ON THE SHARP MAKAI INEQUALITY

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Dedicated to Prof. Giuseppe Buttazzo

Abstract. On a convex bounded open set, we prove that Poincaré–Sobolev constants for functions vanishing at the boundary can be bounded from below in terms of the norm of the distance function in a suitable Lebesgue space. This generalizes a result shown, in the planar case, by E. Makai, for the torsional rigidity. In addition, we compare the sharp Makai constants obtained in the class of convex sets with the optimal constants defined in other classes of open sets. Finally, an alternative proof of the *Hersch–Protter inequality* for convex sets is given.

Contents

1. Introduction

The aim of this paper is to provide a sharp lower bound for the quantity

$$\lambda_{p,q}(\Omega)\,\|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)}, \tag{1.1}$$

where $\Omega \subsetneq \mathbb{R}^N$ is an convex bounded open set, $1 \le q < p < \infty$ or $1 < q = p < \infty$, $\lambda_{p,q}(\Omega)$ is the *generalized principal frequency* defined as

$$\lambda_{p,q}(\Omega) := \inf_{\psi \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla \psi|^p\,dx \,:\, \int_\Omega |\psi|^q\,dx = 1 \right\},$$

and d_Ω is the distance function from the boundary $\partial\Omega$, namely

$$d_\Omega(x) := \inf\big\{|x-y| : y \in \partial\Omega\big\}.$$

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Here and in what follows, $L^{\frac{p\,q}{p-q}}(\Omega)$ stands for $L^\infty(\Omega)$ when $p=q$ and we will write $\lambda_p(\Omega)$ in place of $\lambda_{p,p}(\Omega)$.

Our study is motivated by an old result due to Makai (see [21]) for the torsional rigidity

$$T(\Omega)=\frac{1}{\lambda_{2,1}(\Omega)},$$

which asserts that, for every convex bounded open set $\Omega\subsetneq\mathbb{R}^2$, the following sharp upper bound holds

$$T(\Omega)\le\int_\Omega d_\Omega^2\,dx.$$

1.1. **Optimal lower bound on convex sets: the main result.** Inspired by the above *Makai inequality*, in this paper we will prove the following theorem which extends, to every dimension N and every $1\le q<p<\infty$, the optimal lower bound given by Makai in the case $q=1$ and $p=2$.

Theorem 1.1 (Makai's inequality)**.** *Let $1\le q<p<\infty$ and let $\Omega\subsetneq\mathbb{R}^N$ be a convex bounded open set. Then, the following lower bound holds*

$$\lambda_{p,q}(\Omega)\ge\frac{C_{p,q}}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}},\tag{1.2}$$

where $C_{p,q}$ is the positive constant given by

$$C_{p,q}=\left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}},\tag{1.3}$$

with

$$\pi_{p,q}:=\inf_{u\in C_0^\infty((0,1))}\left\{\|u'\|_{L^p([0,1])}\,:\,\|u\|_{L^q([0,1])}=1\right\}.\tag{1.4}$$

Moreover, the estimate (1.2) is sharp.

The proof of Theorem 1.1 is inspired by the covering argument for polygonal sets exploited by Makai in the planar case. In the N-dimensional case, thanks to a standard approximation argument, we can restrict ourselves to consider the case when $\Omega\subsetneq\mathbb{R}^N$ is the interior of a polytope K (see Section 3). In this case, in order to prove (1.2), the key tool we use is given by Lemma 2.5 where we construct a suitable covering of Ω by means of convex sets Ω_i, every one satisfying the property that $\partial\Omega_i\cap\Omega$ is the graph of a continuous function defined on a facet S_i of the polytope K. The proof of the convexity of each set Ω_i relies on the concavity property of the distance function d_Ω (see [3]).

If we denote by r_Ω the *inradius* of Ω, which coincides with supremum of the distance function d_Ω, as an application of Theorem 1.1, we can give a different proof of the following sharp estimate (1.5), first proved in [8, Theorem 1.1] when $1\le q<2$ and then extended to cover the case $p\ne2$ and $q=1$ in [15, Theorem 4.3]. The general case $1\le q<p<\infty$ was first shown in [9, Theorem 5.7] by means of a comparison argument.

Corollary 1.2 (Hersch-Protter–type inequality)**.** *Let $1\le q<p<\infty$ and $\Omega\subsetneq\mathbb{R}^N$ be a convex bounded open set. Then, the following lower bound holds*

$$\lambda_{p,q}(\Omega)\,|\Omega|^{\frac{p-q}{q}}\ge\left(\frac{\pi_{p,q}}{2}\right)^p\frac{1}{r_\Omega^{\,p}}.\tag{1.5}$$

Moreover, the estimate (1.5) *is sharp.*

A further result that follows from (1.5), simply taking the limit as $q \nearrow p$[1], is the sharp inequality

$$\lambda_p(\Omega) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_\Omega^p}, \tag{1.6}$$

valid for every convex bounded open set $\Omega \subsetneq \mathbb{R}^N$. Here π_p is defined as in (1.4) with $p = q$. Formula (1.6) represents the extension to the case of the p-Laplacian of the *Hersch–Protter inequality*, originally proved by Hersch in [18, Théorème 8.1] for $p = N = 2$ and then generalized to every dimension $N \geq 2$ by Protter in [25, Theorem 2] (see also [12]). The case $p \neq 2$ has been already obtained in [7, 14, 19]. In Section 4, we will give a further alternative proof of (1.6) by exploiting a change of variables formula (4.2), proved in [13, Theorem 7.1] when the domain of integration is a connected open set of class C^2, and then using a suitable approximation result for convex sets.

1.2. **Lower bounds on other classes of open sets.** The final part of this paper is devoted to compare the optimal constant $C_{p,q}$ for convex sets, defined by (1.3), with the infimum of (1.1) in other classes of open sets of $\mathbb{R}^N$. First of all, for every $1 \leq q < p < \infty$ or $1 < q = p < \infty$, we introduce the constant

$$\widetilde{C}_{p,q} = \inf\left\{\lambda_{p,q}(\Omega)\, \|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)} : \Omega \subsetneq \mathbb{R}^N \text{ is an open set, } d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega)\right\}. \tag{1.7}$$

The condition on d_Ω is motivated by the recent results in [10] where, by means of a comparison principle provided in [9] for the sub-homogeneous Lane–Emden equation, it is shown that, when $\Omega \subsetneq \mathbb{R}^N$ is an open set and $1 \leq q < p < \infty$ or $1 < p = q < \infty$, then the following implication holds

$$\lambda_{p,q}(\Omega) > 0 \qquad \Longrightarrow \qquad d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega)$$

(see [10, Theorems 5.1 and 5.4]). Moreover, in the same paper, following an argument used in [5] when $p = 2$, the above implication is shown to be an equivalence in the class of the open sets $\Omega \subsetneq \mathbb{R}^N$ which satisfy the Hardy inequality

$$\int_\Omega |\nabla u|^p\, dx \geq C \int_\Omega \frac{|u|^p}{d_\Omega^p}\, dx, \qquad \text{for every } u \in C_0^\infty(\Omega), \tag{1.8}$$

with $1 < p < \infty$ and $C = C(p, \Omega) > 0$.
Indeed, if

$$d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega) \qquad \text{for } 1 \leq q < p < \infty \text{ or } 1 < q = p < \infty, \tag{1.9}$$

then, the joint application of the Hölder inequality and of (1.8), gives

$$\begin{aligned}\int_\Omega |u|^q\, dx &\leq \left(\int_\Omega \frac{|u|^p}{d_\Omega^p}\, dx\right)^{\frac{q}{p}} \|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)} \\ &\leq \frac{1}{C^{\frac{q}{p}}} \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{q}{p}} \|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)}, \qquad \text{for every } u \in C_0^\infty(\Omega),\end{aligned}$$

that implies

$$\lambda_{p,q}(\Omega) \geq \frac{\mathfrak{h}_p(\Omega)}{\|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)}}, \tag{1.10}$$

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[1] We use here that $q \mapsto \lambda_{p,q}(\Omega)$ is left-continuous at $q = p$ when $\Omega \subset \mathbb{R}^N$ is a bounded open set.

where

$$\mathfrak{h}_p(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\, dx \ : \ \int_\Omega \frac{|u|^p}{d_\Omega^p}\, dx = 1 \right\}.$$

We recall that, when $\Omega \subsetneq \mathbb{R}^N$ is a convex bounded open set, it is well known that

$$\mathfrak{h}_p(\Omega) = \left(\frac{p-1}{p}\right)^p,$$

(for a proof, see [23, Theorem 11]). The resulting estimate (1.10) in the class of the convex bounded open sets is far of being sharp, as the case $q = 1$ and $p = N = 2$ shows, being $C_{2,1} = 1 > \frac{1}{4}$.

Now, the constants $\widetilde{C}_{p,q}$ defined in (1.7) satisfy the following facts:

- when $1 < p \le N$, it holds that

$$\widetilde{C}_{p,q} = 0, \qquad \text{for every } 1 \le q < p \text{ or } q = p. \tag{1.11}$$

  Indeed, if $1 < p \le N$, fixed a bounded open subset $\Omega \subsetneq \mathbb{R}^N$, we remove from it a periodic array of n points and we call Ω_n the open set so constructed. As $p \le N$, points in $\mathbb{R}^N$ have zero p-capacity (see [27, Chapter 17]), then, for every $1 \le q < p \le N$ or $1 < q = p \le N$, it holds

$$\lambda_{p,q}(\Omega_n) = \lambda_{p,q}(\Omega), \qquad \text{for every } n \in \mathbb{N}.$$

  Since r_{Ω_n} tends to 0, as $n \to \infty$, the above equality implies that

$$\widetilde{C}_{p,q} \le \limsup_{n\to\infty} \lambda_{p,q}(\Omega_n)\, \|d_{\Omega_n}\|^p_{L^{\frac{p\,q}{p-q}}(\Omega_n)} \le \lambda_{p,q}(\Omega)\, |\Omega|^{\frac{p-q}{q}} \limsup_{n\to\infty} r^p_{\Omega_n} = 0,$$

  for every $1 \le q < p \le N$ or $1 < q = p \le N$. In particular, in this range, it follows that

$$\widetilde{C}_{p,q} < C_{p,q}; \tag{1.12}$$

- when $p > N$, as shown independently by Lewis in [20] and Wannebo in [29], every open subset $\Omega \subsetneq \mathbb{R}^N$ satisfies the Hardy inequality (1.8) and it holds

$$\mathfrak{h}_p(\Omega) \ge \left(\frac{p-N}{p}\right)^p,$$

  (for the latter, see [4] and [17]). By using the above lower bound in (1.10), we get that

$$\widetilde{C}_{p,q} \ge \left(\frac{p-N}{p}\right)^p > 0, \qquad \text{for every } N < p < \infty \text{ and } 1 \le q < p, \tag{1.13}$$

  and the natural question that arises is whether the strict inequality (1.12) also holds in the case $p > N$, for $1 \le q < p$ or $q = p$. We will address Section 5.1 to discuss this question and, by means of a perturbative argument, we will be able to show that, for every $N \ge 2$ and for every fixed $1 \le q < N$, there exists $\overline{p} = \overline{p}(q) > N$ such that (1.12) holds for every $p \in (q, \overline{p}]$.

Another interesting class of sets where the quantity (1.1) is bounded from below by a positive constant is that one of planar simply connected open sets. Indeed, if $p = N = 2$, thanks to a result due to Ancona (see [2]), every simply connected open set $\Omega \subsetneq \mathbb{R}^2$ verifies the Hardy inequality (1.8) with the optimal Hardy constant satisfying

$$\mathfrak{h}_2(\Omega) \ge \frac{1}{16}.$$

Hence, for $N=2$ and every $1\le q\le 2$, defined
(1.14)
$$\widehat{C}_{2,q}=\inf\left\{\lambda_{2,q}(\Omega)\,\|d_\Omega\|^2_{L^{\frac{2q}{2-q}}(\Omega)}\,:\,\Omega\subsetneq\mathbb{R}^2 \text{ is a simply connected open set}, d_\Omega\in L^{\frac{2q}{2-q}}(\Omega)\right\},$$
the joint application of (1.10) and (1.11) implies that
$$\widehat{C}_{2,q}\ge\frac{1}{16}>0=\widetilde{C}_{2,q},\qquad \text{for every } 1\le q\le 2.$$
By using a different argument, in [22] Makai shows that
$$\frac{1}{4}\le\widehat{C}_{2,2}<\frac{\pi^2}{4}=C_{2,2},$$
and, in order to prove the upper bound, he exhibits a simply connected open set $\Omega\subsetneq\mathbb{R}^2$ satisfying
$$\lambda_2(\Omega)\,r_\Omega^2<\frac{\pi^2}{4}. \tag{1.15}$$
In Section 5.2, we use this fact to show that there exists $1\le\overline{q}<2$ such that it holds
$$\widehat{C}_{2,q}<C_{2,q},\qquad \text{for every } q\in[\overline{q},2].$$
In addition, by exploiting (1.15), we finally prove that, in the case $N=2$, there exists $\overline{p}>2$ such that
$$0<\widetilde{C}_{p,p}<\left(\frac{\pi_p}{2}\right)^p=C_{p,p},\qquad \text{for every } p\in(2,\overline{p}].$$

1.3. **Plan of the paper.** In Section 2 we introduce some basic properties of polytopes in $\mathbb{R}^N$ and we extend, to any dimension $N\ge 2$, the covering argument applied by Makai in [21]. In Section 3, we prove the main results stated in Theorem 1.1 and in Corollary 1.2. In the subsequent Section 4, we give an alternative proof for the Hersch–Protter inequality (1.6), by means of a change of variables formula. Finally, in Section 5, we compare the sharp constant for the Makai inequality on convex sets with the optimal constants for other class of sets; in particular, we consider the class of general open sets whose distance function satisfies (1.9) and that one of planar simply connected sets.

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2. Preliminaries

For every convex set $C\subset\mathbb{R}^N$, we will denote by $\mathrm{relint}(C)$ and $\mathrm{relbd}(C)$ the *relative interior* and the *relative boundary* of C, once we regard it as a subset of its affine hull. We define the dimension of C, and we denote it by $\dim(C)$, as the dimension of its affine hull. Conventionally, the empty set has dimension -1.

Definition 2.1. Let $C \subset \mathbb{R}^N$ be a not empty convex closed set. A convex subset $S \subseteq C$ is a face of C if each segment $[x, y] \subset C$ satisfying $S \cap \operatorname{relint}([x, y]) \neq \emptyset$ is contained in S.
We denote by $\mathcal{F}(C)$ the set of all faces of C and by $\mathcal{F}_i(C)$ the set of all faces of C having dimension i, for every $0 \leq i \leq \dim(C) - 1$. The empty set and C itself are faces of C; the other faces are called *proper*. Every $(\dim(C) - 1)$-dimensional face of C is called a *facet* of C.

We can summarise the main properties of faces in the following theorem (see [26, Section 2.1]).

Theorem 2.2. *Let $C \subset \mathbb{R}^N$ be a not empty convex bounded set. Then*
(1) *the faces of C are closed;*
(2) *if $F \neq C$ is a face of C, then $F \cap \operatorname{relint}(C) = \emptyset$;*
(3) *if G and F are faces of C, then $G \cap F$ is a face of C;*
(4) *if G is a face of F and F is a face of C, then G is a face of C;*
(5) *each point $x \in C$ is contained in $\operatorname{relint}(F)$ for a unique face $F \in \mathcal{F}(C)$.*

Definition 2.3. We say that $K \subset \mathbb{R}^N$ is a polytope if it is the convex hull of finitely many points of $\mathbb{R}^N$.

We recall that, thanks to [26, Theorem 1.1.11 and Theorem 2.4.7], a polytope is a compact convex set. Moreover, each proper face of K is itself a polytope and is contained in some facet of K.

Furthermore, we recall that if K is a polytope and $0 \in \operatorname{int}(K)$, then the following facts hold:
(i) the *polar set* $K^\circ = \{x \in \mathbb{R}^N : \langle x, y\rangle \leq 1, \text{ for every } y \in K\}$ is itself a polytope;
(ii) if F is a face of K then the conjugate set $\widehat{F} = \{x \in K^\circ : \langle x, y\rangle = 1, \text{ for every } y \in F\}$ is itself a polytope such that
$$\dim(\widehat{F}) = N - \dim(F) - 1;$$
(iii) if $F, G \in \mathcal{F}(K)$ are such that $F \subset G$, then $\widehat{F} \supset \widehat{G}$;
(iv) the application $F \mapsto \widehat{F}$ is a bijection from $\mathcal{F}(K)$ to $\mathcal{F}(K^\circ)$.

In the sequel we need the following result.

Lemma 2.4. *Let K be a polytope and assume that $0 \in \operatorname{int}(K)$. Then,*
(1) *if S, S' are facets of K such that $S \neq S'$ then $S \cap \operatorname{relint}(S') = \emptyset$;*
(2) *for every facet S of K, if $z \in \operatorname{relbd}(S)$, then there exists another facet $\widetilde{S} \neq S$ of K such that $z \in \widetilde{S}$. In particular, $z \in \operatorname{relbd}(S) \cap \operatorname{relbd}(\widetilde{S})$.*

Proof. (1) Without loss of generality, we assume that $F = S \cap S' \neq \emptyset$. Since $S \neq S'$, we have that F is a proper face of S'. Then, by Part (2) of Theorem 2.2, we have
$$S \cap \operatorname{relint}(S') \subset S \cap S' \cap \operatorname{relint}(S') = F \cap \operatorname{relint}(S') = \emptyset;$$
(2) let S be a facet of K and let $z \in \operatorname{relbd}(S)$. Then, there exists a facet F of S such that $z \in F$. Then, $F \in \mathcal{F}(K)$ and $\dim(F) = N - 2$. This implies $\dim(\widehat{F}) = 1$. Therefore, there exist exactly two points x', x'', such that $x' \neq x''$ and
$$\widehat{F} = [x', x''].$$
Then, thanks to property (iv) above, there exist two facets $S', S'' \in \mathcal{F}(K)$, with $S' \neq S''$, such that $\widehat{S'} = \{x'\}$ and $\widehat{S''} = \{x''\}$, and, by property (ii) above,
$$z \in F \subseteq \widehat{\{x'\}} \cap \widehat{\{x''\}} = S' \cap S'',$$

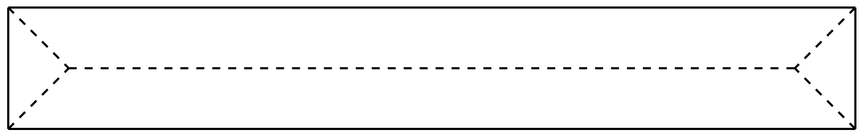

FIGURE 1. A polytope $\Omega \subset \mathbb{R}^2$ divided into subsets Ω_i, with $i \in \{1, \ldots, 4\}$.

namely, there exists at least another facet $\widetilde{S} \neq S$ containing z. Since $z \in \mathrm{relbd}(S) \cap \widetilde{S} \subseteq S \cap \widetilde{S}$ and, thanks to Part (1) of this lemma, $S \cap \mathrm{relint}(\widetilde{S}) = \emptyset$, we can conclude that $z \in \mathrm{relbd}(S) \cap \mathrm{relbd}(\widetilde{S})$.

$\square$

In the next lemma, we extend to every dimension the argument applied by Makai in [21] in order to divide the interior of a polytope K, when $\mathrm{int}(K) \neq \emptyset$, in a finite number of suitable convex open subsets.

Lemma 2.5. *Let $N \geq 2$ and $K \subset \mathbb{R}^N$ be a polytope such that $0 \in \mathrm{int}(K)$. Let $\Omega = \mathrm{int}(K)$ and let $S_1, S_2, \ldots, S_h$ be the facets of K. For every $i \in \{1, \ldots, h\}$, let $\Pi_i : \mathbb{R}^N \to H_i$ be the orthogonal projection on the affine hyperplane H_i containing S_i. Define*

$$\Omega_i = \Big\{ x \in \Omega : d_{S_i}(x) = d_\Omega(x) \Big\}, \tag{2.1}$$

where

$$d_{S_i}(x) = \min_{y \in S_i} |x - y|.$$

Then, for every $i \in \{1, \ldots, h\}$, the following facts hold:

1. *if $x \in \Omega_i$ then $\Pi_i(x) \in S_i$. In particular $\Pi_i(x)$ is the unique minimizer of the problem defining $d_{S_i}(x)$;*
2. *$\Pi_i(x) \in \mathrm{relint}(S_i)$, for every $x \in \Omega_i$;*
3. *Ω_i is a convex set;*
4. *$\mathrm{int}(\Omega_i) \neq \emptyset$;*
5. *Ω_i can be included in a rectangle with base S_i and height r_Ω;*
6. *for every $x_0 \in \mathrm{relint}(S_i)$, there exists a unique $y_0 \in \partial\Omega_i \cap \Omega$ such that $\Pi_i(y_0) = x_0$.*

In particular, the restriction $\Pi_i : \partial\Omega_i \cap \Omega \to \mathrm{relint}(S_i)$ is a continuous bijection.

Proof. First of all, we notice that, since $\mathrm{int}(K) \neq \emptyset$, we have that $\overline{\Omega} = K$. Now, we prove every part separately. Let us fix $i \in \{1, \ldots, h\}$.

1. Let $x \in \Omega_i$. By contradiction, if $\Pi_i(x) \notin K$, then the segment $[x, \Pi_i(x)]$ would intersect $\partial\Omega$ in a point z. Since $z \neq \Pi_i(x)$, then $z \in S_j$, with $S_j \neq S_i$. Hence, we would have that
$$d_\Omega(x) \leq d_{S_j}(x) \leq |x - z| < |x - \Pi_i(x)| \leq d_{S_i}(x) = d_\Omega(x),$$
which is a contradiction. In particular, using the fact that $\Pi_i(x) \in S_i$, we get that
$$d_{S_i}(x) = \min_{y \in S_i} |x - y| \leq |x - \Pi_i(x)| = d_{H_i}(x) \leq d_{S_i}(x)$$
that is, $\Pi_i(x)$ is the unique minimizer of the problem defining $d_{S_i}(x)$;

(2) by contradiction, let us suppose that $\Pi_i(x) \in S_i \setminus \mathrm{relint}(S_i) = \mathrm{relbd}(S_i)$. By Lemma 2.4, Part (2), there exists $S_j \neq S_i$ such that $\Pi_i(x) \in S_j$. Since

$$d_{S_j}(x) \leq |x - \Pi_i(x)| = d_{S_i}(x) = d_\Omega(x) \leq d_{S_j}(x),$$

we get that

$$d_{S_j}(x) = |x - \Pi_i(x)|.$$

Being $\Pi_i(x) \in S_j$, by uniqueness, we would obtain $\Pi_j(x) = \Pi_i(x) \in H_i \cap H_j$. Then $H_i \equiv H_j$, giving the contradiction $S_i = S_j$;

(3) by contradiction, assume that there exist $x, y \in \Omega_i$ and $\lambda \in (0,1)$ such that $z = \lambda\, x + (1 - \lambda)\, y \notin \Omega_i$. Hence $z \in \Omega_j$ for some $j \neq i$, and

$$d_\Omega(z) = d_{S_j}(z) < d_{S_i}(z). \tag{2.2}$$

Since Ω is a convex set, then the distance function d_Ω is a concave function (see [3]), hence, it follows that

$$d_\Omega(z) = d_\Omega(\lambda\, x + (1 - \lambda)\, y) \geq \lambda\, d_\Omega(x) + (1 - \lambda)\, d_\Omega(y). \tag{2.3}$$

On the other hand, by Part (1), we have that $\Pi_i(x), \Pi_i(y) \in S_i$. By linearity, we get that

$$\Pi_i(z) = \lambda\, \Pi_i(x) + (1 - \lambda)\, \Pi_i(y) \in S_i,$$

which implies that

$$d_{S_i}(z) = |z - \Pi_i(z)| \leq \lambda |x - \Pi_i(x)| + (1 - \lambda)|y - \Pi_i(y)| = \lambda\, d_\Omega(x) + (1 - \lambda)\, d_\Omega(y). \tag{2.4}$$

By combining (2.3), (2.4) and (2.2), we obtain a contradiction;

(4) let $x_0 \in \mathrm{relint}(S_i)$, then, by Part(1) of Lemma 2.4, we have that $x_0 \notin S_j$ for every $S_j \neq S_i$. We set $g_j(\,\cdot\,) = d_{S_j}(\,\cdot\,) - d_{S_i}(\,\cdot\,)$, then, it holds that

$$g_j(x_0) = d_{S_j}(x_0) > 0.$$

Being g_j a continuous function on $\mathbb{R}^N$, there exists $B_\rho(x_0)$ such that $g_j > 0$ on $B_\rho(x_0)$, for every $j \neq i$. Hence $d_\Omega = d_{S_i}$ on the open set $B_\rho(x_0) \cap \Omega \neq \emptyset$, which implies that $B_\rho(x_0) \cap \Omega \subset \Omega_i$, giving the desired conclusion;

(5) without loss of generality, suppose that

$$H_i = \{(y, x_N) \in \mathbb{R}^N : \, x_N = 0\}.$$

Hence, thanks to Part (3), one of the following inclusion holds

$$\Omega_i \subset \{(y, x_N) \in \mathbb{R}^N : \, x_N \geq 0\} \qquad \text{or} \qquad \Omega_i \subset \{(y, x_N) \in \mathbb{R}^N : \, x_N \leq 0\}.$$

In both cases, since

$$|x_N| = d_{S_i}(x) = d_\Omega(x) \leq r_\Omega, \qquad \text{for every } x \in \Omega_i,$$

we obtain the claimed conclusion;

(6) let $x_0 \in \mathrm{relint}(S_i)$. Since $\mathrm{int}(\Omega_i) \neq \emptyset$, we can take an open half-line r_{x_0} with origin x_0 such that it is perpendicular to S_i in x_0 and $r_{x_0} \cap \mathrm{int}(\Omega_i) \neq \emptyset$. Being Ω_i bounded set and r_{x_0} a connected set, we obtain that $r_{x_0} \cap \partial\Omega_i \neq \emptyset$. Moreover, as $\partial\Omega_i = (\partial\Omega_i \cap \Omega) \cup S_i$ and $r_{x_0} \cap S_i = \emptyset$, we also have that

$$\Sigma_i(x_0) := r_{x_0} \cap \partial\Omega_i \cap \Omega \neq \emptyset.$$

Now, we will show that $\Sigma_i(x_0)$ consists in exactly a point. Indeed, we note that if $y' \in \Sigma_i(x_0)$, then

$$]x_0, y'[=]x_0, x] \cup [x, y'[\subseteq \mathrm{int}(\Omega_i), \qquad \text{for every } x \in r_{x_0} \cap \mathrm{int}(\Omega_i).$$

In particular, if $y', y'' \in \Sigma_i(x_0)$, with $y' \neq y''$, then we would obtain the contradiction

$$y'' \in]x_0, y'[\subset \mathrm{int}(\Omega_i) \qquad \text{or} \qquad y' \in]x_0, y''[\subset \mathrm{int}(\Omega_i).$$

□

3. Proof of the main result

In proving Theorem 1.1, first we will restrict ourselves to consider the case when the convex open set Ω coincides with the interior of a polytope K. Then, the general result will follow thanks to an approximation argument by means of polytopes, valid for convex sets.

Proof of Theorem 1.1. By following [21], we divide the proof in three parts: first, we prove the lower bound (1.2) when Ω is the interior of a polytope K, then, by applying an approximation argument, we show that such a lower bound holds when $\Omega \subsetneq \mathbb{R}^N$ is a general convex set. Finally, we show that (1.2) is asymptotically sharp for slab-type sequences.

Part 1: Makai's inequality for a polytope. Without loss of generality, let us suppose that $0 \in \Omega$. Moreover, in this step we assume that $\Omega = \mathrm{int}(K)$, where $K \subset \mathbb{R}^N$ is a polytope. According to the notation in Lemma 2.5, we consider the subsets Ω_i given by (2.1), with $i \in \{1, \ldots, h\}$.

Now, we will show that, for every $i \in \{1, \ldots, h\}$ and for every $u \in C_0^\infty(\Omega)$, it holds

$$\int_{\Omega_i} |u|^q \, dx \le \left(\frac{2}{\pi_{p,q}}\right)^q \left(\frac{pq + p - q}{p - q}\right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} d_\Omega^{\frac{p\,q}{p-q}} \, dx\right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} |\nabla u|^p \, dx\right)^{\frac{q}{p}}. \tag{3.1}$$

Indeed, let $i \in \{1, \ldots, h\}$ and, up to translations and rotations, we can assume that

$$H_i = \{(y, t) \in \mathbb{R}^N : t = 0\}$$

is the affine hyperplane containing S_i. Then

$$t = d_{S_i}(y, t) = d_\Omega(y, t), \qquad \text{for every } (y, t) \in \Omega_i. \tag{3.2}$$

Thanks to Lemma 2.5, we have that $\Pi_i : \overline{\partial\Omega_i \cap \Omega} \to S_i$ is a bijective and continuous function between two compact sets. Hence, defining $S_i' = \{y \in \mathbb{R}^{N-1} : (y, 0) \in S_i\}$, we obtain that there exists a continuous function $f_i : S_i' \to [0, +\infty)$ such that

$$(y, t) \in \overline{\partial\Omega_i \cap \Omega} \qquad \Longleftrightarrow \qquad y \in S_i' \quad \text{and} \quad t = f_i(y), \tag{3.3}$$

and it is easy to show that

$$\Omega_i = \Big\{(y, t) \in S_i' \times \mathbb{R} : 0 < t \le f_i(y)\Big\}. \tag{3.4}$$

Indeed, the inclusion "$\subseteq$" follows by using (3.2) and (3.3), while the converse one "$\supseteq$" is an application of Parts (3) and (6) of Lemma 2.5, taking into account that

$$f_i(y) > 0 \qquad \Longleftrightarrow \qquad (y, 0) \in \mathrm{relint}(S_i).$$

Now, we recall that

$$\left(\frac{\pi_{p,q}}{2}\right)^p=\min_{\varphi\in W^{1,p}((0,1))\setminus\{0\}}\left\{\frac{\displaystyle\int_0^1|\varphi'|^p\,dt}{\left(\displaystyle\int_0^1|\varphi|^q\,dt\right)^{\frac{p}{q}}}:\varphi(0)=0\right\},$$

(see [7, Lemma A.1]), which implies that, for every $s>0$

$$\left(\int_0^s|\varphi|^q\,dt\right)^{\frac{p}{q}}\le\left(\frac{2}{\pi_{p,q}}\right)^p s^{\frac{pq+p-q}{q}}\int_0^s|\varphi'|^p\,dt,\qquad\text{for every }\varphi\in C_0^\infty((0,s]).\tag{3.5}$$

Hence, for every $u\in C_0^\infty(\Omega)$ and for every $i\in\{1,\dots,h\}$, thanks to formula (3.4), by using Fubini's Theorem and (3.5) with $s=f_i(y)$, we get

$$\begin{aligned}\int_{\Omega_i}|u(x)|^q\,dx&=\int_{S_i'}\int_0^{f_i(y)}|u(y,t)|^q\,dt\,dy\\&\le\left(\frac{2}{\pi_{p,q}}\right)^q\int_{S_i'}f_i(y)^{\frac{pq+p-q}{p}}\left(\int_0^{f_i(y)}\left|\frac{\partial u}{\partial t}(y,t)\right|^p\,dt\right)^{\frac{q}{p}}\,dy\\&\le\left(\frac{2}{\pi_{p,q}}\right)^q\left(\int_{S_i'}f_i(y)^{\frac{pq+p-q}{p-q}}\,dt\right)^{\frac{p-q}{p}}\left(\int_{S_i'}\int_0^{f_i(y)}\left|\frac{\partial u}{\partial t}(y,t)\right|^p\,dy\,dt\right)^{\frac{q}{p}},\end{aligned}\tag{3.6}$$

where we also apply an Hölder's inequality in the last line. Taking into account (3.2), we have that

$$\begin{aligned}\int_{S_i'}f_i(y)^{\frac{pq+p-q}{p-q}}\,dy&=\left(\frac{pq+p-q}{p-q}\right)\int_{S_i'}\left(\int_0^{f_i(y)}t^{\frac{p\,q}{p-q}}\,dt\right)dy\\&=\left(\frac{pq+p-q}{p-q}\right)\int_{S_i'}\left(\int_0^{f_i(y)}(d_\Omega(y,t))^{\frac{p\,q}{p-q}}\,dt\right)dy\\&=\left(\frac{pq+p-q}{p-q}\right)\int_{\Omega_i}d_\Omega^{\frac{p\,q}{p-q}}\,dx.\end{aligned}\tag{3.7}$$

By combining (3.6) and (3.7), for every $i\in\{1,\dots,h\}$, we obtain (3.1).

Finally, since

$$\Omega=\bigcup_{i=1}^h\Omega_i,$$

by summing with respect to the index $i\in\{1,\dots,h\}$ in (3.1), it follows that

$$\int_\Omega|u|^q\,dx\le\left(\frac{2}{\pi_{p,q}}\right)^q\left(\frac{pq+p-q}{p-q}\right)^{\frac{p-q}{p}}\sum_{i=1}^h\left(\int_{\Omega_i}d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{p}}\left(\int_{\Omega_i}|\nabla u|^p\,dx\right)^{\frac{q}{p}}.$$

By applying Hölder's inequality

$$\|a\,b\|_{\ell^1}\le\|a\|_{\ell^r}\,\|b\|_{\ell^{r'}},$$

with $r=p/q$, we get

$$\int_\Omega|u|^q\,dx\le\left(\frac{2}{\pi_{p,q}}\right)^q\left(\frac{pq+p-q}{p-q}\right)^{\frac{p-q}{p}}\left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{p}}\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{q}{p}}.$$

and by raising to the power p/q on both sides, this in turns implies

$$\frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\left(\displaystyle\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}}} \ge \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}} \frac{1}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}}, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

Taking the infimum on $C_0^\infty(\Omega)$ on the left-hand side, we get that for every polytope $K \subset \mathbb{R}^N$, the set $\Omega = \mathrm{int}(K)$ satisfies the lower bound (1.2), as desired.

Part 2: approximation argument. If $\Omega \subsetneq \mathbb{R}^N$ is a general convex bounded open set, thanks to [26, Theorem 1.8.19], for every $0 < \varepsilon \ll 1$, there exists a polytope K_ε such that

$$K_\epsilon \subseteq \Omega \subseteq \overline{\Omega} \subseteq \frac{1}{1-\varepsilon} K_\varepsilon.$$

In particular, we have that

$$(1-\varepsilon)\,\Omega \subseteq \mathrm{int}(K_\varepsilon).$$

Then, thanks to Part (1) of the proof, we obtain that

$$\frac{\lambda_{p,q}(\Omega)}{(1-\varepsilon)^{p-N+N\frac{p}{q}}} = \lambda_{p,q}((1-\varepsilon)\,\Omega) \ge \lambda_{p,q}(\mathrm{int}(K_\varepsilon)) \ge \frac{C_{p,q}}{\left(\displaystyle\int_{\mathrm{int}(K_\varepsilon)} d_{\mathrm{int}(K_\varepsilon)}^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}} \ge \frac{C_{p,q}}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}},$$

where the last inequality follows from the fact that $d_{\mathrm{int}(K_\varepsilon)}(x) \le d_\Omega(x)$, for every $x \in K_\epsilon$. Then, by sending $\varepsilon \to 0^+$, we get that Ω satisfies (1.2).

Part 3: sharpness. Now we will show that estimate (1.2) is asimptotically sharp for slab-type sequences

$$\Omega_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^{N-1} \times (0,1), \qquad \text{with } L \ge 1.$$

With this aim, we denote by $y = (x_1, \dots, x_{N-1})$ a point of $\mathbb{R}^{N-1}$. Let S_0 and S_1 be the facets of Ω_L contained, respectively, in the hyperplanes $\big\{(y,t) \in \mathbb{R}^N : t = 0\big\}$ and $\big\{(y,t) \in \mathbb{R}^N : t = 1\big\}$, and we define the *lateral surface* $\mathcal{S}_L$ of Ω_L as the following union:

$$\mathcal{S}_L := \bigcup_{i=1}^{N-1} \left\{(y,t) \in \Omega_L : x_i = \pm\frac{L}{2},\, 0 < t < 1\right\}.$$

Then, we define the set Ω_1 as

$$\Omega_1 = \Big\{x \in \Omega_L : d_{\mathcal{S}_L}(x) \ge 1\Big\}.$$

Since $|\Omega_1| = (L-2)^{N-1}$, we obtain that

$$|\Omega_L \setminus \Omega_1| = L^{N-1} - (L-2)^{N-1} \sim L^{N-2}\, C_N, \qquad \text{as } L \to \infty,$$

which implies

$$\int_{\Omega_L \setminus \Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx \le r_{\Omega_L}^{\frac{p\,q}{p-q}} |\Omega_L \setminus \Omega_1| \sim L^{N-2}\, C_N\, \left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}, \qquad \text{as } L \to \infty.$$

Moreover, since, for every $(y,t) \in \Omega_1$, it holds that

$$d_{\Omega_L}(y,t) = d_{S_0 \cup S_1}(y,t) = \min\{t, 1-t\},$$

we obtain

$$\int_{\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dt = \left(\int_{\left(-\frac{L}{2}+1,\frac{L}{2}-1\right)^{N-1}} dy\right)\left(2\int_0^{1/2} t^{\frac{p\,q}{p-q}}\,dt\right) = L^{N-1}\,\left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}\,\frac{p-q}{pq+p-q}.$$

In particular, since

$$\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx = \int_{\Omega_L\setminus\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx + \int_{\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx,$$

we have the following asymptotic behavior

$$\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx \sim L^{N-1}\,\left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}\,\frac{p-q}{pq+p-q}, \qquad \text{as } L\to\infty.$$

Hence, we finally obtain

$$\frac{C_{p,q}}{\left(\displaystyle\int_{\Omega_L} d_{\Omega_L}^{\frac{pq}{p-q}}\,dx\right)^{\frac{p-q}{q}}} = \left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}}\,\frac{1}{\left(\displaystyle\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}} \sim (\pi_{p,q})^p\,\frac{1}{L^{\frac{(p-q)(N-1)}{q}}}, \qquad \text{as } L\to\infty.$$

On the other hand, by [6, Main Theorem], the following upper bound holds

$$\lambda_{p,q}(\Omega) < \left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{P(\Omega)}{|\Omega|^{1-\frac{1}{p}+\frac{1}{q}}}\right)^p$$

and it is asimptotically sharp for slab–type sequences Ω_L. Finally, since

$$P(\Omega_L)\sim 2\,L^{N-1} \qquad \text{and} \qquad |\Omega_L|\sim L^{N-1}, \qquad \text{as } L\to\infty,$$

we obtain

$$\lambda_{p,q}(\Omega_L)\sim(\pi_{p,q})^p\,\frac{1}{L^{\frac{(p-q)(N-1)}{q}}}, \qquad \text{as } L\to\infty.$$

The proof is over. □

As an easy consequence of Theorem 1.1, we can give an alternative proof of the sharp lower bound (1.5).

Proof of Corollary 1.2. We use the same notation as in Theorem 1.1. Thanks to Part (3) of Lemma 2.5, it holds that

$$f_i(y)\le r_\Omega, \qquad \text{for every } y\in S_i'. \tag{3.8}$$

By combining (3.7) and (3.8), we obtain that

$$\begin{aligned}
\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx &= \sum_{i=1}^h \int_{S_i'}\int_0^{f_i(y)} (d_\Omega(y,t))^{\frac{p\,q}{p-q}}\,dt\,dy \\
&= \frac{p-q}{pq+p-q}\sum_{i=1}^h \int_{S_i'} (f_i(y))^{\frac{p\,q}{p-q}+1}\,dy \\
&\le \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} \left(\sum_{i=1}^h \int_{S_i'} f_i(y)\,dy\right) \\
&= \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} \left(\sum_{i=1}^h \int_{S_i'}\int_0^{f_i(y)} dt\,dy\right) \\
&= \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} |\Omega|.
\end{aligned}$$

and, by applying the above estimate in (1.2), we get the desired lower bound (1.5). Finally, we obtain that this inequality is asymptotically sharp by using the slab-type sequences

$$\Omega_L = \left(-\frac{L}{2},\frac{L}{2}\right)^{N-1}\times(0,1), \qquad \text{with } L\ge 1,$$

as it was proved in [9, Theorem 5.7]. □

Remark 3.1. Let $\Omega \subsetneq \mathbb{R}^N$ be a convex bounded open set, then, from the proof of Corollary 1.2, it follows that

$$\int_\Omega d_\Omega^\alpha\,dx \le |\Omega|\,\frac{r_\Omega^\alpha}{\alpha+1}, \qquad \text{for every } \alpha>0.$$

Moreover, such an estimate is sharp and equality is asymptotically attained by slab–type sequences Ω_L, as in Corollary 1.2.

4. Alternative proof for Hersch-Protter-Kajikiya inequality for λ_p

We now focus on the case $1<p=q<\infty$. In this section we provide an alternative proof for the Hersch–Protter-type inequality for λ_p in any dimension $N\ge 2$. More precisely, we show the following result:

Theorem 4.1 (Hersch-Protter-Kajikiya inequality)**.** *Let $1<p<\infty$ and let $\Omega\subsetneq\mathbb{R}^N$ be a convex bounded open set. Then, the following lower bound holds*

$$\lambda_p(\Omega) \ge \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_\Omega^p}.$$

Moreover, the estimate is sharp.

With this aim, we need some preliminary results.

4.1. **Change of variables theorem.** First of all, we recall the change of variables formula which follows from [13, Theorem 7.1] when $K = \overline{B}_1$ (see also [13, Example 5.6]).

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open connected set of class C^2. Let $l : \partial\Omega \to \mathbb{R}$ be the function given by*

$$l(x) = \sup\left\{d_\Omega(z) : z \in \overline{\Omega} \text{ and } x \in \Pi(z)\right\}, \tag{4.1}$$

where

$$\Pi(z) = \{x \in \partial\Omega : d_\Omega(z) = |x - z|\},$$

and let $\Phi = \partial\Omega \times \mathbb{R} \to \mathbb{R}^N$ be the map defined by

$$\Phi(x,t) = x + t\nu(x), \qquad \text{for every } (x,t) \in \partial\Omega \times \mathbb{R},$$

where, for every $x \in \partial\Omega$, $\nu(x)$ is the inward normal unit vector to $\partial\Omega$ at x. Then, for every $h \in L^1(\Omega)$, it holds

$$\int_\Omega h(x)\,dx = \int_{\partial\Omega} \left(\int_0^{l(x)} h(\Phi(x,t)) \prod_{i=1}^{N-1} (1 - tk_i(x))\,dt \right) d\mathcal{H}^{N-1}(x). \tag{4.2}$$

Here $k_1(x), \dots, k_{n-1}(x)$ are the principal curvatures of $\partial\Omega$ at x, i.e. the eigenvalues of the Weingarten map $W(x) : T_x \to T_x$, where T_x denotes the tangent space to $\partial\Omega$ at x. Thanks to the C^2 regularity assumption on Ω, it follows that k_i is continuous on $\partial\Omega$ for every $i \in \{1, \dots, N-1\}$.

Remark 4.3. We note that, when Ω is a polytope, with the notation of Lemma 2.5, we have that $l(x) = |x - \Pi_i^{-1}(x)|$, for every $x \in \operatorname{relint}(S_i)$.

4.2. **Weighted Rayleigh quotients.** Let $w : (0, L) \to \mathbb{R}$ be a monotone non-increasing positive function, $w \not\equiv 0$. For every $1 < p < \infty$, we define the following weighted Rayleigh quotient

$$\mu_p(w, (0,L)) := \inf_{\psi \in C_0^\infty((0,L]) \setminus \{0\}} \left\{ \frac{\displaystyle\int_0^L |\psi'(t)|^p\, w(t)\,dt}{\displaystyle\int_0^L |\psi(t)|^p\, w(t)\,dt} \right\}.$$

When $w \equiv 1$ on $(0, L)$, we will write $\mu_p(w, (0, L)) = \mu_p(1, (0, L))$.
With the aim to show that $\mu_p(1, (0, L)) \le \mu_p(w, (0, L))$ for every monotone non-increasing positive weight $w : (0, L) \to \mathbb{R}$, first we prove that there exists a monotone non-increasing positive minimizer for $\mu_p(1, (0, L))$.

Lemma 4.4. *Let $1 < p < \infty$, then*

$$\mu_p(1, (0, L)) = L^{-p} \left(\frac{\pi_p}{2}\right)^p.$$

In particular, there exists a positive and monotone non-decreasing solution of the minimization problem

$$\mu_p(1, (0,L)) = \inf_{\psi \in W^{1,p}((0,L)) \setminus \{0\}, \psi(0)=0} \left\{ \frac{\displaystyle\int_0^L |\psi'(t)|^p\,dt}{\displaystyle\int_0^L |\psi(t)|^p\,dt} \right\}. \tag{4.3}$$

Proof. First we note that, thanks to the density of $C_0^\infty((0,L])$ in the subspace $\{\varphi \in W^{1,p}(0,L) : \varphi(0)=0\}$, we have that (4.3) holds. Moreover, following the proof of [7, Lemma A.1], we have that

$$\mu_p(1,(0,L)) = L^{-p}\mu_p(1,(0,1)) = L^{-p}\left(\frac{\pi_p}{2}\right)^p,$$

and there exists a positive symmetric function $v \in W_0^{1,p}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ which is monotone non-increasing on $\left(0,\frac{1}{2}\right)$, such that

$$\pi_p^p = \frac{\displaystyle\int_0^{\frac{1}{2}} |v'(t)|^p\,dt}{\displaystyle\int_0^{\frac{1}{2}} |v(t)|^p\,dt}.$$

Then, it is easy to show that the function $\tilde{v} \in W^{1,p}((0,L))$ defined by

$$\tilde{v}(t) := v\left(\frac{L-t}{2L}\right)$$

satisfies $\tilde{v}(0)=0$ and it is a positive monotone non-decreasing minimizer of the problem (4.3). □

Now we are in position to show the following minimization result.

Theorem 4.5. *Let $1<p<\infty$ and let w be a positive and monotone non-increasing function on $(0,L)$, then*

$$\mu_p(w,(0,L)) \geq \mu_p(1,(0,L)). \tag{4.4}$$

Proof. First, we show that, for every positive and monotone non-increasing weight $w \in L^\infty((0,L))$, it holds

$$\mu_p(1,(0,L))\int_0^L |\varphi|^p w\,dt \leq \int_0^L |\varphi'|^p\,w\,dt, \qquad \text{for every } \varphi \in C_0^\infty((0,L]). \tag{4.5}$$

Indeed, let $v \in W^{1,p}((0,L))$ be a positive and monotone non-decreasing eigenfunction for $\mu_p(1,(0,L))$ whose existence is ensured by Lemma 4.4. Then v is a weak solution of

$$\begin{cases} -(|v'|^{p-2}v')' = \mu_p(1,(0,L))\,v^{p-1}, & \text{in } (0,L),\\ v(0)=0. \end{cases}$$

Moreover, fixed a mollifier $\delta \in C_0^\infty(\mathbb{R})$, given by

$$\delta(x) := \begin{cases} e^{\frac{1}{|x|^2-1}}, & \text{if } |x|<1,\\ 0, & \text{if } |x|\geq 1, \end{cases}$$

for $0<\varepsilon<1$, we define

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\,\delta\left(\frac{x}{\varepsilon}\right) \in C^\infty(\mathbb{R}).$$

Then $w_\varepsilon = w * \delta_\varepsilon \in C^\infty((\varepsilon, L-\varepsilon))$ and, as $\varepsilon \to 0$, w_ε pointwise converges to w a. e. on $(0, L)$. Moreover, $w'_\varepsilon \leq 0$. Indeed, for every $t, t' \in (\varepsilon, L-\varepsilon)$ such that $t > t'$, we have that

$$\begin{aligned} w_\varepsilon(t) = (w * \delta_\varepsilon)(t) &= \int_0^L \delta_\varepsilon(t-y)\, w(y)\, dy = \frac{1}{\varepsilon}\int_0^L \delta\left(\frac{t-y}{\varepsilon}\right) w(y)\, dy \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \delta\left(\frac{t-y}{\varepsilon}\right) w(y)\, dy = \int_{-1}^{1} \delta(z)\, w(t-\varepsilon z)\, dz \\ &\leq \int_{-1}^{1} \delta(z)\, w(t'-\varepsilon z)\, dz = w_\varepsilon(t'), \end{aligned}$$

where in the last inequality we use that $\delta(z) > 0$, for every $z \in (-1, 1)$ and w is a monotone non-increasing function. Now, by using that $w'_\varepsilon \leq 0$ and $v' \geq 0$ a. e. on $(0, L)$ (thanks to Lemma 4.4), for every $\varphi \in C_0^\infty((0, L])$, we have that

$$\begin{aligned} \mu_p(1,(0,L)) \int_0^L |\varphi|^p w_\varepsilon\, dt &= \mu_p(1,(0,L)) \int_0^L v^{p-1} \frac{|\varphi|^p}{v^{p-1}} w_\varepsilon\, dt \\ &= \int_0^L |v'|^{p-2}|v'| \left(\frac{|\varphi|^p}{v^{p-1}}\, w_\varepsilon\right)' dt \\ &= \int_0^L |v'|^{p-2} v' \left(\frac{|\varphi|^p}{v^{p-1}}\right)' w_\varepsilon\, dt + \int_0^L |v'|^{p-2} v' \frac{|\varphi|^p}{v^{p-1}} w'_\varepsilon\, dt \\ &\leq \int_0^L |v'|^{p-2} v' \left(\frac{|\varphi|^p}{v^{p-1}}\right)' w_\varepsilon\, dt. \end{aligned}$$

By applying Picone's inequality on the last integral (see [1]), the above inequality implies

$$\mu_p(1,(0,L)) \int_0^L |\varphi|^p w_\varepsilon\, dt \leq \int_0^L |\varphi'|^p\, w_\varepsilon\, dt.$$

Since $\|w_\varepsilon\|_{L^\infty((0,L))} \leq \|w\|_{L^\infty((0,L))}$, as $\varepsilon \to 0$, by using the Dominated Convergence Theorem, we obtain that w satisfies (4.5).

Now, we remove the assumption that w is bounded. For every $M > 0$, we define $w_M := \min\{w, M\} \in L^\infty((0, L))$. By applying (4.5), we have that

$$\mu_p(1,(0,L)) \int_0^L |\varphi|^p w_M\, dt \leq \int_0^L |\varphi'|^p\, w_M\, dt \leq \int_0^L |\varphi'|^p\, w\, dt, \qquad \text{for every } \varphi \in C_0^\infty((0,L])$$

and, sending $M \to \infty$, we get that also w satisfies (4.5).

Finally, passing to the infimum on functions $\varphi \in C_0^\infty((0, L])$ in (4.5), we obtain the desired estimate (4.4). $\square$

Now, by applying the previous results, we are in a position to show Theorem 4.1.

Proof of Theorem 4.1. We divide the proof in two parts.

Part 1: inequality for C^2 convex bounded sets. We first suppose that Ω is a convex bounded open set of class C^2. Let $u \in C_0^\infty(\Omega)$ then, by using formula (4.2), we have that

$$\int_\Omega |u(x)|^p\, dx = \int_{\partial\Omega} \left(\int_0^{l(x)} |u(x + t\nu(x))|^p \prod_{i=1}^{N-1} (1 - t k_i(x))\, dt \right) d\mathcal{H}^{n-1}(x) \tag{4.6}$$

and

$$\int_{\Omega}|\nabla u(x)|^p\,dx=\int_{\partial\Omega}\left(\int_0^{l(x)}|\nabla u(x+t\nu(x))|^p\prod_{i=1}^{N-1}(1-tk_i(x))\,dt\right)d\mathcal{H}^{n-1}(x). \tag{4.7}$$

Now we fix $x\in\partial\Omega$ and let $v\in C_0^\infty((0,l(x)])$ be defined by $v(t):=u(x+t\nu(x))$ for every $t\in(0,l(x)]$. Furthermore we introduce the weight $w_x:(0,l(x))\to\mathbb{R}$ given by

$$w_x(t)=\prod_{i=1}^{N-1}(1-tk_i(x))\in L^\infty(0,l(x)).$$

It is easy to verify that the weight w_x is monotone non-increasing. Moreover, we note that, by [13, Proposition 3.10, Lemma 4.1, Theorem 6.7], l is a positive and continuous function on $\partial\Omega$. In particular, there exists $z=z(x)\in\Omega$, such that $x\in\Pi(z)$ and $l(x)=d_\Omega(z)$. Hence

$$1-tk_i(x)>1-d_\Omega(z)\,k_i(x)\geq 0,\qquad \text{for every } t\in(0,l(x)),$$

where the last inequality follows from [13, Lemma 5.4]. In particular, this implies that $w_x>0$ on $(0,l(x))$. Being satisfied all the hypotheses of Theorem 4.5, we have that

$$\begin{aligned}\int_0^{l(x)}|u(x+t\nu(x))|^p\;w_x(t)\,dt&=\int_0^{l(x)}|v(t)|^p\,w_x(t)\,dt\\&\leq\frac{1}{\mu_p(w_x,(0,l(x)))}\int_0^{l(x)}|v'(t)|^p\,w_x(t)\,dt\\&\leq\frac{1}{\mu_p(1,(0,l(x)))}\int_0^{l(x)}|v'(t)|^p\,w_x(t)\,dt.\end{aligned}$$

Then, applying Lemma 4.4 and taking into account that $l(x)\leq r_\Omega$ for every $x\in\partial\Omega$, we obtain that

$$\begin{aligned}\int_0^{l(x)}|u(x+t\nu(x))|^p\;w_x(t)\,dt&\leq\left(\frac{2}{\pi_p}\right)^p l(x)^p\int_0^{l(x)}|v'(t)|^p\,w_x(t)\,dt\\&\leq\left(\frac{2}{\pi_p}\right)^p r_\Omega^p\int_0^{l(x)}|\nabla u(x+t\nu(x))|^p\,w_x(t)\,dt,\qquad\text{for every } x\in\partial\Omega.\end{aligned}$$

By exploiting the above estimate in (4.6) and then using (4.7), we get

$$\int_\Omega|u|^p\,dx\leq\left(\frac{2}{\pi_p}\right)^p r_\Omega^p\left(\int_{\partial\Omega}\left(\int_0^{l(x)}|\nabla u(x+t\nu(x))|^p\,w_x(t)\,dt\right)d\mathcal{H}^{N-1}(x)\right)=\left(\frac{2}{\pi_p}\right)^p r_\Omega^p\int_\Omega|\nabla u|^p\,dx,$$

that is

$$\frac{\displaystyle\int_\Omega|\nabla u|^p\,dx}{\displaystyle\int_\Omega|u|^p\,dx}\geq\left(\frac{\pi_p}{2}\right)^p\frac{1}{r_\Omega^p},\qquad\text{for every } u\in C_0^\infty(\Omega).$$

Taking the infimum on $C_0^\infty(\Omega)$, we obtain that (1.5) holds when Ω is a convex bounded open set of class C^2.

Part 2: inequality for convex bounded sets. We will apply an approximation argument to show the validity of (1.5) for every convex bounded open set. Let $\Omega\subset\mathbb{R}^N$ be a convex bounded open set, then, thanks to [16, Section 4.3], there exists a sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ of convex bounded closed sets of class C^2 such that

- $C_{k+1} \subseteq C_k \subseteq C_1$, for every $k \in \mathbb{N}$;
- $\overline{\Omega} \subseteq C_k$ and
$$d_{\mathcal{H}}(C_k, \overline{\Omega}) = \min\left\{\lambda \geq 0 : \overline{\Omega} \subseteq C_k + \lambda B_1, C_k \subseteq \overline{\Omega} + \lambda B_1\right\} \leq \frac{1}{k},$$
i. e. $\{C_k\}_{k\in\mathbb{N}}$ converges to $\overline{\Omega}$, as $k \to \infty$, in the sense of Hausdorff.

This implies that, for every $\varepsilon > 0$, there exists $k_1 = k_1(\varepsilon) \in \mathbb{N}$, such that
$$(1-\varepsilon)\,\overline{\Omega} \subseteq C_k, \qquad \text{for every } k \geq k_1,$$
which leads to
$$(1-\varepsilon)\,\Omega \subseteq \operatorname{int}(C_k), \qquad \text{for every } k \geq k_1.$$
Let r_k be the inradius of $\Omega_k := \operatorname{int}(C_k)$, for every $k \in \mathbb{N}$. Then, thanks to the monotonicity of λ_p with respect to the inclusion of sets and by using Part (1) on Ω_k, we obtain that
$$\frac{\lambda_p(\Omega)}{(1-\varepsilon)^p} \geq \lambda_p(\Omega_k) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_k^p}. \tag{4.8}$$
Since
$$\Omega \subseteq \Omega_k \subseteq C_1, \qquad \text{for every } k \in \mathbb{N},$$
we can consider the co-Hausdorff distance between Ω and Ω_k, given by
$$d^{\mathcal{H}}(\Omega_k, \Omega) = d_{\mathcal{H}}(C_1 \setminus \Omega_k, C_1 \setminus \Omega),$$
and, being Ω and Ω_k convex open sets, we also have that
$$d^{\mathcal{H}}(\Omega_k, \Omega) = d_{\mathcal{H}}(\partial\Omega_k, \partial\Omega) = d_{\mathcal{H}}(C_k, \overline{\Omega}) \leq \frac{1}{k}.$$
Hence, as $k \to \infty$, the sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ converges to Ω in the sense of co-Hausdorff. By using [11, Lemma 4.4], we have that
$$r_k \to r_\Omega, \qquad \text{as } k \to \infty.$$
Hence, by sending $k \to \infty$ and then $\varepsilon \to 0$ in (4.8), we finally get (1.5).

Finally, we notice that the inequality is sharp. Indeed, the equality can be attained by different class of sets, for example by infinite slabs, as $\mathbb{R}^{N-1} \times (0,1)$, or asymptotically by the family of *collapsing pyramids* $C_\alpha = \text{convex hull}\left((-1,1)^{N-1} \cup \{(0,\dots,0,\alpha)\}\right)$, as proved in [7, Theorem 1.2]. The proof is over. □

5. Makai's costants for non-convex sets

As pointed out in the introduction, defined $\widetilde{C}_{p,q}$ and $\widehat{C}_{2,q}$ as in (1.7) and (1.14), the natural questions that arise are whether

- $\widetilde{C}_{p,q} < C_{p,q}$, when $p > N$ and $1 \leq q < N$;
- $\widetilde{C}_{p,p} < C_{p,p}$, when $p > N$;
- $\widehat{C}_{2,q} < C_{2,q}$, when $N = 2$ and $1 \leq q < 2$.

In this section, we give some partial answers to the questions above.

5.1. **The case of general open sets for $1 \le q < N < p$.** We focus on the class of general open sets of $\mathbb{R}^N$ with the aim to show that, for every fixed $1 \le q < N$, there exists $\overline{p} = \overline{p}(q) > N$ such that

$$\widetilde{C}_{p,q} < C_{p,q}, \qquad \text{for every } p \in (q, \overline{p}]. \tag{5.1}$$

We consider the *infinite fragile tower* set $\mathcal{T} \subset \mathbb{R}^N$ defined as in [10, Theorem 5.1, (iii)]. By contruction, it satisfies the following properties:

- $d_{\mathcal{T}} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T})$;
- $\lambda_{p,q}(\mathcal{T}) = 0$, for every $1 \le q < p \le N$,

Moreover, thanks to (1.7), $\lambda_{p,q}(\mathcal{T}) > 0$, for every $p > N$ and $1 \le q \le p$. In order to show (5.1), it is sufficient to prove that

$$\lim_{p \searrow N} \lambda_{p,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} < \lim_{p \searrow N} C_{p,q}.$$

With this aim, we observe that the following convergences hold

$$\lim_{p \searrow N} \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} = \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{N\,q}{N-q}}\, dx \right)^{\frac{N-q}{q}} \tag{5.2}$$

and

$$\lim_{p \searrow N} C_{p,q} = \lim_{p \searrow N} \left(\frac{\pi_{p,q}}{2} \right)^p \left(\frac{p-q}{pq+p-q} \right)^{\frac{p-q}{q}} = C_{N,q}. \tag{5.3}$$

The last limit follows by taking into account that, as computed in [28, equation (7)],

$$\pi_{p,q} = \frac{2}{q} \left(1 + \frac{q}{p'} \right)^{\frac{1}{q}} \left(1 + \frac{p'}{q} \right)^{-\frac{1}{p}} B\left(\frac{1}{q}, \frac{1}{p'} \right),$$

where $p' = p/(p-1)$ and B is the *Euler Beta function*, which is continuous on $(0, +\infty)$. If we show that

$$\limsup_{p \searrow N} \lambda_{p,q}(\mathcal{T}) \le \lambda_{N,q}(\mathcal{T}), \tag{5.4}$$

by using (5.2), (5.3) and (5.4), we obtain that

$$\lim_{p \searrow N} \lambda_{p,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} \le \lambda_{N,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{N\,q}{N-q}}\, dx \right)^{\frac{N-q}{q}} = 0 < C_{N,q} = \lim_{p \searrow N} C_{p,q},$$

which gives the desired conclusion.
In order to prove the claim (5.4), we note that, for every $1 \le r < \infty$ and for every open set $\Omega \subseteq \mathbb{R}^N$, it holds

$$\lim_{p \searrow r} \|\nabla \varphi\|_{L^p(\Omega)}^p = \|\nabla \varphi\|_{L^r(\Omega)}^r, \qquad \text{for every } \varphi \in C_0^\infty(\Omega). \tag{5.5}$$

Indeed, if $\varphi \in C_0^\infty(\Omega)$, then, for every $p > r$, it follows that

$$\int_\Omega |\nabla \varphi|^p\, dx = \int_\Omega |\nabla \varphi|^{p-r}\, |\nabla \varphi|^r\, dx \le \|\nabla \varphi\|_{L^\infty(\Omega)}^{p-r} \int_\Omega |\nabla \varphi|^r\, dx,$$

which implies

$$\limsup_{p \searrow r} \|\nabla \varphi\|_{L^p(\Omega)}^p \le \|\nabla \varphi\|_{L^r(\Omega)}^r.$$

On the other hand, by Fatou's Lemma, we also have that

$$\int_\Omega |\nabla\varphi|^r\,dx \le \liminf_{p\searrow r}\int_\Omega |\nabla\varphi|^p\,dx.$$

By applying (5.5) with $r = N$, we obtain

$$\limsup_{p\searrow N}\lambda_{p,q}(\mathcal{T}) \le \limsup_{p\searrow N}\frac{\displaystyle\int_{\mathcal{T}}|\nabla\varphi|^p\,dx}{\left(\displaystyle\int_{\mathcal{T}}|\varphi|^q\,dx\right)^{p/q}} = \frac{\displaystyle\int_{\mathcal{T}}|\nabla\varphi|^N\,dx}{\left(\displaystyle\int_{\mathcal{T}}|\varphi|^q\,dx\right)^{N/q}},\qquad \text{for every } \varphi\in C_0^\infty(\mathcal{T}),$$

and by taking the infimum on $C_0^\infty(\mathcal{T})$, this easily implies (5.4).

Remark 5.1. Let $1 \le q < \infty$. We notice that

$$\lim_{p\to\infty}\left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} = \lim_{p\to\infty}(C_{p,q})^{\frac{1}{p}} = 1.$$

Indeed, by [10, Corollary 6.1], it holds that

$$\lim_{p\to\infty}\frac{\pi_{p,q}}{2} = \frac{1}{2}\frac{1}{\left(\displaystyle\int_0^1(\min\{t,1-t\})^q\,dx\right)^{\frac{1}{q}}} = \frac{1}{(q+1)^{\frac{1}{q}}},$$

which implies

$$\lim_{p\to\infty}(C_{p,q})^{\frac{1}{p}} = \lim_{p\to\infty}\left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}} = 1.$$

Then, taking into account also (1.7), it easily follows that

$$1 \le \liminf_{p\to\infty}\left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} \le \limsup_{p\to\infty}\left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} \le \lim_{p\to\infty}(C_{p,q})^{\frac{1}{p}} = 1.$$

5.2. **The case of simply connected open sets for $p = N = 2$.** In this subsection, we restrict ourselves to the case $N = 2$ and we prove that there exist $1 \le \overline{q} < 2$ and $\overline{p} > 2$ such that

$$\widehat{C}_{2,q} < C_{2,q},\qquad \text{for every } q\in[\overline{q},2], \tag{5.6}$$

and

$$\widetilde{C}_{p,p} < C_{p,p},\qquad \text{for every } p\in[2,\overline{p}]. \tag{5.7}$$

First of all, we give an explicit example of a simply connected open set $\widetilde{\Omega}\subset\mathbb{R}^2$ such that

$$\lambda_2(\widetilde{\Omega})\,r_{\widetilde{\Omega}}^2 < C_{2,2} = \frac{\pi^2}{4}. \tag{5.8}$$

Let $A\subset\mathbb{R}^2$ be an annulus from which we remove a segment, that is

$$A = \left\{(x_1,x_2)\in\mathbb{R}^2 : 1 < \sqrt{x_1^2+x_2^2} < 2\right\}\setminus\left\{(x_1,0)\in\mathbb{R}^2 : 1 < x_1 < 2\right\}.$$

Then it holds that $\lambda_2(A) = \pi^2$ (see [24], page 551). Now, for a fixed $0 < \varepsilon < 1$, we consider the simply connected open set

$$\widetilde{\Omega} = A\cup\left\{(x_1,x_2)\in\mathbb{R}^2 : \sqrt{4-x_2^2}\le x_1 < 2+\varepsilon, -\varepsilon < x_2 < \varepsilon\right\}\setminus\left\{(x_1,0)\in\mathbb{R}^2 : 1 < x_1 < 2+\epsilon\right\}.$$

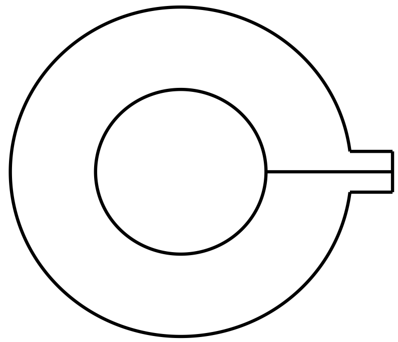

Figure 2. The set $\widetilde{\Omega}$ obtained from an annulus adding a *small tooth* and removing a segment.

Since $A \subset \widetilde{\Omega}$ and $|\widetilde{\Omega} \setminus A| \neq \emptyset$, we get that

$$\lambda_2(\widetilde{\Omega}) < \lambda_2(A) = \pi^2.$$

Being $r_A = r_{\widetilde{\Omega}} = 1/2$, the above inequality implies (5.8). Now, we recall that, by [9, Proposition 2.3], it holds

$$\lim_{q \nearrow 2} \lambda_{2,q}(\widetilde{\Omega}) = \lambda_2(\widetilde{\Omega}) \qquad \text{and} \qquad \lim_{q \nearrow 2} \pi_{2,q} = \pi,$$

hence, by combining the above limits with (5.8), we get

$$\lim_{q \nearrow 2} \lambda_{2,q}(\widetilde{\Omega}) \left(\int_{\widetilde{\Omega}} d_{\widetilde{\Omega}}^{\frac{2\,q}{2-q}} \, dx \right)^{\frac{2-q}{q}} = \lambda_2(\widetilde{\Omega}) \, r_{\widetilde{\Omega}}^2 < \frac{\pi^2}{4} = \lim_{q \nearrow 2} C_{2,q}.$$

This gives the desired conclusion (5.6).

Finally, we note that, by applying (5.5) with $r = 2$, for every $\varphi \in C_0^\infty(\widetilde{\Omega})$, it holds that

$$\limsup_{p \searrow 2} \lambda_p(\widetilde{\Omega}) \le \limsup_{p \searrow 2} \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla \varphi|^p \, dx}{\displaystyle\int_{\widetilde{\Omega}} |\varphi|^p \, dx} \le \limsup_{p \searrow 2} \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla \varphi|^p \, dx}{|\widetilde{\Omega}|^{\frac{2-p}{2}} \left(\displaystyle\int_{\widetilde{\Omega}} |\varphi|^2 \, dx \right)^{\frac{p}{2}}} = \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla \varphi|^2 \, dx}{\displaystyle\int_{\widetilde{\Omega}} |\varphi|^2 \, dx},$$

and, by taking the infimum on $\varphi \in C_0^\infty(\widetilde{\Omega})$, we get that

$$\limsup_{p \searrow 2} \lambda_p(\widetilde{\Omega}) \le \lambda_2(\widetilde{\Omega}).$$

Hence,

$$\limsup_{p \searrow 2} \lambda_p(\widetilde{\Omega}) \, r_{\widetilde{\Omega}}^p \le \lambda_2(\widetilde{\Omega}) \, r_{\widetilde{\Omega}}^2 < \frac{\pi^2}{4} = \lim_{p \searrow 2} C_{p,p},$$

which implies the desired conclusion (5.7).

References

[1] W. Allegretto, Y. X. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal., **32** (1998), 819–830. 16

[2] A. Ancona, On strong barriers and inequality of Hardy for domains in $\mathbb{R}^N$, J. London Math. Soc. (2) **34**, (1986), 274-–290. 4

[3] D. H. Armitage, Ü. Kuran, The Convexity of a Domain and the Superharmonicity of the Signed Distance Function, Proc. Amer. Math. Soc., **93** (1985), 598–600. 2, 8

[4] F. G. Avkhadiev, Hardy type inequalities in higher dimensions with explicit estimate of constants, Lobachevskii J. Math., **21** (2006), 3–31. 4

[5] M. van den Berg, Estimates for the torsion function and Sobolev constants, Potential Anal., **36** (2012), 607–616. 3

[6] L. Brasco, On principal frequencies and isoperimetric ratios in convex sets, Ann. Fac. Sci. Toulouse Math. (6), **29**, (2020), 977–1005. 12

[7] L. Brasco, On principal frequencies and inradius in convex sets, Bruno Pini Math. Anal. Semin. **9**, Univ. Bologna, Alma Mater Stud., Bologna, (2018). 3, 10, 15, 18

[8] L. Brasco, D. Mazzoleni, On principal frequencies, volume and inradius in convex sets, NoDEA Nonlinear Differential Equations Appl., **27**, (2020). 2

[9] L. Brasco, F. Prinari, A. C. Zagati, A comparison principle for the Lane–Emden equation and applications to geometric estimates, Nonlinear Analysis, **220**, (2022), 112847. 2, 3, 13, 21

[10] L. Brasco, F. Prinari, A. C. Zagati, Sobolev embeddings and distance functions, *to appear on* Advances in Calculus of Variations, (2023). 3, 19, 20

[11] L. Briani, G. Buttazzo, F. Prinari, A Shape Optimization Problem on Planar Sets with Prescribed Topology, J Optim Theory Appl **193**, (2022), 760–784. 18

[12] G. Buttazzo, S. Guarino Lo Bianco, M. Marini, Sharp estimates for the anisotropic torsional rigidity and the principal frequency, J. Math. Anal. Appl., **457** (2018), 1153–1172. 3

[13] G. Crasta, A. Malusa, The distance function from the boundary in a Minkowski space, Trans. Amer. Math. Soc. **359**, (2007), 5725–5759. 3, 14, 17

[14] F. Della Pietra, G. Di Blasio N. Gavitone, Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle, Adv. Nonlinear Anal., **9** (2020), 278–291. 3

[15] F. Della Pietra, N. Gavitone, S. Guarino Lo Bianco, On functionals involving the torsional rigidity related to some classes of nonlinear operators, J. Differential Equations, **265** (2018), 6424–6442. 2

[16] H. G. Eggleston, Convexity, Cambridge Tracts in Math and Mathematical Phisics, 47, Cambridge University Press, 1958. 18

[17] D. Goel, Y. Pinchover, G. Psaradakis, On weighted L^p-Hardy inequality on domains in $\mathbb{R}^N$, Special Issue on Analysis and PDE Dedicated to Professor Shmuel Agmon, Pure Appl. Funct. Anal., 7(3), (2022), 1025–1023. 4

[18] J. Hersch, Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum, Z. Angew. Math. Phys., **11**, (1960), 387–413. 3

[19] R. Kajikiya, A priori estimate for the first eigenvalue of the $p-$Laplacian, Differential Integral Equations, **28** (2015), 1011–1028. 3

[20] J. L. Lewis, Uniformly fat sets, Trans. Amer. Math. Soc., **308** (1988), 177–196. 4

[21] E. Makai, On the principal frequency of a membrane and the torsional rigidity of a beam, Studies in Math. Analysis and Related Topics, Stanford Univ. Press, Stanford, (1962), 227–231. 2, 5, 7, 9

[22] E. Makai, A lower estimation of the principal frequencies of simply connected membranes, Acta Mathematica Academiae Scientiarum Hungaricae **16**, (1965), 319–323 . 5

[23] M. Marcus, V. J. Mizel, Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^n$, Trans. Amer. Math. Soc., **350** (1998), 3237–3255. 4

[24] R. Osserman, A note on Hayman's theorem on the bass note of a drum, Commentarii Mathematici Helvetici, **52**, (1977), 545–555. 20

[25] M. H. Protter, A lower bound for the fundamental frequency of a convex region, Proc. Amer. Math. Soc., **81**, (1981), 65–70. 3

[26] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Encyclopedia Math. Appl., Cambridge University Press, Cambridge, **44**, 1993. 6, 11

[27] L. Tartar, An introduction to Sobolev spaces and interpolation spaces, Lecture Notes of the Unione Matematica Italiana, **3**, Springer, Berlin; UMI, Bologna, 2007. 4

[28] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (4), **110** (1976), 353–372. 19

[29] A. Wannebo, Hardy inequalities, Proc. Amer. Math. Soc. **109** (1990), 85–95. 4

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