NON-ABSOLUTELY CONVERGENT INTEGRALS IN METRIC SPACES

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Contents

1. Introduction

In these lecture notes we recall some definitions of non-absolutely convergent integrals and present a new type of integral which works well in the setting of metric spaces.

The theory of Lebesgue integral is satisfactory for a majority of applications, but not for all. If our space admits more structure than just a measure, it is sometimes desirable consider a concept of integral which respects the geometry of the space. The drawback of the Lebesgue integral is easily seen on the example $f(x) = \big(x^2\sin(1/x^2)\big)'$, $f(0)=0$. Although the function f is a derivative, it is not integrable in the Lebesgue sense.

The problem to find a concept of integral which contains the Lebesgue integral and integrates all derivatives arose naturally with the discovery of the Lebesgue integral. It leads to a variety of definitions of the so-called non-absolutely convergent integrals. These integrals differ in the approach to the definition and sometimes (but not always) in the classes of integrable functions.

For further survey and comments we refer to sections. Our plan is first to show a simple, but new, one-dimensional definition of a nonabsolutely convergent integral (Section 2). Then we discuss the problem of multidimensional setting (Sections 3, Section 4) and explain our definition in the metric-valued setting (Section 6). We apply the new integral to a new form of the Gauss-Green-Stokes theorem (Subsection 8.1).

1.1. Notation and conventions.

If X is a metric space, $x \in X$ and $r > 0$, then $B(x,r)$ denotes the open ball with center at x and radius r and $\overline{B}(x,r)$ is the corresponding closed ball.

If $\mathcal{E}(X)$ is a space of scalar functions and Y is another space, we let $\mathcal{E}(X,Y)$ denote the corresponding space of Y-valued functions.

We use the notation Y^* for the dual space of a normed space Y .

If Y and Z are normed linear spaces, $L(Y,Z)$ stands for the space of all continuous linear operators mapping Y to Z. The action of an operator $\ell \in L(Y,Z)$ on $y \in Y$ is denoted simply as ℓy. Sometimes we use $a \in \mathbb{R}$ where $a \in L(Y,Y)$ is expected, then we operate with a as with the operator $y \mapsto ay$.

The symbol $\langle \mathscr{T}, \varphi\rangle$ is used for the action of a functional (possibly vector-valued, in fact operator) $\mathscr{T}$ on a function φ.

All measures are ≥ 0 unless labeled as "signed".

If f is a function on X, then $\operatorname{spt} f$ is the smallest closed set F such that $f = 0$ on $X \setminus F$. If μ is a Borel measure on X, then $\operatorname{spt}\mu$ is the smallest closed set F such that $\mu(X \setminus F) = 0$.

If μ is a measure on X, E is a μ-measurable set with $0 < \mu(E) < \infty$ and f is a μ-integrable function on E, we write

$$⨍_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu.$$

We use symbol C for a generic constant which may change at each step of the computation.

2. One-dimensional integral

2.1. Denjoy-Perron integral.

Recall that we seek for a concept of integral which contains the Lebesgue integral and integrates all derivatives. First such a definition is due to Denjoy [16] in 1912, shortly followed by Luzin [39] and Perron [45]. We present below Luzin's version on the Denjoy integral.

Luzin's definition is *descriptive*, we describe which properties a function should have to deserve the name "indefinite integral".

Throughout this section $I = [a, b]$ will be a bounded closed interval.

Classically, a function $F : I \to \mathbb{R}$ is said to be an *antiderivative* (or *indefinite integral*) of a function $f : I \to \mathbb{R}$ if $F'(x) = f(x)$ for each $x \in I$. The *definite* integral of f over I is then the *increment* of F over I, namely $F(b) - F(a)$. This concept of integral goes back to Newton.

The requirement of everywhere existence of F' is very restrictive, for example, the function $F(x) = |x|$ fails to have derivative at 0, although we would like to consider $|x|$ as the indefinite integral of the function $f(x) = \operatorname{sgn} x$, where

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ 0, & x = 0. \end{cases}$$

A good idea is to neglect sets of measure zero, this means that the property $F'(x) = f(x)$ is required not everywhere, but only for points not belonging to an exceptional set.

However, well known examples (the Cantor ternary function) show that the uniqueness of the indefinite integral is then lost. To get it back, complicated additional assumptions on the function F like absolute continuity or its generalizations must be imposed. We give such a definition here:

Definition 2.1 (ACG$*$)**.** If $E \subset I$, we say that a function $F : I \to \mathbb{R}$ is AC$*$ on E is if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite system $\{[a_j, b_j]\}$ of non-overlapping intervals with endpoints in E we have

$$\sum_j (b_j - a_j) < \delta \implies \sum_j \operatorname{osc}_{[a_j,b_j]} F < \varepsilon.$$

We say that F is ACG$*$ on I if there exists a sequence $\{E_k\}$ of sets such that $I = \bigcup_k E_k$ and F is AC$*$ on each E_k.

Now, we are ready to formulate the definition of the restricted Denjoy integral.

Definition 2.2 (Denjoy-Luzin)**.** We say that a function $F : I \to \mathbb{R}$ is a restricted Denjoy indefinite integral of f if F is ACG$*$ and $F' = f$ almost everywhere. Then the definite integral of f is $F(b) - F(a)$.

Remark 2.3. Following tradition, we say that f is Denjoy-Perron integrable if f is integrable in the sense of definition 2.2, although this definition is not exactly the definition by Denjoy, nor the definition by Perron. However, all the mentioned definitions lead to the same class of integrable functions.

2.2. **Henstock-Kurzweil integral.** In late fifties, a "generalized Riemann integral" has been introduced independently by Kurzweil [35] and Henstock [27], see also [37], [28], [29]. This definition leads to the same family of Denjoy-Perron integrable functions. Its advantage is that it resembles the Riemann definition which is respected as the most transparent and comprehensible definition of integral. In contrast with descriptive definitions, definitions based on approximation and Riemann sums (or something similar) are called *constructive*.

Definition 2.4 (partitions)**.** A tagged partition is a selection of points of the form

$$D: \quad a = x_0 \le t_1 \le x_1 \le t_2 \le \cdots \le t_m = b.$$

The Riemann sum to a function f using the partition D is

$$S(f, D) = \sum_{i=1}^{m} f(t_i)(x_i - x_{i-1}).$$

Let $\delta : I \to (0, \infty)$ be a function (called a *gauge*). Then a tagged partition D is said to be δ*-fine* if $x_i - x_{i-1} < \delta(t_i)$ for each $i = 1, \dots, m$.

Definition 2.5 (Henstock-Kurzweil integral)**.** We say that a number K is the *Henstock-Kurzweil integral* of f over I if for each $\varepsilon > 0$ there exists a function $\delta : I \to (0, \infty)$ such that for each δ-fine partition D we have

$$|S(f, D) - K| < \varepsilon.$$

Notice that the only change in comparison with the Riemann integral is that δ is not a constant but a variable.

2.3. MC**-integral.** We return back to the descriptive idea. We want to avoid the ACG$_*$ condition. We present a definition given in [10]. This is completely elementary, it uses only the simple notion of increasing function. The abbreviation "MC" refers to "monotonically controlled".

Definition 2.6. Let $f, F : I \to \mathbb{R}$ be functions. We say that F is an *MC-antiderivative* (or an *indefinite MC-integral*) of f if there exists an increasing (=strictly increasing) function $\gamma \colon I \to \mathbb{R}$ (the so-called *control function* to the pair (F, f)) such that

$$\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\gamma(y) - \gamma(x)} = 0, \qquad x \in I. \tag{1}$$

Then we define the (definite) *integral* of f over I as

$$\int_a^b f(x)\, dx := F(b) - F(a).$$

Remark 2.7. If F is the ordinary indefinite integral of f, we may use just $\gamma(x) = x$ and recover the classical definition of indefinite integral (primitive function, antiderivative). But we are allowed to choose other control functions to integrate more functions f. This "small" variation of the definition is very powerful. The simplest example how to use it is the pair $F(x) = |x|$, $f(x) = \operatorname{sgn} x$. If we choose $\gamma(x) = x + \operatorname{sgn} x$, then γ controls the pair (F, f), so that F is an antiderivative of f in the new sense (but not in the classical sense). But much more is enabled. In fact, the MC integral covers the Lebesgue integral, whose usual definition is, in comparison with the definition above, quite complicated.

We present here some basic facts, for proofs we refer to [49], [38], [10].

Theorem 2.8. *All (definite) integrals defined in this section are uniquely determined by the integrand.*

Theorem 2.9. *Let $f : I \to \mathbb{R}$ be a function. Then the following properties are equivalent*

(i) *f is Denjoy integrable in the restricted sense,*

(ii) *f is Henstock-Kurzweil integrable,*
(iii) *f is MC-integrable.*

Theorem 2.10. *Let $f : I \to \mathbb{R}$ be a function.*
(a) *If f is Lebesgue integrable, then f is Denjoy-Perron integrable.*
(b) *If f is Denjoy-Perron integrable, then f is Lebesgue measurable.*
(c) *If f is Denjoy-Perron integrable and nonnegative, then f is Lebesgue integrable.*

3. Integration of functions of several variables

The task to develop the multidimensional theory of nonabsolutely convergent integral brings new significant difficulties. The one-dimensional integration can be understood as an inverse process to differentiation. In the multidimensional case, an attempt to develop parallel thoughts leads to differentiation of set functions. If we want to obtain a rich non-absolutely convergent integral, we need to start from a narrow class of sets. The class of intervals is the most natural choice. (One could also think about the class of all balls, but it has inconvenient partitioning properties.) Thus, we observe constructive definitions based on partitions into intervals or descriptive definitions based on differentiation of interval functions.

The Denjoy-Perron integral has been generalized to higher dimension by Bauer [9]. Both Kurzweil and Henstock considered also multidimensional or abstract versions of their integral, [28], [29], [37].

In this section we discuss several definitions of multidimensional nonabsolute convergent integrals. For various other definition of multidimensional integrals, discussion of problems and further bibliography we refer to [11], [15], [30], [32], [33], [38], [44], [47], [48].

3.1. **Multidimensional Henstock-Kurzweil integral.**

In this subsection, let $I \subset \mathbb{R}^n$ be a bounded closed interval (this means, a cartesian product of one-dimensional bounded closed intervals).

Definition 3.1 (partitions)**.** A partition of I is a system $D = (I_1, \dots, I_m)$ of non-overlapping closed subintervals of I such that $I = \bigcup_{i=1}^m I_i$. A tagged partition is a system $D = ((I_1, t_1), \dots, (I_m, t_m))$: a partition $(I_1, \dots, I_m)$ is equipped with a selection of points $t_i \in I_i$, $i = 1, \dots, m$. The Riemann sum to a function f using the tagged partition D is

$$S(f, D) = \sum_{i=1}^{m} f(t_i)\,|I_i|.$$

Let $\delta : I \to (0, \infty)$ be a function (called a *gauge*). Then a tagged partition $D = ((I_1, t_1), \dots, (I_m, t_m))$ is said to be *δ-fine* if $\operatorname{diam} I_i < \delta(t_i)$ for each $i = 1, \dots, m$.

Definition 3.2 (Henstock-Kurzweil integral)**.** We say that a number K is the *Henstock-Kurzweil integral* of f over I if for each $\varepsilon > 0$ there exists a function $\delta : I \to (0, \infty)$ such that for each δ-fine partition D we have

$$|S(f, D) - K| < \varepsilon.$$

Remark 3.3. For the multidimensional Henstock-Kurzweil integral, the Fubini theorem is available, [36], [37]. However, in contrast with the one-dimensional case, this integral does not integrate all derivatives.

Mawhin [41] restricted the class of admissible partitions to obtain an integral which integrates all derivatives. His partitions are *regular*, this means that we control the shape of intervals in the partition (the proportion of the longest edge to the shortest edge should be watched). Such regular integrals integrate all derivatives, however, the Fubini theorem is lost. This is not a symptom of cumbersomeness of the definition but a general feature. The example in [32] shows that the validity of Fubini's theorem is not compatible with the property of integrating all derivatives. It demonstrates the following

Theorem 3.4. *There exists a differentiable function u on $\mathbb{R}^2$ and a set $E \subset \mathbb{R}$ of positive Lebesgue measure such that for each $x \in E$ we have the following behavior:*

$$\frac{\partial u}{\partial x}(x,y) \geq 0, \qquad y \in [0,1],$$

and

$$\int_0^1 \frac{\partial u}{\partial x}(x,y)\,dy = \infty.$$

(this is the Lebesgue integral).

3.2. **Pfeffer integral.** A disadvantage of the theory of integral based on interval functions is that such integral depends on the choice of the coordinate system and we do not obtain reasonable results on change of variables. In more sophisticated theories, the indefinite integral is a function defined on a more general family of sets. The natural choice for such a system of sets is the family of all BV-sets. This direction has been developed by Pfeffer, see e.g. [46], [48]. We present here the descriptive definition, following [48].

The importance of the BV class is that it is one of few natural function spaces which distinguishes between "regular" and "irregular" sets: it contains some nontrivial characteristic functions of bounded measurable sets but not all of them.

Definition 3.5 (BV-set)**.** We say that $E \subset \mathbb{R}^n$ is a BV-set if E is Lebesgue measurable, $0 < |E| < \infty$, and the distributional derivative of χ_E is a vector Radon measure. We denote by $\|E\|$ the total variation of the measure of $D\chi_E$. The family of all BV sets is denoted by $\mathcal{BV}$.

Definition 3.6 (Charge)**.** Charge is an additive function $\mathcal{F}$ on $\mathcal{BV}$ with the following continuity property:

If $E_j \in \mathcal{BV}$, $\sup_j \|E_j\| < \infty$, $|E_j| \to 0$ and $\operatorname{diam} \bigcup_j E_j < \infty$, then $\mathcal{F}(E_j) \to 0$.

Definition 3.7 (Pfeffer gauge)**.** A function $\delta : \mathbb{R}^n \to [0,\infty)$ is called a *Pfeffer gauge* if the set $\{x : \delta(x) = 0\}$ is of σ-finite $(n-1)$-dimensional Hausdorff measure. The collection of all Pfeffer gauges on $\mathbb{R}^n$ is denoted by $\mathscr{P}$.

Definition 3.8 (δ-fine partition)**.** The regularity of a BV set E is the number

$$\operatorname{reg}(E) = \frac{|E|}{\|E\| \operatorname{diam} E},$$

with the convention that $\operatorname{reg}(E) = 0$ if $|E| = 0$. We also denote

$$\operatorname{reg}_x E = \operatorname{reg}(E \cup \{x\}).$$

Let $\delta \in \mathscr{P}$ and $\eta > 0$. An η-regular δ-fine partition on $E \subset \mathbb{R}^n$ is a system $\{(E_1,t_1),\dots,(E_m,t_m)\}$ of disjoint bounded BV-sets accompanied with points, such that $t_i \in E$, $\operatorname{reg}_{t_i}(E_i) > \eta$ and

$$\operatorname{diam}(x_i \cup E_i) \le \delta(t_i), \qquad i = 1, \dots, m.$$

Definition 3.9 (Derivates)**.** If $\mathcal{F}$ is a charge, $x \in \mathbb{R}^n$, $\delta, \eta > 0$, we denote

$$\overline{D}\mathcal{F}(x) = \inf_{\eta>0} \sup_{\delta>0} \sup \Big\{ \frac{\mathcal{F}(E)}{|E|} : \ E \in \mathcal{BV}, \ E \subset B(x,\delta), \ \operatorname{reg}_x(E) > \eta \Big\},$$

and

$$\underline{D}\mathcal{F}(x) = -\overline{D}(-\mathcal{F})(x).$$

If $\underline{D}\mathcal{F}(x) = \overline{D}\mathcal{F}(x) \in \mathbb{R}$, we say that $\mathcal{F}$ is *derivable* at x and the derivate $D\mathcal{F}(x)$ is defined as the common value of $\underline{D}\mathcal{F}(x)$ and $\overline{D}\mathcal{F}(x)$.

Definition 3.10 (Variation)**.** If $\mathcal{F}$ is a charge, $\delta \in \mathscr{P}$ and $\eta > 0$, we denote

$$V_{\delta,\eta}\mathcal{F}(E) = \sup \Big\{ \sum_{i=1}^{m} \mathcal{F}(E_i) : \ \{(E_i, t_i)\} \text{ is a } \eta\text{-regular } \delta\text{-fine partition on } E \Big\}$$

and

$$V_*\mathcal{F}(E) = \sup_{\eta>0} \inf_{\delta \in \mathscr{P}} V_{\delta,\eta}\mathcal{F}(E),$$

The set function

$$V_*\mathcal{F} : \ E \mapsto V_*\mathcal{F}(E)$$

is a Borel measure, which may be infinite even on compact sets.

Definition 3.11 (AC_*-charge)**.** We say that a charge $\mathcal{F}$ is AC_* if $V_*\mathcal{F}$ is absolutely continuous (this means, $|E| = 0 \implies V_*\mathcal{F}(E) = 0$).

Definition 3.12 (Pfeffer integral)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that a AC_* charge $\mathcal{F}$ is an *indefinite Pfeffer integral* of f if $D\mathcal{F}(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$. If $E \in \mathcal{BV}$ and $\mathcal{F}$ is an indefinite Pfeffer integral of f, then the value $\mathcal{F}(E)$ is called the *definite Pfeffer integral* of f over E.

Remark 3.13. The definition of the Pfeffer integral looks fairly complicated, but for some versions of the Gauss-Green theorem it is probably optimal.

4. UC-integral

In our approach, we do not use set functions but functionals. An important step towards this setting was the PU-integral (referring to "partition of unity") introduced by Jarník and Kurzweil [30], [31]. Here the partition of unity to smooth functions replaces the partition into intervals. The PU-integral fits into the generality of manifolds and has been applied to establish a version of the Stokes theorem.

We will work fully in the functional setting. Our method is descriptive. One of advantage of the functional approach is that this falls within the well established theory of distributions. Further, function spaces provides much more possibilities for the choice of test objects than classes of sets. Indeed, characteristic functions of sets are irregular on principle, whereas we use smooth functions to test with.

Some authors think on distributions in connection with nonabsolutely convergent integrals in the sense that an indefinite integration can be applied not only to functions but to distributions, [42], [6], [51]. Our aim is different: the indefinite integral is a distribution, but our integrand is always a function. The idea to integrate with respect to a distribution appears in the work by Burkill [12], where the Stieltjes type integral $\int_0^1 f(d^n g/dx^{n-1})$ is considered.

Also in our case, the integration is an inverse process to differentiation: we differentiate distributions with respect to distributions similarly as the Radon-Nikodým differentiation (in the geometric setting by Lebesgue and Besicovitch) differentiates measures with respect to measures. Our approach, following [40] is new even in the case that the underlying distribution is just the Lebesgue measure.

In what follows, let $\Omega \subset \mathbb{R}^n$ be an open set.

Recall that the space $\mathcal{C}_c^\infty(\Omega)$ of "test functions" is defined as the class of all infinitely differentiable functions φ on Ω with a compact support. This space is often labeled as $\mathcal{D}(\Omega)$; however, we reserve the symbol $\mathcal{D}$ for a different space.

The action of a distribution $\mathcal{F}$ on a test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ is denoted by $\langle \mathcal{F}, \varphi\rangle$.

We denote

$$\mathcal{D}(x,r) = \{\varphi \in \mathcal{C}_c^\infty(\Omega,\mathbb{R}^n),\ \operatorname{spt}\varphi \subset \overline{B}(x,r),\ ;\operatorname{Lip}\varphi \le 1/r\}.$$

Let $\mathscr{T}$ be a distribution on Ω, and $B(x,r) \subset \Omega$ is a ball. We define

$$\|\mathscr{T}\|_{x,r} = \sup\Big\{\langle \mathscr{T}, \varphi\rangle :\ \varphi \in \mathcal{D}(x,r)\Big\}.$$

The definition of UC-integral in Ω is the following (UC refers to "uniformly controlled"; the control is uniform with respect to $\varphi \in \mathcal{D}(x,r)$):

Definition 4.1 (UC integral in Ω)**.** Let $\mathcal{F}$, $\mathscr{G}$ be distributions on Ω and $f\colon \Omega \to \mathbb{R}$ be a function. We say that $\mathcal{F}$ is an *indefinite UC-integral of f with respect to $\mathscr{G}$* if there exist $\sigma \ge 1$ and a Radon measure μ on Ω such that

$$\lim_{r\to 0_+} \frac{\|\mathcal{F} - f(x)\mathscr{G}\|_{x,r}}{\mu(B(x,\sigma r))} = 0$$

for each $x \in \Omega$. If the indefinite UC-integral exists, we say that f is *locally UC-integrable* with respect to $\mathscr{G}$.

Remark 4.2. Another natural idea is to consider a limit process for quotients

$$\frac{\langle \mathcal{F}, \varphi\rangle - f(x)\langle \mathscr{G}, \varphi\rangle}{\int_{\mathbb{R}} \varphi\, d\mu}.$$

This would follow more closely the idea of MC-integration. Our experience indicates that our approach based on Definition 4.1 is more efficient.

Theorem 4.3. *The UC integral is uniquely determined by $\mathscr{G}$ and f.*

Proof. See the general case in Section 6 below. □

Definition 4.4. Let ν be a Radon measure on Ω. Then the UC-integral of a function f with respect to ν is defined by means of the identification of ν with the distribution

$$\varphi \mapsto \int_\Omega \varphi\, d\nu.$$

Theorem 4.5. *Let ν be a Radon measure on Ω and $f : \Omega \to \mathbb{R}$ be a function.*

(a) *If f is locally ν-integrable, then f is locally UC-integrable with respect to ν and*

$$\varphi \mapsto \int_\Omega f\, \varphi\, d\nu$$

is the corresponding indefinite UC-integral.

(b) *If f is locally UC-integrable with respect to ν, then f is ν-measurable.*

(c) *If $f \geq 0$ is locally UC-integrable with respect to ν, then f is locally ν-integrable.*

Proof. See the general case in Section 6 below. □

4.1. **Weakly differentiable functions.** In this subsection, we define a class of "weakly differentiable functions" which is a simultaneous generalization of the classes of Sobolev functions (namely, function whose distributional derivatives are locally Lebesgue integrable) and pointwise differentiable functions.

Definition 4.6. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $\boldsymbol{g} : \Omega \to \mathbb{R}^n$ is said to be a UC-derivative of a locally Lebesgue integrable function $u : \Omega \to \mathbb{R}$ if there exist a constant $\sigma \geq 1$ and a Radon measure μ on Ω such that

$$\lim_{r \to 0+} \sup_{\varphi \in \mathcal{D}(x,r)} \frac{\left| \int_\Omega \Big(u(y) - u(x) - \boldsymbol{g}(x) \cdot (y - x) \Big) \operatorname{div} \varphi(y)\, dy \right|}{\mu(B(x, \sigma r))} = 0, \quad x \in \Omega. \tag{2}$$

Proposition 4.7. *Let $u : \Omega \to \mathbb{R}$ be a locally Lebesgue integrable function.*

(a) *A function $\boldsymbol{g}$ is a UC-derivative of u if and only if the distributional derivative of u becomes the indefinite UC-integral of $\boldsymbol{g}$ with respect to the Lebesgue measure.*
(b) *If $\boldsymbol{g}$, $\boldsymbol{g}^*$ are UC-derivatives of u, then $\boldsymbol{g} = \boldsymbol{g}^*$ a.e. in Ω.*
(c) *If 0 is a UC-derivative of u and Ω is connected, then u is a.e. in Ω equal to a constant.*
(d) *If u is pointwise differentiable, then ∇u is a UC-derivative of u.*
(e) *If u belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega)$, then ∇u is a UC-derivative of u.*

Proof. See [40]. □

5. Function spaces

5.1. **Lipschitz spaces.** If we want to define a version of the UC-integral on metric spaces, we must keep in mind that Lipschitz smoothness is the ultimate smoothness in the generic metric space setting. Therefore, "distributions" will be functionals not on infinitely smooth functions but just on Lipschitz functions.

It is well known that spaces of Lipschitz functions can be represented as dual spaces. The result goes back to Arens and Eells [7], see [52]. The approach below is based on [34].

Let (X, ρ) be a metric space. **In this section, our space X will be separable and boundedly compact.** The latter means that all balls are relatively compact.

Definition 5.1 (Spaces of continuous functions and measures)**.** We use the following notation. $\mathcal{C}_b(X)$ is the Banach space of all bounded continuous functions on X, $\mathcal{C}_c(X)$ is the set of all continuous functions on X with compact support and $\mathcal{C}_0(X)$ is the closure of $\mathcal{C}_c(X)$ in $\mathcal{C}_b(X)$. The norm in the space $\mathcal{C}_b(X)$ is

$$\|u\|_{\mathcal{C}} = \sup\nolimits_{x \in X} |u(x)|.$$

Linear functionals $\mathcal{F}$ on $\mathcal{C}_c(X)$ satisfying the condition

$$u \geq 0 \implies \mathcal{F}(u) \geq 0$$

are called *Radon integrals.* According to the Riesz representation theorem, there is an one-to-one correspondence between Radon integrals and *Radon measures*, this means, nonnegative Borel measures on X such that $\mu(K) < \infty$ for each compact set $K \subset X$. The term "Borel measure" does not mean that we measure only Borel sets, for each fixed measure μ we extend the measure from the Borel σ-algebra to the σ-algebra of *μ-measurable sets* by the completion process.

If possible, we identify Radon integrals with Radon measures.

The dual to the space $\mathcal{C}_0(X)$ is the space $\mathcal{C}_0(X)^*$ of all signed Radon integrals on X. These are integrals with respect to *signed Radon measures.* Signed measures are always real-valued. Hence, there is no inclusion between Radon measures (allowed to attain infinite values) and signed Radon measures.

Definition 5.2 (Lipschitz function)**.** A Lipschitz constant of a mapping between metric spaces $f : (Z, \rho_Z) \to (Y, \rho_Y)$ is the smallest $K \in [0, \infty]$ such that

$$\rho_Y(f(x), f(y)) \le K\rho_Z(x, y), \qquad x, y, \in Z.$$

The Lipschitz constant of f is denoted by $|f|_{\mathrm{Lip}}$. We say that f is Lipschitz if $|f|_{\mathrm{Lip}} < \infty$.

Definition 5.3 (Lipschitz and "co-Lipschitz" spaces)**.** We start from the Banach space $\mathcal{D}(X)$ of all bounded Lipschitz functions u on X ("test functions") endowed with the norm

$$\|u\|_{\mathcal{D}(X)} = \max\{\|u\|_{\mathcal{C}},\ |u|_{\mathrm{Lip}}\}.$$

Recall that $\mathcal{D}(X)^*$ denotes the dual Banach space to $\mathcal{D}(X)$. Each element of the space $\mathcal{C}_0(X)^*$ is identified with the continuous linear functional

$$u \mapsto \int_X u\, d\nu, \qquad u \in \mathcal{C}_b(X),$$

where ν is the signed measure representing the given functional. Since $\mathcal{D}(X)$ is trivially continuously embedded into $\mathcal{C}_b(X)$, by the dual process, $\mathcal{C}_0(X)^*$ is naturally embedded into $\mathcal{D}(X)^*$ (and, in what follows, identified with the corresponding subclass of $\mathcal{D}(X)^*$). The closure of $C_0(X)^*$ in $\mathcal{D}(X)^*$ is the space of *convergent (metric) distributions* on X, it is denoted by $\mathcal{D}'(X)$. The term "convergent" is motivated by the feature that it is possible to determine the definite integral if the indefinite integral is a convergent metric distribution.

Let $u \in \mathcal{D}(X)$. Then the functional

$$\varkappa(u) :\ \mu \mapsto \int_X u\, d\mu, \qquad \mu \in \mathcal{C}_0(X)^*$$

is continuous with respect to the $\mathcal{D}(X)^*$ norm and thus it can be uniquely extended as a continuous linear functional on $\mathcal{D}'(X)$. The mapping

$$\varkappa :\ u \mapsto \varkappa(u) :\ \mathcal{D}(X) \to (\mathcal{D}'(X))^*$$

is called *canonical embedding.*

Definition 5.4 (Localization)**.** The co-Lipschitz spaces can be localized as follows. If $K \subset X$ is compact, let $\mathcal{C}_K(X)$ be the subspace of $\mathcal{C}_0(X)$ consisting of all $\mathcal{C}_0(X)$ function with support in K and $\mathcal{D}_K(X)$ be the subspace of $\mathcal{D}(X)$ consisting of all test functions with support in K. Then $\mathcal{D}'_K(X)$ is defined as the closure of $\mathcal{C}_K(X)^*$ in $\mathcal{D}_K(X)^*$.We define $\mathcal{D}_c(X)$ as the union of all $\mathcal{D}_K(X)$ over all compact $K \subset X$ and $\mathcal{D}'_{\mathrm{loc}}(X)$ as the intersection of $\mathcal{D}'_K(X)$ over all compact $K \subset X$. The elements of $\mathcal{D}'_{\mathrm{loc}}(X)$ will be called *(metric) distributions.* In what follows, we study in detail the "global versions" of spaces.

Theorem 5.5. *The dual of $\mathcal{D}'(X)$ is (isometrically isomorphic to) $\mathcal{D}(X)$. The dual of $\mathcal{D}'_K(X)$ is (isometrically isomorphic to) $\mathcal{D}_K(X)$.*

Proof. We prove only the first assertion. Obviously the canonical embedding $\varkappa : \mathcal{D}(X) \to (\mathcal{D}'(X))^*$ is an isometrical endomorphism, we need only to show that it is onto. Let $\mathbb{T}$ be a continuous linear functional on $\mathcal{D}'(X)$. We set

$$u(x) = \mathbb{T}(\delta_x), \qquad x \in X,$$

where δ_x is the Dirac measure at x. Then u is a function on X. If $x, y \in X$, then

$$|u(y) - u(x)| = |\mathbb{T}(\delta_y - \delta_x)| \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*} \|\delta_y - \delta_x\|_{\mathcal{D}(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*} \rho(x, y)$$

and

$$|u(x)| = |\mathbb{T}(\delta_x)| \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*} \|\delta_x\|_{\mathcal{D}(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}$$

Hence $u \in \mathcal{D}(X)$ and $\|u\|_{\mathcal{D}(X)} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}$. Choose $\mu \in \mathcal{C}_0(X)^*$. As an exercise we find signed measures μ_j such that $\mu_j \to \mu$ in $\mathcal{D}(X)^*$ and μ_j are linear combinations of Dirac measures in X. Then

$$\int_X u \, d\mu_j = \mathbb{T}(\mu_j)$$

and passing to the limit we obtain

$$\int_X u \, d\mu = \mathbb{T}(\mu).$$

Then it is easy to conclude that $\mathbb{T} = \varkappa(u)$. □

Remark 5.6. We identify $\mathcal{D}(X)$ with $(\mathcal{D}'(X)^*$, so that from now, we have weak* topology and convergence on $\mathcal{D}(X)$ (and similarly on $\mathcal{D}_K(X)$) well defined.

Proposition 5.7. *Let u, $u_j \in \mathcal{D}(X)$, $j = 1, 2, \dots$. Then the following assertions are equivalent:*

(i) *$u_j \to u$ weak* in $\mathcal{D}(X)$,*
(ii) *u_j is bounded in $\mathcal{D}(X)$ and $u_j \to u$ pointwise,*
(iii) *u_j is bounded in $\mathcal{D}(X)$ and $u_j \to u$ locally uniformly.*

Proof. (i) $\implies$ (ii). By the Banach-Steinhaus theorem [18, Theorem 3.12]), each weak* convergent sequence is bounded. Since each Dirac measure is in $\mathcal{D}'(X)$, the pointwise convergence follows.

(ii) $\implies$ (iii): Boundedness in $\mathcal{D}(X)$ implies equicontinuity and each equicontinuous pointwise convergent sequence converges locally uniformly (this is related to the Arzelà-Ascoli theorem).

(iii) $\implies$ (i): It follows form the facts that $\varkappa(u_j)(\mu) \to \varkappa(u)(\mu)$ for each measure $\mu \in \mathcal{C}_0(X)^*$, that the measures are dense in $\mathcal{D}'(X)$ and that the sequence $(u_j)_j$ is bounded. □

Proposition 5.8. *The following properties of a linear functional $\mathscr{T} : \mathcal{D}(X) \to \mathbb{R}$ are equivalent:*

(i) *$\mathscr{T} \in \mathcal{D}'(X)$,*
(ii) *$\mathscr{T}$ is weak* continuous on $\mathcal{D}(X)$,*
(iii) *$\mathscr{T}$ is sequentially weak* continuous on $\mathcal{D}(X)$.*

Proof. This follows from Proposition 5.7 using basic facts from functional analysis. □

Proposition 5.9. *$\mathcal{D}_c(X)$ is weak* dense in $\mathcal{D}(X)$*

Proof. It is left as an exercise. Here it is essential that X is boundedly compact. □

Remark 5.10. If $\mathscr{T}$ is a distribution on $\Omega \subset \mathbb{R}^n$, then it can be extended to a functional from $\mathcal{D}'(\Omega)$ (in our notation) if and only if $\mathscr{T}$ is bounded on $\mathcal{C}_c^\infty(\Omega)$ in the $\mathcal{D}(\Omega)$-norm and

$$\langle \mathscr{T}, \varphi_j \rangle \to 0$$

whenever $\varphi_j \in \mathcal{C}_c^\infty(\Omega)$ and $\varphi_j \to 0$ weak* in $\mathcal{D}(\Omega)$.

We can consider the example

$$\langle \mathscr{T}, \varphi \rangle = \varphi'(0), \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{3}$$

Then $\mathscr{T}$ is bounded on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in the $\mathcal{D}(\mathbb{R}^n)$-norm and can be extended as a bounded linear functional on $\mathcal{D}(\mathbb{R}^n)$. However, such an extension is not constructive (the use of Hahn-Banach theorem) and is not weak* continuous on $\mathcal{D}(\mathbb{R}^n)$. Hence $\mathscr{T}$ from (3) is not a metric distribution.

Definition 5.11 (Banach space valued distributions)**.** Let Y be a Banach space. We say that a linear mapping $\mathscr{T} : \mathcal{D}_c(X) \to Y$ is a Y-valued metric distribution if for each compact set $K \subset X$, $\mathscr{T}$ is sequentially weak* continuous from $\mathcal{D}_K(X)$ to Y. The collection of all Y-valued metric distributions on X is denoted by $\mathcal{D}'_{\mathrm{loc}}(X, Y)$.

5.2. **Pointwise BV functions.** In the metric measure space setting, BV-functions can be defined as functions, for which a Poincaré-type inequality holds. In this approach, we say that $u \in L^1(X, \lambda)$ is BV-function if there exist constants $C, \sigma \geq 1$ and a Radon measure ν such that the inequality

$$\int_{B(x,r)} |u - u_{x,r}|\, d\lambda \leq r\nu(B(x, \sigma r)) \tag{4}$$

holds for each ball $B(x, r) \subset X$. Here $u_{x,r}$ is the integral average of u over $B(x, r)$. The measure ν appearing in (4) has the role of an "upper gradient" to u. For the BV theory in the metric space setting see [43], [1], [2], [8], [26].

We obtain a much more wider class of functions if we let (4) hold for small balls only and with the constant C depending on the point.

Definition 5.12. We say that $u \in L^1_{\mathrm{loc}}(X, \lambda)$ is a pointwise BV function (we abbreviate a PBV function) there exist a constant $\sigma \geq 1$ and a Radon measure ν such that the asymptotic behavior

$$\limsup_{r \to 0_+} \frac{\int_{B(x,r)} |u - u_{x,r}|\, d\lambda}{r\nu(B(x, \sigma r))} < \infty \tag{5}$$

holds for each $x \in X$.

The measures serving for (5) are called *pointwise upper gradients* to u. The class of all pointwise upper gradients to u is denoted by $\mathrm{PUG}(u)$.

Example 5.13. A typical example of a PBV function is a Sobolev function on an Euclidean domain. However, it is easily seen that also each pointwise differentiable function is PBV, with the Lebesgue measure serving as an pointwise upper gradient. Of course, not all pointwise gradients are Lebesgue integrable; recall that this was one of main motivation for the theory of nonabsolutely convergent integrals.

Example 5.14. Another typical example of a BV (and thus PBV) function in the Euclidean setting is a characteristic set of a set of finite perimeter. However, if the (topological) boundary of a set $G \subset \mathbb{R}^n$ is countably $(n-1)$-rectifiable, then χ_G is also a PBV function. In particular, we may consider the case that there exists an exterior normal vector at each boundary point x to G, see [3, Theorem 2.61] for the proof of countable rectifiability in such a situation. Note that we do not require the normal vector to depend continuously on x and thus the perimeter of G may happen to be locally infinite.

6. Integration in metric spaces

In this section, based on [34], we introduce our concept of an integral with respect to a metric distribution, prove that this integral makes sense and investigate some basic properties.

In the sequel, we assume that X is a complete separable metric space equipped with a doubling measure λ. This means that λ is a Radon measure and there exists a constant C_2 such that

$$\lambda(B(x,2r)) \le C_2 \lambda(B(x,r))$$

for each $x \in X$ and $r > 0$. Note that such a space X is always boundedly compact.

Notice that for each $\tau > 1$ there exists a constant C_τ such that

$$\lambda(B(x,\tau r)) \le C_\tau \lambda(B(x,r)). \tag{6}$$

For example, we can use $C_\tau = C_2$ for $1 < \tau \le 2$, $C_\tau = C_2^2$ for $2 < \tau \le 4$ and so on.

Definition 6.1. If $x \in X$ and $r > 0$, we denote

$$\mathcal{D}(x,r) = \{\varphi \in \mathcal{D}(X),\ \operatorname{spt}\varphi \in B(x,r),\ |\varphi|_{\mathrm{Lip}} \le 1/r,\ \|\varphi\|_{\mathcal{C}} \le 1.\}$$

If Z is a Banach space and $\mathscr{T} \in \mathcal{D}'(X,Z)$, we write

$$\|\mathscr{T}\|_{x,r} = \sup\Big\{|\langle \mathscr{T}, \varphi\rangle| :\ \varphi \in \mathcal{D}(x,r)\Big\},$$

where $|\cdot|$ is the norm in Z.

Definition 6.2. Let Y, Z be Banach spaces and $L = L(Y,Z)$. Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X,Z)$, $\mathscr{G} \in \mathcal{D}'_{\mathrm{loc}}(X,Y)$ and $f : X \to L$ be a function. We say that $\mathcal{F}$ is an *indefinite UC-integral* of f with respect to $\mathscr{G}$ if there exist $\sigma \ge 1$, and a finite Radon measure μ on X such that

$$\lim_{r\to 0_+} \frac{\|\mathcal{F} - f(x)\mathscr{G}\|_{x,r}}{\mu(B(x,\sigma r))} = 0 \tag{7}$$

holds for each $x \in X$.

We denote the indefinite UC-integral of f with respect to $\mathscr{G}$ (which is unique by Theorem 6.7 below) by

$$\mathscr{G}\lfloor f.$$

We say that f is *UC-integrable* with respect to $\mathscr{G}$ if $\mathscr{G}\lfloor f$ exists and belongs to $\mathcal{D}'(X)$. Then we define the *definite* UC-integral of f with respect to $\mathscr{G}$ as

$$\int_{\mathscr{G}} f = \langle \mathscr{G}\lfloor f, 1\rangle.$$

By Proposition 5.9, the definite integral is determined by the values of $\mathcal{F}$ on $\mathcal{D}_c(X,Z)$).

We develop most of the theory in the scalar case, the vector valued generalization is only a matter of a more careful notation.

Remark 6.3. The purpose of the scaling factor σ is to avoid the dependence of the concept of integral on the geometry of balls. As it is, the integral is invariant under bilipschitz transformations (see Subsection 6.2) and, in particular, it does not depend on the choice of a norm in $\mathbb{R}^n$.

Definition 6.4. Let $\tau > 1$, μ be a Borel measure on X and $x \in X$. We say that $r > 0$ is a τ-*absorbing radius* for μ and x if

$$\mu(B(x,\tau r)) + \lambda(B(x,\tau r)) \le 2C_\tau \big(\mu(B(x,r)) + \lambda(B(x,r))\big),$$

where C_τ is as in (6).

Lemma 6.5. *Let μ be a Radon measure on X, $x \in X$, $\tau > 1$ and $\delta > 0$. Then there exists a τ-absorbing radius $r \in (0,\delta)$ for μ and x.*

Proof. Denote $\bar\mu = \mu+\lambda$. Suppose the assertion is not true. Starting with $R \in (0,\delta)$, by iteration we obtain

$$(2C_\tau)^m \bar\mu(B(x,\tau^{-m}R)) \le \dots \le 2C_\tau\, \bar\mu(B(x,\tau^{-1}R)) \le \bar\mu(B(x,R)),\qquad m = 1,2,\dots.$$

Hence

$$\lambda(B(x,R)) \le C_\tau^m \lambda(B(x,\tau^{-m}R)) \le C_\tau^m \bar\mu(B(x,\tau^{-m}R)) \le 2^{-m}\bar\mu(B(x,R)),\qquad m = 1,2,\dots,$$

which is a contradiction. □

Lemma 6.6 (Partition of unity)**.** *Let μ be a Radon measure on X, $\Delta : X \to \mathbb{R}$ be a strictly positive function, $\sigma \ge 1$ be a constant and $K \subset X$ be a compact set. Then there exist a constant C and a finite system $(\omega_i)_{i=1}^m$ of smooth functions, accompanied with a system $((B(x_i,r_i))_{i=1}^m$ of pairwise disjoint balls such that $\sum_{i=1}^m \omega_i = 1$ on K and the following properties are satisfied for $i = 1,\dots,m$:*

$$\omega_i \ge 0 \text{ on } X, \tag{8}$$

$$\omega_i \in \mathcal{D}_{B(x_i,5r_i)}(X), \tag{9}$$

$$|\omega_i|_{\mathrm{Lip}} \le 2/r_i, \tag{10}$$

$$r_i \le \Delta(x_i), \tag{11}$$

$$\overline{B}(x_i,5\sigma r_i) \subset X \tag{12}$$

and

$$\bar\mu(B(x_i,5\sigma r_i)) \le C\,\bar\mu(B(x_i,r_i)), \tag{13}$$

where $\bar\mu = \mu + \lambda$.

Proof. Using Lemma 6.5, with each $x \in K$ we associate a 5σ-absorbing radius $r(x)$ for μ and x such that $r(x) < \Delta(x)$. . We cover K with the balls $B(x,r(x))$ and using compactness, we select a finite subcovering. Now, the Vitali covering theorem yields a finite sequence $(B(x_i,r_i))_{i=1}^m$ of pairwise disjoint balls from the collection $\{B(x,r(x)\colon x\in K\}$ such that

$$K\subset\bigcup_{i=1}^m B(x_i,3x_i).$$

We may assume that these balls are ordered so that

$$r_1\ge r_1\ge\dots\ge r_m. \tag{14}$$

We set

$$\eta_i(x)=\begin{cases}1, & x\in B(x_i,3r_i),\\ 4-\rho(x,x_i), & x\in B(x_i,4r_i)\setminus B(x_i,3r_i),\\ 0, & x\notin B(x_i,4r_i),\end{cases}$$

$$\omega_i=\max_{j=1,\dots,i}\eta_j-\max_{j=1,\dots,i-1}\eta_j,\quad i=1,\dots,m$$

and (in the notation of (6))

$$C=2C_{5\sigma}.$$

Then all the requirements are satisfied. □

Theorem 6.7. *Let Y,Z be Banach spaces. Let $\mathscr{G}\in\mathcal{D}'_{\mathrm{loc}}(X,Y)$ and $f:X\to L(Y,Z)$ be a function. There exists at most one indefinite integral of f with respect to $\mathscr{G}$.*

Proof. For simplicity we consider the case that $Z=\mathbb{R}$. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are indefinite integrals of f with respect to $\mathscr{G}$. Then $\mathcal{F}=\mathcal{F}_1-\mathcal{F}_2$ is an indefinite integral of 0. Therefore, it is enough to show that there is only the trivial indefinite integral of 0. (Notice that the distribution $\mathscr{G}$ plays no role if we integrate the zero function.) Let $\mathcal{F}$ be an indefinite integral of 0 and $\eta\in\mathcal{D}_c(X)$ be a test function. Let K be the support of η and $U\subset X$ be a relatively compact open set containing all balls $B(x,1)$ with $x\in K$. Let μ be as in Definition 6.2 and $\bar\mu=\mu+\lambda$. For each $x\in K$ we find $\Delta(x)\in(0,1)$ such that

$$|\langle\mathcal{F},\varphi\rangle|\le\varepsilon\mu(B(x,5\sigma r)),\qquad \varphi\in\mathcal{D}(x,5r),\ 0<r\le\Delta(x). \tag{15}$$

Using Lemma 6.6 we find a finite system $(\omega_i)_{i=1}^m$ of smooth functions and a system $((B(x_i,r_i))_{i=1}^m$ of pairwise disjoint balls such that $\sum_{i=1}^m\omega_i=1$ on K and the properties (8)–(13) are satisfied for $i=1,\dots,m$. Since $r_i\le 1$, from (10) we obtain

$$|\omega_i\eta|_{\mathrm{Lip}}\le\frac{3}{r_i}\,\|\eta\|_{\mathcal{D}(X)}\qquad i=1,\dots,m.$$

By (15) and (13),

$$\begin{aligned}\left|\langle\mathcal{F},\ \omega_i\eta\rangle\right|&\le C\|\mathcal{F}\|_{x_i,5r_i}\le C\varepsilon\mu(B(x_i,5\sigma r_i))\\&\le C\varepsilon\bar\mu(B(x_i,r_i)),\qquad i=1,\dots,m.\end{aligned}$$

Since the balls $B(x_i,r_i)$ are pairwise disjoint, summing over i we obtain

$$\left|\langle\mathscr{T},\ \eta\rangle\right|\le C\varepsilon\bar\mu(U).$$

Letting $\varepsilon\to 0$ we conclude the proof. □

Proposition 6.8. *Let f be UC-integrable with respect to $\mathscr{G}$. Then*

$$\int_{\mathscr{G}} f\,\varphi=\langle\mathscr{G}\lfloor f,\varphi\rangle.$$

Proof. The proof is left as an exercise. □

Example 6.9. Let $\mathscr{L}$ be the integration with respect to the Lebesgue measure on $X = (0, \infty)$ and $f(x) = \frac{\sin x}{x}$. Then f is not UC-integrable with respect to $\mathscr{L}$. Indeed, we cannot substitute $\varphi(x) = \sin x$ to the indefinite integral

$$\langle \mathcal{F}, \varphi \rangle = \int_0^\infty \frac{\sin x}{x}\, \varphi(x)\, dx$$

although $\sin \in \mathcal{D}(X)$. If we want to give a reasonable sense to the UC-integral

$$\int_0^\infty \frac{\sin x}{x}\, dx, \tag{16}$$

we change the distance function on X to the "hyperbolic distance"

$$\tilde\rho(x, y) = \Big|\log \frac{y}{x}\Big|.$$

and write $\tilde X = ((0, \infty), \tilde\rho)$. The space $\tilde X$ is obviously equipped with a doubling measure (for example, the measure with density $1/x$). We set

$$\langle \mathcal{F}, \varphi \rangle = \int_0^\infty \frac{1 - \cos x}{x^2} \big(\varphi(x) - x\varphi'(x)\big)\, dx, \qquad \varphi \in \mathcal{D}(\tilde X).$$

If $\varphi \in \mathcal{D}(\tilde X)$, then the Lipschitz condition implies that $|\varphi'(x)| \le \frac{1}{x}$ a.e. and thus

$$\langle \mathcal{F}, \varphi \rangle \le C \|\varphi\|_{\mathcal{D}(\tilde X)}.$$

Assume that $\varphi_j \to 0$ weak* in $\mathcal{D}(\tilde X)$ and $u \in \mathcal{C}_c^\infty((0, \infty))$. Then

$$\int_0^\infty u(x)\, x\, \varphi_j'(x)\, dx = -\int_0^\infty \Big(u(x) + xu'(x)\Big)\varphi_j(x)\, dx \to 0.$$

Since $\mathcal{C}_c^\infty((0, \infty))$ is dense in $L^1((0, \infty))$, it follows that $\psi_j \to 0$ weak* in $L^\infty((0, \infty))$, where $\psi_j(x) := x\varphi_j'(x)$. Since the function

$$x \mapsto \frac{1 - \cos x}{x^2}$$

belongs to $L^1((0, \infty))$, we deduce that $\langle \mathcal{F}, \varphi_j \rangle \to 0$. Hence $\mathcal{F} \in \mathcal{D}'(\tilde X)$. It is easy to verify that $\mathcal{F}$ is a weak* continuous extension of the functional

$$\varphi \mapsto \int_0^\infty \frac{\sin x}{x}\, \varphi(x)\, dx, \qquad \varphi \in \mathcal{D}_{\mathrm{loc}}(\tilde X)$$

to $\mathcal{D}(\tilde X)$ and thus we may define the definite UC-integral (16) as $\langle \mathcal{F}, 1 \rangle$.

We however admit that the ordinary Denjoy-Perron integral is better to handle integrals over unbounded subintervals of the real line than the UC-integral, which covers a wide range of local oscillations, but is not optimized with respect to the global convergence.

We are convinced that for fine studies of the definite integral, it is better to modify the definition according to particular structures than to try to propose universal directives based only on the setting of metric spaces.

6.1. **Integration with respect to a measure.** We prove that the Lebesgue integral with respect to a Radon measure is included in our integral. We also show that each UC-integrable function is measurable, so that the difference between the Lebesgue integral and the UC-integral is a matter of cancellation if the integrand attains large negative and positive values simultaneously.

In Theorem 6.15 and Remark 6.16, we compare our integral with the Denjoy-Perron integral.

Lemma 6.10. *Let μ be a Radon measure on X and $N \subset X$ be a μ-null set. Then there exist a Radon measure μ^* on X which absolutely continuous with respect to μ and a lower semicontinuous function w on X such that $w \geq 1$, $w = \infty$ on N and $d\mu^* = w\,d\mu$.*

Proof. For each $j = 1, 2, \dots$ we find an open set $W_j \subset X$ such that $N \subset W_j$ and $\mu(W_j) < 4^{-j}$. Then the the function

$$w = \sum_j 2^j \chi_{W_j}$$

and μ^* determined by $d\mu^* = w\,d\mu$ have obviously the required properties. □

Lemma 6.11. *Let ν, μ be finite Radon measures on X. Then*

$$\limsup_{r>0} \frac{\mu(B(x,r))}{\nu(B(x,3r))} < \infty \qquad \text{for } \nu\text{-a.e. } x \in \operatorname{spt} \nu.$$

Proof. Let

$$E_s = \Big\{x \in X : \limsup_{r\to 0_+} \frac{\mu(B(x,r))}{\nu(B(x,3r))} > s\Big\}, \qquad s > 0$$

and $K \subset E_s$ be a compact set. Then we use Vitali covering theorem we cover K with open balls $B(x_i, 3r_i)$ such that $B(x_i, r_i)$ are pairwise disjoint and

$$s\nu(B(x_i, 3r_i) < \mu(B(x_i, r_i)), \qquad i = 1, \dots, m.$$

Summing over i and passing to supremum over $K \subset E_s$ (notice that E_s is a Borel set) we obtain

$$s\nu(E_s) \leq \mu(X).$$

Letting $s \to \infty$ we obtain the assertion. □

Theorem 6.12. *Let ν be a Radon measure on X and $\mathscr{G}_\nu$ be the metric distribution induced by ν. Let $f : X \to \mathbb{R}$ be a ν-integrable function. Then UC-integral $\mathscr{G}_\nu \lfloor f$ exists as well and*

$$\big\langle \mathscr{G}_\nu \lfloor f, \varphi \big\rangle = \int_X f\,\varphi\,d\nu, \qquad \varphi \in \mathcal{D}(X).$$

Proof. We will show that

$$\mathcal{F}(\varphi) = \int_X f\,\varphi\,d\nu.$$

is the indefinite UC-integral of f with respect $\mathscr{G}_\nu$ We find a sequence $(f_j)_{j=1}^\infty$ of continuous functions such that

$$\int_X |f_j - f|\,d\nu < 2^{-j-1}$$

and write

$$g=\sum_{j=1}^{\infty} j|f_j-f_{j+1}|.$$

Then g is Lebesgue integrable with respect to ν. Set

$$\mu(E)=\int_E g\,d\nu+\lambda(E),\qquad E\subset X \text{ Borel}.$$

We pick $x\in X$, choose $\varepsilon>0$ and distinguish two cases.

CASE (a): Suppose that $\sum_j |f_j(x)-f_{j+1}(x)|<\infty$. Then $f(x)=\lim f_j(x)$. We find $m\in\mathbb{N}$ such that $\varepsilon m>1$ and $|f_m(x)-f(x)|<\varepsilon$. Further we find $\delta>0$ such that $|f_m-f(x)|<\varepsilon$ on $B(x,\delta)$. Let $0<r<\delta$ and $\varphi\in\mathcal{D}(x,r)$. Then we estimate

$$\begin{aligned}\Big|\langle\mathcal{F},\varphi\rangle-f(x)\langle\mathscr{G}_\nu,\,\varphi\rangle\Big|&=\Big|\int_X (f-f(x))\varphi\,d\nu\Big|\\&\le\int_X |f-f_m|\,|\varphi|\,d\nu+\int_X |f_m-f(x)|\,|\varphi|\,d\nu\\&\le\int_X\Big(\frac{g}{m}+\varepsilon\Big)|\varphi|\,d\nu\le 2\varepsilon\,\mu(B(x,r)).\end{aligned}$$

CASE (b): Suppose that $\sum_j |f_j(x)-f_{j+1}(x)|=\infty$. We find $m\in\mathbb{N}$ such that $\varepsilon m>1$. We observe that $g(x)=\infty$, and since g is lower semicontinuous, there exists $\delta>0$ such that $|f_m-f(x)|<\varepsilon g$ on $B(x,\delta)$. Let $0<r<\delta$ and $\varphi\in\mathcal{D}(x,r)$. Then we estimate

$$\begin{aligned}\Big|\langle\mathcal{F},\varphi\rangle-f(x)\langle\mathscr{G}_\nu,\varphi\rangle\Big|&=\Big|\int_X (f-f(x))\varphi\,d\nu\Big|\\&\le\int_X |f-f_m|\,|\varphi|\,d\nu+\int_X |f_m-f(x)|\,|\varphi|\,d\nu\\&\le\int_X\Big(\frac{g}{m}+\varepsilon g\Big)|\varphi|\,d\nu\le 2\varepsilon\,\mu(B(x,r)).\end{aligned}$$

In any case, we have shown that $\mathcal{F}$ has the required properties. □

Theorem 6.13. *Let ν be a Radon measure on X and $\mathscr{G}_\nu$ be the metric distribution induced by ν. Let a function $f:X\to\mathbb{R}$ be UC-integrable with respect to $\mathscr{G}_\nu$. Then*

(a) *f is ν-measurable,*
(b) *if $\mathscr{G}_\nu\lfloor f=0$, then $f=0$ ν-a.e.,*
(c) *if $f\ge 0$, then*

$$0\le\int_X f\,d\nu<\infty.$$

Proof. We only sketch the proof. Denote $\mathcal{F}=\mathscr{G}_\nu\lfloor f$. Let σ and μ be as in Definition 6.2. We consider the "derivate" $g:\operatorname{spt}\mu\to[-\infty,\infty]$ defined as

$$g(x)=\inf_{\varepsilon>0}\liminf_{r\to 0+} g_{r,\varepsilon}(x),$$

where

$$g_{r,\varepsilon}(x)=\inf\Big\{\frac{\langle\mathcal{F},\varphi\rangle+\varepsilon\mu(B(x,\sigma r))}{\langle\mathscr{G}_\nu,\varphi\rangle}:\ \varphi\in\mathcal{D}(x,r),\varphi\ge 0,\ \langle\mathscr{G}_\nu,\varphi\rangle>0\Big\}.$$

In can be seen that g is Borel measurable and $g \geq f$. It remains to prove that $g \leq f$ ν-a.e. Set

$$E = \Big\{ x \in \operatorname{spt}\nu : \limsup_{r>0} \frac{\overline{\mu}(B(x,r))}{\nu(B(x,3r))} < \infty \Big\} \tag{17}$$

where $\overline{\mu} = \mu + \lambda$. By Lemma 6.5,

$$\liminf_{r\to 0_+} \frac{\overline{\mu}(B(x,\sigma r))}{\overline{\mu}(B(x,r/6))} < \infty., \qquad x \in E.$$

From Lemma 6.11 we infer that $\nu(X \setminus E) = 0$. If $x \in E$ and $\varepsilon > 0$, we find a sequence $r_j \searrow 0$ and a sequence $(\varphi_j)_j$ of nonnegative test functions, $\varphi_j \in \mathcal{D}(x, r_j)$, such that $\varphi_j \geq 1/2$ on $B(x, r_j/2)$ and thus

$$\mu(B(x,\sigma r_j)) \leq C\overline{\mu}(B(x,r_j/6)) \leq C\nu(B(x,r_j/2)) \leq C\langle \mathscr{G}_\nu, \varphi_j\rangle.$$

Then

$$\begin{aligned}\liminf_{j\to\infty} g_{r_j,s} &\leq \liminf_{j\to\infty} \frac{\big(\langle \mathcal{F}, \varphi_j\rangle - f(x)\langle \mathscr{G}_\nu, \varphi_j\rangle\big) + f(x)\langle \mathscr{G}_\nu, \varphi_j\rangle + \varepsilon\mu(B(x,\sigma r_j))}{\langle \mathscr{G}_\nu, \varphi\rangle} \\ &\leq f(x) + \liminf_{j\to\infty} \frac{2\varepsilon\mu(B(x,\sigma r_j))}{\langle \mathscr{G}_\nu, \varphi_j\rangle} \\ &\leq f(x) + C\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain that $g \leq f$ on E, this means ν-a.e. It follows that f is ν-measurable, this proves (a).

From the above reasoning we also see that $f \geq 0$ λ-a.e. on $\{x \in E : g(x) \geq 0\}$. If $\mathcal{F} = 0$, then it follows that $f \geq 0$ λ-a.e. on X and from the symmetry reason we obtain that $f = 0$, which is (b).

Finally, assume that $f \geq 0$ and the indefinite UC-integral $\mathscr{G}_\nu \lfloor f$ exists. Similarly to the proof of Theorem 6.7 we are able to show that nonnegative functions have nonnegative integrals, thus, the integral depends monotonically on the integrand. Since f is a ν-measurable function, there exist ν-integrable functions $f_j \geq 0$ such that $f_j \nearrow f$. The monotonicity arguments and Theorem 6.12 show that

$$\lim_j \int_X f_j \, d\nu = \lim_j \int_{\mathscr{G}_\nu} f_j \leq \int_{\mathscr{G}_\nu} f < \infty.$$

By the monotone convergence theorem, f is ν-integrable. □

Corollary 6.14. *Let ν be a Radon measure on X, $f : X \to \mathbb{R}$ be a function. Then f is ν-integrable if and only if the UC integral of f with respect to $\mathscr{G}_\nu$ "converges absolutely", this means, both f and $|f|$ are UC-integrable.*

Proof. This follows from Theorems 6.12 and 6.13.

□

Theorem 6.15. *Let F be an indefinite Denjoy-Perron integral of $f : (a,b) \to \mathbb{R}$ on (a,b). Let a distribution $\mathcal{F}$ be defined as*

$$\langle \mathcal{F}, \varphi\rangle = -\int_a^b F(x)\varphi'(x)\,dx, \qquad \varphi \in \mathcal{D}((a,b)).$$

Then $\mathcal{F}$ is an indefinite UC-integral of f with respect to the Lebesgue measure.

Proof. We use the coincidence of the Denjoy-Perron integral with the MC-integral. By (1), there exists an increasing function γ on (a,b) such that

$$\lim_{y\to x}\frac{F(y)-F(x)-f(x)(y-x)}{\gamma(y)-\gamma(x)}=0,\qquad x\in(a,b).$$

Let μ be a Lebesgue-Stieltjes measure induced by γ, so that

$$\mu((y,z])=\gamma(z)-\gamma(y),\qquad y,z\in(a,b),\quad y<z.$$

Given $x\in(a,b)$ and $\varepsilon>0$, let $\delta>0$ be so small that $(x-\delta,x+\delta)\subset(a,b)$ and

$$|F(y)-F(x)-f(x)(y-x)|\le\varepsilon|\gamma(y)-\gamma(x)|,\qquad y\in(x-\delta,x+\delta).$$

If $0<r<\delta$ and $\varphi\in\mathcal{D}(x,r)$, then

$$\begin{aligned}\Big|\int_a^b F(y)\varphi'(y)\,dy-f(x)\int_a^b\varphi(y)\,dy\Big|&=\Big|\int_a^b\Big(F(y)-F(x)-f(x)(y-x)\Big)\varphi'(y)\,dy\Big|\\&\le 2r\,\varepsilon\sup_{y\in(a,b)}|\varphi'(y)|\,\mu((x-r,x+r))\\&\le 2\varepsilon\mu(B(x,r)).\end{aligned}$$

This shows that $\mathcal{F}$ is the indefinite UC-integral of f with respect to the Lebesgue measure. □

Remark 6.16. The converse of Theorem 6.15 is not true. Whereas the indefinite Denjoy-Perron integral is always continuous, there exists a UC-integrable function on $\mathbb{R}$ such that its indefinite UC-integral cannot be represented by a continuous function.

6.2. **Change of variables.** In this subsection we describe what we mean by the phrase that "the UC-integral is invariant with respect to a bilipschitz change of variables". We assume that X,Y are locally compact separable metric spaces equipped with doubling measures λ_X, λ_Y, respectively.

Definition 6.17 (Push forward)**.** Let $\mathscr{G}\in\mathcal{D}'(X)$ and $\Phi:X\to Y$ be a bilipschitz mapping. Then we define the *push forward* $\Phi_\sharp\mathscr{G}$ as

$$\big\langle\Phi_\sharp\mathscr{G},\ \psi\big\rangle=\big\langle\mathscr{G},\ \psi\circ\Phi\big\rangle,\qquad\psi\in\mathcal{D}(\Phi(X)).$$

Theorem 6.18. *Let $\mathscr{G}\in\mathcal{D}'(X)$ and $\Phi:X\to Y$ be a bilipschitz mapping. Let $f:\Phi(X)\to\mathbb{R}$ be a function. Suppose that the UC-indefinite integral $\mathscr{G}\lfloor(f\circ\Phi)$ exists. Then there exists a UC-indefinite integral $(\Phi_\sharp\mathscr{G})\lfloor f$ and*

$$(\Phi_\sharp\mathscr{G})\lfloor f=\Phi_\sharp(\mathscr{G}\lfloor(f\circ\Phi)).$$

Proof. It is left as an exercise.

□

7. Application to currents

Throughout this section we consider Banach spaces Y, Z and write $L=L(Y,Z)$. We use $|\cdot|$ for the norms in Y, Z and L. In standard examples, Y is finite-dimensional and $Z=Y^*$ or $\mathbb{R}$. However it can easily happen that a future research will find the general case important.

The idea of currents in metric spaces goes back to De Giorgi [14] and has been developed in the pioneering paper by Ambrosio and Kirchheim [4]. Our motivation is to study integration of "differential forms" with wild coefficients.

L^p and Sobolev differential forms on Lipschitz manifolds have been studied e.g. in [23], [22].

We follow [34] in this section.

Definition 7.1 (Current)**.** Let $\mathcal{D}^k(X)$ be the family of all ordered $(k+1)$-tuples $\vec{\psi} = (\psi_0, \dots, \psi_k)$ of Lipschitz functions on X such that ψ_0 is bounded. The elements of $\mathcal{D}^k(X)$ are called *test differential forms*. The support of a test differential form $\vec{\psi}$ is defined as the support of the product $\psi_0\psi_1 \dots \psi_k$. The family of all compactly supported test differential forms with a compact support is denoted by $\mathcal{D}^k_c(X)$. We say that $\vec{\psi}^{(n)} \overset{*}{\to} \vec{\psi}$ in $\mathcal{D}^k(X)$ if the Lipschitz constants of $\vec{\psi}^{(n)}$ and $\mathcal{C}$-norms of $\psi_0^{(n)}$ form bounded sequences and $\vec{\psi}^{(n)} \to \vec{\psi}$ pointwise. The $\overset{*}{\to}$ convergence in $\mathcal{D}^k_c(X)$ requires in addition that all $\vec{\psi}^{(n)}$ have the same compact support. We say that a mapping $\mathscr{T} : \mathcal{D}^k_c(X) \to Y$ is a Y-valued *k-current* if the following properties are satisfied:

(C-1) $\mathscr{T}$ is $(k+1)$-linear (this means, linear in each variable separately).

(C-2) $\vec{\psi}^{(n)} \overset{*}{\to} \vec{\psi}$ in $\mathcal{D}^k_c(X) \implies \mathscr{T}(\vec{\psi}^{(n)}) \to \mathscr{T}(\vec{\psi})$ in Y.

(C-3) If a linear combination of $\psi_1, \dots, \psi_k$ is constant on $\{x \in X : \psi_0(x) \neq 0\}$, then $\mathscr{T}(\vec{\psi}) = 0$. Specially, $\mathscr{T}$ is alternating in $\psi_1, \dots, \psi_k$.

(C-4) $\mathscr{T}(\psi_0, uv, \psi_2, \dots \psi_k) = \mathscr{T}(u\psi_0, v, \psi_2, \dots \psi_k) + \mathscr{T}(v\psi_0, u, \psi_2, \dots \psi_k)$.

The collection of all Y-valued k-currents on X is denoted by $\mathcal{D}'_{k,\mathrm{loc}}(X, Y)$. The family $\mathcal{D}'_k(X, Y)$ of all *convergent currents* on X is defined analogously, with the difference that they are defined in $\mathcal{D}^k(X)$ and continuous with respect to the $\mathcal{D}^k(X)$-convergence.

We say that a current $\mathscr{T}$ has a *locally finite mass* if there exists a Radon measure μ on X such that

$$|\mathscr{T}(\vec{\psi})| \le |\psi_1|_{\mathrm{Lip}} \dots |\psi_k|_{\mathrm{Lip}} \int_X |\psi_0| \, d\mu, \qquad \vec{\psi} \in \mathcal{D}^k(X). \tag{18}$$

We also say that $\mathscr{T}$ is *dominated by* μ if (18) is satisfied.

The boundary of a k-current $\mathscr{T}$ is defined as

$$\partial\mathscr{T}(\psi_0, \dots, \psi_{k-1}) = \mathscr{T}(1, \psi_0 \dots, \psi_{k-1}),$$

it is a $(k-1)$-current. We say that $\mathscr{T}$ is *boundary-free* if $\partial\mathscr{T} = 0$.

We identify 0-currents with metric distributions.

We use also the alternative (and more intuitive) notation

$$\langle \mathscr{T}, \, \psi_0 \, d\psi_1 \wedge \dots \wedge d\psi_k \rangle$$

for $\mathscr{T}(\psi_0, \psi_1, \psi_2, \dots \psi_k)$.

Definition 7.2 (Integral with respect to a current)**.** $f : X \to L$ be a function and $\mathcal{F} \in \mathcal{D}'_{k,\mathrm{loc}}(X, Z)$, $\mathscr{G} \in \mathcal{D}'_{k,\mathrm{loc}}(X, Y)$ be currents. We say that $\mathcal{F}$ is an indefinite integral of f with respect to $\mathscr{G}$ if for each k-tuple $(\psi_1, \dots \psi_k)$ of Lipschitz functions on X we get that $\mathcal{F}(\cdot, \psi_1, \dots \psi_k)$ is an indefinite integral of f with respect to $\mathscr{G}(\cdot, \psi_1, \dots \psi_k)$. The indefinite integral is uniquely determined by $\mathscr{G}$ and f, it is denoted by $\mathscr{G}\lfloor f$. The definite integral is defined by

$$\int_{\mathscr{G}} f \, d\psi_1 \wedge \dots \wedge d\psi_k = \mathscr{G}\lfloor f(1, \psi_1, \dots, \psi_k)$$

if $\mathscr{G}\lfloor f \in \mathcal{D}'_k(X)$.

Example 7.3. 1. The integration over a 2-dimensional smooth surface $\mathcal{M}$ is customarily expressed by the current

$$\mathscr{T}(\psi_0, \psi_1, \psi_2) = \int_{\mathcal{M}} \psi_0 \, d\psi_1 \wedge d\psi_2.$$

Example 7.4. However, we can model such an integration by the $\mathbb{R}^3$-valued 1-current

$$\mathscr{T}(\varphi, \psi) = \sum_{i=1}^{3} \mathbf{e}_i \int_{\mathcal{M}} \varphi \, d\psi \wedge dx_i.$$

If $\mathbf{f} = (f_1, f_2, f_3) : \mathcal{M} \to \mathbb{R}^3$ is a smooth vector field, $\mathbf{g} = (g_1, g_2, g_3) = \mathbf{curl}\,\mathbf{f}$, and $\varphi \in \mathcal{D}_c(\mathcal{M})$, then we have

$$\begin{aligned}
&\int_{\mathcal{M}} \varphi(g_1 \, dx_2 \wedge dx_3 + g_2 \, dx_3 \wedge dx_2 + g_3 \, dx_1 \wedge dx_2) \\
&\quad = \sum_{i=1}^{3} \int_{\mathcal{M}} \varphi \, df_i \wedge dx_i = -\sum_{i=1}^{3} \int_{\mathcal{M}} f_i \, d\varphi \wedge dx_i \\
&\quad = -(\mathscr{T} \lfloor \mathbf{f})(1, \varphi) = -\partial(\mathscr{T} \lfloor \mathbf{f})(\varphi).
\end{aligned}$$

Example 7.5. Similarly, if $\Omega \subset \mathbb{R}^3$ is an open set, $\mathbf{f} : \Omega \to \mathbb{R}^3$ is a smooth vector field and $\varphi \in \mathcal{D}_c(\Omega)$, then

$$\int_{\Omega} \varphi(x) \operatorname{div} \mathbf{f}(x) \, dx = -\partial(\mathscr{T} \lfloor \mathbf{f})(\varphi), \tag{19}$$

where

$$\mathscr{T}(\varphi, \psi) = \mathbf{e}_1 \int_{\Omega} \varphi \, d\psi \wedge dx_2 \wedge dx_3 + \mathbf{e}_2 \int_{\Omega} \varphi \, d\psi \wedge dx_3 \wedge dx_1 + \mathbf{e}_3 \int_{\Omega} \varphi \, d\psi \wedge dx_1 \wedge dx_2.$$

Remark 7.6. We see from the examples that there is no relation between the "dimension" k of the k-current and the "natural dimension" of the domain of integration. Also we observe that the setting of 1-currents is rich enough to describe the integration over k-dimensional manifolds, but then we are lead to the "vector-valued" setting, see also Remark 7.11.

Lemma 7.7. *Suppose that $\mathscr{T} \in D'_{1,\mathrm{loc}}(X, Y)$ is a 1-current and $f : X \to L$ is a function such that $\mathscr{T} \lfloor f$ exists. Let $\varphi, \psi \in \mathcal{D}_c(X)$. If $f\psi = 0$ on X, then $\langle \mathscr{T} \lfloor f, \, \varphi \, d\psi \rangle = 0$.*

Proof. We may assume that $\psi \geq 0$. If $\operatorname{spt} f \cap \operatorname{spt} \varphi = \emptyset$, then the conclusion is an easy consequence of (C-3). In the general case, we consider the sequence $(\psi_j)_j$, where

$$\psi_j = (\psi - 2^{-j})^+.$$

Then for each j we have $\langle \mathscr{T} \lfloor f, \, \varphi \, d\psi_j \rangle = 0$ by the first part of the proof. Since $(\varphi, \psi_j) \overset{*}{\to} (\varphi, \psi)$ in $\mathcal{D}^1_c(X)$, using the axiom (C-2) we obtain the assertion. ☐

Proposition 7.8. *Suppose that $\mathscr{T} \in D'_{1,\mathrm{loc}}(X, Y)$ is a 1-current dominated by λ, $f : X \to L$ be a locally λ-integrable function and $\varphi, \psi \in \mathcal{D}_c(X)$. Then*

$$\big| \langle \mathscr{T} \lfloor f, \, \varphi \, d\psi \rangle \big| \leq |\psi|_{\mathrm{Lip}} \int_W |f\varphi| \, d\lambda,$$

where $W = \{x \in X : \psi(x) \neq 0\}$.

Proof. We only sketch the proof. By Lemma 7.7 we may assume that $f = 0$ on $X \setminus W$. Therefore, it is enough to prove the assertion with integration over X. We choose $\varepsilon > 0$ and find a lower semicontinuous function $u : X \to [0, \infty]$, $u \geq |f\varphi|$, such that

$$\int_X u \, d\lambda \leq \int_X |f\varphi| \, d\lambda + \varepsilon.$$

Let μ and σ be as in the definition of the integral of f with respect to $\mathscr{T}$. With each $x \in X$ we associate a $\Delta(x) > 0$ such that

$$|f(x)\varphi(y)| \leq u(y), \qquad y \in B(x, \Delta(x))$$

and

$$\|\mathscr{T} \lfloor f(\cdot, \psi) - f(x)\mathscr{T}(\cdot, \psi)\|_{x,5r} \leq \varepsilon\mu(B(x, 5\sigma r)), \qquad 0 < r < \Delta(x).$$

Then we use the partition of unity and estimate the sum as in the proof of Theorem 6.7. □

Definition 7.9 (UC-differential)**.** The *UC-differential* of a function $f : X \to L$ with respect to a Y-valued 1-current $\mathscr{T}$ is defined as the "commutator"

$$(\partial\mathscr{T}) \lfloor f - \partial(\mathscr{T} \lfloor f)$$

and denoted by $\mathscr{T} \lfloor df$. The definite integral has a good sense in the "convergent case" $\mathscr{T} \lfloor df \in \mathcal{D}'_k(X, Z)$, in which we define

$$\int_{\mathscr{T}} df = \langle \mathscr{T} \lfloor df, \, 1 \rangle$$

More generally, if V is a further Banach space, $g : X \to L(Y, V)$, $f : X \to L(V, Z)$, we define

$$\int_{\mathscr{T}} f \, dg = \langle (\mathscr{T} \lfloor dg) \lfloor f, \, 1 \rangle$$

if $(\mathscr{T} \lfloor dg) \lfloor f \in \mathcal{D}'_k(X, Z)$.

Example 7.10. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and $\mathscr{T}$ is the current from Example 7.5, then

$$\langle \mathscr{T} \lfloor d\mathbf{f}, \, \varphi \rangle = \int_\Omega \varphi(x) \operatorname{div} \mathbf{f}(x) \, dx, \qquad \varphi \in \mathcal{D}(\Omega).$$

This works for all test functions, whereas (19) is valid only for compactly supported test functions.

Remark 7.11. If we develop the theory only for scalar 1-currents, we can still define some version of the divergence operator, for example, if $n = 3$, the operator

$$\mathbf{f} \mapsto \mathscr{T}_1 \lfloor df_1 + \mathscr{T}_2 \lfloor df_2 + \mathscr{T}_1 \lfloor df_3,$$

where (in the notation of Example 7.5)

$$\mathscr{T}_1(\varphi, \psi) = \int_\Omega \varphi \, d\psi \wedge dx_2 \wedge dx_3,$$

$$\mathscr{T}_2(\varphi, \psi) = \int_\Omega \varphi \, d\psi \wedge dx_3 \wedge dx_1,$$

$$\mathscr{T}_3(\varphi, \psi) = \int_\Omega \varphi \, d\psi \wedge dx_1 \wedge dx_2.$$

However, then we lose some important applications. Indeed, due to cancellation, divergence of a vector field is often more regular than the individual summands $\frac{\partial f_i}{\partial x_i}$. If we relax smoothness assumptions, it can happen that the currents $\mathscr{T}_1 \lfloor df_1$ do not make sense, but the current from Example 7.10 does.

Remark 7.12 (Stokes formula)**.** The "Stokes formula"

$$\int_{\partial \mathscr{T}} f - \int_{\mathscr{T}} df = 0 \tag{20}$$

is trivial in our setting, as the left hand part of (20) is

$$\partial(\mathscr{T} \lfloor f)(1) = \mathscr{T} \lfloor f(1,1) = 0.$$

A version of Stokes formula which is worth being considered is presented in Subsection 8.1 below.

8. Integration by parts

Throughout this section we assume that X is a locally compact separable metric space equipped with a doubling measure λ. We consider a 1-current $\mathscr{T}$ dominated by λ.

We will present here results from [34] on integration by parts and on the Stokes formula.

The formula on integration by parts

$$\int_{\partial \mathscr{T}} fg = \int_{\mathscr{T}} f\, dg + \int_{\mathscr{T}} g\, df$$

holds under fairly reasonable assumptions. However, this setting is quite technical and the main ideas are already visible in the special situation that $\partial \mathscr{T} = 0$ which we present below. Roughly speaking, if we want to apply our results to the model case of integration over manifolds, the assumption that $\mathscr{T}$ is boundary-free selects manifolds without boundaries.

Definition 8.1 (Concepts of the Lebesgue differentiation theory)**.** By the Lebesgue decomposition theorem, each Radon measure μ on X can be decomposed as $\mu_a + \mu_s$ where μ_a is the absolutely continuous part and μ_s is the singular part, both with respect to λ. Since λ is doubling, the Lebesgue differentiation theory is valid (see e.g. [5, Theorem 5.2.6]) and λ-a.e. point is a Lebesgue point for the Radon-Nikodým derivative $h = \frac{d\mu_a}{d\lambda}$, this means that

$$\lim_{r \to 0+} \fint_{B(x,r)} |h - h(x)|\, d\lambda = 0.$$

Clearly, each Lebesgue point is a weak Lebesgue point, this means that

$$\lim_{r \to 0+} h_{x,r} = h(x), \quad \text{where} \quad h_{x,r} = \fint_{B(x,r)} h\, d\lambda = h(x).$$

Definition 8.2 (Approximate Lipschitz points)**.** Let V be a Banach space. We say that $x \in X$ is an L^1-*approximate Lipschitz point* for a function $f : X \to V$ if there exist $K \in \mathbb{R}$ and a function $h : X \to \mathbb{R}$ such that

$$|f(y) - f(x)| \le K\rho(y,x) + h(y), \qquad y \in X$$

and

$$\lim_{r\to 0+} \frac{1}{r} \fint_{B(x,r)} |h| \, d\lambda = 0.$$

Remark 8.3. Of course, this is weaker than the usual pointwise Lipschitz continuity. By the result of Calderón and Zygmund [13], see also [3, Theorem 3.83], any BV-function on $\mathbb{R}^n$ has L^1-approximate Lipschitz points a.e. However, we conjecture that PBV-condition is not sufficient for this.

Theorem 8.4 (Integration by parts)**.** *Let $f : X \to Y^*$, $g : X \to \mathbb{R}$ be bounded PBV-functions on X, $\alpha \in \mathrm{PUG}(f)$, $\beta \in \mathrm{PUG}(g)$. Suppose that β_s and α_s are mutually singular, that f has weak Lebesgue points β-a.e., g has weak Lebesgue points α-a.e. and f has L^1-approximate Lipschitz points λ-a.e. Let $\mathscr{T}$ be a boundary-free 1-current on X dominated by λ. Then*

$$(\mathscr{T}\lfloor df)\lfloor g + (\mathscr{T}\lfloor dg)\lfloor f = \mathscr{T}\lfloor d(fg) \tag{21}$$

if at least one of these indefinite integrals makes sense.

Proof. Since $\mathscr{T}$ is boundary free, we can rewrite (21) as

$$[\partial(\mathscr{T}\lfloor f)]\lfloor g + [\partial(\mathscr{T}\lfloor g)]\lfloor f = \partial(\mathscr{T}\lfloor fg). \tag{22}$$

Assume that $[\partial(\mathscr{T}\lfloor f)]\lfloor g$ makes sense, we want to show that $\partial(\mathscr{T}\lfloor fg) - [\partial(\mathscr{T}\lfloor f)]\lfloor g$ is an indefinite UC-integral of f with respect to $\partial(\mathscr{T}\lfloor g)$. We have

$$\begin{aligned}
&\|\partial(\mathscr{T}\lfloor fg) - [\partial(\mathscr{T}\lfloor f)]\lfloor g - f(x)\partial(\mathscr{T}\lfloor g)\|_{x,r} \\
&\qquad \le \|\partial(\mathscr{T}\lfloor fg) - g(x)[\partial(\mathscr{T}\lfloor f)] - f(x)\partial(\mathscr{T}\lfloor g)\|_{x,r} \\
&\qquad\quad + \|[\partial(\mathscr{T}\lfloor f)]\lfloor g - g(x)[\partial(\mathscr{T}\lfloor f)]\|_{x,r} \\
&\qquad = \|\partial[\mathscr{T}\lfloor (f - f(x))(g - g(x))]\|_{x,r} + \|[\partial(\mathscr{T}\lfloor f)]\lfloor g - g(x)[\partial(\mathscr{T}\lfloor f)]\|_{x,r}.
\end{aligned}$$

The second term is controlled and we need to control the first one. The same situation occurs if we assume that $[\partial(\mathscr{T}\lfloor g)]\lfloor f$ makes sense. Therefore, our aim is to show that $\partial[\mathscr{T}\lfloor (f - f(x))(g - g(x))]$ is controlled.

By the singularity assumption, there exist Borel sets P, Q, R such that $X = P \cup Q \cup R$ and

$$\alpha(Q) = \beta(P) = \alpha_s(R) = \beta_s(R) = \lambda(P \cup Q) = 0.$$

Further, there exist an α-null set S, a β-null set T and a λ-null set $N \subset R$ such that f has weak Lebesgue points everywhere on $X \setminus T$, g has weak Lebesgue points everywhere on $X \setminus S$, g has Lebesgue points everywhere on $R \setminus N$ and f has L^1-approximate Lipschitz points everywhere on $R \setminus N$.

By Lemma 6.10, there exist measures α^*, β^* λ^* and and lower semicontinuous functions a, b, w such that $a = \infty$ on $Q \cup T \cup N$, $b = \infty$ on $P \cup S \cup N$, $w = \infty$ on N and

$$d\alpha^* = a \, d\alpha, \quad d\beta^* = b \, d\beta, \quad d\lambda^* = w \, d\lambda.$$

Choose $x \in X$, $r > 0$ and $\varphi \in \mathcal{D}(x, r)$. We distinguish several cases according to the position of x.

Case 1. If $x \in Q$, then

$$\begin{aligned}
\langle \mathscr{T}\lfloor (f - f(x))(g - g(x)), d\varphi\rangle &= \langle \mathscr{T}\lfloor (f - f_{x,r}))(g - g(x)), d\varphi\rangle \\
&\quad + \langle \mathscr{T}\lfloor (f_{x,r} - f(x)))(g - g_{x,r})), d\varphi\rangle \\
&\quad + \langle \mathscr{T}\lfloor (f_{x,r} - f(x)))(g_{x,r} - g(x))), d\varphi\rangle
\end{aligned}$$

However, the last term vanishes as $\partial\mathscr{T}=0$. Using Proposition 7.8, for small r we estimate

$$\begin{aligned}
&\big|\langle \mathscr{T}\lfloor(f-f(x))(g-g(x)),\,d\varphi\rangle\big| \\
&\qquad\le \frac{1}{r}\int_{B(x,r)} |(f-f_{x,r})(g-g(x))|\,d\lambda+\frac{1}{r}\int_{B(x,r)} |(f_{x,r}-f(x))(g-g_{x,r})|\,d\lambda \\
&\qquad\le \frac{C}{r}\int_{B(x,r)} |f-f_{x,r}|\,d\lambda+|f_{x,r}-f(x)|\,\frac{C}{r}\int_{B(x,r)} |g-g_{x,r}|\,d\lambda \\
&\qquad\le C\alpha(B(x,\sigma r))+|f_{x,r}-f(x)|\,\beta(B(x,\sigma r))
\end{aligned}$$

with some $\sigma\ge 1$. Since $x\in Q$, we have

$$\lim_{r\to 0_+}\frac{\alpha(B(x,\sigma r))}{\alpha^*(B(x,\sigma r))}=0.$$

For the second term, either x is a weak Lebesgue point for f of $x\in S$, but in any case

$$\lim_{r\to 0+}\frac{|f_{x,r}-f(x)|\,\beta(B(x,\sigma r))}{\beta^*(B(x,\sigma r))}=0$$

Case 2. The case $x\in P$ is similar, we interchange the role of f and g.

Case 3. Let $x\in N$. We estimate as in Case 1 and obtain

$$\big|\langle \mathscr{T}\lfloor(f-f(x))(g-g(x)),\,d\varphi\rangle\big|\le C\alpha(B(x,\sigma r))+C\beta(B(x,\sigma r)),$$

but

$$\lim_{r\to 0_+}\frac{\alpha(B(x,\sigma r))}{\alpha^*(B(x,\sigma r))}=0,\qquad \lim_{r\to 0_+}\frac{\beta(B(x,\sigma r))}{\beta^*(B(x,\sigma r))}=0$$

Case 4. Let $x\in R\setminus N$. We estimate as in Case 1, but for the first term we have

$$\big|\langle \mathscr{T}\lfloor(f-f(x))(g-g(x)),\,d\varphi\rangle\big|\le CK\int_{B(x,r)}|g-g(x)|\,d\lambda+\frac{C}{r}\int_{B(x,r)}h\,d\lambda$$

where K, h are as in Definition 8.2, and we know that

$$\lim_{r\to 0_+}\frac{\int_{B(x,r)}|g-g(x)|\,d\lambda}{\lambda(B(x,r))}=0,\qquad \lim_{r\to 0_+}\frac{\int_{B(x,r)}h\,d\lambda}{r\lambda(B(x,r))}=0.$$

Hence, in all cases we established a control for $\langle \mathscr{T}\lfloor(f-f(x))(g-g(x)),\,d\varphi\rangle$, which concludes the proof. □

Corollary 8.5. *Let $f:X\to Y^*$, $g:X\to\mathbb{R}$ be bounded PBV-functions on X, $\alpha\in\mathrm{PUG}(f)$, $\beta\in\mathrm{PUG}(g)$. Suppose that $fg\in L^1(X,\lambda)$, β_s and α_s are mutually singular, that f has weak Lebesgue points β-a.e., g has weak Lebesgue points α-a.e. and f has L^1-approximate Lipschitz points λ-a.e. Let $\mathscr{T}$ be a boundary-free 1-current on X dominated by λ. Then*

$$\int_{\mathscr{T}} g\,df+\int_{\mathscr{T}} f\,dg=0$$

if at least one of these integrals makes sense.

Proof. Since $fg\in L^1(X,\lambda)$, the current $\partial(\mathscr{T}\lfloor fg)$ is convergent and $\langle\partial(\mathscr{T}\lfloor fg),\,1\rangle=0$. Hence the result follows from Theorem 8.4. □

8.1. **Gauss-Green-Stokes formula.** A prevalent motivation for investigation of nonabsolutely convergent integrals is an effort to find a general setting for formulae of integral calculus like the Gauss-Green (divergence) theorem or Stokes theorem.

Within the framework absolutely convergent integration, the Gauss-Green formula for BV-sets (or, sets of finite perimeter) is the ultimate version. see e.g. [3] [17], [54] for exposition of BV-theory and historical comments.

However, only the non-absolutely convergent integrals can integrate all pointwise derivatives. A version of Stokes formula with nonabsolutely convergent integration over manifolds with boundaries is in [30].

The Pfeffer integral allows to prove Gauss-Green formula for sets of finite perimeter, where the "interior" integral of divergence is non-absolutely convergent [46], [48], [15].

Our aim is to allow also the "boundary integral" be non-absolutely convergent. Various abstract theories allow us to study the Stokes theorem beyond rectifiable sets. For different approaches see [53], [19], [24], [25], [50]. The generalized integrals can be defined by duality or by approximation. Our integrals are not as general, but look more as genuine integrals.

A version of the Gauss-Green theorem for the UC integral in the Euclidean setting is proposed in [40].

Theorem 8.6 (Stokes theorem)**.** *Let $G \subset X$ be PBV set (this means, χ_G is a PBV function). Let $f : X \to Y^*$ be a bounded PBV-functions on X, $\alpha \in \mathrm{PUG}(f)$, $\beta \in \mathrm{PUG}(\chi_G)$. Suppose that β_s and α_s are mutually singular, that f has weak Lebesgue points β-a.e., χ_G has weak Lebesgue points α-a.e. and f has L^1-approximate Lipschitz points λ-a.e. Let $\mathscr{T}$ be a boundary-free 1-current on X dominated by λ. Assume that $\int_G |f|\,d\lambda$ converges. Then*

$$\int_{\mathscr{T}} \chi_G\, df = \int_{\partial(\mathscr{T}\lfloor\chi_G)} f$$

if at least one of these integrals makes sense.

Proof. It is enough to set $g = \chi_G$ in Corollary 8.5. □

Remark 8.7. As a special, "absolutely convergent" case we obtain the following situation:

Let us consider that $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ is a bounded vector field and g is the characteristic function of a set $G \subset \mathbb{R}^n$ of finite perimeter. Then we can take α to be absolutely continuous with respect to the Lebesgue measure (say, with density $|Df|$) and β to be the perimeter measure (=the total variation of $D\chi_G$). By results of Federer, Fleming and Ziemer, [19], [21], [20], see also [17, 4.8., 5.6.3], f has Lebesgue points $\mathscr{H}^{n-1}$-a.e. and thus β-a.e. On the other hand, by the Lebesgue density theorem, a.e. point of G is a density point for G and a.e. point of $\mathbb{R}^n \setminus G$ is a density point for $\mathbb{R}^n \setminus G$. Hence, g has Lebesgue points a.e. with respect to the Lebesgue measure and thus α-a.e. Also, by Remark 8.3, both f and g have L^1-approximate Lipschitz points a.e.

This absolutely convergent case is not new, but in view of Examples 5.13 and 5.14, it is clear that our result covers much more situations.

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