HOMOGENISATION PROBLEMS FOR FREE DISCONTINUITY FUNCTIONALS WITH BOUNDED COHESIVE SURFACE TERMS

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Abstract. We study stochastic homogenisation problems for free discontinuity functionals under a new assumption on the surface terms, motivated by cohesive fracture models. The results are obtained using a characterization of the limit functional by means of the asymptotic behaviour of suitable minimum problems on cubes with very simple boundary conditions. An important role is played by the subadditive ergodic theorem.

Keywords: stochastic homogenisation, free discontinuity problems, Γ-convergence, subadditive ergodic theorem.

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1. Introduction

Several problems in damage and fracture mechanics lead to the study of free discontinuity functionals of the form

$$\int_A f(x, \nabla u)dx + \int_{A\cap J_u} g(x, [u], \nu_u) d\mathcal{H}^{d-1}, \tag{1.1}$$

where A is a bounded open subset of $\mathbb{R}^d$, u is a function defined on A, ∇u denotes its approximate gradient, J_u is the jump set of u, with unit normal ν_u, and $[u]$ is the amplitude of the jump; the second integral is with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$.

A large part of the results regarding Γ-convergence of sequences of functionals of the form (1.1) have been obtained under the hypotheses that $f(x,\xi)$ has p-growth with respect to ξ, for some $p>1$, and that g is larger than a positive constant (see, e.g., [3, 4, 10, 13, 19, 24, 25, 26]). Under these conditions the problem is usually studied in the space of special functions of bounded variation $SBV(A)$, for which we refer to [2].

When $f(x,\xi)$ has linear growth in ξ and $g(x,\zeta,\nu)$ has a linear behaviour in ζ near 0, a functional of type (1.1) cannot be lower semicontinuous. The lower semicontinuity results in [6] and the necessary condition for lower semicontinuity in [7] suggest to consider instead functionals of the form

$$E^{f,g}(u,A) := \int_A f(x,\nabla u)dx + \int_A f^\infty\big(x, \frac{dD^c u}{d|D^c u|}\big) d|D^c u| + \int_{A\cap J_u} g(x,[u],\nu_u) d\mathcal{H}^{d-1}, \tag{1.2}$$

defined for $u \in BV(A)$, where $f^\infty(x,\xi)$ is the recession function of $f(x,\xi)$ with respect to ξ, the measure $D^c u$ is the Cantor part of the distributional gradient Du of u (see [2, Definition 3.91]), and $\frac{dD^c u}{d|D^c u|}$ is the Radon-Nikodym derivative of $D^c u$ with respect to its variation $|D^c u|$.

In this paper we study homogenisation problems for functionals of the form (1.2). The new feature is that we consider surface integrands g of bounded cohesive type, i.e., satisfying an estimate of the form $c_1(|\zeta|\wedge 1) \le g(x,\zeta,\nu) \le c_3(|\zeta|\wedge 1)$ for suitable constants $0<c_1<c_3$,

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where $a \wedge b$ is the minimum between a and b. This hypothesis on the surface integrand is natural in the study of the relaxed version of the Dugdale model for cohesive cracks [18] (see also [8]).

The upper bound $g(x,\zeta,\nu) \leq c_3(|\zeta| \wedge 1)$ implies that we cannot apply the results of [12], for which the assumption $c_1|\zeta| \leq g(x,\zeta,\nu)$ was crucial to study the problem in the space $BV(A)$. The weak inequality $c_1(|\zeta| \wedge 1) \leq g(x,\zeta,\nu)$ forces us to use the larger space $GBV_\star(A)$ introduced in [16], in which the measure $D^c u$ can still be defined.

As usual in homogenisation problems for integral functionals (see, e.g., [5, 9, 21, 23]), given a function $f_1(x,\xi)$ with linear growth in ξ and a function $g_1(x,\zeta,\nu)$ of bounded cohesive type, we consider the rescaled functions

$$f_\varepsilon(x,\xi) := f_1(\tfrac{x}{\varepsilon},\xi) \quad \text{and} \quad g_\varepsilon(x,\zeta,\nu) := g_1(\tfrac{x}{\varepsilon},\zeta,\nu) \quad \text{for} \quad \varepsilon > 0\,, \tag{1.3}$$

and we study the asymptotic behaviour of the functionals $E^{f_\varepsilon,g_\varepsilon}$ as $\varepsilon \to 0+$. This is done by means of Γ-convergence. Since we are mainly interested in stochastic homogenisation problems, we do not assume any periodicity condition with respect to x for f_1 and g_1.

To study the Γ-limit of $E^{f_\varepsilon,g_\varepsilon}$ we use a result obtained in our previous paper [17] concerning sequences E^{f_k,g_k} of functionals of the form (1.2), not necessarily obtained by rescaling. Under suitable hypotheses on f_k and g_k we proved that the Γ-limit can be written in the form

$$E(u,A) = \int_A f(x,\nabla u)dx + E^c(u,A) + \int_{A\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}, \tag{1.4}$$

with suitable integrands f and g, where $E^c(u,\cdot)$ is a measure that is absolutely continuous with respect to $|D^c u|$. This result was obtained by identifying suitable properties of the functionals E^{f_k,g_k} that are inherited by the Γ-limit and by proving that all functionals with these properties can be represented in the form (1.4).

In [17] we also proved that $E^c(u,A)$ can be represented by an integral involving f^∞ under a very strong continuity hypothesis with respect to translations. The drawback of this result is that this continuity condition can be easily obtained for the Γ-limits corresponding to the rescaled integrands in (1.3) only in the periodic case. To deal with the stochastic homogenisation problem we need a different approach.

The first result of the present paper is a characterization of the integrands f and g in (1.4) as limits of suitable minimum values of problems for E^{f_k,g_k} on small cubes with very simple boundary conditions (see Theorem 3.3). Unfortunately, this result is not enough to identify the Γ-limit, unless we can prove that $E^c(u,A)$ can be written as an integral involving f^∞.

An important part of this paper is devoted to the integral representation of $E^c(u,A)$ under the hypothesis that the function $f(x,\xi)$ in (1.4) does not depend on x. To obtain this result we have to strengthen the hypotheses on E^{f_k,g_k}, assuming that they satisfy further properties, which are analysed in Section 4. Among these properties we mention the existence of the limit

$$g_k^0(x,\zeta,\nu) := \lim_{t\to 0+} \frac{1}{t} g_k(x,t\zeta,\nu)\,,$$

which will play an important role in the rest of the paper. Moreover, we need some uniform estimates for

$$\big|\frac{1}{t} f_k(x,t\xi) - f_k^\infty(x,\xi)\big| \quad \text{and} \quad \big|\frac{1}{t} g_k(x,t\zeta,\nu) - g_k^0(x,\zeta,\nu)\big| \tag{1.5}$$

(see Remarks 4.3 and 4.5).

We show (see Theorem 5.1) that, if a functional E is the Γ-limit of a sequence E^{f_k,g_k}, with f_k and g_k satisfying these properties, and if (1.4) holds with f independent of x, then $E^c(u,A)$ can be represented as the second integral in (1.2); consequently, E satisfies the complete integral representation (1.2). This allows us to prove that the Γ-limit is uniquely determined by the behaviour of the minimum values of suitable minimum problems for E^{f_k,g_k} on small cubes with very simple boundary conditions (see Theorem 5.4).

These results are then applied to homogenisation problems, where we fix a sequence $\varepsilon_k \to 0+$ and study the Γ-limit of the functionals $E^{f_{\varepsilon_k},g_{\varepsilon_k}}$, with f_{ε_k} and g_{ε_k} defined by (1.3). Using the natural change of variables $y = x/\varepsilon_k$ and the uniform estimates for (1.5), the auxiliary minimum problems on small cubes for $E^{f_{\varepsilon_k},g_{\varepsilon_k}}$ are transformed into minimum problems on large cubes for the functionals E^{f_1,g_1^0} and $E^{f_1^\infty,g_1}$ (see Lemmas 6.1 and 6.2). Therefore, the previous results imply that the Γ-limit of $E^{f_{\varepsilon_k},g_{\varepsilon_k}}$ exists and can be represented as in (1.2), provided the limits of these minimum values, as the size of the cubes tends to $+\infty$, exist and are independent of the centres of the cubes (see Theorem 6.3).

In the case of stochastic homogenisation, f_1 and g_1 are random functions satisfying suitable stochastic periodicity conditions (see Definitions 6.5 and 6.6), and the existence of the above mentioned limits on large cubes can be obtained by means of the Subadditive Ergodic Theorem [1, Theorem 2.7], arguing as in [11, 12] (see Propositions 6.10 and 6.11). This leads to the almost sure Γ-convergence result for the sequence $E^{f_{\varepsilon_k},g_{\varepsilon_k}}$ (see Theorem 6.12). Finally, we observe that the deterministic periodic case can be obtained as a byproduct of our results (see Remark 6.14).

2. Notation and preliminaries

We begin by recalling the notation used in [17].

(a) Throughout this paper $d \geq 1$ is a fixed integer. The Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. We set $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$ and $\mathbb{S}^{d-1}_\pm := \{\nu \in \mathbb{S}^{d-1} : \pm\nu_{i(\nu)} > 0\}$, where $i(\nu)$ is the largest $i \in \{1, \dots, d\}$ such that $\nu_i \neq 0$. Note that $\mathbb{S}^{d-1} = \mathbb{S}^{d-1}_+ \cup \mathbb{S}^{d-1}_-$.

(b) Given an open set $A \subset \mathbb{R}^d$, let $\mathcal{A}(A)$ be the collection of all open subsets of A and let $\mathcal{A}_c(A) := \{A' \in \mathcal{A}(A) : A' \subset\subset A\}$, where $A' \subset\subset A$ means that A' is relatively compact in A. Given a Borel set $B \subset \mathbb{R}^d$, $\mathcal{B}(B)$ denotes the σ-algebra of all Borel measurable subsets of B.

(c) For every $x \in \mathbb{R}^d$ and $\rho > 0$ let $Q(x,\rho) := \{y \in \mathbb{R}^d : |(y-x)\cdot e_i| < \rho/2,$ for every $i = 1, \dots, d\}$, where $(e_i)_{i=1,\dots,d}$ is the canonical basis in $\mathbb{R}^d$, and $\cdot$ denotes the Euclidean scalar product.

(d) For every $\nu \in \mathbb{S}^{d-1}$ we fix a rotation $R_\nu\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $R_\nu(e_d) = \nu$. We assume that R_{e_d} is the identity, that the restrictions of the function $\nu \mapsto R_\nu$ to the sets $\mathbb{S}^{d-1}_\pm$ are continuous, and that $R_\nu(Q(0,\rho)) = R_{-\nu}(Q(0,\rho))$ for every $\nu \in \mathbb{S}^{d-1}$ and every $\rho > 0$.

(e) For every $\lambda > 0$, $\nu \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$, and $\rho > 0$ let $Q^\lambda_\nu(x,\rho)$ be the rectangle defined by
$$Q^\lambda_\nu(x,\rho) := x + R_\nu\big((-\tfrac{\lambda\rho}{2}, \tfrac{\lambda\rho}{2})^{d-1} \times (-\tfrac{\rho}{2}, \tfrac{\rho}{2})\big)\,. \tag{2.1}$$

(f) For every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$ we define the functions $\ell_\xi\colon \mathbb{R}^d \to \mathbb{R}$ and $u_{x,\zeta,\nu}\colon \mathbb{R}^d \to \mathbb{R}$ by
$$\ell_\xi(y) := \xi \cdot y\,,$$
$$u_{x,\zeta,\nu}(y) := \begin{cases} \zeta & \text{if } (y-x)\cdot\nu \geq 0\,, \\ 0 & \text{if } (y-x)\cdot\nu < 0\,; \end{cases}$$
moreover, we set $\Pi^\nu_x = \{y \in \mathbb{R}^d : (y-x)\cdot\nu = 0\}$.

(g) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and an $\mathcal{L}^d$-measurable function $u\colon A \to \overline{\mathbb{R}}$, we say that $a \in \overline{\mathbb{R}}$ is the approximate limit of u as $y \to x \in A$ if for every neighbourhood U of a we have
$$\lim_{\rho\to 0+} \frac{\mathcal{L}^d(\{y \in A \cap Q(x,\rho) : u(y) \notin U\})}{\rho^d} = 0\,;$$

the same definition is meaningful also if $x \in \partial A$ provided $\lim_{\rho\to 0+} \frac{\mathcal{L}^d(A\cap Q(x,\rho))}{\rho^d} > 0$; moreover, the set of points $x \in A$ where the approximate limit $\tilde u(x)$ exists and is finite is a Borel subset of A, and the function $x \mapsto \tilde u(x)$ is a Borel function defined on it; we say that $\xi \in \mathbb{R}^d$ is the approximate gradient of u at x if the approximate limit of $\frac{u(y)-u(x)-\xi\cdot(y-x)}{|y-x|}$ as $y \to x$ is equal to 0.

(h) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and an $\mathcal{L}^d$-measurable function $u\colon A \to \overline{\mathbb{R}}$, the jump set J_u of u is the set of all points $x \in A$ for which there exist $u^+(x), u^-(x) \in \overline{\mathbb{R}}$, with $u^+(x) \neq u^-(x)$, and $\nu_u(x) \in \mathbb{S}^{d-1}$ such that $u^\pm(x)$ is the approximate limit as $y \to x$ of the restriction of u to the set $\{y \in A : \pm(y-x)\cdot\nu_u(x) > 0\}$. It is easy to see that the triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely defined up to a swap of the first two terms and a change of sign in the third one. For every $x \in J_u$ we set $[u](x) := u^+(x) - u^-(x)$. It can be proved that J_u is a Borel set and that, if we choose ν_u so that $\nu_u(x) \in \mathbb{S}^{d-1}_+$ for every $x \in J_u$, then the functions $u^+, u^-, [u]\colon J_u \to \overline{\mathbb{R}}$ and $\nu_u\colon J_u \to \mathbb{S}^{d-1}$ are Borel functions.

(i) For every $A \in \mathcal{A}(\mathbb{R}^d)$ and $u \in BV(A)$ let Du be the distributional gradient of u, which can be decomposed as the sum of three $\mathbb{R}^d$-valued measures:

$$Du = D^a u + D^c u + D^j u\,,$$

where $D^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, $D^c u$ is singular with respect to the Lebesgue measure and vanishes on all $B \in \mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B) < +\infty$, and $D^j u$ is concentrated on the jump set J_u of u. The approximate gradient of u at x exists for $\mathcal{L}^d$-a.e. $x \in A$ and is denoted by $\nabla u(x)$; it is known that the function ∇u coincides $\mathcal{L}^d$-a.e. in A with the density of $D^a u$ with respect to $\mathcal{L}^d$. Moreover, it is known that $D^j u = [u]\nu_u \mathcal{H}^{d-1} \llcorner J_u$, where for every measure μ the measure $\mu \llcorner E$ is defined by $\mu \llcorner E(B) := \mu(E \cap B)$. For these and related fine properties of BV functions we refer to [2].

(j) For every $A \in \mathcal{A}(\mathbb{R}^d)$ let $L^0(A)$ be the set of $\mathcal{L}^d$-measurable functions $u\colon A \to \mathbb{R}$ endowed with the metrisable topology of convergence in $\mathcal{L}^d$-measure.

Given $B \in \mathcal{B}(\mathbb{R}^d)$, $u\colon B \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and $m > 0$ the truncation $u^{(m)}$ of u is defined as

$$u^{(m)}(x) := (u(x) \wedge m) \vee (-m)\,,$$

where $a \wedge b$ and $a \vee b$ denote the minimum and the maximum between a and b, respectively.

We now recall the definition of the space $GBV_\star(A)$ introduced in [16, Definition 3.1].

Definition 2.1. For every $A \in \mathcal{A}_c(\mathbb{R}^d)$ let $GBV_\star(A)$ be the space of functions $u\colon A \to \mathbb{R}$ such that $u^{(m)} \in BV(A)$ for every $m > 0$ and

$$\sup_{m>0} \Big(\int_A |\nabla u^{(m)}| dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \Big) < +\infty\,.$$

The main properties of these functions are summarized in [17, Theorem 2.2] and the main properties of the space $GBV_\star(A)$ are presented in [17, Theorem 2.3] .

We now introduce a functional that is strictly related to the definition of $GBV_\star$ and will play an important role in this paper.

Definition 2.2. The functional $V\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ is defined in the following way. For every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we set

$$V(u, A) := \int_A |\nabla u| dx + |D^c u|(A) + \int_{A\cap J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} \quad \text{if } u|_A \in GBV_\star(A)\,, \tag{2.2}$$

and $V(u, A) := +\infty$ otherwise; the definition is then extended to $\mathcal{A}(\mathbb{R}^d)$ by setting

$$V(u, A) := \sup\{V(u, A') : A' \in \mathcal{A}_c(\mathbb{R}^d) \cap \mathcal{A}(A)\} \quad \text{for } A \in \mathcal{A}(\mathbb{R}^d)\,, \tag{2.3}$$

and to $\mathcal{B}(\mathbb{R}^d)$ by setting

$$V(u,B) := \inf\{V(u,A) : A \in \mathcal{A}(\mathbb{R}^d),\ B \subset A\} \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d)\,. \tag{2.4}$$

Throughout the paper we fix five constants $c_1,\dots,c_5 \geq 0$ and a bounded continuous function $\sigma\colon [0,+\infty) \to [0,+\infty)$, such that

$$0 < c_1 \leq 1 \leq c_3 \leq c_5\,, \tag{2.5}$$

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(t) \geq c_3(t \wedge 1) \quad \text{for every } t \geq 0\,. \tag{2.6}$$

We recall the definition of the class of free discontinuity functionals introduced in [17, Definition 3.1].

Definition 2.3. Let $\mathfrak{E}$ denote the class of functionals $E\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ that satisfy the following properties:

(a) E is local on $\mathcal{A}(\mathbb{R}^d)$, i.e., $E(u,A) = E(v,A)$ if $A \in \mathcal{A}(\mathbb{R}^d)$, $u,v \in L^0(\mathbb{R}^d)$, and $u = v$ $\mathcal{L}^d$-a.e. in A;

(b) for every $u \in L^0(\mathbb{R}^d)$ the function $E(u,\cdot)\colon \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ is a nonnegative Borel measure and

$$E(u,B) = \inf\{E(u,A) : A \in \mathcal{A}(\mathbb{R}^d),\ B \subset A\} \tag{2.7}$$

for every $B \in \mathcal{B}(\mathbb{R}^d)$;

(c1) for every $u \in L^0(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$c_1 V(u,B) - c_2\mathcal{L}^d(B) \leq E(u,B)\,; \tag{2.8}$$

(c2) for every $u \in L^0(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$E(u,B) \leq c_3 V(u,B) + c_4\mathcal{L}^d(B)\,; \tag{2.9}$$

(d) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $a \in \mathbb{R}$ we have

$$E(u+a,B) = E(u,B)\,; \tag{2.10}$$

(e) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $\xi \in \mathbb{R}^d$ we have

$$E(u+\ell_\xi,B) \leq E(u,B) + c_5|\xi|\mathcal{L}^d(B)\,; \tag{2.11}$$

(f) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$E(u+u_{x,\zeta,\nu},B) \leq E(u,B) + \sigma(|\zeta|)\mathcal{H}^{d-1}(B \cap \Pi^\nu_x)\,; \tag{2.12}$$

(g) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $w_1,w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$, with $w_1 \leq w_2$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$, we have

$$E((u \vee w_1) \wedge w_2, B) \leq E(u,B) + c_3\int_{B^u_{12}} |\nabla w_1| \vee |\nabla w_2|dx + c_4\mathcal{L}^d(B^u_{12})\,, \tag{2.13}$$

where $B^u_{12} = \{x \in B : u(x) \notin [w_1(x), w_2(x)]\}$.

Finally, let $\mathfrak{E}_{sc}$ denote the class of functionals E in $\mathfrak{E}$ that satisfy the following property:

(h) for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the functional $E(\cdot,A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$.

A first example of functional belonging to $\mathfrak{E}_{sc}$ is given by V (see [17, Remark 3.15]).

Remark 2.4. Let $E \in \mathfrak{E}$, $A \in \mathcal{A}(\mathbb{R}^d)$, and $u \in L^0(A)$. For every $B \in \mathcal{B}(A)$ we can define $E(u,B)$ by extending u to a function $v \in L^0(\mathbb{R}^d)$ and setting $E(u,B) := E(v,B)$. The value $E(u,B)$ does not depend on the extension (see [17, Remark 3.2]).

We now recall the definitions of the classes of functions $\mathcal{F}$ and $\mathcal{G}$ introduced in [17].

Definition 2.5. Let $\mathcal{F}$ be the set of functions

$$f\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$$

that satisfy the following conditions:

(f1) f is Borel measurable;
(f2) $c_1|\xi| - c_2 \le f(x,\xi)$ for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$;
(f3) $f(x,\xi) \le c_3|\xi| + c_4$ for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$;
(f4) $|f(x,\xi_1) - f(x,\xi_2)| \le c_5|\xi_1 - \xi_2|$ for every $x \in \mathbb{R}^d$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$.

Definition 2.6. Let $\mathcal{G}$ be the set of functions

$$g\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$$

that satisfy the following conditions:

(g1) g is Borel measurable;
(g2) $c_1(|\zeta| \wedge 1) \le g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g3) $g(x,\zeta,\nu) \le c_3(|\zeta| \wedge 1)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g4) $|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \le \sigma(|\zeta_1 - \zeta_2|)$ for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g5) $g(x,-\zeta,-\nu) = g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g6) for every $x \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ the function $\zeta \mapsto g(x,\zeta,\nu)$ is non-decreasing on $[0,+\infty)$ and non-increasing on $(-\infty,0]$.

We recall the definition of the recession function.

Definition 2.7. For every $f\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ the recession function $f^\infty\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ (with respect to ξ) is defined by

$$f^\infty(x,\xi) := \limsup_{t\to+\infty} \frac{f(x,t\xi)}{t} \tag{2.14}$$

for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$.

We are now in a position to introduce the integral functionals associated with the integrands f and g.

Definition 2.8. Given $f \in \mathcal{F}$ and a Borel function $g\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$ we define the functional $E^{f,g}\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ in the following way: if $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u|_A \in GBV_\star(A)$ we set

$$E^{f,g}(u,A) := \int_A f(x,\nabla u)dx + \int_A f^\infty\big(x, \frac{dD^c u}{d|D^c u|}\big)d|D^c u| + \int_{A\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}, \tag{2.15}$$

while we set $E^{f,g}(u,A) := +\infty$ if $u|_A \notin GBV_\star(A)$. The definition is then extended to $\mathcal{A}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u,A) := \sup\{E^{f,g}(u,A') : A' \in \mathcal{A}_c(\mathbb{R}^d) \cap \mathcal{A}(A)\} \quad \text{for } A \in \mathcal{A}(\mathbb{R}^d), \tag{2.16}$$

and to $\mathcal{B}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u,B) := \inf\{E^{f,g}(u,A) : A \in \mathcal{A}(\mathbb{R}^d),\ B \subset A\} \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d). \tag{2.17}$$

Remark 2.9. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the functionals $E^{f,g}$ belong to $\mathfrak{E}$ (see [17, Proposition 3.11]). Moreover, if $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in GBV_\star(A)$, then

$$E^{f,g}(u,B) := \int_B f(x,\nabla u)dx + \int_B f^\infty\big(x, \frac{dD^c u}{d|D^c u|}\big)d|D^c u| + \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1} \tag{2.18}$$

for every $B \in \mathcal{B}(A)$, where $E^{f,g}(u,B)$ is defined according to Remark 2.4.

The following compactness result is proved in [17, Theorem 3.16].

Theorem 2.10. *Let (E_k) be a sequence in $\mathfrak{E}$. Then there exist a subsequence, not relabelled, and a functional $E \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$.*

In view of the integral representation of a functional E in $\mathfrak{E}_{sc}$ it is useful to consider the measures introduced in the following definition.

Definition 2.11. Let $E\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ be a functional satisfying properties (b) and (c2) in Definition 2.3. Let $u\in L^0(\mathbb{R}^d)$ and $A\in\mathcal{A}_c(\mathbb{R}^d)$ with $u|_A\in GBV_\star(A)$. The measures $E^a(u,\cdot)$, $E^s(u,\cdot)$, $E^c(u,\cdot)$, and $E^j(u,\cdot)$ on $\mathcal{B}(A)$ are defined in the following way:

$$E^a(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d\,, \tag{2.19}$$

$$E^s(u,\cdot) \text{ is the singular part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d\,, \tag{2.20}$$

$$E^c(u,B):=E^s(u,B\setminus J_u) \text{ for every } B\in\mathcal{B}(A)\,, \tag{2.21}$$

$$E^j(u,B):=E^s(u,B\cap J_u)=E(u,B\cap J_u) \text{ for every } B\in\mathcal{B}(A)\,. \tag{2.22}$$

Remark 2.12. The following properties hold (see [17, Remark 4.2]):

$$E(u,\cdot)=E^a(u,\cdot)+E^c(u,\cdot)+E^j(u,\cdot)\quad\text{in } \mathcal{B}(A)\,, \tag{2.23}$$

$$E^c(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } |D^cu|\,, \tag{2.24}$$

$$E^j(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } \mathcal{H}^{d-1}\llcorner J_u\,. \tag{2.25}$$

We now introduce the minimum problems that are used to define the integrands for the integral representation results proved in [17].

Definition 2.13. Let $A\in\mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundary and $w\in BV(A)$. Given an arbitrary functional $E(\cdot,A)\colon BV(A)\to[0,+\infty]$, we define (see [7])

$$m^E(w,A):=\inf\{E(u,A): u\in BV(A)\,,\ \mathrm{tr}_A u=\mathrm{tr}_A w\ \ \mathcal{H}^{d-1}\text{-a.e. on } \partial A\}\,, \tag{2.26}$$

where $\mathrm{tr}_A v$ denotes the trace on ∂A of a function $v\in BV(A)$.

We are now in a position to define the integrands used in the integral representation results for functionals in $\mathfrak{E}_{sc}$.

Definition 2.14. Given $E\in\mathfrak{E}$ we define the integrands $f\colon\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$, and $g\colon\mathbb{R}^d\times\mathbb{R}\times\mathbb{S}^{d-1}\to[0,+\infty)$ by setting

$$f(x,\xi):=\limsup_{\rho\to0+}\frac{m^E(\ell_\xi,Q(x,\rho))}{\rho^d}\,, \tag{2.27}$$

$$g(x,\zeta,\nu):=\limsup_{\rho\to0+}\frac{m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}\,. \tag{2.28}$$

Remark 2.15. In [17, Theorem 5.1] it is proved that for every $E\in\mathfrak{E}$ we have $f\in\mathcal{F}$ and $g\in\mathcal{G}$.

The following integral representation result for E^a and E^j on $GBV_\star(A)$ is proved in [17, Theorem 6.3].

Theorem 2.16. *Let $E\in\mathfrak{E}_{sc}$, let f and g be defined by* (2.27) *and* (2.28)*, respectively, and let $A\in\mathcal{A}_c(\mathbb{R}^d)$. Then*

$$E^a(u,B)=\int_B f(x,\nabla u)dx\,, \tag{2.29}$$

$$E^j(u,B)=\int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,, \tag{2.30}$$

for every $u\in GBV_\star(A)$ and every $B\in\mathcal{B}(A)$.

We shall use frequently the following technical lemma, proved in [17, Lemma 4.16], taking (2.29) into account. For every $\xi\in\mathbb{R}^d$ we set

$$c_\xi:=\frac{c_2+c_4+1}{c_1}d^{1/2}+|\xi|d^{1/2}\,. \tag{2.31}$$

Lemma 2.17. *Let $E \in \mathfrak{E}_{sc}$. Then there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$, $\lambda \geq 1$, $\nu \in \mathbb{S}^{d-1}$, and $\eta > 0$ there exists $\rho^\lambda_{\nu,\eta}(x) > 0$ with the following property: for every $0 < \rho < \rho^\lambda_{\nu,\eta}(x)$ and every $\xi \in \mathbb{R}^d$ there exists $u \in BV(Q^\lambda_\nu(x,\rho)) \cap L^\infty(Q^\lambda_\nu(x,\rho))$ satisfying $\|u - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho))} \leq c_\xi \lambda\rho$, $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)} u = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(x,\rho)$, and*

$$E(u, Q^\lambda_\nu(x,\rho)) \leq m^E(\ell_\xi, Q^\lambda_\nu(x,\rho)) + \eta\lambda^{d-1}\rho^d . \tag{2.32}$$

If, in addition, f is continuous in $\mathbb{R}^d \times \mathbb{R}^d$, then $N = \emptyset$. Finally, if there exists $\hat{f} \colon \mathbb{R}^d \to [0, +\infty)$ such that

$$f(x,\xi) = \hat{f}(\xi) \quad \textit{for } \mathcal{L}^d\textit{-a.e. } x \in \mathbb{R}^d \textit{ and every } \xi \in \mathbb{R}^d ,$$

then $\rho^\lambda_{\nu,\eta}(x) = +\infty$.

3. The integrands of the Γ-limit

Let (E^{f_k,g_k}) be a sequence of integral functionals, with $f_k \in \mathcal{F}$ and $g_k \in \mathcal{G}$, and let $E \in \mathfrak{E}_{sc}$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E^{f_k,g_k}(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the L^0-topology. From Section 2 we know that E^a and E^j can be represented in an integral form (see (2.29) and (2.30)) using two integrands $f \in \mathcal{F}$ and $g \in \mathcal{G}$. The aim of this section is to prove Theorem 3.3, which provides a connection between the integrands f and g and the minimum values of some auxiliary problems for the functionals E^{f_k,g_k} on small cubes.

We begin by showing a relation between $m^E(w, A)$ and the sequence $m^{E^{f_k,g_k}}(w, A')$ for $A' \subset\subset A$, provided $w \in W^{1,1}(A)$.

Proposition 3.1. *Let $(f_k) \subset \mathcal{F}$, $(g_k) \subset \mathcal{G}$, let $E_k := E^{f_k,g_k}$, and let $E \in \mathfrak{E}_{sc}$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Let A', $A \in \mathcal{A}_c(\mathbb{R}^d)$, with Lipschitz boundaries and $A' \subset\subset A$, and let $w \in W^{1,1}(A)$. Then*

$$m^E(w, A) \leq \liminf_{k\to\infty} m^{E_k}(w, A') + \int_{A \setminus A'} (c_3 |\nabla w| + c_4) dx . \tag{3.1}$$

Proof. By the definition of m^{E_k} there exists $u_k \in BV(A')$, with $\mathrm{tr}_{A'} u_k = \mathrm{tr}_{A'} w$ $\mathcal{H}^{d-1}$-a.e. on $\partial A'$, such that

$$E_k(u_k, A') \leq m^{E_k}(w, A') + \tfrac{1}{k} .$$

Let $v_k \in BV(A)$ be defined by $v_k = u_k$ in A' and $v_k = w$ in $A \setminus A'$. Since $w \in W^{1,1}(A)$ and $\mathrm{tr}_{A'} u_k = \mathrm{tr}_{A'} w$ $\mathcal{H}^{d-1}$-a.e. on $\partial A'$, we have $\mathcal{H}^{d-1}(J_{v_k} \cap \partial A') = 0$, which implies that $E_k(v_k, \partial A') = 0$. Therefore, by (f3) we have

$$\begin{aligned} E_k(v_k, A) &= E_k(u_k, A') + E_k(w, A \setminus \overline{A}') \leq m^{E_k}(w, A') + \tfrac{1}{k} + \int_{A \setminus A'} (c_3 |\nabla w| + c_4) dx \\ &\leq E_k(w, A') + \tfrac{1}{k} + \int_{A \setminus A'} (c_3 |\nabla w| + c_4) dx \leq \int_A (c_3 |\nabla w| + c_4) dx + \tfrac{1}{k} . \end{aligned}$$

Recalling (c1) in Definition 2.3, we deduce that $V(v_k, A)$ is bounded. Therefore we can apply [17, Theorem 7.13] which provides a subsequence of (E_k), not relabelled, and a sequence $(y_k) \subset GBV_\star(A)$ such that $y_k = w$ $\mathcal{L}^d$-a.e. in $A \setminus A'$,

$$E_k(y_k, A) \leq E_k(v_k, A) + \tfrac{1}{k} \leq m^{E_k}(w, A') + \tfrac{2}{k} + \int_{A \setminus A'} (c_3 |\nabla w| + c_4) dx ,$$

and y_k converge in $L^0(A)$ to a function $y \in GBV_\star(A)$ with $y = w$ $\mathcal{L}^d$-a.e. in $A \setminus A'$.

By Γ-convergence

$$E(y, A) \leq \liminf_{k\to\infty} E_k(y_k, A) \leq \liminf_{k\to\infty} m^{E_k}(w, A') + \int_{A \setminus A'} (c_3 |\nabla w| + c_4) dx .$$

Since $\mathrm{tr}_A y = \mathrm{tr}_A w$ $\mathcal{H}^{d-1}$-a.e. on ∂A, we have $m^E(w,A) \leq E(y,A)$, which, together with the previous inequalities, gives (3.1). □

The following proposition shows an inequality connecting $m^E(w,A)$ with the sequence $m^{E_k}(w,A)$.

Proposition 3.2. *Let $(E_k) \subset \mathfrak{E}$ and $E \in \mathfrak{E}_{sc}$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Let $A \in \mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundary and let $w \in BV(A) \cap L^\infty(A)$. Then*

$$\limsup_{k\to\infty} m^{E_k}(w,A) \leq m^E(w,A)\,. \tag{3.2}$$

Proof. Let us fix $\eta > 0$. By the definition of m^E there exists $u \in BV(A)$, with $\mathrm{tr}_A u = \mathrm{tr}_A w$ $\mathcal{H}^{d-1}$-a.e. on ∂A, such that

$$E(u,A) < m^E(w,A) + \eta\,.$$

Recalling [17, Proposition 4.3], by a truncation argument based on (g) in Definition 2.3 we see that we may assume $u \in BV(A) \cap L^\infty(A)$. We may regard u as the restriction to A of a function $u \in L^\infty(\mathbb{R}^d)$. By the definition of Γ-convergence there exists a sequence $(v_k) \subset L^0(\mathbb{R}^d)$ converging to u in $L^0(\mathbb{R}^d)$ as $k \to \infty$ such that

$$\lim_{k\to\infty} E_k(v_k,A) = E(u,A) < m^E(w,A) + \eta < +\infty\,. \tag{3.3}$$

By [17, Remark 3.5] we may assume that $v_k|_A \in GBV_\star(A)$ for every $k \in \mathbb{N}$. Replacing v_k with $v_k^{(m)}$ for $m > \|u\|_{L^\infty(\mathbb{R}^d)}$, we may also assume that $\|v_k\|_{L^\infty(\mathbb{R}^d)} \leq m$, since (3.3) continues to hold by [17, Remark 3.4]. Moreover $v_k|_A \in BV(A)$ and $v_k \to u$ in $L^1_{loc}(\mathbb{R}^d)$.

We now fix a compact set K such that $K \subset A$ and

$$c_3 V(u, A\setminus K) + c_4 \mathcal{L}^d(A\setminus K) < \eta\,, \tag{3.4}$$

and we set $B := A \setminus K$ and $w_k = u$ for every $k \in \mathbb{N}$. We also fix two open sets A' and A'', with $K \subset A' \subset\subset A'' \subset\subset A$. We apply [17, Lemma 3.19] and we obtain a sequence $(u_k) \subset L^1_{loc}(\mathbb{R}^d)$, with $u_k|_A \in BV(A)$, converging to u in $L^1_{loc}(\mathbb{R}^d)$ as $k \to +\infty$, such that

$$u_k = v_k \quad \mathcal{L}^d\text{-a.e. in } A' \quad \text{and} \quad u_k = u \quad \mathcal{L}^d\text{-a.e. in } A\setminus A''\,, \tag{3.5}$$

$$\limsup_{k\to\infty} E_k(u_k,A) \leq (1+\eta)\limsup_{k\to\infty}\big(E_k(v_k,A) + E_k(u,A\setminus K)\big) + \eta\,, \tag{3.6}$$

where we used the equality $A = A' \cup (A\setminus K)$. By (c2) in Definition 2.3 and (3.4) we have $E_k(u, A\setminus K) < \eta$. Hence (3.6) together with (3.3) gives

$$\limsup_{k\to\infty} E_k(u_k,A) \leq (1+\eta)\big(m^E(w,A) + 2\eta\big) + \eta\,.$$

Since $\mathrm{tr}_A u_k = \mathrm{tr}_A u$ $\mathcal{H}^{d-1}$-a.e. on ∂A by (3.5), and $\mathrm{tr}_A u = \mathrm{tr}_A w$ $\mathcal{H}^{d-1}$-a.e. on ∂A, we have $m^{E_k}(w,A) \leq E_k(u_k,A)$ for every k. Therefore

$$\limsup_{k\to\infty} m^{E_k}(w,A) \leq (1+\eta)\big(m^E(w,A) + 2\eta\big) + \eta\,.$$

Passing to the limit as $\eta \to 0+$ we obtain (3.2). □

We are now in a position to state and prove the main result of this section.

Theorem 3.3. *Let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$, let $E_k := E^{f_k,g_k}$, let $E \in \mathfrak{E}_{sc}$, and let f and g be as in Definition 2.14. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot,A)$*

Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then for every $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$ we have

$$f(x,\xi)=\limsup_{\rho\to0+}\liminf_{k\to\infty}\frac{m^{E_k}(\ell_\xi,Q(x,\rho))}{\rho^d}=\limsup_{\rho\to0+}\limsup_{k\to\infty}\frac{m^{E_k}(\ell_\xi,Q(x,\rho))}{\rho^d},$$

$$g(x,\zeta,\nu)=\limsup_{\rho\to0+}\liminf_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}=\limsup_{\rho\to0+}\limsup_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}.$$

Proof. Let us fix $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$. By Propositions 3.1 and 3.2 for every $\rho>0$, setting $r:=\rho+\rho^2$ we have

$$\begin{aligned}m^E(\ell_\xi,Q(x,r))-(c_3|\xi|+c_4)\big(r^d-\rho^d\big)&\le\liminf_{k\to\infty}m^{E_k}(\ell_\xi,Q(x,\rho))\\&\le\limsup_{k\to\infty}m^{E_k}(\ell_\xi,Q(x,\rho))\le m^E(\ell_\xi,Q(x,\rho))\,.\end{aligned}$$

Since $r/\rho\to1$ as $\rho\to0+$, from the previous inequalities we obtain

$$\begin{aligned}\limsup_{\rho\to0+}\frac{m^E(\ell_\xi,Q(x,\rho))}{\rho^d}&\le\limsup_{\rho\to0+}\ \liminf_{k\to\infty}\frac{m^{E_k}(\ell_\xi,Q(x,\rho))}{\rho^d}\\\le\limsup_{\rho\to0+}\ \limsup_{k\to\infty}\frac{m^{E_k}(\ell_\xi,Q(x,\rho))}{\rho^d}&\le\limsup_{\rho\to0+}\frac{m^E(\ell_\xi,Q(x,\rho))}{\rho^d}.\end{aligned}$$

The equalities for f in the statement of the theorem follow from (2.27).

To prove the statement for g we introduce a function $w\in W^{1,1}(Q(0,1))\cap L^\infty(Q(0,1))$ such that $\mathrm{tr}_{Q(0,1)}w=\mathrm{tr}_{Q(0,1)}u_{0,1,e_d}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(0,1)$ (it can be constructed by elementary arguments or by applying Gagliardo's Theorem, see [20]). We fix $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$; for every $\rho>0$ we set

$$w_{x,\zeta,\nu,\rho}(y):=\zeta w\big(\frac{R_\nu^{-1}(y-x)}{\rho}\big)\,,$$

where R_ν is the rotation used in the definition of the cube $Q_\nu(x,\rho)$. Since $\mathrm{tr}_{Q_\nu(x,\rho)}w_{x,\zeta,\nu,\rho}=\mathrm{tr}_{Q(x,\rho)}u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$ we have

$$m^F(w_{x,\zeta,\nu,\rho},Q_\nu(x,\rho))=m^F(u_{x,\zeta,\nu},Q_\nu(x,\rho))\,,$$

for every $F\in\mathfrak{E}$ and every $\rho>0$. Hence, by Propositions 3.1 and 3.2 for every $\rho>0$ setting $r:=\rho+\rho^2$ we have

$$\begin{aligned}m^E(u_{x,\zeta,\nu},Q_\nu(x,r))-c_3\int_{Q_\nu(x,r)\setminus Q_\nu(x,\rho)}&|\nabla w_{x,\zeta,\nu,r}|dy-c_4(r^d-\rho^d)\\\le\liminf_{k\to\infty}m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))&\le\limsup_{k\to\infty}m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))\\&\le m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho))\,.\end{aligned}\tag{3.7}$$

By a rotation and a translation we obtain

$$\int_{Q_\nu(x,r)\setminus Q_\nu(x,\rho)}|\nabla w_{x,\zeta,\nu,r}|dy=\int_{Q(0,r)\setminus Q(0,\rho)}|\nabla w_{0,\zeta,e_d,r}|dy\,.$$

Since $|\nabla w_{0,\zeta,e_d,r}(y)|=\frac{1}{r}|\nabla w(\frac{y}{r})|$ by a further change of variables we obtain

$$\int_{Q_\nu(x,r)\setminus Q_\nu(x,\rho)}|\nabla w_{x,\zeta,\nu,r}|dy=r^{d-1}\int_{Q(0,1)\setminus Q(0,\rho/r)}|\nabla w|dy\,.$$

Dividing (3.7) by ρ^{d-1} we obtain

$$\frac{r^{d-1}}{\rho^{d-1}}\frac{m^E(u_{x,\zeta,\nu},Q_\nu(x,r))}{r^{d-1}}-c_3\frac{r^{d-1}}{\rho^{d-1}}\int_{Q(0,1)\setminus Q(0,\rho/r)}|\nabla w|dy-c_4\frac{r^d-\rho^d}{\rho^{d-1}}$$
$$\le\liminf_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}\le\limsup_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}$$
$$\le\frac{m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}\,.$$

Since $r/\rho\to 1$ as $\rho\to 0+$, from the previous inequalities we obtain the equalities for g in the statement. □

For technical reasons, in the characterisation of the volume integrand $f(x,\xi)$ it is convenient to replace $m^{E_k}(\ell_\xi,Q(x,\rho))$ by the minimum value of a different problem, where we impose a constraint on the L^∞-norm of $u-\ell_\xi$. This leads to the following definition.

Definition 3.4. Let $A\in\mathcal{A}_c(\mathbb{R}^d)$, $w\in BV(A)$, and $t>0$. Given an arbitrary functional $E\colon BV(A)\times\mathcal{B}(A)\to[0,+\infty)$, we set

$$m^E_t(w,A)=\inf_{\substack{u\in BV(A)\\ \|u-w\|_{L^\infty(A)}\le t\\ \mathrm{tr}_A u=\mathrm{tr}_A w\ \mathcal{H}^{d-1}\text{-a.e. on }\partial A}}E(u,A)\,. \tag{3.8}$$

The following result provides the analogue of Proposition 3.2 in the case of m^E_t.

Proposition 3.5. *Let $(E_k)\subset\mathfrak{E}$, $E\in\mathfrak{E}_{sc}$, $A\in\mathcal{A}_c(\mathbb{R}^d)$, let $w\in W^{1,1}(A)$, and let $0<t_1<t_2$. Assume that $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then*

$$\limsup_{k\to\infty}m^{E_k}_{t_2}(w,A)\le m^E_{t_1}(w,A)\,. \tag{3.9}$$

Proof. Let us fix $\eta>0$. By (3.8) there exists $u\in BV(A)$, such that $\|u-w\|_{L^\infty(A)}\le t_1$, $\mathrm{tr}_A u=\mathrm{tr}_A w$ $\mathcal{H}^{d-1}$-a.e. on ∂A, and

$$E(u,A)\le m^E_{t_1}(w,A)+\eta<+\infty\,. \tag{3.10}$$

By the definition of Γ-convergence there exists $(z_k)\subset L^0(A)$ converging to u in $L^0(A)$ as $k\to\infty$ such that

$$\lim_{k\to\infty}E_k(z_k,A)=E(u,A)<+\infty\,. \tag{3.11}$$

By [17, Remark 3.5] we may assume that $z_k\in GBV_\star(A)$ for every $k\in\mathbb{N}$. We set $v_k:=(z_k\vee(w-t_2))\wedge(w+t_2)=w+(z_k-w)^{(t_2)}$ and observe that $\|v_k-w\|_{L^\infty(A)}\le t_2$ and $v_k\to u$ in $L^1(A)$. Since $GBV_\star(A)$ is a vector space we have $z_k-w\in GBV_\star(A)$, hence $(z_k-w)^{(t_2)}\in BV(A)$, which implies that $v_k\in BV(A)$. Moreover, by (g) in Definition 2.3 we have

$$E_k(v_k,A)\le E_k(z_k,A)+\varepsilon_k\,, \tag{3.12}$$

where $\varepsilon_k:=c_3\int_{\{|z_k-w|>t_2\}}|\nabla w|dx+c_4\mathcal{L}^d(\{|z_k-w|>t_2\})$. Since $z_k\to u$ in $L^0(\mathbb{R}^d)$ and $\|u-w\|_{L^\infty(A)}\le t_1<t_2$, we conclude that $\varepsilon_k\to 0$. By (3.10), (3.11), and (3.12) we have

$$\limsup_{k\to\infty}E_k(v_k,A)\le m^E_{t_1}(w,A)+\eta\,. \tag{3.13}$$

To conclude the proof we argue as in the second part of the proof of Proposition 3.2, observing that, since $\|v_k-w\|_{L^\infty(A)}\le t_2$ and $\|u-w\|_{L^\infty(A)}\le t_1$, the function u_k satisfies also the estimate $\|u_k-w\|_{L^\infty(A)}\le t_2$. □

The following result shows that in the definition of $f(x,\xi)$ we can replace $m^E(\ell_\xi, Q(x,\rho))$ by $m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))$, where

$$c_\xi := \frac{c_2+c_4+1}{c_1} d^{1/2} + |\xi| d^{1/2}.$$

Lemma 3.6. *Let $E \in \mathfrak{E}_{sc}$ and let f be as in Definition 2.14. Then there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$ and for every $\xi \in \mathbb{R}^d$ we have*

$$f(x,\xi) = \limsup_{\rho\to 0+} \frac{m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d}. \tag{3.14}$$

If, in addition, there exists $\hat f\colon \mathbb{R}^d \to [0,+\infty)$ such that

$$f(x,\xi) = \hat f(\xi) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and every } \xi \in \mathbb{R}^d, \tag{3.15}$$

then (3.14) *holds for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$.*

Proof. Since $m^E \le m^E_{c_\xi\rho}$, we have only to prove that

$$\limsup_{\rho\to 0+} \frac{m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d}. \tag{3.16}$$

By Lemma 2.17 there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, with the following property: for every $x \in \mathbb{R}^d \setminus N$, $\xi \in \mathbb{R}^d$, and $\eta > 0$ there exists $\rho_\eta(x) > 0$ such that for every $0 < \rho < \rho_\eta(x)$ there exists $u \in BV(Q(x,\rho)) \cap L^\infty(Q(x,\rho))$, with $\|u - \ell_\xi\|_{L^\infty(Q(x,\rho))} \le c_\xi\rho$ and $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, satisfying

$$m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho) \le E(u, Q(x,\rho)) \le m^E(\ell_\xi, Q(x,\rho)) + \eta\rho^d. \tag{3.17}$$

This implies

$$\limsup_{\rho\to 0+} \frac{m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d} + \eta.$$

Letting $\eta \to 0$ we obtain (3.16), which gives (3.14).

If, in addition, (3.15) holds, then the conclusion follows from the last sentence of Lemma 2.17. □

We conclude this section by a result which shows that in Theorem 3.3 we can replace $m^{E_k}(\ell_\xi, Q(x,\rho))$ by $m^{E_k}_{\kappa_\xi\rho}(\ell_\xi, Q(x,\rho))$, where

$$\kappa_\xi := c_\xi + 1 = \frac{c_2+c_4+1}{c_1} d^{1/2} + |\xi| d^{1/2} + 1. \tag{3.18}$$

We shall see in Section 6 that this formulation of the result is more convenient in the study of homogenisation problems.

Theorem 3.7. *Let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$, let $E_k := E^{f_k,g_k}$, let $E \in \mathfrak{E}_{sc}$, and let f be as in Definition 2.14. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then there exists $N \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{L}^d(N) = 0$ such that for every $x \in \mathbb{R}^d \setminus N$ and $\xi \in \mathbb{R}^d$ we have*

$$f(x,\xi) = \limsup_{\rho\to 0+} \liminf_{k\to\infty} \frac{m^{E_k}_{\kappa_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d} = \limsup_{\rho\to 0+} \limsup_{k\to\infty} \frac{m^{E_k}_{\kappa_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d}. \tag{3.19}$$

If, in addition, there exists $\hat f\colon \mathbb{R}^d \to [0,+\infty)$ such that

$$f(x,\xi) = \hat f(\xi) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and every } \xi \in \mathbb{R}^d, \tag{3.20}$$

then (3.19) *holds for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$.*

Proof. Let N be the set given by Lemma 3.6. Let us fix $x \in \mathbb{R}^d \setminus N$ and $\xi \in \mathbb{R}^d$. By Propositions 3.1 and 3.5 for every $\rho > 0$ we have

$$m^E(\ell_\xi, Q(x, \rho+\rho^2)) - (c_3|\xi| + c_4)\big((\rho+\rho^2)^d - \rho^d\big) \leq \liminf_{k\to\infty} m^{E_k}(\ell_\xi, Q(x,\rho))$$
$$\leq \limsup_{k\to\infty} m^{E_k}_{\kappa_\xi\rho}(\ell_\xi, Q(x,\rho)) \leq m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))\,.$$

Since $(\rho+\rho^2)^d/\rho^d \to 1$ as $\rho \to 0+$, from the previous inequality we obtain

$$\limsup_{\rho\to 0+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d} \leq \limsup_{\rho\to 0+} \liminf_{k\to\infty} \frac{m^{E_k}(\ell_\xi, Q(x,\rho))}{\rho^d}$$
$$\leq \limsup_{\rho\to 0+} \limsup_{k\to\infty} \frac{m^{E_k}_{\kappa_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d} \leq \limsup_{\rho\to 0+} \frac{m^E_{c_\xi\rho}(\ell_\xi, Q(x,\rho))}{\rho^d}.$$

The conclusion follows from the definition of f and Lemma 3.6.

If, in addition, (3.20) holds we can take $N = \emptyset$ in Lemma 3.6. □

4. A narrower class of local functionals

In the next section we shall prove an integral representation result for the Cantor part of a functional $E \in \mathfrak{E}_{sc}$ without assuming the continuity with respect to translations requested in [17, Theorem 6.7]. Instead we shall assume that E is the Γ-limit of a sequence of integral functionals (E^{f_k,g_k}), with $f_k \in \mathcal{F}$ and $g_k \in \mathcal{G}$, and we shall use the characterization of the integrands f and g of E given by Theorem 3.3. To obtain this result we need slightly stronger hypotheses on the integrands f_k and g_k, which are studied in the present section.

Throughout the rest of the paper we fix two constants $c_6 > 0$ and $0 < \alpha < 1$, and a continuous non-decreasing function $\vartheta\colon [0,+\infty) \to [0,+\infty)$, with $\vartheta(0) = 0$ and

$$\vartheta(\tau) \geq \frac{c_1}{c_3}\tau - 1 \quad \text{for every } \tau \geq 0\,. \tag{4.1}$$

We introduce a new class of functionals which plays an important role in our approach to homogenisation problems.

Definition 4.1. Let $\mathfrak{E}^{\alpha,\vartheta}$ be the class of functionals $E \in \mathfrak{E}$ that satisfy the following inequality:

$$\Big|\frac{E(su,A)}{s} - \frac{E(tu,A)}{t}\Big| \leq \frac{c_6}{s}\mathcal{L}^d(A)^\alpha E(su,A)^{1-\alpha} + \vartheta(sm_A)\frac{E(su,A)}{s} + \frac{c_6}{s}\mathcal{L}^d(A)$$
$$+\frac{c_6}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + \vartheta(tm_A)\frac{E(tu,A)}{t} + \frac{c_6}{t}\mathcal{L}^d(A) \tag{4.2}$$

for every $s,t > 0$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $u \in BV(A)\cap L^\infty(A)$, where $m_A := \operatorname*{osc}_A u := \operatorname{ess\,sup}_A u - \operatorname{ess\,inf}_A u$. We also set $\mathfrak{E}^{\alpha,\vartheta}_{sc} := \mathfrak{E}^{\alpha,\vartheta} \cap \mathfrak{E}_{sc}$.

We now provide an example of integral functionals which belong to $\mathfrak{E}^{\alpha,\vartheta}$. To this end we introduce two new classes of integrands, which are closely related to those considered in [12, Definition 3.1].

Definition 4.2. Let $\mathcal{F}^\alpha$ be the set of functions $f \in \mathcal{F}$ such that

$$\Big|\frac{1}{s}f(x,s\xi) - \frac{1}{t}f(x,t\xi)\Big| \leq \frac{c_6}{s}f(x,s\xi)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t}f(x,t\xi)^{1-\alpha} + \frac{c_6}{t} \tag{4.3}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and every $s,t > 0$ and $\xi \in \mathbb{R}^d$.

Remark 4.3. Inequality (4.3) and (f3) imply that for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$ the function $t \mapsto \frac{1}{t}f(x,t\xi)$ satisfies the Cauchy condition as $t \to +\infty$. Therefore, if $f \in \mathcal{F}^\alpha$, then

$$f^\infty(x,\xi) = \lim_{t\to+\infty} \frac{1}{t}f(x,t\xi) \tag{4.4}$$

and

$$\Big|\frac{1}{t}f(x,t\xi)-f^\infty(x,\xi)\Big|\leq\frac{c_6}{t}+\frac{c_6}{t}f(x,t\xi)^{1-\alpha} \tag{4.5}$$

for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$ and every $t>0$ and $\xi\in\mathbb{R}^d$. Conversely, if the limit in (4.4) exists and (4.5) holds, then f satisfies (4.3)

Definition 4.4. Let $\mathcal{G}^\vartheta$ be the set of functions $g\in\mathcal{G}$ such that

$$|\frac{1}{s}g(x,s\,\zeta,\nu)-\frac{1}{t}g(x,t\,\zeta,\nu)|\leq\vartheta(s|\zeta|)\frac{1}{s}g(x,s\,\zeta,\nu)+\vartheta(t|\zeta|)\frac{1}{t}g(x,t\,\zeta,\nu) \tag{4.6}$$

for every $s,t>0$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$.

Remark 4.5. If (4.6) holds, then using (g3) we obtain

$$|\frac{1}{s}g(x,s\,\zeta,\nu)-\frac{1}{t}g(x,t\,\zeta,\nu)|\leq c_3\vartheta(s|\zeta|)|\zeta|+c_3\vartheta(t|\zeta|)|\zeta|\,. \tag{4.7}$$

This implies that for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$ the function $t\mapsto\frac{1}{t}g(x,t\,\zeta,\nu)$ satisfies the Cauchy condition as $t\to 0+$. Therefore, if $g\in\mathcal{G}^\vartheta$, then the limit

$$g^0(x,\zeta,\nu):=\lim_{t\to 0+}\frac{1}{t}g(x,t\,\zeta,\nu) \tag{4.8}$$

exists and

$$\Big|\frac{1}{t}g(x,t\,\zeta,\nu)-g^0(x,\zeta,\nu)\Big|\leq\vartheta(t|\zeta|)\frac{1}{t}g(x,t\,\zeta,\nu)\leq c_3\vartheta(t|\zeta|)|\zeta| \tag{4.9}$$

for every $t>0$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$. Conversely, if the limit in (4.8) exists and (4.9) holds, then g satisfies (4.6).

Remark 4.6. For every $g\in\mathcal{G}^\vartheta$, using (4.8), (g2), and (g3) we obtain that

$$c_1|\zeta|\leq g^0(x,\zeta,\nu)\leq c_3|\zeta|\,. \tag{4.10}$$

Moreover, using (g1) and (g4) we obtain also that g^0 is a Borel function.

Remark 4.7. Inequality (4.1) is equivalent to $\mathcal{G}^\vartheta\neq\emptyset$. Indeed, if (4.1) holds then the function $(x,\zeta,\nu)\mapsto(c_1|\zeta|)\wedge c_3$ belongs to $\mathcal{G}^\vartheta$. Conversely, if $g\in\mathcal{G}^\vartheta$ then, by (4.9), (4.10), and (g3), for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$ we have

$$c_1|\zeta|-c_3\leq g^0(x,\zeta,\nu)-g(x,\zeta,\nu)\leq\vartheta(|\zeta|)g(x,\zeta,\nu)\leq\vartheta(|\zeta|)c_3\,,$$

which implies (4.1).

Remark 4.8. Given $g\in\mathcal{G}$, assume that the limit in (4.8) exists and is uniform with respect to $x\in\mathbb{R}^d$ and $\nu\in\mathbb{S}^{d-1}$. Then $g\in\mathcal{G}^{\hat\vartheta}$ for a suitable continuous non-decreasing function $\hat\vartheta\colon[0,+\infty)\to[0,+\infty)$ satisfying $\hat\vartheta(0)=0$ and (4.1). Indeed, since the limit in (4.8) exists and is uniform with respect to $x\in\mathbb{R}^d$ and $\nu\in\mathbb{S}^{d-1}$, there exists a continuous non-decreasing function $\omega\colon[0,+\infty)\to[0,+\infty)$, with $\omega(0)=0$, such that

$$\Big|\frac{1}{t}g(x,t\,\zeta,\nu)-g^0(x,\zeta,\nu)\Big|\leq\omega(t) \tag{4.11}$$

for every $x\in\mathbb{R}^d$, $\zeta\in\{1,-1\}$, and $\nu\in\mathbb{S}^{d-1}$. This implies that for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}\setminus\{0\}$, and $\nu\in\mathbb{S}^{d-1}$ we have

$$\Big|\frac{1}{t}g(x,t\,\zeta,\nu)-g^0(x,\zeta,\nu)\Big|=|\zeta|\Big|\frac{1}{t|\zeta|}g(x,t|\zeta|\frac{\zeta}{|\zeta|},\nu)-g^0(x,\frac{\zeta}{|\zeta|},\nu)\Big|\leq|\zeta|\omega(t|\zeta|)\,.$$

If $t|\zeta|\leq 1$, by (g2) we have

$$|\zeta|\omega(t|\zeta|)\leq\frac{\omega(t|\zeta|)}{c_1}\frac{1}{t}g(x,t\,\zeta,\nu)$$

and if $t|\zeta|\geq 1$, by (g2) we have

$$|\zeta|\omega(t|\zeta|)=t|\zeta|\frac{\omega(t|\zeta|)}{c_1 t}c_1\leq t|\zeta|\frac{\omega(t|\zeta|)}{c_1}\frac{g(x,t\,\zeta,\nu)}{t}\,.$$

From these inequalities, setting

$$\hat{\vartheta}(\tau) := (\tau \vee 1)\frac{\omega(\tau)}{c_1} \quad \text{for every } \tau \geq 0\,, \tag{4.12}$$

we obtain that (4.9) holds with ϑ replaced by $\hat{\vartheta}$, hence $g \in \mathcal{G}^{\hat{\vartheta}}$ by the final sentence of Remark 4.5. Inequality (4.1) for $\hat{\vartheta}$ can be obtained from Remark 4.6, or more directly using the lower estimates for ω which follow from (4.11), recalling (4.10), (g2), and (g3).

The following result provides the main examples of functionals in the class $\mathfrak{E}^{\alpha,\vartheta}$.

Proposition 4.9. *Let $f \in \mathcal{F}^\alpha$, let $g \in \mathcal{G}^\vartheta$, and let $E^{f,g}$ be the functional introduced in Definition 2.8. Then $E^{f,g} \in \mathfrak{E}^{\alpha,\vartheta}$.*

Proof. To simplify the notation we set $E := E^{f,g}$. By Remark 2.9 we already know that $E \in \mathfrak{E}$. It remains to prove that E satisfies (4.2). Let us fix s, t, A, and u as in Definition 4.1. Since $\xi \mapsto f^\infty(x,\xi)$ is positively homogeneous of degree one, by (2.15) we have

$$\frac{E(su,A)}{s} = \int_A \frac{f(x,s\nabla u)}{s}dx + \int_A f^\infty\big(x,\frac{dD^c u}{d|D^c u|}\big)d|D^c u| + \int_{A\cap J_u} \frac{g(x,s[u],\nu_u)}{s}d\mathcal{H}^{d-1}\,,$$

$$\frac{E(tu,A)}{t} = \int_A \frac{f(x,t\nabla u)}{t}dx + \int_A f^\infty\big(x,\frac{dD^c u}{d|D^c u|}\big)d|D^c u| + \int_{A\cap J_u} \frac{g(x,t[u],\nu_u)}{t}d\mathcal{H}^{d-1}\,,$$

hence

$$\begin{aligned}\Big|\frac{E(su,A)}{s} - \frac{E(tu,A)}{t}\Big| &\leq \int_A \Big|\frac{f(x,s\nabla u)}{s} - \frac{f(x,t\nabla u)}{t}\Big|dx \\ &+ \int_{A\cap J_u} \Big|\frac{g(x,s[u],\nu_u)}{s} - \frac{g(x,t[u],\nu_u)}{t}\Big|d\mathcal{H}^{d-1}\,.\end{aligned} \tag{4.13}$$

By (4.3) we have

$$\begin{aligned}\int_A \Big|\frac{f(x,s\nabla u)}{s} - \frac{f(x,t\nabla u)}{t}\Big|dx &\leq \frac{c_6}{s}\int_A f(x,s\nabla u)^{1-\alpha}dx + \frac{c_6}{s}\mathcal{L}^d(A) \\ &+\frac{c_6}{t}\int_A f(x,t\nabla u)^{1-\alpha}dx + \frac{c_6}{t}\mathcal{L}^d(A)\,.\end{aligned} \tag{4.14}$$

Since by Hölder's inequality for every $r > 0$ we have

$$\int_A f(x,r\nabla u)^{1-\alpha}dx \leq \mathcal{L}^d(A)^\alpha\Big(\int_A f(x,r\nabla u)dx\Big)^{1-\alpha} \leq \mathcal{L}^d(A)^\alpha E(ru,A)^{1-\alpha},$$

from (4.14) we obtain

$$\begin{aligned}\int_A \Big|\frac{f(x,s\nabla u)}{s} - \frac{f(x,t\nabla u)}{t}\Big|dx &\leq \frac{c_6}{s}\mathcal{L}^d(A)^\alpha E(su,A)^{1-\alpha} + \frac{c_6}{s}\mathcal{L}^d(A) \\ &+\frac{c_6}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(A)\,.\end{aligned} \tag{4.15}$$

By (4.6) we have

$$\begin{aligned}&\int_{A\cap J_u} \Big|\frac{g(x,s[u],\nu_u)}{s} - \frac{g(x,t[u],\nu_u)}{t}\Big|d\mathcal{H}^{d-1} \\ &\leq \vartheta(sm_A)\int_{A\cap J_u} \frac{g(x,s[u],\nu_u)}{s}d\mathcal{H}^{d-1} + \vartheta(tm_A)\int_{A\cap J_u} \frac{g(x,t[u],\nu_u)}{t}d\mathcal{H}^{d-1} \\ &\leq \vartheta(sm_A)\frac{E(su,A)}{s} + \vartheta(tm_A)\frac{E(tu,A)}{t}\,.\end{aligned} \tag{4.16}$$

From (4.13), (4.15), and (4.16) we obtain (4.2). □

We now prove that $\mathfrak{E}^{\alpha,\vartheta}$ is closed with respect to Γ-convergence.

Theorem 4.10. *Let (E_k) be a sequence of functionals in $\mathfrak{E}^{\alpha,\vartheta}$ and let $E \in \mathfrak{E}$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then $E \in \mathfrak{E}^{\alpha,\vartheta}_{sc}$.*

Proof. We begin by observing that since E is a Γ-limit we have $E \in \mathfrak{E}_{sc}$. To prove that $E \in \mathfrak{E}^{\alpha,\vartheta}$ let us fix s, t, A, u, and m_A as in Definition 4.1 and let $m' > m_A$.

Using the continuity of ϑ, and exchanging the roles of s and t, to prove (4.2) it is enough to show that

$$\begin{aligned} &\big(1 - \vartheta(sm')\big)\frac{E(su, A)}{s} - \frac{c_6}{s}\mathcal{L}^d(A)^\alpha E(su, A)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(A) \\ &\le \big(1 + \vartheta(tm')\big)\frac{E(tu, A)}{t} + \frac{c_6}{t}\mathcal{L}^d(A)^\alpha E(tu, A)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(A). \end{aligned} \tag{4.17}$$

If the left-hand side is less than or equal to zero, then the inequality is trivial. We may therefore assume that it is positive.

By the basic property of Γ-convergence and arguing as in the proof of Proposition 3.5 we deduce that there exists a sequence (u_k) in $BV(A) \cap L^\infty(A)$ converging to u in $L^0(\mathbb{R}^d)$ such that $\operatorname{osc}_A u_k \le m'$ for every $k \in \mathbb{N}$ and

$$E(tu, A) = \lim_{k\to\infty} E_k(tu_k, A)\,, \tag{4.18}$$

$$E(su, A) \le \liminf_{k\to\infty} E_k(su_k, A)\,. \tag{4.19}$$

Since $E_k \in \mathfrak{E}^{\alpha,\vartheta}$ we have

$$\begin{aligned} &\big(1 - \vartheta(sm')\big)\frac{E_k(su_k, A)}{s} - \frac{c_6}{s}\mathcal{L}^d(A)^\alpha E_k(su_k, A)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(A) \\ &\le \big(1 + \vartheta(tm')\big)\frac{E_k(tu_k, A)}{t} + \frac{c_6}{t}\mathcal{L}^d(A)^\alpha E_k(tu_k, A)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(A). \end{aligned} \tag{4.20}$$

By (4.18) we have

$$\begin{aligned} &\big(1 + \vartheta(tm')\big)\frac{E(tu, A)}{t} + \frac{c_6}{t}\mathcal{L}^d(A)^\alpha E(tu, A)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(A) \\ &= \lim_{k\to\infty}\Big(\big(1 + \vartheta(tm')\big)\frac{E_k(tu_k, A)}{t} + \frac{c_6}{t}\mathcal{L}^d(A)^\alpha E_k(tu_k, A)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(A)\Big)\,. \end{aligned} \tag{4.21}$$

The left-hand side of (4.17) can be expressed as $\Psi\big(E(su, A)\big)$, where for every $z \in [0, +\infty)$ we set

$$\Psi(z) := \big(1 - \vartheta(sm')\big)\frac{z}{s} - \frac{c_6}{s}\mathcal{L}^d(A)^\alpha z^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(A)\,.$$

Hence, in order to prove (4.17) it is enough to show that

$$\Psi\big(E(su, A)\big) \le \liminf_{k\to\infty} \Psi\big(E_k(su_k, A)\big)\,, \tag{4.22}$$

when

$$\Psi\big(E(su, A)\big) > 0 \quad \text{and hence} \quad \big(1 - \vartheta(sm')\big) > 0\,. \tag{4.23}$$

Note that if $\Psi(z) > 0$ then $(1 - \vartheta(sm'))z > c_6\mathcal{L}^d(A)^\alpha z^{1-\alpha}$, hence $z > z_0 := c_6^{1/\alpha}\big(1 - \vartheta(sm')\big)^{-1/\alpha}\mathcal{L}^d(A)$. Moreover, the function Ψ is increasing in $(z_0, +\infty)$, hence (4.19) and (4.23) give (4.22), which together with (4.21) gives (4.17) and concludes the proof of the theorem. □

We conclude this section with two results that can be considered as a partial converse of Proposition 4.9.

Proposition 4.11. *Let $E \in \mathfrak{E}^{\alpha,\vartheta}_{sc}$ and let f be as in Definition 2.14. Then $f \in \mathcal{F}^\alpha$.*

Proof. By Remark 2.15 we have $f \in \mathcal{F}$ and by Theorem 2.16 for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in GBV_\star(A)$ we have

$$E^a(u,B) = \int_B f(x,\nabla u)dx$$

for every $B \in \mathcal{B}(A)$.

It remains to prove that f satisfies (4.3) for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$. We now fix $\xi \in \mathbb{R}^d$, $s > 0$, and $t > 0$ and we set $c := \frac{c_2+c_4+1}{c_1 t} d^{1/2} + |\xi| d^{1/2}$. By Lemma 2.17 there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$ and $\eta > 0$ there exists $\rho_\eta(x) > 0$ such that for every $0 < \rho < \rho_\eta(x)$ there exists $u \in BV(Q(x,\rho)) \cap L^\infty(Q(x,\rho))$ satisfying $\|u - \ell_\xi\|_{L^\infty(Q(x,\rho))} \le c\rho$, $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, and

$$E(tu, Q(x,\rho)) \le m^E(t\ell_\xi, Q(x,\rho)) + \eta\rho^d\,. \tag{4.24}$$

Then $\mathrm{osc}_{Q(x,\rho)}\, u \le C\rho$, where $C := 2c + |\xi| d^{1/2}$. By (4.2) we have

$$\begin{aligned} &\big(1-\vartheta(sC\rho)\big)\frac{E(su,Q(x,\rho))}{s} - \frac{c_6}{s}\rho^{\alpha d} E(su,Q(x,\rho))^{1-\alpha} - \frac{c_6}{s}\rho^d \\ &\le \frac{E(tu,Q(x,\rho))}{t} + \frac{c_6}{t}\rho^{\alpha d} E(tu,Q(x,\rho))^{1-\alpha} + \vartheta(tC\rho)\frac{E(tu,Q(x,\rho))}{t} + \frac{c_6}{t}\rho^d \end{aligned} \tag{4.25}$$

for every $0 < \rho < \rho_\eta(x)$. Let X be the right-hand side of (4.25).

We want to prove that (4.25) implies

$$\big(1-\vartheta(sC\rho)\big)\frac{m^E(s\ell_\xi,Q(x,\rho))}{s} - \frac{c_6}{s}\rho^{\alpha d} m^E(s\ell_\xi,Q(x,\rho))^{1-\alpha} - \frac{c_6}{s}\rho^d \le X$$

which is equivalent to

$$\Psi_\rho\big(m^E(s\ell_\xi,Q(x,\rho))\big) \le X\,, \tag{4.26}$$

where for every $z \in [0,+\infty)$ we set

$$\Psi_\rho(z) := \big(1-\vartheta(sC\rho)\big)\frac{z}{s} - \frac{c_6}{s}\rho^{\alpha d} z^{1-\alpha} - \frac{c_6}{s}\rho^d.$$

If the left-hand side of (4.26) is negative we have nothing to prove. Hence it is enough to prove (4.26) when

$$\Psi_\rho\big(m^E(s\ell_\xi,Q(x,\rho))\big) > 0 \quad \text{and hence} \quad 1-\vartheta(sC\rho) > 0\,.$$

Arguing as in the proof of Theorem 4.10 we obtain that Ψ_ρ is strictly increasing in the half-line $[m^E(s\ell_\xi,Q(x,\rho)),+\infty)$. Since $\mathrm{tr}_{Q(x,\rho)}(s) = \mathrm{tr}_{Q(x,\rho)}(s\ell_\xi)$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, we have $m^E(s\ell_\xi,Q(x,\rho)) \le E(su,Q(x,\rho))$ which implies that

$$\Psi_\rho\big(m^E(s\ell_\xi,Q(x,\rho))\big) \le \Psi_\rho\big(E(su,Q(x,\rho))\big) \le X\,,$$

where in the last inequality we used (4.25) and the definition of Ψ_ρ. This concludes the proof of (4.26), which, by (4.24) implies

$$\begin{aligned} \frac{m^E(s\ell_\xi,Q(x,\rho))}{s} &\le \frac{c_6}{s}\rho^{\alpha d} m^E(s\ell_\xi,Q(x,\rho))^{1-\alpha} + \vartheta(sC\rho)\frac{m^E(s\ell_\xi,Q(x,\rho))}{s} + \frac{c_6}{s}\rho^d \\ &+ \frac{m^E(t\ell_\xi,Q(x,\rho)) + \eta\rho^d}{t} + \frac{c_6}{t}\rho^{\alpha d}(m^E(t\ell_\xi,Q(x,\rho)) + \eta\rho^d)^{1-\alpha} \\ &+ \vartheta(tC\rho)\frac{m^E(t\ell_\xi,Q(x,\rho)) + \eta\rho^d}{t} + \frac{c_6}{t}\rho^d \end{aligned} \tag{4.27}$$

for every $0 < \rho < \rho_\eta(x)$. Dividing by ρ^d and passing to the limsup as $\rho \to 0+$ we obtain

$$\frac{f(x,s\xi)}{s} \le \frac{c_6}{s} f(x,s\xi)^{1-\alpha} + \frac{c_6}{s} + \frac{f(x,t\xi)+\eta}{t} + \frac{c_6}{t}(f(x,t\xi)+\eta)^{1-\alpha} + \frac{c_6}{t}\,. \tag{4.28}$$

Taking the limit as $\eta \to 0+$ and exchanging the roles of s and t we obtain (4.3) for every $x \in \mathbb{R}^d \setminus N$ and every $\xi \in \mathbb{R}^d$ and $s,t > 0$. □

Proposition 4.12. *Let* $E \in \mathfrak{E}^{\alpha,\vartheta}$ *and let* g *be as in Definition 2.14. Then* $g \in \mathcal{G}^{\vartheta}$ *.*

Proof. By Remark 2.15 we have $g \in \mathcal{G}$ and it remains to prove (4.6). We fix $s, t > 0$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. We give the proof only for $\zeta \geq 0$ since the case $\zeta < 0$ can be obtained with obvious changes.

For every $\rho > 0$ there exists $u_\rho \in BV(Q_\nu(x,\rho))$ with $\mathrm{tr}_{Q_\nu(x,\rho)} u_\rho = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$ such that

$$E(tu_\rho, Q_\nu(x,\rho)) \leq m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d\,.$$

Let $v_\rho := (u_\rho \vee 0) \wedge \zeta$. Then $v_\rho \in BV(Q_\nu(x,\rho)) \cap L^\infty(Q_\nu(x,\rho))$, $\mathrm{tr}_{Q_\nu(x,\rho)} v_\rho = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, and, by (g) in Definition 2.3, we have

$$E(tv_\rho, Q_\nu(x,\rho)) \leq E(tu_\rho, Q_\nu(x,\rho)) + c_4\rho^d \leq m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + (1+c_4)\rho^d\,. \tag{4.29}$$

By (4.2) we have

$$\begin{aligned}&\Big(1-\vartheta(s|\zeta|)\Big)\frac{E(sv_\rho, Q_\nu(x,\rho))}{s} - \frac{c_6}{s}\rho^{\alpha d}E(sv_\rho, Q_\nu(x,\rho))^{1-\alpha} - \frac{c_6}{s}\rho^d \\ &\leq \frac{E(tv_\rho, Q_\nu(x,\rho))}{t} + \frac{c_6}{t}\rho^{\alpha d}E(tv_\rho, Q_\nu(x,\rho))^{1-\alpha} + \vartheta(t|\zeta|)\frac{E(tv_\rho, Q_\nu(x,\rho))}{t} + \frac{c_6}{t}\rho^d\end{aligned} \tag{4.30}$$

for every $\rho > 0$. Let X be the right-hand side of (4.30).

We want to prove that (4.30) implies

$$\big(1-\vartheta(s|\zeta|)\big)\frac{m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} - \frac{c_6}{s}\rho^{\alpha d}m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} - \frac{c_6}{s}\rho^d \leq X\,,$$

which is equivalent to

$$\Psi_\rho\big(m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))\big) \leq X\,, \tag{4.31}$$

where for every $z \in [0,+\infty)$ we set

$$\Psi_\rho(z) := \big(1-\vartheta(s|\zeta|)\big)\frac{z}{s} - \frac{c_6}{s}\rho^{\alpha d}z^{1-\alpha} - \frac{c_6}{s}\rho^d\,.$$

If the left-hand side of (4.31) is negative we have nothing to prove. Hence it is enough to prove (4.31) when

$$\Psi_\rho\big(m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))\big) > 0 \quad \text{and hence} \quad 1-\vartheta(s|\zeta|) > 0\,.$$

Arguing as in the proof of Theorem 4.10 we obtain that Ψ_ρ is strictly increasing in the half-line $[m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho)), +\infty)$. Since $\mathrm{tr}_{Q_\nu(x,\rho)}(sv_\rho) = \mathrm{tr}_{Q_\nu(x,\rho)}(su_{x,\zeta,\nu})$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, we have $m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho)) \leq E(sv_\rho, Q_\nu(x,\rho))$ which implies that

$$\Psi_\rho\big(m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))\big) \leq \Psi_\rho\big(E(sv_\rho, Q_\nu(x,\rho))\big) \leq X\,,$$

where in the last inequality we used (4.30) and the definition of Ψ_ρ. This concludes the proof of (4.31), which, by (4.29) implies

$$\begin{aligned}\frac{m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} &\leq \frac{c_6}{s}\rho^{\alpha d}m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \vartheta(s|\zeta|)\frac{m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} \\ &+\frac{c_6}{s}\rho^d + \frac{m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d}{t} + \frac{c_6}{t}\rho^{\alpha d}(m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d)^{1-\alpha} \\ &+\vartheta(t|\zeta|)\frac{m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d}{t} + \frac{c_6}{t}\rho^d\end{aligned}$$

for every $\rho > 0$. Since $m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le E(su_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le c_3\rho^{d-1} + c_4\rho^d$ by (c2), and a similar inequality holds for t, we have

$$\frac{m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} \le \frac{c_6}{s}(c_3 + c_4\rho)^{1-\alpha}\rho^{d-1+\alpha} + \vartheta(s|\zeta|)\frac{m^E(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s}$$
$$+\frac{c_6}{s}\rho^d + \frac{m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d}{t} + \frac{c_6}{t}(c_3 + c_4\rho + \rho)^{1-\alpha}\rho^{d-1+\alpha}$$
$$+\vartheta(t|\zeta|)\frac{m^E(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d}{t} + \frac{c_6}{t}\rho^d.$$

Dividing by ρ^{d-1} and passing to the limsup as $\rho \to 0+$ we obtain

$$\frac{1}{s}g(x, s\,\zeta, \nu) \le \vartheta(s|\zeta|)\frac{1}{s}g(x, s\,\zeta, \nu) + \frac{1}{t}g(x, t\,\zeta, \nu) + \vartheta(t|\zeta|)\frac{1}{t}g(x, t\,\zeta, \nu)\,.$$

Exchanging the roles of s and t we obtain (4.6) for every $s, t \in (0, +\infty)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. □

5. Integral representation for functionals in $\mathfrak{E}_{sc}^{\alpha,\theta}$

The following theorem provides a complete integral representation for a functional in the class $\mathfrak{E}_{sc}^{\alpha,\theta}$, without assuming the continuity condition with respect to translations required in [17, Theorem 6.7] to deal with the Cantor part. Instead, we assume only that the integrand f introduced in Definition 2.14 does not depend on x, a condition which is satisfied by the functionals obtained in the limit of homogenisation problems considered in Section 6.

Theorem 5.1. *Let $E \in \mathfrak{E}_{sc}^{\alpha,\vartheta}$ and let f and g be the functions introduced in Definition 2.14. Assume that there exists a function $\hat{f}\colon \mathbb{R}^d \to [0, +\infty)$ such that*

$$f(x, \xi) = \hat{f}(\xi) \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d. \tag{5.1}$$

Then $f \in \mathcal{F}^\alpha$, $g \in \mathcal{G}^\vartheta$, and $E = E^{f,g}$.

Note that, even if both f and g do not depend on x, for a functional E in the wider class $\mathfrak{E}_{sc}$ we do not know a proof of the continuity of E with respect to translations, which is needed to apply [17, Theorem 6.7]. Indeed, while the independence of x of the integrands f and g guarantees the translation invariance of E^a and E^j by Theorem 2.16, the same property cannot be obtained for E^c, for which no integral representation is available. We underline that Theorem 5.1 provides an integral representation for E^c, but only when E belongs to the narrower class $\mathfrak{E}_{sc}^{\alpha,\vartheta}$ introduced in this paper.

To prove this result, given a functional $E \in \mathfrak{E}_{sc}$, we express the Radon-Nikodym derivative of E^c with respect to $D^c u$ by means of the functions m^E, introduced in Definition 2.13, computed in rectangles of the form $Q_\nu^\lambda(x,\rho)$.

Lemma 5.2. *Let $E \in \mathfrak{E}_{sc}$, let $A \in \mathcal{A}_c(\mathbb{R}^d)$, and let $u \in BV(A)$. Assume that there exists a function $\hat{f}\colon \mathbb{R}^d \to [0, +\infty)$ satisfying (5.1), where f is the function introduced in Definition 2.14. For $|D^c u|$-a.e. $x \in A$, for every $\lambda \ge 1$, and for every $\rho > 0$ we set*

$$\nu_u(x) := \frac{dD^c u}{d|D^c u|}(x)\,, \quad s_\rho^\lambda(x) := \frac{|D^c u|(Q_{\nu_u(x)}^\lambda(x,\rho))}{\lambda^{d-1}\rho^d}\,, \quad \xi_\rho^\lambda(x) := s_\rho^\lambda(x)\nu_u(x)\,. \tag{5.2}$$

Then

$$\lim_{\rho\to 0+} s_\rho^\lambda(x) = +\infty \quad \text{and} \quad \lim_{\rho\to 0+} \rho s_\rho^\lambda(x) = 0 \quad \text{for every } \lambda \ge 1\,, \tag{5.3}$$

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) = \lim_{\lambda\to+\infty}\limsup_{\rho\to 0+} \frac{m^E(\ell_{\xi_\rho^\lambda(x)}, Q_{\nu_u(x)}^\lambda(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)} \tag{5.4}$$

for $|D^c u|$-a.e. $x \in A$.

Proof. Let E^a be as in Definition 2.11. By Theorem 2.16 we have

$$E^a(u,A) = \int_A \hat{f}(\nabla u)dx \tag{5.5}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and every $u \in BV(A)$. For every $\varepsilon > 0$ we consider the functional $E_{(\varepsilon)}(u,A) := E(u,A) + \varepsilon|Du|(A)$, introduced in [17, Definition 4.7] and defined for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and every $u \in BV(A)$. Applying [7, Lemma 3.9] to $E^c_{(\varepsilon)} := E^c + \varepsilon|D^c u|$, for $|D^c u|$-a.e. $x \in A$ we obtain

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) + \varepsilon = \frac{dE^c_{(\varepsilon)}(u,\cdot)}{d|D^c u|}(x) = \lim_{\lambda\to+\infty} \limsup_{\rho\to 0+} \frac{m^{E_{(\varepsilon)}}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}\,. \tag{5.6}$$

In the same lemma it is shown that (5.3) is satisfied for $|D^c u|$-a.e. $x \in A$.

We now show that (5.3) and (5.6) imply (5.4). Since $E \le E_{(\varepsilon)}$, from (5.6) for $|D^c u|$-a.e. $x \in A$ we obtain

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) + \varepsilon \ge \lim_{\lambda\to+\infty} \limsup_{\rho\to 0+} \frac{m^{E}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}\,. \tag{5.7}$$

Letting $\varepsilon \to 0+$ we obtain the inequality $\ge$ in (5.4).

To prove the opposite inequality it is enough to show that for every $\lambda \ge 1$, for every $\varepsilon > 0$, and for $|D^c u|$-a.e. $x \in A$ we have

$$\limsup_{\rho\to 0+} \frac{m^{E_{(\varepsilon)}}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} \le \big(1 + \frac{\varepsilon}{c_1}\big) \limsup_{\rho\to 0+} \frac{m^{E}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}\,. \tag{5.8}$$

Let us fix $\eta > 0$, $\lambda \ge 1$, and $x \in A$ for which (5.2) are defined and (5.3) holds. We set $a_\lambda := \lambda d^{1/2}$ and $b_\lambda := \frac{c_2+c_4+1}{c_1} d^{1/2}\lambda$. For every $\rho > 0$ let $w^x_\rho(y) := \xi^\lambda_\rho(x)\cdot(y-x)$. Recalling that E is invariant under addition of constants (see Definition 2.3(d)), by the last sentence in Lemma 2.17 for every $\rho > 0$ there exists $u^x_\rho \in BV(Q^\lambda_{\nu_u(x)}(x,\rho)) \cap L^\infty(Q^\lambda_{\nu_u(x)}(x,\rho))$, with $\|u^x_\rho - w^x_\rho\|_{L^\infty(Q^\lambda_{\nu_u(x)}(x,\rho))} \le b_\lambda\rho + a_\lambda\rho s^\lambda_\rho(x)$, and $\operatorname{tr} u^x_\rho = \operatorname{tr} w^x_\rho$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_{\nu_u(x)}(x,\rho)$, such that

$$E(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) \le m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho)) + \eta\lambda^{d-1}\rho^d\,. \tag{5.9}$$

The L^∞-estimate for $u^x_\rho - w^x_\rho$ implies that $|[u^x_\rho]| = |[u^x_\rho - w^x_\rho]| \le 2(b_\lambda\rho + a_\lambda\rho s^\lambda_\rho(x))$ $\mathcal{H}^{d-1}$-a.e. in $J_{u^x_\rho} \cap Q^\lambda_{\nu_u(x)}(x,\rho)$. Hence, setting $J^1_{u^x_\rho} := \{y \in J_{u^x_\rho} : |[u^x_\rho](y)| \ge 1\}$ we have

$$\begin{aligned}\int_{J_{u^x_\rho}\cap Q^\lambda_{\nu_u(x)}(x,\rho)} |[u^x_\rho]| d\mathcal{H}^{d-1} &\le \int_{(J_{u^x_\rho}\setminus J^1_{u^x_\rho})\cap Q^\lambda_{\nu_u(x)}(x,\rho)} |[u^x_\rho]| d\mathcal{H}^{d-1} \\ &\quad + (2b_\lambda\rho + 2a_\lambda\rho s^\lambda_\rho(x))\mathcal{H}^{d-1}(J^1_{u^x_\rho} \cap Q^\lambda_{\nu_u(x)}(x,\rho)) \\ &\le \big(1 + 2a_\lambda\rho s^\lambda_\rho(x) + 2b_\lambda\rho\big) \int_{J_{u^x_\rho}\cap Q^\lambda_{\nu_u(x)}(x,\rho)} |[u^x_\rho]| \wedge 1 d\mathcal{H}^{d-1}\,.\end{aligned}$$

By the definition of $E_{(\varepsilon)}$ this implies that for every $\varepsilon > 0$

$$
\begin{aligned}
E_{(\varepsilon)}(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho)) &= E(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho)) + \varepsilon \int_{Q^\lambda_{\nu_u(x)}(x,\rho)} |\nabla u_\rho^x| dy \\
&\quad + \varepsilon |D^c u_\rho^x|(Q^\lambda_{\nu_u(x)}(x,\rho)) + \varepsilon \int_{J_{u_\rho^x} \cap Q^\lambda_{\nu_u(x)}(x,\rho)} |[u_\rho^x]| d\mathcal{H}^{d-1} \\
&\le E(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho)) + \varepsilon \int_{Q^\lambda_{\nu_u(x)}(x,\rho)} |\nabla u_\rho^x| dy + \varepsilon |D^c u_\rho^x|(Q^\lambda_{\nu_u(x)}(x,\rho)) \\
&\quad + \varepsilon \big(1 + 2a_\lambda \rho s_\rho^\lambda(x) + 2b_\lambda \rho\big) \int_{J_{u_\rho^x} \cap Q^\lambda_{\nu_u(x)}(x,\rho)} |[u_\rho^x]| \wedge 1 d\mathcal{H}^{d-1} \\
&\le \big(1 + \varepsilon \frac{1 + 2a_\lambda \rho s_\rho^\lambda(x) + 2b_\lambda\rho}{c_1}\big) E(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho)) + \varepsilon \frac{c_2}{c_1} \big(1 + 2a_\lambda \rho s_\rho^\lambda(x) + 2b_\lambda\rho\big) \lambda^{d-1}\rho^d,
\end{aligned}
$$

where in the last inequality we used (c1) of Definition 2.3. Recalling again that E is invariant under addition of constants (property (d)), by (2.26) and (5.9) we get

$$
\begin{aligned}
\frac{m^{E_{(\varepsilon)}}(\ell_{\xi_\rho^\lambda(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)} &\le \frac{E_{(\varepsilon)}(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)} \\
&\le \big(1 + \varepsilon \frac{1 + 2a_\lambda \rho s_\rho^\lambda(x) + 2b_\lambda\rho}{c_1}\big) \frac{E(u_\rho^x, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)} \\
&\quad + \varepsilon \frac{c_2}{c_1} \big(\frac{1}{s_\rho^\lambda(x)} + 2a_\lambda\rho + 2b_\lambda \frac{\rho}{s_\rho^\lambda(x)}\big) \\
&\le \big(1 + \varepsilon \frac{1 + 2a_\lambda \rho s_\rho^\lambda(x) + 2b_\lambda\rho}{c_1}\big) \Big(\frac{m^E(\ell_{\xi_\rho^\lambda(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)} + \frac{\eta}{s_\rho^\lambda(x)}\Big) \\
&\quad + \varepsilon \frac{c_2}{c_1} \big(\frac{1}{s_\rho^\lambda(x)} + 2a_\lambda\rho + 2b_\lambda \frac{\rho}{s_\rho^\lambda(x)}\big) .
\end{aligned}
$$

Passing to the limsup as $\rho \to 0+$ and using (5.3) we obtain (5.8), which concludes the proof. □

To prove Theorem 5.1 we need also the following result which will allow us to obtain the representation of the Cantor part using cubes instead of rectangles.

Lemma 5.3. *Let $E \in \mathfrak{E}$, $\xi \in \mathbb{R}^d$, $\lambda \ge 1$, $\nu \in \mathbb{S}^{d-1}$, $\kappa > 0$, and $\mu \in [0, +\infty)$. Assume that for every $x \in \mathbb{R}^d$ and $\rho > 0$ we have*

$$
m^E_{\kappa\rho}(\ell_\xi, Q(x,\rho)) \le \mu \rho^d . \tag{5.10}
$$

Then

$$
m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x,\rho)) \le \mu \lambda^{d-1}\rho^d \tag{5.11}
$$

for every $x \in \mathbb{R}^d$ and $\rho > 0$. If, in addition, for some $x_0 \in \mathbb{R}^d$ we have

$$
\limsup_{\rho \to 0+} \frac{m^E_{\kappa\rho}(\ell_\xi, Q(x_0,\rho))}{\rho^d} = \mu , \tag{5.12}
$$

then

$$
\limsup_{\rho \to 0+} \frac{m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x_0,\rho))}{\lambda^{d-1}\rho^d} = \mu . \tag{5.13}
$$

Proof. Let us fix $x \in \mathbb{R}^d$ and $\rho > 0$. We cover $\mathcal{L}^d$-almost all of $Q^\lambda_\nu(x,\rho)$ with a countable union of pairwise disjoint cubes $Q(x_i,\rho_i)$ contained in $Q^\lambda_\nu(x,\rho)$. For every $j \in \mathbb{N}$ we set

$$
R_j = Q^\lambda_\nu(x,\rho) \setminus \bigcup_{i=1}^j Q(x_i,\rho_i) .
$$

We observe that $\mathcal{L}^d(R_j) \to 0$ as $j \to \infty$.

Given $\eta > 0$, for every $i \in \mathbb{N}$ let $u_i \in BV(Q(x_i,\rho_i)) \cap L^\infty(Q(x_i,\rho_i))$, with $\mathrm{tr}_{Q(x_i,\rho_i)} u_i = \mathrm{tr}_{Q(x_i,\rho_i)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x_i,\rho_i)$ and $\|u_i - \ell_\xi\|_{L^\infty(Q(x_i,\rho_i))} \le \kappa\rho_i$, such that

$$E(u_i, Q(x_i,\rho_i)) \le m^E_{\kappa\rho_i}(\ell_\xi, Q(x_i,\rho_i)) + \frac{\eta}{2^i}\,.$$

Let $v_j \in BV(Q^\lambda_\nu(x,\rho)) \cap L^\infty(Q^\lambda_\nu(x,\rho))$ be defined by $v_j := u_i$ in $Q(x_i,\rho_i)$ for $i \le j$, and by $v_j := \ell_\xi$ in R_j. Then $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)} v_j = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(x,\rho)$ and $\|v_j - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho))} \le \kappa\lambda\rho$, since $\rho_i \le \lambda\rho$.

By (a), (b), and (c2) in Definition 2.3 we have

$$m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x,\rho)) \le E(v_j, Q^\lambda_\nu(x,\rho)) \le \sum_{i=1}^{j} E(u_i, Q(x_i,\rho_i)) + E(v_j, R_j)$$

$$\le \sum_{i=1}^{j} m^E_{\kappa\rho_i}(\ell_\xi, Q(x_i,\rho_i)) + \eta + (c_3|\xi| + c_4)\mathcal{L}^d(R_j)\,.$$

Passing now to the limit as $j \to \infty$ and $\eta \to 0+$, from (5.10) we obtain

$$m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x,\rho)) \le \sum_{i=1}^{\infty} m^E_{\kappa\rho_i}(\ell_\xi, Q(x_i,\rho_i)) \le \mu \sum_{i=1}^{\infty} \rho_i^d = \mu\lambda^{d-1}\rho^d\,,$$

where in the last equality we used the fact that the cubes are pairwise disjoint and cover almost all of the rectangle $Q^\lambda_\nu(x,\rho)$. This concludes the proof of (5.11).

Assume now (5.12). Then there exists $\sigma_d \colon (0,+\infty) \to (0,+\infty)$ with

$$\liminf_{\rho\to 0+} \sigma_d(\rho)/\rho^d = 0$$

such that

$$m^E_{\kappa\rho}(\ell_\xi, Q(x_0,\rho)) \ge \mu\rho^d - \sigma_d(\rho) \quad \text{for every } \rho > 0\,. \tag{5.14}$$

Given $\rho > 0$ we set $R := d^{1/2}\lambda\rho$, so that $Q^\lambda_\nu(x_0,\rho) \subset Q(x_0,R)$. We cover $\mathcal{L}^d$-almost all of $Q(x_0,R)$ with a countable union of pairwise disjoint rectangles $Q^\lambda_\nu(y_i,r_i)$ with $y_1 = x_0$, $r_1 = \rho$, and $\lambda r_i \le R$ for every $i \in \mathbb{N}$. For every $j \in \mathbb{N}$ we set

$$S_j = Q(x_0,R) \setminus \bigcup_{i=1}^{j} Q^\lambda_\nu(y_i,r_i)\,.$$

We observe that $\mathcal{L}^d(S_j) \to 0$ as $j \to \infty$.

Given $\eta > 0$, for every $i \in \mathbb{N}$ let $w_i \in BV(Q^\lambda_\nu(y_i,r_i)) \cap L^\infty(Q^\lambda_\nu(y_i,r_i))$ be such that $\mathrm{tr}_{Q^\lambda_\nu(y_i,r_i)} w_i = \mathrm{tr}_{Q^\lambda_\nu(y_i,r_i)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(y_i,r_i)$, $\|w_i - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(y_i,r_i))} \le \kappa\lambda r_i$, and

$$E(w_i, Q^\lambda_\nu(y_i,r_i)) \le m^E_{\kappa\lambda r_i}(\ell_\xi, Q^\lambda_\nu(y_i,r_i)) + \frac{\eta}{2^i}\,.$$

Let $z_j \in BV(Q(x_0,R)) \cap L^\infty(Q(x_0,R))$ be defined by $z_j := w_i$ in $Q^\lambda_\nu(y_i,r_i)$ for $i \le j$, and by $z_j := \ell_\xi$ in S_j. Then $\mathrm{tr}_{Q(x_0,R)} z_j = \mathrm{tr}_{Q(x_0,R)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x_0,R)$ and $\|z_j - \ell_\xi\|_{L^\infty(Q(x_0,R))} \le \kappa R$, since $\lambda r_i \le R$.

By (a), (b), and (c2) in Definition 2.3 we have

$$m^E_{\kappa R}(\ell_\xi, Q(x_0,R)) \le E(z_j, Q(x_0,R)) \le \sum_{i=1}^{j} E(w_i, Q^\lambda_\nu(y_i,r_i)) + E(z_j, S_j)$$

$$\le m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x_0,\rho)) + \sum_{i=2}^{j} m^E_{\kappa\lambda r_i}(\ell_\xi, Q^\lambda_\nu(y_i,r_i)) + \eta + (c_3|\xi| + c_4)\mathcal{L}^d(S_j)\,.$$

Passing now to the limit as $j \to \infty$ and $\eta \to 0+$, from (5.14) we obtain

$$\mu R^d - \sigma_d(R) \le m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x_0,\rho)) + \sum_{i=2}^\infty m^E_{\kappa\lambda r_i}(\ell_\xi, Q^\lambda_\nu(y_i,r_i))\,. \tag{5.15}$$

We claim that

$$m^E_{\kappa\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x_0,\rho)) \ge \mu\lambda^{d-1}\rho^d - \sigma_d(d^{1/2}\lambda\rho)\,. \tag{5.16}$$

We argue by contradiction. If (5.16) is not satisfied, since $R^d = \sum_{i=1}^\infty \lambda^{d-1} r_i^d = \lambda^{d-1}\rho^d + \sum_{i=2}^\infty \lambda^{d-1} r_i^d$, from (5.15) and (5.11), applied with $x = y_i$ and $\rho = r_i$, we obtain

$$\mu R^d - \sigma_d(R) < \mu\lambda^{d-1}\rho^d - \sigma_d(d^{1/2}\lambda\rho) + \mu\sum_{i=2}^\infty \lambda^{d-1} r_i^d = \mu R^d - \sigma_d(R)\,.$$

This contradiction proves (5.16), which, together with (5.11), implies (5.13). □

Proof of Theorem 5.1. Let E^a, E^c, E^j be the functionals introduced in Definition 2.11. By Propositions 4.11 and 4.12 we have $\hat f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\vartheta$. By Theorem 2.16 for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have

$$E^a(u,B) = \int_B \hat f(\nabla u)dx\,, \tag{5.17}$$

$$E^j(u,B) = \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,, \tag{5.18}$$

for every $u \in GBV_\star(A)$ and every $B \in \mathcal{B}(A)$.

Let us fix $A \in \mathcal{A}_c(\mathbb{R}^d)$. We now want to prove that

$$E^c(u,B) = \int_B \hat f^\infty\Big(\frac{dD^c u}{d|D^c u|}\Big)d|D^c u| \tag{5.19}$$

for every $u \in GBV_\star(A)$ and every $B \in \mathcal{B}(A)$.

Taking ℓ_ξ in the minimum problem which defines $m^E(\ell_\xi, Q(x,\rho))$ and using (5.17), for every $\xi \in \mathbb{R}^d$ we obtain

$$m^E(\ell_\xi, Q(x,\rho)) \le \hat f(\xi)\rho^d \quad \text{for every } x \in \mathbb{R}^d \text{ and } \rho > 0\,. \tag{5.20}$$

Since by (5.1) we have

$$\limsup_{\rho\to 0+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d} = \hat f(\xi) \quad \text{for every } x \in \mathbb{R}^d,$$

we can apply Lemma 5.3 and for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\lambda \ge 1$, and $\nu \in \mathbb{S}^{d-1}$ we obtain

$$\limsup_{\rho\to 0+} \frac{m^E(\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d} = \hat f(\xi)\,.$$

In particular, for $\xi = s\nu$ we have

$$\limsup_{\rho\to 0+} \frac{m^E(s\ell_\nu, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s} = \frac{\hat f(s\nu)}{s}\,. \tag{5.21}$$

for every $s > 0$, $\nu \in \mathbb{S}^{d-1}$, $\lambda \ge 1$, $x \in \mathbb{R}^d$.

Let us fix $\nu \in \mathbb{S}^{d-1}$, $\lambda \ge 1$, and $x \in \mathbb{R}^d$. We claim that

$$\limsup_{\rho\to 0+} \frac{m^E(s_\rho\ell_\nu, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} = \hat f^\infty(\nu) \tag{5.22}$$

when $s_\rho \to +\infty$ and $\rho s_\rho \to 0+$ as $\rho \to 0+$.

We observe that by taking $s_\rho\ell_\nu$ in the minimum problem defining $m^E(s_\rho\ell_\nu, Q^\lambda_\nu(x,\rho))$ we obtain that

$$m^E(s_\rho\ell_\nu, Q^\lambda_\nu(x,\rho)) \le \hat f(s_\rho\nu)\lambda^{d-1}\rho^d\,,$$

hence

$$\limsup_{\rho\to 0+}\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\leq \limsup_{\rho\to 0+}\frac{\hat f(s_\rho\nu)}{s_\rho}\leq \hat f^\infty(\nu)\,. \tag{5.23}$$

Therefore, in order to prove (5.22) it is enough to show that

$$\hat f^\infty(\nu)\leq \limsup_{\rho\to 0+}\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} \tag{5.24}$$

when $s_\rho\to+\infty$ and $\rho s_\rho\to 0+$ as $\rho\to 0+$.

We set $a_\lambda := 2^{\frac{c_2+c_4+1}{c_1}}d^{1/2}\lambda + 2d^{1/2}\lambda$. For every $s\geq 1$, by (5.1) we can apply the last statement of Lemma 2.17, which implies that $m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))=\inf E(su,Q^\lambda_\nu(x,\rho))$ among all $u\in BV(Q^\lambda_\nu(x,\rho))\cap L^\infty(Q^\lambda_\nu(x,\rho))$ with $\|u-\ell_\nu\|_{L^\infty(Q^\lambda_\nu(x,\rho))}\leq a_\lambda\rho/2$ and $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}u=\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_\nu$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(x,\rho)$.

Since $E\in\mathfrak{E}^{\alpha,\vartheta}$ for every such u and for every $s,t\geq 1$ we have

$$\Big|\frac{E(su,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}-\frac{E(tu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}\Big|\leq \vartheta(a_\lambda\rho s)\frac{E(su,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}+\frac{c_6}{s^\alpha}\Big(\frac{E(su,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\Big)^{1-\alpha}$$
$$+\frac{c_6}{s}+\vartheta(a_\lambda\rho t)\frac{E(tu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}+\frac{c_6}{t^\alpha}\Big(\frac{E(tu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}\Big)^{1-\alpha}+\frac{c_6}{t}\,.$$

Arguing as in the proof of Theorem 4.10 we obtain

$$\Big|\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}-\frac{m^E(t\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}\Big|\leq\frac{c_6}{s}+\frac{c_6}{t}+\vartheta(a_\lambda\rho s)\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}$$
$$+\frac{c_6}{s^\alpha}\Big(\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\Big)^{1-\alpha}+\vartheta(a_\lambda\rho t)\frac{m^E(t\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}+\frac{c_6}{t^\alpha}\Big(\frac{m^E(t\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t}\Big)^{1-\alpha}.$$

Taking $t=s_\rho$ in the previous inequality, we obtain

$$\Big|\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}-\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big|\leq\frac{c_6}{s}+\frac{c_6}{s_\rho}+\vartheta(a_\lambda\rho s)\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}$$
$$+\frac{c_6}{s^\alpha}\Big(\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\Big)^{1-\alpha}+\vartheta(a_\lambda\rho s_\rho)\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}+\frac{c_6}{s_\rho^\alpha}\Big(\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha}. \tag{5.25}$$

Let $\varepsilon>0$. By the definition of $\hat f^\infty$ there exists $s\geq 1$ such that

$$\Big|\frac{\hat f(s\nu)}{s}-\hat f^\infty(\nu)\Big|<\varepsilon\quad\text{and}\quad\frac{c_6}{s}+\frac{c_6}{s^\alpha}\Big(\hat f^\infty(\nu)+\varepsilon\Big)^{1-\alpha}<\varepsilon\,. \tag{5.26}$$

By (5.21) and the first inequality in (5.26) we get

$$\hat f^\infty(\nu)-\varepsilon\leq\limsup_{\rho\to 0+}\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\leq\hat f^\infty(\nu)+\varepsilon\,, \tag{5.27}$$

which by (5.25) gives

$$\hat f^\infty(\nu)-\varepsilon\leq\limsup_{\rho\to 0+}\Big[\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}+\frac{c_6}{s}+\frac{c_6}{s_\rho}+\vartheta(a_\lambda\rho s)\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}$$
$$+\frac{c_6}{s^\alpha}\Big(\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\Big)^{1-\alpha}+\vartheta(a_\lambda\rho s_\rho)\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}+\frac{c_6}{s_\rho^\alpha}\Big(\frac{m^E(s_\rho\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha}\Big].$$

By the second inequality in (5.26) and (5.27) we have

$$\limsup_{\rho\to 0+}\Big[\frac{c_6}{s}+\vartheta(a_\lambda\rho s)\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}+\frac{c_6}{s^\alpha}\Big(\frac{m^E(s\ell_\nu,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s}\Big)^{1-\alpha}\Big]$$
$$\leq\varepsilon+\lim_{\rho\to 0+}\vartheta(a_\lambda\rho s)(\hat f^\infty(\nu)+\varepsilon)=\varepsilon\,.$$

Since $s_\rho\to+\infty$ and $\rho s_\rho\to 0+$, by (5.23) we deduce that

$$\lim_{\rho\to 0+}\Big[\frac{c_6}{s_\rho}+\frac{c_6}{s_\rho^\alpha}\Big(\frac{m^E(s_\rho\ell_\nu,Q_\nu^\lambda(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha}+\vartheta(a_\lambda\rho s_\rho)\frac{m^E(s_\rho\ell_\nu,Q_\nu^\lambda(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big]=0\,.$$

Combining the last three displayed formulas we obtain

$$\hat f^\infty(\nu)-2\varepsilon\le\limsup_{\rho\to 0+}\frac{m^E(s_\rho\ell_\nu,Q_\nu^\lambda(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\,,$$

which implies (5.24) by the arbitrariness of $\varepsilon>0$. This concludes the proof of (5.22).

Let us fix $u\in BV(A)$. By (5.3) and (5.22) for every $\lambda>0$ $|D^cu|$-a.e. $x\in A$ we have

$$\limsup_{\rho\to 0+}\frac{m^E(\ell_{\xi_\rho^\lambda(x)},Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s_\rho^\lambda(x)}=\hat f^\infty(\nu_u(x))\,,$$

where

$$\xi_\rho^\lambda(x):=s_\rho^\lambda(x)\nu_u(x)\quad\text{and}\quad\nu_u(x):=\frac{dD^cu}{d|D^cu|}(x)\,.$$

By (5.4) this gives

$$\frac{dE^c(u,\cdot)}{d|D^cu|}(x)=\hat f^\infty\Big(\frac{dD^cu}{d|D^cu|}(x)\Big)\,.$$

Since the measure $E^c(u,\cdot)$ is absolutely continuous with respect to $|D^cu|$, we conclude that

$$E^c(u,B)=\int_B\hat f^\infty\Big(\frac{dD^cu}{d|D^cu|}\Big)d|D^cu| \tag{5.28}$$

for every $u\in BV(A)$ and every $B\in\mathcal B(A)$.

To prove that (5.28) holds also for every $u\in GBV_\star(A)$ we approximate u by truncations and we pass to the limit repeating the arguments of the last part of the proof of [17, Theorem 6.7]. This concludes the proof of (5.19). Together with (5.17) and (5.18) this gives $E(u,B)=E^{f,g}(u,B)$ for every $u\in GBV_\star(A)$ and every $B\in\mathcal B(A)$. This implies that $E=E^{f,g}$ on $L^0(\mathbb R^d)\times\mathcal B(\mathbb R^d)$. ☐

The following result shows that, for integrands in $\mathcal F^\alpha$ and $\mathcal G^\vartheta$, the Γ-convergence of a sequence of integral functionals can be deduced from the asymptotic behaviour of the minimum values of some auxiliary minimum problems on small cubes.

Theorem 5.4. *Let $(f_k)_k\subset\mathcal F^\alpha$, let $(g_k)_k\subset\mathcal G^\vartheta$, and let $E_k:=E^{f_k,g_k}$ as in Definition 2.8. Assume there exist two functions $\hat f\colon\mathbb R^d\to[0,+\infty)$ and $\hat g\colon\mathbb R^d\times\mathbb R\times\mathbb S^{d-1}\to[0,+\infty)$ such that*

$$\hat f(\xi)=\limsup_{\rho\to 0+}\liminf_{k\to\infty}\frac{m^{E_k}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))}{\rho^d}=\limsup_{\rho\to 0+}\limsup_{k\to\infty}\frac{m^{E_k}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))}{\rho^d},$$

$$\hat g(x,\zeta,\nu)=\limsup_{\rho\to 0+}\liminf_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}=\limsup_{\rho\to 0+}\limsup_{k\to\infty}\frac{m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}},$$

for every $x\in\mathbb R^d$, $\xi\in\mathbb R^d$, $\zeta\in\mathbb R$, and $\nu\in\mathbb S^{d-1}$, where κ_ξ is the constant defined in (3.18). *Then $\hat f\in\mathcal F^\alpha$, $\hat g\in\mathcal G^\vartheta$, and for every $A\in\mathcal A_c(\mathbb R^d)$ the sequence $E_k(\cdot,A)$ Γ-converges to $E^{\hat f,\hat g}$ with respect to the topology of $L^0(\mathbb R^d)$.*

Proof. By Theorem 2.10 there exist a subsequence (not relabelled) and a functional $E\in\mathfrak E_{sc}$, such that $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb R^d)$ for every $A\in\mathcal A_c(\mathbb R^d)$. By Theorem 4.10 we have $E\in\mathfrak E_{sc}^{\alpha,\vartheta}$. Let E^a,E^c,E^j be the functionals introduced in Definition 2.11 and let f,g be the functions introduced in Definition 2.14. By our hypotheses and Theorems 3.3 and 3.7 we have that $f(x,\xi)=\hat f(\xi)$ and $g(x,\zeta,\nu)=\hat g(x,\zeta,\nu)$ for every $x\in\mathbb R^d$, $\xi\in\mathbb R^d$, $\zeta\in\mathbb R$, and $\nu\in\mathbb S^{d-1}$. Moreover, by Propositions 4.11 and 4.12 we have $\hat f\in\mathcal F^\alpha$ and $\hat g\in\mathcal G^\vartheta$. Therefore, by Theorem 5.1 we conclude that $E = E^{\hat{f},\hat{g}}$ on $L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d)$. By the Urysohn property of Γ-convergence, see [14, Proposition 8.3], we deduce that for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ the whole sequence $E_k(\cdot,A)$ Γ-converges to $E^{\hat{f},\hat{g}}(\cdot,A)$. □

6. Stochastic homogenisation

In this section we apply the results obtained in the previous sections to study homogenisation problems for integral functionals with integrands in $\mathcal{F}^\alpha$ and $\mathcal{G}^\vartheta$. We first study integral functionals obtained by rescaling and after a natural change of variables we reformulate the results of Theorem 5.4 in term of limits of minimum problems on large cubes. This characterization is then used to study stochastic homogenisation problems by means of the Subadditive Ergodic Theorem.

6.1. **Functionals defined by rescaling.** In this subsection we fix $f\in\mathcal{F}^\alpha$ and $g\in\mathcal{G}^\vartheta$; for every $\varepsilon>0$ we define $f_\varepsilon(x,\xi):=f(x/\varepsilon,\xi)$, $g_\varepsilon(x,\zeta,\nu):=g(x/\varepsilon,\zeta,\nu)$, and $E_\varepsilon:=E^{f_\varepsilon,g_\varepsilon}$, see Definition 2.8. We also consider the function $g^0\colon\mathbb{R}^d\times\mathbb{R}\times\mathbb{S}^{d-1}\to\mathbb{R}$ introduced in (4.8) and the functionals E^{f,g^0} and $E^{f^\infty,g}$. By Remark 4.6 the functional E^{f,g^0} is well-defined. Finally, for every $\xi\in\mathbb{R}^d$ let κ_ξ be the constant introduced in (3.18). Our aim is to prove a condition which implies the Γ-convergence of the sequence E_{ε_k} when $\varepsilon_k\to 0+$ (see Theorem 6.3 below).

We begin with two lemmas related to the change of variables $z=y/\varepsilon$.

Lemma 6.1. *Let $0<\varepsilon<1$, $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, and $\rho>0$. Then*

$$\big|m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))-\varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))\big|\le C_\xi\vartheta(2\kappa_\xi\rho)\rho^d\,, \tag{6.1}$$

where $C_\xi:=(c_3|\xi|+c_4)c_3/c_1$.

Proof. Let $u\in BV(Q(x,\rho))\cap L^\infty(Q(x,\rho))$ with

$$\mathrm{tr}_{Q(x,\rho)}u=\mathrm{tr}_{Q(x,\rho)}\ell_\xi\quad \mathcal{H}^{d-1}\text{-a.e. on }\partial Q(x,\rho)\text{ and }|u-\ell_\xi|\le\kappa_\xi\rho\text{ in }Q(x,\rho)\,, \tag{6.2}$$

and let $v\in BV(Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))\cap L^\infty(Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$ be the function defined by $v(z):=\frac{1}{\varepsilon}u(\varepsilon z)$. Note that (6.2) is equivalent to

$$\mathrm{tr}_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}v=\mathrm{tr}_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\ell_\xi\quad \mathcal{H}^{d-1}\text{-a.e. on }\partial Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})\text{ and }|v-\ell_\xi|\le\kappa_\xi\tfrac{\rho}{\varepsilon}\text{ in }Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})\,. \tag{6.3}$$

By a change of variables we obtain

$$\int_{Q(x,\rho)}f_\varepsilon(y,\nabla u)dy=\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f(z,\nabla v)dz\,, \tag{6.4}$$

$$\int_{Q(x,\rho)}f_\varepsilon^\infty(y,\frac{dD^cu}{d|D^cu|})d|D^cu|=\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f^\infty(z,\frac{dD^cv}{d|D^cv|})d|D^cv|\,, \tag{6.5}$$

$$\int_{Q(x,\rho)\cap J_u}g_\varepsilon(y,[u],\nu_u)d\mathcal{H}^{d-1}=\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v}\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)d\mathcal{H}^{d-1}\,. \tag{6.6}$$

Hence by (g3), (4.9), and (4.10) we have

$$|\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)-g^0(z,[v],\nu_v)|\le\vartheta(2\kappa_\xi\rho)\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)\,,$$
$$|\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)-g^0(z,[v],\nu_v)|\le\tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho)g^0(z,[v],\nu_v)\,.$$

This implies that

$$g^0(z,[v],\nu_v)\le(1+\vartheta(2\kappa_\xi\rho))\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)\,,$$
$$\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)\le(1+\tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho))g^0(z,[v],\nu_v)\,.$$

Integrating we obtain

$$\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v} g^0(z,[v],\nu_v)d\mathcal{H}^{d-1} \le (1+\vartheta(2\kappa_\xi\rho))\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v}\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)d\mathcal{H}^{d-1},$$

$$\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v}\tfrac{1}{\varepsilon}g(z,\varepsilon[v],\nu_v)d\mathcal{H}^{d-1} \le (1+\tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho))\varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v} g^0(z,[v],\nu_v)d\mathcal{H}^{d-1}.$$

Adding the absolutely continuous and the Cantor part, and recalling (6.4)-(6.6), leads to

$$\varepsilon^d E^{f,g^0}(v,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \le (1+\vartheta(2\kappa_\xi\rho))E_\varepsilon(u,Q(x,\rho)),$$

$$E_\varepsilon(u,Q(x,\rho)) \le (1+\tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho))\varepsilon^d E^{f,g^0}(v,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).$$

By the equivalence between (6.2) and (6.3) we obtain

$$\varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \le (1+\vartheta(2\kappa_\xi\rho))m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)),$$

$$m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)) \le (1+\tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho))\varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).$$

Using ℓ_ξ in the minimum problems defining $m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))$ and $m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$ we obtain $m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)) \le (c_3|\xi|+c_4)\rho^d$ and $\varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})) \le (c_3|\xi|+c_4)\rho^d$, which together with the previous inequalities give

$$\varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \le m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)) + \vartheta(2\kappa_\xi\rho)(c_3|\xi|+c_4)\rho^d,$$

$$m^{E_\varepsilon}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)) \le \varepsilon^d m^{E^{f,g^0}}_{\kappa_\xi\frac{\rho}{\varepsilon}}(\ell_\xi,Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + \tfrac{c_3}{c_1}\vartheta(2\kappa_\xi\rho)(c_3|\xi|+c_4)\rho^d,$$

which prove (6.1). ☐

Lemma 6.2. *Let* $0<\varepsilon<1/(2c_6)$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, $\nu\in\mathbb{S}^{d-1}$, *and* $0<\rho<1$. *Then*

$$\Big|m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) - \varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))\Big| \le C_\zeta\rho^{d-1+\alpha}, \tag{6.7}$$

where C_ζ *is a constant depending only on* ζ *and on the structural constants* α, c_3, *and* c_6, *but independent of* ε, x, *and* ρ.

Proof. Let $u\in BV(Q_\nu(x,\rho))$ and let $v\in BV(Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$ be defined by $v(z):=u(\varepsilon z)$. By a change of variables we see that

$$\int_{Q_\nu(x,\rho)} f_\varepsilon(y,\nabla u)dy = \varepsilon^d\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z,\tfrac{1}{\varepsilon}\nabla v)dz, \tag{6.8}$$

$$\int_{Q_\nu(x,\rho)} f^\infty_\varepsilon(y,\frac{dD^cu}{d|D^cu|})d|D^cu| = \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty(z,\frac{dD^cv}{d|D^cv|})d|D^cv|, \tag{6.9}$$

$$\int_{Q_\nu(x,\rho)\cap J_u} g_\varepsilon(y,[u],\nu_u)d\mathcal{H}^{d-1} = \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\cap J_v} g(z,[v],\nu_v)d\mathcal{H}^{d-1}. \tag{6.10}$$

By Remark 4.3 for $\mathcal{L}^d$-a.e. $z\in Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$ we have

$$\big|\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v) - f^\infty(z,\nabla v)\big| \le c_6\varepsilon + c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v)^{1-\alpha}. \tag{6.11}$$

This implies that

$$f^\infty(z,\nabla v) \le \varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v) + c_6\varepsilon + c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v)^{1-\alpha} \tag{6.12}$$

and

$$f^\infty(z,\nabla v) \ge \varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v) - c_6\varepsilon - c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v)^{1-\alpha} \ge \tfrac{\varepsilon}{2}f(z,\tfrac{1}{\varepsilon}\nabla v) - c_6\varepsilon - c_6\varepsilon(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}},$$

where we used the inequality $\tau^{1-\alpha} \le \frac{1}{2c_6}\tau + (2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$ for $\tau\ge 0$ together with $0<\varepsilon<1/(2c_6)$. This implies that

$$\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v) \le 2f^\infty(z,\nabla v) + C\varepsilon,$$

where $C := 2c_6 + 2c_6(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$. This inequality together with (6.11) gives

$$\begin{aligned} \varepsilon f(z, \tfrac{1}{\varepsilon}\nabla v) &\le f^\infty(z,\nabla v) + c_6\varepsilon + c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\nabla v)^{1-\alpha} \\ &\le f^\infty(z,\nabla v) + c_6\varepsilon + c_6\varepsilon^\alpha\big(2f^\infty(z,\nabla v) + C\varepsilon\big)^{1-\alpha}. \end{aligned} \tag{6.13}$$

Integrating on $Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$ and using the Hölder inequality, from (6.12) we obtain

$$\begin{aligned} \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty(z,\nabla v)dz &\le \varepsilon^d \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z,\tfrac{1}{\varepsilon}\nabla v)dz + c_6\rho^d \\ &+ c_6\Big(\varepsilon^d\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z,\tfrac{\nabla v}{\varepsilon})dz\Big)^{1-\alpha}\rho^{\alpha d}, \end{aligned} \tag{6.14}$$

and from (6.13) we get

$$\begin{aligned} \varepsilon^d\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z,\tfrac{1}{\varepsilon}\nabla v)dz &\le \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty(z,\nabla v)dz + c_6\rho^d \\ &+ c_6\Big(2\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty(z,\nabla v)dz + C\rho^d\Big)^{1-\alpha}\rho^{\alpha d}. \end{aligned} \tag{6.15}$$

Adding the jump and the Cantor part and recalling (6.8)-(6.10), from (6.14) it follows that

$$\varepsilon^{d-1}E^{f^\infty,g}(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \le E_\varepsilon(u,Q_\nu(x,\rho)) + c_6\rho^d + c_6E_\varepsilon(u,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d},$$

while from (6.15) we obtain

$$\begin{aligned} E_\varepsilon(u,Q_\nu(x,\rho)) &\le \varepsilon^{d-1}E^{f^\infty,g}(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d \\ &+ c_6\Big(2\varepsilon^{d-1}E^{f^\infty,g}(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + C\rho^d\Big)^{1-\alpha}\rho^{\alpha d}. \end{aligned}$$

Since $\mathrm{tr}_{Q_\nu(x,\rho)}u = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$ if and only if $\mathrm{tr}_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}v = \mathrm{tr}_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}u_{\frac{x}{\varepsilon},\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$, from the last two inequalities we deduce that

$$\begin{aligned} \varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) &\le m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d \\ &+ c_6\big(m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))\big)^{1-\alpha}\rho^{\alpha d}, \end{aligned} \tag{6.16}$$

$$\begin{aligned} m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho) &\le \varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d \\ &+ c_6\Big(2\varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + C\rho^d\Big)^{1-\alpha}\rho^{\alpha d}. \end{aligned} \tag{6.17}$$

Taking $u_{x,\zeta,\nu}$ in the minimum problem that defines $m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))$ and using the estimate (g3) we obtain $m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \le c_3(|\zeta|\wedge 1)\rho^{d-1}$. Similarly we prove that $m^{E^{f^\infty,g}}(u_{x/\varepsilon,\zeta,\nu},Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})) \le c_3(|\zeta|\wedge 1)\rho^{d-1}/\varepsilon^{d-1}$. Hence from (6.16) and (6.17) we obtain

$$\varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{x/\varepsilon,\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \le m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d + c_6(c_3(|\zeta|\wedge 1))^{1-\alpha}\rho^{d-1+\alpha},$$

$$\begin{aligned} m^{E_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) &\le \varepsilon^{d-1}m^{E^{f^\infty,g}}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d \\ &+ c_6(2c_3(|\zeta|\wedge 1)) + C\big)^{1-\alpha}\rho^{d-1+\alpha}. \end{aligned}$$

From these inequalities we deduce (6.7). □

Thanks to the following result, the Γ-convergence of functionals defined by rescaling can be deduced from the convergence of the minimum values of some auxiliary minimum problems on large cubes.

Theorem 6.3. *Assume that there exist two functions* $\hat f\colon \mathbb{R}^d \to [0,+\infty)$ *and* $\hat g\colon \mathbb{R}\times\mathbb{S}^{d-1}\times(0,+\infty)\to[0,+\infty)$ *such that for every* $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, *and* $\nu\in\mathbb{S}^{d-1}$ *we have*

$$\lim_{r\to+\infty}\frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r))}{r^d} = \hat f(\xi)\,, \tag{6.18}$$

$$\lim_{r\to+\infty}\frac{m^{E^{f^\infty,g}}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} = \hat g(\zeta,\nu)\,, \tag{6.19}$$

where κ_ξ *is defined in* (3.18)*.*

Then $\hat f\in\mathcal{F}^\alpha$, $\hat g\in\mathcal{G}^\vartheta$, *and for every* $\varepsilon_k\to 0+$ *and* $A\in\mathcal{A}_c(\mathbb{R}^d)$ *the sequence* $E_{\varepsilon_k}(\cdot,A)$ Γ*-converges to* $E^{\hat f,\hat g}(\cdot,A)$ *with respect to the topology of* $L^0(\mathbb{R}^d)$*.*

Proof. Given $\varepsilon_k\to 0+$, let us check that the hypotheses of Theorem 5.4 are satisfied by $f_k:=f_{\varepsilon_k}$ and $g_k:=g_{\varepsilon_k}$. For a given $\rho>0$ we take $r_k:=\rho/\varepsilon_k$. By (6.18), applied with x replaced by x/ρ, and Lemma 6.1 we have

$$\begin{aligned}\hat f(\xi)\rho^d - C_\xi\vartheta(2\kappa_\xi\rho)\rho^d &= \lim_{k\to\infty}(\rho/r_k)^d\, m^{E^{f,g^0}}_{\kappa_\xi r_k}(\ell_\xi, Q(r_kx/\rho,r_k)) - C_\xi\vartheta(2\kappa_\xi\rho)\rho^d\\ &\le \liminf_{k\to\infty} m^{E_k}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho)) \le \limsup_{k\to\infty} m^{E_k}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))\\ \le \lim_{k\to\infty}(\rho/r_k)^d m^{E^{f,g^0}}_{\kappa_\xi r_k}&(\ell_\xi,Q(r_kx/\rho,r_k)) + C_\xi\vartheta(2\kappa_\xi\rho)\rho^d = \hat f(\xi)\rho^d + C_\xi\vartheta(2\kappa_\xi\rho)\rho^d.\end{aligned}$$

This shows that the hypothesis of Theorem 5.4 concerning $m^{E_k}_{\kappa_\xi\rho}(\ell_\xi,Q(x,\rho))$ is satisfied.

By (6.19), applied with x replaced by x/ρ, and Lemma 6.2 we have

$$\begin{aligned}\hat g(\zeta,\nu)\rho^{d-1} - C_\zeta\rho^{d-1+\alpha} &= \lim_{k\to\infty}(\rho/r_k)^{d-1} m^{E^{f^\infty,g}}(u_{r_kx/\rho,\zeta,\nu}, Q_\nu(r_kx/\rho,r_k)) - C_\zeta\rho^{d-1+\alpha}\\ &= \liminf_{k\to\infty} m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \le \limsup_{k\to\infty} m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))\\ \le \lim_{k\to\infty}(\rho/r_k)^{d-1}m^{E^{f^\infty,g}}&(u_{r_kx/\rho,\zeta,\nu},Q_\nu(r_kx/\rho,r_k)) + C_\zeta\rho^{d-1+\alpha} = \hat g(\zeta,\nu)\rho^{d-1} + C_\zeta\rho^{d-1+\alpha}\,.\end{aligned}$$

Since $\alpha>0$, this shows that the hypothesis of Theorem 5.4 concerning $m^{E_k}(u_{x,\zeta,\nu},Q_\nu(x,\rho))$ is satisfied with $\hat g$ independent of x.

Therefore, all hypotheses of Theorem 5.4 are satisfied and the conclusion follows. ☐

The following lemma shows that in order to apply Theorem 6.3 it is enough to check that (6.18) holds when ξ is rational.

Lemma 6.4. *Assume that for every* $x\in\mathbb{R}^d$ *and* $\xi\in\mathbb{Q}^d$ *the limit*

$$\lim_{r\to+\infty}\frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi,Q(rx,r))}{r^d} \tag{6.20}$$

exists and does not depend on x, *where* κ_ξ *is defined in* (3.18)*. Then there exists a continuous function* $\hat f\colon\mathbb{R}^d\to[0,+\infty)$ *such that*

$$\lim_{r\to+\infty}\frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi,Q(rx,r))}{r^d} = \hat f(\xi) \tag{6.21}$$

for every $x\in\mathbb{R}^d$ *and* $\xi\in\mathbb{R}^d$*.*

Proof. Given $r>0$, $x\in\mathbb{R}^d$, and $\xi_1,\xi_2\in\mathbb{R}^d$, we set $a:=1+d^{1/2}|\xi_1-\xi_2|$ and we claim that

$$m^{E^{f,g^0}}_{\kappa_{\xi_1}ar}(\ell_{\xi_1},Q(rx,ar)) \le m^{E^{f,g^0}}_{\kappa_{\xi_2}r}(\ell_{\xi_2},Q(rx,r)) + c_5|\xi_1-\xi_2|r^d + (c_3|\xi_1|+c_4)(a^d-1)r^d\,. \tag{6.22}$$

Given $\eta > 0$, let $u_2 \in BV(Q(rx,r)) \cap L^\infty(Q(rx,r))$ be such that $\mathrm{tr}_{Q(rx,r)} u_2 = \mathrm{tr}_{Q(rx,r)} \ell_{\xi_2}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(rx,r)$, $\|u_2 - \ell_{\xi_2}\|_{L^\infty(Q(rx,r))} \le \kappa_{\xi_2} r$, and

$$E^{f,g^0}(u_2, Q(rx,r)) \le m^{E^{f,g^0}}_{\kappa_{\xi_2} r}(\ell_{\xi_2}, Q(rx,r)) + \eta\,,$$

and extend it as ℓ_{ξ_2} to $Q(rx,ar)$. Then $u_1 := u_2 - \ell_{\xi_2} + \ell_{\xi_1}$ satisfies $\mathrm{tr}_{Q(rx,ar)} u_1 = \mathrm{tr}_{Q(rx,ar)} \ell_{\xi_1}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(rx,ar)$,

$$|u_1 - \ell_{\xi_1}| \le \kappa_{\xi_2} r = \kappa_{\xi_1} r + (|\xi_2| - |\xi_1|) d^{1/2} r \le \kappa_{\xi_1} r + |\xi_1 - \xi_2| d^{1/2} r \le \kappa_{\xi_1} a r\,,$$

where we used $\kappa_{\xi_1} > 1$. By (a), (b), (c2), and (e) in Definition 2.3 we have

$$\begin{aligned} m^{E^{f,g^0}}_{\kappa_{\xi_1} ar}(\ell_{\xi_1}, Q(rx,ar)) &\le E^{f,g^0}(u_1, Q(rx,ar)) \le E^{f,g^0}(u_2, Q(rx,r)) \\ &\quad + c_5 |\xi_1 - \xi_2| r^d + (c_3|\xi_1| + c_4)(a^d - 1) r^d \\ &\le m^{E^{f,g^0}}_{\kappa_{\xi_2} r}(\ell_{\xi_2}, Q(rx,r)) + \eta + c_5|\xi_1 - \xi_2| r^d + (c_3|\xi_1| + c_4)(a^d-1) r^d\,, \end{aligned} \tag{6.23}$$

which, by the arbitrariness of η gives (6.22).

Let us now fix $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$. To prove the existence of $\hat f(\xi)$ such that (6.21) holds it is enough to show that the function $r \mapsto \frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r))}{r^d}$ satisfies the Cauchy condition for $r \to +\infty$.

Let $r, s > 0$, let $\delta > 0$, and $\xi_\delta \in \mathbb{Q}^d$ with $|\xi_\delta - \xi| < \delta$. We set $a_\delta := 1 + |\xi_\delta - \xi| d^{1/2}$ and observe that $a_\delta \to 1$ as $\delta \to 0+$. We also set $r_\delta := a_\delta r$, $s_\delta := s/a_\delta$, $x_\delta := a_\delta x$, and $\hat x_\delta := x/a_\delta$. By (6.22) applied with $\xi_1 = \xi$, $\xi_2 = \xi_\delta$, $x = x_\delta$, and $r = s_\delta$, we have

$$m^{E^{f,g^0}}_{\kappa_\xi s}(\ell_\xi, Q(sx,s)) \le m^{E^{f,g^0}}_{\kappa_{\xi_\delta} s_\delta}(\ell_{\xi_\delta}, Q(s_\delta x_\delta, s_\delta)) + c_5 \delta s_\delta^d + (c_3|\xi| + c_4)(a_\delta^d - 1) s_\delta^d\,.$$

By (6.22) applied with $\xi_1 = \xi_\delta$ and $\xi_2 = \xi$ we have

$$m^{E^{f,g^0}}_{\kappa_{\xi_\delta} r_\delta}(\ell_{\xi_\delta}, Q(r_\delta \hat x_\delta, r_\delta)) \le m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r)) + c_5 \delta r^d + (c_3|\xi_\delta| + c_4)(a_\delta^d - 1) r^d\,.$$

Recalling that $a_\delta \ge 1$, these inequalities give

$$\frac{m^{E^{f,g^0}}_{\kappa_\xi s}(\ell_\xi, Q(sx,s))}{s^d} \le \frac{m^{E^{f,g^0}}_{\kappa_{\xi_\delta} s_\delta}(\ell_{\xi_\delta}, Q(s_\delta x_\delta, s_\delta))}{s_\delta^d} + c_5\delta + (c_3|\xi| + c_4)(a_\delta^d - 1) \tag{6.24}$$

$$\frac{m^{E^{f,g^0}}_{\kappa_{\xi_\delta} r_\delta}(\ell_{\xi_\delta}, Q(r_\delta \hat x_\delta, r_\delta))}{r_\delta^d} \le \frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r))}{r^d} + c_5\delta + (c_3|\xi_\delta| + c_4)(a_\delta^d - 1)\,. \tag{6.25}$$

Let us fix $\varepsilon > 0$ and choose $\delta > 0$ such that $c_5\delta + (c_3(|\xi| + \delta) + c_4)(a_\delta^d - 1) < \varepsilon$. By (6.20) there exists $M_\delta > 0$ such that

$$\Big| \frac{m^{E^{f,g^0}}_{\kappa_{\xi_\delta} s_\delta}(\ell_{\xi_\delta}, Q(s_\delta x_\delta, s_\delta))}{s_\delta^d} - \frac{m^{E^{f,g^0}}_{\kappa_{\xi_\delta} r_\delta}(\ell_{\xi_\delta}, Q(r_\delta \hat x_\delta, r_\delta))}{r_\delta^d} \Big| < \varepsilon\,, \tag{6.26}$$

provided $r_\delta, s_\delta > M_\delta$. Therefore, if $r, s > a_\delta M_\delta$, by (6.24)-(6.26) we obtain

$$\frac{m^{E^{f,g^0}}_{\kappa_\xi s}(\ell_\xi, Q(sx,s))}{s^d} \le \frac{m^{E^{f,g^0}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r))}{r^d} + 3\varepsilon\,. \tag{6.27}$$

Exchanging the roles of r and s we obtain that the Cauchy condition is satisfied and the proof of the existence of the limit in (6.21) is concluded.

To prove the continuity of $\hat f$, using the inequality $a \ge 1$ we deduce from (6.22) that

$$\frac{m^{E^{f,g^0}}_{\kappa_{\xi_1} ar}(\ell_{\xi_1}, Q(rx,ar))}{a^d r^d} \le \frac{m^{E^{f,g^0}}_{\kappa_{\xi_2} r}(\ell_{\xi_2}, Q(rx,r))}{r^d} + c_5|\xi_1 - \xi_2| + (c_3|\xi_1| + c_4)(a^d - 1)\,.$$

Passing to the limit as $r \to +\infty$ we obtain

$$\hat{f}(\xi_1) \leq \hat{f}(\xi_2) + c_5|\xi_1 - \xi_2| + (c_3|\xi_1| + c_4)(a^d - 1)$$

with $a := 1 + d^{1/2}|\xi_1 - \xi_2|$. Exchanging the roles of ξ_1 and ξ_2 we obtain an inequality which together with the previous one entails the continuity of $\hat{f}$. □

6.2. **Stochastic homogenisation.** To study stochastic homogenisation problems we fix a probability space $(\Omega, \mathcal{T}, P)$ and a group $(\tau_z)_{z\in\mathbb{Z}^d}$ of P-preserving transformations on $(\Omega, \mathcal{T}, P)$, i.e., a family $(\tau_z)_{z\in\mathbb{Z}^d}$ of maps $\tau_z\colon \Omega \to \Omega$ with the following properties

(a) (measurability) τ_z is $\mathcal{T}$-measurable for every $z \in \mathbb{Z}^d$;
(b) (bijectivity) τ_z is bijective for every $z \in \mathbb{Z}^d$;
(c) (invariance) $P(\tau_z(E)) = P(E)$ for every $E \in \mathcal{T}$ and every $z \in \mathbb{Z}^d$;
(d) (group property) $\tau_0 = \mathrm{id}_\Omega$ (the identity map on Ω) and $\tau_{z+z'} = \tau_z \circ \tau_{z'}$ for every $z, z' \in \mathbb{Z}^d$.

We recall that the group is called ergodic if every set $E \in \mathcal{T}$ with $\tau_z(E) = E$ for every $z \in \mathbb{Z}^d$ has probability 0 or 1.

We introduce now the classes of random integrands we are going to consider.

Definition 6.5 (Stochastically periodic random volume integrands)**.** Let $\mathcal{SF}^\alpha$ be the collection of functions $f\colon \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ satisfying the following properties

(a) f is measurable for the product of $\mathcal{T}$ and the Borel σ-algebra of $\mathbb{R}^d \times \mathbb{R}^d$;
(b) setting $f(\omega) := f(\omega, \cdot, \cdot)$, we have $f(\omega) \in \mathcal{F}^\alpha$ for every $\omega \in \Omega$;
(c) f is stochastically periodic with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$, i.e.,

$$f(\omega, x + z, \xi) = f(\tau_z(\omega), x, \xi) \tag{6.28}$$

for every $\omega \in \Omega$, $x \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$, and $\xi \in \mathbb{R}^d$.

Definition 6.6 (Stochastically periodic random surface integrands)**.** Let $\mathcal{SG}^\vartheta$ be the collection of functions $g\colon \Omega \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0, +\infty)$ satisfying the following properties

(a) g is measurable for the product of $\mathcal{T}$ and the Borel σ-algebra of $\mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1}$;
(b) setting $g(\omega) := g(\omega, \cdot, \cdot, \cdot)$, we have $g(\omega) \in \mathcal{G}^\vartheta$ for every $\omega \in \Omega$;
(c) g is stochastically periodic with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$, i.e.,

$$g(\omega, x + z, \zeta, \nu) = g(\tau_z(\omega), x, \zeta, \nu) \tag{6.29}$$

for every $\omega \in \Omega$, $x \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$.

Let $(\Omega, \widehat{\mathcal{T}}, \widehat{P})$ be the completion of $(\Omega, \mathcal{T}, P)$. It is obvious that for every $z \in \mathbb{Z}^d$ the function $\tau_z\colon \Omega \to \Omega$ is also $\widehat{\mathcal{T}}$-measurable. We now introduce the notion of subadditive process. Let $\mathcal{R}$ be the collection of all rectangles of the form

$$[a, b) := \{x \in \mathbb{R}^d : a_i \leq x_i < b_i \text{ for } i = 1, \cdots, d\} \quad \text{with } a, b \in \mathbb{R}^d.$$

Definition 6.7 (Subadditive process)**.** A subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$ is a function $\mu\colon \Omega \times \mathcal{R} \to \mathbb{R}$ with the following properties:

(a) (measurability) the function $\mu(\cdot, R)$ is $\widehat{\mathcal{T}}$-measurable for every $R \in \mathcal{R}$;
(b) (covariance) $\mu(\omega, R + z) = \mu(\tau_z(\omega), R)$ for every $\omega \in \Omega$, $R \in \mathcal{R}$, and $z \in \mathbb{Z}^d$;
(c) (subadditivity) if $R \in \mathcal{R}$ and $(R_i)_{i\in I} \subset \mathcal{R}$ is a finite partition of R then

$$\mu(\omega, R) \leq \sum_{i\in I} \mu(\omega, R_i) \quad \text{for every } \omega \in \Omega;$$

(d) (boundedness) there exists $c > 0$ such that $0 \leq \mu(\omega, R) \leq c\mathcal{L}^d(R)$ for every $\omega \in \Omega$ and $R \in \mathcal{R}$.

We shall use the following variant of the Subadditive Ergodic Theorem [1, Theorem 2.7], see also [15] and [22].

Theorem 6.8. *Let μ be a subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$. Then there exist a set $\Omega'\in\mathcal{T}$ with $P(\Omega')=1$ and a $\mathcal{T}$-measurable function $\varphi\colon\Omega'\to[0,+\infty)$ such that*

$$\lim_{r\to+\infty}\frac{\mu(\omega,Q(rx,r))}{r^d}=\varphi(\omega) \tag{6.30}$$

for every $x\in\mathbb{R}^d$ and every $\omega\in\Omega'$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then φ is constant P-a.e.

Given $f\in\mathcal{SF}^\alpha$, $g\in\mathcal{SG}^\vartheta$, and $\omega\in\Omega$ let $f^\infty(\omega)$ and $g^0(\omega)$ be the functions defined by

$$f^\infty(\omega)(x,\xi):=\lim_{r\to+\infty}\frac{f(\omega,x,r\xi)}{r}\quad\text{ and }\quad g^0(\omega,x,r\zeta,\nu):=\lim_{r\to0+}\frac{1}{r}g(\omega,x,r\zeta,\nu)$$

for every $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$.

Lemma 6.9. *Let $f\in\mathcal{SF}^\alpha$, $g\in\mathcal{SG}^\vartheta$, and let $\xi\in\mathbb{R}^d$. For every $R\in\mathcal{R}$ let $\rho(R)$ be the maximum length of its sides. Then the function $\Phi_\xi\colon\Omega\times\mathcal{R}\to[0,+\infty)$ defined by*

$$\Phi_\xi(\omega,R):=m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi\rho(R)}(\ell_\xi,\mathring{R})$$

is a subadditive process.

Proof. The $\widehat{\mathcal{T}}$-measurability of $\Phi_\xi(\cdot,R)$ can be obtained by adapting the proof in [12, Appendix].

To prove the covariance property, we fix $z\in\mathbb{Z}^d$, $R\in\mathcal{R}$, and $\omega\in\Omega$. By (6.28) and (6.29) we have

$$E^{f(\tau_z(\omega)),g^0(\tau_z(\omega))}(u,\mathring{R})=E^{f(\omega),g^0(\omega)}(\tau_z u,\mathring{R}+z)\,.$$

Since $\tau_z\ell_\xi=\ell_\xi-\ell_\xi(z)$, by the invariance property (d) in Definition 2.3, we deduce that $\Phi(\tau_z(\omega),R)=\Phi(\omega,R+z)$.

We now prove subadditivity. Let us fix $\omega\in\Omega$, $R\in\mathcal{R}$, a finite partition $(R_i)_{i=1}^n$ of R, and $\eta>0$. For every $i=1,\dots,n$ there exists $u_i\in BV(\mathring{R}_i)\cap L^\infty(R_i)$ with $\mathrm{tr}_{R_i}u_i=\mathrm{tr}_{R_i}\ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on ∂R_i and $\|u_i-\ell_\xi\|_{L^\infty(R_i)}\le\kappa_\xi\rho(R_i)$ such that

$$E^{f(\omega),g^0(\omega)}(u_i,\mathring{R}_i)\le m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi\rho(R_i)}(\ell_\xi,\mathring{R}_i)+\tfrac{\eta}{n}\,.$$

Let u be the function defined $\mathcal{L}^d$-a.e. in $\mathring{R}$ by $u=u_i$ on $\mathring{R}_i$. Then $u\in BV(\mathring{R})$ and $\mathrm{tr}_R u=\mathrm{tr}_R\ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on ∂R.

Since $\rho(R_i)\le\rho(R)$ for every $i=1,\cdots,n$ we have also that $\|u-\ell_\xi\|_{L^\infty(R)}\le\kappa_\xi\rho(R)$. This implies that

$$\begin{aligned}\Phi(\omega,R)&=m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi\rho(R)}(\ell_\xi,\mathring{R})\le E^{f(\omega),g^0(\omega)}(u,\mathring{R})\\&\le\sum_{i=1}^n E^{f(\omega),g^0(\omega)}(u_i,\mathring{R}_i)\le\sum_{i=1}^n m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi\rho(R_i)}(\ell_\xi,\mathring{R}_i)+\eta=\sum_{i=1}^n\Phi(\omega,R_i)+\eta\,,\end{aligned} \tag{6.31}$$

and passing to the limit as $\eta\to0+$ we obtain the subadditivity of Φ.

Moreover, taking ℓ_ξ in the minimum problem defining $m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi\rho(R)}(\ell_\xi,\mathring{R})$ we obtain

$$\Phi(\omega,R)\le(c_4|\xi|+c_5)\mathcal{L}^d(R)\,,$$

which concludes the proof. □

Proposition 6.10. *Let $f\in\mathcal{SF}^\alpha$ and $g\in\mathcal{SG}^\vartheta$. Then there exist $\Omega'\in\mathcal{T}$ with $P(\Omega')=1$ and a function $\hat f:\Omega'\times\mathbb{R}^d\to[0,+\infty)$, with $\hat f(\cdot,\xi)$ $\mathcal{T}$-measurable for every $\xi\in\mathbb{R}^d$ and $\hat f(\omega,\cdot)$ continuous for every $\omega\in\Omega'$, such that*

$$\lim_{r\to+\infty}\frac{m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi r}(\ell_\xi,Q(rx,r))}{r^d}=\hat f(\omega,\xi) \tag{6.32}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^d$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $\hat{f}$ is independent of ω and

$$\hat{f}(\xi) = \lim_{r\to+\infty} \frac{1}{r^d} \int_\Omega m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi r}(\ell_\xi, Q(0,r))\, dP(\omega)\,. \tag{6.33}$$

Proof. By Lemma 6.9, for every $\xi \in \mathbb{Q}^d$ the map

$$(\omega, R) \mapsto m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi \rho(R)}(\ell_\xi, \mathring{R})$$

is a subadditive process on $(\Omega, \widehat{\mathcal{T}}, \widehat{P})$. By the Subadditive Ergodic Theorem 6.8 there exist $\Omega' \in \mathcal{T}$ with $P(\Omega') = 1$ and a function $\hat{f}\colon \Omega' \times \mathbb{Q}^d \to [0,+\infty)$, with $\hat{f}(\cdot,\xi)$ $\mathcal{T}$-measurable for every $\xi \in \mathbb{Q}^d$, such that

$$\lim_{r\to+\infty} \frac{m^{E^{f(\omega),g^0(\omega)}}_{\kappa_\xi r}(\ell_\xi, Q(rx,r))}{r^d} = \hat{f}(\omega,\xi) \tag{6.34}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{Q}^d$. By Lemma 6.4 we can extend $\hat{f}$ to a function $\hat{f}\colon \Omega' \times \mathbb{R}^d \to [0,+\infty)$, with $\hat{f}(\cdot,\xi)$ $\mathcal{T}$-measurable for every $\xi \in \mathbb{R}^d$ and $\hat{f}(\omega,\cdot)$ continuous for every $\omega \in \Omega'$, such that (6.32) holds for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^d$.

If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then the function $\hat{f}(\omega,\xi)$ for $\xi \in \mathbb{Q}^d$ does not depend on ω and consequently the same property holds for its extension to $\xi \in \mathbb{R}^d$. □

Proposition 6.11. *Let $f \in \mathcal{SF}^\alpha$ and $g \in \mathcal{SG}^\vartheta$. Then there exist $\Omega' \in \mathcal{T}$ with $P(\Omega') = 1$ and a measurable function $\hat{g} : \Omega' \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$ for the product of $\mathcal{T}$ and the Borel σ-algebra of $\mathbb{R} \times \mathbb{S}^{d-1}$, such that*

$$\lim_{r\to+\infty} \frac{m^{E^{f^\infty(\omega),g(\omega)}}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} = \hat{g}(\omega,\zeta,\nu) \tag{6.35}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $\hat{g}$ is independent of ω and

$$\hat{g}(\zeta,\nu) = \lim_{r\to+\infty} \frac{1}{r^{d-1}} \int_\Omega m^{E^{f^\infty(\omega),g(\omega)}}(u_{0,\zeta,\nu}, Q_\nu(0,r)) dP(\omega)\,. \tag{6.36}$$

Proof. The result can be obtained by adapting all arguments of the proofs of [12, Propositions 9.3, 9.4, and 9.5] and of [11, Theorem 6.1]. □

The following theorem summarizes the results of this section.

Theorem 6.12. *Let $f \in \mathcal{SF}^\alpha$ and $g \in \mathcal{SG}^\vartheta$. For every $\varepsilon > 0$ and $\omega \in \Omega$ let $f_\varepsilon(\omega) \in \mathcal{F}^\alpha$ and $g_\varepsilon(\omega) \in \mathcal{G}^\vartheta$ be defined by*

$$f_\varepsilon(\omega,x,\xi) := f(\omega,x/\varepsilon,\xi) \quad \text{and} \quad g_\varepsilon(\omega,x,\zeta,\nu) := g(\omega,x/\varepsilon,\zeta,\nu) \tag{6.37}$$

for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. Finally, let $E_\varepsilon(\omega) := E^{f_\varepsilon(\omega),g_\varepsilon(\omega)} \in \mathfrak{E}^{\alpha,\vartheta}$. Then there exist a set $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and two functions $\hat{f}\colon \Omega' \times \mathbb{R}^d \to [0,+\infty)$ and $\hat{g} : \Omega' \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that

- (a) *$\hat{f}$ is measurable for the product of $\mathcal{T}$ and the Borel σ-algebra of $\mathbb{R}^d$, and $\hat{f}(\omega,\cdot) \in \mathcal{F}^\alpha$ for every $\omega \in \Omega'$;*
- (b) *$\hat{g}$ is measurable for the product of $\mathcal{T}$ and the Borel σ-algebra of $\mathbb{R} \times \mathbb{S}^{d-1}$, and $\hat{g}(\cdot,\zeta,\nu)$ is $\mathcal{T}$-measurable for every $\zeta \in \mathbb{R}$ and $\nu \in \mathbb{S}^{d-1}$;*
- (c) *for every $\omega \in \Omega'$, $\varepsilon_k \to 0+$, and $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_{\varepsilon_k}(\omega)(\cdot,A)$ Γ-converges to $E^{\hat{f}(\omega),\hat{g}(\omega)}(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$.*

If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $\hat{f}$ and $\hat{g}$ are independent of ω.

Proof. The result follows from Theorem 6.3 and Propositions 6.10 and 6.11. □

Remark 6.13. Thanks to Theorem 6.12 the convergence results for minimum values and quasi-minimisers considered in [17, Theorems 7.1 and 7.14, Corollary 7.15] can be obtained for the functionals $E_\varepsilon(\omega)$ for P-a.e. $\omega \in \Omega$.

The deterministic periodic case is considered in the following remark.

Remark 6.14. If Ω consists of a single point, and consequently $\tau_z = id_\Omega$ for every $z \in \mathbb{R}^d$, then stochastic periodicity (see Definitions 6.5 and 6.6) reduces to periodicity of period 1 in each coordinate. Therefore all results of this subsection are valid in the deterministic periodic case.

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