STABLE DOMAINS FOR HIGHER ORDER ELLIPTIC OPERATORS

by

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Abstract. — This paper is devoted to prove that any domain satisfying a $(\delta_0, r_0)-$capacitary condition of first order is automatically $(m,p)-$stable for all $m \geqslant 1$ and $p > 1$, and for any dimension $N \geqslant 1$. In particular, this includes regular enough domains such as $\mathscr{C}^1-$domains, Lipchitz domains, Reifenberg flat domains, but is sufficiently weak to also include cusp points. Our result extends some of the results of Hayouni and Pierre valid only for $N = 2, 3$, and partially extends the results of Bucur and Zolésio for higher order operators, with a different and simpler proof.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded and open set. Following [10, 4, 5], we say that Ω is $(m,p)-$stable if

$$W^{m,p}(\mathbb{R}^N) \cap \left\{u = 0 \text{ a.e. in } \overline{\Omega}^c\right\} = W_0^{m,p}(\Omega).$$

This notion is related to the continuity of a $2m-$order elliptic PDE with respect to domain perturbation. In particular, if Ω is $(m,2)-$stable, then it implies that for any sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ converging to Ω in a certain Hausdorff sense, one has that $(u_n)_{n\in\mathbb{N}}$ converges strongly in H^m to u, where u_n is the unique solution in $H_0^m(\Omega_n)$ for the equation $(-\Delta)^m(u_n) = f$ in Ω_n, and u is the solution of the same problem in Ω. It is also equivalent to the convergence of $(W_0^{m,p}(\Omega_n))_{n\in\mathbb{N}}$ to $W_0^{m,p}(\Omega)$ in the sense of Mosco (see Section 6).

In the literature, a lot of attention has been devoted to the case $m = 1$ and $p = 2$ because of its relation to the Laplace operator. On the other hand, very few results are available for the higher order spaces $H_0^m(\Omega)$, related to bi-harmonic or more generally poly-harmonic equations, that have a lot of applications. The objective of this paper is to give a short and elementary proof of the fact that any domain which is "regular enough" is always $(m,p)-$stable for all m, p and all dimensions N.

Notice that in general, the stability for $W_0^{m,p}(\Omega)$ does not simply reduce to the one of $W_0^{1,p}(\Omega)$. To enlight this fact we recall that for every open set $\Omega \subset \mathbb{R}^N$, we have the characterisation (see for instance [1, Chapter 9])

$$W_0^{m,p}(\Omega) = W^{m,p}(\mathbb{R}^N) \cap \left\{\nabla^k u|_{\Omega^c} = 0 \quad (m-k,p) - \text{q.e. for all } k \leqslant m-1 \right\},$$

where $\nabla^k u := (\partial^\alpha u)_{|\alpha|=k}$ and $\partial^\alpha u$ is the $(m-k,p)-$quasicontinuous representative, which is in particular defined pointwise $(m-k,p)-$q.e. If Ω is $(1,p)-$stable, then for any $|\alpha| \leqslant m-1$ and from the assumption $\partial^\alpha u = 0$ a.e. in $\overline{\Omega}^c$ we would only deduce that $\partial^\alpha u = 0$ $(1,p)-$q.e. on Ω^c, whereas in order to prove that $u \in W_0^{m,p}(\Omega)$ we would need the stronger condition $\partial^\alpha u = 0$ $(m-|\alpha|,p)-$q.e. on Ω^c.

In [7], Hayouni and Pierre exploited the compact embedding of H^2 into continuous functions in dimensions 2 and 3, in order to get some stability results for the space H_0^2. In particular, they proved that, in dimension 2 and 3, any $(1,2)-$stable domain is automatically $(2,2)-$stable (see [7] or [10]). They also proved in the same paper that, in dimensions 2 and 3, any sufficiently smooth domain is a $(2,2)-$stable domain.

In the present paper, we show that there is no true restriction on the dimension N to obtain $(m,p)-$stability. Our main result asserts that any domain that satisfies a variant of the classical $(1,p)-$capacitary condition, will be automatically $(m,p)-$stable, in any dimension, and for any m. This includes a large class or "regular" domains such as $\mathscr{C}^1-$domains, Lipschitz domains, Reifenberg-flat domains, domains satisfying the so-called external corkscrew condition (see Definition 5.1), ε-cone property, or even domains with the segment property which allows domains with cusps, or more generally domains with the so called flat cone property [3].

In the sequel, we restrict ourselves to open subset of a fixed ball $D \subset \mathbb{R}^N$, and we denote the set of admissible domains by

$$\mathscr{O}(D) := \{\Omega \mid \Omega \subseteq D \text{ is open}\} .$$

In the following definition, the precise definition of capacity has no real importance since they are most often equivalent up to a constant. In this paper we will work with the Bessel capacity, that will be precisely defined in the next section.

Definition 1.1. — *Let $r_0 > 0$, $\delta_0 > 0$ and $p \in (1, +\infty)$. An open set $\Omega \subseteq \mathbb{R}^N$ satisfies the $(r_0, \delta_0)-$capacitary condition if for all $x \in \partial\Omega$ and for all $r \leqslant r_0$,*

$$\operatorname{Cap}_{1,p}\left(\frac{1}{r}\left(\overline{\Omega}^c \cap B(x,r) - x\right)\right) \geqslant \delta_0, \tag{1.1}$$

where $\operatorname{Cap}_{1,p}$ is the Bessel capacity of first order. The class of open subset of D having the $(r_0, \delta_0)-$capacity condition is denoted by $\mathscr{O}_{\text{cap}}^{\delta_0, r_0}(D)$.

Here is our main statement.

Theorem 1.1. — *If $\Omega \in \mathscr{O}_{\text{cap}}^{\delta_0, r_0}(D)$ satisfies $|\partial\Omega| = 0$, then Ω is $(m,p)-$stable for any $m \geqslant 1$ and $p \in (1, +\infty)$.*

Let us provide some comments about the result. One of the main feature and somewhat surprising is that the condition involves only the $(1,p)-$capacity even if the conclusion yields $(m,p)-$stability for all $m \geqslant 1$.

Our main statement reminds the classical one in [5], where Bucur and Zolésio proved that a domain is $(1,2)-$stable under a weaker condition with $(1,2)-$capacity. More precisely, in [5] the authors prove that the following is enough

$$\forall x \in \partial\Omega, \forall r \leq r_0, \quad \frac{\operatorname{Cap}_{1,2}(\Omega^c \cap B(x,r), B(x,2r))}{\operatorname{Cap}_{1,2}(B(x,r), B(x,2r))} \geqslant \delta_0. \tag{1.2}$$

In (1.2), the notation $\operatorname{Cap}_{1,2}(A \cap B(x,r), B(x,2r))$ refers to the so called "condenser capacity" (see Section 4 for a definition). Actually, it can be proved (see Remark 4.1 in Section 4) that (1.2) is equivalent to

$$\forall x \in \partial\Omega, \forall r \leq r_0, \quad \operatorname{Cap}_{1,2}\left(\frac{1}{r}\left(\Omega^c \cap B(x,r) - x\right)\right) \geqslant \delta_0. \tag{1.3}$$

Observe that (1.3) looks similar to (1.1) but without a bar over Ω, which stands for a substantial difference. Since (1.1) clearly implies (1.3), the main result of [5] is stronger than ours.

The typical example of domain that satisfies (1.3) but not (1.1) is a domain with a crack. For instance the unit ball of $\mathbb{R}^2$ minus a radius, i.e. $B(0,1)\setminus([0,1]\times\{0\})$. The capacity of Ω^c in a small ball $B(x,r)$ centered at the boundary has a positive $(1,2)-$capacity, because it always contains a segment of length at least a radius, so (1.2) is satisfied. On the other hand for a small ball centered on the crack, the complement of $\overline{\Omega}$ in the ball is empty, thus our condition (1.1) is not satisfied.

In other words for the special case of $m=1$ and $p=2$ we only partially recover the result in [5]. In contrast, under the similar and slightly stronger $(1,2)-$capacitary condition (1.1), we obtain $(m,2)-$stability for all $m\geqslant 1$, which is interesting. It is worth mentioning that our proof is different and simpler than the one [5], thus provides an alternative argument for many standard classes of domains (such as Lipschitz domains or domain satisfying the uniform flat cone property, or corkscrew property) which is new even for the standard case $m=1$, and works similarly for $p\neq 2$.

Let us further emphasis that the difference between Ω and $\overline{\Omega}$ has a crucial importance. Indeed, under our assumption $|\partial\Omega|=0$, the question of (m,p)-stability is equivalent to asking whether

$$W^{m,p}(\mathbb{R}^N)\cap\{u=0 \text{ a.e. in } \Omega^c\}=W^{m,p}_0(\Omega).$$

For instance for $m=1$ and $p=2$, as already mentioned before, it is known that

$$H^1_0(\Omega)=H^1(\mathbb{R}^N)\cap\{u=0 \text{ q.e. in } \Omega^c\},$$

thus the main question for stability is whether $\{u=0 \text{ a.e in } \Omega^c\}\Rightarrow\{u=0 \text{ q.e in } \Omega^c\}$. If we add a bar on Ω then the simliar question becomes trivial because $\overline{\Omega}^c$ is open and therefore we always have, for a precise representative $u\in H^1(\mathbb{R}^N)$,

$$\{u=0 \text{ a.e in } \overline{\Omega}^c\}\Leftrightarrow\{u=0 \text{ q.e in } \overline{\Omega}^c\}.$$

This fact will play a major role in the proof of our main result. Another fact that we use in our proof is the following, valid for any $u\in W^{m,p}(\mathbb{R}^N)$,

$$\{u=0 \text{ a.e in } \overline{\Omega}^c\}\Rightarrow\{\nabla^k u=0 \text{ a.e. in } \overline{\Omega}^c \text{ for all } k\leqslant m\}.$$

Again, this follows from the fact that $\overline{\Omega}^c$ is open and explains why we need a bar over Ω in the capacitary condition (1.1) for our proof to work.

Another difference with [5] is the assumption $|\partial\Omega|=0$ that we need in our main statement Theorem 1.1 (here $|\partial\Omega|$ denotes the Lebesgue measure of $\partial\Omega$). In practice this assumption is not very restrictive since it will be easily satisfied by all standard classes of domains. For instance it holds true as soon as a corckscrew condition is satisfied (see Proposition 5.1). We do not know whether the capacity condition (1.1) directly implies $|\partial\Omega|=0$, in which case this assumption would be redundant.

As a consequence of our main result we get a capacity condition which implies stability for the polyharmonic equation along a Hausdorff converging sequence of domains. We refer to Section 6 for the definition of Hausdorff convergence, Mosco convergence and γ_m-convergence, and we give here in the introduction two different statements. In the first one (Corollary 1.1) we assume only the limiting domain Ω to be "regular" while in the second (Theorem 1.2) we assume the whole sequence to be "regular".

Corollary 1.1. — *Let $\Omega \in \mathscr{O}_{\text{cap}}^{\delta_0, r_0}(D)$ and $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{O}(D)$. If $|\partial\Omega| = 0$, $(\overline{\Omega_n})_{n\in\mathbb{N}}$ d_H-converges to $\overline{\Omega}$, and $(\Omega_n)_{n\in\mathbb{N}}$ $d_{H^c}-$converges to Ω, then the sequence $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω, or equivalently, $(H_0^m(\Omega_n))_{n\in\mathbb{N}}$ converges to $H_0^m(\Omega)$ in the sense of Mosco.*

Corollary 1.1 follows from gathering together Proposition 6.3 and Theorem 1.1. Let us now mention a few remarks.

1. The interesting feature of Corollary 1.1 is that only the limiting domain Ω is assumed to be stable (thus somehow "regular") and nothing is assumed on the sequence $(\Omega_n)_{n\in\mathbb{N}}$, which could be arbitrary open sets.
2. It is worth mentioning that in [5] the authors assumed only $\Omega_n \xrightarrow{d_{H^c}} \Omega$ to obtain the γ_m-convergence of a sequence $(\Omega_n)_{n\in\mathbb{N}}$. On the other hand they assumed that every term Ω_n along the sequence satisfies a capacitary condition with uniform constants. A similar statement will be given later in Theorem 1.2.
3. It is easy to construct an example of stable domain Ω (even smooth) and a sequence $(\Omega_n)_{n\in\mathbb{N}}$ such that $\Omega_n \xrightarrow{d_{H^c}} \Omega$ and $(\Omega_n)_{n\in\mathbb{N}}$ does not γ_m-converges to Ω. This shows that without any other assumption on the sequence, the second assumption $\overline{\Omega_n} \xrightarrow{d_H} \overline{\Omega}$ is pivotal for the result to hold true. The construction is rather classical : consider the sequence made from an enumeration $x_i \in B(0,1)$ of points with rational coordinates. Then define
$$\Omega_n := B(0,2) \setminus \bigcup_{i=0}^{n} \{x_i\}.$$
It is easy to see that $(\Omega_n)_{n\in\mathbb{N}}$ converges to $\Omega := B(0,2) \setminus \overline{B}(0,1)$ for the complementary Hausdorff distance, which is clearly a $(m,2)-$stable domain because the boundary is smooth. On the other hand, for dimension $N \geqslant 2m$ we know that $\text{Cap}_{m,2}(\{x_i\}) = 0$, so it is classical that $(\Omega_n)_{n\in\mathbb{N}}$ does not γ_m-converge to Ω (see [10, Section 3.2.6, page 80] for the case $m = 1$). On the other hand $\overline{\Omega_n} = \overline{B}(0,2)$ clearly does not Hausdorff converge to $\overline{\Omega} = \overline{B}(0,2) \setminus B(0,1)$, which explains why Theorem 6.3 does not apply.

Next, in order to get existence of shape optimisation problems for higher order equations under geometrical constraints, the following variant is more usefull. Notice that here we suppose (1.1) on the whole sequence and by this way we can avoid the assumption

$\overline{\Omega_n} \xrightarrow{d_H} \overline{\Omega}$ but $\Omega_n \xrightarrow{d_{H^c}} \Omega$ suffices.

Theorem 1.2. — *Let $\Omega \in \mathscr{O}(D)$ and $(\Omega_n)_{n\in\mathbb{N}}$ all belonging to $\mathscr{O}^{\delta_0,r_0}_{\text{cap}}(D)$. If $|\partial\Omega| = 0$ and $(\Omega_n)_{n\in\mathbb{N}}$ $d_{H^c}-$converges to Ω, then $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω, or equivalently, $(H^m_0(\Omega_n))_{n\in\mathbb{N}}$ converges to $H^m_0(\Omega)$ in the sense of Mosco.*

Since complementary Hausdorff topology is relatively compact, it is easy to get existence results for shape optimisation problems using Theorem 1.2, with additional geometrical constraints on the domain. This applies to various standard classes of domains such as uniformly Lipschitz domains, Reifenberg-flat, corkscrew, or $\varepsilon-$cone, as described in the last section of the paper (see Theorem 8.1).

2. Preliminaries

The term domain and the symbol Ω will be reserved for an open and bounded set in the $N-$dimensional euclidean space $\mathbb{R}^N$. We will denote the Lebesgue measure of a set $A \subset \mathbb{R}^N$ by $|A|$. The norm of a point $x \in \mathbb{R}^N$ is denoted by $|x| := (\sum_{i=1}^N x_i^2)^{1/2}$. If α is a multi-indice, i.e. $\alpha \in \mathbb{N}^N$, then the norm of α is $|\alpha| := \sum_{i=1}^N \alpha_i$ and we define the partial derivative operator

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_k}},$$

and the vector $\nabla^k := (\partial^\alpha)_{|\alpha|=k}$. The notations $\partial\Omega$ and $\overline{\Omega}$ stand for the boundary and the closure of Ω, respectively. Let $\mathscr{C}^\infty_c(\Omega)$ be the space of smooth functions with compact support in Ω. The ball of radius $r \geqslant 0$ and centered at $x \in \mathbb{R}^N$ is denoted by $B(x,r)$. For $m \in \mathbb{N}$ and $p \in (1,+\infty)$, we consider the usual Sobolev space $W^{m,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{k=0}^m \|\nabla^k u\|^p_{L^p(\Omega)}\right)^{1/p},$$

where

$$\|\nabla^k u\|^p_{L^p(\Omega)} := \int_\Omega |\nabla^k u|^p \, dx.$$

Finally, the space $W^{m,p}_0(\Omega)$ is the completion of $\mathscr{C}^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

When the dimension $N < mp$, elements of $W^{m,p}(\mathbb{R}^N)$ can be represented as continuous functions. However, if $N \geqslant mp$, this is no longer the case and the natural way of measuring by how much the functions deviate from continuity is by means of capacity.

In this paper we will work with the Bessel capacity defined for instance in [13, Chapter 2.6]. If $K \subset \mathbb{R}^N$ is any set, then we define the $(m,p)-$capacity of K by

$$\text{Cap}_{m,p}(K) := \inf\left\{\|f\|_p^p \;\middle|\; f \geq 0 \text{ and } g_m * f \geqslant 1 \text{ on } K\right\}, \tag{2.1}$$

where g_m with $m \geq 1$ is the Bessel kernel, defined as being the function whose Fourier transform is $\hat{g}_\alpha(x) = (2\pi)^{-N/2}(1+|x|^2)^{-m/2}$. We refer to [13, Chapter 2.6] for more details and several basic properties of the Bessel Capacity.

In this paper, while considering a function $u \in W^{1,p}(\mathbb{R}^N)$, we will always tacitly mean that u is a quasicontinuous representative, without mentioning it explicitly (see for instance [1] for a definition).

The proof of the main result will use the following Poincaré type inequality that can be found for instance in [13, Corollary 4.5.2, page 195]. To be more precise, we can use the inequality stated for $B(0,1)$ in [13, Corollary 4.5.2, page 195], and apply it to the function $x \longmapsto u(Rx)$ to get (2.2). A similar statement can also be found in [9, Theorem 4.1]. According to [9], Lemma 2.1 was first proved in [8] by Hedberg in 1978.

Lemma 2.1. — [13, Corollary 4.5.2, page 195] *Let $r > 0$, $p \in (1,+\infty)$ and $u \in W^{1,p}(B(0,r))$. We define $Z(u) := \{x \in B(0,r) \mid u(x) = 0\}$. If $\mathrm{Cap}_{1,p}(Z(u)) > 0$, then*

$$\int_{B(0,r)} |u|^p \, dx \leqslant C \frac{r^p}{\mathrm{Cap}_{1,p}(r^{-1}Z(u))} \int_{B(0,r)} |\nabla u|^p \, dx, \tag{2.2}$$

where $C > 0$ depends only on p and N.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. — Let Ω be a bounded domain satisfying the assumptions of Theorem 1.1 and let $u \in W^{m,p}(\mathbb{R}^N)$ be given satisfying $u = 0$ almost everywhere in $\overline{\Omega}^c$. To prove the theorem it suffice to prove that u can be approximated in the $W^{m,p}(\mathbb{R}^N)$ norm by a sequence of functions in $\mathscr{C}_c^\infty(\Omega)$. To do so we will first truncate u near the boundary of Ω as follows. For all $n \in \mathbb{N}$, we consider

$$K_n := \left\{x \in \Omega \,\middle|\, d(x, \partial\Omega) \geqslant 2^{-n}\right\}.$$

The exhaustive family of compact $(K_n)_{n\in\mathbb{N}}$ satisfies $K_n \subseteq K_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n$. Then take a test function $\rho \in \mathscr{C}_c^\infty(B(0,1))$ such that $\rho \geqslant 0$ and

$$\int_{\mathbb{R}^N} \rho(x) dx = 1.$$

We define $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$ and

$$\theta_{n,\varepsilon}(x) := \mathbf{1}_{K_n} * \rho_\varepsilon(x) = \varepsilon^{-N} \int_{K_n} \rho\left(\frac{x-y}{\varepsilon}\right) dy,$$

which satisfies Supp $(\theta_{n,\varepsilon}) \subseteq K_n + \overline{B}(0,\varepsilon)$. We take $\varepsilon_n := 2^{n+1}$ and denote now $\theta_n := \theta_{n,\varepsilon_n}$ so that $\theta_n \in \mathscr{C}_c^\infty(\Omega)$, $\theta_n = 1$ on K_{n-1}, $\theta_n = 0$ on K_{n+1}^c,

$$\mathrm{Supp}\ (\nabla^k \theta_n) \subseteq K_{n+1} \setminus \mathrm{Int}(K_{n-1}).$$

To prove the theorem it suffices to prove that

$$u_n := u\theta_n \xrightarrow[n\longrightarrow+\infty]{} u \text{ in } W^{m,p}(\mathbb{R}^N),$$

because then we can conclude by using the density of $\mathscr{C}_c^\infty(\Omega)$ into $W^{m,p}(\mathrm{Int}(K_{n+2}))$, and a diagonal argument. Let $k \leqslant m$ be a positive integer. To prove the claim we first estimate the L^p norm :

$$\|u_n - u\|^p_{L^p(\mathbb{R}^N)} \leqslant \int_{\overline{\Omega}\backslash K_{n-1}} |u|^p \, dx.$$

Using the fact that $(\Omega\backslash K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Lebesgue measurable sets, and thanks to the condition $|\overline{\Omega} \setminus \Omega| = 0$, we know that $|\overline{\Omega}\backslash K_n| \longrightarrow 0$ as $n \longrightarrow +\infty$ and therefore $u_n \longrightarrow u$ in $L^p(\mathbb{R}^N)$.

Next for the norm of gradients we will use a covering of $\partial\Omega$. More precisely, the infinite family $(B(x, 2^{-(n-2)}))_{x\in\partial\Omega}$ is a cover of Supp $(\nabla^k\theta_n)$ and by the famous $5B$-covering lemma (see for instance [2, Theorem 2.2.3]) there exists a countably subcover indexed by $(x_i)_{i\in\mathbb{N}} \subseteq \partial\Omega$ such that $(B(x_i, 2^{-(n-2)}))_{i\in\mathbb{N}}$ is a disjoint family,

$$\text{Supp } (\nabla^k\theta_n) \subseteq \bigcup_{i\in\mathbb{N}} B(x_i, 5\cdot 2^{-(n-2)}), \text{ and } \sum_{i\in\mathbb{N}} \mathbf{1}_{B(x_i,5\cdot2^{-(n-2)})} \leqslant N_0,$$

for a universal constant $N_0 \in \mathbb{N}$. In the sequel, we simply write $B_n(x_i)$ instead of $B(x_i, 5\cdot 2^{-(n-2)})$. Afterwards, we estimate

$$\|\nabla^k u_n - \nabla^k u\|^p_{L^p(\mathbb{R}^N)} \leqslant C\int_{\overline{\Omega}\backslash K_{n-1}} |\nabla^k u|^p \, dx + C\sum_{\substack{k=|\beta|+|\gamma|\\ \gamma\neq 0}} \int_{\overline{\Omega}\backslash K_{n-1}} |\partial^\beta u|^p |\partial^\gamma\theta_n|^p \, dx.$$

The first term converges to 0 as $n \longrightarrow +\infty$ for the same reasons as before. For the other term we use the following estimate

$$|\partial^\gamma\theta_n(x)|^p \leqslant \varepsilon_n^{-pN}\int_{K_n} \varepsilon_n^{-p|\gamma|}\left|\partial^\gamma\rho\left(\frac{x-y}{\varepsilon_n}\right)\right|^p dy \leqslant C\varepsilon_n^{-p|\gamma|}.$$

The function u vanishes almost everywhere on the open set $\overline{\Omega}^c$, so $\partial^\beta u$ is zero in $\mathscr{D}'(\overline{\Omega}^c)$ and vanishes almost everywhere on this open set. Hence the Poincaré inequality (2.2) applies to all the $\partial^\beta u$ for $|\beta| < m$, and for all ball $B_n(x_i)$ such that $2^{-(n-2)} \leqslant r_0$, thanks to our capacitary condition (1.1) we get

$$\mathrm{Cap}_{1,p}(5^{-1}\cdot 2^{n-2}(Z(\partial^\beta u) - x_i)) \geqslant \mathrm{Cap}_{1,p}(5^{-1}\cdot 2^{n-2}(\overline{\Omega}^c \cap B(x_i, 5\cdot 2^{2-n}) - x_i)) \geqslant \delta_0.$$

Therefore,

$$\int_{B_n(x_i)} |\partial^\beta u|^p \, dx \leqslant C\delta_0^{-1}\varepsilon_n^p \int_{B_n(x_i)} |\nabla\partial^\beta u|^p \, dx, \tag{3.1}$$

and using successively $(k-|\beta|)-$times the Poincaré inequality and the covering of $\partial\Omega$, we get

$$\begin{aligned}\int_{\overline{\Omega}\setminus K_{n-1}} |\partial^\beta u|^p |\partial^\gamma \theta_n|^p \, dx &\leqslant C\varepsilon_n^{-p|\gamma|} \int_{\overline{\Omega}\setminus K_{n-1}} |\partial^\beta u|^p \, dx \\ &\leqslant C\varepsilon_n^{-p|\gamma|} \sum_{i\in\mathbb{N}} \int_{B_n(x_i)} |\partial^\beta u|^p \, dx \\ &\leqslant C \sum_{i\in\mathbb{N}} \int_{B_n(x_i)} |\nabla^k u|^p \, dx \\ &\leqslant CN_0 \int_{\overline{\Omega}\setminus K_{n-5}} |\nabla^k u|^p \, dx\end{aligned}$$

and this tends to zero as $n \longrightarrow +\infty$ so follows the proof. □

4. Equivalent condition with condencer capacity

In this section we give an equivalent condition to (1.1) with the notion of "condenser capacity" which allows us to compare it with the condition in [5]. More precisely, in [5] the condition involves the condenser capacity defined for any compact subset $K \subset \mathbb{R}^N$ and $p \in (1, +\infty)$ by

$$\mathrm{Cap}_{1,p}(K\cap B(x,r), B(x,2r)) := \inf\left\{ \int_{B(x,2r)} |\nabla\varphi|^p \, dx \quad | \quad \varphi \in C_0^\infty(B(x,2r)),\ \varphi \geq 1 \text{ on } K\cap B(x,r) \right\}.$$

This notion can then be extended to arbitrary sets K by approximation (see for instance [1, Definition 2.2.4]). The difference with the Bessel capacity defined in (2.1) can be seen through the following well known equivalent definition of Bessel Capacity (See for instance [9, Section 2]), for a closed set K,

$$\mathrm{Cap}_{1,p}(K) = \inf\left\{ \int_{\mathbb{R}^N} |\varphi|^p + |\nabla\varphi|^p \, dx \quad | \quad \varphi \in C_0^\infty(\mathbb{R}^N),\ \varphi \geq 1 \text{ on } K \right\}.$$

The next proposition says in particular that our condition implies the one in [5].

Proposition 4.1. — *Let $\Omega \subset \mathbb{R}^N$ be open. Then the condition*

$$\exists \delta_0, r_1 > 0 \text{ s.t. } \forall x \in \partial\Omega, \forall r \leq r_1, \quad \mathrm{Cap}_{1,p}\left(\frac{1}{r}\left(\overline{\Omega}^c \cap B(x,r) - x\right)\right) \geqslant \delta_0, \tag{4.1}$$

is equivalent to the following one

$$\exists \delta_0, r_1 > 0 \text{ s.t. } \forall x \in \partial\Omega, \forall r \leq r_1, \quad \frac{\mathrm{Cap}_{1,p}(\overline{\Omega}^c \cap B(x,r), B(x,2r))}{\mathrm{Cap}_{1,p}(B(x,r), B(x,2r))} \geqslant \delta_0. \tag{4.2}$$

Proof. — The condenser capacity enjoys a nice scaling property. Assume for simplicity and without loss of generality, that $x = 0$. It is easy to prove by a simple change of variables that for all $\lambda > 0$,

$$\text{Cap}_{1,p}((\lambda K) \cap B(0,\lambda r), B(0,2\lambda r)) = \lambda^{N-p}\text{Cap}_{1,p}(K \cap B(0,r), B(0,2r)). \tag{4.3}$$

In particular,

$$\text{Cap}_{1,p}(B(0,r), B(0,2r)) = C_0 r^{N-p},$$

with $C_0 := \text{Cap}_{1,p}(B(0,1), B(0,2))$, and it follows that

$$\begin{aligned}\frac{\text{Cap}_{1,p}(\overline{\Omega}^c \cap B(0,r), B(0,2r))}{\text{Cap}_{1,p}(B(0,r), B(0,2r))} &= \frac{r^{N-p}\text{Cap}_{1,p}\left((\frac{1}{r}\overline{\Omega}^c) \cap B(0,1), B(0,2)\right)}{\text{Cap}_{1,p}(B(0,r), B(0,2r))}.\\ &= \frac{1}{C_0}\text{Cap}_{1,p}\left((\frac{1}{r}\overline{\Omega}^c) \cap B(0,1), B(0,2)\right).\end{aligned}$$

Therefore, to prove the Proposition it suffices to prove that there exist some constants $C_1, C_2 > 0$ such that for all sets $K \subset B(0,1)$,

$$C_1\text{Cap}_{1,p}(K) \leq \text{Cap}_{1,p}(K \cap B(0,1), B(0,2)) \leq C_2\text{Cap}_{1,p}(K).$$

But this is a well known fact about relative capacity. A proof can be found for instance in [10, Proposition 3.3.17]. □

Remark 4.1. — *Of course arguing as in the proof of Proposition 4.1 we can also prove that the condition in [5],*

$$\exists \delta_0, r_1 > 0 \text{ s.t. } \forall x \in \partial\Omega, \forall r \leq r_1, \quad \frac{\text{Cap}_{1,p}(\Omega^c \cap B(x,r), B(x,2r))}{\text{Cap}_{1,p}(B(x,r), B(x,2r))} \geqslant \delta_0, \tag{4.4}$$

is equivalent to the following one

$$\exists \delta_0, r_1 > 0 \text{ s.t. } \forall x \in \partial\Omega, \forall r \leq r_1, \quad \text{Cap}_{1,p}\left(\frac{1}{r}(\Omega^c \cap B(x,r) - x)\right) \geqslant \delta_0. \tag{4.5}$$

5. Examples of domains satisfying our condition

As we said in the introduction, any smooth enough domain will satisfy our condition. For instance domains satisfying an external corkscrew condition as defined below.

Definition 5.1. — *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $a \in (0,1)$, and $r_0 > 0$. We say that Ω satisfies an $(a, r_0)-$external corkscrew condition if for every $x \in \partial\Omega$ and $r \leqslant r_0$, one can find a ball $B(y, ar)$ such that*

$$B(y, ar) \subset B(x,r) \cap \overline{\Omega}^c.$$

We give a non-exhaustive list of class of domains includes in $\mathscr{O}(D)$:

— $\mathscr{O}_{\text{convex}}(D) := \{\Omega \subseteq D \mid \Omega \text{ open and convex}\}$.

— $\mathscr{O}^{r_0}_{\text{seg}}(D) := \{\Omega \subseteq D \mid \Omega$ open and has the r_0-external segment property$\}$. We say Ω satisfies the r_0-external segment property if for every $x \in \partial\Omega$, there exists a vector $y_x \in \mathbb{S}^{N-1}(0, r_0)$ such that $x + ty_x \in \Omega^c$ for $t \in (0, 1)$. This notion can also be generalized by the "flat cone" condition as in [5, Definition 5.2] (see also [3]).

— $\mathscr{O}^{\lambda}_{\text{Lip}}(D) := \{\Omega \subseteq D \mid \Omega$ open and is a Lipschitz domain$\}$.

— $\mathscr{O}^{\delta_0,r_0}_{\text{Reif flat}}(D) := \{\Omega \subseteq D \mid \Omega$ open and is $(\varepsilon_0, \delta_0)$ – Reifenberg flat$\}$. We say Ω is $(\varepsilon_0, \delta_0)$ – Reifenberg flat for $\varepsilon_0 \in (0, 1/2)$ and $\delta_0 \in (0, 1)$ if for all $x \in \partial\Omega$ and $\delta \in (0, \delta_0]$, there exists an hyperplan $\mathscr{P}_x(\delta)$ of $\mathbb{R}^N$ such that $x \in \mathscr{P}_x(\delta)$ and

$$d_H\left(\partial\Omega \cap \overline{B}(x, \delta), \mathscr{P}_x(\delta) \cap \overline{B}(x, \delta)\right) \leqslant \delta\varepsilon_0.$$

Moreover for all $x \in \partial\Omega$, the set

$$B(x, \delta_0) \cap \left\{x \in \mathbb{R}^N \;\middle|\; d(x, \mathscr{P}_x(\delta_0)) \geqslant 2\delta_0\varepsilon_0\right\}$$

has two connected components: one is contained in Ω, the other one in $\mathbb{R}^N \backslash \Omega$.

— $\mathscr{O}^{\varepsilon}_{\text{cone}}(D) := \{\Omega \subseteq D \mid \Omega$ open and has the external ε – cone condition$\}$. We say Ω has the external $\varepsilon-$cone condition if there exists a cone C of angle ε such that for every $x \in \partial\Omega$, there exists a cone C_x with non empty interior, congruent to C by rigid motion and such that x is the vertex of C_x and $C_x \subset \Omega^c$.

— $\mathscr{O}^{a,r_0}_{\text{corks}}(D) := \{\Omega \subseteq D \mid \Omega$ open and has the $(a, r_0)-$external corkscrew condition$\}$, see Definition 5.1.

— $\mathscr{O}^{\delta_0,r_0}_{\text{cap}}(D) := \{\Omega \subseteq D \mid \Omega$ open and has the $(\delta_0, r_0)-$capacity condition (1.1)$\}$.

It is easy to see that for some fixed parameters we have the inclusions

$$\mathscr{O}^{\varepsilon}_{\text{cone}}(D) \subseteq \mathscr{O}^{a,r_0}_{\text{corks}}(D),$$

and

$$\mathscr{O}_{\text{convex}}(D) \subseteq \mathscr{O}^{\lambda}_{\text{Lip}}(D) \subseteq \mathscr{O}^{\delta_0,r_0}_{\text{Reif flat}}(D) \subseteq \mathscr{O}^{a,r_1}_{\text{corks}}(D) \subseteq \mathscr{O}^{\delta_1,r_2}_{\text{cap}}(D).$$

Since a segment has positive $(1, p)$-Capacity provided $p > N - 1$ (see [6, Proposition 2.5]) we have

$$\mathscr{O}^{r_0}_{\text{seg}}(D) \subseteq \mathscr{O}^{\delta_0,r_1}_{\text{cap}}(D) \qquad \text{for } p > N - 1.$$

Any C^1 domain or Lipschitz domain satisfies an external corkscrew condition. It also follows from porosity estimates that the crokscrew condition implies $|\overline{\Omega} \backslash \Omega| = 0$, as stated in the following useful proposition.

Proposition 5.1. — *If $\Omega \in \mathscr{O}^{a,r_0}_{\text{corks}}(D)$, then $|\partial\Omega| = 0$.*

Proof. — For all $x \in \partial\Omega$ and $r \leqslant r_0$, there exists $y \in \mathbb{R}^N$ such that

$$B(y, ar) \subset B(x, r) \cap \overline{\Omega}^c \subset \mathbb{R}^N \setminus \partial\Omega.$$

In other words, $\partial\Omega$ is a $\sigma-$porous set in $\mathbb{R}^N$, in the sense of [12, Definition 2.22], with $\sigma = 2a$. In virtue of [12] (see the last paragraph at the bottom of page 321 in [12], or see also [11, Proposition 3.5]), we conclude that $|\partial\Omega| = 0$. □

Corollary 5.1. — *If Ω belongs to one of the following classes : $\mathscr{O}^{\varepsilon}_{\text{cone}}(D)$, $\mathscr{O}_{\text{convex}}(D)$, $\mathscr{O}^{\lambda}_{\text{Lip}}(D)$, $\mathscr{O}^{\delta_0, r_0}_{\text{Reif flat}}(D)$, or $\mathscr{O}^{a, r_0}_{\text{corks}}(D)$, then Ω is $(m,p)-$stable for any $m \geqslant 1$ and $1 < p < +\infty$.*

6. Stability with respect to domain perturbation

As before, we consider a fixed bounded domain $D \subset \mathbb{R}^N$. Let Ω and $(\Omega_n)_{n\in\mathbb{N}}$ be bounded subdomains of D such that $\overline{\Omega_n} \longrightarrow \overline{\Omega}$ and $\overline{D}\setminus\Omega \longrightarrow \overline{D}\setminus\Omega$ as $n \longrightarrow +\infty$ for the Hausdorff convergence. In particular, this implies the convergence "in the sense of compacts" (see [10, Section 2.2.4]). In this section we verify that the $(m, 2)-$stability of Ω implies the Mosco convergence of the sequence $(H^m_0(\Omega_n))_{n\in\mathbb{N}}$ towards $H^m_0(\Omega)$. This will follow from the same argument as for the classical case of H^1_0 (see for instance [10, Proposition 3.5.4]), but for the sake of completeness we give here the full details. For this purpose, we first prove the equivalence between γ_m-convergence and Mosco convergence (Proposition 6.1). Then we show that $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω when Ω is $(m, 2)-$stable (Proposition 6.3).

Definition 6.1. — *The sequence $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω if for all $f \in H^{-m}(D)$, the sequence $(u_n)_{n\in\mathbb{N}}$ strongly converges in $H^m_0(D)$ to u, where u_n (resp. u) is the unique solution of the Dirichlet problem $(-\Delta)^m u_n = f$ (resp. $(-\Delta)^m u = f$) in $H^m_0(\Omega_n)$ (resp. $H^m_0(\Omega)$).*

Definition 6.2. — *The sequence $(H^m_0(\Omega_n))_{n\in\mathbb{N}}$ converges to $H^m_0(\Omega)$ in the sense of Mosco if the following holds :*

1. *If $(v_{n_k})_{k\in\mathbb{N}}$ is a subsequence, where $v_{n_k} \in H^m_0(\Omega_{n_k})$, and weakly converges to $v \in H^m_0(D)$, then $v \in H^m_0(\Omega)$.*
2. *For all $v \in H^m_0(\Omega)$, there exists a sequence $(v_n)_{n\in\mathbb{N}}$, where $v_n \in H^m_0(\Omega_n)$, which strongly converges to v in $H^m_0(D)$.*

Proposition 6.1. — *The sequence $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω if, and only if, $(H^m_0(\Omega_n))_{n\in\mathbb{N}}$ converges to $H^m_0(\Omega)$ in the sense of Mosco.*

Proof. — Assume that the sequence $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converge to Ω. Let $(v_{n_k})_{k\in\mathbb{N}}$ be a subsequence which weakly converges to $v \in H_0^m(D)$, where $v_{n_k} \in H_0^m(\Omega_{n_k})$. Consider the distribution $f := (-\Delta)^m v$. Then $f \in H^{-m}(D)$ and v is the unique solution of the Dirichlet problem in $H_0^1(D)$. To prove the first item in the definition of Mosco convergence we need to prove that $v \in H_0^1(\Omega)$.

The γ_m-convergence implies that $(u_{n_k})_{k\in\mathbb{N}}$ strongly converges in $H_0^m(D)$ to $u \in H_0^m(\Omega)$ where u_{n_k} satisfies $(-\Delta)^m u_{n_k} = f$ in Ω_{n_k} and u satisfies $(-\Delta)^m u = f$ in $H_0^1(\Omega)$. So it suffices to show that $u = v$. For all $k \in \mathbb{N}$,

$$\begin{aligned}\int_D \nabla^m u_{n_k} : \nabla^m (u_{n_k} - v_{n_k})\, dx &= \int_{\Omega_{n_k}} \nabla^m u_{n_k} : \nabla^m (u_{n_k} - v_{n_k})\, dx \\ &= \langle f, (u_{n_k} - v_{n_k})\rangle.\end{aligned}$$

Then as $k \longrightarrow +\infty$, we use the strong convergence of $(u_{n_k})_{k\in\mathbb{N}}$ and the weak convergence of $(v_{n_k})_{k\in\mathbb{N}}$ to obtain

$$\int_D \nabla^m u : \nabla^m (u - v)\, dx = \langle f, (u - v)\rangle.$$

Then according to equality $f := (-\Delta)^m v$, we have

$$\begin{aligned}\int_D |\nabla^m (u - v)|^2 dx &= \int_D \nabla^m (u - v) : \nabla^m u\, dx - \int_D \nabla^m (u - v) : \nabla^m v\, dx \\ &= \langle f, (u - v)\rangle - \langle f, (u - v)\rangle = 0,\end{aligned}$$

thus $u = v$ and the first point of Mosco convergence follows.

The proof of the second item of the Mosco convergence is simpler. Consider $v \in H_0^m(\Omega) \subseteq H_0^m(D)$ and let $f := (-\Delta)^m v$. We need to find a recovery sequence $u_n \in H_0^m(\Omega_n)$ that converges strongly to v. It suffice to define u_n being the solution of $-\Delta u_n = f$ in $H_0^1(\Omega_n)$. By γ_m-convergence we directly have $u_n \to v$ strongly in $H^m(D)$.

Now we prove the converse. Namely, we suppose that $(H_0^m(\Omega_n))_{n\in\mathbb{N}}$ converges in the sense of Mosco to $H_0^m(\Omega)$. Then we want to prove the γ_m-convergence. Consider $f \in H^{-m}(D)$ and the associated solutions u_n of the Dirichlet problem in Ω_n. For all $n \in \mathbb{N}$,

$$\int_D |\nabla^m u_n|^2 dx = \int_{\Omega_n} \nabla^m u_n : \nabla^m u_n\, dx = \langle f, u_n\rangle.$$

We infer that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^m(D)$ since

$$|\langle f, u_n\rangle| \leqslant \|f\|_{H^{-m}(D)} \|u_n\|_{H_0^m(D)}.$$

Let $(u_{n_k})_{k\in\mathbb{N}}$ be a subsequence which weakly converges to a function $v \in H_0^m(D)$. Using the Mosco convergence, $v \in H_0^m(\Omega)$ and for all $\varphi \in H_0^m(\Omega)$, there exists a sequence

$(\varphi_k)_{k\in\mathbb{N}}$, with $\varphi_k \in H_0^m(\Omega_{n_k})$, strongly converging to φ in $H_0^m(D)$. Hence, for all $k \in \mathbb{N}$,

$$\int_D \nabla^m u_{n_k} : \nabla^m \varphi_k \, dx = \int_{\Omega_{n_k}} \nabla^m u_{n_k} : \nabla^m \varphi_k \, dx = \langle f, \varphi_k \rangle,$$

and using the strong convergence of $(\varphi_k)_{k\in\mathbb{N}}$ and the weak convergence of $(u_{n_k})_{n\in\mathbb{N}}$ as $k \longrightarrow +\infty$, we obtain

$$\int_\Omega \nabla^m v : \nabla^m \varphi \, dx = \langle f, \varphi \rangle.$$

The uniqueness of the solution of the Dirichlet problem proves $u = v$. Moreover,

$$\int_D |\nabla^m u_{n_k}|^2 dx = \langle f, u_{n_k} \rangle$$

and

$$\langle f, u_{n_k} \rangle \xrightarrow[k\to+\infty]{} \langle f, u \rangle = \int_D |\nabla^m u|^2 dx.$$

This yields

$$\|u_{n_k}\|_{H_0^m(D)} \xrightarrow[k\to+\infty]{} \|u\|_{H_0^m(D)},$$

and the convergence of the subsequence is strong. By uniqueness of the limit, the whole sequence is strongly converging to u in $H_0^m(D)$, and this achieves the proof of the γ_m-convergence, so follows the Proposition. □

Definition 6.3. — *For two closed sets $A, B \subset \mathbb{R}^N$, the Hausdorff distance $d_H(A,B)$ is defined by*

$$d_H(A,B) := \max_{x\in A} \operatorname{dist}(x,B) + \max_{x\in B} \operatorname{dist}(x,A).$$

A sequence of closed sets $(A_n)_{n\in\mathbb{N}}$ converges to A for the Hausdorff distance if $d_H(A_n, A) \longrightarrow 0$ as $n \longrightarrow +\infty$. In this case, we will write $A_n \xrightarrow{d_H} A$.

Next, we define the complementary Hausdorff distance over $\mathscr{O}(D)$ by

$$d_{H^c}(\Omega_1, \Omega_2) := d_H(\overline{D}\backslash\Omega_1, \overline{D}\backslash\Omega_2),$$

and one can show that the topolgy induced on $\mathscr{O}(D)$ is compact. In the sequel we will use the following well known result.

Proposition 6.2. — *If $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{O}(D)$ such that $\Omega_n \xrightarrow{d_{H^c}} \Omega \in \mathscr{O}(D)$, then for any compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ depending on K such that $K \subset \Omega_n$ for all $n \geqslant n_0$.*

Proof. — Since K is compact and Ω is open, we know that

$$\inf_{x\in K} \operatorname{dist}(x,\Omega^c) =: a > 0.$$

By Hausdorff convergence of the complements, there exists $n_0(a) \in \mathbb{N}$ such that for all $n \geqslant n_0(a)$,

$$\Omega_n^c \subset \{y \in \mathbb{R}^N \mid \operatorname{dist}(y,\Omega^c) < a/2\}.$$

We deduce from the triangle inequality that $\inf_{x\in K} \operatorname{dist}(x,\Omega_n^c) > 0$ for n large enough and in particular $K \subset \Omega_n$. □

We are now ready to state the following result that will directly imply Corollary 1.1 stated in the introduction.

Proposition 6.3. — *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{O}(D)$ such that $\overline{\Omega_n} \xrightarrow{d_H} \overline{\Omega}$ and $\Omega_n \xrightarrow{d_{H^c}} \Omega$ where $\Omega \in \mathscr{O}(D)$. If Ω is $(m,2)-$stable, then the sequence $(\Omega_n)_{n\in\mathbb{N}}$ γ_m-converges to Ω or equivalently, $(H_0^m(\Omega_n))_{n\in\mathbb{N}}$ converges to $H_0^m(\Omega)$ in the sense of Mosco.*

Proof of Proposition 6.3. — Under the assumptions of the Proposition we will prove the γ_m-convergence. Consider then $f \in H^{-m}(D)$. We know that the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to the Dirichlet problem associated with f in $H_0^m(\Omega_n)$, is bounded in $H_0^m(D)$. Therefore there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that weakly converges to a function $v \in H_0^m(D)$. Let $\varphi \in \mathscr{C}_c^\infty(\Omega)$ be a test function. By complementary Hausdorff convergence, there exists an integer $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$,

$$\operatorname{Supp}(\varphi) \subseteq \Omega_{n_k}.$$

Thus, for all $k \geqslant k_0$,

$$\int_{\Omega_{n_k}} \nabla^m u_{n_k} : \nabla^m \varphi \, dx = \langle f, \varphi\rangle,$$

and by weak convergence of $(u_{n_k})_{k\geqslant k_0}$ in $H_0^m(D)$,

$$\int_{\Omega} \nabla^m v : \nabla^m \varphi \, dx = \langle f, \varphi\rangle. \tag{6.1}$$

Let us now prove that $v \in H_0^m(\Omega)$. Up to a subsequence, we can assume that $(u_{n_k})_{k\in\mathbb{N}}$ converges almost everywhere to v. The functions u_{n_k} vanishes $(m,2)-$quasi everywhere on $\overline{\Omega}_{n_k}^c$ so almost everywhere. By Hausdorff convergence of the adherence we know that for all compact $K \subset \overline{\Omega}^c$ then $K \subset \overline{\Omega}_n^c$ for n large enough thus finaly $v = 0$ almost everywhere in $\overline{\Omega}^c$. Using the definition of $(m,2)-$stability, we conclude that $v \in H_0^m(\Omega)$. Therefore $v = u$, the unique solution to the Dirichlet problem associated to f in $H_0^m(\Omega)$.

To conclude the proof it remains to show the strong convergence in $H_0^m(D)$. By density the equality (6.1) stays true for $\varphi \in H_0^m(\Omega)$. In particular for $\varphi = v$ we get

$$\int_\Omega |\nabla^m v|^2 \, dx = \langle f, v \rangle.$$

On the other hand by weak convergence we have

$$\int_{\Omega_{n_k}} |\nabla^m u_{n_k}|^2 \, dx = \langle f, u_{n_k} \rangle \xrightarrow[k \to +\infty]{} \langle f, v \rangle.$$

In other words $\|\nabla^m u_{n_k}\|_{L^2(D)} \to \|\nabla^m v\|_{L^2(D)}$ which together with the weak convergence, proves the strong convergence of u_{n_k} to v in $H^m(D)$. At the end, from the uniqueness of the possible limit we infer that the whole sequence converges to v, not only a subsequence. This achieves the proof of γ_m-convergence. □

7. Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. — Let $\Omega, \Omega_n \subset D$ be bounded domains as in the statement of Theorem 1.2 that satisfies

$$\Omega_n \xrightarrow{d_{H^c}} \Omega,$$

and such that (1.1) holds true for all Ω_n with the same $\delta_0 > 0$ and $r_0 > 0$. We want to prove that Ω_n γ_m-converges to Ω. To this aim we start with a similar argument as in the proof of Proposition 6.3. Consider $f \in H^{-m}(D)$. We know that the sequence $(u_n)_{n \in \mathbb{N}}$ of the solutions to the Dirichlet problem associated to f in Ω_n is bounded in $H_0^m(D)$. There exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ which weakly converges to a function $v \in H_0^m(D)$. Let $\varphi \in \mathscr{C}_c^\infty(\Omega)$ be test function. By complementary Hausdorff convergence, there exists an integer $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$,

$$\mathrm{Supp}(\varphi) \subseteq \Omega_{n_k}.$$

Thus, for all $k \geqslant k_0$,

$$\int_\Omega \nabla^m u_{n_k} : \nabla^m \varphi \, dx = \int_{\Omega_{n_k}} \nabla^m u_{n_k} : \nabla^m \varphi \, dx = \langle f, \varphi \rangle$$

and by weak convergence of $(u_{n_k})_{k \geqslant k_0}$ in $H_0^m(D)$,

$$\int_\Omega \nabla^m v : \nabla^m \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Now argument as in the proof of Proposition 6.3, in order to conclude it suffices to prove that $v \in H_0^m(\Omega)$. In particular, the strong convergence would then follow by use of the same argument as in the proof of Proposition 6.3.

Thus let us prove that $v \in H^m_0(\Omega)$. Since Ω satisfies the $(\delta_0, r_0)-$capacitary condition we know from Theorem 1.1 that Ω is a $(m,2)-$stable domain. Thus we are left to prove that $v = 0$ a.e. in $\overline{\Omega}^c$. From here the proof differs from the one of Proposition 6.3 because we do not know anymore that $\overline{\Omega_n} \xrightarrow{d_H} \overline{\Omega}$. Instead, we shall benefit from the fact that (1.1) holds true for the whole sequence Ω_n and we will use a construction similar to the one used in the proof of Theorem 1.1, but on the functions u_n. From now on we will simply denote by n instead of n_k for the subsequence $u_n \to v$ in $H^m(D)$ as $n \longrightarrow +\infty$. Let $K \subset \overline{\Omega}^c$ be an arbitrary compact set and let $\varepsilon > 0$ be given. Our goal is to prove that $v = 0$ a.e. on K. For a general closed set $F \subset \mathbb{R}^N$ and $\lambda > 0$ we denote by $(F)_\lambda$ the $\lambda-$enlargement of F, namely,

$$F_\lambda := \left\{ x \in \mathbb{R}^N \;\middle|\; \operatorname{dist}(x, F) \leqslant \lambda \right\}.$$

By the Hausdorff convergence of Ω_n^c to Ω^c we know that there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geqslant n_0(\varepsilon)$,

$$\Omega^c \subset (\Omega_n^c)_\varepsilon, \text{ and } \Omega_n^c \subset (\Omega^c)_\varepsilon.$$

From the above we deduce that

$$K \subset \Omega^c \subset (\Omega_n^c)_\varepsilon \subset (\Omega^c)_{2\varepsilon}. \tag{7.1}$$

Next, we want to construct a test function in $\mathscr{C}_c^\infty(\Omega_n)$ which is very close to u_n in L^2 and equal to 0 on K. Let us consider the following subset of Ω_n,

$$A_{n,\varepsilon} := \{ x \in \Omega_n \mid d(x, \Omega_n^c) \geqslant 10\varepsilon \},$$

and the function

$$w_{n,\varepsilon} := u_n \mathbf{1}_{A_{n,\varepsilon}}.$$

The main point being that $w_{n,\varepsilon} = 0$ in $(\Omega_n^c)_\varepsilon$ and in virtue of (7.1) we deduce that $w_{n,\varepsilon} = 0$ on K. Now we estimate the difference $w_{n,\varepsilon} - u_n$ in $L^2(\mathbb{R}^N)$ using a covering of $\partial\Omega_n$. More precisely, the infinite family $(B(x, 20\varepsilon))_{x \in \partial\Omega_n}$ is a cover of $\Omega_n \setminus A_{n,\varepsilon}$ and by the *5B*-covering Lemma there exists a countably subcover indexed by $(x_i)_{i \in \mathbb{N}} \subset \partial\Omega$ such that $(B(x_i, 20\varepsilon))_{i \in \mathbb{N}}$ is a disjoint family,

$$\Omega_n \setminus A_{n,\varepsilon} \subset \bigcup_{i \in \mathbb{N}} B(x_i, 100\varepsilon), \text{ and } \sum_{i \in \mathbb{N}} \mathbf{1}_{B(x_i, 100\varepsilon)} \leqslant N_0,$$

for a universal constant $N_0 \in \mathbb{N}$. Then we can estimate,

$$\int_D |w_{n,\varepsilon} - u_n|^2 \, dx \leqslant \int_{\Omega_n \setminus A_{n,\varepsilon}} |u_n|^2 \, dx.$$

The functions $\partial^\beta u_n$ vanishes almost everywhere on the open set $\overline{\Omega_n}^c$, so thanks to our capacitary condition (1.1) we have for ε small enough

$$(100\varepsilon)^{-(N-p)} \operatorname{Cap}_{1,2}(Z(\partial^\beta u_n)) \geqslant C\varepsilon^{-(N-p)} \operatorname{Cap}_{1,2}(\overline{\Omega_n}^c \cap B(0, 100\varepsilon)) \geqslant C\delta_0.$$

Therefore, the Poincaré inequality (2.2) applied to $\partial^\beta u_n$ in all ball $B(x_i, 100\varepsilon)$ gives

$$\int_{B(x_i,100\varepsilon)} |\partial^\beta u_n|^2 \, dx \leqslant C\delta_0^{-1}\varepsilon^2 \int_{B(x_i,100\varepsilon)} |\nabla\partial^\beta u_n|^2 \, dx. \tag{7.2}$$

We deduce that

$$\begin{aligned}\int_{\Omega_n\setminus A_{n,\varepsilon}} |u_n|^2 \, dx &\leqslant \sum_{i\in\mathbb{N}} \int_{B(x_i,100\varepsilon)} |u_n|^2 \, dx \\ &\leqslant C\sum_{i\in\mathbb{N}} \varepsilon^{2m} \int_{B(x_i,100\varepsilon)} |\nabla^m u_n|^2 \, dx \\ &\leqslant CN_0\varepsilon^{2m} \int_D |\nabla^m u_n|^2 \, dx \\ &\leqslant C\varepsilon^2,\end{aligned}$$

since the sequence u_n is uniformly bounded in $H^1(D)$. In conclusion we have proved the following : for each $\varepsilon > 0$, we have $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geqslant n_0(\varepsilon)$, there exists $w_{n,\varepsilon} \in L^2(D)$ such that $\|w_{n,\varepsilon} - u_n\|_{L^2} \leqslant C\varepsilon$ and $w_{n,\varepsilon} = 0$ on K. Now for n sufficiently large let $\varepsilon = 2^{-n}$ and let $w_n := w_{n_0(2^{-n}),2^{-n}}$. We can assume that $n_0(2^{-n}) \to +\infty$. The function w_n converges to v in L^2 because u_n converges to v in L^2, and $w_n = 0$ on K for all $n \in \mathbb{N}$. Therefore, up to a subsequence, w_n converges a.e. on K and this shows that $u = 0$ a.e. on K. Since K is arbitrary, this shows that $v = 0$ a.e. on $\overline{\Omega}^c$, hence $u \in H_0^m(\Omega)$ because Ω is $(m, 2)-$stable. This achieves the proof. □

8. Existence for shape optimisation problems under geometrical constraints

Let $D \subset \mathbb{R}^N$ be a fixed bounded open set and let $\mathscr{O}_D := \{\Omega \subseteq D \mid \Omega \text{ is open}\}$ denote all open subsets of D. For a shape functional $F : \mathscr{O}_D \longrightarrow \mathbb{R}^+$, it is a natural question to ask if there exists extremal points. In order to answer this question, we introduce a subfamily of $\mathscr{O}_D$ which is compact for the γ_m-convergence and satisfies the capacitary condition (1.1). If F is lower semi-continuous for the γ_m-convergence, then we use Theorem 1.2 to conclude.

Proposition 8.1. — *Any class of the following list is compact for the complementary Hausdorff convergence :* $\mathscr{O}_{\text{convex}}(D)$, $\mathscr{O}^{\varepsilon}_{\text{cone}}(D)$, $\mathscr{O}^{a,r_0}_{\text{corks}}(D)$.

Proof. —
1. *Case* $\mathscr{O} = \mathscr{O}_{\text{convex}}(D), \mathscr{O}^{\varepsilon}_{\text{cone}}(D)$. The proof can be founded in [4, Proposition 5.1.1, page 126].
2. *Case* $\mathscr{O} = \mathscr{O}^{a,r_0}_{\text{corks}}(D)$. Suppose $(\Omega_n)_{n\in\mathbb{N}}$ is sequence of $(a, r_0)-$corkscrew domains which converges to an open set $\Omega \subset D$. Let $x \in \partial\Omega$ and $r \leqslant r_0$. By Hausdorff complementary convergence properties, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \partial\Omega_n$ and $x_n \longrightarrow x$ as $n \longrightarrow +\infty$. By corkscrew conditions, one finds

$B(y_n, ar) \subset \overline{\Omega}_n^c \cap B(x_n, r)$ with, up to a subsequence, $y_n \longrightarrow y$ as $n \longrightarrow +\infty$. First of all, it is obvious that $B(y, ar) \subset B(x, r)$, it remains to prove that $B(y, ar) \subset \overline{\Omega}^c$. Let $\varepsilon > 0$, from the enlargement characterisation of Hausdorff convergence, there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $n \geqslant N(\varepsilon)$, $\overline{D}\backslash\Omega_n \subset (\overline{D}\backslash\Omega)_\varepsilon$ where

$$(\overline{D}\backslash\Omega)_\varepsilon := \left\{x \in \mathbb{R}^N \;\middle|\; \operatorname{dist}(x, \overline{D}\backslash\Omega) \leqslant \varepsilon\right\}.$$

Thus $B(y_n, ar) \subset (\overline{D}\backslash\Omega)_\varepsilon$ and passing to the limit as $n \to +\infty$, then taking the intersection in ε, we get

$$B(y, ar) \subset \bigcap_{\varepsilon>0} (\overline{D}\backslash\Omega)_\varepsilon = \overline{D}\backslash\Omega.$$

The ball $B(y, ar)$ is open so we conclude $B(y, ar) \subset \overline{\Omega}^c \cap B(x, r)$.

□

Proposition 8.2. — *Any class of the following list is compact for the γ_m−convergence: $\mathscr{O}_{\text{convex}}(D)$, $\mathscr{O}^{\varepsilon}_{\text{cone}}(D)$, $\mathscr{O}^{a,r_0}_{\text{corks}}(D)$.*

Proof. — Let $\mathscr{O}$ be one of the class of domains listed above and let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{O}$. Because of the compactness of the complementary Hausdorff convergence in $\mathscr{O}(D)$, there exists a subsequence $(\Omega_n)_{n\in\mathbb{N}}$ denoted by the same indices which d_{H^c}−converges to $\Omega \in \mathscr{O}(D)$. Using Theorem 1.2 and Proposition 5.1, it is sufficient to prove that $\Omega \in \mathscr{O}$, i.e. $\mathscr{O}$ is closed for d_{H^c}−convergence. Proposition 8.1 concludes the proof. □

Theorem 8.1. — *Let $\mathscr{O}$ be a γ_m−compact class of subset listed in Proposition 8.2. Let $F : \mathscr{O} \longrightarrow \overline{\mathbb{R}}$ be a lower semi-continuous functional for the γ_m−convergence. There exists $\Omega \in \mathscr{O}$ such that*

$$F(\Omega) = \inf\left\{F(\omega) \mid \omega \in \mathscr{O}\right\}.$$

Proof. — Let $(\Omega_n)_{n\in\mathbb{N}}$ be a minimising sequence in $\mathscr{O}$, i.e. $F(\Omega_n)$ converges to $\inf\left\{F(\omega) \mid \omega \in \mathscr{O}\right\}$ as $n \longrightarrow +\infty$. Using Proposition 8.2, up to a subsequence there exists an open set $\Omega \in \mathscr{O}$ such that

$$\Omega_n \xrightarrow[n\longrightarrow+\infty]{\gamma_m} \Omega.$$

Then by lower semi-continuity of the functional we get

$$F(\Omega) \leqslant \liminf_{n\to+\infty} F(\Omega_n) = \inf\left\{F(\omega) \mid \omega \in \mathscr{O}\right\},$$

which finishes the proof. □

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