In the last three decades the result of Cioranescu and Murat [12] has been extended in a number of directions, ranging from the case of general *nonlinear* elliptic operators in periodically perforated domains [3, 15] to the case where a *random* distribution of holes is also allowed [7, 8, 9, 18, 20, 21], just to mention a few examples.

As far as a nonlinear variant of [12] is concerned, in [3] Ansini and Braides proved a *nonlinear vectorial* version of the Cioranescu and Murat result when the Dirichlet boundary value problem (1.1) is of variational nature; *i.e.*, when $\mathcal{A}[u] = 0$ in D_ε is the Euler-Lagrange system associated to an integral functional defined on D_ε. In [3] the corresponding limit problem is obtained by resorting to a direct Γ-convergence approach instead of the more classical PDE approach. The variational methods used in [3] allow for minimal assumptions on the integral functionals and hence on the nonlinearity; these assumptions are the same considered in this paper and will be discussed below. On the other hand, as far as the geometry of the perforated domain is concerned, in [21] Giunti, Höfer, and Velázquez proved a stochastic counterpart of [12] for the linear problem

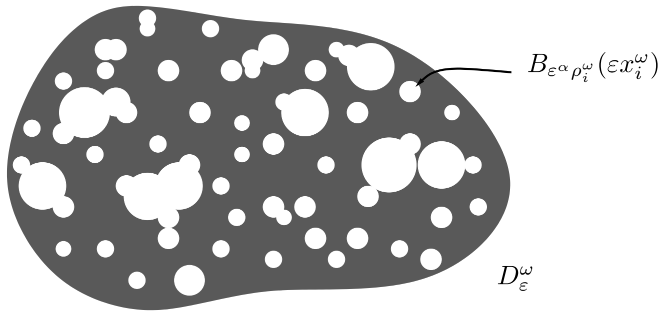

Figure 1. Realisation of the random domain D_ε^ω.

(1.4), where $H_\varepsilon = H_\varepsilon^\omega$ is a *random set* given by the union of small balls with random centres and radii, for which clusters occur with overwhelming probability (see Figure 1 for an illustration). In fact, the assumptions on H_ε^ω formulated in [21] are shown to be the minimal ones in order to have homogenisation.

The aim of this work is to combine the two general frameworks described above to devise *minimal assumptions* both on the nonlinearity and on the random set of the spherical perforations H_ε^ω, for which, almost surely, the corresponding Dirichlet problems (1.1) admit a homogenised limit of the type (1.2).

For the sake of the exposition, to illustrate our main result we consider here a prototypical random geometry for the set H_ε^ω, while we refer the reader to Section 2.3 for the more general probabilistic framework considered in the paper.

In what follows $(\Omega, \mathcal{T}, \mathbb{P})$ denotes a given underlying probability space. We consider a marked point process $(\Phi, \mathcal{R})$ on $\mathbb{R}^n \times \mathbb{R}_+$ where Φ is a Poisson point process, $\Phi(\omega) := (x_i)_i$, of constant intensity $0 < \lambda < +\infty$; *i.e.*, the average number of points of the process per unit volume satisfies $\langle N(Q)\rangle = \lambda$, where $N(Q) = \#\big(\Phi \cap Q\big)$. For the marks we assume that $\mathcal{R}(\omega) = (\rho_i)_{x_i \in \Phi(\omega)}$, with ρ_i identical and independently distributed *unbounded* random variables.

The random set of perforations associated to $(\Phi, \mathcal{R})$ is defined as

$$H_\varepsilon^\omega := \bigcup_{x_i \in \Phi(\omega)} \overline{B}_{\varepsilon^\alpha \rho_i}(\varepsilon x_i)\,, \tag{1.7}$$

where $\alpha = n/(n-2)$ is the critical scaling for problems with quadratic growth. The analogue of (1.6) follows from the strong law of large numbers, which guarantees that *almost surely*

$$\lim_{\varepsilon \to 0^+} \mathrm{Cap}(H_\varepsilon^\omega \cap Q, \mathbb{R}^n) \le \lim_{\varepsilon \to 0^+} \varepsilon^n \sum_{x_i \in \Phi(\omega) \cap (\varepsilon^{-1} Q)} \mathrm{Cap}(B_{\rho_i}(\varepsilon x_i), \mathbb{R}^n) = \lambda \langle \mathrm{Cap}(B_\rho(0), \mathbb{R}^n)\rangle\,. \tag{1.8}$$

Moreover, an explicit calculation gives

$$\widetilde{\mu}_0 = \lambda \langle \mathrm{Cap}(B_\rho(0), \mathbb{R}^n)\rangle = (n-2)\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\lambda\langle \rho^{n-2}\rangle\,, \tag{1.9}$$

where clearly $\widetilde{\mu}_0$ reduces to μ_0 if $\Phi = \mathbb{Z}^n$ and $\rho_i = \rho$, constant and deterministic.

We observe that in view of (1.9), the minimal condition for $\widetilde{\mu}_0$ to be well-defined is that the following stochastic integrability condition

$$\langle \rho^{n-2}\rangle < +\infty \tag{1.10}$$

holds. We note, however, that (1.10) does not prevent the balls generating H_ε^ω from overlapping. Indeed, it is easy to check that the expected number of holes that may overlap (that is, for which $\varepsilon^{n/(n-2)}\rho_i > \varepsilon$, with $\lambda\varepsilon$ being the typical distance between two points in Φ) is of the order ε^{-n+2}, while the expected total number of holes is of order ε^{-n}. In [21] Giunti, Höfer, and Velázquez proved that, even though with probability one there are regions where the holes cluster, the moment condition (1.10) is indeed sufficient to ensure that almost surely these regions have a vanishing capacity, as $\varepsilon \to 0^+$. Moreover, this moment condition allows to extend the Cioranescu and Murat construction of the "oscillating test functions" to this random setting and to prove that the stochastic analogue of [12] holds true, almost surely, up to replacing in (1.5) μ_0 with $\widetilde{\mu}_0$.

In this paper we extend the result by Giunti, Höfer, and Velázquez to the nonlinear vectorial setting. In the same spirit as in [3], we work with functionals rather than with the associated Euler-Lagrange systems. Therefore, we consider $1 < q < n$, $m \in \mathbb{N}$, and a Borel-measurable function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, with $f(0) = 0$, satisfying a q-growth and coercivity condition; *i.e.*,

$$c_1(|\xi|^q - 1) \le f(\xi) \le c_2(|\xi|^q + 1) \quad \forall\, \xi \in \mathbb{R}^{m\times n}$$

with $0 < c_1 < c_2$. Then, we introduce the vectorial, random functionals defined on $W^{1,q}_0(D;\mathbb{R}^m)$ as

$$\mathcal{F}^\omega_\varepsilon(u) := \int_{D\setminus H^\omega_\varepsilon} f(\nabla u)\,\mathrm{d}x, \quad \text{ if } \ u = 0 \ \text{ in } \ H^\omega_\varepsilon \cap D\,, \tag{1.11}$$

and $+\infty$ otherwise. In (1.11), H^ω_ε is as in (1.7) with α being the critical scaling in the case of q-growth, namely $\alpha = n/(n-q)$. Moreover, for every $\xi \in \mathbb{R}^{m\times n}$ we set

$$g(\xi) = \lim_{\varepsilon\to 0^+} \varepsilon^{nq/(n-q)} Qf(\varepsilon^{-n/(n-q)}\xi)\,,$$

where Qf denotes the quasiconvex envelope of f. Upon passing to a subsequence the function g is always well-defined (see Section 3 for more details), moreover the limit in ε becomes redundant when f (and hence Qf) is positively homogeneous of degree q.

Eventually, for every $z \in \mathbb{R}^m$ we define the random variables

$$\varphi_{\rho_i}(z) := \inf\left\{ \int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x \colon \zeta - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m),\ \zeta \equiv 0 \text{ on } \overline{B}_{\rho_i}(0) \right\}.$$

We observe that for $\omega \in \Omega$ fixed, $\varphi_\rho(z)$ can be interpreted as a *nonlinear g-capacity density* of the ball $B_\rho(0)$ in $\mathbb{R}^n$; furthermore, for $f(\xi) = |\xi|^q$ we have

$$\varphi_\rho(z) = \mathrm{Cap}_q(B_\rho(0), \mathbb{R}^n)|z|^q\,,$$

where $\mathrm{Cap}_q(B_\rho(0), \mathbb{R}^n)$ denotes the classical q-capacity of $B_\rho(0)$ relative to $\mathbb{R}^n$ (cf. (3.11)). In general, however, φ_ρ is not positively q-homogeneous as a function of z (as observed by Casado-Diaz and Garroni in [10]), hence in the definition of $\varphi_\rho(z)$ (and in the case $m = 1$) we cannot reduce to the case $z = 1$ just by unscaling.

Thanks to the growth conditions of order q satisfied by f, it is easy to show that

$$C_1|z|^q\rho^{n-q} \le \varphi_\rho(z) \le C_2|z|^q\rho^{n-q}\,,$$

for some $0 < C_1 < C_2$ only depending on n, q, c_1, and c_2 (see Lemma 4.1). Hence, in analogy to (1.9)-(1.10), in this nonlinear framework the homogenised problem is well-defined up to assuming the stochastic integrability condition

$$\langle \rho^{n-q}\rangle < +\infty \tag{1.12}$$

for the perforations radii.

The main result of this paper is Theorem 3.2, which establishes an *almost sure* Γ-convergence result for the random functionals $\mathcal{F}^\omega_\varepsilon$. Namely, Theorem 3.2 states that if $(\Phi, \mathcal{R})$ is a marked point process as above whose marks additionally satisfy the moment condition (1.12), then, almost surely, the functionals $\mathcal{F}^\omega_\varepsilon$ Γ-converge, as $\varepsilon \to 0^+$, to the *deterministic* functional given by

$$\mathcal{F}_0(u) := \int_D Qf(\nabla u)\,\mathrm{d}x + \lambda \int_D \langle\varphi_\rho(u)\rangle\,\mathrm{d}x, \quad \text{if} \quad u \in W^{1,q}_0(D;\mathbb{R}^m)\,.$$

In particular, if f is convex and differentiable, by the fundamental property of Γ-convergence we deduce that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the unique solution $u^\omega_\varepsilon \in W^{1,q}_0(D;\mathbb{R}^m)$ to

$$\begin{cases} -\mathrm{div} Df(\nabla u^\omega_\varepsilon) = \psi & \text{in } D^\omega_\varepsilon \\ u^\omega_\varepsilon = 0 & \text{on } \partial D^\omega_\varepsilon \end{cases}$$

converges weakly in $W^{1,q}(D;\mathbb{R}^m)$ to the unique solution $u \in W_0^{1,q}(D;\mathbb{R}^m)$ of the boundary value problem

$$\begin{cases} -\mathrm{div} Df(\nabla u) + \lambda\varphi'(u) = \psi & \text{in } D \\ u = 0 & \text{on } \partial D\,, \end{cases}$$

where $\varphi(z) := \langle \varphi_\rho(z) \rangle$ and $\psi \in W^{-1,\frac{q}{q-1}}(D;\mathbb{R}^m)$. Thus, in particular, for $f(\xi) = |\xi|^q$ we obtain a convergence result for the q-Laplace equation in randomly perforated domains of general geometry.

To prove the Γ-convergence result in Theorem 3.2 we follow a proof methodology which is similar in spirit to that of Ansini and Braides [3]. This approach is purely variational and is based on a so-called "joining lemma on perforated domains" which allows to replace any sequence with equibounded energy $\mathcal{F}_\varepsilon^\omega$ with a sequence which is constant in a spherical layer surrounding each perforation, without essentially changing the energy. In [3] this construction is then pivotal both in the proof of the lower-bound and in the upper-bound inequalities. Indeed, when proving the lower-bound inequality the joining lemma allows to estimate separately the energy close to and far from the perforations. Moreover, using the modified sequence it is possible to recover the nonlinear capacitary term as the limit of some suitable discrete energy densities. On the other hand, in the proof of the upper-bound inequality the joining lemma enters in the construction of the recovery sequence.

Crucially, in [3] to prove the joining lemma and thus the Γ-convergence result, it is of fundamental importance that the perforations are well-separated from one another. In the periodic setting this is a straightforward consequence of the regular arrangement of the holes on an ε-scale together with the $\varepsilon^\alpha \ll \varepsilon$ scaling for their (constant) radius. Whereas, as already observed, in our framework for almost every realisation we have to deal with the presence of large radii or centres very close to each other and thus of clustering holes. This technical issue is tackled similarly as in [21] by showing that, almost surely, the set of perforations H_ε^ω can be partitioned into two sets: a set of "good" holes $H_{\varepsilon,g}^\omega$ and a set of "bad" holes $H_{\varepsilon,b}^\omega$. In the set of good holes we identify a subset of "small" balls which are "ε-separated" from one another, in which then a stochastic variant of the joining lemma holds. The set of bad holes $H_{\varepsilon,b}^\omega$ contains, among others, all those balls which overlap with probability one. We then show that $H_{\varepsilon,b}^\omega$ can be enclosed into a "safety layer" $D_{\varepsilon,b}^\omega$ which is still well-separated from $H_{\varepsilon,g}^\omega$, and such that the nonlinear g-capacity of $H_{\varepsilon,b}^\omega$ relative to $D_{\varepsilon,b}^\omega$ vanishes as $\varepsilon \to 0^+$. In other words, the set of bad holes is well-separated from $H_{\varepsilon,g}^\omega$ and is asymptotically negligible. Then, one of the main difficulties of this work is to show that the energy contribution relative to the "good" perforations is actually enough to reconstruct the nonlinear capacitary term in $\mathcal{F}_0$ (cf. Proposition 6.1).

Similarly, in the construction of the recovery sequences for the Γ-limit the only energy contribution that is relevant for the capacitary term is the one carried by the balls in $H_{\varepsilon,g}^\omega$. However, since a recovery sequence needs to be admissible for $\mathcal{F}_\varepsilon^\omega$, it has to vanish also in the bad balls and hence in particular in the clusters. This constraint makes for a rather delicate proof of the upper bound which also relies on a corrector-like construction in the bad holes for the q-Laplace equation (cf. Lemma 7.4).

This paper is organized as follows. In Section 2 we recall the basics of marked point processes $(\Phi, \mathcal{R})$, and we list the assumptions that the process generating the holes H_ε^ω needs to satisfy. These assumptions are quite mild: our analysis is valid for rather general stationary point processes Φ whose associated marks in $\mathcal{R}$ need not be independent, as long as their correlation-range is suitably controlled (cf. (2.11) and (2.12)). In Section 3 we state the main result of this paper, Theorem 3.2. To prove it, we need a number of technical results of both analytical and probabilistic nature. The analytical preliminaries are collected in Section 4 which contains, among other results, a variant of the joining lemma which is relevant in our case (cf. Lemma 4.7). Section 5 is instead entirely devoted to some probabilistic auxiliary results. In this section we prove, in particular, a version of the strong law of large numbers for correlated marked point processes in the nonlinear setting (cf. Proposition 5.6). In Section 6 we build upon sections 4 and 5 to prove a discrete approximation result for the nonlinear capacitary term (cf. Proposition 6.1). Section 7 is devoted to the proof of the Γ-convergence result Theorem 3.2 (cf. Proposition 7.1 and Proposition 7.2).

2. Notation and setup

In this section we collect some useful notation and we introduce the probabilistic setup.

2.1. **Notation.** We denote with $\mathcal{B}(\mathbb{R}^n)$ the σ-algebra of Borel subsets of $\mathbb{R}^n$. For every $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote by $\tau_x A$ the shift of the set A by x; *i.e.*, $\tau_x A := x + A$. The diameter of A is denoted with $\mathrm{diam}(A)$ and the characteristic function of A with χ_A. Given $A, B \subset \mathbb{R}^n$ we write $A \subset\subset B$ iff $\overline{A} \subset B$. By $\#A$ we denote the cardinality of a discrete set A. For $n, k \in \mathbb{N}$ we denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively.

Given $\rho > 0$ and $x \in \mathbb{R}^n$, we denote with $B_\rho(x)$ the open ball centred at x with radius ρ. (We also use the notation $B^n_\rho(x)$ to clarify the dimension, if needed.) We denote with $Q_\rho(x)$ the half-open cube centred at x with side-length $\rho > 0$, namely

$$Q_\rho(x) := x + \rho[-\tfrac{1}{2}, \tfrac{1}{2})^n \,,$$

and we omit the subscript when $\rho = 1$, so that $Q(x) = Q_1(x)$. The unit sphere in $\mathbb{R}^n$ is denoted with $\mathbb{S}^{n-1}$. Moreover, we use the notation $\beta_n := \mathcal{L}^n(B_1(0))$ for the volume of the unit ball in $\mathbb{R}^n$.

For every $a, b \in \mathbb{R}$ we use the standard notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. From time to time we use the notation $\overline{\lim}$ and $\underline{\lim}$ to indicate the lim sup and lim inf respectively. The symbols $\sim_{M_1,M_2,\dots}, \lesssim_{M_1,M_2,\dots}$ indicate that the corresponding equality, inequality is valid up to a (positive multiplicative) constant that depends only on the parameters $M_1, M_2, \dots$ and the space dimension, but is allowed to vary from line to line.

Let $(\Omega, \mathcal{T}, \mathbb{P})$ denote an underlying given probability space; the expected value of a random variable $X : \Omega \to \mathbb{R}$ with respect to the probability measure $\mathbb{P}$ is denoted by $\langle X \rangle$; *i.e.*,

$$\langle X \rangle := \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega) =: \int_{\mathbb{R}} x \,\mathrm{d}(X_* \mathbb{P})(x) \,,$$

where $X_*\mathbb{P}$ is the push-forward measure of $\mathbb{P}$ onto $\mathbb{R}$, or the probability distribution of X, defined via $(X_*\mathbb{P})(B) := \mathbb{P}(X^{-1}(B))$, for every $B \in \mathcal{B}(\mathbb{R})$.

2.2. **Marked point processes.** We refer the reader to [14, Chapter 9, Definitions 9.1.I - 9.1.IV], [25, Section 3.5] and [11, Chapter 4] for a systematic treatment of marked point processes.

Throughout the paper $(\Phi, \mathcal{R})$ denotes a marked point-process (in short, m.p.p.) where Φ is a point process (unmarked, called ground process) in $\mathbb{R}^n$, and $\mathcal{R}$ is the associated mark space, with marks in $\mathbb{R}_+$. For a fixed realisation $\omega \in \Omega$, the set $\Phi^\omega := \Phi(\omega) = \{x_i\}_{i\in\mathbb{N}}$ is a locally finite countable collection of points in $\mathbb{R}^n$; *i.e.*, $\Phi^\omega \cap B$ is a *finite* set for every bounded $B \in \mathcal{B}(\mathbb{R}^n)$. Similarly, for $\omega \in \Omega$, $\mathcal{R}^\omega := \mathcal{R}(\omega) = \{\rho_i\}_{i\in\mathbb{N}}$, where, for every $i \in \mathbb{N}$, $\rho_i \in \mathbb{R}_+$ is the mark associated to the point $x_i \in \mathbb{R}^n$.

The first moment measure of the point process Φ is the measure $\mu^{(1)}$ defined on $\mathcal{B}(\mathbb{R}^n)$ by $\mu^{(1)}(B) = \big\langle N(B) \big\rangle$, where $N(B) := \#\Phi(B)$, with $\Phi(B) := \Phi \cap B$ is the number of points of the process in B. This is also called the intensity measure of Φ. The Campbell Theorem connects the integration with respect to the probability measure $\mathbb{P}$ with integration in $\mu^{(1)}$, since

$$\Big\langle \sum_{x_i \in \Phi} g(x_i, \rho_i) \Big\rangle = \int_{\mathbb{R}^n \times \mathbb{R}_+} g(x, \rho)\, \mathrm{d}\mu^{(1)}(x)\, \mathrm{d}\mathcal{P}^{\mathcal{R}}_x(\rho) \tag{2.1}$$

for every nonnegative measurable function g on $\mathbb{R}^n \times \mathbb{R}_+$, where $\mathcal{P}^{\mathcal{R}}_x$ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, which can be interpreted as the distribution of the mark of a point at x. We assume that $\mathcal{P}^{\mathcal{R}}_x$ is absolutely continuous with respect to the one-dimensional Lebesgue measure and denote its density with $f_1(x, \rho)$, namely

$$\mathcal{P}^{\mathcal{R}}_x(B) = \int_B f_1(x, \rho)\, \mathrm{d}\rho \quad \text{for all } \; B \in \mathcal{B}(\mathbb{R}_+) \,. \tag{2.2}$$

Hence (2.1) can be written as

$$\Big\langle \sum_{x_i \in \Phi} g(x_i, \rho_i) \Big\rangle = \int_{\mathbb{R}^n \times \mathbb{R}_+} g(x, \rho)\, f_1(x, \rho)\, \mathrm{d}\mu^{(1)}(x)\, \mathrm{d}\rho. \tag{2.3}$$

The second moment measure of Φ is the measure $\mu^{(2)}$ defined on $\mathcal{B}(\mathbb{R}^{2n})$ by $\mu^{(2)}(B_1 \times B_2) = \big\langle N(B_1)N(B_2)\big\rangle$. The second-order factorial measure $\Theta^{(2)}$ of Φ is defined on $\mathcal{B}(\mathbb{R}^{2n})$ by

$$\Theta^{(2)}(B_1 \times B_2) = \mu^{(1)}(B_1 \cap B_2) + \mu^{(2)}(B_1 \times B_2). \tag{2.4}$$

In particular, $\Theta^{(2)}(B_1 \times B_2) = \mu^{(2)}(B_1 \times B_2)$ if $B_1 \cap B_2 = \emptyset$. A more refined version of the Campbell Theorem connects the integration with respect to the probability measure $\mathbb{P}$ with integration in $\Theta^{(2)}$, since

$$\Bigg\langle \sum_{x_i\in\Phi} \sum_{\substack{x_j\in\Phi \\ x_j\neq x_j}} g(x_i,\rho_i,x_j,\rho_j) \Bigg\rangle = \int_{(\mathbb{R}^n\times\mathbb{R}_+)^2} g(x,x',\rho,\rho')\,\mathrm{d}\Theta^{(2)}(x,x')\,\mathrm{d}\mathcal{P}^{\mathcal{R}}_{x,x'}(\rho,\rho') \tag{2.5}$$

for every nonnegative measurable function g on $(\mathbb{R}^n \times \mathbb{R}_+)^2$, where $\mathcal{P}^{\mathcal{R}}_{x,x'}$ is a probability measure on $((\mathbb{R}_+)^2, \mathcal{B}((\mathbb{R}_+)^2)$, which can be interpreted as the two-point mark distribution. It gives the joint distribution of the marks at the two locations x and x', under the condition that there are points of Φ at x and x'. We assume that $\mathcal{P}^{\mathcal{R}}_{x,x'}$ is absolutely continuous with respect to the two-dimensional Lebesgue measure and denote its density with $f_2\big((x,\rho),(x',\rho')\big)$, namely

$$P^{\mathcal{R}}_{x,x'}(B\times C) = \int_B\int_C f_2\big((x,\rho),(x',\rho')\big)\mathrm{d}\rho\,\mathrm{d}\rho' \qquad \text{for all } B, C \in \mathcal{B}(\mathbb{R}_+)\,. \tag{2.6}$$

Hence (2.5) can be written as

$$\Bigg\langle \sum_{x_i\in\Phi} \sum_{\substack{x_j\in\Phi \\ x_j\neq x_j}} g(x_i,\rho_i,x_j,\rho_j) \Bigg\rangle = \int_{(\mathbb{R}^n\times\mathbb{R}_+)^2} g(x,x',\rho,\rho')\, f_2\big((x,\rho),(x',\rho')\big)\,\mathrm{d}\Theta^{(2)}(x,x')\,\mathrm{d}\rho\,\mathrm{d}\rho'. \tag{2.7}$$

2.3. **Assumptions on the m.p.p. $(\Phi,\mathcal{R})$.** Below we list the assumptions we require on the m.p.p. $(\Phi,\mathcal{R})$; these are in the same spirit as the ones formulated in [21, Section 2].

(H1) The point process Φ is *stationary*; *i.e.*, for every $x \in \mathbb{R}^n$ the processes $\tau_x\Phi := \{x + x_i\}_{x_i\in\Phi}$ and Φ have the same probability distribution. This implies in particular that the intensity measure $\mu^{(1)}$ of Φ is a multiple of the Lebesgue measure, namely

$$\mu^{(1)}(B) = \big\langle N(B)\big\rangle = \lambda\mathcal{L}^n(B),$$

where $\lambda > 0$ is called the intensity of the process and it is possibly infinite.

(H2) The point process Φ has finite intensity $0 < \lambda < +\infty$. In particular it is *locally square integrable*; *i.e.*, for every unitary cube $Q \subset \mathbb{R}^n$,

$$\big\langle (N(Q))^2\big\rangle \le \lambda^2\,. \tag{2.8}$$

We note that the stationarity of Φ ensures that the bound in (2.8) is independent of the centre of the cube Q.

(H3) The point process Φ satisfies the following *strong mixing condition.* For $A \in \mathcal{B}(\mathbb{R}^n)$, let $\mathcal{T}(A)$ denote the smallest σ-algebra with respect to which the random variables $N(B)$ are $\mathbb{P}$-measurable for every Borel subset $B \subset A$. We assume that there exist constants $C > 0$ and $\gamma > n$ with the following property. For every $A \in \mathcal{B}(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ with $|x| > \mathrm{diam}(A)$, and for every random variables Z_1, Z_2 measurable with respect to $\mathcal{T}(A), \mathcal{T}(\tau_x A)$ respectively, there holds

$$|\langle Z_1Z_2\rangle - \langle Z_1\rangle\langle Z_2\rangle| \le \frac{C}{1 + (|x| - \mathrm{diam}(A))^\gamma}\langle Z_1^2\rangle^{1/2}\langle Z_2^2\rangle^{1/2}\,. \tag{2.9}$$

We observe that (2.9) in particular ensures the ergodicity of Φ, cf. [14, Paragraph 12.3].

(H4) Let f_1 and f_2 be as in (2.2) and (2.6), respectively. In view of the stationarity of Φ, the density f_1 is independent of x; *i.e.*, for every $x \in \mathbb{R}^n$ we have $f_1(x,\rho) = h(\rho)$, for some $h \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ with $\int_{\mathbb{R}_+} h(\rho)\,\mathrm{d}\rho = 1\,.$

We assume that h satisfies the following integrability condition:

$$\int_0^{+\infty} \rho^{n-q} h(\rho)\,\mathrm{d}\rho < +\infty\,, \tag{2.10}$$

which is equivalent to asking $\langle \rho^{n-q}\rangle < +\infty$. For the density f_2 the stationarity of Φ implies that its dependence on x, x' is only via $x - x'$. Moreover we assume that

$$f_2\big((x,\rho),(x',\rho')\big) = h(\rho)h(\rho') + K(|x-x'|,\rho,\rho')\,, \tag{2.11}$$

for some function K satisfying

$$\int_{\mathbb{R}_+\times\mathbb{R}_+} K(r,\rho,\rho')\,\mathrm{d}\rho\,\mathrm{d}\rho' = 0 \quad \text{for every } r\geq 0\,,$$

and

$$|K(r,\rho,\rho')| \leq \frac{C}{(1+r^\gamma)(1+\rho^s)(1+(\rho')^s)}\,, \tag{2.12}$$

for some $C>0$ and $s > n-q$, and where $\gamma > n$ is the constant in (H3).

The interested reader is referred to [21, Subsection 2.1] for some explicit examples of m.p.p. $(\Phi,\mathcal{R})$ satisfying (H1)–(H4).

Remark 2.1 (Independent marking)**.** Under our assumptions the marks have the same distribution, but they are not independent. If the m.p.p. is in addition independently marked, then the expression of f_2 simplifies to

$$f_2\big((x,\rho),(x',\rho')\big) = h(\rho)h(\rho')\,.$$

The additional (location-dependent) term in (2.11) introduces a *short-range correlation* between the marks conditioned on the point positions, thus giving a measure of the lack of independence of the marks. We note that if, conditional to the point process Φ, all the marks are independent but not necessarily identically distributed, then the density of the 2-point mark distribution still factorises as

$$f_2\big((x,\rho),(x',\rho')\big) = f_1(x,\rho)\, f_1(x',\rho')\,,$$

but is location-dependent.

Remark 2.2. As in [21], the assumptions (H1)–(H4) guarantee the validity of the *strong law of large numbers-type results* stated in Section 5 (see Lemmata 5.4–5.6 therein), which will play an important role in what follows.

3. Statement of the main result

In this section we state the main result of the paper. To this end, we need to introduce some additional notation.

Let $n\in\mathbb{N}$ and $1<q<n$, and let $(\Phi,\mathcal{R})$ be a m.p.p. satisfying (H1)–(H4). For $\omega\in\Omega$ fixed, we consider a countable family of points $\Phi^\omega = (x_i)_i$ and the corresponding marks $\mathcal{R}^\omega = (\rho_i)_i$. For fixed $\varepsilon>0$, we associate to $(x_i,\rho_i)_i$ the family of open balls $\big(B_{\alpha_\varepsilon\rho_i}(\varepsilon x_i)\big)_i$, where

$$\alpha_\varepsilon := \varepsilon^{n/(n-q)}\,. \tag{3.1}$$

Let $D\subset\mathbb{R}^n$ be an open, bounded, Lipschitz set, star-shaped with respect to the origin. The set of random spherical perforations in D is given by

$$H_\varepsilon^\omega := \bigcup_{x_i\in\Phi^\omega\cap(\varepsilon^{-1}D)} \overline{B}_{\alpha_\varepsilon\rho_i}(\varepsilon x_i)\,; \tag{3.2}$$

note that the sets $(\varepsilon^{-1}D)_{\varepsilon>0}$ are nested as $\varepsilon\to 0^+$, since D is star-shaped with respect to the origin. We finally define the randomly perforated domain as

$$D_\varepsilon^\omega := D\setminus H_\varepsilon^\omega\,.$$

Now, let $m\in\mathbb{N}$, and let $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ be a Borel-measurable function of q-growth; *i.e.*, there exist two constants $0<c_1<c_2$, such that

$$c_1(|\xi|^q - 1) \leq f(\xi) \leq c_2(|\xi|^q+1)\ \forall\,\xi\in\mathbb{R}^{m\times n}\,. \tag{3.3}$$

Without loss of generality we assume that $f(0)=0$.

Finally, we introduce the nonlinear vectorial (random) functionals $\mathcal{F}^\omega_\varepsilon : L^1(D;\mathbb{R}^m) \longrightarrow \mathbb{R}\cup\{+\infty\}$ defined as

$$\mathcal{F}^\omega_\varepsilon(u) := \begin{cases} \displaystyle\int_{D^\omega_\varepsilon} f(\nabla u)\,\mathrm{d}x & \text{if } u\in W^{1,q}_0(D;\mathbb{R}^m) \text{ and } u=0 \text{ in } H^\omega_\varepsilon\cap D\,,\\ +\infty & \text{otherwise in } L^1(D;\mathbb{R}^m)\,. \end{cases} \tag{3.4}$$

We recall that the α_ε-scaling (3.1) for the radii of the perforations is the critical one in the case of energies with q-growth, under Dirichlet boundary conditions (cf. [3]).

The aim of this paper is to study the limit behaviour of the functionals $\mathcal{F}^\omega_\varepsilon$ as $\varepsilon\to 0^+$ (see Theorem 3.2).

Remark 3.1. Without loss of generality we can additionally assume that $H^\omega_\varepsilon \subset\subset D$ for $\varepsilon>0$ small enough. Indeed, we will see that the holes intersecting ∂D have both negligible volume and capacity (cf. Section 5).

We now define the nonlinear capacitary term which appears in the limit functional of $\mathcal{F}^\omega_\varepsilon$.

Let $(\varepsilon_j)\searrow 0$; we define the functions $g_j:\mathbb{R}^{m\times n}\to\mathbb{R}$ as

$$g_j(\xi) := \alpha^q_{\varepsilon_j} Qf(\alpha^{-1}_{\varepsilon_j}\xi) \quad \forall\,\xi\in\mathbb{R}^{m\times n}\,, \tag{3.5}$$

where α_ε is as in (3.1) and Qf denotes the *quasiconvex envelope* of f; *i.e.*,

$$Qf(\xi) := \inf\left\{\int_{(0,1)^n} f(\xi+\nabla\psi)\,\mathrm{d}x:\ \psi\in W^{1,q}_0((0,1)^n;\mathbb{R}^m)\right\}.$$

We note that g_j is quasiconvex for every $j\in\mathbb{N}$; moreover, by (3.3),

$$c_1(|\xi|^q-\alpha^q_{\varepsilon_j}) \le g_j(\xi) \le c_2(|\xi|^q+\alpha^q_{\varepsilon_j}) \tag{3.6}$$

for every $\xi\in\mathbb{R}^{m\times n}$. Invoking, *e.g.*, [5, Remark 4.13], we then get that (g_j) are locally equi-Lipschitz continuous, namely

$$|g_j(\xi_1)-g_j(\xi_2)| \le L(\alpha^{q-1}_{\varepsilon_j}+|\xi_1|^{q-1}+|\xi_2|^{q-1})|\xi_1-\xi_2|\,, \tag{3.7}$$

for every $\xi_1,\xi_2\in\mathbb{R}^{m\times n}$, every $j\in\mathbb{N}$, and for some constant $L:=L(c_1,c_2,q)>0$. Consequently, up to a subsequence (not relabelled), for every $\xi\in\mathbb{R}^{m\times n}$ there exists the limit

$$g(\xi) := \lim_{j\to+\infty} g_j(\xi)\,. \tag{3.8}$$

The function g is quasiconvex, and in view of (3.6) and (3.7) it satisfies the growth conditions

$$c_1|\xi|^q \le g(\xi) \le c_2|\xi|^q \quad \forall\,\xi\in\mathbb{R}^{m\times n}\,, \tag{3.9}$$

as well as the bound

$$|g(\xi_1)-g(\xi_2)| \le L(|\xi_1|^{q-1}+|\xi_2|^{q-1})|\xi_1-\xi_2| \quad \forall\,\xi_1,\xi_2\in\mathbb{R}^{m\times n}\,. \tag{3.10}$$

Given two open sets $A\subset\subset B\subset\mathbb{R}^n$, with A bounded, we define the q-capacity of A relative to B as

$$\mathrm{Cap}_q(A,B) := \inf\left\{\int_{\mathbb{R}^n}|\nabla v|^q\,\mathrm{d}x\colon v\in W^{1,q}_0(B;\mathbb{R}),\ v\equiv 1 \text{ on } \overline{A}\right\}. \tag{3.11}$$

In the definition above $\mathrm{Cap}_q(A,B)$ depends on A only via its closure $\overline{A}$, which is compact since A is bounded. Hence (3.11) agrees with the classical definition of q-capacity for compact sets, see *e.g.*, [17]. Note that, if $A\subset A'\subset\subset B\subset B'$, then

$$\mathrm{Cap}_q(A,B') \le \mathrm{Cap}_q(A,B) \le \mathrm{Cap}_q(A',B)\,. \tag{3.12}$$

Moreover, Cap_q is countably subadditive with respect to the first entry (see, e.g. [17], Section 3), namely

$$\text{if } A\subset\bigcup_{i=1}^\infty A_i, \quad \text{then } \mathrm{Cap}_q(A,B) \le \sum_{i=1}^\infty \mathrm{Cap}_q(A_i,B)\,. \tag{3.13}$$

Given $A \subset\subset B \subset \mathbb{R}^n$ and $z \in \mathbb{R}^m$, we define the *g-capacity* of A relative to B at z as

$$\mathrm{Cap}_g(A, B; z) := \inf\left\{ \int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x \colon \zeta - z \in W_0^{1,q}(B;\mathbb{R}^m),\ \zeta \equiv 0 \text{ on } \overline{A} \right\}.$$

As in the deterministic setting [3], the g-capacity of the ball (with respect to $\mathbb{R}^n$) appears in the definition of the limit functional. For this reason we consider

$$\varphi_\rho(z) := \mathrm{Cap}_g(B_\rho(0), \mathbb{R}^n; z) = \inf\left\{ \int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x \colon \zeta - z \in W_0^{1,q}(\mathbb{R}^n;\mathbb{R}^m),\ \zeta \equiv 0 \text{ on } \overline{B}_\rho(0) \right\}, \tag{3.14}$$

which is well-defined for every $z \in \mathbb{R}^m$ and every $\rho > 0$. Throughout the paper the function φ_ρ is referred to as the *nonlinear g-capacity density.*

We note that if $\rho \in \mathcal{R}$, namely $\rho : \Omega \to \mathbb{R}_+$ is a random variable, then $\omega \mapsto \varphi_{\rho^\omega}(z)$ is also a random variable. Indeed, for every $z \in \mathbb{R}^m$ the function $\rho \mapsto \varphi_\rho(z)$ is continuous (see Lemma 4.1 below and the estimates in (4.2)), hence the composite function $\omega \mapsto \varphi_{\rho^\omega}(z)$ is $\mathcal{T}$-measurable. We can then define the *average g-capacity density* at $z \in \mathbb{R}^m$ as the expected value of $\varphi_{\rho^\omega}(z)$. That is, we set

$$\varphi(z) := \langle \varphi_\rho(z) \rangle := \int_0^{+\infty} \varphi_\rho(z) h(\rho)\,\mathrm{d}\rho\,, \tag{3.15}$$

where $h \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ satisfies (2.10).

We are now in a position to state the main result of this paper.

Theorem 3.2. *Let $(\Phi, \mathcal{R})$ be a m.p.p. satisfying (H1)–(H4) and let $(\mathcal{F}_\varepsilon^\omega)_{\varepsilon>0}$ be the functionals defined in (3.4), for $\omega \in \Omega$. Then there exists a sequence $(\varepsilon_j) \searrow 0$ and a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$, such that for every $\omega \in \Omega'$*

$$\mathcal{F}_{\varepsilon_j}^\omega \xrightarrow{\Gamma} \mathcal{F}_0 \text{ with respect to the strong } L^1(D;\mathbb{R}^m)\text{-topology},$$

where $\mathcal{F}_0 : L^1(D;\mathbb{R}^m) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the deterministic functional defined as

$$\mathcal{F}_0(u) := \begin{cases} \displaystyle\int_D Qf(\nabla u)\,\mathrm{d}x + \lambda \int_D \varphi(u)\,\mathrm{d}x & \text{if } u \in W_0^{1,q}(D;\mathbb{R}^m)\,, \\ +\infty & \text{otherwise in } L^1(D;\mathbb{R}^m)\,, \end{cases} \tag{3.16}$$

with $N(Q)$ and φ as in (2.8) and (3.15), respectively.

In general the Γ-convergence result in Theorem 3.2 holds true only up to subsequences. In fact the limit density g appearing in the definition of φ clearly depends on the choice of the subsequence (cf. (3.5) and (3.8)). This phenomenon is typical of the nonlinear setting and is also observed in [3] (see Remark 2.7 therein).

Remark 3.3 (Convergence of the Euler-Lagrange equations)**.** Let (ε_j) be the vanishing sequence and $\Omega' \in \mathcal{T}$ be the set of probability one whose existence is established by Theorem 3.2. Let $\psi \in W^{-1,\frac{q}{q-1}}(D;\mathbb{R}^m)$ be fixed; we consider the functionals defined for $u \in L^1(D;\mathbb{R}^m)$ as

$$\mathcal{F}_{\varepsilon_j}^\omega(u) + \int_D \psi : u\,\mathrm{d}x\,, \tag{3.17}$$

where $\int_D \psi : u\,\mathrm{d}x$ denotes the duality pairing between $W_0^{1,q}(D;\mathbb{R}^m)$ and $W^{-1,\frac{q}{q-1}}(D;\mathbb{R}^m)$. By continuity it is immediate to check that for every $\omega \in \Omega'$ the functionals in (3.17) Γ-converge, with respect to the strong $L^1(D;\mathbb{R}^m)$-topology, to the deterministic functional defined for $u \in L^1(D;\mathbb{R}^m)$ as

$$\mathcal{F}_0(u) + \int_D \psi : u\,\mathrm{d}x\,.$$

Let now f be convex and differentiable; by the $L^1(D;\mathbb{R}^m)$ equi-coerciveness of (3.17) and the fundamental property of Γ-convergence we can deduce a convergence result for the corresponding

Euler-Lagrange equations. That is, for every $\omega \in \Omega'$, the sequence $(u^\omega_{\varepsilon_j})$, where $u^\omega_{\varepsilon_j} \in W^{1,q}_0(D;\mathbb{R}^m)$ is the unique solution to

$$\begin{cases} -\mathrm{div} Df(\nabla u^\omega_{\varepsilon_j}) = \psi & \text{in } D^\omega_{\varepsilon_j} \\ u^\omega_{\varepsilon_j} = 0 & \text{on } \partial D^\omega_{\varepsilon_j}\,, \end{cases}$$

converges weakly in $W^{1,q}(D;\mathbb{R}^m)$ to the unique solution $u_0 \in W^{1,q}_0(D;\mathbb{R}^m)$ of the following deterministic problem

$$\begin{cases} -\mathrm{div} Df(\nabla u_0) + \lambda\varphi'(u_0) = \psi & \text{in } D \\ u_0 = 0 & \text{on } \partial D\,. \end{cases}$$

Finally, we observe that when f is q-homogeneous, the convergence in (3.8) defining g holds true for the whole sequence. Hence the function g (and, consequently, φ in (3.15)) is independent of the subsequence. Therefore both the convergence of the functionals in Theorem 3.2 and the convergence of the optimality conditions above hold true for the whole sequence. Our result then provides an extension of [21, Theorem 2.1] to the nonlinear q-homogeneous vectorial setting.

4. Analytical building blocks

In this section we collect some analytical technical results which will be used in the proof of Theorem 3.2.

We start by establishing some properties of the nonlinear capacity density φ_ρ defined in (3.14). Moreover, we introduce some auxiliary capacity densities whose role will become apparent in the next sections. Finally, we state and prove a so-called *joining lemma*, Lemma 4.7, in the same spirit of [3, Lemma 3.1].

Lemma 4.1. *There exist two constants $C_1 := C_1(n,q,c_1), C_2 := C_2(n,q,c_2) > 0$ such that*

$$C_1|z|^q\rho^{n-q} \le \varphi_\rho(z) \le C_2|z|^q\rho^{n-q}\,, \tag{4.1}$$

for every $z \in \mathbb{R}^m$ and every $\rho > 0$, where $c_1, c_2 > 0$ are the constants in (3.3)*. Moreover, for every $0 < \rho_1 < \rho_2$ and every $z \in \mathbb{R}^m$ we have that*

$$\varphi_{\rho_1}(z) \le \varphi_{\rho_2}(z) \le \varphi_{\rho_1}(z)\Big(\frac{\rho_2}{\rho_1}\Big)^n\left(1 + \frac{L}{c_1\rho_2^q}\Big(\rho_1^{q-1} + \rho_2^{q-1}\Big)(\rho_2 - \rho_1)\right), \tag{4.2}$$

where L is the constant in (3.10).

Proof. Let $z \in \mathbb{R}^m$ and $\rho > 0$ be fixed; let

$$\widehat{\varphi}_\rho(z) := \inf\Big\{\int_{\mathbb{R}^n} |\nabla\zeta|^q\,\mathrm{d}x \colon \zeta - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m),\ \zeta \equiv 0 \text{ on } \overline{B}_\rho(0)\Big\} \tag{4.3}$$

be the nonlinear capacity density corresponding to the q-Dirichlet energy; namely $\widehat{\varphi}_\rho$ is the density defined as (3.14) in the model case $f(\xi) = g(\xi) = |\xi|^q$. The unique solution $\widehat{\zeta}_{z,\rho}$ of (4.3) satisfies the following q-Laplace boundary value problem

$$\begin{cases} \mathrm{div}(|\nabla\widehat{\zeta}_{z,\rho}|^{q-2}\nabla\widehat{\zeta}_{z,\rho}) = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}_\rho(0)\,, \\ \\ \widehat{\zeta}_{z,\rho} - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m), & \widehat{\zeta}_{z,\rho}|_{\overline{B}_\rho(0)} \equiv 0\,. \end{cases}$$

This can be computed explicitly and is given by the radially symmetric function

$$\widehat{\zeta}_{z,\rho}(x) = z\big(-(\rho/|x|)^{(n-q)/(q-1)} + 1\big)\chi_{\mathbb{R}^n\setminus\overline{B}_\rho(0)}.$$

A direct calculation yields

$$\widehat{\varphi}_\rho(z) = C_{n,q}|z|^q\rho^{n-q}\,, \tag{4.4}$$

where

$$C_{n,q} := \Big(\frac{n-q}{q-1}\Big)^{q-1}\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\,.$$

Therefore, gathering the upper bound in (3.9), (3.14), and (4.4), we get

$$\varphi_\rho(z) \le \int_{\mathbb{R}^n} g(\nabla\widehat{\zeta}_{z,\rho})\,\mathrm{d}x \le c_2\widehat{\varphi}_\rho(z) = c_2C_{n,q}|z|^q\rho^{n-q}\,,$$

which gives the second inequality in (4.1) with $C_2 := c_2 C_{n,q}$.

Similarly, for every ζ such that $\zeta - z \in W_0^{1,q}(\mathbb{R}^n;\mathbb{R}^m)$ and $\zeta \equiv 0$ on $\overline{B_\rho}(0)$, again using (3.9), (3.14), and (4.4), we obtain

$$\int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x \geq c_1 \int_{\mathbb{R}^n} |\nabla\zeta|^q\,\mathrm{d}x \geq c_1\widehat{\varphi}_\rho(z) = c_1 C_{n,q}|z|^q\rho^{n-q}\,.$$

Taking the infimum in ζ we get the first inequality in (4.1) with $C_1 := c_1 C_{n,q}$. This concludes the proof of (4.1).

For (4.2), note that the first inequality follows by monotonicity, since for $0 < \rho_1 < \rho_2$ every competitor ζ for the minimisation problem defining $\varphi_{\rho_2}(z)$ is also a competitor for the minimisation problem defining $\varphi_{\rho_1}(z)$. To prove the second inequality, let ζ_1 be the minimiser of the problem defining $\varphi_{\rho_1}(z)$. Set $\zeta_2(x) := \zeta_1(\rho_1 x/\rho_2)$; then ζ_2 is a competitor for the minimisation problem defining $\varphi_{\rho_2}(z)$. Moreover, by (3.10) and (3.9) we get

$$\begin{aligned}
g(\nabla\zeta_2(x)) &\leq g(\nabla\zeta_1(\rho_1 x/\rho_2)) + L\big(|\nabla\zeta_2(x)|^{q-1} + |\nabla\zeta_1(\rho_1 x/\rho_2)|^{q-1}\big)|\nabla\zeta_2(x) - \nabla\zeta_1(\rho_1 x/\rho_2)| \\
&\leq g(\nabla\zeta_1(\rho_1 x/\rho_2)) + L\left(1 + \Big(\frac{\rho_1}{\rho_2}\Big)^{q-1}\right)\left(1 - \frac{\rho_1}{\rho_2}\right)|\nabla\zeta_1(\rho_1 x/\rho_2)|^q \\
&\leq g(\nabla\zeta_1(\rho_1 x/\rho_2))\left(1 + \frac{L}{c_1\rho_2^q}\Big(\rho_1^{q-1} + \rho_2^{q-1}\Big)(\rho_2 - \rho_1)\right).
\end{aligned}$$

By integrating over $\mathbb{R}^n$, a change of variables and the definition of ζ_1 give

$$\int_{\mathbb{R}^n} g(\nabla\zeta_2)\,\mathrm{d}x \leq \varphi_{\rho_1}(z)\Big(\frac{\rho_2}{\rho_1}\Big)^n\left(1 + \frac{L}{c_1\rho_2^q}\Big(\rho_1^{q-1} + \rho_2^{q-1}\Big)(\rho_2 - \rho_1)\right).$$

Eventually, since ζ_2 is a competitor for the minimization problem defining $\varphi_{\rho_2}(z)$, we obtain

$$\varphi_{\rho_2}(z) \leq \varphi_{\rho_1}(z)\Big(\frac{\rho_2}{\rho_1}\Big)^n\left(1 + \frac{L}{c_1\rho_2^q}\Big(\rho_1^{q-1} + \rho_2^{q-1}\Big)(\rho_2 - \rho_1)\right)$$

and thus (4.2). □

Remark 4.2. As an immediate corollary of Lemma 4.1 (cf. the estimates in (4.1)) we have that $\varphi_\rho(u) \in L^1(D)$ whenever $u \in L^q(D;\mathbb{R}^m)$ and $0 < \rho < +\infty$. Moreover, again by Lemma 4.1 and by (2.10) and (3.15) we have also that $\varphi(u) \in L^1(D)$ whenever $u \in L^q(D;\mathbb{R}^m)$, where φ is defined in (3.15).

Let $\theta \in (0,1)$ and $\rho > 0$ be fixed. Let $(\varepsilon_j) \searrow 0$; we set

$$K_j := \frac{\varepsilon_j}{\alpha_{\varepsilon_j}} = \varepsilon_j^{-q/(n-q)}\,, \tag{4.5}$$

where α_{ε_j} is defined as in (3.1), and assume that $j \in \mathbb{N}$ is large enough to guarantee that

$$\theta K_j \geq 2\rho\,. \tag{4.6}$$

Moreover, for $z \in \mathbb{R}^m$ we define the class of functions

$$X^j_{\theta,\rho,z} := \Big\{\zeta\colon \zeta - z \in W_0^{1,q}(B_{\theta K_j}(0);\mathbb{R}^m),\ \zeta \equiv 0 \text{ on } \overline{B_\rho}(0)\Big\}\,,$$

and the auxiliary capacity densities

$$\varphi^j_{\theta,\rho}(z) := \inf\left\{\int_{B_{\theta K_j}(0)} g_j(\nabla\zeta)\,\mathrm{d}x\colon \zeta \in X^j_{\theta,\rho,z}\right\}. \tag{4.7}$$

We observe that the function $\rho \mapsto \varphi^j_{\theta,\rho}(z)$ is increasing. Indeed, if $0 < \rho_1 \leq \rho_2$, then every competitor for the minimisation problem defining $\varphi^j_{\theta,\rho_2}(z)$ is also a competitor for the minimisation problem defining $\varphi^j_{\theta,\rho_1}(z)$.

The next lemma is the analogue of Lemma 4.1 for the functions $\varphi^j_{\theta,\rho}$.

Lemma 4.3. *Let $(\varepsilon_j) \searrow 0$, $\theta \in (0,1)$ and $\rho > 0$ be fixed, and let K_j be defined as in (4.5). Assume that (4.6) is satisfied. Let $\varphi^j_{\theta,\rho}$ be as in (4.7); then*

(i) there exist constants $C_1, \dots, C_4 > 0$ depending only on n, q, c_1, c_2, such that for every $z \in \mathbb{R}^m$

$$\varphi^j_{\theta,\rho}(z) \geq C_1|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q} - C_2\theta^n \tag{4.8}$$

and

$$\varphi^j_{\theta,\rho}(z) \leq C_3|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q} + C_4\theta^n. \tag{4.9}$$

(ii) Let $M > 0$, and assume in addition that $\rho \in (0, M]$, and that $j \in \mathbb{N}$ is large enough so that

$$\theta K_j \geq 2M\,. \tag{4.10}$$

Then, there exists a constant $C_M > 0$, with $C_M \to +\infty$ as $M \to +\infty$, so that

$$|\varphi^j_{\theta,\rho}(z) - \varphi^j_{\theta,\rho}(w)| \leq C_M\big(\theta^{n(q-1)/q} + \alpha^{q-1}_{\varepsilon_j} + |z|^{q-1} + |w|^{q-1}\big)|z-w|\,, \tag{4.11}$$

for every $z, w \in \mathbb{R}^m$.

Proof. We start proving (i). To this end, for every $z \in \mathbb{R}^m$ we set

$$\widehat{\varphi}^j_{\theta,\rho}(z) := \inf\bigg\{\int_{B_{\theta K_j}(0)} |\nabla\zeta|^q\,\mathrm{d}x \colon \zeta \in X^j_{\theta,\rho,z}\bigg\}. \tag{4.12}$$

Similarly as in the proof of (4.1), the unique solution $\widehat{\zeta}^j_{\theta,\rho,z}$ to (4.12) can be computed explicitly. Moreover, a direct calculation gives

$$\widehat{\varphi}^j_{\theta,\rho}(z) = c_{n,q}|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q}\,, \tag{4.13}$$

where the constant $c_{n,q} > 0$ can be computed explicitly.

To prove (4.8), let $\zeta \in X^j_{\theta,\rho,z}$; note that ζ is a competitor for the minimisation problems defining $\varphi^j_{\theta,\rho}(z)$ and $\widehat{\varphi}^j_{\theta,\rho}(z)$. By combining (3.6) and (4.13), also recalling (3.1), (4.5), we find

$$\begin{aligned}\int_{B_{\theta K_j}(0)} g_j(\nabla\zeta)\,\mathrm{d}x &\geq c_1\bigg(\int_{B_{\theta K_j}(0)} |\nabla\zeta|^q\,\mathrm{d}x - \beta_n(\theta K_j)^n\alpha^q_{\varepsilon_j}\bigg) \geq c_1\big(\widehat{\varphi}^j_{\theta,\rho}(z) - \beta_n\theta^n\big)\\ &= c_1\big(c_{n,q}|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q} - \beta_n\theta^n\big)\,.\end{aligned}$$

Then, by taking the infimum over $\zeta \in X^j_{\theta,\rho,z}$ we obtain (4.8) with $C_1 := c_1c_{n,q} > 0$ and $C_2 := c_1\beta_n > 0$.

To prove (4.9) we use the fact that $\widehat{\zeta}^j_{\theta,\rho,z}$ is a competitor for the minimisation problem defining $\varphi^j_{\theta,\rho}(z)$, which together with (3.6) and (4.13) gives

$$\begin{aligned}\varphi^j_{\theta,\rho}(z) &\leq \int_{B_{\theta K_j}(0)} g_j(\nabla\widehat{\zeta}^j_{\theta,\rho,z})\,\mathrm{d}x \leq c_2\bigg(\int_{B_{\theta K_j}(0)} |\nabla\widehat{\zeta}^j_{\theta,\rho,z}|^q\,\mathrm{d}x + \beta_n(\theta K_j)^n\alpha^q_{\varepsilon_j}\bigg)\\ &= c_2\big(\widehat{\varphi}^j_{\theta,\rho}(z) + \beta_n\theta^n\big) = c_2\big(c_{n,q}|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q} + \beta_n\theta^n\big)\,,\end{aligned}$$

which proves (4.9) with $C_3 := c_2c_{n,q} > 0$ and $C_4 := c_2\beta_n > 0$.

We now prove (ii). By the quasiconvexity and coercivity of g_j (cf. (3.5) and (3.6)), the Direct Method of the Calculus of Variations ensure the existence of a function $\zeta^j_{\theta,\rho,z} \in X^j_{\theta,\rho,z}$ such that

$$\varphi^j_{\theta,\rho}(z) = \int_{B_{\theta K_j}(0)} g_j(\nabla\zeta^j_{\theta,\rho,z})\,\mathrm{d}x\,. \tag{4.14}$$

We now modify $\zeta^j_{\theta,\rho,z}$ to construct a competitor for the minimization problem defining $\varphi^j_{\theta,\rho}(w)$. To this end, we consider a radial cut-off function $\eta_M \in C^\infty_c(B_{2M}(0))$ satisfying

$$0 \leq \eta_M \leq 1\,, \quad \eta_M|_{\overline{B}_\rho(0)} \equiv 1\,, \quad \|\nabla\eta_M\|_{L^\infty} \leq \frac{1}{2M-\rho} \leq \frac{1}{M}\,, \tag{4.15}$$

where in the last inequality we used that $\rho \in (0, M]$. We now define the function $\zeta^j_{\theta,\rho,w}$ as

$$\zeta^j_{\theta,\rho,w} := \zeta^j_{\theta,\rho,z} + (1-\eta_M)(w-z)\,.$$

In view of (4.15) and (4.10), for $j \in \mathbb{N}$ large enough we have that $\eta_M|_{\partial B_{\theta K_j}} \equiv 0$, and therefore $\zeta^j_{\theta,\rho,w} \in X^j_{\theta,\rho,w}$, thus, it is an admissible competitor for the minimisation problem defining $\varphi^j_{\theta,\rho}(w)$. Note that

$$\nabla\zeta^j_{\theta,\rho,w} - \nabla\zeta^j_{\theta,\rho,z} = \nabla\eta_M \otimes (z-w) \in C^\infty_c(B_{2M}(0);\mathbb{R}^{m\times n})\,. \tag{4.16}$$

By (4.7), (4.14), (3.7), (3.6), (4.10) (4.15) and (4.16), we obtain

$$\begin{aligned}
\varphi^j_{\theta,\rho}(w) - \varphi^j_{\theta,\rho}(z) &\le \int_{B_{\theta K_j}(0)} \big|g_j(\nabla\zeta^j_{\theta,\rho,w}) - g_j(\nabla\zeta^j_{\theta,\rho,z})\big|\,\mathrm{d}x \\
&\le L\int_{B_{2M}(0)} \Big(\alpha^{q-1}_{\varepsilon_j} + |\nabla\zeta^j_{\theta,\rho,w}|^{q-1} + |\nabla\zeta^j_{\theta,\rho,z}|^{q-1}\Big)|\nabla\zeta^j_{\theta,\rho,w} - \nabla\zeta^j_{\theta,\rho,z}|\,\mathrm{d}x \\
&\le \frac{L}{M}\bigg(\beta_n(2M)^n\alpha^{q-1}_{\varepsilon_j} + \int_{B_{2M}(0)}\Big(|\nabla\zeta^j_{\theta,\rho,w}|^{q-1} + |\nabla\zeta^j_{\theta,\rho,z}|^{q-1}\Big)\,\mathrm{d}x\bigg)|w-z| \\
&\le \frac{C}{M}\bigg(M^n\alpha^{q-1}_{\varepsilon_j} + \frac{M^n}{M^{q-1}}|z-w|^{q-1} + M^{n/q}\Big(\int_{B_{2M}(0)}|\nabla\zeta^j_{\theta,\rho,z}|^q\Big)^{\frac{q-1}{q}}\bigg)|w-z| \\
&\le \frac{CM^n}{M}\left(\alpha^{q-1}_{\varepsilon_j} + |z|^{q-1} + |w|^{q-1} + \Big(\varphi^j_{\theta,\rho}(z) + M^n\alpha^q_{\varepsilon_j}\Big)^{\frac{q-1}{q}}\right)|w-z|\,,
\end{aligned} \tag{4.17}$$

where C is a positive constant depending only on c_1, c_2, n, q, L. Now, by (4.9)

$$\begin{aligned}
\varphi^j_{\theta,\rho}(z) &\le C_3|z|^q\big(\rho^{(q-n)/(q-1)} - (\theta K_j)^{(q-n)/(q-1)}\big)^{1-q} + C_4\theta^n \\
&\le C_3|z|^q\big(1 - 2^{(q-n)/(q-1)}\big)^{1-q}\rho^{n-q} + C_4\theta^n\,,
\end{aligned} \tag{4.18}$$

where the last inequality follows from (4.10), for $j \in \mathbb{N}$ large enough (depending on θ, M). Finally, (4.17) and (4.18) imply that

$$\begin{aligned}
\varphi^j_{\theta,\rho}(w) - \varphi^j_{\theta,\rho}(z) &\le CM^{n-1}\left(\alpha^{q-1}_{\varepsilon_j} + |z|^{q-1} + |w|^{q-1} + \Big(|z|^q\rho^{n-q} + \theta^n + M^n\alpha^q_{\varepsilon_j}\Big)^{\frac{q-1}{q}}\right)|w-z| \\
&\le CM^{2n-1}\Big(\theta^{n(q-1)/q} + \alpha^{q-1}_{\varepsilon_j} + |z|^{q-1} + |w|^{q-1}\Big)|w-z|\,.
\end{aligned}$$

Then, interchanging the role of z and w in the previous argument we obtain (4.11), and this concludes the proof. □

Note that, as detailed in the remark below, sequences of minimisers of $(\varphi^j_{\theta,\rho})_j$ are pre-compact, modulo a straightforward extension.

Remark 4.4 (Compactness after extension)**.** Let $\theta \in (0,1)$, $M > 0$, $\rho \in (0, M]$, $z \in \mathbb{R}^m$ be fixed, and let $j \in \mathbb{N}$ be so large that (4.10) holds. Let $(\zeta^j) \subset X^j_{\theta,\rho,z}$ be such that

$$\sup_{j\in\mathbb{N}} \int_{B_{\theta K_j}(0)} g_j(\nabla\zeta^j)\,\mathrm{d}x =: C_{\theta,\rho,z} < +\infty\,, \tag{4.19}$$

where g_j is as in (3.5). We set

$$\widetilde{\zeta}^j := \begin{cases} \zeta^j & \text{in } B_{\theta K_j}(0)\,, \\ z & \text{in } \mathbb{R}^n \setminus B_{\theta K_j}(0)\,. \end{cases}$$

Clearly $(\widetilde{\zeta}^j) \subset W^{1,q}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$, and $\widetilde{\zeta}^j - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m)$. Moreover, in view of (3.6), (3.1), (4.5), and (4.19) we have

$$\begin{aligned}\sup_{j\in\mathbb{N}}\int_{\mathbb{R}^n}|\nabla(\widetilde{\zeta}^j - z)|^q\,\mathrm{d}x = \sup_{j\in\mathbb{N}}\int_{\mathbb{R}^n}|\nabla\widetilde{\zeta}^j|^q\,\mathrm{d}x &= \sup_{j\in\mathbb{N}}\int_{B_{\theta K_j(0)}}|\nabla\zeta^j|^q\,\mathrm{d}x \\ &\lesssim \sup_{j\in\mathbb{N}}\Big(\int_{B_{\theta K_j(0)}} g_j(\nabla\zeta^j)\,\mathrm{d}x + (\theta K_j)^n\alpha^q_{\varepsilon_j}\Big) \\ &\lesssim C_{\theta,\rho,z} + \theta^n. \end{aligned}\tag{4.20}$$

Then, by the Sobolev Embedding Theorem we deduce the existence of $c_{n,q} > 0$ such that

$$\sup_{j\in\mathbb{N}}\int_{\mathbb{R}^n}|\widetilde{\zeta}^j - z|^{q^*}\,\mathrm{d}x \le c_{n,q}\sup_{j\in\mathbb{N}}\int_{\mathbb{R}^n}|\nabla\widetilde{\zeta}^j|^q\,\mathrm{d}x \lesssim C_{\theta,\rho,z} + \theta^n,\tag{4.21}$$

where $q^* := nq/(n-q)$ denotes the conjugate Sobolev exponent of q. The estimates (4.20) and (4.21) then guarantee that

$$\widetilde{\zeta}^j - z \rightharpoonup \widetilde{\zeta} \ \text{ weakly in } \ W^{1,q}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m) \text{ as } j \to +\infty\,,$$

for some $\widetilde{\zeta} \in W^{1,q}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$. Equivalently, setting $\zeta := \widetilde{\zeta} + z$, we have that $\zeta - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m)$, $\zeta|_{\overline{B}_\rho(0)} \equiv 0$, and

$$\widetilde{\zeta}^j \rightharpoonup \zeta \text{ weakly in } W^{1,q}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m) \text{ as } j \to +\infty\,.$$

The following result is an immediate consequence of Lemma 4.3.

Corollary 4.5. *Let $M > 0$ and $\rho \in (0, M]$ be fixed. Let $\theta \in (0,1)$ and $j \in \mathbb{N}$ satisfy (4.6) and let $\varphi^j_{\theta,\rho}$ be as in (4.7). Then we have the following.*

(i) There exists a subsequence (not relabelled) and a measurable function $\varphi_{\theta,\rho}\colon \mathbb{R}^m \to \mathbb{R}$ such that

$$\varphi^j_{\theta,\rho} \longrightarrow \varphi_{\theta,\rho} \ \text{ in } \ L^\infty_{\rm loc}(\mathbb{R}^m), \text{ as } j \to +\infty\,.\tag{4.22}$$

Moreover, the function $\varphi_{\theta,\rho}$ satisfies

$$C_1|z|^q\rho^{n-q} - C_2\theta^n \le \varphi_{\theta,\rho}(z) \le C_3|z|^q\rho^{n-q} + C_4\theta^n \quad \forall\, z \in \mathbb{R}^m\,,\tag{4.23}$$

and

$$|\varphi_{\theta,\rho}(z) - \varphi_{\theta,\rho}(w)| \le C_M\big(\theta^{n(q-1)/q} + |z|^{q-1} + |w|^{q-1}\big)|z-w| \quad \forall\, z,w \in \mathbb{R}^m\,,\tag{4.24}$$

where $C_1, \dots, C_4 > 0$ and $C_M > 0$ are as in Lemma 4.3. Additionally, for $\theta \in (0,1)$ and $z \in \mathbb{R}^m$ fixed, the function $\rho \mapsto \varphi_{\theta,\rho}(z)$ is increasing.

(ii) There exists a subsequence (not relabelled) and a measurable function $\widetilde{\varphi}_\rho\colon \mathbb{R}^m \to \mathbb{R}$ such that

$$\varphi_{\theta,\rho} \longrightarrow \widetilde{\varphi}_\rho \ \text{ in } \ L^\infty_{\rm loc}(\mathbb{R}^m), \quad \text{as } \theta \to 0^+\,.\tag{4.25}$$

Moreover, the function $\widetilde{\varphi}_\rho$ satisfies

$$C_1|z|^q\rho^{n-q} \le \widetilde{\varphi}_\rho(z) \le C_3|z|^q\rho^{n-q} \quad \forall\, z \in \mathbb{R}^m\,,\tag{4.26}$$

and

$$|\widetilde{\varphi}_\rho(z) - \widetilde{\varphi}_\rho(w)| \le C_M\big(|z|^{q-1} + |w|^{q-1}\big)|z-w| \quad \forall\, z,w \in \mathbb{R}^m\,,\tag{4.27}$$

where $C_1, C_3 > 0$ and $C_M > 0$ are as in Lemma 4.3. Additionally, for $z \in \mathbb{R}^m$ fixed the function $\rho \mapsto \widetilde{\varphi}_\rho(z)$ is increasing.

Our next goal is to prove that the abstract limit density $\widetilde{\varphi}_\rho$ given by Corollary 4.5 (ii) coincides with φ_ρ defined in (3.14). To do so, we follow a similar approach as in [2, Section 7].

Proposition 4.6. *Let $M > 0$, and $\rho \in (0, M]$ be fixed. Let φ_ρ be as in (3.14) and let $\widetilde{\varphi}_\rho$ be given by Corollary 4.5 (ii). Then, $\widetilde{\varphi}_\rho \equiv \varphi_\rho$. Therefore, in particular, φ_ρ satisfies the estimates (4.26) and (4.27).*

Proof. We divide the proof into two main steps.

Step 1: $\widetilde{\varphi}_\rho \geq \varphi_\rho$. Let $z \in \mathbb{R}^m$, $\theta \in (0,1)$ and $K \geq 2M$ be fixed. Moreover, let $(\varepsilon_j) \searrow 0$, and take $j \in \mathbb{N}$ so large that $\theta K_j > K$ holds, where K_j is defined in (4.5).

In view of the quasiconvexity and coercivity of g_j (cf. (3.5) and (3.6)), there exists $\zeta^j \in X^j_{\theta,\rho,z}$ so that

$$\varphi^j_{\theta,\rho}(z) = \int_{B_{\theta K_j(0)}} g_j(\nabla \zeta^j)\,\mathrm{d}x\,.$$

Hence, up to a subsequence, (4.22) ensures that

$$\sup_{j\in\mathbb{N}} \int_{B_{\theta K_j}(0)} g_j(\nabla \zeta^j)\,\mathrm{d}x \leq C_{\theta,\rho,z}\,,$$

for some constant $C_{\theta,\rho,z} > 0$. As observed in Remark 4.4 we can extend (ζ^j) to obtain a new sequence $(\widetilde{\zeta}^j)$ and a function $\zeta \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\widetilde{\zeta}^j \rightharpoonup \zeta$ weakly in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ as $j \to +\infty$. Then, in particular using (3.6), (4.5), (3.1), and since

$$\int_{B_{\theta K_j}(0)\setminus B_K(0)} g_j(\nabla \widetilde{\zeta}_j) \geq -c_1\alpha^q_{\varepsilon_j}\mathcal{L}^n(B_{\theta K_j}(0)\setminus B_K(0)) \geq -\beta_n c_1 \theta^n\,,$$

we get

$$\varphi_{\theta,\rho}(z) = \lim_{j\to+\infty} \varphi^j_{\theta,\rho}(z) = \lim_{j\to+\infty} \int_{B_{\theta K_j}(0)} g_j(\nabla\zeta^j)\,\mathrm{d}x \geq \varliminf_{j\to+\infty} \int_{B_K(0)} g_j(\nabla\widetilde{\zeta}^j)\,\mathrm{d}x - \beta_n c_1\theta^n\,. \tag{4.28}$$

Now consider the auxiliary integral functionals defined as

$$\mathcal{G}^j_K(\zeta) := \begin{cases} \displaystyle\int_{B_K(0)} g_j(\nabla\zeta)\,\mathrm{d}x & \text{if } \zeta \in W^{1,q}(B_K(0);\mathbb{R}^m)\,, \\ +\infty & \text{otherwise in } L^1(B_K(0);\mathbb{R}^m)\,, \end{cases}$$

and

$$\mathcal{G}_K(\zeta) := \begin{cases} \displaystyle\int_{B_K(0)} g(\nabla\zeta)\,\mathrm{d}x & \text{if } \zeta \in W^{1,q}(B_K(0);\mathbb{R}^m)\,, \\ +\infty & \text{otherwise in } L^1(B_K(0);\mathbb{R}^m)\,, \end{cases}$$

where g is as in (3.8). Then, we can invoke [5, Proposition 12.8] to deduce that the functionals $(\mathcal{G}^j_K)_j$ Γ-converge to $\mathcal{G}_K$ with respect to the strong $L^1(B_K(0);\mathbb{R}^m)$-topology and the weak-$W^{1,q}(B_K(0);\mathbb{R}^m)$-topology. Since $\widetilde{\zeta}^j \rightharpoonup \zeta$ weakly in $W^{1,q}(B_K(0);\mathbb{R}^m)$, in particular we can deduce that

$$\liminf_{j\to+\infty} \int_{B_K(0)} g_j(\nabla\widetilde{\zeta}^j)\,\mathrm{d}x \geq \int_{B_K(0)} g(\nabla\zeta)\,\mathrm{d}x\,. \tag{4.29}$$

Therefore, gathering (4.28) and (4.29) gives

$$\varphi_{\theta,\rho}(z) \geq \int_{B_K(0)} g(\nabla\zeta)\,\mathrm{d}x - \beta_n c_1\theta^n\,.$$

Passing to the limit as $K \to +\infty$ and using the Dominated Convergence Theorem, we obtain

$$\varphi_{\theta,\rho}(z) \geq \int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x - \beta_n c_1\theta^n \geq \varphi_\rho(z) - \beta_n c_1\theta^n\,,$$

where we also used that ζ is a competitor for the minimisation problem defining $\varphi_\rho(z)$. Eventually, passing to the limit as $\theta \to 0^+$ and recalling (4.25), we get that $\widetilde{\varphi}_\rho(z) \geq \varphi_\rho(z)$.

Step 2: $\widetilde{\varphi}_\rho \leq \varphi_\rho$. Let $z \in \mathbb{R}^m$ be fixed; by the quasiconvexity and coercivity of g (cf. (3.8) and (3.9)), there exists ζ with $\zeta - z \in W^{1,q}_0(\mathbb{R}^n;\mathbb{R}^m)$, $\zeta|_{\overline{B}_\rho(0)} \equiv 0$, such that

$$\varphi_\rho(z) = \int_{\mathbb{R}^n} g(\nabla\zeta)\,\mathrm{d}x\,. \tag{4.30}$$

Let $\theta \in (0,1)$ and $K \geq 2M$ be fixed, and let $(\varepsilon_j) \searrow 0$; we take $j \in \mathbb{N}$ so large that $\theta K_j > K$ holds. Consider a smooth radial cut-off function $\eta_K \in C_c^\infty(B_K(0))$ satisfying

$$0 \leq \eta_K \leq 1\,, \quad \eta_K|_{\overline{B}_{K/2}(0)} \equiv 1\,, \quad \|\nabla \eta_K\|_{L^\infty} \leq \frac{2}{K}\,,$$

and define

$$\zeta_K := \eta_K \zeta + (1-\eta_K) z\,. \tag{4.31}$$

Then ζ_K is an admissible test function for the following minimisation problem

$$\widetilde{\varphi}^j_{K,\rho}(z) := \min\left\{ \int_{B_K(0)} g_j(\nabla \widetilde{\zeta})\,\mathrm{d}x \colon \widetilde{\zeta} - z \in W^{1,q}_0(B_K(0);\mathbb{R}^m)\,,\ \widetilde{\zeta}|_{\overline{B}_\rho(0)} \equiv 0 \right\}.$$

By the Γ-convergence of $(\mathcal{G}^j_K)_j$ to $\mathcal{G}_K$, and by [5, Proposition 11.7], there exists a recovery sequence for $\mathcal{G}^j_K$ converging to ζ_K. More precisely, there is $(\zeta^j_K)_j$, with $\zeta^j_K \rightharpoonup \zeta_K$ weakly in $W^{1,q}(B_K(0);\mathbb{R}^m)$,

$$\zeta^j_K - z \in W^{1,q}_0(B_K(0);\mathbb{R}^m)\,, \quad \zeta^j_K|_{\overline{B}_\rho(0)} \equiv 0\,,$$

and for which the energies converge, namely

$$\lim_{j\to+\infty} \int_{B_K(0)} g_j(\nabla \zeta^j_K)\,\mathrm{d}x = \int_{B_K(0)} g(\nabla \zeta_K)\,\mathrm{d}x\,. \tag{4.32}$$

We extend ζ^j_K to $B_{\theta K_j}(0)$ by setting

$$\widetilde{\zeta}^j_K := \begin{cases} \zeta^j_K & \text{in } B_K(0) \\ z & \text{in } B_{\theta K_j}(0) \setminus B_K(0)\,. \end{cases} \tag{4.33}$$

Then $\widetilde{\zeta}^j_K \in X^j_{\theta,\rho,z}$, so $\widetilde{\zeta}^j_K$ is an admissible competitor for the minimisation problem defining $\varphi^j_{\theta,\rho}(z)$ in (4.7). Therefore, by (4.33) we have

$$\varphi^j_{\theta,\rho}(z) \leq \int_{B_{\theta K_j}(0)} g_j(\nabla \widetilde{\zeta}^j_K)\,\mathrm{d}x = \int_{B_K(0)} g_j(\nabla \zeta^j_K)\,\mathrm{d}x\,.$$

Taking the limit as $j \to +\infty$ (possibly passing to a subsequence), and using (4.22), (4.32), (3.9) and (4.31), we obtain

$$\begin{aligned} \varphi_{\theta,\rho}(z) &\leq \int_{B_K(0)} g(\nabla \zeta_K)\,\mathrm{d}x = \int_{B_K(0)\setminus B_{K/2}(0)} g(\nabla\zeta_K)\,\mathrm{d}x + \int_{B_{K/2}(0)} g(\nabla \zeta_K)\,\mathrm{d}x \\ &\leq c_2 \int_{B_K(0)\setminus B_{K/2}(0)} |\nabla \zeta_K|^q\,\mathrm{d}x + \int_{B_{K/2}(0)} g(\nabla\zeta)\,\mathrm{d}x\,. \end{aligned} \tag{4.34}$$

We claim that

$$\varlimsup_{K\to+\infty} \int_{B_K(0)\setminus B_{K/2}(0)} |\nabla \zeta_K|^q\,\mathrm{d}x = 0\,. \tag{4.35}$$

By (4.31), and since by the Sobolev Embedding Theorem $\zeta - z \in L^{q^*}(\mathbb{R}^n;\mathbb{R}^m)$, we have

$$\begin{aligned} \int_{B_K(0)\setminus B_{K/2}(0)} |\nabla\zeta_K|^q\,\mathrm{d}x &= \int_{B_K(0)\setminus B_{K/2}(0)} \big|\nabla\eta_K \otimes (\zeta - z) + \eta_K \nabla \zeta\big|^q\,\mathrm{d}x \\ &\lesssim \|\nabla \eta_K\|^q_{L^\infty} \int_{B_K(0)\setminus B_{K/2}(0)} \big|\zeta - z\big|^q\,\mathrm{d}x + \|\eta_K\|^q_{L^\infty} \int_{B_K(0)\setminus B_{K/2}(0)} \big|\nabla\zeta\big|^q\,\mathrm{d}x \\ &\lesssim K^{-q} K^{n(1-\frac{q}{q^*})} \Big(\int_{B_K(0)\setminus B_{K/2}(0)} |\zeta - z|^{q^*}\,\mathrm{d}x\Big)^{\frac{q}{q^*}} + \int_{B_K(0)\setminus B_{K/2}(0)} |\nabla\zeta|^q\,\mathrm{d}x \\ &\lesssim \Big(\int_{\mathbb{R}^n\setminus B_{K/2}(0)} |\zeta - z|^{q^*}\,\mathrm{d}x\Big)^{\frac{q}{q^*}} + \int_{\mathbb{R}^n\setminus B_{K/2}(0)} |\nabla\zeta|^q\,\mathrm{d}x\,, \end{aligned}$$

where we have used that $K^{-q}K^{n(1-\frac{q}{q^*})}=1$. Hence, passing to the limsup as $K\to+\infty$ yields (4.35). Taking the limsup as $K\to+\infty$ in (4.34), by (4.30), (4.35), and the Dominated Convergence Theorem we get

$$\varphi_{\theta,\rho}(z)\le\varphi_\rho(z)\,.$$

Eventually, by taking the limit as $\theta\to0^+$ in the above inequality, and by (4.25), we have that $\widetilde{\varphi}_\rho(z)\le\varphi_\rho(z)$. □

The following result, Lemma 4.7, is an adaptation to the non-periodic setting of the so-called "Joining Lemma" (cf. [3, Lemma 3.1]). This is a technical tool which allows to modify sequences of functions near "good perforations" without increasing the energy too much, and will be crucial in the proof of Theorem 3.2. First, we need to define the class of the good perforations, consisting of balls which are well-separated from one another and not too large, in the sense specified below.

Let $\varepsilon>0$ and $M>0$ be fixed, and let $D\subset\mathbb{R}^n$ be an open, bounded, Lipschitz set, star-shaped with respect to the origin. We define $\mathscr{G}_{\varepsilon,M}$ as the collection of points in $\mathbb{R}^n$ satisfying the following two properties:

(a) $|x_i-x_j|\ge 2/M$ for every $i\neq j$;

(b) $\bigcup_{x_i\in\mathscr{G}_{\varepsilon,M}}\overline{B}_{\varepsilon/M}(\varepsilon x_i)\subset D$.

We note that by (a) and (b) the family of balls $(B_{\varepsilon/M}(\varepsilon x_i))_{x_i\in\mathscr{G}_{\varepsilon,M}}$ consists of pairwise disjoint subsets of D; therefore we immediately get that

$$(\beta_nM^{-n}\varepsilon^n)\#\mathscr{G}_{\varepsilon,M}\le\mathcal{L}^n(D)\,. \tag{4.36}$$

For $\theta\in(0,1)$ fixed, we refer to $(B_{\theta\varepsilon/M}(\varepsilon x_i))_{x_i\in\mathscr{G}_{\varepsilon,M}}$ as the family of "truncated good perforations" in D. By (b) we have that $B_{\theta\varepsilon/M}(\varepsilon x_i)\subset\subset D$, for every $x_i\in\mathscr{G}_{\varepsilon,M}$.

For $x_i\in\mathscr{G}_{\varepsilon,M}$ and $l\in\mathbb{N}$ we define the annulus

$$C^l_{\varepsilon,\theta,M}(\varepsilon x_i):=\Big\{x\in\mathbb{R}^n\colon 2^{-(l+1)}\theta\varepsilon/M<|x-\varepsilon x_i|<2^{-l}\theta\varepsilon/M\Big\}\subseteq B_{\theta\varepsilon/M}(\varepsilon x_i)\,. \tag{4.37}$$

If $(\varepsilon_j)\searrow0$ we adopt the shorthand notation $\mathscr{G}_{j,M}:=\mathscr{G}_{\varepsilon_j,M}$.

We are now ready to state and prove the following variant of the Joining Lemma.

Lemma 4.7 (Joining Lemma). *Let $(\varepsilon_j)\searrow0$, $M>0$, $\theta\in(0,1)$ and $k\in\mathbb{N}$ be fixed. Let $(u_j)\subset W^{1,q}_0(D;\mathbb{R}^m)$ be such that*

$$u_j\rightharpoonup u\quad\text{weakly in } W^{1,q}(D;\mathbb{R}^m)\,, \tag{4.38}$$

for some $u\in W^{1,q}_0(D;\mathbb{R}^m)$, and let $\mathscr{G}_{j,M}$ be a collection of points in $\mathbb{R}^n$ satisfying (a)-(b). Then for every $x_{j,i}\in\mathscr{G}_{j,M}$ there exists $k_{j,i}\in\{0,\dots,k-1\}$ and a corresponding annulus $C^{k_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_jx_{j,i})$ (defined as in (4.37) with ε, l, and x_i replaced by $\varepsilon_j, k_{j,i}$, and $x_{j,i}$, respectively), such that we can construct a sequence $(w_j)\subset W^{1,q}_0(D;\mathbb{R}^m)$ satisfying the following properties:

(i) $w_j\equiv u_j$ *in* $D\setminus\bigcup_{x_{j,i}\in\mathscr{G}_{j,M}}C^{k_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_jx_{j,i})$;

(ii) $w_j\equiv\bar{u}_{j,i}$ *on* $\partial B_{\bar\sigma_{j,i}}(\varepsilon_jx_{j,i})$, *where*

$$\bar u_{j,i}:=\fint_{C^{k_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_jx_{j,i})}u_j\,\mathrm{d}x\,,\qquad \bar\sigma_{j,i}:=\frac34 2^{-k_{j,i}}\frac{\theta\varepsilon_j}{M}\,; \tag{4.39}$$

(iii) $w_j\rightharpoonup u$ *weakly in* $W^{1,q}(D;\mathbb{R}^m)$;

(iv) $\Big|\int_D f(\nabla w_j)\,\mathrm{d}x-\int_D f(\nabla u_j)\,\mathrm{d}x\Big|\le\dfrac{C}{k}$, *for some $C>0$ depending on c_2, n, m, q, D, and* $\sup_{j\in\mathbb{N}}\|\nabla u_j\|_{L^q(D;\mathbb{R}^{m\times n})}$.

If, additionally, the sequence $(|\nabla u_j|^q)$ is equi-integrable, then also $(|\nabla w_j|^q)$ is equi-integrable, and one can take $k_{j,i}=0$ for all $x_{j,i}\in\mathscr{G}_{j,M}$, up to replacing (iv) with the following estimate

$$\Big|\int_D f(\nabla w_j)\,\mathrm{d}x-\int_D f(\nabla u_j)\,\mathrm{d}x\Big|\le C_k\theta^n+\frac{C}{k}\,, \tag{4.40}$$

where $C_k>0$ can blow up as $k\to+\infty$.

Proof. The proof is an adaptation of that of [3, Lemma 3.1], and we present it in detail for the convenience of the readers.

For every $j \in \mathbb{N}$, $x_{j,i} \in \mathscr{G}_{j,M}$, and $l \in \{0, \dots, k-1\}$ we define the shorthand

$$C^l_{j,i} := C^l_{\varepsilon_j,\theta,M}(\varepsilon_j x_{j,i})$$

and we denote with $\bar{u}^l_{j,i}, \bar{\sigma}^l_{j,i}$ the quantities defined as in (4.39), with l replacing $k_{j,i}$. Let $\psi^l_{j,i} \in C^\infty_c(C^l_{j,i})$ be a cut-off function satisfying

$$\psi^l_{j,i}\big|_{\partial B_{\bar{\sigma}^l_{j,i}}} \equiv 1\,, \quad 0 \le \psi^l_{j,i} \le 1\,, \quad \|\nabla \psi^l_{j,i}\|_{L^\infty(C^l_{j,i})} \le \frac{c}{\bar{\sigma}^l_{j,i}}\,, \tag{4.41}$$

with $c > 0$, and set

$$w^l_{j,i} := u_j + \psi^l_{j,i}(\bar{u}^l_{j,i} - u_j)\,. \tag{4.42}$$

Note that $w^l_{j,i} \equiv u_j$ outside the annulus $C^l_{j,i}$ and $w^l_{j,i} = \bar{u}^l_{j,i}$ on $\partial B_{\bar{\sigma}^l_{j,i}}$. By (3.3) and (4.41) we have

$$\begin{aligned}\int_{C^l_{j,i}} f(\nabla w^l_{j,i})\,\mathrm{d}x &\le c_2 \int_{C^l_{j,i}} \Big(\big|\nabla \psi^l_{j,i} \otimes (\bar{u}^l_{j,i} - u_j) + (1-\psi^l_{j,i})\nabla u_j\big|^q + 1\Big)\,\mathrm{d}x \\ &\lesssim (\bar{\sigma}^l_{j,i})^{-q} \int_{C^l_{j,i}} |u_j - \bar{u}^l_{j,i}|^q\,\mathrm{d}x + \int_{C^l_{j,i}} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x \lesssim \int_{C^l_{j,i}} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x\,,\end{aligned} \tag{4.43}$$

where to conclude we have used the Poincaré inequality in the annulus $C^l_{j,i}$. Since the sets $(C^l_{j,i})_{l=0,\dots,k-1}$ are pairwise disjoint and

$$\bigcup_{l=0}^{k-1} C^l_{j,i} \subset B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}) \setminus \overline{B}_{2^{-k}\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}) \subset B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})\,,$$

we obtain for every $j \in \mathbb{N}$ and $x_{j,i} \in \mathscr{G}_{j,M}$,

$$\sum_{l=0}^{k-1} \int_{C^l_{j,i}} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x \le \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x\,.$$

In particular, for fixed $j \in \mathbb{N}$ and $x_{j,i} \in \mathscr{G}_{j,M}$, there exists $k_{j,i} \in \{0, \dots, k-1\}$ so that

$$\int_{C^{k_{j,i}}_{j,i}} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x \le \frac{1}{k} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x\,.$$

Setting $C_{j,i} := C^{k_{j,i}}_{j,i}$, $\bar{u}_{j,i} := \bar{u}^{k_{j,i}}_{j,i}$, $\bar{\sigma}_{j,i} := \bar{\sigma}^{k_{j,i}}_{j,i}$ and $w_{j,i} := w^{k_{j,i}}_{j,i}$, we define the sequence (w_j) as

$$w_j := \begin{cases} u_j & \text{in } D \setminus \bigcup\limits_{x_{j,i} \in \mathscr{G}_{j,M}} C_{j,i}\,, \\ w_{j,i} & \text{in } C_{j,i}\,. \end{cases} \tag{4.44}$$

From the definition and by (4.41) and (4.42) we have that w_j satisfies properties (i)-(ii). Moreover by (4.43) - (4.44), and the fact that the balls $(B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}))_{x_{j,i} \in \mathscr{G}_{j,M}}$ are pairwise disjoint subsets of D, we have

$$\begin{aligned}\Big|\int_D f(\nabla w_j)\,\mathrm{d}x - \int_D f(\nabla u_j)\,\mathrm{d}x\Big| &\le \sum_{x_{j,i} \in \mathscr{G}_{j,M}} \int_{C_{j,i}} |f(\nabla w_{j,i}) - f(\nabla u_j)|\,\mathrm{d}x \\ &\lesssim \frac{1}{k} \sum_{x_{j,i} \in \mathscr{G}_{j,M}} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x \\ &\lesssim \frac{1}{k} \int_D \big(1 + |\nabla u_j|^q\big)\,\mathrm{d}x\,.\end{aligned}$$

Then, (iv) follows immediately by (4.38).

It only remains to check that w_j satisfies (iii). To do so, we start by showing that $w_j \to u$ strongly in $L^q(D;\mathbb{R}^m)$, as $j \to +\infty$. Indeed, by (4.41), (4.42), (4.44), and by using the Poincaré inequality in the annuli $C_{j,i}$, we get

$$\begin{aligned}\int_D |w_j - u|^q \,\mathrm{d}x &= \int_{D\setminus \bigcup\limits_{x_{j,i}\in\mathscr{G}_{j,M}} C_{j,i}} \big|u_j - u\big|^q \,\mathrm{d}x + \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{C_{j,i}} \big|u_j + \psi_{j,i}^{k_{j,i}}(\bar u_{j,i} - u_j) - u\big|^q \,\mathrm{d}x \\ &\lesssim \int_D \big|u_j - u\big|^q \,\mathrm{d}x + \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{C_{j,i}} \big|u_j - \bar u_{j,i}\big|^q \,\mathrm{d}x \\ &\lesssim \int_D \big|u_j - u\big|^q \,\mathrm{d}x + (\theta\varepsilon_j/M)^q \sup_{j\in\mathbb{N}} \int_D |\nabla u_j|^q \,\mathrm{d}x\,, \end{aligned} \tag{4.45}$$

which is infinitesimal as $j \to +\infty$ by (4.38), and since $(\varepsilon_j) \searrow 0$. By combining (4.44), (3.3), and (4.43) we also obtain

$$\begin{aligned}\int_D |\nabla w_j|^q \,\mathrm{d}x &= \int_{D\setminus \bigcup\limits_{x_{j,i}\in\mathscr{G}_{j,M}} C_{j,i}} |\nabla u_j|^q \,\mathrm{d}x + \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{C_{j,i}} |\nabla w_{j,i}|^q \,\mathrm{d}x \\ &\lesssim \int_D |\nabla u_j|^q \,\mathrm{d}x + \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{C_{j,i}} (f(\nabla w_{j,i}) + 1) \,\mathrm{d}x \\ &\lesssim \int_D |\nabla u_j|^q \,\mathrm{d}x + \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} (1 + |\nabla u_j|^q) \,\mathrm{d}x \lesssim \int_D (1 + |\nabla u_j|^q) \,\mathrm{d}x\,,\end{aligned}$$

which by (4.38) and (4.45) yields the desired convergence (iii).

Suppose now that the sequence $(|\nabla u_j|^q)$ is equi-integrable. In this case, for each $x_{j,i} \in \mathscr{G}_{j,M}$ we set

$$C_{j,i} := \{\theta\varepsilon_j/(2M) < |x - \varepsilon_j x_{j,i}| < \theta\varepsilon_j/M\}\,, \quad \bar u_{j,i} := \fint_{C_{j,i}} u_j \,\mathrm{d}x\,, \quad \bar\sigma_{j,i} := \frac{3}{4}\frac{\theta\varepsilon_j}{M}\,,$$

and

$$w_{j,i} := u_j + \psi_{j,i}(\bar u_{j,i} - u_j)\,,$$

where $\psi_{j,i} \in C_c^\infty(C_{j,i})$ is a cut-off function such that

$$0 \le \psi_{j,i} \le 1\,, \quad \psi_{j,i}\big|_{\partial B_{\bar\sigma_{j,i}}} \equiv 1\,, \quad \|\nabla\psi_{j,i}\|_{L^\infty(C_{j,i})} \le \frac{c}{\bar\sigma_{j,i}}\,,$$

with $c > 0$. Similarly to (4.43), also in this case we have

$$\int_{C_{j,i}} f(\nabla w_{j,i}) \,\mathrm{d}x \lesssim \int_{C_{j,i}} \big(1 + |\nabla u_j|^q\big) \,\mathrm{d}x\,.$$

Setting

$$w_j := \begin{cases} u_j & \text{in } D \setminus \bigcup\limits_{x_{j,i}\in\mathscr{G}_{j,M}} C_{j,i}\,, \\ w_{j,i} & \text{in each } C_{j,i}\,, \end{cases}$$

one can easily check that (i),(ii) and (iii) are satisfied. To prove (4.40) note that for every $T > 0$ large enough,

$$\begin{aligned}\Big|\int_D f(\nabla w_j) \,\mathrm{d}x - \int_D f(\nabla u_j) \,\mathrm{d}x\Big| &\lesssim \sum_{x_{j,i}\in\mathscr{G}_{j,M}} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} (1 + |\nabla u_j|^q) \,\mathrm{d}x \\ &\lesssim (1 + T^q)(\theta\varepsilon_j/M)^n \#\mathscr{G}_{j,M} + \int_{D\cap\{|\nabla u_j| > T\}} |\nabla u_j|^q \,\mathrm{d}x\,.\end{aligned}$$

By (4.36) we get

$$\Big|\int_D f(\nabla w_j) \,\mathrm{d}x - \int_D f(\nabla u_j) \,\mathrm{d}x\Big| \lesssim (1 + T^q)\theta^n + \int_{D\cap\{|\nabla u_j| > T\}} |\nabla u_j|^q \,\mathrm{d}x\,.$$

For $k \in \mathbb{N}$, taking first $T = T_k > 0$ large enough, using the equi-integrability assumption on $(|\nabla u_j|^q)$ we have

$$\sup_{j\in\mathbb{N}} \int_{D\cap\{|\nabla u_j|>T_k\}} |\nabla u_j|^q \,\mathrm{d}x \leq \frac{1}{k}$$

and hence (4.40). □

5. Probabilistic building blocks

In this section we collect some probabilistic results that we use in the proof of Theorem 3.2. Preliminarily we recall that for a bounded set $E \subset \mathbb{R}^n$ we have set

$$\Phi(E) := \Phi \cap E, \quad N(E) := \#\Phi(E) .$$

Moreover for $\varepsilon > 0$ we also define the ε-dependent random variables

$$\Phi_\varepsilon(E) := \Phi(\varepsilon^{-1}E), \quad N_\varepsilon(E) := \#\Phi_\varepsilon(E) \tag{5.1}$$

and for $\delta > 0$, we introduce the *thinning process* Φ^δ of δ-isolated centres, defined by

$$\Phi^{\delta,\omega} := \Big\{x \in \Phi^\omega \colon \min_{y\in\Phi^\omega,\ y\neq x} |y - x| \geq \delta\Big\} . \tag{5.2}$$

Analogously, we define

$$\begin{aligned} \Phi^\delta(E) &:= \Phi^\delta \cap E\,, & \Phi^\delta_\varepsilon(E) &:= \Phi^\delta\big(\varepsilon^{-1}E\big)\,, \\ N^\delta(E) &:= \#\Phi^\delta(E)\,, & N^\delta_\varepsilon(E) &:= \#\Phi^\delta_\varepsilon(E)\,. \end{aligned} \tag{5.3}$$

Lemma 5.1 below is a statement on the asymptotic random geometry of the perforations and is a straightforward adaptation of [21, Lemma 4.2] to our setting. Since we deal with functionals with q-growth (rather than quadratic), the critical scale of the perforations for us is $\varepsilon^{n/(n-q)}$ (rather than $\varepsilon^{n/(n-2)}$). This difference in the scale causes some minor changes in the statement of the result, but is of no consequence in its proof for which we refer to [21] and omit here.

In what follows $\Omega' \in \mathcal{T}$ denotes a set with $\mathbb{P}(\Omega') = 1$ that may vary from line to line and depends only on the m.p.p. $(\Phi, \mathcal{R})$. If a property holds true for every $\omega \in \Omega'$ we may equivalently write that it holds $\mathbb{P}$-a.e. in Ω or, in short, *almost surely*.

Lemma 5.1. *Let $(\Phi, \mathcal{R})$ be a m.p.p. satisfying the assumptions (H1)–(H4), and let H^ω_ε, for $\varepsilon > 0$ and $\omega \in \Omega$, be the family of random holes associated to the m.p.p. defined as in* (3.2).

There exist $\varepsilon_0 := \varepsilon_0(n, m, q) > 0$, random variables (r_ε) with $r_\varepsilon : \Omega \to \mathbb{R}_+$, and a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ with the following properties. For every $\omega \in \Omega'$,

$$\lim_{\varepsilon\to 0^+} r^\omega_\varepsilon = 0\,, \tag{5.4}$$

and for every $\omega \in \Omega'$ and every $\varepsilon \in (0, \varepsilon_0]$ there exists a set $I^\omega_{\varepsilon,b} \subset \Phi^\omega_\varepsilon(D)$ such that, defining $I^\omega_{\varepsilon,g} := \Phi^\omega_\varepsilon(D) \setminus I^\omega_{\varepsilon,b}$ and

$$H^\omega_{\varepsilon,b} := \bigcup_{x_i\in I^\omega_{\varepsilon,b}} \overline{B}_{\alpha_\varepsilon\rho_i}(\varepsilon x_i)\,, \quad D^\omega_{\varepsilon,b} := \bigcup_{x_i\in I^\omega_{\varepsilon,b}} \overline{B}_{2\alpha_\varepsilon\rho_i}(\varepsilon x_i)\,, \quad H^\omega_{\varepsilon,g} := \bigcup_{x_i\in I^\omega_{\varepsilon,g}} \overline{B}_{\alpha_\varepsilon\rho_i}(\varepsilon x_i)\,, \tag{5.5}$$

we have

$$\operatorname{dist}(H^\omega_{\varepsilon,g}, D^\omega_{\varepsilon,b}) \geq \frac{\varepsilon r^\omega_\varepsilon}{2}\,, \quad \lim_{\varepsilon\to 0^+} \varepsilon^n \# I^\omega_{\varepsilon,b} = 0\,, \quad \lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{x_i\in I^\omega_{\varepsilon,b}} (\rho_i)^{n-q} = 0\,, \tag{5.6}$$

$$\min_{\substack{x_l,x_i\in I^\omega_{\varepsilon,g}\\ x_l\neq x_i}} |x_l - x_i| \geq 2r^\omega_\varepsilon\,, \quad \max_{x_i\in I^\omega_{\varepsilon,g}} \alpha_\varepsilon\rho_i \leq \frac{\varepsilon r^\omega_\varepsilon}{2}\,, \quad \lim_{\varepsilon\to 0^+} \varepsilon^n \# I^\omega_{\varepsilon,g} = \lambda\mathcal{L}^n(D)\,. \tag{5.7}$$

Finally, if $\delta > 0$, for the thinning process defined in (5.2)*, for every $\omega \in \Omega'$ there holds*

$$\lim_{\varepsilon\to 0^+} \varepsilon^n \#\{x_i \in \Phi^{2\delta,\omega}_\varepsilon(D)\colon \operatorname{dist}(\varepsilon x_i, D^\omega_{\varepsilon,b}) \leq \delta\varepsilon\} = 0\,. \tag{5.8}$$

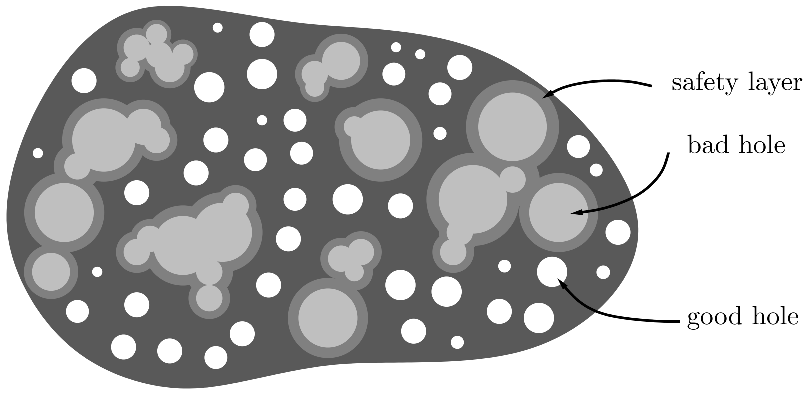

Figure 2. Domain decomposition into good holes $H^\omega_{\varepsilon,g}$, bad holes $H^\omega_{\varepsilon,b}$, and safety layer $D^\omega_{\varepsilon,b}$.

In what follows we refer to the sets $H^\omega_{\varepsilon,g}$ and $H^\omega_{\varepsilon,b}$ in (5.5) as *good* and *bad perforations*, respectively, while $D^\omega_{\varepsilon,b}$ is referred to as the *safety layer*. Note that $H^\omega_\varepsilon = H^\omega_{\varepsilon,g} \cup H^\omega_{\varepsilon,b}$. In short, the good perforations are $(\varepsilon r^\omega_\varepsilon/2)$-separated from the safety layer, they are $\varepsilon r^\omega_\varepsilon$-separated from one another, their radii are bounded by $(\varepsilon r^\omega_\varepsilon)/2$, and asymptotically they are the only relevant set of perforations. An illustration of the geometry is shown in Figure 2.

Note that, by (3.12)-(3.13), proceeding as in the proof of (4.13), we have that for every $\omega \in \Omega'$ and $z \in \mathbb{R}^m$

$$\begin{aligned}
\mathrm{Cap}_q(H^\omega_{\varepsilon,b}, D^\omega_{\varepsilon,b}) &= \mathrm{Cap}_q\Big(\bigcup_{x_i \in I^\omega_{\varepsilon,b}} B_{\alpha_\varepsilon \rho_i}(\varepsilon x_i), D^\omega_{\varepsilon,b}\Big) \le \sum_{x_i \in I^\omega_{\varepsilon,b}} \mathrm{Cap}_q\Big(B_{\alpha_\varepsilon \rho_i}(\varepsilon x_i), D^\omega_{\varepsilon,b}\Big) \\
&\le \sum_{x_i \in I^\omega_{\varepsilon,b}} \mathrm{Cap}_q\big(B_{\alpha_\varepsilon \rho_i}(\varepsilon x_i), B_{2\alpha_\varepsilon \rho_i}(\varepsilon x_i)\big) \\
&\sim \sum_{x_i \in I^\omega_{\varepsilon,b}} \big((\alpha_\varepsilon \rho_i)^{(q-n)/(q-1)} - (2\alpha_\varepsilon \rho_i)^{(q-n)/(q-1)}\big)^{1-q} \\
&\sim \varepsilon^n \sum_{x_i \in I^\omega_{\varepsilon,b}} (\rho_i)^{n-q} \to 0 \quad \text{as } \varepsilon \to 0^+ ,
\end{aligned} \tag{5.9}$$

where we have used (5.6). Condition (5.9) (with Cap_q replaced by the classical harmonic capacity) is explicitly stated in [21, Lemma 4.2] instead of the last equality in (5.6); however the analogue of (5.6) can be found in the proof of their result (cf. equation (4.58) and the one above it therein). Moreover by following the steps in the proof of [21, Lemma 4.2] it is easy to verify that in our case the random variables (r^ω_ε) can be chosen as

$$r^\omega_\varepsilon := \big(\varepsilon^{n/(n-q)} \max_{x_i \in \Phi^\omega_\varepsilon(D)} \rho_i\big)^{\frac{1}{n}} \vee \varepsilon^{\alpha/4} \text{ for some } \alpha \in (0, \tfrac{q}{n-q})\,.$$

Following [21], we now introduce for $\omega \in \Omega'$ the subset of $I^\omega_{\varepsilon,g}$ given by the (centres of the) balls that are deterministically spaced apart from one another and from the safety layer, and have uniformly bounded rescaled radii ρ_i. More precisely, for $M \in \mathbb{N}$ fixed we define

$$G^\omega_{\varepsilon,M} := \big\{x_i \in I^\omega_{\varepsilon,g} \colon d^\omega_{\varepsilon,i} \ge \varepsilon/M \ \text{ and } \rho_i \le M\big\}\,, \tag{5.10}$$

where, for $x_i \in \Phi^\omega_\varepsilon(D)$, we set

$$d^\omega_{\varepsilon,i} := \min\Big\{\mathrm{dist}(\varepsilon x_i, D^\omega_{\varepsilon,b}), \tfrac{1}{2}\min_{x_l \neq x_i} \varepsilon|x_l - x_i|, \varepsilon\Big\}\,. \tag{5.11}$$

If $(\varepsilon_j) \searrow 0$ we adopt the shorthand notation $G^\omega_{j,M} := G^\omega_{\varepsilon_j,M}$. Without loss of generality, in all that follows we can assume that $\overline{B}_{\varepsilon/M}(\varepsilon x_i) \subset D$ for every $x_i \in G^\omega_{\varepsilon,M}$. Indeed, set

$$G^{\partial,\omega}_{\varepsilon,M} := \{x_i \in G^\omega_{\varepsilon,M} \colon \overline{B}_{\varepsilon/M}(\varepsilon x_i) \cap \partial D \neq \emptyset\}\,;$$

then

$$\bigcup_{x_i \in G^{\partial,\omega}_{\varepsilon,M}} \overline{B}_{\varepsilon/M}(\varepsilon x_i) \subset (\partial D)_{2\varepsilon/M} := \big\{x \in \mathbb{R}^n \colon \operatorname{dist}(x, \partial D) \le 2\varepsilon/M\big\},$$

from which we infer that

$$\mathcal{L}^n\Big(\bigcup_{x_i \in G^{\partial,\omega}_{\varepsilon,M}} \overline{B}_{\varepsilon/M}(\varepsilon x_i)\Big) \lesssim \mathcal{H}^{n-1}(\partial D)\frac{\varepsilon}{M} \to 0\,, \text{ as } \varepsilon \to 0^+\,.$$

Moreover, since the balls $\{\overline{B}_{\varepsilon/M}(\varepsilon x_i)\}_{x_i \in G^{\partial,\omega}_{\varepsilon,M}}$ are pairwise disjoint, we also obtain

$$\# G^{\partial,\omega}_{\varepsilon,M} \lesssim_D (\varepsilon/M)^{1-n}\,.$$

Consequently, we have

$$\varepsilon^n \sum_{x_i \in G^{\partial,\omega}_{\varepsilon,M}} \rho_i^{n-q} \le M^{n-q}\varepsilon^n \cdot \big(\# G^{\partial,\omega}_{\varepsilon,M}\big) \lesssim_D M^{2n-(q+1)}\varepsilon \to 0\,, \text{ as } \; \varepsilon \to 0^+\,.$$

so that the capacitary contribution of these balls is negligible as well.

Compared to the good centres $I^\omega_{\varepsilon,g}$, where the same scale $\varepsilon r^\omega_\varepsilon$ controlled both the size of the perforations and their separation, for the balls centred at $G^\omega_{\varepsilon,M}$ the separation is of order ε, while the size is much smaller, of the critical order α_ε.

Below we show that the family of points $G^\omega_{\varepsilon,M}$ satisfies, almost surely, properties (a)-(b) before (4.36). Hence we can deduce a probabilistic version of Lemma 4.7 for $\mathbb{P}$-a.e. $\omega \in \Omega$, where sequences will be modified around balls with centres in $G^\omega_{\varepsilon,M}$.

Lemma 5.2 (Probabilistic Joining Lemma)**.** *There exists a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ satisfying the following property. Let $(\varepsilon_j) \searrow 0$, $M \in \mathbb{N}$, $\theta \in (0,1)$ (with $\frac{1}{\theta} \in \mathbb{N}$) and $k \in \mathbb{N}$ be fixed. Let $(u_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ be such that*

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,q}(D;\mathbb{R}^m)\,,$$

for some $u \in W^{1,q}_0(D;\mathbb{R}^m)$, and let $(G^\omega_{j,M})_j$ be collections of points in $\mathbb{R}^n$ defined as in (5.10), with $\omega \in \Omega$. Then for every $\omega \in \Omega'$ and for $x_{j,i} \in G^\omega_{j,M}$ there exists $k^\omega_{j,i} \in \{0, \dots, k-1\}$ and corresponding annuli $C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_j x_{j,i})$, such that we can construct a sequence $(w^\omega_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ satisfying the following properties:

- *(i)* $w^\omega_j \equiv u_j$ *in* $D \setminus \bigcup_{x^\omega_{j,i} \in G^\omega_{j,M}} C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_j x_{j,i})$*;*
- *(ii)* $w^\omega_j \equiv \bar{u}^\omega_{j,i}$ *on* $\partial B_{\bar\sigma^\omega_{j,i}}(\varepsilon_j x_{j,i})$*, where*

$$\bar{u}^\omega_{j,i} := \fint_{C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_j x_{j,i})} u_j \,\mathrm{d}x\,, \qquad \bar\sigma^\omega_{j,i} := \frac{3}{4} 2^{-k^\omega_{j,i}} \frac{\theta\varepsilon_j}{M}\,;$$

- *(iii)* $w^\omega_j \rightharpoonup u$ *weakly in* $W^{1,q}(D;\mathbb{R}^m)$*;*
- *(iv)* $\Big|\int_D f(\nabla w^\omega_j)\,\mathrm{d}x - \int_D f(\nabla u_j)\,\mathrm{d}x\Big| \le \frac{C}{k}$*, for some $C > 0$ depending on c_2, n, m, q, D, and* $\sup_{j\in\mathbb{N}} \|\nabla u_j\|_{L^q(D;\mathbb{R}^{m\times n})}$.

If, additionally, the sequence $(|\nabla u_j|^q)$ is equi-integrable, then also $(|\nabla w^\omega_j|^q)$ is equi-integrable, and in the definition of $\bar\sigma^\omega_{j,i}$ one can take $k^\omega_{j,i} = 0$ for all $x_{j,i} \in G^\omega_{j,M}$, up to replacing (iv) with the following estimate

$$\Big|\int_D f(\nabla w^\omega_j)\,\mathrm{d}x - \int_D f(\nabla u_j)\,\mathrm{d}x\Big| \le C_k\theta^n + \frac{C}{k}\,,$$

where $C_k > 0$ can blow up as $k \to +\infty$.

Proof. Let $M \in \mathbb{N}$ be fixed; by Lemma 5.1, there exists $\Omega_M \in \mathcal{T}$ with $\mathbb{P}(\Omega_M) = 1$ such that for every $\omega \in \Omega_M$ the collection of points $G^\omega_{j,M}$ satisfies properties (a)-(b) (note that (a) and (b) follow immediately from (5.10) and (5.11), and the discussion right after them. Hence, as in (4.36), we get that for every $\omega \in \Omega_M$

$$\sup_{j\in\mathbb{N}} (\beta_n M^{-n}\varepsilon_j^n) \# G^\omega_{j,M} \le \mathcal{L}^n(D)\,. \tag{5.12}$$

Finally, set $\Omega' := \bigcap_{M\in\mathbb{N}} \Omega_M$; clearly $\mathbb{P}(\Omega') = 1$ and for $\omega \in \Omega'$ fixed, Lemma 4.7 applied to the family of the *truncated good perforations* $(B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}))_{x_{j,i}\in G^\omega_{j,M}}$ provides us with a sequence $(w^\omega_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ enjoying the desired properties. $\square$

Remark 5.3. We observe that, by (5.10) and (5.11), the points in $I^\omega_{\varepsilon,g}$ whose distance from the safety layer $D^\omega_{\varepsilon,b}$ is smaller than ε/M do not belong to the set $G^\omega_{\varepsilon,M}$. This guarantees that

$$\Big(\bigcup_{x_{j,i}\in G^\omega_{j,M}} C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_j x_{j,i})\Big) \cap \Big(\bigcup_{x_{j,i}\in I^\omega_{\varepsilon_j,b}} B_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i})\Big) = \emptyset\,,$$

namely that the annuli around the truncated good perforations where the sequence is modified do not touch the bad balls.

This request is also present in [21, Equation (4.65)] to ensure that the correctors provided by [21, Lemma 3.1] are well-defined.

Note, however, that the annuli around the truncated good perforations, might in principle intersect balls centred at $I^\omega_{\varepsilon_j,g} \setminus G^\omega_{j,M}$.

5.1. **Strong laws of large numbers for marked point processes.** In this section we state three generalizations of the strong law of large numbers for marked point processes which are relevant for our problem. The first two results, Lemma 5.4 and Lemma 5.5, were originally stated and proven in [21, Section 5]. We recall their statements here for the readers' convenience.

First, we need to introduce some notation. Let $(\Phi,\mathcal{Y})$ be a m.p.p. in $\mathbb{R}^n\times\mathbb{R}_+$, with Φ satisfying the assumptions (H1)–(H3) of Subsection 2.3 and $\mathcal{Y} := (Y_i)_{x_i\in\Phi^\omega}$, with $Y_i : \Omega \to \mathbb{R}_+$ measurable, satisfying (H4) therein, with (2.10) replaced by

$$\langle Y\rangle := \int_0^{+\infty} y h(y)\,\mathrm{d}y < +\infty\,, \tag{5.13}$$

and (2.12) replaced by

$$|K(r,y_1,y_2)| \le \frac{C}{(1+r^\gamma)(1+y_1^{s/(n-q)})(1+y_2^{s/(n-q)})}\,. \tag{5.14}$$

(Think of $\mathcal{Y} := (\rho_i^{n-q})$ for our application.)

Lemma 5.4. *Let $Q \subset \mathbb{R}^n$ be a unit cube, $(\Phi,\mathcal{Y})$ a m.p.p. as above and $B \subset \mathbb{R}^n$ a bounded set star-shaped with respect to the origin. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0^+} \varepsilon^n N^\omega_\varepsilon(B) = \lambda\mathcal{L}^n(B) \tag{5.15}$$

and

$$\lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{x_i\in\Phi^\omega_\varepsilon(B)} Y^\omega_i = \lambda\langle Y\rangle\mathcal{L}^n(B)\,. \tag{5.16}$$

Moreover, for any bounded set $A \subset \mathbb{R}^n$, the thinning process Φ^δ defined in (5.2) satisfies

$$\lim_{\delta\to 0^+} \langle N^\delta(A)\rangle = \langle N(A)\rangle = \lambda\mathcal{L}^n(A)\,. \tag{5.17}$$

A by-product of Lemma 5.4 (cf. [21, Section 5]) is the following.

Lemma 5.5. *Let $(\Phi,\mathcal{Y})$ be a m.p.p. as above and $B \subset \mathbb{R}^n$ a bounded set star-shaped with respect to the origin. Let $I^\omega_\varepsilon \subset \Phi^\omega_\varepsilon(B)$ be such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0^+} \varepsilon^n \# I^\omega_\varepsilon = 0\,.$$

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^n \sum_{x_i\in I^\omega_\varepsilon} Y^\omega_i = 0\,.$$

We conclude this section with a technical result which can be seen as the nonlinear counterpart of [21, Lemma 5.3]. This result will then be used to prove a stochastic Riemann-sum approximation for the capacitary term appearing in the Γ-limit $\mathcal{F}_0$ in (3.16).

Let $M \in \mathbb{N}$ be fixed and let $\kappa : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ be a Borel function bounded from below, such that

$$\kappa(0,\cdot) = 0, \quad \kappa(z,y) \leq C_\kappa(|z|^q y^r + 1) \ \text{ for } 0 < q, r < n\,, \tag{5.18}$$

and

$$|\kappa(z_1,y) - \kappa(z_2,y)| \leq C_{\kappa,M}(1 + |z_1|^{q-1} + |z_2|^{q-1})|z_1 - z_2| \ \text{ for every } y \in [0,M]\,, \tag{5.19}$$

for some $C_\kappa, C_{\kappa,M} > 0$. We observe that $\kappa(\cdot,y) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)$, with norm uniformly bounded in $y \in [0,M]$; *i.e.*, for every open and bounded set $E \subset\subset \mathbb{R}^m$ and for any $y \in [0,M]$,

$$|\kappa(z_1,y) - \kappa(z_2,y)| \leq C_{\kappa,M,E}|z_1 - z_2| \ \ \forall\, z_1, z_2 \in \overline{E}\,, \tag{5.20}$$

for some $C_{\kappa,M,E} > 0$.

Let $(\Phi, \mathcal{Y})$ be a m.p.p. as defined above; we furthermore assume that

$$Y_i : \Omega \mapsto [0,M]\,. \tag{5.21}$$

Finally, let $u \in C^\infty_c(D;\mathbb{R}^m)$ be fixed and let $X^u_i : D \times \Omega \to \mathbb{R}$ be defined as

$$X^u_i(x,\omega) := \kappa(u(x), Y^\omega_i)\,. \tag{5.22}$$

Then, $X^u_i(x,\cdot)$ is $(\mathcal{T}, \mathcal{B}(\mathbb{R}))$-measurable and $X^u_i(\cdot,\omega)$ is continuous. Let $\mathcal{X}^u := (X^u_i)_{x_i \in \Phi^\omega}$ denote this family of *space-dependent* marks, and let $(\Phi, \mathcal{X}^u)$ be the corresponding marked point process in $\mathbb{R}^n \times \mathbb{R}_+$. Moreover, we assume for the average function $x \mapsto \langle X^u(x,\cdot)\rangle := \int_0^M \kappa(u(x),y)h(y)\,\mathrm{d}y$ that

$$\langle X^u \rangle = \int_0^M \kappa(u(\cdot),y)h(y)\,\mathrm{d}y \in L^\infty(D)\,. \tag{5.23}$$

Proposition 5.6. *Let $Q \subset \mathbb{R}^n$ be a unit cube and $(\Phi, \mathcal{X}^u)$ be the m.p.p. in $\mathbb{R}^n \times \mathbb{R}_+$ defined above, where Φ satisfies* (H1)–(H3)*, and X^u_i are defined in* (5.22)*. Let $r_\varepsilon > 0$ be such that*

$$\lim_{\varepsilon \to 0^+} r_\varepsilon = 0\,. \tag{5.24}$$

Then there exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ and a subsequence in $\varepsilon > 0$ (not relabelled) such that

$$\lim_{\varepsilon \to 0^+} \varepsilon^n \sum_{x_i \in \Phi^{2/M,\omega}_\varepsilon(D)} \fint_{B_{r_\varepsilon}(\varepsilon x_i)} X^u_i(x,\omega)\,\mathrm{d}x = \langle N^{2/M}(Q)\rangle \int_D \langle X^u(x,\cdot)\rangle\,\mathrm{d}x\,, \tag{5.25}$$

for every $\omega \in \Omega'$, every $M \in \mathbb{N}$, and every $u \in C^\infty_c(D;\mathbb{R}^m)$.

Proof. In what follows, by $\lim_{\varepsilon \to 0^+}$ we mean limits taken up to an ω-independent subsequence.

Let $u \in C^\infty_c(D;\mathbb{R}^m)$ and $M \in \mathbb{N}$ be fixed. We split the proof into a number of steps.

Step 1: Properties of the space-dependent marks. We observe that $(X^u_i)_{x^\omega_i \in \Phi^\omega_\varepsilon(D)}$ satisfy the following properties.

(P1) For every $\omega \in \Omega$ and $x_i \in \Phi^\omega_\varepsilon(D)$ we have

$$X^u_i(x,\omega) = 0 \ \text{ for every } \ x \in \partial D\,.$$

(P2) There exists $\Lambda := \Lambda(q,r,\kappa,\|u\|_\infty,M) \in (0,+\infty)$ such that

$$\sup_{\omega \in \Omega} \sup_{x_i \in \Phi^\omega_\varepsilon(D)} \|X^u_i(\cdot,\omega)\|_{L^\infty(D)} \leq \Lambda\,. \tag{5.26}$$

(P3) For every $\omega \in \Omega$ the functions $X^u_i(\cdot,\omega)$ belong to $W^{1,\infty}(D)$, with Lipschitz norm uniformly bounded in i (and similarly for their expected value $\langle X^u(x,\cdot)\rangle$).

Property (P1) follows immediately from $u|_{\partial D} \equiv 0$, since for every $\omega \in \Omega$, $x_i \in \Phi_\varepsilon^\omega(D)$, and $x \in \partial D$, by (5.18) we have

$$X_i^u(x,\omega) = \kappa(u(x), Y_i^\omega) = \kappa(0, Y_i^\omega) = 0\,.$$

Property (P2) is a consequence of (5.18), (5.21), and the fact that $u \in C_c^\infty(D;\mathbb{R}^m)$. Indeed, for every $\omega \in \Omega$, $x_i \in \Phi_\varepsilon^\omega(D)$, and $x \in D$ we have

$$|X_i^u(x,\omega)| = |\kappa(u(x), Y_i^\omega)| \le C_\kappa(|u(x)|^q (Y_i^\omega)^r + 1) \le C_\kappa(\|u\|_\infty^q M^r + 1)\,,$$

thus (5.26) is satisfied with $\Lambda := C_\kappa(\|u\|_\infty^q M^r + 1)$, which is independent of both ω and i.

Finally, by (5.20) and (5.21), for every $\omega \in \Omega$, $x_i \in \Phi_\varepsilon^\omega(D)$, and $x, x' \in D$ we deduce that

$$\begin{aligned}|X_i^u(x,\omega) - X_i^u(x',\omega)| &= |\kappa(u(x), Y_i^\omega) - \kappa(u(x'), Y_i^\omega)| \\ &\le C_{\kappa,M,\|u\|_\infty}|u(x) - u(x')| \le C_{\kappa,M,\|u\|_\infty}\|\nabla u\|_\infty |x - x'|\,,\end{aligned} \tag{5.27}$$

and therefore (P3). We observe that by passing to the expected value, the analogue of (5.27) holds true for $\langle X^u(x,\cdot)\rangle$ as well.

Step 2: Replacing r_ε with ε in (5.25). If r_ε satisfies (5.24), by a change of variables and by (5.27) and (5.3) we get

$$\begin{aligned}&\Bigg|\varepsilon^n \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \fint_{B_{r_\varepsilon}(\varepsilon x_i)} X_i^u(x,\omega)\,\mathrm{d}x - \varepsilon^n \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \fint_{B_\varepsilon(\varepsilon x_i)} X_i^u(x,\omega)\,\mathrm{d}x\Bigg| \\ &\quad = \frac{\varepsilon^n}{\beta_n}\Bigg|\sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_1(0)} \big(X_i^u(\varepsilon x_i + r_\varepsilon z, \omega) - X_i^u(\varepsilon x_i + \varepsilon z, \omega)\big)\,\mathrm{d}z\Bigg| \\ &\quad \le C|r_\varepsilon - \varepsilon|\varepsilon^n N_\varepsilon^{2/M,\omega}(D) \le C\varepsilon\big(\varepsilon^n N_\varepsilon^\omega(D)\big) \underset{\varepsilon\to 0^+}{\longrightarrow} 0\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, thanks to (5.15). Therefore, to prove (5.25) it suffices to show that there exists $\Omega_M \in \mathcal{T}$ with $\mathbb{P}(\Omega_M) = 1$ such that, up to subsequences

$$\lim_{\varepsilon\to 0^+} \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} X_i^u(x,\omega)\,\mathrm{d}x = \beta_n \langle N^{2/M}(Q)\rangle \int_D \langle X^u(x,\cdot)\rangle\,\mathrm{d}x\,, \tag{5.28}$$

for every $\omega \in \Omega_M$ and every $u \in C_c^\infty(D;\mathbb{R}^m)$.

Step 3: Reducing (5.28) *to a dense subset $\mathcal{D} \subset C_c^\infty(D;\mathbb{R}^m)$.* Let $\mathcal{D} \subset C_c^\infty(D;\mathbb{R}^m)$ be countable and dense with respect to the strong $L^q(D;\mathbb{R}^m)$-topology. Assume that there exists $\Omega_M \in \mathcal{T}$ with $\mathbb{P}(\Omega_M) = 1$ such that, up to subsequences, (5.28) holds true for every $\omega \in \Omega_M$ and every $u \in \mathcal{D}$.

Let now $\omega \in \Omega_M$ and $u \in C_c^\infty(D;\mathbb{R}^m)$, and let $(u_k) \subset \mathcal{D}$ be such that $u_k \to u$ strongly in $L^q(D;\mathbb{R}^m)$ as $k \to +\infty$. We have

$$\begin{aligned}&\Bigg|\sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} X_i^u(x,\omega)\,\mathrm{d}x - \beta_n\langle N^{2/M}(Q)\rangle \int_D \langle X^u(x,\cdot)\rangle\,\mathrm{d}x\Bigg| \\ &\le \Bigg|\sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} X_i^{u_k}(x,\omega)\,\mathrm{d}x - \beta_n\langle N^{2/M}(Q)\rangle \int_D \langle X^{u_k}(x,\cdot)\rangle\,\mathrm{d}x\Bigg| \\ &\quad + \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} \Big|X_i^u(x,\omega) - X_i^{u_k}(x,\omega)\Big|\,\mathrm{d}x \\ &\quad + \beta_n\langle N^{2/M}(Q)\rangle \int_D \Big|\langle X^u(x,\cdot)\rangle - \langle X^{u_k}(x,\cdot)\rangle\Big|\,\mathrm{d}x\,.\end{aligned} \tag{5.29}$$

The first term in the right-hand side of (5.29) converges to zero as $\varepsilon \to 0^+$ by assumption, since $u_k \in \mathcal{D}$. For the second term, by the definition of X_i^u and by (5.19) we find that for every

$x_i \in \Phi_\varepsilon^{2/M,\omega}(D)$

$$\begin{aligned}\int_{B_\varepsilon(\varepsilon x_i)} \Big|X_i^u(x,\omega) - X_i^{u_k}(x,\omega)\Big| \,\mathrm{d}x &\leq C_{\kappa,M} \int_{B_\varepsilon(\varepsilon x_i^\omega)} \Big(1 + |u|^{q-1} + |u_k|^{q-1}\Big)|u-u_k| \,\mathrm{d}x \\ &\lesssim \Big(\varepsilon^{n/q} + \|u\|_{L^q(B_\varepsilon(\varepsilon x_i))} + \|u_k\|_{L^q(B_\varepsilon(\varepsilon x_i))}\Big)^{q-1} \|u-u_k\|_{L^q(B_\varepsilon(\varepsilon x_i))} \\ &\lesssim (\varepsilon^{n/q} + \|u\|_{L^q(B_\varepsilon(\varepsilon x_i))})^{q-1} \|u-u_k\|_{L^q(B_\varepsilon(\varepsilon x_i))}\,,\end{aligned}$$

where in the last inequality we used the fact that $u_k \to u$ strongly in $L^q(D;\mathbb{R}^m)$. Adding up the previous inequality over all $x_i \in \Phi_\varepsilon^{2/M,\omega}(D)$, using the fact that each ball $B_\varepsilon(\varepsilon x_j)$ overlaps with at most a finite number (depending only on M) of other balls of the family $(B_\varepsilon(\varepsilon x_i))_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)}$, and by a discrete Hölder inequality, we deduce

$$\begin{aligned}&\sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} \Big|X_i^u(x,\omega) - X_i^{u_k}(x,\omega)\Big| \\ &\lesssim \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} (\varepsilon^{n/q} + \|u\|_{L^q(B_\varepsilon(\varepsilon x_i))})^{q-1} \|u-u_k\|_{L^q(B_\varepsilon(\varepsilon x_i))} \\ &\leq \Big(\sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} (\varepsilon^{n/q} + \|u\|_{L^q(B_\varepsilon(\varepsilon x_i))})^q\Big)^{\frac{q-1}{q}} \Big(\sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} |u_k-u|^q\Big)^{\frac{1}{q}} \\ &\lesssim \Big(\varepsilon^n N_\varepsilon^{2/M,\omega}(D) + \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i^\omega)} |u|^q\Big)^{\frac{q-1}{q}} \|u_k - u\|_{L^q(D)} \\ &\leq \Big((\varepsilon^n N_\varepsilon^\omega(D))^{q-1/q} + \|u\|^{q-1}_{L^q(D)}\Big)\|u_k-u\|_{L^q(D)}\,. \end{aligned} \tag{5.30}$$

Hence, by (5.30), the second term in the right-hand side of (5.29) converges to zero as $k \to +\infty$. To conclude, we show that the third term in the right-hand side of (5.29) converges to zero as $k \to +\infty$ as well. By the definition (5.23) and (5.19) we estimate

$$\begin{aligned}\int_D \Big|\langle X^u(x,\cdot)\rangle - \langle X^{u_k}(x,\cdot)\rangle\Big| \,\mathrm{d}x &\leq \int_D \int_0^M \Big|\kappa(u(x),y) - \kappa(u_k(x),y)\Big| h(y) \,\mathrm{d}y \,\mathrm{d}x \\ &\leq C_{\kappa,M} \int_D (1 + |u_k|^{q-1} + |u|^{q-1})|u-u_k| \leq C_{\kappa,M}(1 + \|u\|^{q-1}_{L^q(D)})\|u_k-u\|_{L^q(D)}\,,\end{aligned} \tag{5.31}$$

where we used that $\int_{\mathbb{R}_+} h(y)\,\mathrm{d}y = 1$ (see (H4)), and that $u_k \to u$ strongly in L^q. This term is indeed infinitesimal as $k \to +\infty$, so the proof of this step is complete.

Step 4: Proving (5.28) *in* $\mathcal{D}$. Let $v \in \mathcal{D}$. We estimate

$$\Big| \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} X_i^v(x,\omega)\,\mathrm{d}x - \beta_n \langle N^{2/M}(Q)\rangle \int_D \langle X^v(x,\cdot)\rangle \,\mathrm{d}x\Big| \leq a_\varepsilon^v(\omega) + b_\varepsilon^v(\omega)\,,$$

where

$$a_\varepsilon^v(\omega) := \Big| \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x - \beta_n \langle N^{2/M}(Q)\rangle \int_D \langle X^v(x,\cdot)\rangle \,\mathrm{d}x\Big|\,, \tag{5.32}$$

and

$$b_\varepsilon^v(\omega) := \Big| \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(D)} \int_{B_\varepsilon(\varepsilon x_i)} \big(X_i^v(x,\omega) - \langle X^v(x,\cdot)\rangle\big) \,\mathrm{d}x\Big|\,. \tag{5.33}$$

We claim that there exists $\Omega_M^v \in \mathcal{T}$ with $\mathbb{P}(\Omega_M^v) = 1$, so that up to a deterministic subsequence in $\varepsilon > 0$ (independent of $v \in \mathcal{D}$),

$$\lim_{\varepsilon\to 0^+} a_\varepsilon^v(\omega) = \lim_{\varepsilon\to 0^+} b_\varepsilon^v(\omega) = 0 \quad \forall\, \omega \in \Omega_M^v\,. \tag{5.34}$$

Note that since $\mathcal{D}$ is countable, by setting $\Omega_M := \bigcap_{v\in\mathcal{D}} \Omega_M^v$, we obtain directly (5.28) in $\mathcal{D}$.

In what follows, to not overburden the notation, we omit the possible extraction of a subsequence in $\varepsilon > 0$, as long as this can be chosen independently of the realisation $\omega \in \Omega$.

Substep 4.1: Rewriting (5.34). We start by rewriting a_ε^v. Since, by (P1), for every $\omega \in \Omega$ and every $x_i \in \Phi_\varepsilon^\omega(D)$ we have $X_i^v(x,\omega) = \langle X^v(x,\cdot)\rangle = 0$ for every $x \in \partial D$, by setting $X_i^v(x,\omega) = \langle X^v(x,\cdot)\rangle = 0$ for every $x \in \mathbb{R}^n \setminus D$, we obtain functions defined in the whole of $\mathbb{R}^n$. We now tessellate $\mathbb{R}^n$ into unitary cubes $\{Q_j\}_{j\in\mathbb{N}}$ with $Q_j := Q(z_j)$ and $\{z_j\}_{j\in\mathbb{N}} \equiv \mathbb{Z}^n$, and observe that

$$|x - \varepsilon z_j| \leq (1 + \sqrt{n}/2)\varepsilon, \quad \text{for every } x \in B_\varepsilon(\varepsilon x_i),\, x_i \in \Phi^{2/M,\omega}(Q_j)\,. \tag{5.35}$$

Set $\mathbb{N}_\varepsilon(D) := \{j \in \mathbb{N} \colon \varepsilon Q_j \cap D \neq \emptyset\}$; noticing that $\varepsilon Q_j = Q_\varepsilon(\varepsilon z_j)$, by (5.32) we split

$$a_\varepsilon^v(\omega) = \Big| \sum_{j\in\mathbb{N}_\varepsilon(D)} \Big(\sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \int_{B_\varepsilon(\varepsilon x_i)} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x - \beta_n \langle N^{2/M}(Q)\rangle \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x \Big) \Big|\,, \tag{5.36}$$

where we have used that $\langle X^v(x,\cdot)\rangle = 0$ for every $x \in \mathbb{R}^n \setminus D$. Now, for every $j \in \mathbb{N}_\varepsilon(D)$ we write

$$\begin{aligned}
&\sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \int_{B_\varepsilon(\varepsilon x_i)} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x - \beta_n \langle N^{2/M}(Q)\rangle \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x \\
&= \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \bigg(\int_{B_\varepsilon(\varepsilon x_i)} \Big(\langle X^v(x,\cdot)\rangle - \langle X^v(\varepsilon z_j,\cdot)\rangle \Big) - \beta_n \int_{\varepsilon Q_j} \Big(\langle X^v(x,\cdot)\rangle - \langle X^v(\varepsilon z_j,\cdot)\rangle \Big) \,\mathrm{d}x \bigg) \\
&\quad + \beta_n \big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle \big) \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x\,.
\end{aligned} \tag{5.37}$$

By (5.27) (for $\langle X^v\rangle$) and (5.35) we estimate for every $x \in B_\varepsilon(\varepsilon x_i)$ with $x_i \in \Phi^{2/M,\omega}(Q_j) = \Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)$, and similarly for $x \in \varepsilon Q_j$,

$$|\langle X^v(x,\cdot)\rangle - \langle X^v(\varepsilon z_j,\cdot)\rangle| \leq C_{\kappa,M,\|v\|_\infty} \|\nabla v\|_\infty |x - \varepsilon z_j| \leq C_{\kappa,M,\|v\|_\infty} \|\nabla v\|_\infty (1 + \sqrt{n}/2)\varepsilon\,. \tag{5.38}$$

Hence, from (5.36), (5.37) and (5.38), we can estimate

$$\begin{aligned}
a_\varepsilon^v(\omega) &\lesssim C_{\kappa,M,\|v\|_\infty} \|\nabla v\|_\infty \varepsilon \sum_{j\in\mathbb{N}_\varepsilon(D)} \varepsilon^n N_\varepsilon^{2/M,\omega}(\varepsilon Q_j) \\
&\quad + \Big| \sum_{j\in\mathbb{N}_\varepsilon(D)} \big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle \big) \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x \Big|\,, \\
&\lesssim C(M,v)\varepsilon(\varepsilon^n N_\varepsilon^{2/M,\omega}(D)) + \Big| \sum_{j\in\mathbb{N}_\varepsilon(D)} \big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle \big) \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x \Big|\,.
\end{aligned}$$

In view of (5.15), to prove (5.34) for α_ε^v, it then suffices to show that there exists $\Omega_M^v \in \mathcal{T}$ with $\mathbb{P}(\Omega_M^v) = 1$, such that

$$\lim_{\varepsilon\to 0^+} S_{1,\varepsilon}(\omega) = 0 \quad \forall\, \omega \in \Omega_M^v\,, \tag{5.39}$$

where

$$S_{1,\varepsilon}(\omega) := \sum_{j\in\mathbb{N}_\varepsilon(D)} \alpha_\varepsilon^j \big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle \big), \quad \alpha_\varepsilon^j := \int_{\varepsilon Q_j} \langle X^v(x,\cdot)\rangle \,\mathrm{d}x\,. \tag{5.40}$$

We now rewrite (5.34) for b_ε^v in (5.33). First we split

$$b_\varepsilon^v(\omega) = \Big| \sum_{j\in\mathbb{N}_\varepsilon(D)} \sum_{x_i \in \Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \int_{B_\varepsilon(\varepsilon x_i)} \widetilde{X}_i^v(x,\omega) \,\mathrm{d}x \Big|\,,$$

where we set

$$\widetilde{X}_i^v(x,\omega) := X_i^v(x,\omega) - \langle X^v(x,\cdot)\rangle\,. \tag{5.41}$$

Then, proceeding as for $a_\varepsilon^v(\omega)$, by (5.27) for $\widetilde{X}_i^v$ and (5.35), by a similar inequality as (5.38) we get,

$$b_\varepsilon^v(\omega) \lesssim C(M,v)(\varepsilon^n N_\varepsilon^{2/M,\omega}(D))\varepsilon + \varepsilon^n \Big| \sum_{j\in\mathbb{N}_\varepsilon(D)} \sum_{x_i\in\Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \widetilde{X}_i^v(\varepsilon z_j,\omega)\Big| .$$

Hence, again by (5.15), to prove (5.34) for b_ε^v, it suffices to show that there exists $\Omega_M^v \in \mathcal{T}$ with $\mathbb{P}(\Omega_M^v) = 1$, such that

$$\lim_{\varepsilon\to 0^+} S_{2,\varepsilon}(\omega) = 0 \quad \forall\, \omega \in \Omega_M^v\,, \tag{5.42}$$

where

$$S_{2,\varepsilon}(\omega) := \varepsilon^n \sum_{j\in\mathbb{N}_\varepsilon(D)} \sum_{x_i^\omega\in\Phi_\varepsilon^{2/M,\omega}(\varepsilon Q_j)} \widetilde{X}_i^v(\varepsilon z_j,\omega)\,. \tag{5.43}$$

Substep 4.2: Rewriting (5.39) *and* (5.42). Assume that there exists a subsequence $(\varepsilon_k)_k$ independent of the realisation $\omega\in\Omega$, with $\varepsilon_k \searrow 0$ as $k\to+\infty$, such that

$$\sum_{k=1}^{\infty} \langle S_{l,\varepsilon_k}^2\rangle < +\infty\,, \text{ for } l=1,2\,. \tag{5.44}$$

We now show that then (5.39) and (5.42) follow. Indeed, for $l=1,2$, (5.44) implies in particular that

$$\Big\langle \sum_{k=1}^{\infty} S_{l,\varepsilon_k}^2 \Big\rangle < +\infty \implies \sum_{k=1}^{\infty} S_{l,\varepsilon_k}^2(\omega) < +\infty \text{ for } \mathbb{P}\text{−a.e. } \omega\in\Omega\,.$$

In particular, $\lim_{k\to+\infty} S_{l,\varepsilon_k}(\omega) = 0$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. As already remarked, the set of events where the last assertion holds depends on $M\in\mathbb{N}$ and $v\in\mathcal{D}$, and hence it is an admissible Ω_M^v as in the claims.

Substep 4.3: Proving (5.44) *for* $l=1$. We note that, by (P2), the deterministic (v-dependent) coefficients (α_ε^j) defined in (5.40) satisfy

$$|\alpha_\varepsilon^j| \le \|\langle X^v\rangle\|_{L^\infty(D)}\varepsilon^n \le \Lambda\,\varepsilon^n\,, \tag{5.45}$$

with $\Lambda>0$ as in (5.26). Recalling that $\langle X^v\rangle \in W^{1,\infty}(D)$ (recall (P3)) and $\langle X^v\rangle = 0$ in $\mathbb{R}^n\setminus D$, we get

$$\begin{aligned} S_{1,\varepsilon}^2(\omega) = &\sum_{j\in\mathbb{N}_\varepsilon(D)} (\alpha_\varepsilon^j)^2\big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle\big)^2 \\ &+ \sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)} \alpha_\varepsilon^j\alpha_\varepsilon^{j'}\big(N^{2/M,\omega}(Q_j) - \langle N^{2/M}(Q)\rangle\big)\big(N^{2/M,\omega}(Q_{j'}) - \langle N^{2/M}(Q)\rangle\big)\,. \end{aligned} \tag{5.46}$$

Note that the thinning process $\Phi^{2/M}$ inherits the properties (H1)–(H3) in Subsection 2.3 from Φ. In particular, the stationarity condition (H1) for $\Phi^{2/M}$ and (2.8) yield

$$\langle N^{2/M}(Q_j)\rangle = \langle N^{2/M}(Q)\rangle, \quad \langle (N^{2/M}(Q_j))^2\rangle = \langle (N^{2/M}(Q))^2\rangle \quad \text{for every } j\in\mathbb{N}_\varepsilon(D)\,,$$

and

$$\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle \le \langle (N^{2/M}(Q))^2\rangle \le \lambda^2 \quad \text{for every } j,j'\in\mathbb{N}_\varepsilon(D)\,. \tag{5.47}$$

Then, taking the expected value in (5.46), by (5.45), (5.47) and (2.8), we have

$$\begin{aligned} \langle S_{1,\varepsilon}^2\rangle &\lesssim \varepsilon^{2n}\Big(\lambda^2\#\mathbb{N}_\varepsilon(D) + \sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)} |\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle - \langle N^{2/M}(Q)\rangle^2|\Big) \\ &\lesssim_\lambda \varepsilon^n + \varepsilon^{2n} \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|\le 2\sqrt{n}}} |\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle - \langle N^{2/M}(Q)\rangle^2| \\ &\quad + \varepsilon^{2n} \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} |\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle - \langle N^{2/M}(Q)\rangle^2|\,, \end{aligned}$$

where we have also used the fact that $\#\mathbb{N}_\varepsilon(D) \lesssim \mathcal{L}^n(D)\varepsilon^{-n}$, and split the sum over $j \neq j' \in \mathbb{N}_\varepsilon(D)$ into contributions "close to the diagonal" and "far from the diagonal". For the "close" contribution, by (5.47) and (2.8) we estimate

$$\sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|\leq 2\sqrt{n}}} |\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle - \langle N^{2/M}(Q)\rangle^2| \lesssim \lambda^2 \sum_{j\in\mathbb{N}_\varepsilon(D)} \#\{j'\colon |z_j-z_{j'}|\leq 2\sqrt{n}\} \lesssim_\lambda \#\mathbb{N}_\varepsilon(D)\,.$$

For the "far" contribution, by (2.9) applied, by stationarity, to the random variables $N^{2/M}(Q(z_j - z_{j'})$ and $N^{2/M}(Q(0))$, we get

$$\begin{aligned}\sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} |\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle - \langle N^{2/M}(Q)\rangle^2| &\lesssim \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} \frac{C\langle (N^{2/M}(Q))^2\rangle}{1+(|z_j-z_{j'}|-\sqrt{n})^\gamma}\\ &\lesssim_\lambda \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} \frac{1}{|z_j-z_{j'}|^\gamma}\,,\end{aligned}$$

where we have used the elementary fact that $\mathrm{diam}(Q) = \sqrt{n}$. Hence, we estimate

$$\langle S_{1,\varepsilon}^2\rangle \lesssim_\lambda \varepsilon^n + \varepsilon^{2n} \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} \frac{1}{|z_j-z_{j'}|^\gamma}\,. \tag{5.48}$$

To conclude the proof of (5.44) for $l = 1$, we are now left to estimate the second summand in (5.48). To this end, note that for every $j \in \mathbb{N}$,

$$\{j' \in \mathbb{N}\colon |z_j - z_{j'}| > 2\sqrt{n}\} \subset \bigcup_{\ell=1}^{\infty}\{k \in \mathbb{N}\colon z_k \in \partial Q_{2\ell}(z_j)\}\,.$$

Since $|z_j - z_k| \geq \ell$ for every $k \in \mathbb{N}$ such that $z_k \in \partial Q_{2\ell}(z_j)$, and

$$\#\{k \in \mathbb{N}\colon z_k \in \partial Q_{2\ell}(z_j)\} \leq (2\ell)^n - (2(\ell-1))^n \lesssim \ell^{n-1}\,,$$

we can estimate

$$\begin{aligned}\sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}} \frac{1}{|z_j-z_{j'}|^\gamma} &\leq \sum_{j\in\mathbb{N}_\varepsilon(D)}\Big(\sum_{\ell\geq 1}\sum_{\substack{k\in\mathbb{N}\\ z_k\in\partial Q_{2\ell}(z_j)}}\frac{1}{|z_j-z_k|^\gamma}\Big)\\ &\leq \sum_{j\in\mathbb{N}_\varepsilon(D)}\sum_{\ell\geq 1}\frac{1}{\ell^\gamma}\#\{k\in\mathbb{N}\colon z_k\in\partial Q_{2\ell}(z_j)\}\\ &\lesssim \left(\sum_{\ell\geq 1}\frac{1}{\ell^{1+(\gamma-n)}}\right)\#\mathbb{N}_\varepsilon(D) \lesssim_\gamma \mathcal{L}^n(D)\varepsilon^{-n}\,,\end{aligned} \tag{5.49}$$

where we used the assumption that $\gamma > n$. Gathering (5.48) and (5.49), if we choose a deterministic $(\varepsilon_k) \searrow 0$ such that $\sum_{k\in\mathbb{N}} \varepsilon_k^n < +\infty$, we obtain

$$\sum_{k=1}^{\infty}\langle S_{1,\varepsilon_k}^2\rangle \lesssim_{\Lambda,\lambda,D,\gamma} \sum_{k=1}^{\infty}\varepsilon_k^n < +\infty\,,$$

hence the desired summability condition (5.44) for S_{1,ε_k}.

Substep 4.4: Proving (5.44) *for* $l = 2$. To simplify the presentation, it is convenient to introduce some shorthand notation for $S_{2,\varepsilon}$. Namely, for $j \in \mathbb{N}$ and $\omega \in \Omega$, set

$$Z_j(\omega) := \sum_{x_i\in\Phi^{2/M,\omega}(Q_j)} \widetilde{X}_i^v(\varepsilon z_j,\omega)$$

so that, by (5.43), we have

$$S_{2,\varepsilon}^2(\omega) = \varepsilon^{2n}\bigg(\sum_{j\in\mathbb{N}_\varepsilon(D)} Z_j(\omega)\bigg)^2 = \varepsilon^{2n}\sum_{j\in\mathbb{N}_\varepsilon(D)} Z_j^2(\omega) + \varepsilon^{2n}\sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)} Z_j(\omega)Z_{j'}(\omega)\,.$$

By the definition of $\widetilde{X}_i^v$ after (5.40) and by (5.26) for $\widetilde{X}_i^v$, we can estimate the diagonal term in $S_{2,\varepsilon}^2(\omega)$ as

$$\sum_{j\in\mathbb{N}_\varepsilon(D)} Z_j^2(\omega) \leq \sum_{j\in\mathbb{N}_\varepsilon(D)}\bigg(\sum_{x_i\in\Phi^{2/M,\omega}(Q_j)}\|\widetilde{X}_i^v(\cdot,\omega)\|_{L^\infty}\bigg)^2 \lesssim_\Lambda \sum_{j\in\mathbb{N}_\varepsilon(D)}(N^{2/M,\omega}(Q_j))^2\,.$$

Therefore, for the expected value of $S_{2,\varepsilon}^2$, by (2.8), we get

$$\langle S_{2,\varepsilon}^2\rangle \lesssim_\Lambda \lambda^2\varepsilon^{2n}\#\mathbb{N}_\varepsilon(D) + \varepsilon^{2n}\sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)}\langle Z_jZ_{j'}\rangle \lesssim_{\Lambda,\lambda,D} \varepsilon^n + \varepsilon^{2n}\sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)}\langle Z_jZ_{j'}\rangle\,, \tag{5.50}$$

where we have used again that $\#\mathbb{N}_\varepsilon(D)\lesssim\mathcal{L}^n(D)\varepsilon^{-n}$. We now estimate the last sum in the right-hand side of (5.50). We start by splitting again the sum into contributions "close to the diagonal" and "far from the diagonal", as

$$\sum_{j\neq j'\in\mathbb{N}_\varepsilon(D)}\langle Z_jZ_{j'}\rangle = \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|\leq 2\sqrt{n}}}\langle Z_jZ_{j'}\rangle + \sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}}\langle Z_jZ_{j'}\rangle\,. \tag{5.51}$$

By the definition of Z_j and by (P2), we can estimate

$$Z_jZ_{j'} = \sum_{x_i\in\Phi^{2/M,\omega}(Q_j)}\ \sum_{x_{i'}\in\Phi^{2/M,\omega}(Q_{j'})}\widetilde{X}_i^v(\varepsilon z_j,\cdot)\widetilde{X}_{i'}^v(\varepsilon z_{j'},\cdot) \leq \Lambda^2 N^{2/M}(Q_j)N^{2/M}(Q_{j'})\,,$$

hence, by (5.47),

$$\langle Z_jZ_{j'}\rangle \leq \Lambda^2\langle N^{2/M}(Q_j)N^{2/M}(Q_{j'})\rangle \leq (\Lambda\lambda)^2\,. \tag{5.52}$$

For the contribution close to the diagonal in (5.51), using (5.52) we obtain

$$\begin{aligned}
\sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|\leq 2\sqrt{n}}}\langle Z_jZ_{j'}\rangle &= \sum_{j\in\mathbb{N}_\varepsilon(D)}\ \sum_{\substack{j'\in\mathbb{N}_\varepsilon(D)\\ j'\neq j,\ |z_j-z_{j'}|\leq 2\sqrt{n}}}\langle Z_jZ_{j'}\rangle\\
&\leq (\Lambda\lambda)^2\sum_{j\in\mathbb{N}_\varepsilon(D)}\#\{z_{j'}\in\mathbb{Z}^n\colon |z_{j'}-z_j|\leq 2\sqrt{n}\} \lesssim_{\Lambda,\lambda,n}\#\mathbb{N}_\varepsilon(D)\,.
\end{aligned} \tag{5.53}$$

Hence, from (5.50), by (5.51) and (5.53), we have

$$\langle S_{2,\varepsilon}^2\rangle \lesssim_{\Lambda,\lambda,n,D} \varepsilon^n + \varepsilon^{2n}\sum_{\substack{j\neq j'\in\mathbb{N}_\varepsilon(D)\\ |z_j-z_{j'}|> 2\sqrt{n}}}\langle Z_jZ_{j'}\rangle\,. \tag{5.54}$$

We are now left to estimate the sum in the right-hand side of (5.54), namely the contribution far from the diagonal. Let then $j,j'\in\mathbb{N}_\varepsilon(D)$ be such that $|z_j-z_{j'}|>2\sqrt{n}$ and recall that

$$\langle Z_jZ_{j'}\rangle = \bigg\langle\sum_{x_i\in\Phi^{2/M,\omega}(Q_j)}\ \sum_{x_{i'}\in\Phi^{2/M,\omega}(Q_{j'})}\widetilde{X}_i^v(\varepsilon z_j,\cdot)\widetilde{X}_{i'}^v(\varepsilon z_{j'},\cdot)\bigg\rangle. \tag{5.55}$$

Let $\Theta_M^{(2)}(\cdot,\cdot)$ denote the second-order factorial moment measure of the point process $\Phi^{2/M}$. Then, by the Campbell Theorem (cf. (2.7)), (5.41), (5.55), and (2.11) we get

$$\begin{aligned}
\langle Z_jZ_{j'}\rangle &= \int_{(Q_j\times Q_{j'})\times[0,M]^2}\widetilde{\kappa}(v(\varepsilon z_j),y)\,\widetilde{\kappa}(v(\varepsilon z_j),y')f_2((x,y),(x',y'))\,\mathrm{d}\Theta_M^{(2)}(x,x')\,\mathrm{d}y\,\mathrm{d}y'\\
&= \int_{(Q_j\times Q_{j'})\times[0,M]^2}\widetilde{\kappa}(v(\varepsilon z_j),y)\widetilde{\kappa}(v(\varepsilon z_{j'}),y')K(|x-x'|,y,y')\,\mathrm{d}\Theta_M^{(2)}(x,x')\,\mathrm{d}y\,\mathrm{d}y'\,,
\end{aligned} \tag{5.56}$$

where

$$\widetilde{\kappa}(v(\varepsilon z_j), Y_i^\omega) := \kappa(v(\varepsilon z_j), Y_i^\omega) - \langle \kappa(v(\varepsilon z_j), Y_i) \rangle \,.$$

To estimate (5.56), we note that for every $x \in Q_j$ and $x' \in Q_{j'}$ there holds

$$|z_j - z_{j'}| \le |z_j - x| + |x - x'| + |x' - z_{j'}| \le \sqrt{n} + |x - x'| \le \tfrac{1}{2}|z_j - z_{j'}| + |x - x'| \,, \tag{5.57}$$

since $|z_j - z_{j'}| > 2\sqrt{n}$. Hence, for every $x \in Q_j$ and $x' \in Q_{j'}$, by (5.14) and (5.57) we have that,

$$\begin{aligned} |K(|x-x'|, y, y')| &\le \frac{C}{(1+|x-x'|^\gamma)(1+y^{s/(n-q)})(1+(y')^{s/(n-q)})} \\ &\lesssim_\gamma \frac{C}{(1+|z_j - z_{j'}|^\gamma)(1+y^{s/(n-q)})(1+(y')^{s/(n-q)})} \,. \end{aligned} \tag{5.58}$$

Appealing to (5.58), (P2), (2.8), and recalling that $s > n-q$, we have

$$\begin{aligned} \big|\langle Z_j Z_{j'} \rangle\big| &\lesssim_\gamma \frac{C\Lambda^2}{1+|z_j - z_{j'}|^\gamma} \Theta_M^{(2)}(Q_j \times Q_{j'}) \bigg(\int_0^M \frac{1}{1+y^{s/(n-q)}} \,\mathrm{d}y \bigg)^2 \\ &\lesssim_\gamma \frac{C\Lambda^2}{1+|z_j - z_{j'}|^\gamma} \langle N^{2/M}(Q_j) N^{2/M}(Q_{j'}) \rangle \bigg(1 + \int_1^{+\infty} \frac{\mathrm{d}y}{y^{s/(n-q)}} \bigg)^2 \\ &\lesssim_\gamma \frac{C(\Lambda\lambda)^2}{1+|z_j - z_{j'}|^\gamma} \bigg(1 + \int_1^{+\infty} \frac{\mathrm{d}y}{y^{s/(n-q)}} \bigg)^2 \\ &\lesssim_{\Lambda,\lambda,\gamma} \frac{1}{|z_j - z_{j'}|^\gamma} \,. \end{aligned} \tag{5.59}$$

Finally, gathering (5.54) and (5.59), we get

$$\langle S_{2,\varepsilon}^2 \rangle \lesssim_{\Lambda,\lambda,n,D} \varepsilon^n + \varepsilon^{2n} \sum_{\substack{j \neq j' \in \mathbb{N}_\varepsilon(D) \\ |z_j - z_{j'}| > 2\sqrt{n}}} \frac{1}{|z_j - z_{j'}|^\gamma} \,,$$

therefore the claim follows by (5.49), arguing as in the end of substep 4.3. □

Remark 5.7. Proposition 5.6 is the key result in the identification of the limit capacitary term in the stochastic Γ-convergence result Theorem 3.2. This identification will be done via successive approximations, and Section 6 (in particular Proposition 6.1) will be devoted to this.

We want to flag up that while the statement of Proposition 5.6 contains the thinning process $\Phi_\varepsilon^{2/M}$, the results in Section 6 are formulated for centres in the set $G_{\varepsilon,M}$ in (5.10), and $G_{\varepsilon,M} \subset \Phi_\varepsilon^{2/M}$. The technical reason for using $\Phi_\varepsilon^{2/M}$ in Proposition 5.6 is that the thinning process inherits the properties (H1)–(H3) from Φ, while $G_{\varepsilon,M}$, which depends further on the random safety layer $D^\omega_{\varepsilon,b}$, does not. However, by (5.8), this choice is of no consequence in the proof of the results of the next sections.

6. Discrete approximation of the limit capacitary term

The main result of this section is Proposition 6.1 below, where we state that the capacitary term in (3.16) can be obtained as the limit of a "random" Riemann sum of the auxiliary capacities (4.7), where the sum is restricted to the perforations centred in the set defined in (5.10).

Proposition 6.1. *Let $(\varepsilon_j) \searrow 0$, and let $(u_j), u \in W_0^{1,q}(D;\mathbb{R}^m) \cap L^\infty(D;\mathbb{R}^m)$ satisfy*

$$L := \sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(D;\mathbb{R}^m)} < +\infty \,, \tag{6.1}$$

and $u_j \rightharpoonup u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$. Then, there exist $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ we have (possibly along ω-independent subsequences)

$$\lim_{M\to+\infty} \lim_{\theta\to 0^+} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i} \in G^\omega_{j,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i}) = \lambda \int_D \varphi(u) \,\mathrm{d}x \,, \tag{6.2}$$

where $G^\omega_{j,M}$ *is as in* (5.10), $\varphi^j_{\theta,\rho_{j,i}}$ *as in* (4.7), *with* ρ *replaced by* $\rho_{j,i}$ *(namely the mark associated to* $x_{j,i}$*),* $\bar u^\omega_{j,i}$ *as in Lemma 5.2, and* φ *is defined in* (3.15).

The rest of this section is devoted to the proof of Proposition 6.1. This will be carried out in a number of intermediate steps, via successive approximations.

In what follows we assume that the sequence (u_j) converges to u pointwise $\mathcal{L}^n$-a.e. in D, which can be achieved up to passing to a subsequence. Moreover, throughout this section we always assume that $1/\theta \in \mathbb{N}$, and that $j \in \mathbb{N}$ is so large that (4.10) holds true. Finally, we do not relabel the (ω-independent) subsequences along which the limits as $j \to +\infty$ and as $\theta \to 0^+$ are taken.

Our first step, Lemma 6.2 below, shows that in the discrete approximation (6.2), the auxiliary capacity $\varphi^j_{\theta,\rho}$ can be replaced by its limit in j, namely by the capacity $\varphi_{\theta,\rho}$ defined in (4.22).

Lemma 6.2. *Under the same assumptions and notational conventions as in Proposition 6.1, there exists* $\Omega' \in \mathcal{T}$ *with* $\mathbb{P}(\Omega') = 1$ *such that for every* $\omega \in \Omega'$*, we have*

$$\lim_{\theta\to 0^+}\lim_{j\to+\infty}\varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \big|\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) - \varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})\big| = 0\,, \tag{6.3}$$

for every $M \in \mathbb{N}$*, where* $\varphi_{\theta,\rho_{j,i}}$ *is defined via* (4.22)*, with* ρ *replaced by* $\rho_{j,i}$*.*

Proof. Let $M \in \mathbb{N}$ be fixed. For $j \in \mathbb{N}$ and $\theta \in (0,1)$, we define the function $\beta^j_\theta : (0,M] \mapsto \mathbb{R}_+$ as

$$\beta^j_\theta(\rho) := \|\varphi^j_{\theta,\rho} - \varphi_{\theta,\rho}\|_{L^\infty(B^m_L(0))}\,. \tag{6.4}$$

In view of Corollary 4.5 and the fact that in (6.4) an essential sup is involved, the functions β^j_θ are measurable. Moreover, by (4.22), $\beta^j_\theta \to 0$ pointwise in $(0,M]$ as $j \to +\infty$.

Let now $\tau \in (0,1)$ be fixed. By the Egoroff Theorem there exists a measurable set $J_{\theta,M,\tau} \subset (0,M]$ such that

$$\mathcal{L}^1(J_{\theta,M,\tau}) > M - \tau \quad \text{and} \quad \|\beta^j_\theta\|_{L^\infty(J_{\theta,M,\tau})} \to 0 \ \text{ as } \ j \to +\infty\,. \tag{6.5}$$

We now start estimating the sum in (6.3). For fixed $j \in \mathbb{N}$ and $\omega \in \Omega$, we get

$$\begin{aligned}\sum_{x_{j,i}\in G^\omega_{j,M}} |\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) - \varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})| \le &\sum_{x_{j,i}\in G^\omega_{j,M}} \big(|\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})| + |\varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})|\big)\chi_{(0,M]\setminus J_{\theta,M,\tau}}(\rho_{j,i})\\ &+ \sum_{x_{j,i}\in G^\omega_{j,M}} \big|\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) - \varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})\big|\chi_{J_{\theta,M,\tau}}(\rho_{j,i})\,,\end{aligned} \tag{6.6}$$

since by (5.10) $\rho_{j,i} \le M$ for $x_{j,i} \in G^\omega_{j,M}$. Moreover, for every $x_{j,i} \in G^\omega_{j,M}$, and for $j \in \mathbb{N}$ large enough, by (4.8), (4.9), (4.18) and (4.23) we obtain

$$|\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})| + |\varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})| \lesssim |\bar u^\omega_{j,i}|^q(\rho_{j,i})^{n-q} + \theta^n\,, \tag{6.7}$$

for some $C > 0$. Hence, from (6.6), using (6.7), (6.1) and (6.4), and since $G^\omega_{j,M} \subset \Phi^\omega_{\varepsilon_j}(D)$, we get

$$\begin{aligned}&\sum_{x_{j,i}\in G^\omega_{j,M}} |\varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) - \varphi_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})|\\ &\lesssim \sum_{x_{j,i}\in G^\omega_{j,M}} \big(L^q(\rho_{j,i})^{n-q} + \theta^n\big)\chi_{(0,M]\setminus J_{\theta,M,\tau}}(\rho_{j,i}) + \sum_{x_{j,i}\in G^\omega_{j,M}} \beta^j_\theta(\rho_{j,i})\chi_{J_{\theta,M,\tau}}(\rho_{j,i})\\ &\lesssim L^q \sum_{x_{j,i}\in \Phi^\omega_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q}\chi_{(0,M]\setminus J_{\theta,M,\tau}}(\rho_{j,i}) + \big(\theta^n + \|\beta^j_\theta\|_{L^\infty(J_{\theta,M,\tau})}\big)(\#G^\omega_{j,M})\,.\end{aligned} \tag{6.8}$$

Define now the random variables $Y_{j,i} : \Omega \to \mathbb{R}_+$ as

$$Y_{j,i} := (\rho_{j,i})^{n-q}\chi_{(0,M]\setminus J_{\theta,M,\tau}}(\rho_{j,i});$$

then Lemma 5.4 (see (5.16)), applied to $(Y_{j,i})$, ensures that there exists $\Omega_{M,\theta,\tau} \in \mathcal{T}$ with $\mathbb{P}(\Omega_{M,\theta,\tau}) = 1$ such that for every $\omega \in \Omega_{M,\theta,\tau}$,

$$\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^\omega_{\varepsilon_j}(D)} Y^\omega_{j,i} = \langle N(Q)\rangle\mathcal{L}^n(D)\langle Y\rangle = \langle N(Q)\rangle\mathcal{L}^n(D)\int_0^{+\infty}\rho^{n-q}\chi_{(0,M]\setminus J_{\theta,M,\tau}}(\rho)h(\rho)\,\mathrm{d}\rho\,. \tag{6.9}$$

Note that, by (2.10), the function $y : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $y(\rho) := \rho^{n-q}h(\rho)$ satisfies $y \in L^1(\mathbb{R}_+)$.

Let $\Omega' \in \mathcal{T}$ be defined as

$$\Omega' := \bigcap_{M,\theta^{-1},\tau^{-1}\in\mathbb{N}} \Omega_{M,\theta,\tau}; \tag{6.10}$$

clearly $\mathbb{P}(\Omega') = 1$, since Ω' is a countable intersection of sets of probability 1. By (6.5), (6.8), (6.9) and (5.15), for every $\omega \in \Omega'$ and for every $T > 0$, we find

$$\begin{aligned}\varlimsup_{j\to+\infty}\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}|\varphi^j_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i})-\varphi_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i})| &\lesssim_{L,\lambda,D}\int_{(0,M]\setminus J_{\theta,M,\tau}}\rho^{n-q}h(\rho)\,\mathrm{d}\rho+\theta^n\\ &\lesssim T\mathcal{L}^1((0,M]\setminus J_{\theta,M,\tau})+\int_{\{y(\rho)>T\}}y(\rho)\,\mathrm{d}\rho+\theta^n\\ &\lesssim_L T\tau+\int_{\{y(\rho)>T\}}y(\rho)\,\mathrm{d}\rho+\theta^n\,,\end{aligned}$$

which holds true for every $\tau \in (0, M)$, and $\theta \in (0, 1)$. By taking first the limit as $\tau \to 0^+$, then as $T \to +\infty$, by using the fact that $y \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ and finally by taking the limit as $\theta \to 0^+$, we obtain (6.3) in the set Ω' defined in (6.10), and hence the claim. □

In the following lemma we derive a qualitative result for the capacity $\varphi_{\theta,\rho}$.

Lemma 6.3. *Under the same assumptions and notational conventions as in Proposition 6.1, there exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, we have*

$$\lim_{j\to+\infty}\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\Big|\varphi_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i})-\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\varphi_{\theta,\rho_{j,i}}(u_j)\,\mathrm{d}x\Big| = 0\,, \tag{6.11}$$

and

$$\lim_{j\to+\infty}\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\Big|\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\varphi_{\theta,\rho_{j,i}}(u_j)\,\mathrm{d}x-\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\varphi_{\theta,\rho_{j,i}}(u)\,\mathrm{d}x\Big| = 0\,, \tag{6.12}$$

for every fixed $M \in \mathbb{N}$ and $\theta \in (0, 1)$ with $1/\theta \in \mathbb{N}$.

Proof. Let $j \in \mathbb{N}$ be large enough so that (4.10) holds true, and $k \in \mathbb{N}$ be fixed. Then, by (4.24) and by the bound (6.1) that extends to $(\bar{u}^\omega_{j,i})$, for every $\omega \in \Omega$ we can estimate

$$\begin{aligned}&\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\Big|\varphi_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i})-\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\varphi_{\theta,\rho_{j,i}}(u_j)\,\mathrm{d}x\Big|\\ &\le\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\big|\varphi_{\theta,\rho_{j,i}}(u_j)-\varphi_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i})\big|\,\mathrm{d}x\\ &\le\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}C_M\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\big(\theta^{n(q-1)/q}+|u_j|^{q-1}+|\bar{u}^\omega_{j,i}|^{q-1}\big)|u_j-\bar{u}^\omega_{j,i}|\,\mathrm{d}x\\ &\lesssim C_M\big(L^{q-1}+\theta^{n(q-1)/q}\big)\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}|u_j-\bar{u}^\omega_{j,i}|\,\mathrm{d}x\,,\end{aligned} \tag{6.13}$$

where we used that, by (5.10), $\rho_{j,i} \le M$ for every $x_{j,i} \in G^\omega_{j,M}$. Let now $C^\omega_{j,i} := C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon x_{j,i})$ be the annulus defined as in Lemma 5.2 and, for every $x_{j,i} \in G^\omega_{j,M}$, set

$$\widetilde{u}_{j,i} := \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} u_j \,\mathrm{d}x\,.$$

By Hölder's inequality, and since $C^\omega_{j,i} \subset B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})$, we can estimate the integral in the right-hand side of (6.13) as

$$\begin{aligned}
&\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \bar{u}^\omega_{j,i}| \,\mathrm{d}x \le \Big(\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \bar{u}^\omega_{j,i}|^q \,\mathrm{d}x\Big)^{1/q} \\
&\lesssim \Big(\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \widetilde{u}_{j,i}|^q \,\mathrm{d}x + \fint_{C^\omega_{j,i}} |\widetilde{u}_{j,i} - \bar{u}^\omega_{j,i}|^q \,\mathrm{d}x\Big)^{1/q} \\
&\lesssim \Big(\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \widetilde{u}^\omega_{j,i}|^q + \frac{1}{\mathcal{L}^n(C^\omega_{j,i})} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \widetilde{u}_{j,i}|^q + \fint_{C^\omega_{j,i}} |u_j - \bar{u}^\omega_{j,i}|^q\Big)^{1/q}.
\end{aligned}$$

Hence, by Poincaré's inequality, and since $\mathcal{L}^n(C^\omega_{j,i})$ and $\mathrm{diam}(C^\omega_{j,i})$ differ from the corresponding quantities of $B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})$ only by a multiplicative constant (which can be bounded from above and below uniformly in i, j and ω), we have

$$\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \bar{u}^\omega_{j,i}| \,\mathrm{d}x \lesssim_k (\theta\varepsilon_j/M)^{1-\frac{n}{q}} \Big(\int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |\nabla u_j|^q\Big)^{1/q}.$$

By adding up the estimate above over all i such that $x_{j,i} \in G^\omega_{j,M}$ and by the discrete Hölder inequality, we have

$$\begin{aligned}
\sum_{x_{j,i}\in G^\omega_{j,M}} \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - \bar{u}^\omega_{j,i}| &\lesssim_k (\theta\varepsilon_j/M)^{1-\frac{n}{q}} \sum_{x_{j,i}\in G^\omega_{j,M}} \Big(\int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |\nabla u_j|^q\Big)^{1/q} \\
&\lesssim_k (\theta\varepsilon_j/M)^{1-\frac{n}{q}} (\#G^\omega_{j,M})^{1-\frac{1}{q}} \Big(\sum_{x_{j,i}\in G^\omega_{j,M}} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |\nabla u_j|^q\Big)^{\frac{1}{q}} \\
&\lesssim_k (\theta\varepsilon_j/M)^{1-\frac{n}{q}} (\#G^\omega_{j,M})^{1-\frac{1}{q}} \|\nabla u_j\|_{L^q(D)}\,, \qquad (6.14)
\end{aligned}$$

where in the last step we used the fact that the good perforations $(B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}))_{x_{j,i}\in G^\omega_{j,M}}$ are pairwise disjoint subsets of D. By (6.13) and (6.14) we then obtain the bound

$$\varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \Big|\varphi_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i}) - \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \varphi_{\theta,\rho_{j,i}}(u_j)\Big| \lesssim_{\theta,M,L,k} \varepsilon_j \big(\varepsilon_j^n \#G^\omega_{j,M}\big)^{1-\frac{1}{q}} \|\nabla u_j\|_{L^q(D)}\,. \qquad (6.15)$$

In view of (5.12) and since (u_j) is weakly convergent in $W^{1,q}(D;\mathbb{R}^m)$, by taking in (6.15) the limit (superior) as $j \to +\infty$, we deduce (6.11).

The proof of (6.12) follows similarly. In particular, using again (4.24) and (6.1), we estimate

$$\begin{aligned}
&\varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \Big|\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \Big(\varphi_{\theta,\rho_{j,i}}(u_j) - \varphi_{\theta,\rho_{j,i}}(u)\Big)\Big| \,\mathrm{d}x \\
&\le C_M \varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \big(\theta^{n(q-1)/q} + |u_j|^{q-1} + |u|^{q-1}\big)|u_j - u| \,\mathrm{d}x \\
&\lesssim C_M \big(L^{q-1} + \theta^{n(q-1)/q}\big)\varepsilon_j^n (\theta\varepsilon_j/M)^{-n} \sum_{x_{j,i}\in G^\omega_{j,M}} \int_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} |u_j - u| \,\mathrm{d}x \\
&\lesssim_{L,M,\theta,D} \int_D |u_j - u| \,\mathrm{d}x \lesssim_{L,M,\theta,D} \|u_j - u\|_{L^q(D)}\,.
\end{aligned}$$

Then, since u_j is weakly convergent to u in $W^{1,q}(D;\mathbb{R}^m)$, taking again the limit (superior) as $j\to+\infty$ in the above estimate, we obtain (6.12). □

We are now ready for the proof of Proposition 6.1.

Proof of Proposition 6.1. In view of Lemmata 6.2 and 6.3 it suffices to show that there exists $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega')=1$ and subsequences in $j\in\mathbb{N}$ and $\theta\in(0,1)$ (not relabelled), such that for all $\omega\in\Omega'$, and every $u\in W^{1,q}_0(D;\mathbb{R}^m)\cap L^\infty(D;\mathbb{R}^m)$,

$$\lim_{M\to+\infty}\lim_{\theta\to0^+}\lim_{j\to+\infty}\varepsilon_j^n\sum_{x_{j,i}\in G^\omega_{j,M}}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})}\varphi_{\theta,\rho_{j,i}}(u)\,\mathrm{d}x=\langle N(Q)\rangle\int_D\varphi(u)\,\mathrm{d}x\,. \tag{6.16}$$

We will prove (6.16) in a number of steps.

Step 1: Extending the sum in (6.16) *to a thinning process with well-separated perforations upon truncation.* For $M\in\mathbb{N}$, $j\in\mathbb{N}$ and $\omega\in\Omega$, let $\Phi^{2/M,\omega}_{\varepsilon_j}(D)$ denote the thinning process defined in (5.3). For every $x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)$ we define the truncated radius of the corresponding perforation as $\rho_{j,i,M}:=\min\{\rho_{j,i},M\}$. (Note that for $x_{j,i}\in G^\omega_{j,M}$ we have that $\rho_{j,i}\le M$, so the truncation is not necessary.) We claim that there exists $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega')=1$ such that for every $\omega\in\Omega'$,

$$\lim_{\substack{j\to+\infty\\ \theta\to0^+\\ M\to+\infty}}\varepsilon_j^n\Bigg|\sum_{x_{j,i}\in G^\omega_{j,M}}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})}\varphi_{\theta,\rho_{j,i}}(u)-\sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})}\varphi_{\theta,\rho_{j,i,M}}(u)\Bigg|=0\,, \tag{6.17}$$

where it is important to mention that the order with which the limits are taken is $j\to+\infty,\theta\to0^+$ and finally $M\to+\infty$ (as stated also in (6.16)). To prove (6.17), we start by estimating the sum over the centres in the thinning process $\Phi^{2/M,\omega}_{\varepsilon_j}(D)$ which are not in the good set $G^\omega_{j,M}$. For fixed $j\in\mathbb{N}$, by (4.23), (6.1), (5.1), Lemma 5.1, and recalling the definitions of $I^\omega_{\varepsilon_j,b}$ and $I^\omega_{\varepsilon_j,g}$ therein, we have

$$\begin{aligned}
&\varepsilon_j^n\Bigg|\sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\setminus G^\omega_{j,M}}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})}\varphi_{\theta,\rho_{j,i,M}}(u)\,\mathrm{d}x\Bigg|\\
&\lesssim\varepsilon_j^n\sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\setminus G^\omega_{j,M}}\Big((\rho_{j,i,M})^{n-q}\fint_{B_{\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})}|u|^q\,\mathrm{d}x+\theta^n\Big)\\
&\lesssim L^q\varepsilon_j^n\sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\setminus G^\omega_{j,M}}(\rho_{j,i,M})^{n-q}+\theta^n(\varepsilon_j^nN^\omega_{\varepsilon_j}(D))\\
&\lesssim_L\varepsilon_j^n\sum_{x_{j,i}\in I^\omega_{\varepsilon_j,b}}(\rho_{j,i})^{n-q}+\varepsilon_j^n\sum_{x_{j,i}\in(\Phi^{2/M,\omega}_{\varepsilon_j}(D)\cap I^\omega_{\varepsilon_j,g})\setminus G^\omega_{j,M}}(\rho_{j,i,M})^{n-q}+\theta^n(\varepsilon_j^nN^\omega_{\varepsilon_j}(D))\,.
\end{aligned} \tag{6.18}$$

By (5.6), (5.15) (and the fact that we will send $\theta\to0^+$), we just need to focus on the second term in the right-hand side of (6.18) and show it is infinitesimal as $j\to+\infty$, $\theta\to0^+$ and $M\to+\infty$. We split

$$(\Phi^{2/M,\omega}_{\varepsilon_j}(D)\cap I^\omega_{\varepsilon_j,g})\setminus G^\omega_{j,M}=I^\omega_{j,M}\cup J^\omega_{j,M}\,,$$

where (recalling (5.10), (5.11))

$$\begin{aligned}
&I^\omega_{j,M}:=\{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\cap I^\omega_{\varepsilon_j,g}\colon\rho_{j,i}>M\}\,,\\
&J^\omega_{j,M}:=\{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\cap I^\omega_{\varepsilon_j,g}\colon\rho_{j,i}\le M,\quad\operatorname{dist}(\varepsilon_jx_{j,i},D^\omega_{\varepsilon_j,b})<\varepsilon_j/M\}\,,
\end{aligned} \tag{6.19}$$

so that

$$\varepsilon_j^n\sum_{x_{j,i}\in(\Phi^{2/M,\omega}_{\varepsilon_j}(D)\cap I^\omega_{\varepsilon_j,g})\setminus G^\omega_{j,M}}(\rho_{j,i,M})^{n-q}=\varepsilon_j^n\sum_{x_{j,i}\in I^\omega_{j,M}}(\rho_{j,i,M})^{n-q}+\varepsilon_j^n\sum_{x_{j,i}\in J^\omega_{j,M}}(\rho_{j,i,M})^{n-q}. \tag{6.20}$$

For the sum over $I^\omega_{j,M}$, by (5.16) (applied to the random variables $Y_{j,i} := (\rho_{j,i})^{n-q}\chi_{(M,+\infty)}(\rho_{j,i})$) there exists $\Omega_M \in \mathcal{T}$ with $\mathbb{P}(\Omega_M) = 1$, such that for every $\omega \in \Omega_M$ we have

$$\begin{aligned}\varlimsup_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in I^\omega_{j,M}} (\rho_{j,i})^{n-q} &\le \varlimsup_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^\omega_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q}\chi_{(M,+\infty)}(\rho_{j,i}) \\ &= \langle N(Q)\rangle \mathcal{L}^n(D) \int_M^{+\infty} \rho^{n-q} h(\rho)\,\mathrm{d}\rho\,. \end{aligned} \tag{6.21}$$

Let $\widetilde{\Omega} := \bigcap_{M\in\mathbb{N}} \Omega_M$; note that $\mathbb{P}(\widetilde{\Omega}) = 1$. Let $\omega \in \widetilde{\Omega}$; by (6.21) and (2.10) we deduce that

$$\varlimsup_{M\to+\infty} \varlimsup_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in I^\omega_{j,M}} (\rho_{j,i})^{n-q} \le \langle N(Q)\rangle \mathcal{L}^n(D) \varlimsup_{M\to+\infty} \int_M^{+\infty} \rho^{n-q} h(\rho)\,\mathrm{d}\rho = 0\,. \tag{6.22}$$

For the sum over $J^\omega_{j,M}$ in (6.20), we first note that by (5.8) (with $\delta := M^{-1}$), there exists $\Omega^M \in \mathcal{T}$ with $\mathbb{P}(\Omega^M) = 1$, such that for every $\omega \in \Omega^M$,

$$\lim_{j\to+\infty} \varepsilon_j^n \# J^\omega_{j,M} = 0\,. \tag{6.23}$$

Hence, we can apply Lemma 5.5 to the m.p.p. $(\Phi, \mathcal{Y})$, with $Y_{j,i} = (\rho_{j,i})^{n-q}$, over the set of indices $J^\omega_{j,M}$ in (6.19), to obtain that for every $\omega \in \Omega^M$

$$\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in J^\omega_{j,M}} (\rho_{j,i})^{n-q} = 0\,. \tag{6.24}$$

Now we set $\widetilde{\Omega}' := \widetilde{\Omega} \cap \big(\bigcap_{M\in\mathbb{N}} \Omega^M\big)$; then $\widetilde{\Omega}' \in \mathcal{T}$, $\mathbb{P}(\widetilde{\Omega}') = 1$, and $\widetilde{\Omega}'$ depends only on the m.p.p. $(\Phi, \mathcal{R})$, but is independent of all the parameters ε_j, θ, M and the function u. For every $\omega \in \widetilde{\Omega}'$, by taking the limit (superior) in (6.18) as $j \to +\infty$, $\theta \to 0^+$ and then $M \to +\infty$ in this order, and by (6.20), (6.22), (6.24), (5.6), and (5.15), we arrive at

$$\varlimsup_{M\to+\infty} \varlimsup_{\theta\to0^+} \varlimsup_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)\setminus G^\omega_{j,M}} \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \varphi_{\theta,\rho_{j,i,M}}(u)\,\mathrm{d}x = 0\,,$$

which proves (6.17).

Step 2: Proof of a simplified version of (6.16) *by* (6.17). In view of (6.17), to prove (6.16) it suffices to show that there exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ and subsequences in $j \in \mathbb{N}$ and $\theta \in (0,1)$ (not relabelled), such that for all $\omega \in \Omega'$, and every $u \in W^{1,q}_0(D;\mathbb{R}^m) \cap L^\infty(D;\mathbb{R}^m)$

$$\lim_{M\to+\infty} \lim_{\theta\to0^+} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)} \fint_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \varphi_{\theta,\rho_{j,i,M}}(u)\,\mathrm{d}x = \langle N(Q)\rangle \int_D \varphi(u)\,\mathrm{d}x\,. \tag{6.25}$$

First of all, by a standard density argument (along the very same lines as the arguments between (5.28)–(5.31), using the bounds (4.23)– (4.27)), and by the Dominated Convergence Theorem we may suppose without loss of generality that $u \in C^\infty_c(D;\mathbb{R}^m)$.

To prove (6.25) we apply Proposition 5.6 to the m.p.p. $(\Phi, \mathcal{X}^u)$, for the space-dependent marks

$$X^u_{j,i}(x,\omega) := \varphi_{\theta,\rho^\omega_{j,i,M}}(u(x)) - \varphi_{\theta,\rho^\omega_{j,i,M}}(0)\,, \tag{6.26}$$

and for the radii $r_{\varepsilon_j} := \theta\varepsilon_j/M$, which satisfy (5.24) for fixed $M \in \mathbb{N}$, $\theta \in (0,1)$. We first check that the marks satisfy the assumptions in the proposition. Comparing (6.26) with (5.22), we have that in our case

$$\kappa(z,y) := \varphi_{\theta,y}(z) - \varphi_{\theta,y}(0), \quad \text{for } (z,y) \in \mathbb{R}^m \times \mathbb{R}_+, \tag{6.27}$$

and that $Y_{j,i} = \rho_{j,i,M}$. Due to the truncation to M the marks $Y_{j,i}$ satisfy (5.13), since $h \in L^1(\mathbb{R}_+;\mathbb{R}_+)$. Moreover, due to the truncation to M they also satisfy (5.21). We now check that the function κ in (6.27) satisfies the requirements for Proposition 5.6. First of all, κ is Carathéodory and hence measurable, since by Corollary 4.5 the function $z \mapsto \varphi_{\theta,y}(z)$ is continuous, while $y \mapsto \varphi_{\theta,y}(z)$ is increasing, and hence measurable. Moreover, $\kappa(0,y) = 0$ and κ is bounded from below by (4.23), which implies that $\kappa(z,y) \ge -(C_2 + C_4)$ for every $y \in [0,M]$ and $\theta \in (0,1)$.

By (4.23) it is immediate to see that (5.18) holds true with $C_\kappa = \max\{C_2+C_4, C_3\}$ and $r := n-q$, for every $\theta \in (0,1)$; also, (4.24) guarantees the validity of (5.19), again for every $\theta \in (0,1)$. Finally, by (5.18) we have that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}|\langle X^u(x,\cdot)\rangle| &= \int_0^{+\infty} |\kappa(u(x),y)|h(y)\,\mathrm{d}y \le C_\kappa |u(x)|^q \int_0^{+\infty} y^{n-q}h(y)\,\mathrm{d}y + C_\kappa \\ &\le C_\kappa |u(x)|^q M^{n-q} + C_\kappa,\end{aligned}$$

where we have used that $\int_{\mathbb{R}_+} h = 1$. Hence (5.23) is also satisfied.

By applying (5.25) to the m.p.p. $(\Phi, \mathcal{X}^u)$ and (5.16) to the m.p.p. $(\Phi^{2/M}, \varphi_{\theta,\rho\wedge M}(0))$, we have that for fixed $M \in \mathbb{N}$ and $\theta \in (0,1)$ (with $\frac{1}{\theta} \in \mathbb{N}$), there exists $\Omega_{M,\theta} \in \mathcal{T}$ with $\mathbb{P}(\Omega_{M,\theta}) = 1$ and a subsequence in $j \in \mathbb{N}$ (not relabelled) such that for every $\omega \in \Omega_{M,\theta}$,

$$\begin{aligned}\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)} \Big(⨍_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} (\varphi_{\theta,\rho_{j,i,M}}(u) - \varphi_{\theta,\rho_{j,i,M}}(0))\,\mathrm{d}x\Big) \\ = \langle N^{2/M}(Q)\rangle \int_D \big(\langle\varphi_{\theta,\rho\wedge M}(u)\rangle - \langle\varphi_{\theta,\rho\wedge M}(0)\rangle\big)\,\mathrm{d}x\,,\end{aligned}$$

and by Lemma 5.4 applied to the process $\Phi^{2/M}$ for the marks $\big(\varphi_{\theta,\rho_{j,i,M}}(0)\big)_i$,

$$\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)} \varphi_{\theta,\rho_{j,i,M}}(0) = \langle N^{2/M}(Q)\rangle \mathcal{L}^n(D)\langle\varphi_{\theta,\rho\wedge M}(0)\rangle\,,$$

which gives

$$\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^{2/M,\omega}_{\varepsilon_j}(D)} ⨍_{B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})} \varphi_{\theta,\rho_{j,i,M}}(u(x))\,\mathrm{d}x = \langle N^{2/M}(Q)\rangle \int_D \langle\varphi_{\theta,\rho\wedge M}(u)\rangle(x)\,\mathrm{d}x\,. \tag{6.28}$$

Finally, we set

$$\Omega' := \bigcap_{M,\frac{1}{\theta}\in\mathbb{N}} \Omega_{M,\theta}\,,$$

which by its definition only depends on the m.p.p. $(\Phi, \mathcal{R})$, and satisfies $\mathbb{P}(\Omega') = 1$. Then for every $\omega \in \Omega'$ using (4.25), Proposition 4.6, the fact that for $\mathcal{L}^n$-a.e. $x \in D$ and $\mathcal{L}^1$-a.e. $\rho > 0$,

$$\varphi_{\rho\wedge M}(u(x)) \to \varphi_\rho(u(x)) \ \text{ as } M \to +\infty\,,$$

and passing to the limit as $\theta \to 0^+$ and then $M \to +\infty$ in (6.28), by the Dominated Convergence Theorem and (5.17) we obtain (6.25) and conclude the proof. □

Remark 6.4. In the periodic setting, an important assumption for the validity of the analogue of Proposition 6.1, i.e., [3, Proposition 4.3], was that the corresponding radii for the application of the joining lemma therein were all equal to a constant multiple of the lattice spacing (with constant less that 1/2) (cf. the Erratum [4]). Such a fact is also reflected here in the proof of Proposition 6.1, where the radii of all the *auxiliary spherical perforations* are equal to $\theta\varepsilon_j/M$, $M \in \mathbb{N}$, $\theta \in (0,1)$, for all $x_{j,i} \in G^\omega_{j,M}$.

7. Proof of Theorem 3.2

7.1. **The Γ-liminf inequality.** In this subsection we prove the following result.

Proposition 7.1. *Let $(\varepsilon_j) \searrow 0$. There exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$, and a deterministic subsequence of (ε_j) (not relabelled), such that for every $\omega \in \Omega'$ and every $(u_j), u \in W^{1,q}_0(D;\mathbb{R}^m)$ satisfying $u_j \to u$ in $L^1(D;\mathbb{R}^m)$, we have*

$$\liminf_{j\to+\infty} \mathcal{F}^\omega_{\varepsilon_j}(u_j) \ge \mathcal{F}_0(u)\,, \tag{7.1}$$

where $\mathcal{F}^\omega_{\varepsilon_j}$ and $\mathcal{F}_0$ are defined in (3.4) and (3.16) respectively.

Proof. To prove (7.1) we follow closely the arguments in the periodic case (cf. [3, Section 5], and [4]), and adapt them to our stochastic setting.

Without loss of generality, we assume that $\liminf_{j\to+\infty}\mathcal{F}^\omega_{\varepsilon_j}(u_j)<+\infty$. Then

$$u_j\equiv 0 \quad \text{on } (H^\omega_{\varepsilon_j}\cap D)\cup\partial D\,,$$

where $H^\omega_{\varepsilon_j}$ is defined as in (3.2), and by (3.3) and up to a non-relabelled subsequence,

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,q}(D;\mathbb{R}^m) \text{ as } j\to+\infty\,. \tag{7.2}$$

Step 1: Truncating (u_j) *and* u. Using [6, Lemma 3.5], for every $L\in\mathbb{N}$ and $\eta>0$, there exists $R_L\geq L$ and a Lipschitz function $\Psi_L:\mathbb{R}^m\mapsto\mathbb{R}^m$ with Lipschitz constant at most 1, satisfying

$$\Psi_L(z):=\begin{cases} z & \text{if } |z|<R_L\,,\\ 0 & \text{if } |z|>2R_L\,,\end{cases}$$

and such that for every $\omega\in\Omega$,

$$\liminf_{j\to+\infty}\mathcal{F}^\omega_{\varepsilon_j}(u_j)\geq\liminf_{j\to+\infty}\mathcal{F}^\omega_{\varepsilon_j}(\Psi_L(u_j))-\eta\,. \tag{7.3}$$

Moreover, setting

$$u^L_j:=\Psi_L(u_j)\,,\quad u^L:=\Psi_L(u)\,, \tag{7.4}$$

we also have that

$$\begin{aligned} &(i)\quad u^L_j\equiv 0 \text{ on } (H^\omega_{\varepsilon_j}\cap D)\cup\partial D\,, \qquad \sup_{j\in\mathbb{N}}\|u^L_j\|_{L^\infty(D)}\leq 2R_L\,,\\ &(ii)\quad u^L_j \underset{j\to+\infty}{\rightharpoonup} u^L \quad\text{and}\quad u^L \underset{L\to+\infty}{\rightharpoonup} u \quad \text{weakly in } W^{1,q}(D;\mathbb{R}^m)\,. \end{aligned} \tag{7.5}$$

Step 2: Applying the Probabilistic Joining Lemma to (u^L_j). Let now $M,k\in\mathbb{N}$, $\theta\in(0,1)$ (with $\frac{1}{\theta}\in\mathbb{N}$) be fixed, and take $j\in\mathbb{N}$ large enough so that

$$\theta K_j>2^kM^2\,, \tag{7.6}$$

where K_j is as in (4.5). For $\omega\in\Omega$, let $G^\omega_{j,M}$ be the set defined in (5.10), with ε replaced by ε_j. By Lemma 5.2 there exists $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega')=1$ such that for every $\omega\in\Omega'$ and every $x_{j,i}\in G^\omega_{j,M}$ there exists $k^\omega_{j,i}\in\{0,\dots,k-1\}$ and corresponding annuli $C^\omega_{j,i}:=C^{k^\omega_{j,i}}_{\varepsilon_j,\theta,M}(\varepsilon_jx_{j,i})$, such that we can construct a sequence $(w^\omega_j)\subset W^{1,q}_0(D;\mathbb{R}^m)$ satisfying properties (i) -(iv), with $\bar u^{L,\omega}_{j,i}$ and $\bar\sigma^\omega_{j,i}$ defined as in (ii) therein.

Note that the modified sequence (w^ω_j) vanishes on the perforations centred in $G^\omega_{j,M}$, and on ∂D. Indeed, by Lemma 5.2 (i), we have that

$$w^\omega_j\equiv u^L_j \quad\text{in } D\setminus\bigcup_{x_{j,i}\in G^\omega_{j,M}}C^\omega_{j,i}\,. \tag{7.7}$$

Moreover, by (3.1), (4.5), (7.6), for $j\in\mathbb{N}$ large enough (with respect to k,M,θ) and for every $x_{j,i}\in G^\omega_{j,M}$ we have

$$\alpha_{\varepsilon_j}\rho_{j,i}\leq(\varepsilon_j/K_j)M\leq 2^{-k}\theta\varepsilon_j/M\,,$$

and hence

$$B_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_jx_{j,i})\subset B_{2^{-k}\theta\varepsilon_j/M}(\varepsilon_jx_{j,i})\quad\forall\, x_{j,i}\in G^\omega_{j,M}\,.$$

It then follows that

$$\bigcup_{x_{j,i}\in G^\omega_{j,M}}\overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_jx_{j,i})\subset D\setminus\bigcup_{x_{j,i}\in G^\omega_{j,M}}C^\omega_{j,i}$$

and so by (7.7) and by (7.5)(i),

$$w^\omega_j\equiv u^L_j\equiv 0 \text{ on } \Bigg(\bigcup_{x_{j,i}\in G^\omega_{j,M}}B_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_jx_{j,i})\cap D\Bigg)\cup\partial D\,. \tag{7.8}$$

Moreover, since $w_j^\omega \underset{j\to\infty}{\rightharpoonup} u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$,

$$\sup_{j\in\mathbb{N}} \|w_j^\omega\|_{W^{1,q}(D;\mathbb{R}^m)} \leq C < +\infty\,, \tag{7.9}$$

where the constant $C > 0$ might depend on L, u but not on $\omega \in \Omega'$.

Step 3: Splitting of the energy. By Lemma 5.2 (iv) we have

$$\mathcal{F}_{\varepsilon_j}^\omega(u_j^L) = \int_D f(\nabla u_j^L)\,\mathrm{d}x \geq \int_{D\setminus E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x + \int_{E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x - \frac{C}{k}, \tag{7.10}$$

where we set

$$E_j^{\theta,M,\omega} := \bigcup_{x_{j,i}\in G_{j,M}^\omega} B_{\bar\sigma_{j,i}^\omega}(\varepsilon_j x_{j,i})\,.$$

In what follows we deal with the two integrals in the right-hand side of (7.10) separately. We will show that the integral outside $E_j^{\theta,M,\omega}$ will result in the first integral in $\mathcal{F}_0$, while the integral on $E_j^{\theta,M,\omega}$ will give the capacitary term in the limit.

Step 3.1: The energy contribution outside $E_j^{\theta,M,\omega}$. For every $\omega \in \Omega'$ we define the auxiliary sequence

$$v_j^\omega := \begin{cases} w_j^\omega & \text{in } \ D\setminus E_j^{\theta,M,\omega}\,,\\ \bar u_{j,i}^{L,\omega} & \text{in } \overline{B}_{\bar\sigma_{j,i}^\omega} \ \text{ for each } x_{j,i}\in G_{j,M}^\omega\,. \end{cases}$$

Note that v_j^ω is a modification of w_j^ω in D at no additional cost in terms of f, since v_j^ω is piecewise constant on $E_j^{\theta,M,\omega}$ and $f(0) = 0$ by assumption, so that

$$\int_{D\setminus E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x = \int_D f(\nabla v_j^\omega)\,\mathrm{d}x\,. \tag{7.11}$$

Moreover, by construction the sequence (v_j^ω) is bounded in $W^{1,q}(D;\mathbb{R}^m)$ independently of $\omega\in\Omega'$, since by (7.5) and (7.9),

$$\sup_{j\in\mathbb{N}} \|v_j^\omega\|_{W^{1,q}(D;\mathbb{R}^m)} \lesssim R_L + \sup_{j\in\mathbb{N}} \|w_j^\omega\|_{W^{1,q}(D;\mathbb{R}^m)} \leq C < +\infty\,. \tag{7.12}$$

We now show that there exists a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ (which may be different from the one in Step 2, but depends only on the m.p.p. $(\Phi, \mathcal{R})$, and hence will not be relabelled) such that for every $\omega \in \Omega'$

$$v_j^\omega \rightharpoonup u^L \quad \text{weakly in } W^{1,q}(D;\mathbb{R}^m) \text{ as } j\to+\infty\,. \tag{7.13}$$

Note that any subsequence (v_j^ω) is bounded in $W^{1,q}(D;\mathbb{R}^m)$ by (7.12), and hence admits a (not-relabelled) convergent subsequence, converging weakly in $W^{1,q}(D;\mathbb{R}^m)$ to a function v^ω, that a priori might be probabilistic. It therefore suffices to show that there exists $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega\in\Omega'$,

$$v^\omega \equiv u^L \quad \mathcal{L}^n-\text{a.e. in } D\,.$$

We have that

$$v_j^\omega = w_j^\omega = u_j^L \quad \text{in } \ D\setminus \bigcup_{x_{j,i}\in\Phi_j^{2/M,\omega}(D)} B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i}) \subset D\setminus \bigcup_{x_{j,i}\in G_{j,M}^\omega} B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})\,.$$

In particular, this means that

$$(v_j^\omega - u_j^L)\chi_j^\omega = 0, \quad \text{where} \quad \chi_j^\omega := \chi_{D\setminus \bigcup_{x_{j,i}\in\Phi_j^{2/M,\omega}(D)} B_{\theta\varepsilon_j/M}(\varepsilon_j x_{j,i})}\,. \tag{7.14}$$

By the stationarity assumption (H1) and (H3) of Subsection 2.3 for $\Phi^{2/M}$ (these properties being transferred to it by the corresponding ones of Φ), we have that $\Phi^{2/M}$ is ergodic. Hence, Birkhoff's Ergodic Theorem (see, e.g., [26, Property 2.10]) guarantees that, since $\theta\in(0,1)$, there exists a set $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega\in\Omega'$

$$\chi_j^\omega \rightharpoonup K > 0 \quad \text{weakly* in } L^\infty\,, \tag{7.15}$$

where $K > 0$ is a deterministic constant. From (7.14), (7.15), (7.5)(ii) and by the convergence (up to subsequences) of v_j^ω to v^ω, we then conclude that

$$0 = \int_D \chi_j^\omega |v_j^\omega - u_j^L| \,\mathrm{d}x \underset{j\to+\infty}{\to} K \int_D |v^\omega - u^L| \,\mathrm{d}x \,,$$

and hence $v^\omega \equiv u^L$ $\mathcal{L}^n$-a.e. in D, as desired. This proves (7.13).

Then, by taking the lim inf as $j \to +\infty$ in (7.11), we obtain that for every ω in a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ (which depends only on the m.p.p. $(\Phi, \mathcal{R})$, but is independent of all the parameters and the functions involved in the arguments),

$$\liminf_{j\to+\infty} \int_{D\setminus E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x = \liminf_{j\to+\infty} \int_D f(\nabla v_j^\omega)\,\mathrm{d}x \geq \int_D Qf(\nabla u^L)\,\mathrm{d}x \,, \tag{7.16}$$

where we have used (7.13), and the fact that the functional $\int_D Qf(\nabla\cdot)\,\mathrm{d}x$ is the lower semicontinuous envelope of $\int_D f(\nabla\cdot)\,\mathrm{d}x$ with respect to the weak $W^{1,q}(D;\mathbb{R}^m)$-topology (cf. [3]).

Step 3.2: The energy contribution in $E_j^{\theta,M,\omega}$. Let $j \in \mathbb{N}$ be large enough so that (7.6) is satisfied, where K_j is as in (4.5), and let $x_{j,i} \in G_{j,M}^\omega$. We define $\zeta_{j,i}^\omega : B_{\theta K_j}(0) \to \mathbb{R}^m$ as

$$\zeta_{j,i}^\omega(y) := \begin{cases} w_j^\omega(\varepsilon_j x_{j,i} + \alpha_{\varepsilon_j} y) & \text{in } B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)\,, \\ \bar u_{j,i}^{L,\omega} & \text{in } B_{\theta K_j}(0) \setminus B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)\,, \end{cases} \tag{7.17}$$

where $\bar\sigma_{j,i}^\omega$ and $\bar u_{j,i}^{L,\omega}$ are defined as in Lemma 5.2(ii). In particular,

$$\zeta_{j,i}^\omega|_{\partial B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)} \equiv w_j^\omega|_{\partial B_{\bar\sigma_{j,i}^\omega}(\varepsilon_j x_{j,i})} \equiv \bar u_{j,i}^{L,\omega}\,, \tag{7.18}$$

and $B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0) = B_{\frac{3}{4}2^{-k_{j,i}^\omega}\theta K_j/M}(0) \subset B_{\theta K_j}(0)$. Moreover, since $x_{j,i} \in G_{j,M}^\omega$, by (7.6) we have that

$$\alpha_{\varepsilon_j}\rho_{j,i} \leq \alpha_{\varepsilon_j} M \leq \frac{\varepsilon_j}{K_j} M < \varepsilon_j M \frac{\theta 2^{-k}}{M^2} \leq \frac{3}{4} 2^{-(k-1)} \frac{\theta\varepsilon_j}{M} \leq \bar\sigma_{j,i}^\omega\,.$$

By (7.17), (7.18), and (7.8), we have

$$\zeta_{j,i}^\omega - \bar u_{j,i}^{L,\omega} \in W_0^{1,q}(B_{\theta K_j}(0);\mathbb{R}^m)\,, \quad \zeta_{j,i}^\omega|_{\overline{B}_{\rho_{j,i}^\omega}(0)} \equiv w_j^\omega|_{\overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i})} \equiv 0\,, \tag{7.19}$$

i.e., $\zeta_{j,i}^\omega$ is a competitor in the minimisation problem defining $\varphi_{\theta,\rho_{j,i}}^j(\bar u_{j,i}^{L,\omega})$ (see (4.7)).

By a change of variables we rewrite the energy contribution in (7.10) relative to the set $E_j^{\theta,M,\omega}$, as

$$\begin{aligned} \int_{E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x &= \sum_{x_{j,i}\in G_{j,M}^\omega} \int_{B_{\bar\sigma_{j,i}^\omega}(\varepsilon_j x_{j,i})} f(\nabla w_j^\omega(x))\,\mathrm{d}x \\ &= \alpha_{\varepsilon_j}^n \sum_{x_{j,i}\in G_{j,M}^\omega} \int_{B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)} f\big(\nabla w_j^\omega(\varepsilon_j x_{j,i} + \alpha_{\varepsilon_j} y)\big)\,\mathrm{d}y \\ &= \varepsilon_j^n \sum_{x_{j,i}\in G_{j,M}^\omega} \int_{B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)} \alpha_{\varepsilon_j}^q f\big(\alpha_{\varepsilon_j}^{-1}\nabla\zeta_{j,i}^\omega(y)\big)\,\mathrm{d}y\,. \end{aligned}$$

Recalling that $f(0) = 0$, since $\nabla\zeta_{j,i}^\omega \equiv 0$ in $B_{\theta K_j}(0) \setminus B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)$, for every $x_{j,i} \in G_{j,M}^\omega$ we have

$$\begin{aligned} \int_{B_{\bar\sigma_{j,i}^\omega/\alpha_{\varepsilon_j}}(0)} \alpha_{\varepsilon_j}^q f\big(\alpha_{\varepsilon_j}^{-1}\nabla\zeta_{j,i}^\omega(y)\big)\,\mathrm{d}y &= \int_{B_{\theta K_j}(0)} \alpha_{\varepsilon_j}^q f\big(\alpha_{\varepsilon_j}^{-1}\nabla\zeta_{j,i}^\omega(y)\big)\,\mathrm{d}y \\ &\geq \int_{B_{\theta K_j}(0)} g_j(\nabla\zeta_{j,i}^\omega(y))\,\mathrm{d}y \geq \varphi_{\theta,\rho_{j,i}}^j(\bar u_{j,i}^{L,\omega})\,, \end{aligned}$$

where we have used that $f \geq Qf$, (3.5), (4.7), and (7.19). In conclusion,

$$\int_{E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x \geq \varepsilon_j^n \sum_{x_{j,i}\in G_{j,M}^\omega} \varphi_{\theta,\rho_{j,i}}^j(\bar u_{j,i}^{L,\omega})\,. \tag{7.20}$$

Taking now in (7.20) the liminf as $j \to +\infty$, then (up to a not-relabelled subsequence) $\theta \to 0^+$ and then $M \to +\infty$, and using (7.5) and Proposition 6.1, we deduce that there exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ (depending only on $(\Phi, \mathcal{R})$) such that for every $\omega \in \Omega'$,

$$\lim_{M\to+\infty} \lim_{\theta\to 0^+} \liminf_{j\to+\infty} \int_{E_j^{\theta,M,\omega}} f(\nabla w_j^\omega)\,\mathrm{d}x \geq \langle N(Q)\rangle \int_D \varphi(u^L)\,\mathrm{d}x\,. \tag{7.21}$$

Step 4: Conclusion. Taking in (7.10) the liminf as $j \to +\infty$, and then the (subsequential) limits as $\theta \to 0^+$ and $M \to +\infty$, and since the left-hand side of (7.10) is independent of θ, M, in view of (7.16) and (7.21) we obtain that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\liminf_{j\to+\infty} \mathcal{F}^\omega_{\varepsilon_j}(u_j^L) \geq \int_D Qf(\nabla u^L)\,\mathrm{d}x + \langle N(Q)\rangle \int_D \varphi(u^L) - \frac{C}{k}. \tag{7.22}$$

Combining (7.22), (7.3) and (7.4) we conclude that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\liminf_{j\to+\infty} \mathcal{F}^\omega_{\varepsilon_j}(u_j) \geq \int_D Qf(\nabla u^L)\,\mathrm{d}x + \langle N(Q)\rangle \int_D \varphi(u^L)\,\mathrm{d}x - \frac{C}{k} - \eta\,. \tag{7.23}$$

By the arbitrariness of $k, \eta, L > 0$ we first let $k \to +\infty$ in (7.23), then $\eta \to 0^+$, and finally $L \to +\infty$. Then, (7.5)(ii), the lower semicontinuity of the functional $\int_D Qf(\nabla\cdot)\,\mathrm{d}x$ with respect to the weak $W^{1,q}(D;\mathbb{R}^m)$-topology and the continuity of $\int_D \varphi(\cdot)\,\mathrm{d}x$ with respect to strong $L^q(D;\mathbb{R}^m)$-topology, guarantee the existence of a set $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ (depending only on $(\Phi, \mathcal{R})$) such that for every $\omega \in \Omega'$ and every $(u_j), u \in W^{1,q}_0(D;\mathbb{R}^m)$ with $u_j \underset{j\to+\infty}{\rightharpoonup} u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$,

$$\liminf_{j\to+\infty} \mathcal{F}^\omega_j(u_j) \geq \int_D Qf(\nabla u)\,\mathrm{d}x + \langle N(Q)\rangle \int_D \varphi(u)\,\mathrm{d}x = \mathcal{F}_0(u)\,,$$

which concludes the proof of the proposition. □

7.2. **The Γ-limsup inequality.** In this subsection we prove the following result.

Proposition 7.2. *Let $(\varepsilon_j) \searrow 0$. Let $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ be such that the conclusions of Proposition 7.1 hold true for every $\omega \in \Omega'$, and let $u \in W^{1,q}_0(D;\mathbb{R}^m)$. Then there exists a sequence $(u_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ satisfying $u_j \to u$ in $L^1(D;\mathbb{R}^m)$, and such that*

$$\limsup_{j\to+\infty} \mathcal{F}^\omega_{\varepsilon_j}(u_j) \leq \mathcal{F}_0(u)\,, \tag{7.24}$$

where $\mathcal{F}^\omega_{\varepsilon_j}$ and $\mathcal{F}_0$ are defined in (3.4) *and* (3.16) *respectively.*

Before embarking on the proof of the proposition, we present some auxiliary lemmata, that can be thought of as partially constructing "correctors" in the spirit of [21, Lemma 3.1]. These auxiliary results will be crucial in the construction of an admissible recovery sequence, namely that vanishes on the entire set of perforations $H^\omega_\varepsilon \cap D$.

We first introduce the set of *very good perforations* as follows. Let $\varepsilon > 0$, $M \in \mathbb{N}$, $\theta \in (0,1)$ (with $1/\theta \in \mathbb{N}$) be fixed, and assume that

$$0 < \theta < \frac{3}{8M}\,. \tag{7.25}$$

For every $\omega \in \Omega$ we define

$$I^\omega_{\varepsilon,g,\theta} := \{x_i \in I^\omega_{\varepsilon,g} \colon \exists\, x_k \in I^\omega_{\varepsilon,g} \setminus \{x_i\} \text{ with } \partial B_{\theta\varepsilon}(\varepsilon x_i) \cap \overline{B}_{\varepsilon r^\omega_\varepsilon}(\varepsilon x_k) \neq \emptyset\}\,,$$

where (r^ω_ε) and $I^\omega_{\varepsilon,g}$ are as in Lemma 5.1. These are the good centres that are not $(\theta + r^\omega_\varepsilon)$-separated from other good centres. We recall that points in $I^\omega_{\varepsilon,g}$ are at least $2r_\varepsilon$-separated from one another (cf. (5.7)). Recalling (5.10), we define the *mildly good centers* as

$$(MG)^\omega_{\varepsilon,\theta,M} := (I^\omega_{\varepsilon,g} \setminus G^\omega_{\varepsilon,M}) \cup (G^\omega_{\varepsilon,M} \cap I^\omega_{\varepsilon,g,\theta})\,, \tag{7.26}$$

and the *very good centers* as

$$(VG)^\omega_{\varepsilon,\theta,M} := G^\omega_{\varepsilon,M} \setminus (MG)^\omega_{\varepsilon,\theta,M}\,. \tag{7.27}$$

In short, the very good centres are the subset of $G^{\omega}_{\varepsilon,M}$ for which the corresponding balls are deterministically separated, at scale ε, also from other good balls in $I^{\omega}_{\varepsilon,g}$, which is a priori not guaranteed (see Remark 5.3).

We also use the shorthand notations $(MG)^{\omega}_{j,\theta,M} := (MG)^{\omega}_{\varepsilon_j,\theta,M}$ and $(VG)^{\omega}_{j,\theta,M} := (VG)^{\omega}_{\varepsilon_j,\theta,M}$ for a sequence $(\varepsilon_j) \searrow 0$.

A fundamental property that we will use is that the mildly good perforations do not contribute to the limit capacity, as made precise in the following lemma.

Lemma 7.3. *Let $(\varepsilon_j) \searrow 0$. There exists $\Omega' \in \mathcal{T}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ we have (possibly along ω-independent subsequences)*

$$\lim_{M\to+\infty} \lim_{\theta\to 0^+} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in(MG)^{\omega}_{j,\theta,M}} (\rho_{j,i})^{n-q} = 0\,. \tag{7.28}$$

Proof. We show separately that

$$\lim_{M\to+\infty} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in(I^{\omega}_{\varepsilon_j,g}\setminus G^{\omega}_{j,M})} (\rho_{j,i})^{n-q} = 0 \tag{7.29}$$

and

$$\lim_{M\to+\infty} \lim_{\theta\to 0^+} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in(G^{\omega}_{j,M}\cap I^{\omega}_{\varepsilon_j,g,\theta})} (\rho_{j,i})^{n-q} = 0\,, \tag{7.30}$$

so that (7.28) follows by (7.26).

Let $\Omega' \in \mathcal{T}$ be as in Lemma 5.1, and let $M \in \mathbb{N}$, $j \in \mathbb{N}$ and $\theta \in (0,1)$ (with $1/\theta \in \mathbb{N}$) be such that (by (7.25))

$$0 < r^{\omega}_{\varepsilon_j} < \theta < \frac{3}{8M}\,. \tag{7.31}$$

To prove (7.29), we first decompose in Ω',

$$\begin{aligned} I^{\omega}_{\varepsilon_j,g} &= (I^{\omega}_{\varepsilon_j,g} \cap \Phi^{2/M,\omega}_{\varepsilon_j}(D)) \cup (I^{\omega}_{\varepsilon_j,g} \setminus \Phi^{2/M,\omega}_{\varepsilon_j}(D)) \\ &= (G^{\omega}_{j,M} \cup I^{\omega}_{j,M} \cup J^{\omega}_{j,M}) \cup (I^{\omega}_{\varepsilon_j,g} \setminus \Phi^{2/M,\omega}_{\varepsilon_j}(D))\,, \end{aligned}$$

where $I^{\omega}_{j,M}$ and $J^{\omega}_{j,M}$ are defined in (6.19), so that we can write

$$I^{\omega}_{\varepsilon_j,g} \setminus G^{\omega}_{j,M} = I^{\omega}_{j,M} \cup J^{\omega}_{j,M} \cup (I^{\omega}_{\varepsilon_j,g} \setminus \Phi^{2/M,\omega}_{\varepsilon_j}(D))\,. \tag{7.32}$$

Hence, by (6.22) and (6.24), the claim (7.29) follows if we show that

$$\lim_{M\to+\infty} \lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in I^{\omega}_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q} = 0\,. \tag{7.33}$$

Let $R \in \mathbb{N}$ be fixed. Then we can write

$$\begin{aligned} &\sum_{x_{j,i}\in I^{\omega}_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q} \\ &\quad\le \sum_{x_{j,i}\in I^{\omega}_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q}\chi_{[0,R]}(\rho_{j,i}) + \sum_{x_{j,i}\in I^{\omega}_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q}\chi_{(R,+\infty)}(\rho_{j,i}) \\ &\quad\le R^{n-q}\#\big(I^{\omega}_{\varepsilon_j,g} \setminus \Phi^{2/M,\omega}_{\varepsilon_j}(D)\big) + \sum_{x_{j,i}\in\Phi^{\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q}\chi_{(R,+\infty)}(\rho_{j,i}). \end{aligned} \tag{7.34}$$

Since by (5.6) we have that $\varepsilon_j^n \# I^{\omega}_{\varepsilon_j,b} \to 0$ as $j \to +\infty$, we deduce that there exists a set $\Omega_M \in \mathcal{T}$, $\Omega_M \subset \Omega'$, with $\mathbb{P}(\Omega_M) = 1$, such that

$$\lim_{j\to+\infty} \varepsilon_j^n \#\big(I^{\omega}_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)\big) = \lim_{j\to+\infty} \varepsilon_j^n \#\big(\Phi^{\omega}_{\varepsilon_j}(D)\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)\big) = (\langle N(Q)\rangle - \langle N^{2/M}(Q)\rangle)\mathcal{L}^n(D)\,, \tag{7.35}$$

for $\omega \in \Omega_M$, where we have used (5.15) and (5.17). Moreover, by Lemma 5.4 applied to $Y_{j,i} := (\rho_{j,i})^{n-q}\chi_{(R,+\infty)}(\rho_{j,i})$ we have that

$$\lim_{j\to+\infty} \varepsilon_j^n \sum_{x_{j,i}\in\Phi^\omega_{\varepsilon_j}(B)} (\rho_{j,i})^{n-q}\chi_{(R,+\infty)}(\rho_{j,i}) = \langle N(Q)\rangle\langle\rho^{n-q}\chi_{(R,+\infty)}(\rho)\rangle\mathcal{L}^n(D) \tag{7.36}$$

in a set of full probability $\Omega_{M,R}$. Hence, by (7.34), (7.35) and (7.36) it follows that $\mathbb{P}$-a.e. in Ω

$$\begin{aligned}\overline{\lim_{j\to+\infty}}\, \varepsilon_j^n \sum_{x_{j,i}\in I^\omega_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q} &\le R^{n-q}(\langle N(Q)\rangle - \langle N^{2/M}(Q)\rangle)\mathcal{L}^n(D) \\ &\quad + \langle N(Q)\rangle\langle\rho^{n-q}\chi_{(R,+\infty)}(\rho)\rangle\mathcal{L}^n(D)\,.\end{aligned}$$

Now, by letting $M\to+\infty$, thanks to (5.17), we have

$$\overline{\lim_{M\to+\infty}}\,\overline{\lim_{j\to+\infty}}\, \varepsilon_j^n \sum_{x_{j,i}\in I^\omega_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)} (\rho_{j,i})^{n-q} \le \langle N(Q)\rangle\langle\rho^{n-q}\chi_{(R,+\infty)}(\rho)\rangle\mathcal{L}^n(D)\,.$$

Finally, by letting $R\to+\infty$, since $\langle\rho^{n-q}\rangle < +\infty$ by (2.10), we obtain (7.29), for every ω in a set of probability 1, where the latter can be chosen independently of $M\in\mathbb{N}$ and $\theta\in(0,1)$ (with $1/\theta\in\mathbb{N}$). For the proof of (7.30), we claim that by the choice (7.31)

$$G^\omega_{j,M}\cap I^\omega_{\varepsilon_j,g,\theta} \subset I^\omega_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)\,. \tag{7.37}$$

If (7.37) is proved, by proceeding as above for the proof of (7.33), we can conclude. So it remains to prove (7.37).

To this aim, let $x_{j,i}\in G^\omega_{j,M}\cap I^\omega_{\varepsilon_j,g,\theta}$. Then there exists $x_{j,k}\in I^\omega_{\varepsilon_j,g}\setminus\{x_{j,i}\}$ (and in fact a posteriori $x_{j,k}\in I^\omega_{\varepsilon_j,g}\setminus G^\omega_{j,M}$) and $y\in\mathbb{R}^n$ such that

$$y\in\partial B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})\cap\overline{B}_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,k})\,.$$

This and (7.31) imply that

$$|\varepsilon_j x_{j,i}-\varepsilon_j x_{j,k}| \le |\varepsilon_j x_{j,i}-y| + |y-\varepsilon_j x_{j,k}| \le (\theta + r^\omega_{\varepsilon_j})\varepsilon_j < 2\theta\varepsilon_j < \frac{3\varepsilon_j}{4M} < \frac{2\varepsilon_j}{M}\,,$$

i.e., $x_{j,i}\in I^\omega_{\varepsilon_j,g}\setminus\Phi^{2/M,\omega}_{\varepsilon_j}(D)$ as we claimed in (7.37). □

We now present a construction of "partial correctors" in the spirit of [21, Lemma 3.1].

Lemma 7.4. *There exists a set $\Omega'\in\mathcal{T}$ with $\mathbb{P}(\Omega')=1$ with the following property. Let $(\varepsilon_j)\searrow 0$, $M\in\mathbb{N}$, and $\theta\in(0,1)$ (with $1/\theta\in\mathbb{N}$) such that (7.31) holds true. For every $\omega\in\Omega'$ there exists a corrector function $\phi^\omega_{j,\theta,M}\in W^{1,q}(D;[0,1])$ satisfying the following properties:*

$$\begin{aligned}&(i)\quad \phi^\omega_{j,\theta,M}\equiv 0 \quad\text{on}\quad H^\omega_{\varepsilon_j,b}\cup\Big(\bigcup_{x_{j,i}\in(MG)^\omega_{j,\theta,M}}\overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i})\Big)\,,\\ &(ii)\quad \phi^\omega_{j,\theta,M}\equiv 1 \quad\text{on}\quad D\setminus\left(D^\omega_{\varepsilon_j,b}\cup\bigcup_{x_{j,i}\in(MG)^\omega_{j,\theta,M}}\overline{B}_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})\right)\,,\\ &(iii)\qquad \lim_{M\to+\infty}\lim_{\theta\to0^+}\lim_{j\to+\infty}\|\phi^\omega_{j,\theta,M}-1\|_{W^{1,q}(D)}=0\,.\end{aligned} \tag{7.38}$$

Proof. First of all, for every $\omega\in\Omega'\in\mathcal{T}$ as in Lemma 5.1, by the definition of the q-capacity in (3.11) there exists $\phi^\omega_{0,j}\in W^{1,q}_0(D^\omega_{\varepsilon_j,b};[0,1])$ such that

$$\phi^\omega_{0,j}|_{H^\omega_{\varepsilon_j,b}}\equiv 1\,,\quad \int_{D^\omega_{\varepsilon_j,b}}|\nabla\phi^\omega_{0,j}|^q\,\mathrm{d}x \le 2\mathrm{Cap}_q(H^\omega_{\varepsilon_j,b},D^\omega_{\varepsilon_j,b})\,.$$

By setting $\phi^\omega_{1,j} := 1-\phi^\omega_{0,j}$ (extended to 1 in $D\setminus D^\omega_{\varepsilon_j,b}$) we obtain a function $\phi^\omega_{1,j}\in W^{1,q}(D;[0,1])$ such that

$$\phi^\omega_{1,j}|_{H^\omega_{\varepsilon_j,b}}\equiv 0\,,\quad \phi^\omega_{1,j}|_{D\setminus D^\omega_{\varepsilon_j,b}}\equiv 1\,,\quad \int_D|\nabla\phi^\omega_{1,j}|^q\,\mathrm{d}x \le 2\mathrm{Cap}_q(H^\omega_{\varepsilon_j,b},D^\omega_{\varepsilon_j,b})\,. \tag{7.39}$$

Analogously, for every $x_{j,i} \in (MG)^\omega_{j,\theta,M}$, let $\phi^\omega_{0,j,i} \in W^{1,q}_0(B_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i});[0,1])$ be such that

$$\phi^\omega_{0,j,i}|_{\overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i})} \equiv 1\,, \quad \int_{B_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})} |\nabla \phi^\omega_{0,j,i}|^q \,\mathrm{d}x \le 2\mathrm{Cap}_q(B_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i}), B_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i}))\,.$$

Note that, all the functions $\phi^\omega_{0,i,j}$ have disjoint supports. Proceeding as in the proof of (5.9), we estimate

$$\begin{aligned}\mathrm{Cap}_q\big(B_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i}), B_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})\big) &\le c_{n,q}\big((\alpha_{\varepsilon_j}\rho_{j,i})^{(q-n)/(q-1)} - (\varepsilon_j r^\omega_{\varepsilon_j})^{(q-n)/(q-1)}\big)^{1-q} \\ &\lesssim \varepsilon_j^n(\rho_{j,i})^{n-q}.\end{aligned}$$

Set

$$\phi^\omega_{2,j,\theta,M} := 1 - \sum_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} \phi^\omega_{0,j,i}\,;$$

then $\phi^\omega_{2,j,\theta,M} \in W^{1,q}(D;[0,1])$,

$$\phi^\omega_{2,j,\theta,M} \equiv 0 \quad \text{on} \bigcup_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} \overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon_j x_{j,i})\,,\ \phi^\omega_{2,j,\theta,M} \equiv 1 \text{ on } D\setminus \bigcup_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} \overline{B}_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})\,,$$

and, since the functions $\phi^\omega_{0,j,i}$ have disjoint supports (cf. (5.7)),

$$\int_D |\nabla \phi^\omega_{2,j,\theta,M}|^q \,\mathrm{d}x = \sum_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} \int_{B_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})} |\nabla \phi^\omega_{0,j,i}|^q \,\mathrm{d}x \lesssim \varepsilon_j^n \sum_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} (\rho_{j,i})^{n-q}\,. \tag{7.40}$$

Setting now

$$\phi^\omega_{j,\theta,M} := \phi^\omega_{1,j} \wedge \phi^\omega_{2,j,\theta,M} \in W^{1,q}(D;[0,1])\,, \tag{7.41}$$

it is clear by (7.39)-(7.41) that $\phi^\omega_{j,\theta,M}$ satisfies the claims (i) and (ii). To prove (iii), since $\phi^\omega_{j,\theta,M}|_{\partial D} \equiv 1$, by the Poincaré inequality in D it suffices to verify that

$$\lim_{M\to+\infty}\lim_{\theta\to 0^+}\lim_{j\to+\infty} \int_D |\nabla \phi^\omega_{j,\theta,M}|^q \,\mathrm{d}x = 0\,. \tag{7.42}$$

By (7.39) and (7.40) it is straightforward to estimate

$$\begin{aligned}\int_D |\nabla \phi^\omega_{j,\theta,M}|^q \,\mathrm{d}x &\le \int_D |\nabla \phi^\omega_{1,j}|^q \,\mathrm{d}x + \int_D |\nabla \phi^\omega_{2,j,\theta,M}|^q \,\mathrm{d}x \\ &\lesssim \mathrm{Cap}_q(H^\omega_{\varepsilon_j,b}, D^\omega_{\varepsilon_j,b}) + \varepsilon^n \sum_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} (\rho_{j,i})^{n-q}\,.\end{aligned}$$

The claim (7.42) then follows from (5.9) and (7.28). □

We are now ready for the proof of the limsup inequality.

Proof of Proposition 7.2. For the proof of (7.24) we follow broadly the approach developed for the periodic setting in [3, Section 6] and [4]. We note, however, that extra care has to be taken here for the construction of an admissible recovery sequence to guarantee that it vanishes in the set H^ω_ε, due to the lack of separation of the perforations. We give the whole argument in detail for the sake of completeness.

Let $(u_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ be such that

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,q}(D;\mathbb{R}^m) \text{ as } j\to+\infty\,, \tag{7.43}$$

and

$$\lim_{j\to+\infty} \int_D f(\nabla u_j)\,\mathrm{d}x = \int_D Qf(\nabla u)\,\mathrm{d}x\,. \tag{7.44}$$

Note that the existence of a sequence (u_j) satisfying (7.43) and (7.44) follows by the characterization of $\int_D Qf(\nabla\cdot)\,\mathrm{d}x$ as the lower semicontinuous envelope of $\int_D f(\nabla\cdot)\,\mathrm{d}x$ with respect to the weak $W^{1,q}(D;\mathbb{R}^m)$-topology. By [5, Lemma C.5, Remark C.6] and [19, Lemma 1.2] we may also suppose, without loss of generality, that $(u_j) \subset W^{1,\infty}_0(D;\mathbb{R}^m)$ and that $(|\nabla u_j|^q)$ are equi-integrable.

We now modify the sequence (u_j) to make it admissible for the energy $\mathcal{F}^{\omega}_{\varepsilon_j}$. In particular, the functions u_j need additionally to be set to zero on the perforations.

Let $j \in \mathbb{N}$ be large enough so that (4.10) is satisfied, where K_j is defined as in (4.5). Let $k, M \in \mathbb{N}$, $\theta \in (0,1)$ (with $1/\theta \in \mathbb{N}$) be fixed and assume that also (7.25) holds true. Let now $\theta' := \frac{4\theta M}{3}$; note that $\theta' \in (0, 1/2)$ by (7.25). We apply Lemma 5.2, with the θ' defined above, to the sequence (u_j), but where $G^{\omega}_{j,M}$ is replaced by the smaller set $(VG)^{\omega}_{j,\theta,M}$ defined in (7.27). Then we find a modified sequence $(w^{\omega}_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ satisfying

(i) $w^{\omega}_j \equiv u_j$ in $D \setminus \bigcup_{x_{j,i} \in (VG)^{\omega}_{j,\theta,M}} C^0_{\varepsilon_j,\theta',M}(\varepsilon_j x_{j,i})$;

(ii) $w^{\omega}_j \equiv \bar{u}^{\omega}_{j,i}$ on $\partial B_{\bar{\sigma}^{\omega}_{j,i}}(\varepsilon_j x_{j,i})$, for every $x_{j,i} \in (VG)^{\omega}_{j,\theta,M}$, where

$$\bar{u}^{\omega}_{j,i} := \fint_{C^0_{\varepsilon_j,\theta',M}(\varepsilon_j x_{j,i})} u_j \,\mathrm{d}x\,, \qquad \bar{\sigma}^{\omega}_{j,i} := \frac{3}{4}\frac{\theta'\varepsilon_j}{M} = \theta\varepsilon_j\,; \tag{7.45}$$

(iii) $w^{\omega}_j \rightharpoonup u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$;

(iv) $\left|\int_D f(\nabla w^{\omega}_j)\,\mathrm{d}x - \int_D f(\nabla u_j)\,\mathrm{d}x\right| \le C_k(\theta')^n + \frac{C}{k}$, where $C_k > 0$ can blow up as $k \to +\infty$;

(v) $(w^{\omega}_j) \subset W^{1,\infty}(D;\mathbb{R}^m)$ and $\|w^{\omega}_j\|_{L^\infty(D)} \le \|u_j\|_{L^\infty(D)}$.

Note that, since $(|\nabla u_j|^q)$ are equi-integrable, one can choose $k^{\omega}_{j,i} = 0$ in (i)-(ii) above for every $x_{j,i} \in (VG)^{\omega}_{j,\theta,M} \subset G^{\omega}_{j,M}$. Moreover, following the explicit construction of the modified sequence in the proof of Lemmata 4.7 and 5.2, one can see that also (w^{ω}_j) has equi-integrable q-gradients, and that it satisfies (v).

We now proceed with the proof of the lim sup inequality (7.24) in two steps.

Step 1: Assuming that $\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(D)} =: L < +\infty$. Let $\eta > 0$ be fixed. By the characterization of $\int_D Qf(\nabla\cdot)\,\mathrm{d}x$ as the lower semicontinuous envelope of $\int_D f(\nabla\cdot)\,\mathrm{d}x$ with respect to the weak $W^{1,q}(D;\mathbb{R}^m)$-topology, and by (3.5) and (4.7), for every $x_{j,i} \in (VG)^{\omega}_{j,\theta,M} \subset G^{\omega}_{j,M}$ we can pick a test function $\zeta^{\omega}_{j,i}$ satisfying

$$\zeta^{\omega}_{j,i} - \bar{u}^{\omega}_{j,i} \in W^{1,q}_0(B_{\theta K_j}(0);\mathbb{R}^m)\,, \quad \zeta^{\omega}_{j,i} \equiv 0 \ \text{ on } \overline{B}_{\rho^{\omega}_{j,i}}(0)\,, \tag{7.46}$$

such that

$$\int_{B_{\theta K_j}(0)} \alpha^q_{\varepsilon_j} f(\alpha^{-1}_{\varepsilon_j}\nabla\zeta^{\omega}_{j,i})\,\mathrm{d}x \le \varphi^j_{\theta,\rho_{j,i}}(\bar{u}^{\omega}_{j,i}) + \eta\,. \tag{7.47}$$

We now set

$$v^{\omega}_{j,\theta,M}(x) := \begin{cases} w^{\omega}_j(x) & \text{for } \ x \in D \setminus \bigcup_{x_{j,i}\in(VG)^{\omega}_{j,\theta,M}} B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})\,, \\ \zeta^{\omega}_{j,i}\big(\alpha^{-1}_{\varepsilon_j}(x-\varepsilon_j x_{j,i})\big) & \text{for } \ x \in B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})\,, x_{j,i} \in (VG)^{\omega}_{j,\theta,M}\,. \end{cases} \tag{7.48}$$

Recall that by property (7.45) and (7.46) we have that $w^{\omega}_j \equiv \bar{u}^{\omega}_{j,i}$ on $\partial B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})$ for every $x_{j,i} \in (VG)^{\omega}_{j,\theta,M}$. Hence $v^{\omega}_{j,\theta,M} \in W^{1,q}_0(D;\mathbb{R}^m)$, since $(w^{\omega}_j) \subset W^{1,q}_0(D;\mathbb{R}^m)$ (and actually in $W^{1,\infty}_0(D;\mathbb{R}^m)$), and with no loss of generality we can assume that $\overline{B}_{\theta\varepsilon_j}(\varepsilon_j x_{j,i}) \subset D$ for every $x_{j,i} \in (VG)^{\omega}_{j,\theta,M}$ by repeating a similar argument as the one presented right after (5.11) (see also the proof of Proposition (5.2)). Moreover, by (7.46) and (7.48) we have that

$$v^{\omega}_{j,\theta,M} = 0 \quad \text{in} \quad \bigcup_{x_{j,i}\in(VG)^{\omega}_{j,\theta,M}} \overline{B}_{\alpha_{\varepsilon_j}\rho_{j,i}}(\varepsilon x_{j,i})\,. \tag{7.49}$$

We now further modify $v^{\omega}_{j,\theta,M}$ to obtain a sequence that vanishes on all perforations. To this aim, we set

$$U^{\omega}_{j,\theta,M} := \phi^{\omega}_{j,\theta,M} v^{\omega}_{j,\theta,M}\,, \tag{7.50}$$

where the corrector $\phi^{\omega}_{j,\theta,M}$ is defined as in Lemma 7.4.

By (7.38)(i), (7.26), (7.27) and (7.49) we have that $U^{\omega}_{j,\theta,M} = 0$ in $H^{\omega}_{\varepsilon_j}$; moreover, by (7.50) and that $v^{\omega}_{j,\theta,M} = 0$ on ∂D, we also have that $U^{\omega}_{j,\theta,M} = 0$ on ∂D. We next show that $U^{\omega}_{j,\theta,M} \rightharpoonup u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$, as $j \to +\infty$, $\theta \to 0+$ and $M \to +\infty$. We do so by proving a uniform bound for the energy (3.4) of $(U^\omega_{j,\theta,M})$, which guarantees a bound on its q-gradients by (3.3), and hence a bound on the $W^{1,q}$-norm of $(U^\omega_{j,\theta,M})$ by the Poincaré inequality on D.

First we set for notational convenience

$$\widetilde{H}^\omega_{j,\theta,M} := D^\omega_{\varepsilon_j,b} \cup \bigcup_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} \overline{B}_{\varepsilon_j r^\omega_{\varepsilon_j}}(\varepsilon_j x_{j,i})\,, \qquad \widetilde{D}^\omega_{j,\theta,M} := \bigcup_{x_{j,i}\in(VG)^\omega_{j,\theta,M}} B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})\,. \quad (7.51)$$

Note that, by (5.5) and(5.7),

$$\begin{aligned}\mathcal{L}^n(\widetilde{H}^\omega_{j,\theta,M}) &\lesssim \sum_{x_{j,i}\in I^\omega_{\varepsilon_j,b}} (\alpha_{\varepsilon_j}\rho_{j,i})^n + \sum_{x_{j,i}\in(MG)^\omega_{j,\theta,M}} (\varepsilon_j r^\omega_{\varepsilon_j})^n \\ &\le \sum_{x_{j,i}\in I^\omega_{\varepsilon_j,b}} (\varepsilon_j^n(\rho_{j,i})^{n-q})^{\frac{n}{n-q}} + \sum_{x_{j,i}\in\Phi^\omega_{\varepsilon_j}(D)} (\varepsilon_j r^\omega_{\varepsilon_j})^n \\ &\le \Big(\varepsilon_j^n \sum_{x_{j,i}\in I^\omega_{\varepsilon_j,b}} (\rho_{j,i})^{n-q}\Big)^{\frac{n}{n-q}} + (\varepsilon_j^n N^\omega_{\varepsilon_j}(D))(r^\omega_{\varepsilon_j})^n\,, \end{aligned} \quad (7.52)$$

where we used that, for $\alpha\ge 1$ and $t_i\ge 0$ for every $i\in\mathbb{N}$,

$$\sum_{i\in\mathbb{N}} t_i^\alpha \le \Big(\sum_{i\in\mathbb{N}} t_i\Big)^\alpha\,.$$

By (5.4)–(5.7) we have that $\mathcal{L}^n(\widetilde{H}^\omega_{j,\theta,M})\to 0$ as $j\to+\infty$. Analogously,

$$\mathcal{L}^n(\widetilde{D}^\omega_{j,\theta,M}) \le \theta^n(\varepsilon_j^n N^\omega_{\varepsilon_j}(D))\,. \quad (7.53)$$

Finally we estimate the energy (3.4) of $(U^\omega_{j,\theta,M})$. Using (7.48)–(7.50) and (3.3), we estimate

$$\begin{aligned}\int_D f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x &= \int_{D\setminus\widetilde{D}^\omega_{j,\theta,M}} f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x + \int_{\widetilde{D}^\omega_{j,\theta,M}} f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \\ &\le \int_{D\setminus(\widetilde{D}^\omega_{j,\theta,M}\cup\widetilde{H}^\omega_{j,\theta,M})} f(\nabla w^\omega_j)\,\mathrm{d}x + \sum_{x^\omega_{j,i}\in(VG)^\omega_{j,\theta,M}} \int_{B_{\theta\varepsilon_j}(\varepsilon_j x^\omega_{j,i})} f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \\ &\quad + c_2\int_{\widetilde{H}^\omega_{j,\theta,M}} \big(|\phi^\omega_{j,\theta,M}\nabla v^\omega_{j,\theta,M} + \nabla\phi^\omega_{j,\theta,M}\otimes v^\omega_{j,\theta,M}|^q + 1|\big)\,. \end{aligned} \quad (7.54)$$

We deal with each term on the right-hand side of (7.54) separately. For the first term, by (3.3) and by property (iv) of (w^ω_j),

$$\begin{aligned}\int_{D\setminus(\widetilde{D}^\omega_j\cup\widetilde{H}^\omega_{j,\theta,M})} f(\nabla w^\omega_j)\,\mathrm{d}x &\le \int_D f(\nabla w^\omega_j)\,\mathrm{d}x + c_1\int_{\widetilde{D}^\omega_{j,\theta,M}\cup\widetilde{H}^\omega_{j,\theta,M}} (1-|\nabla w^\omega_j|^q)\,\mathrm{d}x \\ &\le \int_D f(\nabla u_j)\,\mathrm{d}x + C_k(\theta')^n + \frac{C}{k} + c_1\mathcal{L}^n(\widetilde{D}^\omega_{j,\theta,M}\cup\widetilde{H}^\omega_{j,\theta,M})\,. \end{aligned} \quad (7.55)$$

For the second term on the right-hand side of (7.54), by a simple change of variables, (7.38)(ii), (7.47) and (7.48), for every $x_{j,i}\in(VG)^\omega_{j,\theta,M}$ we have

$$\begin{aligned}\int_{B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})} f(\nabla U^\omega_{j,\theta,M}(x))\,\mathrm{d}x &= \int_{B_{\theta\varepsilon_j}(\varepsilon_j x_{j,i})} f(\alpha_{\varepsilon_j}^{-1}\nabla\zeta^\omega_{j,i}\big(\alpha_{\varepsilon_j}^{-1}(x-\varepsilon_j x_{j,i})\big))\,\mathrm{d}x \\ &= \varepsilon_j^n\int_{B_{\theta K_j}(0)} \alpha^q_{\varepsilon_j} f(\alpha_{\varepsilon_j}^{-1}\nabla\zeta^\omega_{j,i}(y))\,\mathrm{d}y \le \varepsilon_j^n\varphi^j_{\theta,\rho_{j,i}}(\bar{u}^\omega_{j,i}) + \eta\varepsilon_j^n\,. \end{aligned}$$

Moreover, by (7.27) and (4.8),

$$\varepsilon_j^n \sum_{x_{j,i}\in (VG)^\omega_{j,\theta,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) = \varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) - \varepsilon_j^n \sum_{x_{j,i}\in (MG)^\omega_{j,\theta,M}\cap G^\omega_{j,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})$$
$$\leq \varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i}) + C_2\theta^n(\varepsilon_j^n N^\omega_{\varepsilon_j}(D)).$$

Finally, we consider the third term on the right-hand side of (7.54). By the fact that $0 \leq \phi^\omega_{j,\theta,M} \leq 1$, and that $v^\omega_{j,\theta,M} = w^\omega_j$ in $\widetilde H^\omega_{j,\theta,M}$ by (7.48) and (7.51), we have

$$\int_{\widetilde H^\omega_{j,\theta,M}} \left(|\phi^\omega_{j,\theta,M}\nabla v^\omega_{j,\theta,M} + \nabla\phi^\omega_{j,\theta,M}\otimes v^\omega_{j,\theta,M}|^q + 1\right)\mathrm{d}x \lesssim \int_{\widetilde H^\omega_{j,\theta,M}} \left(|\nabla w^\omega_j|^q + L^q|\nabla\phi^\omega_{j,\theta,M}|^q + 1\right)\mathrm{d}x\,, \tag{7.56}$$

where we have used that $\|w^\omega_j\|_{L^\infty(D)} \leq \|u_j\|_{L^\infty(D)} \leq L < +\infty$, by property (v) of (w^ω_j) and by the assumption in this step.

The upper bound estimate (7.54), together with (7.55)–(7.56), (7.53), yields

$$\begin{aligned}\int_D f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \leq &\int_D f(\nabla u_j) + \varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \varphi^j_{\theta,\rho_{j,i}}(\bar u^\omega_{j,i})\\ &+ C\int_{\widetilde H^\omega_{j,\theta,M}} |\nabla w^\omega_j|^q\,\mathrm{d}x + CL^q\int_D |\nabla\phi^\omega_{j,\theta,M}|^q\,\mathrm{d}x\\ &+ C\mathcal{L}^n(\widetilde H^\omega_{j,\theta,M}) + C(\eta+\theta^n)(\varepsilon_j^n N^\omega_{\varepsilon_j}(D)) + C_k(\theta')^n + \frac{C}{k}.\end{aligned} \tag{7.57}$$

By (7.44), (4.18), the growth condition (3.3), property (iv) for (w^ω_j), (7.42), (7.52), and (5.4)–(5.7) it follows that

$$\sup_{M,\theta,j} \int_D f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \leq C_L < +\infty\,,$$

and hence

$$\sup_{M,\theta,j} \|U^\omega_{j,\theta,M}\|_{W^{1,q}(D;\mathbb{R}^m)} \leq C_L < +\infty\,,$$

for a constant $C_L > 0$ independent of $\omega \in \Omega$. By (7.43), proceeding as for the proof of (7.13), we deduce that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$U^{\theta,M,\omega}_j \rightharpoonup u \text{ weakly in } W^{1,q}(D;\mathbb{R}^m) \text{ as } j\to+\infty,\ \theta\to 0^+,\ M\to+\infty\,.$$

Finally, we show that the admissible sequence $(U^{\theta,M,\omega}_j)$ is a recovery sequence, up to a diagonal procedure. Indeed, taking in (7.57) the limsup as $j \to +\infty$, and using the subadditivity of the limsup, (7.44), the equi-integrability of $(|\nabla w^\omega_j|^q)$, (7.52) and (5.15), we deduce

$$\begin{aligned}\varlimsup_{j\to+\infty}\int_D f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \quad\leq\quad &\int_D Qf(\nabla u)\,\mathrm{d}x + \varlimsup_{j\to+\infty}\ \varepsilon_j^n \sum_{x_{j,i}\in G^\omega_{j,M}} \varphi^j_{\theta,\rho^\omega_{j,i}}(\bar u^\omega_{j,i})\\ &+CL^q\varlimsup_{j\to+\infty}\int_D |\nabla\phi^\omega_{j,\theta,M}|^q\,\mathrm{d}x + C(\eta+\theta^n)\langle N(Q)\rangle\mathcal{L}^n(D)\\ &+C_k(\theta M)^n + \frac{C}{k}.\end{aligned}$$

Passing further to the limit superior in the above inequality in $\theta \to 0^+$, and $M \to +\infty$, using Proposition 6.1 and (7.42) we get

$$\varlimsup_{M\to+\infty}\varlimsup_{\theta\to 0^+}\varlimsup_{j\to+\infty}\int_D f(\nabla U^\omega_{j,\theta,M})\,\mathrm{d}x \leq \int_D Qf(\nabla u)\,\mathrm{d}x + \langle N(Q)\rangle\int_D \varphi(u)\,\mathrm{d}x + C(\eta + 1/k)\,.$$

By finally choosing a diagonal sequence $(\widetilde{u}_j^\omega) \subset W_0^{1,q}(D;\mathbb{R}^m)$, with $\widetilde{u}_j^\omega := U^\omega_{j,\theta_j,M_j}$, we conclude that for $\mathbb{P}$-a.e. $\omega \in \Omega$ $(\widetilde{u}_j^\omega)$ is admissible; *i.e.*, $\widetilde{u}_j^\omega|_{H^\omega_{\varepsilon_j}\cup\partial D} \equiv 0$, and

$$\limsup_{j\to+\infty} \int_D f(\nabla \widetilde{u}_j^\omega)\,\mathrm{d}x \leq \int_D Qf(\nabla u)\,\mathrm{d}x + \langle N(Q)\rangle \int_D \varphi(u)\,\mathrm{d}x + C(\eta + 1/k)\,,$$

where we have used that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the sequence $(\widetilde{u}_j^\omega)$ converges to u weakly in $W^{1,q}(D;\mathbb{R}^m)$, and hence also in $L^1(D;\mathbb{R}^m)$.

Hence, (7.24) follows by the arbitrariness of $k \in \mathbb{N}$ and $\eta \in (0,1)$, with $(\widetilde{u}_j^\omega)$ as a recovery sequence.

Step 2: Removing the uniform L^∞-bound on (u_j). This is done by following verbatim the final argument in [3]. Indeed, assume first that $u \in W_0^{1,q}(D;\mathbb{R}^m)\cap L^\infty(D;\mathbb{R}^m)$, and let $L := 4\|u\|_{L^\infty(D)}$. Let $\Psi_L\colon \mathbb{R}^m \mapsto \mathbb{R}^m$ be a Lipschitz function with Lipschitz constant at most 1 such that

$$\Psi_L(z) := \begin{cases} z & \text{if } |z| \leq L/2\,,\\ 0 & \text{if } |z| \geq L\,. \end{cases}$$

Let again $(u_j) \in W_0^{1,q}(D;\mathbb{R}^m)$ be such that (7.43)-(7.44) hold true. Without loss of generality we assume that $u_j \to u$ pointwise $\mathcal{L}^n$-a.e in D as $j \to +\infty$, and that $(|\nabla u_j|^q)$ is equi-integrable on D. We now set $u_j^L := \Psi_L(u_j)$. By the $\mathcal{L}^n$-a.e. pointwise convergence u_j to u we have that

$$\mathcal{L}^n\big(\{u_j^L \neq u_j\}\big) \leq \mathcal{L}^n\big(\{|u_j| > \|u\|_{L^\infty(D)}\}\big) \longrightarrow 0 \text{ as } \ j \to +\infty\,,$$

hence, we also have that $u_j^L \rightharpoonup u$ weakly in $W^{1,q}(D;\mathbb{R}^m)$ as $j \to +\infty$. Furthermore, by the equi-integrability of $(|\nabla u_j|^q)$ we obtain

$$\lim_{j\to+\infty} \int_D f(\nabla u_j^L)\,\mathrm{d}x = \lim_{j\to+\infty} \int_D f(\nabla u_j)\,\mathrm{d}x = \int_D Qf(\nabla u)\,\mathrm{d}x\,,$$

and we can therefore repeat all the reasonings of Step 1, with (u_j^L) in the place of (u_j) in order to obtain the desired limsup inequality.

Finally, given an arbitrary $u \in W_0^{1,q}(D;\mathbb{R}^m)$, we can approximate it by a sequence $(u^L) \subset W_0^{1,q}(D;\mathbb{R}^m) \cap L^\infty(D;\mathbb{R}^m)$ with respect to the strong $W^{1,q}(D,\mathbb{R}^m)$-topology. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ by the lower semicontinuity of the functional

$$\mathcal{F}_0^{''\omega} := \Gamma\text{-}\limsup_{j\to+\infty} \mathcal{F}^\omega_{\varepsilon_j}$$

with respect to the strong $L^q(D;\mathbb{R}^m)$-topology (cf. [5, Remark 7.8]), we obtain

$$\mathcal{F}_0^{''\omega}(u) \leq \liminf_{L\to+\infty} \mathcal{F}_0^{''\omega}(u^L) = \lim_{L\to+\infty} \mathcal{F}_0(u^L) = \mathcal{F}_0(u)\,,$$

which, by the definition of $\mathcal{F}_0^{''\omega}$, is just another way of writing the desired limsup inequality in the general case. The proof is now complete. □

Acknowledgements

L. Scardia was supported by the EPSRC grants EP/V00204X/1 and EP/V008897/1. The work of K. Zemas and C. I. Zeppieri was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure. K. Zemas also acknowledges current support by the Sonderforschungsbereich 1060 and the Hausdorff Center for Mathematics (HCM) under Germany's Excellence Strategy EXC 2047/1-390685813.

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

Declaration

Conflict of interest. The authors declare no conflict of interest.

References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Non-critical size of the holes for a volume distribution and a surface distribution of holes. *Arch. Rational Mech. Anal.*, **113(3)** (1990), 261–298.

[2] N. Ansini, J. F. Babadjian, C. I. Zeppieri. The Neumann sieve problem and dimension reduction: a multiscale approach. *Math. Models Methods Appl. Sci.*, **17(05)** (2007), 681–735.

[3] N. Ansini, A. Braides. Asymptotic analysis of periodically-perforated nonlinear media. *J. Math. Pures Appl.*, **81(5)** (2002), 439–451.

[4] N. Ansini, A. Braides. Erratum to Asymptotic analysis of periodically-perforated nonlinear media. *J. Math. Pures Appl.*, **81.5** (2002), 439–451.

[5] A. Braides, A. Defranceschi. *Homogenization of multiple integrals*. Oxford University Press, **Vol. 12**, New York (1998).

[6] A. Braides, A. Defranceschi, E. Vitali. Homogenization of free discontinuity problems. *Arch. Rational Mech. Anal.*, **135(4)** (1996), 297–356.

[7] L. A. Caffarelli, A. Mellet. Random homogenisation of an obstacle problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26(2)** (2009), 375–395.

[8] C. Calvo-Jurado, J. Casado-Diaz, M. Luna-Laynez. Homogenization of nonlinear Dirichlet problems in random perforated domains. *Nonlinear Anal.*, **133** (2016), 250–274.

[9] C. Calvo-Jurado, J. Casado-Diaz, M. Luna-Laynez. Homogenization of the Poisson equation with Dirichlet conditions in random perforated domains. *J. Comp. App. Math.*, **275** (2016), 375–381.

[10] J. Casado-Diaz, A. Garroni. Asymptotic behaviour of nonlinear elliptic systems on varying domains. *SIAM J. Math. Anal.*, **31** (2000), 581–624.

[11] S. Chiu, D. Stoyan, W. Kendall, J. Mecke. *Stochastic geometry and its applications*. Wiley, New York (2013).

[12] D. Cioranescu, F. Murat. Un terme étrange venu d'ailleurs II. *Nonlinear partial differential equations and their applications, Collége de France Seminar, Res. Notes in Math.* **Vol. III** (1982).

[13] D. J. Daley, D. Vere-Jones. *An introduction to the theory of point processes. Vol.I: Elementary theory and methods*. Probability and its applications, Springer-Verlag New York (2003).

[14] D. J. Daley, D. Vere-Jones. *An introduction to the theory of point processes. Vol.II: General theory and structures*. Probability and its applications, Springer-Verlag New York (2008).

[15] G. Dal Maso, A. Defranceschi. Limits of nonlinear Dirichlet problems in varying domains. *Manuscripta Math.*, **61** (1988), 251–278.

[16] G. Dal Maso, A. Garroni. New results on the asymptotic behaviour of Dirichlet problems in perforated domains. *Math. Models Methods Appl. Sci.*, **4(3)** (1994), 373–407.

[17] H. Federer, W. Ziemer. The Lebesgue set of a function whose distribution derivatives are p-th power summable. *Indiana Univ. Math. J.*, **22(2)** (1972), 139–158.

[18] M. Focardi. Homogenisation of random fractional obstacle problems via Γ-convergence. *Comm. in PDEs*, **34(12)** (2009), 1607–1631.

[19] I. Fonseca, S. Müller, P. Pedregal. Analysis of concentration and oscillation effects generated by gradients. *SIAM J. Math. Anal.*, **29(3)** (1996), 736–756.

[20] A. Giunti. Convergence rates for the homogenization of the Poisson problem in randomly perforated domains. *Netw. Heterog. Media*, **16(3)** (2021), 341–375.

[21] A. Giunti, R. Höfer, J. Velázquez. Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations*, **43(9)** (2018), 1377–1412.

[22] A. V. Marchenko, E. Y. Khruslov. *Boundary Value Problems in Domains with Fine-Granulated Boundaries*. Naukova Dumka, Kiev (1974).

[23] A. V. Marchenko, E. Y. Khruslov. *Homogenization of partial differential equations*. Progress in Mathematical Physics **46**, Birkhäuser Boston, Inc., MA (2006).

[24] G. C. Papanicolaou, S. R. S. Varadhan. Diffusion in regions with many small holes. *Springer Berlin Heidelberg* (1980), 190–206.

[25] R. Schneider, W. Weil. *Stochastic and Integral Geometry*. Probability and its applications, Springer-Verlag New York (2008).

[26] X. Pellet, L. Scardia, C. I. Zeppieri. Stochastic homogenisation of free-discontinuity functionals in random perforated domains. Accepted in *Adv. Calc. Var.* (2023).

(Lucia Scardia) Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland
Email address: L.Scardia@hw.ac.uk

(Konstantinos Zemas) Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, Bonn 53115, Germany
Email address: zemas@iam.uni-bonn.de

(Caterina Ida Zeppieri) Applied Mathematics Münster, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany
Email address: caterina.zeppieri@uni-muenster.de