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On the closability of differential operators

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Abstract. We discuss the closability of directional derivative operators with respect to a general Radon measure μ on $\mathbb{R}^d$; our main theorem completely characterizes the vectorfields for which the corresponding operator is closable from the space of Lipschitz functions $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$, for $1 \leq p \leq \infty$. We also discuss the closability of the same operators from $L^q(\mu)$ to $L^p(\mu)$, and give necessary and sufficient conditions for closability, but we do not have an exact characterization.
As a corollary we obtain that classical differential operators such as gradient, divergence and Jacobian determinant are closable from $L^q(\mu)$ to $L^p(\mu)$ only if μ is absolutely continuous with respect to the Lebesgue measure.
We finally consider the closability of a certain class of multilinear differential operators; these results are then rephrased in terms of metric currents.

Keywords: closable operators, directional derivative operators, Lipschitz functions, Sobolev spaces, normal currents, metric currents.

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1. Introduction

One way of defining the Sobolev spaces $W_0^{1,p}(\Omega)$ for an open set Ω in $\mathbb{R}^d$ is taking the completion of the space $C_c^1(\Omega)$ of functions of class C^1 with compact support on Ω with respect to the Sobolev norm $\|\cdot\|_{W^{1,p}}$.

This construction can be made more precise as follows: we consider the graph of the gradient operator $\nabla : C_c^1(\Omega) \to C_c^0(\Omega; \mathbb{R}^d)$ as a subset of the product space $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^d)$, take its closure Γ, and show that Γ is still a graph, that is, for every $u \in L^p(\Omega)$ there exists *at most one* $v \in L^p(\Omega; \mathbb{R}^d)$ such that $(u, v) \in \Gamma$. We then consider the operator whose graph is Γ: the domain is the Sobolev space $W_0^{1,p}(\Omega)$ and the operator is the gradient for Sobolev functions.[1]

Note that the essential ingredient in this construction is that the closure of the graph of the gradient is still a graph. The extension of this construction to more general operators leads to the following abstract definition:

Closable operators. Given X, Y topological spaces, D subset of X, and a map $T : D \to Y$, we denote by Γ the closure of the graph $\{(x, T(x)) : x \in D\}$ in $X \times Y$, and we say that T is *closable* (from X to Y) if Γ is also a graph, that is, for every $x \in X$ there exists *at most one* $y \in Y$ such that $(x, y) \in \Gamma$.

In this paper, we study the closability of certain first-order differential operators, and we focus in particular on directional derivative operators of the form (**??**). The

[1] Moreover the norm of every $(u, v) \in \Gamma$ as element of $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^d)$ agrees with $\|u\|_{W^{1,p}}$.

spaces X and Y are always spaces of functions on $\mathbb{R}^d$, taken from those listed in Paragraph **??**.

Last but not least, we always intend continuity, closure and closability in the sequential sense. However, in many instances the sequential notions can be replaced by the topological ones for simple reasons (for example because the spaces are metrizable, or because we are dealing with a non-closability result).

1.1. Functions spaces. Through this paper μ is a Radon measure on $\mathbb{R}^d$ and the space Y is one of the following:

- $L^p(\mu)$ with $1 \le p \le \infty$, endowed with the strong topology;
- $L^p_w(\mu)$, which denotes the space $L^p(\mu)$ endowed with the weak topology if $p < \infty$, and with the weak* topology (as dual of $L^1(\mu)$) if $p = \infty$.

The space X is one of the following:

- $\mathrm{Lip}(\mathbb{R}^d)$, namely the space of Lipschitz functions on $\mathbb{R}^d$ endowed with the following notion of convergence: $u_n \to u$ in $\mathrm{Lip}(\mathbb{R}^d)$ if $u_n \to u$ uniformly and the Lipschitz constants $\mathrm{Lip}(u_n)$ are uniformly bounded.[2]
- $L^q(\mu)$ or $L^q_w(\mu)$ with $1 \le q \le \infty$.

Finally the set D is always $C^1_c(\mathbb{R}^d)$.

1.2. Directional derivative operators. Let v be a Borel vector field on $\mathbb{R}^d$; we denote by T_v the directional derivative operator on $C^1_c(\mathbb{R}^d)$ associated to v, that is,

$$T_v u := \frac{\partial u}{\partial v} \quad \text{for every } u \in C^1_c(\mathbb{R}^d). \tag{1.1}$$

The next theorem is our main result. The statement involves the notion of decomposability bundle $V(\mu,\cdot)$ of a measure μ; the precise definition is given in Paragraph **??**, but for a first reading it suffices to know that $V(\mu,x)$ is a linear subspace of $\mathbb{R}^d$ for every $x \in \mathbb{R}^d$.

1.3. Theorem. *Let v and T_v be as above and assume that $v \in L^p(\mu)$ for some $1 \le p \le \infty$.*

(i) *If $v(x) \in V(\mu,x)$ for μ-a.e. x, then every $u \in \mathrm{Lip}(\mathbb{R}^d)$ is differentiable at μ-a.e. $x \in \mathbb{R}^d$ in the direction $v(x)$, and the linear operator $\widetilde{T}_v : \mathrm{Lip}(\mathbb{R}^d) \to L^p_w(\mu)$ defined by*

$$\widetilde{T}_v u(x) := \frac{\partial u}{\partial v}(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d \tag{1.2}$$

is a continuous extension of T_v.
It follows that T_v is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$.

(ii) *Conversely, if $\mu(\{x\colon v(x) \notin V(\mu,x)\}) > 0$ then T_v is nowhere continuous as an operator from $C^1_c(\mathbb{R}^d)$ (endowed with Lip-convergence) to $L^p_w(\mu)$. More precisely, for every $u \in C^1_c(\mathbb{R}^d)$ and every $\varepsilon > 0$ there exist a sequence (u_n) in $C^1_c(\mathbb{R}^d)$ and $w \in L^p(\mu)$ with $w \ne T_v u$ such that*
- *$u_n \to u$ uniformly;*
- *$\mathrm{Lip}(u_n) \le \mathrm{Lip}(u) + \varepsilon$ for every n;*

[2] Equivalently, $u_n \to u$ uniformly and $\nabla u_n \to \nabla u$ in $L^\infty_w(\mathbb{R}^d)$. Therefore this notion of convergence is induced by a topology.

- *$T_v u_n \to w$ in $L^p(\mu)$ if $p < \infty$, and in $L^\infty_w(\mu)$ if $p = \infty$.*

It follows that T_v is not closable from $\mathrm{Lip}(\mathbb{R}^d)$ *to $L^p_w(\mu)$, and not even from* $\mathrm{Lip}(\mathbb{R}^d)$ *to $L^p(\mu)$ if $p < \infty$.*

1.4. Remarks. (i) Since the space $L^p_w(\mu)$ embeds continuously in $L^p(\mu)$, the conclusion that T_v is closable in Theorem **??**(i) holds all the more so if we replace $L^p_w(\mu)$ by $L^p(\mu)$, but clearly $\widetilde{T}_v$ is not continuous from $\mathrm{Lip}(\mathbb{R}^d)$ in $L^p(\mu)$.

(ii) The non-closability part of Theorem **??**(ii) holds even if replace $\mathrm{Lip}(\mathbb{R}^d)$ by $L^q(\mu)$ with $1 \le q \le \infty$ (or $L^q_w(\mu)$); this assertion is immediate when μ is a finite measure, because the fact that $u_n \to u$ uniformly implies that $u_n \to u$ in $L^q(\mu)$; if μ is only locally finite one should use that the sequence (u_n) can be chosen so that the functions $u_n - u$ have uniformly bounded supports.

We now turn our attention to the closability of classical differential operators such as gradient, divergence and Jacobian determinant.[3] Let indeed T be any of these three operators: it is well known that T can be extended using a distributional definition to a continuous operator $\widetilde{T}$ from $\mathrm{Lip}(\mathbb{R}^d)$ in $L^\infty_w(\mathscr{L}^d)$,[4] and this implies that T is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^\infty_w(\mathscr{L}^d)$.

It is natural to ask what happens if we replace the Lebesgue measure $\mathscr{L}^d$ by a general Radon measure μ. The complete answer is contained in the following corollary of Theorem **??**:

1.5. Corollary. *Let T be any of the following operators on $C^1_c(\mathbb{R}^d)$: gradient, divergence, Jacobian determinant, and let $1 \le p \le \infty$.*

(i) *If μ is absolutely continuous with respect to the Lebesgue measure ($\mu \ll \mathscr{L}^d$) then T is closable from* $\mathrm{Lip}(\mathbb{R}^d)$ *to $L^p_w(\mu)$ and then also to $L^p(\mu)$.*

(ii) *If μ is not absolutely continuous with respect to the Lebesgue measure then T is not closable from* $\mathrm{Lip}(\mathbb{R}^d)$ *to $L^p_w(\mu)$, and not even to $L^p(\mu)$ if $p < +\infty$.*

The next corollary answers a question posed by M. Fukushima about the closability of the gradient (see [**?**, Section 2.6] and [**?**]):

1.6. Corollary. *Let T be as in Corollary* **??** *and take $1 \le p < \infty$ and $1 \le q \le \infty$. Then T is closable from $L^q(\mu)$ to $L^p(\mu)$ only if $\mu \ll \mathscr{L}^d$.*

Structure of this paper. In Section **??** we collect some definition and preliminary results that are widely used through the rest of the paper, while Section **??** is devoted to the proofs of the results stated above.

As pointed out in Remark **??**(ii), Theorem **??**(ii) gives a necessary condition for the closability of the directional derivative operator T_v in (**??**) from $L^p(\mu)$ to $L^p(\mu)$, but this condition is not sufficient (cf. Remark **??**(iii)). In Section **??**, and more specifically in Theorem **??**, we give a sufficient condition for the closability of T_v; we do not know if this condition is also necessary.

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[3] The Jacobian determinant of $u \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$ is $Ju := \det(\nabla u)$.

[4] Here $\mathscr{L}^d$ is the Lebesgue measure, and, depending on which T we consider, $\mathrm{Lip}(\mathbb{R}^d)$ and $L^\infty_w(\mathscr{L}^d)$ may denote spaces of $\mathbb{R}^d$-valued functions.

In Section **??** we discuss the closability of a general class of alternating k-linear differential operators akin to the Jacobian determinant. In Theorem **??** we give necessary and sufficient conditions for closability from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^\infty_w(\mu)$; we do not know if these conditions match (unlike those in Theorem **??**). In the second part of Section **??** we reformulate these results in terms of metric currents in $\mathbb{R}^d$; among other things we obtain a reformulation of the *Flat Chain Conjecture* (see [**?**, Section 11]) both in terms of the k-tangent bundle of the measure associated to a metric current and in terms of closability/continuity of a suitably defined alternating k-linear differential operator (Theorem **??**).

1.7. Concluding remarks. (i) Theorem **??** has been used in [**?**] to give a new proof of the chain rule for BV functions first proved in [**?**]; this new proof can be adapted to finite dimensional RCD spaces.

(ii) Through this paper, we consider for simplicity functional spaces defined on the domain $\mathbb{R}^d$. However, all results are essentially local in nature, thus the domain $\mathbb{R}^d$ can be easily replaced by any open subset of $\mathbb{R}^d$, and even by more general domains.

(iii) Regarding Corollary **??**, it is well known that the assumption $\mu \ll \mathscr{L}^d$ alone does not imply the closability of the gradient operator from $L^q(\mu)$ to $L^p(\mu)$, not even in dimension $d = 1$. For example, let μ be the restriction of the Lebesgue measure $\mathscr{L}^1$ to a totally disconnected compact subset of $\mathbb{R}$; then it is easy to prove that the derivative is not closable from $L^q(\mu)$ to $L^p(\mu)$ for any $1 \le p, q \le \infty$.

A precise characterization of the (absolutely continuous) measures μ on $\mathbb{R}$ such that the derivative is closable from $L^2(\mu)$ to $L^2(\mu)$ has been given in [**?**, Theorem 2.2] (see also [**?**, Theorem 3.1.6]).

(iv) Theorem **??** can be easily extended to more general *first order* differential operators than just directional derivatives, thus obtaining a statement that includes Corollaries **??** and **??** as particular cases (see Remark **??**).

However, it seems that *second order differential operators* are not easily included in our analysis. For instance, one would guess that the Laplace operator (defined on $C^2_c(\mathbb{R}^d)$) should be closable from $L^p(\mu)$ to $L^q(\mu)$ only if μ is absolutely continuous with respect to $\mathscr{L}^d$, but such statement does not seem to follow from any of the results in this paper (at least, not easily).

(v) At the core of the proof of Theorem **??**(ii), and of the "only if" part of Corollaries **??** and **??**, are the so-called "width functions", which are taken from [**?**, Lemma 4.12]. This notion was introduced by David Preiss while studying the differentiability of Lipschitz functions (cf. [**?**], [**?**], and references therein); he also used it to give a first (unpublished) proof of Corollary **??**(ii) in dimension $d = 2$.

(vi) Given a Radon measure μ on $\mathbb{R}^d$, in [**?**, Section 3] the authors define for μ-a.e. $x \in \mathbb{R}^d$ a tangent space $T_\mu(x)$ in such a way that the corresponding tangential gradient operator is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^\infty_w(\mu)$. Theorem **??**(ii) shows that $T_\mu(x) \subset V(\mu, x)$ for μ-a.e. x, and with some additional work one can prove that equality holds. This remark gives a way to compute T_μ effectively; moreover, thanks to [**?**, Theorem 1.1], every function $u \in \mathrm{Lip}(\mathbb{R}^d)$, is differentiable in the directions in $T_\mu(x)$ for μ-a.e. x.

(vii) Given a Radon measure μ on $\mathbb{R}^d$ and $1 \le p \le \infty$, in [**?**, Section 2] the authors define for μ-a.e. $x \in \mathbb{R}^d$ a tangent space $T^p_\mu(x)$ in such a way that the corresponding tangential gradient operator is closable from $L^p(\mu)$ to $L^p(\mu)$. Using Theorem **??**(ii) one can prove that $T^p_\mu(x) \subset V(\mu, x)$ for μ-a.e. x. However, this inclusion may be strict (cf. the example in remark (iii) above).

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2. Notation and preliminary results

Through this paper, sets and functions are always Borel; measures are Borel and positive (unless stated otherwise) and, with the notable exception of Hausdorff measures, they are also locally bounded (that is, Radon). If μ is a real- or vector-valued measure, $|\mu|$ denotes the variation of μ.

In the next paragraph we recall the basic notation about currents in the Euclidean setting. For more details about currents see for instance [**?**], [**?**].

2.1. Classical currents. A *k-dimensional current* T in $\mathbb{R}^d$ is a continuous linear functional on the space $\mathscr{D}^k(\mathbb{R}^d)$ of smooth k-forms on $\mathbb{R}^d$ with compact support. The *boundary* of T is the $(k-1)$-current ∂T defined by $\langle \partial T\,;\,\omega\rangle := \langle T\,;\,d\omega\rangle$ for every $\omega \in \mathscr{D}^{k-1}(\mathbb{R}^d)$.

The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T\,;\,\omega\rangle$ over all $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ such that $|\omega| \le 1$ everywhere. A current T is called *normal* if both T and ∂T have finite mass.

By Riesz theorem a k-current T with finite mass can be viewed as a finite measure with values in the space k-vectors on $\mathbb{R}^d$, and therefore it can be written in the form $T = \tau\mu$ where μ is a finite positive measure and τ is a k-vector field in $L^1(\mu)$. Thus the action of T on a k-form ω is given by

$$\langle T\,;\,\omega\rangle = \int_{\mathbb{R}^d} \langle \tau(x)\,;\,\omega(x)\rangle\, d\mu(x)\,.$$

Given a Lipschitz path $\gamma : [a, b] \to \mathbb{R}^d$, we denote by $[\gamma]$ the associated current, namely the 1-dimensional current defined by

$$\big\langle [\gamma]\,;\,\omega\big\rangle = \int_a^b \big\langle \gamma'(t)\,;\,\omega(\gamma(t))\big\rangle\, dt\,.$$

2.2. Decomposability bundle. Given a measure μ on $\mathbb{R}^d$, its *decomposability bundle* is a Borel map $V(\mu, \cdot)$ on $\mathbb{R}^d$ whose values are linear subspaces of $\mathbb{R}^d$ defined as follows: a vector $v \in \mathbb{R}^d$ belongs to $V(\mu, x)$ if and only if there exists a 1-dimensional normal current N in $\mathbb{R}^d$ with $\partial N = 0$ such that

$$\lim_{r\to 0} \frac{|N - v\mu|(B(x,r))}{\mu(B(x,r))} = 0\,. \tag{2.1}$$

The decomposability bundle $V(\mu, x)$ was introduced in [**?**], Paragraph 2.6; in Paragraph 6.1 of the same paper, the authors introduced also an auxiliary bundle $N(\mu, x)$, and then they proved that these two bundles agree [**?**, Theorem 6.4]. The definition above is actually that of the auxiliary bundle $N(\mu, x)$, which is simpler to state than the original definition of the decomposability bundle.

In the ensuing sections we will use several results from [**?**] concerning the decomposability bundle. We state here the most relevant ones.

2.3. Theorem (see [**?**, Theorem 1.1]). *Let μ be a measure on $\mathbb{R}^d$. Then the following statements hold:*

(i) *Every Lipschitz function f on $\mathbb{R}^d$ is differentiable at μ-a.e. x with respect to the linear subspace $V(\mu, x)$, that is, there exists a linear function from $V(\mu, x)$ to $\mathbb{R}$, denoted by $d_V f(x)$, such that*

$$f(x+h) = f(x) + \langle d_V f(x)\,;\, h\rangle + o(|h|) \quad \text{for } h \in V(\mu, x).$$

(ii) *The previous statement is optimal, meaning that there exists a Lipschitz function f on $\mathbb{R}^d$ such that for μ-a.e. x and every $v \notin V(\mu, x)$ the derivative of f at x in the direction v does not exist.*

2.4. Theorem. *Let μ be a measure on $\mathbb{R}^d$. Then $V(\mu, x) = \mathbb{R}^d$ for μ-a.e. $x \in \mathbb{R}^d$ if and only if $\mu \ll \mathscr{L}^d$.*

Proof. The "if" implication is contained in [**?**, Proposition 2.9(iii)]. The hard part is the "only if" implication, which is a direct consequence of the results in [**?**] and [**?**]; for instance, it can be obtained by combining Theorem **??** and [**?**, Theorem 1.14]. □

3. Proof of Theorem ?? and Corollaries ??, ??

Proof of Theorem ??(i), case $p = \infty$. The operator $\widetilde{T}_v : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_w(\mu)$ given by formula (**??**) is well defined thanks to the assumption $v(x) \in V(\mu, x)$ for μ-a.e. x and Theorem **??**(i), and it is obviously an extension of the operator T_v defined in (**??**).

It remains to prove that $\widetilde{T}_v$ is continuous. By [**?**, Theorem 6.3] there exists a normal 1-current $N = \tilde{v}\tilde{\mu}$ on $\mathbb{R}^d$ such that

- $\partial N = 0$;
- $\tilde{v} \in L^\infty(\tilde{\mu})$ and $\tilde{v}(x) \in V(\tilde{\mu}, x)$ for μ-a.e. x;
- $\tilde{v}$ and $\tilde{\mu}$ extend v and μ in the following sense: $\tilde{\mu} = \mu + \sigma$ with σ and μ mutually singular, and $\tilde{v}(x) = v(x)$ for μ-a.e. x.

Let $\widetilde{T}_{\tilde{v}} : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_w(\tilde{\mu})$ be the operator defined by formula (**??**) with v replaced by $\tilde{v}$; one easily checks that the continuity of $\widetilde{T}_v$ follows from that of $\widetilde{T}_{\tilde{v}}$.

To prove the continuity of $\widetilde{T}_{\tilde{v}}$, we note that for every $u \in \mathrm{Lip}(\mathbb{R}^d)$ the boundary of the current uN is given by

$$\partial(uN) = -\widetilde{T}_{\tilde{v}}\, u\, \tilde{\mu} \tag{3.1}$$

(use Proposition 5.13 in [?] and the fact that $\partial N = 0$) and then

$$\int_{\mathbb{R}^d} \varphi\, \widetilde{T}_{\tilde{v}} u \, d\tilde{\mu} = -\langle uN\,;\, d\varphi \rangle \quad \text{for every } \varphi \in \mathscr{D}(\mathbb{R}^d), \tag{3.2}$$

where $\mathscr{D}(\mathbb{R}^d)$ is the space of smooth test functions with compact support on $\mathbb{R}^d$.

Consider now a sequence (u_n) such that $u_n \to u$ in $\mathrm{Lip}(\mathbb{R}^d)$. Since $u_n \to u$ uniformly, the currents $u_n N$ converge to uN with respect to the mass norm $\mathbb{M}$, and therefore also in the sense of currents; then formula (??) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi\, \widetilde{T}_{\tilde{v}} u_n \, d\tilde{\mu} = \int_{\mathbb{R}^d} \varphi\, \widetilde{T}_{\tilde{v}} u \, d\tilde{\mu} \quad \text{for every } \varphi \in \mathscr{D}(\mathbb{R}^d). \tag{3.3}$$

Since $\mathscr{D}(\mathbb{R}^d)$ is dense in $L^1(\tilde{\mu})$ and the functions $\widetilde{T}_{\tilde{v}} u_n$ are uniformly bounded in $L^\infty(\tilde{\mu})$, (??) holds also for every $\varphi \in L^1(\tilde{\mu})$, which means that

$$\widetilde{T}_{\tilde{v}} u_n \to \widetilde{T}_{\tilde{v}} u \quad \text{in } L^\infty_w(\tilde{\mu}),$$

and the continuity of $\widetilde{T}_{\tilde{v}}$ is proved. □

Proof of Theorem ??(i), case $p < \infty$. As for the case $p = \infty$, the operator $\widetilde{T}_v : \mathrm{Lip}(\mathbb{R}^d) \to L^p_w(\mu)$ is well defined and extends T_v. To prove that $\widetilde{T}_v$ is continuous, we consider the vector field $\widehat{v}$ on $\mathbb{R}^d$ defined by

$$\widehat{v}(x) := \begin{cases} \frac{v(x)}{|v(x)|} & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0. \end{cases} \tag{3.4}$$

Then $\widehat{v}$ is bounded and belongs to $V(\mu, x)$ for μ-a.e. x because so does v; hence $\widetilde{T}_{\widehat{v}} : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_w(\mu)$ is continuous by Theorem ??(i), case $p = \infty$.

Moreover the identity $v = |v| \widehat{v}$ implies that $\widetilde{T}_v u = |v|\, \widetilde{T}_{\widehat{v}}\, u$ for every u, and therefore, for every $\varphi \in L^q(\mu)$ where q is the Hölder conjugate of the exponent p, there holds

$$\int_{\mathbb{R}^d} \big(\widetilde{T}_v u\big)\, \varphi \, d\mu = \int_{\mathbb{R}^d} \big(\widetilde{T}_{\widehat{v}}\, u\big)\, |v| \varphi \, d\mu\,.$$

The continuity of $\widetilde{T}_{\widehat{v}}$ and the fact that $|v|\varphi$ belongs to $L^1(\mu)$ imply the continuity of the right-hand side (with respect to u), and therefore also of the left-hand side, which in turn implies the continuity of $\widetilde{T}_v$. □

Proof of Theorem ??(ii), case $p = \infty$. Given $y, y' \in \mathbb{R}^d$ and W subspace of $\mathbb{R}^d$, we write $\theta(y, y')$ for the angle between y and y', and $\theta(y, W)$ for the angle between y and W (both angles are set to be 0 if any of the vectors involved is null).

Let $\vartheta(x) := \theta(v(x), V(\mu(x))$ for every $x \in \mathbb{R}^d$: by assumption ϑ is strictly positive on a set of positive μ-measure, and therefore we can find $\bar{x}$ such that $\vartheta(\bar{x}) > 0$ and both ϑ and v are approximately continuous at $\bar{x}$. Then, having set $\beta := \frac{1}{3}\vartheta(\bar{x})$ and $w := v(\bar{x})$, for every α with $0 < \alpha < \beta$ the set

$$E_\alpha := \big\{ x \in \mathbb{R}^d \colon \vartheta(x) \geq 2\beta\,,\ \theta(v(x), w) \leq \alpha \big\}$$

has positive μ-measure. Moreover, for every $x \in E_\alpha$ there holds

$$\theta(w, V(\mu, x)) \geq \theta(v(x), V(\mu, x)) - \theta(v(x), w) \geq 2\beta - \alpha > \beta\,,$$

which means that the intersection of $V(\mu, x)$ and the cone $C := \{y \colon \theta(w, y) \leq \beta\}$ contains only 0. Therefore, by [?, Lemma 7.5], E_α contains a compact set F_α with positive μ-measure which is C-null in the sense of [**?**, Paragraph 4.11], and then, by [**?**, Lemma 4.12], there exists a sequence of smooth functions $f_n : \mathbb{R}^d \to \mathbb{R}$ such that, for every $x \in \mathbb{R}^d$,

(a) $0 \leq f_n(x) \leq \frac{1}{n}$;
(b) $0 \leq d_w f_n(x) \leq 1$, and $d_w f_n(x) = 1$ if $x \in F_\alpha$;
(c) $|d_W f_n(x)| \leq 1/\tan\beta$ where d_W is the restriction of the differential d to the subspace $W := w^\perp$ (and $|\cdot|$ is the operator norm).

Using statements (b) and (c) we obtain that:

(d) there exists a finite constant L such that $\mathrm{Lip}(f_n) \leq L$ for every n;
(e) for α small enough there exists a constant $\delta > 0$ such that $T_v f_n(x) \geq \delta$ for every $x \in F_\alpha$ and every n.

We finally consider the functions $u_n := u + L^{-1}\varepsilon f_n$. Statement (a) implies that $u_n \to u$ uniformly; (d) implies $\mathrm{Lip}(u_n) \leq \mathrm{Lip}(u) + \varepsilon$; (e) implies $T_v u_n \geq T_v u + \delta'$ on F_α, where $\delta' := L^{-1}\varepsilon\delta$, and therefore, possibly passing to a subsequence, we have that $T_v u_n \to w$ in $L^\infty_w(\mu)$ for some $w \neq T_v u$. □

To prove Theorem **??**(ii) for $p < \infty$ we need the following lemma.

3.1. Lemma. *Let $1 \leq p < \infty$ and $T : C^1_c(\mathbb{R}^d) \to L^p(\mu)$ be a linear operator. Let (u_n) be a sequence of functions in $C^1_c(\mathbb{R}^d)$ such that $u_n \to u$ uniformly and $Tu_n \to w$ in $L^p_w(\mu)$, that is, weakly. Then there exists a sequence $(\tilde{u}_n)$ of convex combinations of the elements of (u_n) such that* (a) *$\tilde{u}_n \to u$ uniformly, and* (b) *$T\tilde{u}_n \to w$ in $L^p(\mu)$, that is, strongly.*

Proof. Since $Tu_n \to w$ weakly in $L^p(\mu)$, by the version of Mazur's lemma stated in [**?**, Lemma 10.19], for every $n = 1, 2, \dots$ there exist an integer $N(n) \geq n$ and for every k with $n \leq k \leq N(n)$ there exist real numbers $\alpha^n_k \geq 0$ with sum equal to 1 such that the functions

$$w_n := \sum_{k=n}^{N(n)} \alpha^n_k \, Tu_k$$

converge to w strongly in $L^p(\mu)$. To conclude we set

$$\tilde{u}_n := \sum_{k=n}^{N(n)} \alpha^n_k \, u_k \, .$$

Indeed (b) follows from the identity $T\tilde{u}_n = w_n$ and the fact that $w_n \to w$ in $L^p(\mu)$. Moreover, since $\tilde{u}_n - u$ is a convex combination of the functions $u_k - u$ with $k \geq n$, by the convexity of the supremum norm $\|\cdot\|$ there holds

$$\|\tilde{u}_n - u\| \leq \delta_n := \sup_{k \geq n} \|u_k - u\| \, ,$$

and $\delta_n \to 0$ because $u_n \to u$ uniformly. Thus (a) is proved. □

Proof of Theorem ??(ii), case $p < \infty$. Let $\widehat{v}$ be the vector field defined in (**??**). Then $\widehat{v}$ satisfies the assumptions of Theorem **??**(ii), case $p = \infty$, and we take (u_n) to be the sequence given in that statement.

Using the identity $T_v u = |v| T_{\hat{v}} u$ and the fact that $T_{\hat{v}} u_n \to w$ in $L^\infty_w(\mu)$ with $w \neq T_{\hat{v}} u$, we obtain

$$T_v u_n = |v|\, T_{\hat{v}} u_n \to w' := |v| w \text{ in } L^p_w(\mu), \text{ and } w' \neq T_v u.$$

Finally we use Lemma **??** to construct from (u_n) a new sequence $(\tilde{u}_n)$ such that $T_v u_n \to w'$ strongly in $L^p(\mu)$. It is easy to check that the sequence $(\tilde{u}_n)$ satisfies all requirements. □

Proof of Corollary ??(i). It is well known that each of the operators T can be extended using a suitable distributional definition to a continuous operator from the Sobolev space $W^{1,\infty}(\mathbb{R}^d)$ to $L^\infty_w(\mathscr{L}^d)$; this means that T can be extended to a continuous operator $\widetilde{T} : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_w(\mathscr{L}^d)$.

Since $\mu \ll \mathscr{L}^d$, the space $L^\infty_w(\mathscr{L}^d)$ embeds continuously in $L^\infty_w(\mu)$ and therefore $\widetilde{T}$ is also a continuous operator from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^\infty_w(\mu)$, and this implies that T is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^\infty_w(\mu)$.

Moreover, $L^\infty_w(\mu)$ embeds continuously also in $(L^p_w(\mu))_{\mathrm{loc}}$,[5] thus $\widetilde{T}$ is also a continuous operator from $\mathrm{Lip}(\mathbb{R}^d)$ to $(L^p_w(\mu))_{\mathrm{loc}}$, and this implies that T is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $(L^p_w(\mu))_{\mathrm{loc}}$.

Finally both $L^p_w(\mu)$ and $L^p(\mu)$ embed continuously in $(L^p_w(\mu))_{\mathrm{loc}}$, and this implies that T is also closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$ and $L^p(\mu)$. □

Proof of Corollary ??(ii). By the assumption on μ, Theorem **??** implies that $V(\mu, x) \neq \mathbb{R}^d$ on a set of positive μ-measure. Then there exists an element e_k of the canonical base of $\mathbb{R}^d$ such that $e_k \notin V(\mu, x)$ on a set of positive μ-measure.

Thus the constant vector field e_k satisfies the assumption of Theorem **??**(ii), and therefore there exist a sequence of functions $u_n \in C^1_c(\mathbb{R}^d)$ such that $u_n \to 0$ in $\mathrm{Lip}(\mathbb{R}^d)$ and the k-th partial derivatives $\partial_k u_n$ converge to some $w \neq 0$ in $L^p(\mu)$ if $p < \infty$, and in $L^\infty_w(\mu)$ if $p = \infty$.

To conclude we notice that the features of the sequence (u_n) imply the non-closability of the gradient from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$, and even to $L^p(\mu)$ if $p < \infty$; the sequence of vector fields $(u_n\, e_k)$ implies the non-closability of the divergence, and finally the sequence of maps (U_n) given by $U_n(x) := x + u_n(x)\, e_k$ implies the non-closability of the Jacobian determinant. □

Proof of Corollary ??. This statement follows from Corollary **??**(ii) arguing as in Remark **??**(ii). □

3.2. Remark. At this point it should be clear that Theorem **??** and Corollaries **??** and **??** can be extended to more general differential operators. More precisely, let T be an operator from $C^1_c(\mathbb{R}^d; \mathbb{R}^m)$ in $L^p(\mu; \mathbb{R}^n)$ of the form

$$(Tu)_k := \sum_{j=1}^m T_{v_{jk}} u_j \tag{3.5}$$

where v_{jk} are vector fields in $L^p(\mu)$ for every $1 \leq j \leq m$ and $1 \leq k \leq n$, and $T_{v_{jk}}$ is the directional derivative defined in (**??**). (Note that for $p = \infty$ this class includes all linear first order differential operator with constant coefficients.)

[5] And even in $L^p_w(\mu)$ if μ is a finite measure.

Then one can prove the following result: if $v_{jk}(x) \in V(\mu, x)$ for μ-a.e. x and every j, k, then T can be extended to a continuous operator $\widetilde{T} : \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^m) \to L^p(\mu; \mathbb{R}^n)$, namely the one obtained by replacing each $T_{v_{ij}}$ in formula (**??**) by the extension $\widetilde{T}_{v_{ij}}$ given in Theorem **??**(i). In particular T is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$.

Conversely, if $v_{jk}(x) \notin V(\mu, x)$ for a set of positive μ-measure of points x and for at least one couple of indices j, k, then T is not closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$, nor from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$ with $p < \infty$, nor from $L^q(\mu)$ to $L^p(\mu)$.

4. Closability of directional derivatives from L^q to L^p

4.1. Theorem. *Let $1 \leq p, q \leq \infty$ and let p', q' denote the corresponding Hölder conjugates. Let v be a vector field in $L^p(\mu)$, and let $T_v : C^1_c(\mathbb{R}^d) \to L^p(\mu)$ be the directional derivative operator defined in* (**??**). *Assume that there exists a Borel function α on $\mathbb{R}^d$ such that*

- *$\alpha \neq 0$ μ-a.e.;*
- *$\alpha \in L^{p'}(\mu)$ and $\alpha v \in L^{q'}(\mu)$;*
- *$N := \alpha v \mu$ is a normal 1-current.*

Then T_v is closable from $L^q(\mu)$ to $L^p(\mu)$.

4.2. Remark. Unlike Theorem **??**(i), the closability result above is not accompanied by any differentiability result for the operator $\widetilde{T}_v$ obtained by closing the graph of T_v from L^q to L^p.

For example, let μ be the restriction of the Lebesgue measure to any bounded open set Ω in $\mathbb{R}^2$ and let $v(x) := (1, 2x_1)$ and $\alpha(x) := 1$ for every $x \in \mathbb{R}^2$. Then the assumptions in Theorem **??** are satisfied for every $1 \leq p, q \leq \infty$, and $\widetilde{T}_v u = 0$ for every function $u : \mathbb{R}^2 \to \mathbb{R}$ of the form $u(x_1, x_2) := g(x_2 - x_1^2)$ with $g : \mathbb{R} \to \mathbb{R}$ bounded and Borel; moreover there exist plenty of u of this form which are not differentiable at any point, in any direction.

The rest of this section is devoted to the proof Theorem **??**. Unspecified measures (as the expression ds that appears in some integrals) are always the Lebesgue measure $\mathscr{L}^1$. The proof relies on the next two lemmas.

4.3. Lemma. *Let $N = \hat{\tau}\lambda$ be a normal 1-current with $|\hat{\tau}| = 1$ λ-a.e. Then N can be decomposed as follows:*

$$N = \int_0^m [\gamma_s] \, ds \tag{4.1}$$

where m is a suitable positive number and

(a) *for every $s \in [0, m]$, $\gamma_s : [0, L_s] \to \mathbb{R}^d$ is a Lipschitz path parametrized by arc-length, that is, L_s is the length of γ_s and $|\gamma'_s(t)| = 1$ for a.e. $t \in [0, L_s]$;*

(b) *$|N| = \lambda = \int_0^m \lambda_s \, ds$ with $\lambda_s := \big|[\gamma_s]\big|$;*

(c) *$\gamma'_s(t) = \hat{\tau}(\gamma_s(t))$ for a.e. $s \in [0, m]$ and a.e. $t \in [0, L_s]$;*

(d) *λ_s is the push-forward according to γ_s of the Lebesgue measure on $[0, L_s]$.*

Proof. This decomposition is a variant of a well-know result by S. Smirnov [**?**]. The result as stated can be found in [**?**, Theorem 3.1], except for statement (c), which can be proved as in [**?**, Theorem 5.5(ii)], and statement (d), which is a direct consequence of (c) and the area formula. We refer to [**?**] or [**?**] for more details.[6] □

4.4. Lemma. *Let λ be a finite positive measure on $\mathbb{R}^d$ that can be decomposed as $\lambda = \int_0^m \lambda_s \, ds$, and let (v_n) be a sequence of functions such that $v_n \to v$ in $L^1(\lambda)$. Then there exists a subsequence (n_k) such that $v_{n_k} \to v$ in $L^1(\lambda_s)$ for a.e. $s \in [0,m]$.*

Proof. Let $g_n(s) := \int |v_n - v| \, d\lambda_s$. Since $\int_0^m g_n(s) \, ds = \|v_n - v\|_{L^1(\lambda)}$, we have that $g_n \to 0$ in $L^1([0,m])$, and therefore there exists a subsequence (n_k) such that $g_{n_k}(s) \to 0$ for a.e. $s \in [0,m]$. □

Proof of Theorem ??. Since T_v is linear, it suffices to prove that T_v is "closable at 0", namely that given a sequence $(u_n) \subset C_c^1(\mathbb{R}^d)$ and $w \in L^p(\mu)$ such that $u_n \to 0$ in $L^q(\mu)$ and $T_v u_n \to w$ in $L^p(\mu)$, then $w = 0$.

Having set $E := \{x \colon v(x) = 0\}$, there holds $T_v u_n(x) = 0$ for every $x \in E$ and every n, and therefore $w(x) = 0$ for μ-a.e. E. This means that it suffices to prove the statement above when μ is replaced by its restriction to $\mathbb{R}^d \setminus E$.

In other words, we can freely assume that $v \neq 0$ μ-a.e.

We set $\tau := \alpha v$, and then $\tau \neq 0$ μ-a.e. by the previous assumption.

We also set $\hat{\tau} := \tau/|\tau|$; then $|N| = |\tau|\mu$ and $N = \hat{\tau}|N|$.

Since $u_n \to 0$ in $L^q(\mu)$ and $|\tau| \in L^{q'}(\mu)$, then $u_n|\tau| \to 0$ in $L^1(\mu)$ or, equivalently, $u_n \to 0$ in $L^1(|\tau|\mu = |N|)$. Using the decomposition $|N| = \int_0^m \lambda_s \, ds$ in Lemma **??**(b) and Lemma **??** we have that, possibly passing to a subsequence in n, $u_n \to 0$ in $L^1(\lambda_s)$ for a.e. $s \in [0,m]$, and then, thanks to Lemma **??**(d),

$$u_n \circ \gamma_s \to 0 \quad \text{in } L^1([0, L_s]) \text{ for a.e. } s \in [0,m],$$

which in turn implies

$$(u_n \circ \gamma_s)' \to 0 \quad \text{in } \mathscr{D}'(0, L_s) \text{ for a.e. } s \in [0,m], \tag{4.2}$$

where $\mathscr{D}'(0, L_s)$ denotes the space of distributions on the interval $(0, L_s)$.

On the other hand, $T_v u_n = \nabla u_n \cdot v$ converges to w in $L^p(\mu)$ by assumption, and since $\alpha \in L^{p'}(\mu)$, then $\nabla u_n \cdot (\alpha v) \to \alpha w$ in $L^1(\mu)$. Recalling that $\alpha v = \tau = \hat{\tau}|\tau|$ we rewrite the last convergence as $(\nabla u_n \cdot \hat{\tau})|\tau| \to \alpha w$ in $L^1(\mu)$ or, equivalently, $\nabla u_n \cdot \hat{\tau} \to \alpha w/|\tau|$ in $L^1(|\tau|\mu = |N|)$, and arguing as above we obtain that, possibly passing to a subsequence in n,

$$(\nabla u_n \cdot \hat{\tau}) \circ \gamma_s \to (\alpha w/|\tau|) \circ \gamma_s \quad \text{in } L^1([0, L_s]) \text{ for a.e. } s \in [0,m]. \tag{4.3}$$

Thanks to Lemma **??**(c) we obtain that, for every n and a.e. $s \in [0,m]$,

$$(\nabla u_n \cdot \hat{\tau}) \circ \gamma_s = (\nabla u_n \circ \gamma_s) \cdot \gamma_s' = (u_n \circ \gamma_s)' \quad \text{a.e. in } [0, L_s],$$

and then (**??**) becomes

$$(u_n \circ \gamma_s)' \to (\alpha w/|\tau|) \circ \gamma_s \quad \text{in } L^1([0, L_s]) \text{ for a.e. } s \in [0,m]\,. \tag{4.4}$$

From (**??**) and (**??**) we infer that

$$(\alpha w/|\tau|) \circ \gamma_s = 0 \quad \text{a.e. in } [0, L_s] \text{ for a.e. } s \in [0,m].$$

[6] For example, the precise meaning of the integrals of in formula (**??**) and statement (b), and the correct measurability assumption on the map $s \mapsto [\gamma_s]$.

By Lemma **??**(d) this means that $\alpha w/|\tau| = 0$ λ_s-a.e. and for a.e. $s \in [0,m]$, which, by Lemma **??**(b), implies

$$\alpha w/|\tau| = 0 \quad |N|\text{-a.e., that is, } (|\tau|\mu)\text{-a.e.}$$

Finally we use that $\alpha \neq 0$ and $\tau \neq 0$ μ-a.e. to conclude that $w = 0$ μ-a.e. □

5. Closability of multilinear operators and metric currents

In Theorem **??** we extend Theorem **??** to the class of alternating k-linear differential operators defined in Paragraph **??**.[7] We then study some closely related objects, namely metric k-currents in $\mathbb{R}^d$. The key point is that metric k-currents can be naturally viewed as alternating k-linear differential operators (Remark **??**(iii)).

Before stating Theorem **??** we recall the definition of k-tangent bundle of a measure and then define the class of k-linear operators we are interested in.

5.1. k-tangent bundle. (See [**?**, Section 4] for more details.) Given a measure μ on $\mathbb{R}^d$ and an integer k with $1 \le k \le d$, the k-tangent bundle $V_k(\mu,\cdot)$ is a Borel map $V_k(\mu,\cdot)$ on $\mathbb{R}^d$ whose values are vector subspaces of the space $\wedge_k(\mathbb{R}^d)$ of k-vectors in $\mathbb{R}^d$ defined as follows: a k-vector v belongs to $V_k(\mu,x)$ if and only if there exists a normal k-current N in $\mathbb{R}^d$ with $\partial N = 0$ such that (**??**) holds.[8]

Given a k-vector v in $\mathbb{R}^d$, we denote by $\mathrm{span}(v)$ its supporting plane (or span), that is, the smallest subspace W of $\mathbb{R}^d$ such that v agrees with a k-vector on W (cf. [**?**], Paragraph 5.8 and Proposition 5.9).

Note that if v belongs to $V_k(\mu,x)$, then $\mathrm{span}(v)$ is contained in $V(\mu,x)$ ([**?**, Proposition 5.6]). We do not know if the converse holds – namely if every k-vector v in $\mathbb{R}^d$ such that $\mathrm{span}(v) \subset V(\mu,x)$ belongs to $V_k(\mu,x)$ – except for the trivial cases $k=1$ and $k=d$ (see [**?**] for a more detailed discussion).

5.2. Alternating k-linear differential operators. Let τ be a k-vector field on $\mathbb{R}^d$ (that is, a map from $\mathbb{R}^d$ to $\wedge_k(\mathbb{R}^d)$) which is bounded and Borel. We denote by J_τ the alternating k-linear differential operator defined by

$$J_\tau u := \langle \tau\,;\, du_1 \wedge \cdots \wedge du_k \rangle \tag{5.1}$$

for every $u = (u_1,\dots,u_k) \in (C^1_c(\mathbb{R}^d))^k$.[9]

In the next theorem we show that the closability and continuity properties of J_τ are connected to the following two assumptions on τ:

$$\tau(x) \in V_k(\mu,x) \quad \text{for } \mu\text{-a.e. } x, \tag{5.2}$$

which in turn implies (cf. Paragraph **??**)

$$\mathrm{span}(\tau(x)) \subset V(\mu,x) \quad \text{for } \mu\text{-a.e. } x. \tag{5.3}$$

5.3. Theorem. *Take τ and J_τ be as above.*

[7] For $k = n$ this class includes the Jacobian determinant; for that particular operator Theorem **??** reduces to Corollary **??**.

[8] In other words, we have replaced the vector v and the normal 1-current N in the definition of $V(\mu,x)$ by a k-vector and a normal k-current, respectively. In particular $V_1(\mu,x) = V(\mu,x)$.

[9] Here dv denotes the differential of the function v, intended as a 1-form, and $\langle\cdot\,;\,\cdot\rangle$ is the duality pairing of k-vectors and k-covectors.

(i) *If (**??**) holds, then every $u \in (\mathrm{Lip}(\mathbb{R}^d))^k$ is differentiable at μ-a.e. $x \in \mathbb{R}^d$ with respect to $W(x) := \mathrm{span}(\tau(x))$, and the operator $\widetilde{J}_\tau : (\mathrm{Lip}(\mathbb{R}^d))^k \to L^\infty_w(\mu)$ given by*

$$\widetilde{J}_\tau u(x) := \big\langle \tau(x) \,;\, d_W u_1(x) \wedge \cdots \wedge d_W u_k(x) \big\rangle \quad \text{for } \mu\text{-a.e. } x, \tag{5.4}$$

is well defined and extends J_τ. Moreover $\widetilde{J}_\tau(u_1, \dots, u_k)$ is separately continuous in each variable u_i.

(ii) *Conversely, if $J_\tau(u_1, \dots, u_k)$ is separately continuous in each variable u_i as a map from $C^1_c(\mathbb{R}^d)$ endowed with* Lip*-convergence to $L^\infty_w(\mu)$, then (**??**) holds. Accordingly, if (**??**) does not hold, then J_τ is not closable from $(\mathrm{Lip}(\mathbb{R}^d))^k$ to $L^\infty_w(\mu)$.*

(iii) *If (**??**) holds, then $\widetilde{J}_\tau$ is continuous; it follows that J_τ is closable from $(\mathrm{Lip}(\mathbb{R}^d))^k$ to $L^\infty_w(\mu)$.*

5.4. Remark. As pointed out in Paragraph **??**, we do not know if the sufficient condition for closability (**??**) agrees with the necessary condition (**??**). Moreover we do not know if the necessary condition (**??**) is also sufficient.

Proof of Theorem ??(i). Assumption (**??**) implies the existence of the differential $d_W u(x)$ for every $u \in \mathrm{Lip}(\mathbb{R}^d)$ and μ-a.e. x, and therefore $\widetilde{J}_\tau u$ is well defined.

To prove the separate continuity of $\widetilde{J}_\tau$, we first rewrite it in terms of the directional derivative operators defined in (**??**). Given $\overline{u}_i \in \mathrm{Lip}(\mathbb{R}^d)$ for $i = 1, \dots, k-1$, let v be the (1-) vector field on $\mathbb{R}^d$ given by

$$v(x) := \tau(x) \llcorner \big(d_W \overline{u}_1 \wedge \cdots \wedge d_W \overline{u}_{k-1} \big) \tag{5.5}$$

where $\llcorner$ is the interior product of k-vectors and $(k-1)$-covectors (in this specific case, vectors and covectors in the linear space $W(x)$), and define $\widetilde{T}_v$ according to (**??**). Then, for every $u \in \mathrm{Lip}(\mathbb{R}^d)$,

$$\begin{aligned}\widetilde{J}_\tau(\overline{u}_1, \dots, \overline{u}_{k-1}, u) &= \big\langle \tau \,;\, d_W \overline{u}_1 \wedge \cdots \wedge d_W \overline{u}_{k-1} \wedge d_W u \big\rangle \\ &= \big\langle \tau \llcorner (d_W \overline{u}_1 \wedge \cdots \wedge d_W \overline{u}_{k-1}) \,;\, d_W u \big\rangle = \langle v \,;\, d_W u \rangle = T_v u\,.\end{aligned}$$

Thus $\widetilde{J}_\tau(\overline{u}_1, \dots, \overline{u}_{k-1}, u)$ is continuous in u by Theorem **??**(i). □

Proof of Theorem ??(ii). We exploit again the connection between the operator J_τ and the directional derivative operators defined in (**??**). Given 1-covectors α_i with $i = 1, \dots, k-1$, let v be the (1-) vector field on $\mathbb{R}^d$ given by

$$v(x) := \tau(x) \llcorner (\alpha_1 \wedge \cdots \wedge \alpha_{k-1})\,, \tag{5.6}$$

and for every i let $\overline{u}_i$ be the linear function on $\mathbb{R}^d$ such that $d\overline{u}_i = \alpha_i$. Then for every $u \in C^1_c(\mathbb{R}^d)$ there holds

$$\begin{aligned}J_\tau(\overline{u}_1, \dots, \overline{u}_{k-1}, u) &= \big\langle \tau \,;\, \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge du \big\rangle \\ &= \big\langle \tau \llcorner (\alpha_1 \wedge \cdots \wedge \alpha_{k-1}) \,;\, du \big\rangle = \langle v \,;\, du \rangle = T_v u\,.\end{aligned}$$

Since $J_\tau(\overline{u}_1, \dots, \overline{u}_{k-1}, u)$ is continuous in u, then T_v is also continuous, and then Theorem **??**(ii) implies $v(x) \in V(\mu, x)$ for μ-a.e. x. Recalling (**??**) and using the fact that every $(k-1)$-covector α is a linear combination of simple covectors of the form $\alpha_1 \wedge \cdots \wedge \alpha_{k-1}$ we obtain that

$$\tau(x) \llcorner \alpha \in V(\mu, x) \quad \text{for every } \alpha \in \wedge^{k-1}(\mathbb{R}^d) \text{ and } \mu\text{-a.e. } x,$$

and now (**??**) follows from the fact that $\mathrm{span}(\tau(x))$ consists of all vectors of the form $\tau(x) \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(\mathbb{R}^d)$ (see [**?**, Proposition 5.9]).
Finally, the non-closability of J_τ follows from the lack of continuity and the weak* pre-compactness of bounded subsets of $L^\infty(\mu)$. □

Proof of Theorem ??(iii). Thanks to Theorems 1.1 and 1.2 in [**?**], assumption (**??**) implies that there exists a normal k-current $N = \tilde{\tau}\tilde{\mu}$ in $\mathbb{R}^d$ such that $\partial N = 0$, and $\tilde{\tau}$ and $\tilde{\mu}$ extend τ and μ in the sense specified in the proof of Theorem **??**(i), case $p = \infty$; proof that we follow almost verbatim to obtain the continuity of $\widetilde{J}_\tau$. □

We recall now the definition of metric currents; see [**?**], [**?**] for more details.

5.5. Metric currents. Let (X, d) be a complete metric space, and let $\mathrm{Lip}_b(X, \mathbb{R})$ be the space of bounded Lipschitz functions on X. Given an $k \in \mathbb{N}$, a k-dimensional *metric current* on X is a functional $T : \mathrm{Lip}_b(X) \times (\mathrm{Lip}(X))^k \to \mathbb{R}$ that satisfies the following assumptions:

(i) *linearity*: $T(f, \pi_1, \ldots, \pi_k)$ is linear in each variable;

(ii) *continuity*: for every f, $T(f, \pi_1, \ldots, \pi_k)$ is sequentially continuous in the variables $\pi_1, \ldots, \pi_k$ with respect to pointwise convergence with uniformly bounded Lipschitz constants;

(iii) *locality*: $T(f, \pi_1, \ldots, \pi_k) = 0$ whenever there exists $i \in \{1, \ldots, k\}$ such that π_i is constant on a neighbourhood of $\mathrm{supp}(f)$;

(iv) *finite mass*: there exists a finite measure μ on X such that, for every f and every $\pi_i, \ldots, \pi_k$,

$$\big|T(f, \pi_1, \ldots, \pi_k)\big| \leq \mathrm{Lip}(\pi_1) \cdots \mathrm{Lip}(\pi_k) \cdot \|f\|_{L^1(\mu)} \,. \tag{5.7}$$

Finally, we define the support of T as the smallest closed set $C \subset X$ such that $T(f, \pi_1, \ldots, \pi_k) = 0$ whenever $C \cap \mathrm{supp}(f) = \varnothing$.

5.6. Remarks. (i) Note that a metric current T is *alternating* in the variables $\pi_1, \ldots, \pi_k$, that is, the value of $T(f, \pi_1, \ldots, \pi_k)$ changes sign if we swap π_i and π_j for any i, j (this follows from the chain-rule in [**?**, Theorem 3.5]).

(ii) If T has compact support, the continuity assumption (ii) is equivalent to say that $T(f, \pi_1, \ldots, \pi_k)$ is sequentially continuous in the variables $\pi_1, \ldots, \pi_k$ with respect to the usual Lip-convergence.

(iii) Estimate (**??**) implies that T can be extended by continuity to all $f \in L^1(\mu)$, and therefore T can be viewed as a (k-linear, alternating) operator that to every $(\pi_1, \ldots, \pi_k) \in (\mathrm{Lip}(X))^k$ associate an element of $(L^1(\mu))' = L^\infty(\mu)$. If T has compact support, this operator is continuous from $(\mathrm{Lip}(X))^k$ to $L^\infty_w(\mu)$.

In the rest of this section X is $\mathbb{R}^d$ endowed with the usual Euclidean distance. In this setting it is natural to compare metric currents and classical ones: the basic connection between these two notions is described in Paragraph **??**; in Theorem **??** we give a new and more detailed description of such connection.

5.7. From metric to classical currents. To every metric k-current T on $\mathbb{R}^d$ with compact support one can associate a classical k-current $\widetilde{T}$ defined as follows (cf. [**?**, Theorem 11.1]):

$$\langle \widetilde{T}\,;\,\omega\rangle := \sum_{\mathbf{i}\in I(d,k)} T(\omega_{\mathbf{i}}, x_{i_1}, \dots, x_{i_k}) \tag{5.8}$$

for every k-form ω of class $C^1_c(\mathbb{R}^d)$, written in coordinates as

$$\omega = \sum_{\mathbf{i}\in I(d,k)} \omega_{\mathbf{i}}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}\,,$$

where $I(d,k)$ is the set of multi-indices $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \le i_1 < \dots < i_k \le d$. One easily checks that for every $f, \pi_1, \dots, \pi_K \in C^1_c(\mathbb{R}^d)$ there holds

$$T(f, \pi_1, \dots, \pi_k) = \langle \widetilde{T}\,;\, f\, d\pi_1 \wedge \dots \wedge d\pi_k\rangle\,. \tag{5.9}$$

5.8. Theorem. *The following statements hold:*

(i) *Let T be a metric k-current on $\mathbb{R}^d$ with compact support. Then there exists a finite measure μ with compact support and a bounded k-vector field τ on $\mathbb{R}^d$ such that (**??**) holds, that is, $W(x) := \mathrm{span}(\tau(x))$ is contained in $V(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^d$, and*

$$T(f, \pi_1, \dots, \pi_k) = \int_{\mathbb{R}^d} f\, \langle \tau\,;\, d_W\pi_1 \wedge \dots \wedge d_W\pi_k\rangle\, d\mu \tag{5.10}$$

for all $f \in \mathrm{Lip}_b(\mathbb{R}^d)$ and all $\pi_1, \dots, \pi_k \in \mathrm{Lip}(\mathbb{R}^d)$.

(ii) *On the other hand, given a finite measure μ with compact support and a bounded k-vector field τ such that (**??**) holds, that is, $\tau(x) \in V_k(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^d$, then formula (**??**) defines a metric k-current T with compact support.*

5.9. Remarks. (i) The assumption that T and μ have compact support in statements (i) and (ii) above is only needed to express the continuity assumption in the definition of metric currents (see Paragraph **??**) in terms to the usual Lip-convergence (cf. Remark **??**(ii)), and it can be removed with some care.

(ii) A slight modification of the proof gives the following generalization of statement (i): a functional $T : \mathrm{Lip}_b(\mathbb{R}^n) \times (\mathrm{Lip}(\mathbb{R}^d))^k \to \mathbb{R}$ admits an integral representation as in (**??**) if and only if T satisfies all assumptions in the definition of metric currents except continuity, which is replaced by the following weaker assumption: $T(f, \pi_1, \dots, \pi_k)$ is separately continuous in each variable π_i with respect to Lip-convergence.

(iii) The assumption $\tau(x) \in V_k(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^d$ in statement (ii) is equivalent to say that $\tau\mu$ is a flat chain with finite mass (see Proposition **??**(ii) below). Thus formula (**??**) defines a map from the space of flat chains with finite mass into metric k-currents,[10] which is clearly a right inverse of the map $T \mapsto \widetilde{T}$ defined in (**??**). The Flat Chain Conjecture (see [**?**, Section 11], and Theorem **??**) states that this map is a bijection, that is, $\widetilde{T}$ is a flat chain with finite mass for every metric current T.

[10] This map has been already defined in [**?**, Theorem 5.5], but in a less explicit form.

Proof of Theorem ??(i). Let $\widetilde{T}$ be the (classical) current defined in (??). Estimate (??) yields

$$\left|\langle \widetilde{T}\,;\,\omega\rangle\right| \leq C\|\omega\|_{L^1(\mu)}$$

where $C := \#(I(k,d)) = \binom{d}{k}$ and μ is the measure in Paragraph ??(iv). Thus $\widetilde{T}$ has finite mass and can be written as $\widetilde{T} = \tau\mu$ where τ is a bounded k-vector field. Recalling (??) we obtain that for every $f, \pi_1, \dots, \pi_k \in C^1_c(\mathbb{R}^d)$ there holds

$$\begin{aligned} T(f,\pi_1,\dots,\pi_k) &= \langle \widetilde{T}\,;\, f\,d\pi_1\wedge\cdots\wedge d\pi_k\rangle \\ &= \int_{\mathbb{R}^d} \langle \tau\,;\, d\pi_1\wedge\cdots\wedge d\pi_k\rangle\, f\,d\mu = \int_{\mathbb{R}^d} J_\tau(\pi_1,\dots\pi_k)\, f\,d\mu\,, \end{aligned} \tag{5.11}$$

where $J_\tau : (C^1_c(\mathbb{R}^n))^k \to L^\infty(\mu)$ is defined in (??).

Since T is continuous in the variables $\pi_1, \dots, \pi_k$ with respect to Lip-convergence (cf. Paragraph ??(ii) and Remark ??(ii)), identity (??) implies that J_τ is a continuous operator from $(C^1_c(\mathbb{R}^n))^k$, endowed with Lip-convergence, to $L^\infty_w(\mu)$.

Then Theorem ??(ii) implies that $\mathrm{span}(\tau(x)) \subset V(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^d$, and therefore the integral at the right-hand side of (??) is well defined.

To conclude we notice that identity (??) holds for $f, \pi_1, \dots, \pi_k \in C^1_c(\mathbb{R}^d)$ by (??), and can be extended to $f \in \mathrm{Lip}_b(\mathbb{R}^d)$ and $\pi_1, \dots, \pi_k \in \mathrm{Lip}(\mathbb{R}^d)$ by continuity.

Indeed, the continuity of the left-hand side of (??) follows from the definition of metric currents, while the separate continuity of the right-hand side with respect to each of the variables $f, \pi_1, \dots, \pi_k$ follows from the separate continuity of the operator $\widetilde{J}_\tau$ proved in Theorem ??(i). □

Proof of Theorem ??(i). Formula (??) can be re-written as

$$T(f,\pi_1,\dots,\pi_k) = \int_{\mathbb{R}^d} \widetilde{J}_\tau(\pi_1,\dots,\pi_k)\, f\,d\mu \tag{5.12}$$

where $\widetilde{J}_\tau$ is defined as in (??). Then T satisfies the continuity assumption in the definition of metric currents because $\widetilde{J}_\tau$ is continuous from $(\mathrm{Lip}(\mathbb{R}^n))^k$ to $L^\infty_w(\mu)$ (Theorem ??(iii)). It is immediate to check that T satisfies all other assumptions in the definition of metric currents. □

We conclude this section by pointing out that the Flat Chain Conjecture is equivalent to the converse of Theorem ??(iii) and Theorem ??(ii). Before giving a precise statement we recall the definition of flat chains and some useful characterizations.

Flat chains. The space of k-dimensional flat chains in $\mathbb{R}^d$ is defined as the closure of k-normal currents with respect to the flat norm. The next statement gives some characterizations of flat chains *with finite mass.*

5.10. Proposition. *Let $T = \tau\mu$ be a k-current with finite mass in $\mathbb{R}^d$, where μ is a finite measure and τ a k-vector field such that $\tau \neq 0$ μ-a.e.*

(i) *If $k = d$, T is a flat chain if and only if $\mu \ll \mathscr{L}^d$.*
(ii) *If $k \leq d$, T is a flat chain if and only if (??) holds.*
(iii) *if $k < d$, T is a flat chain if and only if it can be written as the restriction of a normal k-current to a Borel set.*

Proof. The proof of statement (i) is immediate. Statement (ii) follows from [?, Theorem 1.2]. Statement (iii) follows from [?, Theorem 1.1]. □

5.11. Theorem. *Let $k = 1, \dots, d$. The following statements are equivalent:*

(i) (Flat Chain Conjecture) *Let T be a metric k-current. Then the current $\widetilde{T}$ defined in* (**??**) *is a flat chain.*

(ii) (Converse of Theorem **??**(ii)) *Let T be a metric k-current, and let μ and τ be as in Theorem **??**(i). Then* (**??**) *holds.*

(iii) (Converse of Theorem **??**(iii)) *Let μ be a finite measure, τ a bounded k-vector field, and J_τ the operator defined in* (**??**)*. If J_τ is closable from $(\mathrm{Lip}(\mathbb{R}^d))^k$ to $L^\infty_w(\mu)$ then* (**??**) *holds.*

Proof. The equivalence of statements (i) and (ii) is an immediate consequence of Theorem **??** and Proposition **??**(ii). The equivalence of statements (ii) and (iii) is an immediate consequence of the following facts: J_τ is closable if and only if $\widetilde{J}_\tau$ is well-defined and continuous (Theorem **??**), and the continuity of $\widetilde{J}_\tau$ is equivalent to that of T because of identity (**??**). □

The following corollary is well known (see [?, Theorem 1.6] and [?, Theorem 1.15], and the more recent proofs [?], [?]); we simply remark that it follows from our previous results as well.

5.12. Corollary. *The Flat Chain Conjecture is true for $k = 1$ and $k = d$.*

Proof. For $k = 1$ and $k = d$ properties (**??**) and (**??**) are equivalent (the case $k = 1$ is trivial, while the case $k = d$ follows from Theorem **??**). This means that Theorem **??**(i) implies statement (ii) in Theorem **??**, which implies the Flat Chain Conjecture by the very same theorem. □

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