WASSERSTEIN ASYMPTOTICS FOR BROWNIAN MOTION ON THE FLAT TORUS AND BROWNIAN INTERLACEMENTS

MAURO MARIANI AND DARIO TREVISAN

ABSTRACT. We study the large time behavior of the optimal transportation cost towards the uniform distribution, for the occupation measure of a stationary Brownian motion on the flat torus in d dimensions, where the cost of transporting a unit of mass is given by a power of the flat distance. We establish a global upper bound, in terms of the limit for the analogue problem concerning the occupation measure of the Brownian interlacement on $\mathbb{R}^d$. We conjecture that our bound is sharp and that our techniques may allow for similar studies on a larger variety of problems, e.g. general diffusion processes on weighted Riemannian manifolds.

1. INTRODUCTION

Given a stochastic process $(X_t)_{t \geq 0}$, its occupation measure up to a time $T > 0$ can be defined as the (random) measure

$$\mu_T^X = \int_0^T \delta_{X_s} ds, \quad \mu_T^X(A) = \int_0^T I_{\{X_s \in A\}} ds, \tag{1.1}$$

If renormalized to be a probability measure, it is also known as the empirical measure of X. From the simplest case of pure jump process associated to i.i.d. random variables $(Y_n)_{n=0}^\infty$, i.e., $X_t = Y_{\lfloor t \rfloor}$, to that of diffusion processes on manifolds, the occupation measure has a plethora of applications, ranging from non-parametric statistics, to Monte Carlo methods and mean field theory.

Under natural assumptions on X, such as stationarity and ergodicity, limit theorems can be established for the empirical measure, as the time horizon T increases, showing convergence towards its invariant measure. It is then a relevant question in applications to quantify such convergence, using a suitable metric between measures defined on the state space of the process. A particularly compelling choice, assuming that the state spaces is already equipped with a distance, e.g. if X takes values in a Riemannian manifold, is given by the optimal transport (Wasserstein) cost of order p, for some $p > 0$. Also known as the earth mover's distances, the metric is defined as the minimum total cost of moving a source mass distribution μ towards the target distribution λ, where the cost of transporting a unit of mass is given by the p-th power of the distance in the underlying state space. Classically, the case $p = 1$ was the preferred choice, also because of the celebrated Kantorovich dual formulation, which represents the minimum cost as a maximum discrepancy between the integrals with respect to μ and λ over all the 1-Lipschitz functions. In recent years, other choices of p, particularly $p = 2$, have been the subject of intense investigation, see the monographs [2, 39]. Also the "concave" case $p < 1$ yields interesting features, as studied in [19, 32].

The problem of establishing rates of convergence for the Wasserstein cost of the occupation measure for pure jump process associated to i.i.d. random variables $(Y_n)_{n=0}^\infty$ is strongly related to the so-called random assignment (or bipartite matching) problem, of

combinatorial nature. Indeed, if instead of a single process one considers two (independent) families of variables $(Y_n)_{n=i}^{\infty}$, $(Z_n)_{n=i}^{\infty}$. Then, for every $T = n$ one can study the assignment cost

$$\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} \mathsf{d}(Y_i, Z_{\sigma(i)})^p \qquad (1.2)$$

where $\mathcal{S}_n$ denotes the set of permutations over n elements. By Birkhoff's theorem on doubly stochastic matrices, it is well-known that the cost above equals the Wasserstein cost of order p between the two occupation measures

$$\mu_n = \sum_{i=1}^{n} \delta_{Y_i}, \quad \lambda_n = \sum_{i=1}^{n} \delta_{Z_i}. \qquad (1.3)$$

From this point of view, it is natural to expand the techniques developed for the assignment problem to the study of asymptotic rates of convergence for more general stochastic processes, e.g. diffusions on Riemannian manifolds. Indeed, the literature on the random assignment problem is vast and growing: stemming from the seminal works [18], it stimulated powerful functional analytic techniques [37, 38] and combinatorial/geometrical ones [1, 16, 10, 7]. A renewed interest due to powerful predictions by the statistical physics community [14, 12, 35, 13, 8] lead recently to the development of novel methods [5, 30, 22, 23] that have found several applications [9, 3, 4, 21, 6, 20, 27, 24], even beyond the case of i.i.d. points [29, 15] and also for other combinatorial optimization problems [11, 25].

In the case of continuous processes on manifolds, F.-Y. Wang and collaborators pioneered systematic exploitation of the PDE tools from [5, 30] to the exploration of asymptotic rates for occupation measure [47, 42, 40, 43, 45, 44, 41]. Let us mention that similar techniques have been also extended to non-Markov processes such as the fractional Brownian motion [28, 31].

However, a known issue that afflicts the original PDE methods [5, 30] (but also in some sense [22]) is that when the dimension of the underlying manifold grows (i.e., for $d \geq 3$ in the random assignment problem and $d \geq 5$ for diffusion processes) the upper and lower bounds resulting from a "global" application of the methods become less precise and yield (conjectured) non-optimal constants, although they still match the correct rates. To overcome this, in the setting of the assignment problem, it was first put forward in [23] and later developed in [6, 25] the need for further "localize" the problem, using geometric decompositions of Whitney-type.

1.1. **Main results.** Aim of this paper is to show for the first time that it is possible to adapt these localization arguments in the setting of diffusion processes on Riemannian manifolds, obtaining (conjectured) sharp results. We focus on the simplest of all cases, namely that of Brownian motion on the flat torus, where the occupation measure convergence towards the uniform (Lebesgue) distribution. It turns out that the limit of the Wasserstein cost of the occupation measure on sufficiently small scales is related to the limit of a suitably defined notion of occupation measure associated to the Brownian (continuous time) interlacement process on $\mathbb{R}^d$, as introduced in [36]. Such a connection is quite natural, in view of similar results about scaling limits of the support of a Brownian process in the torus (for an exposition in the discrete time case, see [17]). Intuitively, the Brownian interlacement plays here the role of a Poisson point process in the case of i.i.d. points.

Our first main result can be stated as follows. All the notation and basic notions (including the Newtonian capacity $\mathrm{Cap}(\Omega)$ and normalized equilibrium measure for a set $\Omega \subseteq \mathbb{R}^d$) will be precisely given in Section 2. We only anticipate that the notation $W_{\Omega}^p(\mu)$ denotes the Wasserstein cost of order p between the restriction of a measure μ on Ω towards the uniform measure on Ω, with the same total mass $\mu(\Omega)$.

Theorem 1.1. *Let $d \in \{3, 4\}$ and $p \in (0, (d-2)/2)$, or $d \geq 5$ and $p > 0$. Then, there exists a constant $\mathsf{c}(\mathcal{I}, d, p) \in (0, \infty)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded connected domain with C^2 boundary (or $\Omega = Q$ be a cube) and let $(B^i)_{i \geq 1}$ be independent Brownian motions on $\mathbb{R}^d$, with initial law given by the normalized equilibrium measure on Ω. Then, it holds*

$$\lim_{n \to \infty} \mathbb{E}\left[W_{\Omega}^p \left(\sum_{i=1}^n \int_0^\infty \delta_{B_s^i} ds \right) \right] / \left(n \operatorname{Cap}(\Omega) \right)^{1-p/(d-2)} = \mathsf{c}(\mathcal{I}, d, p)|\Omega|. \qquad (1.4)$$

The notation $\mathcal{I}$ in $\mathsf{c}(\mathcal{I}, d, p)$ stands for "interlacement". Although strictly speaking in the statement above there is no Brownian interlacement process, its existence is deduced by showing first (Theorem 4.4) that a similar limit holds for the Brownian interlacement, by exploiting its stronger self-similarity properties.

Our second main result links the $\mathsf{c}(\mathcal{I}, d, p)$ with the occupation measure of a stationary Brownian motion on the flat torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Theorem 1.2. *Let $d \in \{3, 4\}$ and $p \in (0, (d-2)/2)$, or $d \geq 5$ and $p > 0$. Let $(B_t)_{t \geq 0}$ be a stationary Brownian motion on $\mathbb{T}^d$, i.e. with uniform initial law. Then,*

$$\limsup_{T \to \infty} \mathbb{E}\left[W_{\mathbb{T}^d}^p \left(\int_0^T \delta_{B_t} dt \right) \right] / T^{1-p/(d-2)} \leq \mathsf{c}(\mathcal{I}, d, p). \qquad (1.5)$$

with $\mathsf{c}(\mathcal{I}, d, p)$ as in Theorem 1.1.

Although both our main results investigate the expected value of the Wasserstein cost of order p, we are able to give a concentration result in Proposition 6.1 which can be used to improve to a.s. convergence for a certain range of p's. This could be relevant in view of applications, e.g. in Monte Carlo methods, where one usually simulates a single sample path instead of averaging over an independent family of paths.

1.2. **Comments on the proof technique.** As already mentioned, our results adapt, for the first time in the study of occupation measures for diffusion processes, the combination of geometric and analytic techniques developed for the assignment problem in the recent works [23, 6, 25]. In particular, in order to establish Theorem 1.1, we actually prove a rather general result, Theorem A.1, that applies to stationary random measures under suitable concentration assumptions, and we believe could be of interest also in other settings. However, since the proof is indeed a generalization of the arguments already employed for the assignment problem, we defer it to Appendix A.

The main technical novelty instead, we believe, comes from its application to prove Theorem 1.2. Indeed, we are able to argue that, as T increases, on a sufficiently small scale, but larger than the critical scale $T^{-1/(d-2)}$, the "local" Wasserstein cost of order p converges exactly (after a suitable renormalization) towards the same constant $\mathsf{c}(\mathcal{I}, d, p)$ of the Brownian interlacement. This is the content of Proposition 5.1, which constitutes the main step in the proof of Theorem 1.2. However, when transferring this exact limit from the local problems to that on the whole $\mathbb{T}^d$, we obtain only an inequality, due to the sub-additivity properties of the transportation cost.

In the proof of Proposition 5.1, which is split into small steps for the reader's convenience, we employ several technical tools, not only from the random assignment problem literature, but also from the random interlacement theory [36, 17]. In particular, we also make use of quantitative mixing rates in the total variation distance for Brownian motion on $\mathbb{T}^d$, in order to efficiently split a single trajectory into a large number of independent ones, which will constitute an approximation of Brownian interlacements. However, it turns out that we also need some fine asymptotics for the hitting distribution of a small ball (in addition to the hitting probability). This is achieved in Proposition 3.7, via a combination of the functional analytic methods for estimating the Wasserstein cost, with classical potential-theoretic arguments.

1.3. **Further questions and conjectures.** Our result raises some further natural questions:

(1) Is the lim sup actually a limit (and equals the right hand side) in (1.5)? When compared with the assignment problem, the situation looks very similar, for sharp (conjectured) constants associated to upper and lower limits are known to exist, see e.g. [7, 25], but in general it is not known whether they coincide. A few exceptions are given by the case $d = p = 2$ [6], and the concave case $d = 1$, $0 < p < 1/2$ [24], but the proofs rely on rather special properties of the assignment in such cases.

(2) Does Theorem 1.2 extend to the case of more general diffusion processes? Let us consider for example the setting of a weighted Riemannian manifolds (M, g, σ), where σ is a (smooth, bounded from above and below) probability density with respect to the Riemannian volume measure, and B denotes the diffusion generated by the corresponding weighted Laplacian (so that the invariant measure is σ). Then, we conjecture that the right hand side in (1.5) should be multiplied by an additional integral term:

$$\limsup_{T \to \infty} \mathbb{E}\left[W_{\mathbb{T}^d}^p \left(\int_0^T \delta_{B_t} dt \right) \right] / T^{1-p/(d-2)} \leq \mathsf{c}(\mathcal{I}, d, p) \int_M f^{1-p/(d-2)}, \qquad (1.6)$$

where

$$f(x) = \lim_{\varepsilon \to 0} \frac{\mathrm{Cap}_M\left(B_\varepsilon(x) \right)}{\mathrm{Cap}_{\mathbb{R}^d}(B_\varepsilon(0))}. \qquad (1.7)$$

It would be already interesting to prove (or disprove) this conjecture in the case of $M = \mathbb{R}^d$ endowed with a Gaussian weight σ, so that the Ornstein-Uhlenbeck process is the resulting diffusion, and p is small enough, e.g. $0 < p < d - 2$.

(3) How should our results be modified in the cases $d \leq 2$, or $d \in \{3, 4\}$ but $p \geq (d-2)/2$? Let us mention that, for a stationary Brownian motion on $\mathbb{T}^d$ it is proved in [28] that, for any $p \geq 1$,

$$\mathbb{E}\left[W_{\mathbb{T}^d}^p \left(\int_0^T \delta_{B_t} dt \right) \right] \sim \begin{cases} T \cdot T^{-p/2} & \text{if } d \leq 3, \\ T^{1-p/2} \cdot (\log T)^{p/2} & \text{if } d = 4, \\ T^{1-p/(d-2)} & \text{if } d \geq 5, \end{cases} \qquad (1.8)$$

where the notation $f(T) \sim g(T)$ means that for every T (large enough), it holds $c^{-1} \leq f(T)/g(T) \leq c$ for some constant $c \in (0, \infty)$. For the concave case $0 < p < 1$, we conjecture, by analogy with the assignment problem [9, 24], that

$$\mathbb{E}\left[W_{\mathbb{T}^d}^p \left(\int_0^T \delta_{B_t} dt \right) \right] \sim T \cdot T^{-p/2} \quad \text{if } d \leq 2 \text{ or } d = 3 \text{ and } p > 1/2. \qquad (1.9)$$

In the remaining case $d = 3$, $p = 1/2$, we conjecture instead

$$\mathbb{E}\left[W_{\mathbb{T}^3}^{1/2} \left(\int_0^T \delta_{B_t} dt \right) \right] \sim \sqrt{T \log T}. \qquad (1.10)$$

(4) In Proposition 6.1, we show that the random variable

$$W_{\mathbb{T}^d}^p \left(\int_0^T \delta_{B_t} dt \right) / T^{1-p/(d-2)} \qquad (1.11)$$

concentrates around its expectation if p is sufficiently small (in particular, if $p < (d-2)/2$). It is natural to conjecture that such concentration holds for every $p > 0$ (if $d \geq 5$). Let us point out that a similar issue arises for the assignment problem, where the corresponding concentration results are known to hold for $0 < p < d$ (if $d \geq 3$) but one expects that it should be true for every p.

1.4. **Structure of the paper.** In Section 2, we introduce all the relevant notation and recall some basic facts, in particular concerning optimal transportation theory. In Section 3 we focus on Brownian motion on $\mathbb{R}^d$ and the torus $\mathbb{T}^d$, establishing, among various useful bounds, Proposition 3.7 about small-ball hitting (probabilities and) hitting distributions. Section 4 is devoted to the proof of Theorem 1.1 although, as already mentioned, most of the general tools from the assignment problem literature are actually recalled and elaborated in Appendix A and Appendix B. The main body of the article ends with section Section 5, devoted to the proof of Theorem 1.2 and Section 6, focusing on the concentration properties of the random Wasserstein transport cost and (non-sharp) lower bounds.

2. Notation and Basic facts

2.1. **General notation.** Given two real-valued functions f, g (e.g. of a real variable x), we write $f(x) \lesssim g(x)$ if there exists a strictly positive and finite constant c (independent of x) such that $f(x) \leq cg(x)$ for every x (notice that we do not require f, g to be positive) and $f(x) \gtrsim g(x)$ if $g(x) \lesssim f(x)$. We also write $f(x) \sim g(x)$ if $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$. The notation $f(x) \ll g(x)$ means that $\lim_x f(x)/g(x) = 0$, for $x \to \infty$ or $x \to 0$ (it will be clear from the context). Since we are going to use functions with several parameters, such as exponents p, q or additional positive parameters ε, δ etc., we may stress the dependence of the constant c with respect to these parameters by adding them as subscripts, e.g. $f_p(x) \lesssim_p g_p(x)$ means that the constant $c = c(p)$ may depend on p. Despite this, we warn the reader that in order to keep the notation as light as possible, we will not stress such dependence in many statements. Again, to keep the statements simple, we do not always stress the fact that the various constants appearing are strictly positive and finite.

Given a set S, we write $\sharp S$ for the number of its elements and χ_S for its indicator function. In case $S \in \mathcal{A}$ is an event on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we use the notation $I_A = \chi_A$. For $L > 0$, we let $Q_L = [-L/2, L/2]^d$ for the cube of side length L and $D_L(y) = \{x : |x - y| \leq L\} \subseteq \mathbb{R}^d$ for the ball of radius L centred at y, where we denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^d$. We consider $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ endowed with the flat distance

$$\mathsf{d}_{\mathbb{T}^d}(x,y) = \inf_{z \in \mathbb{Z}^d} |x - y - z|, \tag{2.1}$$

and identify it (as a set) with $Q_1 = [-1/2, 1/2]^d \subseteq \mathbb{R}^d$. We use the same notation

$$D_L(y) = \{x : \mathsf{d}_{\mathbb{T}^d}(x,y) \leq L\}, \tag{2.2}$$

so that the ball $D_L(0)$ on $\mathbb{T}^d$ is naturally identified (as a set) with the ball in $\mathbb{R}^d$.

We use the notation $|A|$ for the Lebesgue measure of a Borel set $A \subseteq \mathbb{R}^d$ or $A \subseteq \mathbb{T}^d$, $\mathsf{d}(\cdot, A)$ for the distance function from A, $\mathrm{diam}(A)$ for its diameter, and $\int_A f = \int_A f(x)dx$ for the Lebesgue integral of a function f on A. Given a Borel measure μ on $\mathbb{R}^d$ or $\mathbb{T}^d$, we write $\mathrm{trans}_x \mu$ for its translation by $x \in \mathbb{R}^d$, i.e., $\mathrm{trans}_x \mu(A) = \mu(A - x)$. In case of $\mathbb{R}^d$, we write $\mathrm{dil}_\rho \mu$ for the dilation by a factor $\rho \in (0, \infty)$, i.e., $\mathrm{dil}_\rho \mu(A) = \mu(A/\rho)$.

For a (sufficiently smooth) function ϕ on $\mathbb{R}^d$ or $\mathbb{T}^d$, we use the notation $\nabla\phi$ for its gradient, $\nabla \cdot \phi$ for the divergence, $\Delta\phi$ for the Laplacian.

2.2. **Occupation measures and hitting times.** Given a continuous curve $x = (x_t)_{t \geq 0}$ with values in $\mathbb{R}^d$ or $\mathbb{T}^d$, we write

$$\mu_T^x = \int_0^T \delta_{x_t} dt \tag{2.3}$$

for its occupation measure up to time $T \in [0, \infty]$. For a (compact) set $K \subseteq \mathbb{R}^d$, define its first hitting time

$$\tau_K x := \inf \{t > 0 : x_t \in K\}. \tag{2.4}$$

When x is understood, we simply write τ_K, in particular we use often the notation $\mu_{\tau_K}^x = \mu_{\tau_K x}^x$. We also write $\theta = (\theta_s)_{s \geq 0}$ for the shift operator acting on continuous curves as $\theta_s x = (x_{t+s})_{t \geq 0}$ for every $s \geq 0$. We apply θ also when $s = s(x)$, i.e., $\theta_{s(x)} x = (x_{t+s(x)})_{t \geq 0}$. This notation is particularly useful in order to define the first hitting time of a set K' after τ_K, which is simply given by $\tau_K + \tau_{K'} \circ \theta_{\tau_K}$. We notice the identity, valid for any x and K,

$$\mu_T^x \llcorner K = \mu_{T-\tau_K}^{\theta_{\tau_K} x} \llcorner K, \tag{2.5}$$

with $\tau_K = \tau_K x$ (valid also for $T = \infty$ with the convention that $\infty - \infty = 0$).

2.3. **Negative Sobolev norms.** Given a bounded domain $\Omega \subseteq \mathbb{R}^d$ (or $\Omega \subseteq \mathbb{T}^d$) with Lipschitz boundary and $p \in (1, \infty)$, with Hölder conjugate $p' = p/(p-1)$, we write $\|f\|_{L^p(\Omega)}$ for the Lebesgue norm of f, and

$$\|f\|_{W^{-1,p}(\Omega)} = \sup_{\|\nabla \phi\|_{L^q(\Omega)} \leq 1} \int_\Omega f\phi \tag{2.6}$$

for the negative Sobolev norm. Notice in particular that if $\|f\|_{W^{-1,p}(\Omega)} < \infty$ then $\int_\Omega f = 0$. In this case we may and will restrict the supremum to functions ϕ having also average zero.

Let us recall that we can bound the $W^{-1,p}$ norm by the L^p norm via the following Poincaré inequality.

Lemma 2.1. *Let Ω be a bounded domain with Lipschitz boundary and let $f : \Omega \to \mathbb{R}$ such that $\int_\Omega f = 0$. Then, for every $p > 1$,*

$$\|f\|_{W^{-1,p}(\Omega)} \lesssim |\Omega|^{\frac{1}{d}} \|f\|_{L^p(\Omega)}. \tag{2.7}$$

Moreover, the implicit constant is invariant by dilations of Ω.

2.4. **Total variation distance.** Given two random variables X, Y taking values in a measurable space $(E, \mathcal{E})$ (possibly defined on different spaces) with laws $\mathbb{P}_X$, $\mathbb{P}_Y$, their total variation distance is

$$\mathsf{d}_{TV}(\mathbb{P}_X, \mathbb{P}_Y) = \sup_{A \in \mathcal{E}} \mathbb{P}(X \in A) - \mathbb{P}(Y \in A) \in [0, 1] \tag{2.8}$$

Recall that, for any $f : E \to \mathbb{R}$ measurable and bounded, it holds

$$\mathbb{E}[f(X)] - \mathbb{E}[f(Y)] \leq \sup_{x \in E} |f(x)| \, \mathsf{d}_{TV}(X, Y). \tag{2.9}$$

We use the following well-known contraction property for the total variation distance: if N is a Markov kernel from E to a measurable space $(F, \mathcal{F})$, then

$$\mathsf{d}_{TV}(N_\sharp \mathbb{P}_X, N_\sharp \mathbb{P}_Y) \leq \mathsf{d}_{TV}(\mathbb{P}_X, \mathbb{P}_Y), \tag{2.10}$$

where $N_\sharp \mathbb{P}_X$ denotes the (push-forward) probability given by $N_\sharp \mathbb{P}_X(A) = \mathbb{E}[N(X, A)]$ for $A \in \mathcal{F}$.

In particular, we will use the following *data-processing* inequality (for the total variation distance): (U, V, W) are random variables that form a Markov chain

$$U \to V \to W \tag{2.11}$$

(possibly taking values in different measurable spaces $(E, \mathcal{E})$, $(F, \mathcal{F})$, $(G, \mathcal{G})$ with transition kernels $N_{F \to E}$, $N_{F \to G}$ and $\tilde{V}$ is any random variable taking values in F, then

$$\mathsf{d}_{TV}\left(\mathbb{P}_{(U,V,W)}, N_{F \to E} \otimes \mathbb{P}_{\tilde{V}} \otimes N_{F \to G}\right) \leq \mathsf{d}_{TV}(V, \tilde{V}), \tag{2.12}$$

where $N_{F \to E} \otimes \mathbb{P}_{\tilde{V}} \otimes N_{F \to G}$ is a notation for the probability measure corresponding to a Markov chain

$$\tilde{U} \to \tilde{V} \to \tilde{W} \tag{2.13}$$

with $\tilde{U}$ having the kernel $N_{F \to E}$ as conditional law given $\tilde{V}$, and $\tilde{W}$ having kernel $N_{F \to G}$ as conditional law given $\tilde{V}$.

2.5. Optimal transport. Given $p > 0$, a Borel subset $\Omega \subseteq \mathbb{R}^d$ (or $\Omega \subseteq \mathbb{T}^d$) and two positive Borel measures μ, λ with $\mu(\Omega) = \lambda(\Omega) \in (0, \infty)$ and finite p-th moments, the optimal transportation (or Wasserstein) cost of order p between μ and λ is defined as the quantity

$$W_\Omega^p(\mu, \lambda) = \min_{\pi \in \mathcal{C}(\mu, \lambda)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathsf{d}(x, y)^p d\pi(x, y) \tag{2.14}$$

where d denotes the Euclidean distance in case $\Omega \subseteq \mathbb{R}^d$, and the flat distance in the case $\Omega \subseteq \mathbb{T}^d$. The set $\mathcal{C}(\mu, \lambda)$ is the set of couplings between μ and λ i.e., positive Borel measures on $\Omega \times \Omega$ such that the first and second marginals are respectively μ and λ. Moreover, if $\mu(\Omega) = \lambda(\Omega) = 0$, we define $W_\Omega^p(\mu, \lambda) = 0$, while if $\mu(\Omega) \neq \lambda(\Omega)$, we let $W_\Omega^p(\mu, \lambda) = \infty$.

Let us collect some basic facts related to the Wasserstein cost. Proofs of the simpler ones can be found in any of the standard references on the subject [2, 39, 33].

Let us notice that if, $\Omega \subseteq \mathbb{T}^d$ and $\mathrm{diam}(\Omega) < 1/2$, the Wasserstein cost can be equivalently computed on $\mathbb{T}^d$ or on $\mathbb{R}^d$, by identify Ω with a subset of $[0, 1]^d$.

For every constant $a > 0$,

$$W_\Omega^p(a\mu, a\lambda) = a W_\Omega^p(\mu, \lambda), \tag{2.15}$$

which always allows to reduce to the case of probability measures $\mu(\Omega) = \lambda(\Omega) = 1$. For $r \geq 1$, by Hölder inequality,

$$W_\Omega^p(\mu, \lambda) \leq \mu(\Omega)^{1-1/r} \left(W_\Omega^{pr}(\mu, \lambda) \right)^{1/r}. \tag{2.16}$$

If $\mu(\Omega) = \lambda(\Omega)$,

$$W_\Omega^p(\mu, \lambda) \leq \mathrm{diam}(\Omega)^p \mu(\Omega), \tag{2.17}$$

which can be slightly improved to

$$W_\Omega^p(\mu, \lambda) \leq \mathrm{diam}(\Omega)^p \mathsf{d}_{TV}(\mu, \lambda). \tag{2.18}$$

Notice also the lower bound

$$W_\Omega^p(\mu, \lambda) \leq \int_\Omega \mathsf{d}(x, \mathrm{supp}(\lambda))^p d\mu(x). \tag{2.19}$$

For $p \geq 1$, the quantity $(W_\Omega^p(\mu, \lambda))^{1/p}$ is a distance, while for $p \in (0, 1)$, $W_\Omega^p(\mu, \lambda)$ is already a distance, hence enjoying the triangle inequality. We will also use the following sub-additivity inequality

$$W_\Omega^p \left(\int_E \mu_z d\sigma(z), \int_E \lambda_z d\sigma(z) \right) \leq \int_E W_\Omega^p(\mu_z, \lambda_z) d\sigma(z) \tag{2.20}$$

where $(E, \mathcal{E}, \sigma)$ is a measure space and $(\mu_z)_{z \in E}$, $(\lambda_z)_{z \in E}$ are measurable families. Notice that, for the right hand side to be finite, it is necessary that $\mu_z(\Omega) = \lambda_z(\Omega)$ for σ-a.e. $z \in E$.

To keep the notation simple, given a Borel subset $\tilde{\Omega} \subseteq \Omega$, we write

$$W_{\tilde{\Omega}}^p(\mu, \lambda) = W_{\tilde{\Omega}}^p(\mu \llcorner \tilde{\Omega}, \lambda \llcorner \tilde{\Omega}), \tag{2.21}$$

which also coincides with $W_\Omega^p(\chi_{\tilde{\Omega}} \mu, \chi_{\tilde{\Omega}} \lambda)$. Moreover, if a measure is absolutely continuous with respect to Lebesgue measure, we only write its density. For example,

$$W_\Omega^p(\mu, \mu(\Omega)/|\Omega|) \tag{2.22}$$

denotes the transportation cost between $\mu \llcorner \Omega$ to the uniform measure on Ω with total mass $\mu(\Omega)$. To further simplify the notation, in this special case of transporting towards the uniform measure, we write

$$W_\Omega^p(\mu) = W_\Omega^p(\mu, \mu(\Omega)/|\Omega|). \tag{2.23}$$

Let us recall some lemmas from [25]. The first one is a "geometric" sub-additivity result, which follows straightforwardly combining the triangle inequality, (2.20) and the elementary inequality, for $p > 0$, there exists $c = c(p)$ such that

$$(a+b)^p \le (1+\varepsilon)a^p + \frac{c}{\varepsilon^{(p-1)^+}}b^p \qquad \forall a, b \ge 0 \text{ and } \varepsilon \in (0,1). \tag{2.24}$$

Lemma 2.2. *Let $(E, \mathcal{E}, \sigma)$ be a measure space, let $\Omega \subset \mathbb{R}^d$ (or $\Omega \subseteq \mathbb{T}^d$) and $(\Omega_z)_{z \in E}$ be a measurable collection of Borel subsets of Ω. Let μ, λ be measures on Ω such that*

$$\int_E \chi_{\Omega_z} d\sigma(z) = 1, \quad \mu + \lambda\text{-a.e.} \tag{2.25}$$

Then, for every $p > 0$, there exists $c = c(p)$ such that, for every $\varepsilon \in [0, 1)$,

$$\begin{aligned}
W_\Omega^p\left(\mu, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right) &\le (1+\varepsilon) \int_E W_{\Omega_z}^p\left(\mu, \frac{\mu(\Omega_z)}{\lambda(\Omega_z)}\lambda\right) d\sigma(z) \\
&\quad + \frac{C}{\varepsilon^{(p-1)^+}} W_\Omega^p\left(\int_E \frac{\mu(\Omega_z)}{\lambda(\Omega_z)}\chi_{\Omega_z} d\sigma(z)\lambda, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right).
\end{aligned} \tag{2.26}$$

Letting in particular λ be the Lebesgue measure, we have, for $p > 0$,

$$W_\Omega^p(\mu) \le (1+\varepsilon) \int_E W_{\Omega_z}^p(\mu) d\sigma(z) + \frac{C}{\varepsilon^{(p-1)^+}} W_\Omega^p\left(\int_E \frac{\mu(\Omega_z)}{|\Omega_z|}\chi_{\Omega_z} d\sigma(z)\right). \tag{2.27}$$

The second lemma relies on a PDE argument and applies to sufficiently nice domains.

Lemma 2.3. *Assume that $\Omega \subseteq \mathbb{R}^d$ (or $\Omega \subseteq \mathbb{T}^d$) is bounded, connected and with Lipschitz boundary. If μ and λ are measures on Ω with $\mu(\Omega) = \lambda(\Omega)$, absolutely continuous with respect to the Lebesgue measure and $\inf_\Omega \lambda > 0$, then, for every $p > 1$,*

$$W_\Omega^p(\mu, \lambda) \lesssim \frac{1}{\inf_\Omega \lambda^{p-1}} \|\mu - \lambda\|_{W^{-1,p}(\Omega)}^p. \tag{2.28}$$

In the lemma above, $\inf_\Omega \lambda$ is the infimum of the density of λ with respect to Lebesgue measure.

Combining the above result with Lemma 2.1, we obtain that, for $p > 1$,

$$W_\Omega^p(\mu, \lambda) \lesssim \frac{|\Omega|^{p/d}}{\inf_\Omega \lambda^{p-1}} \|\mu - \lambda\|_{L^p(\Omega)}^p, \tag{2.29}$$

where the implicit constant depends is invariant with respect to dilations of Ω.

Remark 2.4. In fact, Lemma 2.3 holds as well in case of compact smooth connected Riemannian manifolds, see [4]. In particular, given the unit sphere $\Omega = \partial D_1 \subseteq \mathbb{R}^d$ and letting $\lambda = \tilde{e}_{D_1}$ be the uniform (probability) distribution on the sphere (the notation will be clarified below), it holds

$$W_{\partial D_1}^p(\mu, \tilde{e}_{D_1}) \lesssim \sup\left\{\int_{\partial D_1} \phi d\mu : \int_{\partial D_1} \phi = 0, \quad \int_{\partial D_1} |\nabla \phi|^{p'} \le 1\right\}. \tag{2.30}$$

A simple scaling argument shows also that, for $\ell > 0$,

$$W_{\partial D_\ell}^p(\mu, \tilde{e}_{D_\ell}) \lesssim \ell^p \sup\left\{\int_{\partial D_\ell} \phi(x/\ell) d\mu(x) : \int_{\partial D_\ell} \phi = 0, \quad \int_{\partial D_1} |\nabla \phi|^{p'} \le 1\right\}. \tag{2.31}$$

where $\tilde{e}_{D_\ell}$ denotes the uniform probability distribution on ∂D_ℓ.

Let us notice that, by (2.20), it always holds

$$W_\Omega^p\left(\lambda+\mu,\tilde\lambda+\mu\right) \le W_\Omega^p\left(\lambda,\tilde\lambda\right) + W_\Omega^p(\mu,\mu) = W_\Omega^p\left(\lambda,\tilde\lambda\right). \tag{2.32}$$

It is natural to ask whether under suitable smoothness assumptions on μ one can improve this bound. The following result is a special case of [25, Proposition 2.9] for the case where μ and $\tilde\lambda$ have constant densities.

Proposition 2.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, λ be any finite measure on Ω and $h > 0$. Then, for every $p > d/(d-1)$, it holds*

$$W_\Omega^p\left(\lambda + \frac{h}{|\Omega|}\right) \lesssim \operatorname{diam}(\Omega)^p \lambda(\Omega)\left(\frac{\lambda(\Omega)}{h}\right)^{p/d}, \tag{2.33}$$

where the implicit constant depends on Ω only, but is invariant by rescaling of Ω.

The condition $p > d/(d-1)$ appears in (2.5) because of an application of the Sobolev embedding in Ω. If $p \le d/(d-1)$ one can deduce suitable modifications of (2.33), as discussed in [25, Remark 2.10]. However, we may dispense of these cases by a simple application of (2.16): if $0 < p \le d/(d-1)$ and r is chosen sufficiently large so that $pr > d/(d-1)$, then

$$\begin{aligned}
W_\Omega^p\left(\lambda + \frac{h}{|\Omega|}\right) &\lesssim (\lambda(\Omega) + h)^{1-1/r} \operatorname{diam}(\Omega)^p \frac{\lambda(\Omega)^{1/r+p/d}}{h^{p/d}} \\
&\lesssim \operatorname{diam}(\Omega)^p \lambda(\Omega)\left(\frac{\lambda(\Omega)}{h}\right)^{p/d} + \operatorname{diam}(\Omega)^p h\left(\frac{\lambda(\Omega)}{h}\right)^{1/r+p/d}.
\end{aligned} \tag{2.34}$$

It is also natural to ask whether (2.20) can be reversed. Indeed, given two measures μ, λ, and setting

$$u := \frac{\lambda(\Omega)}{\mu(\Omega) + \lambda(\Omega)}, \tag{2.35}$$

then applying the triangle inequality, (2.24), (2.20) and (2.17) easily yield

$$\begin{aligned}
W_\Omega^p(\mu) &\le (1+\varepsilon)W_\Omega^p\left((1-u)(\mu+\lambda)\right) + \frac{c}{\varepsilon^{(p-1)^+}} W_\Omega^p\left(\mu,(1-u)(\mu+\lambda)\right) \\
&\le (1+\varepsilon)W_\Omega^p\left(\mu+\lambda\right) + \frac{c}{\varepsilon^{(p-1)^+}} \operatorname{diam}(\Omega)^p\lambda(\Omega).
\end{aligned} \tag{2.36}$$

Combining this derivation with Proposition 2.5 leads to the following bounds.

Lemma 2.6. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, let $p > d/(d-1)$ and $\varepsilon \in (0,1)$. Then there exists $c = c(\Omega, d, p, \varepsilon)$ such that the following holds. If μ and λ are measures on Ω, then*

$$W_\Omega^p(\mu+\lambda) \le (1+\varepsilon)W_\Omega^p(\mu) + c\operatorname{diam}(\Omega)^p\lambda(\Omega)\left(\frac{\lambda(\Omega)}{\mu(\Omega)}\right)^{p/d}, \tag{2.37}$$

and

$$W_\Omega^p(\mu) \le (1+c\delta)(1+\varepsilon)W_\Omega^p(\mu+\lambda) + c\operatorname{diam}(\Omega)^p\lambda(\Omega)\left(\frac{\lambda(\Omega)}{\delta\mu(\Omega)}\right)^{p/d} \tag{2.38}$$

provided that δ satisfies

$$\lambda(\Omega)/\mu(\Omega) \le \delta \le 1/c. \tag{2.39}$$

Moreover, the constant c and is invariant by rescaling of Ω.

Proof. The first inequality is straightforward from the triangle inequality and (2.24),

$$W_\Omega^p(\mu+\lambda) \le (1+\varepsilon)W_\Omega^p(\mu+\lambda, \mu(\Omega)/|\Omega| + \lambda) + c(p,\varepsilon)W_\Omega^p(\mu(\Omega)/|\Omega| + \lambda), \tag{2.40}$$

and an application of Proposition 2.5. Let us focus on the second one. Without loss of generality, we can assume that $c \geq 2$, hence $\delta \leq 1/2$. Write

$$u := \frac{\lambda(\Omega)}{\mu(\Omega) + \lambda(\Omega)} < \delta. \tag{2.41}$$

By the triangle inequality and (2.24), we have, for $c = c(p, \varepsilon)$,

$$W_\Omega^p(\mu) \leq (1 + \varepsilon)W_\Omega^p((1 - u)(\mu + \lambda)) + cW_\Omega^p(\mu, (1 - u)(\mu + \lambda)). \tag{2.42}$$

Since

$$W_\Omega^p((1 - u)(\mu + \lambda)) = (1 - u)W_\Omega^p(\mu + \lambda) \leq W_\Omega^p(\mu + \lambda), \tag{2.43}$$

we focus on the second term. By (2.20) and repeated applications of the triangle inequality and (2.24) with $\varepsilon = 1/2$, we find

$$
\begin{aligned}
W_\Omega^p(\mu, (1 - u)(\mu + \lambda)) &\leq W_\Omega^p((1 - 2\delta)\mu, (1 - 2\delta)\mu) + W_\Omega^p(2\delta\mu, (2\delta - u)\mu + (1 - u)\lambda) \\
&\leq W_\Omega^p(2\delta\mu, (2\delta - u)\mu + (1 - u)\lambda) \\
&\lesssim W_\Omega^p(2\delta\mu, (2\delta - u)\mu(\Omega)/|\Omega| + (1 - u)\lambda) \\
&\quad + W_\Omega^p((2\delta - u)\mu(\Omega)/|\Omega| + (1 - u)\lambda, (2\delta - u)\mu + (1 - u)\lambda) \\
&\lesssim W_\Omega^p(2\delta\mu) + W_\Omega^p((2\delta - u)\mu(\Omega)/|\Omega| + (1 - u)\lambda) \\
&\quad + W_\Omega^p((2\delta - u)\mu),
\end{aligned}
\tag{2.44}
$$

where the implicit constants depend on p only. By Proposition 2.5, we obtain

$$
\begin{aligned}
W_\Omega^p\left(\frac{(2\delta - u)\mu(\Omega)}{(1 - u)|\Omega|} + \lambda\right) &\lesssim \operatorname{diam}(\Omega)^p\lambda(\Omega)\left(\frac{\lambda(\Omega)(1 - u)}{(2\delta - u)\mu(\Omega)}\right)^{p/d} \\
&\lesssim \operatorname{diam}(\Omega)^p\lambda(\Omega)\left(\frac{\lambda(\Omega)}{\delta\mu(\Omega)}\right)^{p/d}.
\end{aligned}
\tag{2.45}
$$

where the implicit constants depend here also on Ω (but are invariant with respect to rescaling). Collecting all the terms, for some constant $\bar{c} = \bar{c}(\Omega, d, p, \varepsilon)$

$$W_\Omega^p(\mu) \leq (1 + \varepsilon)W_\Omega^p(\mu + \lambda) + \bar{c}\delta W_\Omega^p(\mu) + \bar{c}\operatorname{diam}(\Omega)^p\lambda(\Omega)\left(\frac{\lambda(\Omega)}{\delta\mu(\Omega)}\right)^{p/d}. \tag{2.46}$$

Subtracting both sides the term $c\delta W_\Omega^p(\mu)$ yields the thesis, by estimating $1/(1 - \bar{c}\delta) \leq 1 + 2\bar{c}\delta$ provided that $\delta\bar{c}$ is sufficiently small, which can be easily achieved by choosing a final constant c in the thesis that is sufficiently large. $\square$

Remark 2.7. If $0 < p \leq d/(d - 1)$, we cannot apply directly Proposition 2.5, but we may rather use (2.34), by choosing r sufficiently large such tha $pr > d/(d - 1)$. This leads to modified bounds, respectively by adding the additional terms

$$\operatorname{diam}(\Omega)^p\mu(\Omega)\left(\frac{\lambda(\Omega)}{\mu(\Omega)}\right)^{1/r + p/d} \quad \text{and} \quad \operatorname{diam}(\Omega)^p(\delta\mu(\Omega))\left(\frac{\lambda(\Omega)}{\delta\mu(\Omega)}\right)^{1/r + p/d} \tag{2.47}$$

in the right hand sides.

3. Brownian motion

3.1. **Brownian motion on $\mathbb{R}^d$.** Throughout this section, we let $d \geq 3$. Given $x \in \mathbb{R}^d$ write P_x for the law of a d-dimensional Brownian motion $(B_t)_{t \geq 0}$ staring at x, i.e., such that the law of $B_0 = x$. Similarly, write E_x for the expectation with respect to P_x. We extend such notations for a probability measure ν on $\mathbb{R}^d$, writing

$$P_\nu = \int_{\mathbb{R}^d} P_x d\nu(x), \quad E_\nu = \int_{\mathbb{R}^d} E_x d\nu(x). \tag{3.1}$$

The Brownian motion process $(B_t)_{t\geq 0}$ is Markov with generator $\frac{1}{2}\Delta$, and transition density

$$p_t(x,y) = \exp\left(-\frac{|x-y|^2}{2t}\right)\frac{1}{(2\pi t)^{d/2}}. \tag{3.2}$$

The associated Green function is

$$g(x,y) = \int_0^\infty p_t(x,y)\,dt = \frac{c}{|x-y|^{d-2}}, \tag{3.3}$$

for a suitable constant $c = c(d) \in (0,\infty)$. It is elementary to check that for every $\rho > 0$, given a Brownian motion $(B_t)_{t\geq 0}$ starting at 0, its rescaled process $B^\rho := (\rho B_{t/\rho^2})_{t\geq 0}$ has the same law of $(B_t)_{t\geq 0}$. This e.g. yields the following identity in law between occupation measures:

$$\mu_T^B \overset{law}{=} \mathrm{dil}_\rho\,\mu_{T/\rho^2}^B. \tag{3.4}$$

For a compact $K \subseteq \mathbb{R}^d$ with Lipschitz boundary, denote with e_K its equilibrium measure, which is defined as the Borel measure μ concentrated on K such that its Newtonian potential

$$u_\mu(x) := \int_{\mathbb{R}^d} g(x,y)d\mu(y), \tag{3.5}$$

is identically 1 for every $x \in K$, where g is Green's function (3.3) It is well-known [34] that such a measure exists and is unique, also for sets X with less regular boundaries (but it requires the notion of regular points). Its total mass $e_K(\mathbb{R}^d)$ is called the capacity $\mathrm{Cap}(K)$ of K. We write $\tilde{e}_K = \mathrm{Cap}(K)^{-1}e_K$ for its normalization to a probability measure, which we call the normalized equilibrium measure. With this notation, we have the following result.

Proposition 3.1. Let $K \subseteq \mathbb{R}^d$. Then, $\mathrm{Cap}(uK) = u^{d-2}\mathrm{Cap}(K)$ for every $u > 0$, and

$$E_{\tilde{e}_K}\left[\mu_\infty^B(K)\right] = |K|. \tag{3.6}$$

Let $\tilde{K} \subseteq K \subseteq \mathbb{R}^d$. Then,

$$P_{e_K}(\tau_{\tilde{K}} < \infty, B_{\tau_{\tilde{K}}} \in A) = P_{e_{\tilde{K}}}(B_0 \in A). \tag{3.7}$$

In particular,

$$P_{\tilde{e}_K}(\tau_{\tilde{K}} < \infty) = \mathrm{Cap}(\tilde{K})/\mathrm{Cap}(K). \tag{3.8}$$

We will use throughout that in case of balls $\tilde{K} = D_\ell$, $K = D_L$, with $\ell < L$, the normalized equilibrium measures are the uniform probability distributions on the respective boundaries. Moreover, given any $x \in \partial D_L$, one can prove that

$$P_x(\tau_{D_\ell} < \infty) = (\ell/L)^{d-2}. \tag{3.9}$$

We end this section collecting some elementary upper and lower bounds on the occupation measures.

Lemma 3.2. Let $d \geq 3$. For every bounded Borel set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, it holds

$$E_x\left[\mu_\infty^B(A)\right] \lesssim \min\left\{\mathrm{diam}(A)^2, \frac{|A|}{\mathsf{d}(x,A)^{d-2}}\right\}. \tag{3.10}$$

For every $q \geq 1$,

$$E_x\left[\mu_\infty^B(A)^q\right] \lesssim_q \mathrm{diam}(A)^{2q}. \tag{3.11}$$

Proof. Without loss of generality, we can assume that A is closed. Consider first the case $q = 1$. Assume first that $\mathsf{d}(x,A) = 0$. Then, $A \subseteq \{|y| \leq \mathrm{diam}(A)\}$, hence

$$\int_0^\infty P_x(B_t \in A)dt = \int_A \frac{c(d)}{|x-y|^{d-2}}dy \lesssim \int_0^{\mathrm{diam}(A)} \frac{r^{d-1}}{r^{d-2}}dr = \mathrm{diam}(A)^2. \tag{3.12}$$

If instead $\mathsf{d}(x, A) > 0$, write

$$\mu_\infty^B = \mu_\infty^{\theta_{\tau_A} B} \tag{3.13}$$

and use that $\theta_{\tau_A} B$ is a Brownian motion (by the strong Markov property) with initial law ν, concentrated on A (actually, on ∂A). Hence,

$$E_x \left[\mu_\infty^B(A) \right] = \int_{\partial A} E_y \left[\mu_\infty^B(A) \right] d\nu(y) \lesssim \mathrm{diam}(A)^2. \tag{3.14}$$

To obtain the estimate

$$E_x \left[\mu_\infty^B(A) \right] \lesssim \frac{|A|}{\mathsf{d}(x, A)^{d-2}}, \tag{3.15}$$

simply bound from above the green function $g(x, y) \lesssim \mathsf{d}(x, A)^{2-d}$ for $y \in A$.

For the general case, it is sufficient to assume that $q \in \mathbb{N}$. Arguing by induction, write

$$E_x \left[\mu_\infty^B(A)^q \right] = \int_{(0,\infty)^q} P_x(B_{t_i} \in A \ \forall i) dt_1 dt_2 \ldots dt_q \tag{3.16}$$

and let $S_q := \{ (t_i)_{i=1}^q \in (0,\infty)^q \ : \ t_1 \leq \ldots \leq t_q \}$. Then,

$$\int_{[0,\infty)^q} P_x(B_{t_i} \in A \ \forall i) dt_1 dt_2 \ldots dt_q = q! \int_{S_q} P_x(B_{t_i} \in A \ \forall i). \tag{3.17}$$

Using the Markov property, we have

$$\int_{S_q} P_x(B_{t_i} \in A \ \forall i) = \int_{S_{q-1}} \left(\int_{t_{q-1}}^\infty dt_q P_y(B_{t_q} \in A | B_{t_{q-1}} = y) \right) P_x(B_{t_i} \in A \ \forall i)$$

$$\lesssim \mathrm{diam}(A)^2 \int_{S_{q-1}} P_x(B_{t_i} \in A \ \forall i). \qquad \square$$

We can also easily estimate from below the time spent in D_1 before exiting D_L, i.e., the variable $\mu_{\tau_{D_L^c}}^B(D_1)$ for a Brownian motion starting at $x \in D_1$.

Lemma 3.3. *If $d \geq 3$, there exists $c = c(d)$ such that, if $\min\{L, T\} \geq c$ it holds*

$$\inf_{x \in D_1} E_x \left[\mu_{T \wedge \tau_{D_L^c}}^B(D_1) \right] \geq 1/c. \tag{3.18}$$

Proof. Notice first that, for every $x \in D_1$,

$$E_x \left[\mu_\infty^B(D_1) \right] = \int_0^\infty P_t(B_t \in D_1) dt = \int_{D_1} \frac{c(d)}{|x-y|^{d-2}} dy \gtrsim \int_{1/2}^{3/2} \frac{r^{d-1}}{r^{d-2}} dr \gtrsim 1. \tag{3.19}$$

where the first lower bound follows by intersecting D_1 with a suitable cone with vertex at x. Using the strong Markov property (3.9) and (3.11), we also find that

$$E_x \left[\mu_\infty^{\theta_{\tau_{D_L^c}} B}(D_1) \right] \lesssim L^{2-d} \sup_{y \in \partial D_1} E_y \left[\mu_\infty^B(D_1) \right] \to 0 \tag{3.20}$$

as $L \to \infty$, and convergence is uniform with respect to $x \in \bar{D}_1$. Moreover, denoting with Z a standard Gaussian variable on $\mathbb{R}^d$, by (3.10), we have

$$E_x \left[\mu_\infty^{\theta_T B}(D_1) \right] = \mathbb{E} \left[E_{x+\sqrt{T} Z} \left[\mu_\infty^B(D_1) \right] \right]$$
$$\lesssim \mathbb{E} \left[\min \left\{ 1, \mathsf{d}(x + \sqrt{T} Z, D_1)^{2-d} \right\} \right] \to 0 \tag{3.21}$$

as $T \to \infty$, and convergence is uniform with respect to $x \in D_1$. Thus, we trivially bound from above

$$E_x \left[\mu_\infty^{\theta_{T \wedge \tau_{D_L^c}} B}(D_1) \right] \leq E_x \left[\mu_\infty^{\theta_{D_L^c} B}(D_1) \right] + E_x \left[\mu_\infty^{\theta_T B}(D_1) \right] \to 0. \tag{3.22}$$

By difference,

$$E_x\left[\mu^B_{\tau_{D_L^c} \wedge T}(D_1)\right] = E_x\left[\mu^B_\infty(D_1)\right] - E_x\left[\mu^{\theta_{\tau_{D_L^c} \wedge T} B}_\infty(D_1)\right] \gtrsim 1 \tag{3.23}$$

if L and T are both sufficiently large. $\qquad\square$

Remark 3.4. A simple scaling argument, yields that, if $0 < \ell < L < \infty$, $T > 0$ are such that $\min\{L/\ell, T/\ell^2\} \geq c$, then

$$\inf_{x \in D_\ell} E_x\left[\mu^B_{T \wedge \tau_{D_L^c}}(D_\ell)\right] \geq \ell^2/c. \tag{3.24}$$

3.2. **Brownian motion on $\mathbb{T}^d$.** The above notions have their counterparts on Riemannian manifolds or even more generally on certain classes of metric measure spaces. We focus here on the flat torus $\mathbb{T}^d$, where the transition density is given by

$$p_t(x, y) = \sum_{z \in \mathbb{Z}^d} \exp\left(-\frac{|x - y - z|^2}{2t}\right)\frac{1}{(2\pi t)^{d/2}}. \tag{3.25}$$

We write P_x, E_x as in the case of $\mathbb{R}^d$. The (uniform) Lebesgue measure on $\mathbb{T}^d$ is the invariant measure for the Brownian motion, and we say that $(B_t)_{t \geq 0}$ is stationary if B_0 has uniform law. For every $p \geq 1$, there exists $c = c(p, d)$ such that

$$E_x\left[(\mathsf{d}_{\mathbb{T}^d}(x, B_t))^p\right] \leq C(t \wedge 1)^{p/2}, \quad \text{for } t \geq 0. \tag{3.26}$$

Moreover, the ultra-contractivity inequality holds: for every $t \geq 0$,

$$\sup_{x \in \mathbb{T}^d} E_x\left[f(B_t)\right] \lesssim t^{-d/2}\int_{\mathbb{T}^d} f. \tag{3.27}$$

for every non-negative Borel $f : \mathbb{T}^d \to [0, \infty]$. We also have, for some $c = c(d) > 0$,

$$\sup_{x \in \mathbb{T}^d} \mathsf{d}_{TV}(p_t(x, \cdot), 1) \lesssim e^{-ct} \quad \text{for all } t > 0, \tag{3.28}$$

where with a slight abuse of notation write densities with respect to Lebesgue measure on $\mathbb{T}^d$ instead of measures. This implies, by (2.10) that for a Brownian motion $(B_t)_{t \geq 0}$ on $\mathbb{T}^d$ with any initial law,

$$\mathsf{d}_{TV}\left(\mathbb{P}_{(B_0, B_t)}, \mathbb{P}_{B_0} \otimes 1\right) \lesssim e^{-ct} \quad \text{for all } t > 0. \tag{3.29}$$

The validity of (3.28) can be seen in many ways, e.g. by using the identity

$$p_t(x, y) = \sum_{z \in \mathbb{Z}^d} e^{i2\pi(x-y)\cdot z - t|z|^2/2} \tag{3.30}$$

and estimating, for $t \geq 1$ (the case $t < 1$ is trivial, since $\mathsf{d}_{TV} \leq 1$ anyway)

$$\begin{aligned}\sup_{x \in \mathbb{T}^d} \mathsf{d}_{TV}(p_t(x, \cdot), 1) &\leq \sum_{z \in \mathbb{Z}^d \setminus \{0\}} e^{-ct|z|^2} \lesssim \int_1^\infty r^{d-1}e^{-ctr^2}dr\\ &\lesssim t^{-d/2}\int_{\sqrt{t}}^\infty u^{d-1}e^{-cu^2}du \lesssim e^{-c't}.\end{aligned} \tag{3.31}$$

A kind of counterpart of Lemma 3.2 is the following.

Lemma 3.5. *Let $d \geq 1$ and let B a stationary Brownian motion on $\mathbb{T}^d$. Then, for every Borel $A \subseteq \mathbb{T}^d$, it holds*

$$\mathbb{E}\left[\mu^B_T(A)\right] = |A|T, \quad \text{for } T \geq 0, \tag{3.32}$$

and, for every $q \geq 1$, there exists $c = c(d, q)$ such that, for every $T \geq 1$,

$$\mathbb{E}\left[\left|\mu^B_T(A) - |A|T\right|^q\right]^{1/q} \leq C|A|^{1/q^*}T^{1/2}, \tag{3.33}$$

and $1/q^* := \min\{1, 1/p + 1/d\}$.

Proof. The first statement is straightforward, since

$$\mathbb{E}\left[\mu_T^B(A)\right] = \int_0^T \mathbb{P}(B_t \in A)dt = \int_0^T |A|dt = |A|T. \tag{3.34}$$

To prove (3.33), since $L^q(\mathbb{P})$ norms are increasing with respect to q, we may assume that $q \geq d/(d-1)$, so that $\min\{1, 1/q + 1/d\} = 1/q + 1/d$. Without loss of generality, we can also assume that A is closed. Given a smooth function g on $\mathbb{T}^d$, with $\int_{\mathbb{T}^d} g = 0$, let f denote the solution to the elliptic equation

$$\frac{1}{2}\Delta f = g, \quad \text{on } \mathbb{T}^d. \tag{3.35}$$

By standard regularity theory, also f is smooth, hence Itô's formula applies yielding

$$\int g d\mu_T^B = \frac{1}{2}\int_0^T \Delta f(B_s)ds = f(B_T) - f(B_0) - \int_0^T \nabla f(B_t)dB_t. \tag{3.36}$$

Taking the $L^q(\mathbb{P})$ norm, we obtain by the triangle inequality and Burkholder-Davis-Gundy inequality that

$$\begin{aligned}
\left\|\int g d\mu_T^B\right\|_{L^q(\mathbb{P})} &\lesssim \|f(B_T)\|_{L^q(\mathbb{P})} + \|f(B_0)\|_{L^q(\mathbb{P})} + \left\|\int_0^T \nabla f(B_t)dB_t\right\|_{L^q(\mathbb{P})} \\
&\lesssim \|f(B_T)\|_{L^q(\mathbb{P})} + \left\|\sqrt{\int_0^T |\nabla f|^2 (B_t)dt}\right\|_{L^p(\mathbb{P})} \\
&\lesssim \|f\|_{L^q(\mathbb{T}^d)} + \sqrt{T}\|\nabla f\|_{L^q(\mathbb{T}^d)}.
\end{aligned} \tag{3.37}$$

By considering a sequence of smooth functions $(g_n)_n$ with $\int_{\mathbb{T}^d} g_n = 0$ for every $n \geq 1$ and such that $g_n \to \chi_A - |A|$ in $L^q(\mathbb{T}^d)$ but also $\int g_n d\mu_T^B \to \mu_T(A) - T|A|$ $\mathbb{P}$-a.s., then by Calderon-Zygmund theory, we have that the induced solutions f_n converge in the Sobolev space $W^{2,q}(\mathbb{T}^d)$ to the solution f to $\frac{1}{2}\Delta f = \chi_A - |A|$, and

$$\left\|\nabla^2 f\right\|_{L^q(\mathbb{T}^d)} \lesssim \|\chi_A - |A|\|_{L^q(\mathbb{T}^d)} \lesssim |A|^{1/q}. \tag{3.38}$$

By the Sobolev embedding on $\mathbb{T}^d$, we obtain

$$\|f\|_{L^p(\mathbb{T}^d)} + \|\nabla f\|_{L^p(\mathbb{T}^d)} \lesssim |A|^{1/q}, \tag{3.39}$$

which leads to the thesis. $\qquad\square$

3.3. Hitting probabilities.
In this section, we consider a stationary Brownian motion B on $\mathbb{T}^d$ and we estimate the hitting probability

$$\mathbb{P}(\sigma < \tau_{D_\ell} \leq \rho) \tag{3.40}$$

as well as the conditional hitting law ν_ρ, defined as

$$\nu_\rho(A) := \mathbb{P}(B_{\tau_{D_\ell}} \in A | 0 < \tau_{D_\ell} \leq \rho), \tag{3.41}$$

for $\ell \to 0$ and $\rho \to \infty$ in the regime $\rho \ll \ell^{2-d}$, and $0 \leq \sigma < \rho$. These assumptions will entail that the event has a small probability, and that $\nu_{\sigma,\rho}$ is very close to the uniform distribution on ∂D_ℓ (i.e., the normalized equilibrium measure $\tilde{e}_{D_\ell}$ for $D_\ell \subseteq \mathbb{R}^d$).

The problem of estimating the hitting probabilities of small sets much studied on manifolds, but we are not aware of bounds for the hitting law, hence we review and expand some results from [26, 17]. Consider a Brownian motion $(B_t)_{t \geq 0}$ on $M = \mathbb{R}^d$ or $\mathbb{T}^d$ and transition density $p(t, x, y)$.

Given a compact $K \subseteq M$ with smooth boundary, write $p_{K^c}(t, x, y)$ for the transition density of the Brownian motion killed upon hitting K. Recall that p_{K^c} is symmetric, i.e., $p_{K^c}(t, x, y) = p_{K^c}(t, y, x)$ for every $x, y \in M$. In particular, on $\mathbb{T}^d$ this yields the identity

$$\int_{\mathbb{T}^d} p_{K^c}(t, x, y)dx = \int_{\mathbb{T}^d} p_{K^c}(t, y, x)dx = P_y(\tau_K > t). \tag{3.42}$$

Given a bounded Borel function on ∂K, we are interested in estimating the function

$$(t, x) \mapsto \Phi(t, x) := E_x \left[\phi(B_{\tau_K}) I_{\{0 < \tau_K \le t\}} \right], \tag{3.43}$$

which is a solution to the equation

$$\begin{cases} \partial_t \Phi = \frac{1}{2} \Delta \Phi & \text{in } (0, \infty) \times K^c, \\ u(0, x) = 0 & \text{for } x \in K^c, \\ u(t, x) = \phi(x) & \text{for } (0, \infty) \times \partial K. \end{cases} \tag{3.44}$$

Moreover, one has the representation

$$\partial_t \Phi(t, x) = \int_{\partial K} \phi(y) \partial_n p_{K^c}(t, x, y) \sigma(dy), \tag{3.45}$$

where σ denotes the surface measure on ∂K and n the inward normal. Combining these facts, the following representation theorem holds (see also [26, Lemma 3.1]).

Lemma 3.6. *Let $K \subseteq K' \subseteq M$ have smooth boundaries and let χ be (sufficiently smooth) and such that*

$$\chi = \phi \quad \text{on } \partial K, \quad \text{and} \quad \partial_n \chi = 0 \quad \text{on } \partial K'. \tag{3.46}$$

Then, for every $t > 0$, $x \in K^c$,

$$\partial_t \Phi(t, x) = \int_{K' \setminus K} p_{K^c}(t, x, y) \Delta \chi(y) dy ds - \int_{K' \setminus K} \partial_t p_{K^c}(t, x, y) \chi(y) dy. \tag{3.47}$$

Assume that $M = \mathbb{T}^d$ and that B is a stationary Brownian motion. Then, integrating (3.47) over $x \in \mathbb{T}^d$ and using (3.42) yields

$$\partial_t \mathbb{E} \left[\phi(B_{\tau_K}) I_{\{0 < \tau_K \le t\}} \right] = \int_{K' \setminus K} P_y(\tau_K > t) \Delta \chi(y) dy - \int_{K' \setminus K} \partial_t P_y(\tau_K > t) \chi(y) dy. \tag{3.48}$$

Further integration with respect to $t \in (\sigma, \rho]$ leads to the identity

$$\mathbb{E} \left[\phi(B_{\tau_K}) I_{\{\sigma < \tau_K \le \rho\}} \right] = \int_\sigma^\rho \int_{K' \setminus K} P_y(\tau_K > t) \Delta \chi(y) dy \\ + \int_{K' \setminus K} P_y(\sigma < \tau_K \le \rho) \chi(y) dy \tag{3.49}$$

which we crucially use to establish the following result.

Proposition 3.7. *Let $d \ge 3$, $0 < \gamma < d - 2$ and for assume that $0 \le \sigma < \rho$, with $\rho \sim \ell^{-\gamma}$ as $\ell \to 0$. Given a stationary Brownian motion B on $\mathbb{T}^d$, it holds*

$$\left| \mathbb{P}(\sigma < \tau_{D_\ell}(B) \le \rho) - (\rho - \sigma)\ell^{d-2} \operatorname{Cap}(D_1) \right| \lesssim (\rho - \sigma)\rho \ell^{2(d-2)} |\log \ell| + \ell^d. \tag{3.50}$$

Moreover,

$$W_{D_\ell}^p(\nu_{\sigma, \rho}, \tilde{e}_{D_\ell}) \lesssim \ell^p \cdot \left(\rho \ell^{d-2} |\log \ell| + \ell^2/(\rho - \sigma) \right), \tag{3.51}$$

where $\tilde{e}_{D_\ell}$ denote the uniform probability measure on ∂D_ℓ, and the implicit constants depend on d, γ and the implicit constant in the condition $\rho \sim \ell^{-\gamma}$ only.

Proof. We start from (3.49) in the case $K = D_\ell$, $K' = D_{2\ell}$. Recall that we identify $D_\ell \subseteq \mathbb{T}^d$ with a subset of $[-1/2, 1/2)^d$, and similarly $y \in \mathbb{T}^d$ with $y \in [-1/2, 1/2)^d$. We further define

$$\tilde{D}_\ell = \bigcup_{z \in \mathbb{Z}^d} D_\ell(z) \subseteq \mathbb{R}^d \tag{3.52}$$

and write $\tilde{P}_y$ for the law of a Brownian motion on $\mathbb{R}^d$ starting at y. With this notation, we notice that

$$P_y(\tau_{D_\ell} > t) = \tilde{P}_y(\tau_{\tilde{D}_\ell} > t) = \tilde{P}_y(\tau_{D_\ell} > t) - \tilde{P}_y(\tau_{D_\ell} > t, \tau_{\tilde{D}_\ell \setminus D_\ell} \leq t). \tag{3.53}$$

Therefore,

$$\left| P_y(\tau_{D_\ell} > t) - \tilde{P}_y(\tau_{D_\ell} > t) \right| \leq \tilde{P}_y(\tau_{\tilde{D}_\ell \setminus D_\ell} \leq t), \tag{3.54}$$

Our next aim is to bound from above the probability in the right hand side above. By the exponential maximal inequality for martingales, it holds, for every $M > 0$ and $t > 0$,

$$\tilde{P}_y \left(\sup_{0 \leq s \leq t} |B_s - y| \geq \sqrt{t}M \right) \lesssim \exp\left(-M^2/2 \right). \tag{3.55}$$

Therefore, we focus on bounding from above the probability

$$\tilde{P}_y \left(\tau_{\tilde{D}_\ell \setminus D_\ell} \leq t, \quad \sup_{s \leq t} |B_s - y| \leq \sqrt{t}M \right). \tag{3.56}$$

By the triangle inequality, in the event above it must hold that $\tau_{D_\ell(z)} < \infty$ for some $z \in \mathbb{Z}^d \setminus \{0\}$, such that

$$|z| \leq 3\ell + \sqrt{t}M. \tag{3.57}$$

Notice that, as $\ell \to 0$, we have that the right hand side is < 1 e.g. if $tM < 1/2$. In such a case the event has null probability. In any case, we define

$$\bar{k} := \lfloor 3\ell + \sqrt{t}M \rfloor, \tag{3.58}$$

then, the probability is bounded from above by

$$\sum_{0 < |z| \leq \bar{k}} P_y(\tau_{D_\ell(z)} < \infty) \lesssim \sum_{k=1}^{\bar{k}} \left(\frac{k}{\ell} \right)^{2-d} \sharp\left\{ z \in \mathbb{Z}^d : k - 1 < |z| \leq k \right\}$$

$$\lesssim \sum_{k=1}^{\bar{k}} \left(\frac{k}{\ell} \right)^{2-d} k^{d-1} \lesssim \ell^{d-2} \bar{k}^2 \lesssim \ell^d + \ell^{d-2} tM \tag{3.59}$$

We find therefore the estimate

$$\sup_{y \in D_{2\ell} \setminus D_\ell} \left| P_y(\tau_K > t) - \tilde{P}_y(\tau_K > t) \right| \lesssim \exp\left(-M/2 \right) + \ell^{d-2} tM + \ell^d \tag{3.60}$$

Letting $M = -2\log(\ell^{d-2}\rho) \sim |\log \ell|$, we obtain

$$\sup_{y \in D_{2\ell} \setminus D_\ell} \left| P_y(\tau_K > t) - \tilde{P}_y(\tau_K > t) \right| \lesssim \ell^{d-2} \rho |\log \ell| + \ell^d. \tag{3.61}$$

Using this bound in (3.49) for $\sigma \leq t \leq \rho$, we obtain

$$\left| \mathbb{E}\left[\phi(B_{\tau_{D_\ell}}) I_{\{\sigma < \tau_{D_\ell}(B) \leq \rho\}} \right] - \int_\sigma^\rho \int_{D_{2\ell} \setminus D_\ell} \tilde{P}_y(\tau_{D_\ell} > t) \Delta\chi(y) dy \right|$$

$$\lesssim \rho \ell^{d-2} |\log \ell| (\rho - \sigma) \int_{D_{2\ell} \setminus D_\ell} |\Delta\chi(y)| dy + \int_{D_{2\ell} \setminus D_\ell} |\chi(y)| \, dy. \tag{3.62}$$

We are now in a position to establish (3.50). We set $\phi(x) = 1$ for every $x \in \partial D_\ell$ and we let χ be any smooth cut-off function v on D_2 such that $v = 1$ on ∂D_1 and $v = \partial_n v = 0$ on ∂D_2, and letting $\chi(x) = v(x/\ell)$. We obtain that $|\chi(y)| \lesssim 1$, $|\Delta\chi| \lesssim \ell^{-2}$, so that

$$\int_{D_{2\ell} \setminus D_\ell} |\chi(y)| \, dy \lesssim \ell^d, \quad \int_{D_{2\ell} \setminus D_\ell} |\Delta\chi(y)| \, dy \lesssim \ell^{d-2}, \tag{3.63}$$

so the second line in (3.62) is bounded from above:

$$\rho(\rho - \sigma)\ell^{d-2} |\log \ell| \int_{D_{2\ell} \setminus D_\ell} |\Delta\chi(y)| dy + \int_{D_{2\ell} \setminus D_\ell} |\chi(y)| \, dy \lesssim \rho(\rho - \sigma)\ell^{2(d-2)} |\log \ell| + \ell^d \tag{3.64}$$

For the first line, we use the scaling properties of Brownian motion on $\mathbb{R}^d$ so that

$$\int_0^\rho \int_{D_{2\ell} \setminus D_\ell} \tilde{P}_y(\tau_{D_\ell} > t)\Delta\chi(y) dy dt = \int_0^\rho \int_{D_{2\ell} \setminus D_\ell} \tilde{P}_{y/\ell}(\tau_{D_1} > t/\ell^2)\Delta\chi(y) dy dt$$

$$= \ell^d \int_0^\rho \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} > t/\ell^2)\Delta\chi(\ell z) dz dt \tag{3.65}$$

$$= \rho\ell^{d-2} \int_0^1 \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} > s\rho/\ell^2)\Delta v(z) dz ds.$$

As $\ell \to 0$, we have by dominated convergence that

$$\int_0^1 \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} > s\rho/\ell^2)\Delta v(z) dz \to \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} = \infty)\Delta v(z) dz. \tag{3.66}$$

Finally, using Green's identity and the properties of v and $\tilde{P}_z(\tau_{D_1} = \infty)$ we obtain that

$$\int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} = \infty)\Delta v(z) dz = \int_{\partial D_1} \partial_n P_z(\tau_{D_1} = \infty) dz = \mathrm{Cap}(D_1). \tag{3.67}$$

The convergence above can be made quantitative e.g. by a classical result by Port [34, Theorem 2.3] (see also [26] for similar bounds on manifolds):

$$\tilde{P}_z(t < \tau_{D_1} < \infty) \lesssim t^{1-d/2} \wedge 1. \tag{3.68}$$

as $t \to \infty$, uniformly on $z \in D_2 \setminus D_1$. As a consequence, we obtain

$$\left| \int_0^1 \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} > s\rho/\ell^2)\Delta v(z) dz ds - \int_{D_2 \setminus D_1} \tilde{P}_z(\tau_{D_1} = \infty)\Delta v(z) dz \right|$$

$$\lesssim \int_0^1 (s\rho/\ell^2)^{1-d/2} \wedge 1 ds \int_{D_2 \setminus D_1} |\Delta v(z)| \, dz$$

$$\lesssim \frac{\ell^2}{\rho} \int_{D_2 \setminus D_1} |\Delta v(z)| \, dz \lesssim \frac{\ell^2}{\rho}. \tag{3.69}$$

If $\sigma = 0$, this concludes the proof of (3.50). Otherwise, we argue similarly and obtain

$$\left| \int_0^\sigma \int_{D_{2\ell} \setminus D_\ell} \tilde{P}_y(\tau_{D_\ell} > t)\Delta\chi(y) dy dt - \sigma\ell^{d-2} \mathrm{Cap}(D_1) \right| \lesssim \ell^d, \tag{3.70}$$

and by difference (3.50).

Next, we address the proof of (3.51). In view (3.50) and using Remark 2.4, it is sufficient to consider $\phi \in H^{1,p'}(\partial D_1)$ with

$$\int_{\partial D_1} \phi d\sigma = 0, \quad \text{and} \quad \int_{\partial D_1} |\nabla\phi|^{p'} d\sigma \leq 1, \tag{3.71}$$

and establish the inequality

$$\mathbb{E}\left[\phi(B_{\tau_\ell}/\ell)I_{\{0<\tau_{D_\ell}\le\rho\}}\right] \lesssim \left(\rho\ell^{d-2}\right)^2 |\log\ell| + \ell^d \tag{3.72}$$

(where the implicit constant does not depend on ϕ). To this aim, we choose a variant of the function χ in (3.62) by letting again $\chi(x) = v(x/\ell)$, where in this case v (defined on D_2) enjoys the following properties:

$$v = \phi \quad \text{on } \partial D_1, \quad v = \partial_n v = 0 \quad \text{on } \partial D_2, \quad \text{and} \quad \int_{D_2\setminus D_1} |v| + |\Delta v| \lesssim 1. \tag{3.73}$$

Granted that such a function indeed exists, we obtain that the argument goes exactly in the same way starting from (3.62) as in the previous case, in particular the second line in (3.62) is bounded from above as in (3.64) and we are finally lead to (3.67). In this case, however, integrating by parts we conclude that

$$\int_{D_2\setminus D_1} \tilde{P}_z(\tau_{D_1} = \infty)\Delta v(z)dz = \int_{\partial D_1} \phi(z)\partial_{\mathfrak{n}} P_z(\tau_{D_1} = \infty)dz$$
$$= \text{Cap}(D_1) \int_{\partial D_1} \phi(z)dz = 0. \tag{3.74}$$

Thus, we only need to show that a v satisfying (3.73) exists. We build it in the following way: first, we consider an extension of ϕ in D_2 by letting

$$\tilde{\phi}(x) = \phi(x/|x|) \tag{3.75}$$

This extension has zero average $\int_{D_2\setminus D_1} \tilde{\phi} = 0$ and satisfies

$$\int_{D_2\setminus D_1} |\nabla\tilde{\phi}|^{p'} \lesssim \int_{\partial D_1} |\nabla\phi|^{p'} d\sigma \lesssim 1 \tag{3.76}$$

by integrating in radial coordinates. In particular, by Poincaré inequality on $D_2 \setminus D_1$,

$$\int_{D_2\setminus D_1} \tilde{\phi} \lesssim 1. \tag{3.77}$$

Next, we solve the problem

$$\begin{cases} \Delta u = \text{div}(\nabla\tilde{\phi}) & \text{in } D_2 \setminus D_1 \\ u = 0 & \text{on } \partial D_1 \cup \partial D_2 \end{cases} \tag{3.78}$$

obtaining a function u with zero average $\int_{D_2\setminus D_1} u = 0$ and (by global Calderon-Zygmund theory) such that

$$\int |\nabla u|^{p'} \lesssim \int_{D_2\setminus D_1} |\nabla\tilde{\phi}|^{p'} \lesssim 1. \tag{3.79}$$

We can then extend u to a Sobolev function identically null on D_1. Therefore, by Poincaré inequality for functions on D_2 that are null on D_1, we obtain

$$\int_{D_2\setminus D_1} |u| \lesssim \int |\nabla u| \lesssim 1. \tag{3.80}$$

We finally define the function

$$v = (\tilde{\phi} - u)\eta, \tag{3.81}$$

where $\eta(x) = \eta(|x|) \in [0,1]$ is a cut-off function with $\eta(1) = 1$, $\eta(2) = \eta'(2) = 0$. Clearly $v = \phi$ on ∂D_1, $v = \partial_n v = 0$ on ∂D_2 and

$$\int_{D_2\setminus D_1} |v| \lesssim \int_{D_2\setminus D_1} |\tilde{\phi}| + |u| \lesssim 1. \tag{3.82}$$

Finally, using that $\Delta\tilde{\phi} = \Delta u$ (in distributional sense), we obtain that

$$\begin{aligned} \Delta v &= \eta\Delta(\tilde{\phi} - u) + (\tilde{\phi} - u)\Delta\eta + 2\nabla(\tilde{\phi} - u)\nabla\eta \\ &= (\tilde{\phi} - u)\Delta\eta + 2\nabla(\tilde{\phi} - u)\nabla\eta, \end{aligned} \tag{3.83}$$

which eventually leads to

$$\int_{D_2\setminus D_1} |\Delta v| \lesssim 1, \tag{3.84}$$

hence the thesis is settled. $\qquad\square$

Next, we recall the following simple bound for exit times (for Brownian motion on $\mathbb{T}^d$ or equivalently on $\mathbb{R}^d$)

Lemma 3.8. *Let $d \geq 1$ and $0 < \ell < L/2 < 1/2$. Then, there exists $c = c(d) > 0$ such that*

$$\sup_{x \in D_\ell} P_x\left(\tau_{D_L^c}(B) > t\right) \lesssim e^{-ct/L^2}. \tag{3.85}$$

Proof. A simple union bound yields that the probability in dimension d is estimated from above (a sum) of the similar probabilities in dimension $d = 1$. But then this is a well-known consequence of the gambler's ruin problem. $\qquad\square$

We end this section with estimates for iterated hitting times of D_ℓ. Precisely, given $0 < \ell < L < 1/2$ and a Brownian motion B on $\mathbb{T}^d$, we define $\tau_\ell^1 = \tau_{D_\ell} B$, and iteratively

$$\begin{aligned} \tau_{k,L} &:= \inf\left\{t \geq \tau_{k-1,\ell} \, : \, B_t \in D_L^c\right\} \\ \tau_{k+1,\ell} &:= \inf\left\{t \geq \tau_{k,L} \, : \, B_t \in D_\ell\right\}. \end{aligned} \tag{3.86}$$

With this notation, we have the following result.

Corollary 3.9. *Let $0 \leq \gamma_L < 1$ and $0 < \gamma_\rho < \gamma_L(d-2)$ and set for $\ell \in (0, 1/2)$, $\rho \sim \ell^{-\gamma}$, $L \sim \ell^{\gamma_L}$. Given a stationary Brownian motion B on $\mathbb{T}^d$, it holds, for every $k \geq 1$,*

$$\mathbb{P}(\tau_{k,\ell} \leq \rho) \lesssim_k (\rho\ell^{d-2})^k, \quad as \ \ell \to 0. \tag{3.87}$$

Proof. With the notation of the previous proof, we first argue that, if $\sigma \sim \ell^{-\gamma_\sigma}$ for some $0 < \gamma_\sigma < \gamma_\rho$, then

$$\sup_{y \in \partial D_L} P_y(\tau_{D_\ell} B \leq \sigma) \lesssim \rho\ell^{d-2}. \tag{3.88}$$

Indeed, for every $\alpha > 0$,

$$\tilde{P}_y\left(\sup_{0 \leq s \leq \sigma} |B_s - y| \geq \sqrt{\alpha\sigma |\log\sigma|}\right) \lesssim \sigma^{-\alpha^2/2}, \tag{3.89}$$

If $B_0 = y \in \partial D_L$, in the event

$$\left\{\tau_{\tilde{D}_\ell} \leq \sigma, \quad \sup_{s \leq \sigma} |B_s - y| \leq \sqrt{\alpha\sigma |\log\sigma|}\right\}, \tag{3.90}$$

by the triangle inequality, it must hold that $\tau_{D_\ell(z)} < \infty$ for some $z \in \mathbb{Z}^d$, such that

$$|z| \leq \ell + L + \sqrt{\alpha\sigma |\log\sigma|}. \tag{3.91}$$

We define

$$\bar{j} := \lfloor \ell + L + \sqrt{\alpha\sigma |\log\sigma|} \rfloor, \tag{3.92}$$

then, the probability of the event (3.90) is bounded from above by

$$\sum_{|z|\leq \bar{j}} P_y(\tau_{D_\ell(z)} < \infty) \lesssim P_y(\tau_{D_\ell(0)} < \infty) + \sum_{j=1}^{\bar{j}} \left(\frac{j}{\ell}\right)^{2-d} \sharp\left\{z \in \mathbb{Z}^d : j-1 < |z| \leq j\right\}$$

$$\lesssim \left(\frac{L}{\ell}\right)^{2-d} + \sum_{k=1}^{\bar{k}} \left(\frac{k}{\ell}\right)^{2-d} k^{d-1} \lesssim \ell^{d-2}(L^{2-d} + \bar{k}^2)$$

$$\lesssim \ell^{d-2}\left(L^{2-d} + \alpha\sigma |\log\sigma|\right).$$
(3.93)

We find therefore the estimate

$$\sup_{y\in\partial D_L} P_y(\tau_K > t) \lesssim \sigma^{-\alpha^2/2} + \ell^{d-2}\left(L^{2-d} + \alpha\sigma |\log\sigma|\right) \lesssim \ell^{d-2}\left(L^{2-d} + \sigma |\log\sigma|\right) \quad (3.94)$$

provided that we chose α sufficiently large. Having settled (3.88), we argue by induction upon k, the case $k = 1$ being already settled in Proposition 3.7. We write, for $k > 1$, the inequality

$$\mathbb{P}(\tau_\ell^k \leq \rho) \leq \mathbb{P}(\tau_\ell^k - \tau_L^{k-1} \leq \sigma, \tau_\ell^k \leq \rho) + \mathbb{P}(\tau_\ell^k - \tau_L^{k-1} > \sigma, \tau_\ell^k \leq \rho). \quad (3.95)$$

for some $\sigma = \ell^{-\gamma_\sigma}$ with $0 < \gamma_\sigma < \gamma_\rho$. For the first term, we apply (3.88) and argue that

$$\mathbb{P}(\tau_\ell^k - \tau_L^{k-1} \leq \sigma, \tau_\ell^k \leq \rho) \leq \mathbb{P}(\tau_\ell^k - \tau_L^{k-1} \leq \sigma, \tau_\ell^{k-1} \leq \rho)$$

$$\leq \mathbb{E}\left[I_{\{\tau_\ell^{k-1}\leq\rho\}}\mathbb{P}\left(\tau_\ell^k - \tau_L^{k-1} \leq \sigma \mid \mathcal{F}_{\tau_\ell^{k-1}}\right)\right]$$

$$\leq \mathbb{P}(\tau_\ell^{k-1} \leq \rho) \sup_{y\in\partial D_L} P_y(\tau_\ell(B) \leq \sigma)]$$

$$\lesssim (\rho\ell^{d-2})^{k-1}\ell^{d-2}\left(L^{2-d} + \sigma|\log\sigma|\right) \ll (\rho\ell^{d-2})^k \quad (3.96)$$

provided that $\gamma_\sigma < \gamma_\rho$ and using also the condition $\gamma_\rho < (d-2)\gamma_L$. For the second term in the right hand side of (3.95), we write

$$\mathbb{P}\left(\tau_\ell^k - \tau_L^{k-1} > \sigma, \tau_\ell^k \leq \rho\right) \leq \mathbb{P}\left(\tau_\ell^{k-1} \leq \rho, \tau_{D_\ell}\theta_{\tau_L^{k-1}+\sigma}B \leq \rho\right). \quad (3.97)$$

We introduce the variables

$$U = (B_{t\wedge\tau_L^{k-1}})_{t\geq0}, \quad V = (B_{\tau_L^{k-1}}, B_{\tau_L^{k-1}+\sigma}) \quad \text{and} \quad W = (B_{\tau_L^{k-1}+\sigma+t})_{t\geq0}, \quad (3.98)$$

which define a Markov chain by the strong Markov property. By (2.12) and (3.28), we deduce that

$$\left|\mathbb{P}(\tau_\ell^{k-1} \leq \rho, \tau_{D_\ell}\theta_{\tau_L^{k-1}+\sigma}B) - \mathbb{P}(\tau_\ell^{k-1} \leq \rho, \tau_{D_\ell}\tilde{B} \leq \rho)\right| \lesssim e^{-c\sigma} \ll_k (\rho\ell^{d-2})^k \quad (3.99)$$

where $\tilde{B}$ is a stationary Brownian motion on $\mathbb{T}^d$, independent from B, and we used the fact that $\sigma = \ell^{-\gamma_\sigma}$ and $\gamma_\sigma > 0$. By induction and Proposition 3.7,

$$\mathbb{P}(\tau_\ell^{k-1} \leq \rho, \tau_{D_\ell}\tilde{B} \leq \rho) = \mathbb{P}(\tau_\ell^{k-1} \leq \rho)\mathbb{P}(\tau_{D_\ell}\tilde{B} \leq \rho) \sim (\rho\ell^{d-2})^k. \quad (3.100)$$

This settles the thesis (3.87). □

4. Brownian interlacement occupation measure

Given an (intensity) parameter $u > 0$ and a compact $K \subseteq \mathbb{R}^d$, we introduce the following random measure on the Borel subsets of K: we consider independent $((B_t^i)_{t\geq0})_{i=1}^\infty$ Brownian motions, all with initial law $\tilde{e}_K$ and a further independent Poisson random variable N, with mean $u\,\mathrm{Cap}(K)$, and let

$$\mathcal{I}_u \llcorner K = \sum_{i=1}^N \mu_\infty^{B^i} \llcorner K, \quad (4.1)$$

i.e., for $A \subseteq K$ Borel,

$$\mathcal{I}_u \llcorner K(A) = \sum_{i=1}^{N} \mu_\infty^{B^i}(A) = \sum_{i=1}^{N} \int_0^\infty I_{\{B_t^i \in A\}} dt. \tag{4.2}$$

Notice that, by (3.6), it holds

$$\mathbb{E}\left[\mathcal{I}_u \llcorner K(K)\right] = u|K|. \tag{4.3}$$

4.1. **Basic facts.** The restriction notation "$\llcorner K$" is justified by the following result.

Lemma 4.1. *Let $u > 0$, $\tilde{K} \subseteq K \subseteq \mathbb{R}^d$. Then,*

$$(\mathcal{I}_u \llcorner K) \llcorner \tilde{K}, \quad and \quad \mathcal{I}_u \llcorner \tilde{K} \tag{4.4}$$

have the same law.

Proof. We have, by (2.5),

$$(\mathcal{I}_u \llcorner K) \llcorner \tilde{K} = \sum_{i=1}^{N} \mu_\infty^{B^i} \llcorner \tilde{K} = \sum_{i=1}^{N} I_{\{\tau^i < \infty\}} \mu_\infty^{\theta_{\tau^i}(B^i)} \llcorner \tilde{K}. \tag{4.5}$$

where we write for brevity $\tau^i = \tau_{\tilde{K}}^{B^i}$. Each variable $I_{\{\tau^i < \infty\}}$ has Bernoulli law with parameter $\mathrm{Cap}(\tilde{K})/\mathrm{Cap}(K)$ by Proposition 3.1 and are all independent, hence the summation is the same (in law) as performed over a Poisson variable $\tilde{N}$ with mean $u\,\mathrm{Cap}(\tilde{K})$. Moreover, by the strong Markov property, conditionally upon $\{\tau^i < \infty\}$, the process $\theta_{\tau^i}(B^i)$ is a Brownian motion with initial law $\tilde{e}_{\tilde{K}}$, and are all independent. Thus,

$$\sum_{i=1}^{N} I_{\{\tau^i < \infty\}} \mu_\infty^{\theta_{\tau^i}(B^i)} \llcorner \tilde{K} \quad and \quad \mathcal{I}_u \llcorner \tilde{K} = \sum_{i=1}^{\tilde{N}} \mu_\infty^{\tilde{B}^i} \llcorner \tilde{K} \tag{4.6}$$

have the same law. $\square$

In view of the result above, we may "glue" together all the measures $\mathcal{I}^u \llcorner K$, and define a random Borel measure $\mathcal{I}_u$ on $\mathbb{R}^d$. Such a measure plays the role of a Poisson point process with intensity u in our setting. Notice that, by (4.3) and the fact that $A \mapsto \mathbb{E}\left[\mathcal{I}_u(A)\right]$ is a measure, it follows that $\mathbb{E}\left[\mathcal{I}_u(A)\right] = u|A|$ for every $A \subseteq \mathbb{R}^d$ Borel.

Lemma 4.2. *For every $u > 0$, $\rho > 0$ and $x \in \mathbb{R}^d$, the following identities in law hold:*

$$\mathrm{trans}_x \mathcal{I}_u = \mathcal{I}_u, \quad and \quad \mathrm{dil}_\rho \mathcal{I}_u = \rho^{-2} \mathcal{I}_{u/\rho^{d-2}}. \tag{4.7}$$

In particular,

$$\mathrm{dil}_{u^{-1/(d-2)}} \mathcal{I}_1 = u^{2/(d-2)} \mathcal{I}_u. \tag{4.8}$$

Proof. The first identity follows by translation invariance of the equilibrium measure, i.e., the fact that $\mathrm{trans}_x \tilde{e}_K = \tilde{e}_{x+K}$ for every compact $K \subseteq \mathbb{R}^d$. In particular, $\mathrm{Cap}(K) = \mathrm{Cap}(x + K)$. Moreover, if $B = (B_t)_{t \geq 0}$ is a Brownian motion with initial law $\tilde{e}_K$, the process $x + B = (x + B_t)_t$ is a Brownian motion with initial law $\tilde{e}_{x+K}$. Thus,

$$\mathrm{trans}_x(\mathcal{I}_u \llcorner K) = \sum_{i=1}^{N} \mathrm{trans}_x \mu_\infty^{B^i} = \sum_{i=1}^{N} \mu_\infty^{x+B^i}, \tag{4.9}$$

which clearly has the law of $\mathcal{I}_u \llcorner (x + K)$.

For the second identity, we notice first that

$$\tilde{e}_{\rho K} = \mathrm{dil}_\rho \tilde{e}_K, \quad and \quad \mathrm{Cap}(\rho K) = \rho^{d-2} \mathrm{Cap}(K). \tag{4.10}$$

so that a Poisson variable N with mean

$$u\,\mathrm{Cap}(K) = \frac{u}{\rho^{d-2}} \mathrm{Cap}(\rho K). \tag{4.11}$$

Moreover, if $B = (B_t)_{t \geq 0}$ is a Brownian motion with initial law $\tilde{e}_K$, the process $\tilde{B}^\rho = (\rho B_{t/\rho^2})_{t \geq 0}$ has the same law of a Brownian motion with initial law $\text{dil}_\rho \tilde{e}_K$. Therefore,

$$\text{dil}_\rho(\mathcal{I}_u \llcorner K) = \text{dil}_\rho \sum_{i=1}^{N} \mu_\infty^{B^i} = \sum_{i=1}^{N} \int_0^\infty \delta_{\rho B_t^i} dt = \rho^2 \sum_{i=1}^{N} \int_0^\infty \delta_{\rho B_{t/\rho^2}^i} dt \qquad (4.12)$$

has the same law as $\rho^2 \mathcal{I}_{u/\rho} \llcorner \rho K$. $\qquad \square$

In the next lemma, we consider the concentration properties for $\mathcal{I}_1(A)$ (the general case follows from (4.8)).

Lemma 4.3. *Let $d \geq 3$. For every $q \geq 1$, there exists $C = C(d, q) < \infty$ such that, if $\text{diam}(A) \geq 1$,*

$$\| \mathcal{I}_1(A) - \mathbb{E}\left[\mathcal{I}_1(A)\right] \|_q \leq C \, \text{diam}(A)^{(d+2)/2}. \qquad (4.13)$$

Proof. Let K be a ball of radius $\text{diam}(A)$ such that $A \subseteq K$. Then, $\mathcal{I}_1(K) = \sum_{i=1}^{N} \mu_\infty^{B^i}(A)$ is the sum of a Poisson number (with mean $\nu = \text{Cap}(K) \lesssim \text{diam}(A)^{d-2}$) of i.i.d. variables, each with finite moments of all orders. By (**??**), it follows that $\mathcal{I}_1(A)$ has finite moments of all orders, and

$$\| \mathcal{I}_1(K) - \mathbb{E}\left[\mathcal{I}_1(A)\right] \|_q \lesssim_q \left(\text{diam}(A)^{(d-2)/q} + \text{diam}(A)^{(d-2)/2} \right) \text{diam}(A)^2$$
$$\lesssim \text{diam}(A)^{(d+2)/2}. \qquad \square$$

4.2. **Limit results.** We are now in a position to establish the convergence as $u \to \infty$ of $\mathbb{E}\left[W_\Omega(\mathcal{I}_u)\right]$. The arguments are a modification of those originally devised for the random matching problem in [7, 23, 6, 25] and employ only basic invariance properties of $\mathcal{I}_u$. Thus, we provide a complete derivation in Appendix A in a more abstract setting and here specialize to the interlacement occupation measure.

Theorem 4.4. *Let $d \in \{3, 4\}$ and $p \in (0, (d-2)/2)$, or $d \geq 5$ and $p > 0$. Then, there exists a constant $\mathsf{c}(\mathcal{I}, d, p) \in (0, \infty)$ such that, for every bounded connected domain Ω with C^2 boundary (or $\Omega = Q$ a cube) it holds*

$$\lim_{u \to \infty} \mathbb{E}\left[W_\Omega^p(\mathcal{I}_u)\right] / u^{1-p/(d-2)} = \mathsf{c}(\mathcal{I}, d, p)|\Omega|. \qquad (4.14)$$

Proof. If we apply Theorem A.1 with $\nu = \mathcal{I}_1$, which satisfies the conditions i) (translation invariance), ii) (integrability) and iii) (concentration) with $\alpha = 2$, we obtain

$$\lim_{n \to \infty} \mathbb{E}\left[W_\Omega^p(\text{dil}_{n^{-1/d}} \mathcal{I}_1)\right] / n^{1-p/d} = \mathsf{c}(\mathcal{I}, d, p)|\Omega|, \qquad (4.15)$$

for some constant $\mathsf{c}(\mathcal{I}, d, p) \in [0, \infty)$. Setting $n = n(u) = u^{d/(d-2)}$ so that $n^{-1/d} = u^{-1/(d-2)}$, hence $\text{dil}_{n^{-1/d}} \mathcal{I}_1 = u^{2/(d-2)} \mathcal{I}_u$ by (4.8), we obtain (4.14). The fact that $\mathsf{c}(\mathcal{I}, d, p)$ is strictly positive will follow as a consequence of Theorem 1.2 and Remark 6.2. $\qquad \square$

We then obtain Theorem 1.1, which is nothing but a *de-Poissonized* version of (4.14), where the number of Brownian motions is deterministic. Since this type of arguments are by now standard, and here we follow closely [23, 25], we prefer to obtain it as a consequence of a more general result, see Proposition B.1 in Appendix B.

We end instead this section by showing how the limit behaves if we assume that the Brownian motions do not start exactly with the equilibrium measure. We restrict ourselves to the case of a ball, although we conjecture that similar bounds should hold true for general domains.

In the next lemma, write $\mathcal{M}$ for the σ-algebra on $C([0, \infty); \mathbb{R}^d)^{\otimes n}$ generated by the map

$$((x_t^i)_{t \geq 0})_{i=1}^n \mapsto ((|x_t^i|)_{t \geq 0})_{i=1}^n, \qquad (4.16)$$

Lemma 4.5. *Let $d \geq 1$ and ν, $\tilde{\nu}$ be probability measures on ∂D_1. Let, $(\tau^i)_{i=1}^n$ be $\mathcal{M}$-measurable functions with values in $[0, \infty]$. Let $B = (B^i)_{i=1}^n$ be independent Brownian motions, with an initial law ν and $\tilde{B} = (\tilde{B}^i)_{i=1}^n$ be independent Brownian motions with initial law $\tilde{\nu}$. If $p > 0$, then there exists $c = c(p) < \infty$ such that, for every $\varepsilon \in [0, 1)$,*

$$
\mathbb{E}\left[W_{D_1}^p\left(\sum_{i=1}^n \mu_{\tau^i(B)}^{B^i}\right)\right] \leq (1 + \varepsilon)\mathbb{E}\left[W_{D_1}^p\left(\sum_{i=1}^n \mu_{\tau^i(\tilde{B})}^{\tilde{B}^i}\right)\right]
$$
$$
+ \frac{C}{\varepsilon^{(p-1)^+}}\mathbb{E}\left[\sum_{i=1}^n \mu_{\tau^i(B)}^{B^i}(D_1)\right]W_{D_1}^p(\nu, \tilde{\nu}).
\tag{4.17}
$$

Moreover, in the second line one can also replace B with $\tilde{B}$.

Proof. Consider an optimal transport plan between ν and $\tilde{\nu}$, and assume for simplicity that it is induced by a map $\Psi : \partial D \to \partial D$. We use Ψ to induce a coupling between a Brownian motion B with initial law ν and a Brownian motion $\tilde{B}$ with initial law $\tilde{\nu}$ in the following way. We let $U = U(\tilde{B}_0) \in \mathbb{R}^{d \times d}$ be the orthogonal transformation which acts as a rotation, on the plane spanned by $\{B_0, \Psi(B_0)\}$, mapping B_0 into $\Psi(B_0)$ (if $B_0 = \Psi(B_0)$ we simply let U be the identity). We then define

$$
\tilde{B}_t = U B_t, \quad \text{for } t \geq 0,
\tag{4.18}
$$

which is a Brownian motion, with $\tilde{B}_0 = \Psi(B_0)$, so that its initial law is $\tilde{e}_D$. Moreover, since $(|\tilde{B}_t|)_{t \geq 0} = (|B_t|)_{t \geq 0}$, we have, for every $T \geq 0$,

$$
U_\sharp \mu_T^B \llcorner D = \mu_T^{\tilde{B}} \llcorner D,
\tag{4.19}
$$

In addition, we have $\tau^B = \tau^{\tilde{B}}$ and $\mu_T^B(D) = \mu_T^{\tilde{B}}(D)$ and because each process $(|B_t|)_{t \geq 0}$ is independent of B_0 (for $|B_0| = 1$) we have that $(\tau^B, \mu_{\tau^B}^B(D))$ and $|B_0 - \Psi(B_0)|$ are independent random variables. Since

$$
\left|\tilde{B}_t - B_t\right| \leq \|I - U\| |B_t| = |B_0 - \Psi(B_0)|, \quad \text{for every } t \geq 0 \text{ such that } B_t \in D,
\tag{4.20}
$$

we conclude that, for every $T \geq 0$,

$$
W_D^p\left(\mu_T^B, \mu_T^{\tilde{B}}\right) \leq \mu_T^{\tilde{B}}(D) |B_0 - \Psi(B_0)|^p,
\tag{4.21}
$$

having used the coupling induced by U. We apply this construction to each B^i and notice that $\tau_i = \tilde{\tau}_i$ and the variables

$$
(\tau^i, \mu_{\tau^i}^{B^i}(D))_{i=1}^n, \quad \text{and} \quad \left(\left|B_0^i - \Psi(B_0^i)\right|\right)_{i=1}^n
\tag{4.22}
$$

are independent. Using this independence, (2.20) and (4.21), we thus obtain

$$
\mathbb{E}\left[W_{D_1}^p\left(\sum_{i=1}^n \mu_{\tau^i}^{B^i}, \sum_{i=1}^n \mu_{\tilde{\tau}^i}^{\tilde{B}^i}\right)\right] \leq \mathbb{E}\left[\sum_{i=1}^n \mu_{\tau_i}^{B^i}(D) \left|B_0^i - \Psi(B_0^i)\right|^p\right]
$$
$$
\leq \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^n \mu_{\tau_i}^{B^i}(D) \left|B_0^i - \Psi(B_0^i)\right|^p \left| (\tau^i, \mu_{\tau^i}^{B^i}(D))_{i=1}^n \right.\right]\right]
$$
$$
= \mathbb{E}\left[\sum_{i=1}^n \mu_{\tau_i}^{B^i}(D)\right] W_{D_1}^p(\nu, \tilde{\nu}).
\tag{4.23}
$$

The thesis then follows from the triangle inequality and an application of (2.24). $\qquad \square$

5. Occupation measure for the Brownian motion on the torus

In this section, we study the asymptotics for the occupation measure of a Brownian motion on $\mathbb{T}^d$ and establish Theorem 1.2. Most of the argument is in fact contained in the following "local" result, showing that by focusing only the cost $W_{D_\ell}^p$ for a small ball D_ℓ, with $\ell = T^{-\gamma}$ for γ smaller (but sufficiently close) to $1/(d-2)$, the constant $\mathsf{c}(\mathcal{I}, d, p)$ associated to the Brownian interlacement of the previous section will appear in a suitably renormalized limit.

Proposition 5.1. *Let $d \in \{3, 4\}$ and $p \in (0, (d-2)/2)$, or $d \geq 5$ and $p > 0$. Let $(B_t)_{t \geq 0}$ be a stationary Brownian motion on $\mathbb{T}^d$. There exists $\bar{\gamma} = \bar{\gamma}(d, p) \in (0, 1/(d-2))$ such that the following holds. For every $\gamma \in (\bar{\gamma}, 1/(d-2))$, letting $\ell = T^{-\gamma}$, it holds*

$$\lim_{T \to \infty} \mathbb{E}\left[W_{D_\ell}^p\left(\mu_T^B\right)\right] / \left(T^{1-p/(d-2)}|D_\ell|\right) = \mathsf{c}(\mathcal{I}, d, p). \tag{5.1}$$

with $\mathsf{c}(\mathcal{I}, d, p)$ as in Theorem 1.1.

Proof. The strategy is to couple the given Brownian motion on $\mathbb{T}^d$ with a family of independent Brownian motions on $\mathbb{R}^d$, with common initial law sufficiently close to $\tilde{e}_{D_1}$, and use Theorem 1.1. Of course the crux of the argument is to take into account the several error terms due to this approximation. We split the proof into several steps.

Step 1 (Time splitting). We introduce two additional parameters γ_ρ, γ_σ, such that

$$0 < \gamma_\sigma < \gamma_\rho < \gamma(d-2) \tag{5.2}$$

and to be specified below: infact, we are going to collect several constraints and only check at the final step that they can be all satisfied. To guide the intuition, we may think of γ_σ and γ_ρ to be very close to 0. We define the quantities

$$n := \lfloor T^{1-\gamma_\rho} \rfloor, \quad \sigma := T^{\gamma_\sigma}, \quad \rho := \frac{T}{n} - \sigma. \tag{5.3}$$

Notice that, by our choices of the parameters, it holds

$$1 \ll \sigma \ll \rho \ll \ell^{2-d} \quad \text{and} \quad T = n(\rho + \sigma). \tag{5.4}$$

For $i = 0, \ldots, n-1$, we consider the intervals

$$I_i = [i(\rho+\sigma), i(\rho+\sigma)+\rho), \quad J_i = [i(\rho+\sigma)+\rho, (i+1)(\rho+\sigma)), \tag{5.5}$$

and decompose

$$\mu_T = \sum_{i=0}^{n-1} \left(\int_{I_i} \delta_{B_t} dt + \int_{J_i} \delta_{B_t} dt\right) = \sum_{i=0}^{n-1} \left(\mu_\rho^{\theta_{i(\rho+\sigma)}B} + \mu_\sigma^{\theta_{i(\rho+\sigma)+\rho}B}\right) \tag{5.6}$$

All the processes appearing in the expression above are stationary Brownian motions on the torus, hence have the same law as B (but they are not independent). Therefore, by (3.32), we have

$$\operatorname{diam}(D_\ell)^p \mathbb{E}\left[\sum_{i=0}^{n-1} \mu_\sigma^{\theta_{i(\rho+\sigma)+\rho}B}(D_\ell)\right] \leq n\sigma\ell^{p+d} \leq \left(T^{1-\gamma_\rho+\gamma_\sigma}\right) T^{-p\gamma}\ell^d \tag{5.7}$$

$$\ll T^{1-p/(d-2)}\ell^d,$$

provided that the following condition is satisfied:

$$\gamma_\rho - \gamma_\sigma > p(1/(d-2) - \gamma). \tag{5.8}$$

Starting from (5.7), we easily deduce that

$$\left|\mathbb{E}\left[W_{D_\ell}^p\left(\mu_T^B\right)\right] - \mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i=1}^{n} \mu_\rho^{\tilde{B}^i}\right)\right]\right| \ll T^{1-p/(d-2)}\ell^d. \tag{5.9}$$

Indeed, by (2.20) and (2.17), we obtain that

$$\mathbb{E}\left[W^p_{D_\ell}(\mu^B_T)\right] - \mathbb{E}\left[W^p_{D_\ell}\left(\sum_{i=0}^{n-1}\mu_\rho^{\theta_{i(\rho+\sigma)}B}\right)\right] \ll T^{1-p/(d-2)}\ell^d. \tag{5.10}$$

Similarly, by (2.36), for every $\varepsilon > 0$,

$$\mathbb{E}\left[W^p_{D_\ell}\left(\sum_{i=0}^{n-1}\mu_\rho^{\theta_{i(\rho+\sigma)}B}\right)\right] - (1+\varepsilon)\mathbb{E}\left[W^p_{D_\ell}(\mu^B_T)\right] \ll_\varepsilon T^{1-p/(d-2)}\ell^d, \tag{5.11}$$

so letting $T \to \infty$ and then $\varepsilon \to 0$, we see that (5.9) holds. Hence, the thesis follows if we prove that

$$\lim_{T\to\infty}\mathbb{E}\left[W^p_{D_\ell}\left(\sum_{i=1}^{n}\mu_\rho^{\tilde{B}^i}\right)\right] / \left(T^{1-p/(d-2)}|D_\ell|\right) = \mathsf{c}(\mathcal{I},d,p), \tag{5.12}$$

where we write, for brevity,

$$\tilde{B}^i = (B_{(i-1)(\rho+\sigma)+t})_{t\in[0,\rho]}. \tag{5.13}$$

Step 2 (Breaking dependence). The processes $(\tilde{B}^i)^n_{i=1}$ are stationary Brownian motions on $\mathbb{T}^d$, but of course they are not independent. In this step we argue that in (5.22) is actually possible to replace the processes them with n independent (and stationary) Brownian motions $(B^i)^n_{i=1}$ on $\mathbb{T}^d$. This will follow from the bound

$$\left|\mathbb{E}\left[W^p_{D_\ell}\left(\sum_{i=0}^{n-1}\mu_\rho^{\tilde{B}^i}\right)\right] - \mathbb{E}\left[W^p_{D_\ell}\left(\sum_{i=1}^{n}\mu_\rho^{B^i}\right)\right]\right| \ll T^{1-p/(d-2)}|D_\ell|, \tag{5.14}$$

which in turn is proved by repeated applications of (2.12) and (3.29). Precisely, consider the following non-negative function of n continuous curves $((x^i_t)_{t\in[0,\rho]})^n_{i=1}$ on $\mathbb{T}^d$:

$$(x^i)^n_{i=1} \mapsto F\left((x^i)^n_{i=1}\right) = W^p_{D_\ell}\left(\sum_{i=1}^{n}\mu_\rho^{x^i}\right). \tag{5.15}$$

By (2.17),

$$\sup_{(x^i)^n_{i=1}} F\left((x^i)^n_{i=1}\right) \le n\rho\ell^p. \tag{5.16}$$

Then, (5.14) follows from the inequality

$$\left|\mathbb{E}\left[F\left((\tilde{B}^i)^n_{i=1}\right)\right] - \mathbb{E}\left[F((B^i)^n_{i=1})\right]\right| \lesssim ne^{-c\sigma}n\rho\ell^p \ll T^{1-p/(d-2)}\ell^d, \tag{5.17}$$

where $c > 0$ is as in (3.29), and the second bound easily follows since $\gamma_\sigma > 0$.

To obtain (5.17), we use telescopic summation on the following set of inequalities, valid for $k = 1, \ldots, n-1$,

$$\left|\mathbb{E}\left[F\left(B^1, \ldots, B^{k-1}, \tilde{B}^1, \ldots \tilde{B}^{n-k+1}\right)\right] - \mathbb{E}\left[F(B^1, \ldots, B^k, \tilde{B}^1, \ldots \tilde{B}^{n-k})\right]\right| \le e^{-c\sigma}n\rho\ell^p. \tag{5.18}$$

For each k, the above inequality is an application of (2.12), with the variables

$$U = (B^1, \ldots, B^{k-1}, \tilde{B}^1), \quad V = (\tilde{B}^1_\rho, \tilde{B}^2_0), \quad W = (\tilde{B}^2, \ldots, \tilde{B}^{n-k+1}). \tag{5.19}$$

Notice that they define a Markov chain because, conditionally upon

$$V = (\tilde{B}^1_{\rho+\sigma}, \tilde{B}^2_0) = (B_\rho, B_{\rho+\sigma}), \tag{5.20}$$

U and V are independent. By (3.29), we estimate

$$\mathsf{d}_{TV}\left(\mathbb{P}_{(B_\rho, B_{\rho+\sigma})}, \mathcal{L}^d_{\mathbb{T}^d} \otimes \mathcal{L}^d_{\mathbb{T}^d}\right) \lesssim e^{-c\sigma}. \tag{5.21}$$

Thus, using (2.12), we can turn U (and in particular $\tilde{B}^1$) and $W = (\tilde{B}^2, \ldots, \tilde{B}^{n-k+1})$ independent, and let the initial distribution of $\tilde{B}^2$ be uniform on $\mathbb{T}^d$, effectively replacing

the joint law of W with that of $(\tilde{B}^1, \ldots, \tilde{B}^{n-k})$, with an error in total variation distance that is $\lesssim e^{-c\sigma}$. Because of (6.23), we see that (5.18) follows.

Step 3 (Series of visits). Our aim is to prove

$$\lim_{T \to \infty} \mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i=1}^n \mu_\rho^{B^i}\right)\right] / \left(T^{1-p/(d-2)}|D_\ell|\right) = \mathsf{c}(\mathcal{I}, d, p), \tag{5.22}$$

where $(B^i)_{i=1}^n$ are independent and stationary Brownian motions on $\mathbb{T}^d$. In this step, we decompose each measure $\mu_\rho^{B^i}$ into a series of contributions due to the sequence of "visits" to D_ℓ. We introduce an additional parameter

$$0 < \gamma_L < \gamma \tag{5.23}$$

and set $L = T^{-\gamma_L}$. For each $i = 1, \ldots, n$, we introduce the sequence of hitting times letting $\tau_{1,\ell}^i = \tau_{D_\ell} B^i$, and iteratively

$$\begin{aligned}
\tau_{k,L}^i &:= \inf\left\{t \geq \tau_{k-1,\ell}^i \ : \ B_t^i \in D_L^c\right\} \\
\tau_{k+1,\ell}^i &:= \inf\left\{t \geq \tau_{k,L}^i \ : \ B_t^i \in D_\ell\right\}.
\end{aligned} \tag{5.24}$$

Notice that between the times $\tau_{k,\ell}^i$ and $\tau_{k,L}^i$, B^i can be identified with a Brownian motion on $\mathbb{R}^d$. We write

$$K^i := \inf\left\{k \ : \ \tau_{k,\ell}^i > \rho\right\} \tag{5.25}$$

so that

$$\mu_\rho^{B^i} \llcorner D_\ell = \sum_{k=1}^{K^i-1} \int_{\tau_{k,\ell}^i}^{\tau_{k,L}^i \wedge \rho} \delta_{B_s^i} ds, \tag{5.26}$$

hence we have the series representation

$$\sum_{i=1}^n \mu_\rho^{B^i} \llcorner D_\ell = \sum_{k=1}^\infty \lambda_k, \tag{5.27}$$

where each term is defined as follows:

$$\lambda_k = \sum_{i \in S_k} \int_{\tau_{k,\ell}^i}^{\tau_{k,L}^i \wedge \rho} \delta_{B_s^i} ds \tag{5.28}$$

and $S_k = \{i : K_i > k\}$. In the next steps, we prove that the main contribution in the sum above comes from the term λ^1. Actually, we further analyze by splitting also the contribution of λ^1 into two parts: we let $S \subseteq S_1$ denote the (random) subset of $i \in \{1, \ldots, n\}$ such that the event

$$A^i := \left\{0 < \tau_{1,\ell}^i \leq \rho\right\} \tag{5.29}$$

holds, and set $S_0 = S_1 \setminus S_0$. We write $\lambda_1 = \lambda_0 + \tilde{\lambda}_1$, with

$$\lambda_0 = \sum_{i \in S_0} \int_{\tau_{1,\ell}^i}^{\tau_{1,L}^i \wedge \rho} \delta_{B_s^i} ds, \quad \tilde{\lambda}_1 = \sum_{i \in S} \int_{\tau_{1,\ell}^i}^{\tau_{1,L}^i \wedge \rho} \delta_{B_s^i} ds. \tag{5.30}$$

In the following steps, our aim is to prove that

$$\left|\mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i=1}^n \mu_\rho^{B^i}\right)\right] - \mathbb{E}\left[W_{D_\ell}^p\left(\tilde{\lambda}_1\right)\right]\right| \ll T^{1-p/(d-2)}\ell^d. \tag{5.31}$$

Step 4 (pper and lower bounds for visiting times). We notice first that, by Proposition 3.7,

$$\mathbb{P}(0 < \tau_{1,\ell}^i < \rho/2) \gtrsim \rho\ell^{d-2}. \tag{5.32}$$

By identifying $\tilde{B}^i = \theta_{\tau^i_{1,\ell}} B^i$ up to the time $\tau_{D^c_L} = \tau^i_{1,L} - \tau^i_{1,\ell}$ with a Brownian motion on $\mathbb{R}^d$, and using the strong Markov property and Remark 3.4, it holds

$$\mathbb{E}\left[I_{A^i} \int_{\tau^i_{1,\ell}}^{\rho \wedge \tau^i_{1,L}} I_{\{B_s \in D_\ell\}} ds\right] \geq \mathbb{E}\left[I_{\{0 < \tau^i_{1,\ell} < \rho/2\}} \mu^{\tilde{B}^i}_{(\rho/2) \wedge \tau_{D^c_L}}(D_\ell)\right] \gtrsim \rho \ell^{d-2} \cdot \ell^2. \tag{5.33}$$

Summation upon $i = 1, \ldots, n$ leads to the lower bound

$$\mathbb{E}\left[\tilde{\lambda}_1(D_\ell)\right] \gtrsim n \rho \ell^d \gtrsim T \ell^d. \tag{5.34}$$

On the other side, again by Corollary 3.9 and the strong Markov property, but using Lemma 3.2, it holds for every $q \geq 1$, the upper bound

$$\mathbb{E}\left[\left(I_{A^i} \int_{\tau^i_{k,\ell}}^{\rho \wedge \tau^i_{k,L}} I_{\{B_s \in D_\ell\}} ds\right)^q\right] \lesssim \rho \ell^{d-2} \cdot \ell^{2q}. \tag{5.35}$$

By Rosenthal inequality, it follows that, for every $q \geq 1$,

$$\mathbb{E}\left[\left|\tilde{\lambda}_1(D_\ell) - \mathbb{E}\left[\tilde{\lambda}_1(D_\ell)\right]\right|^q\right] \lesssim n \cdot \rho \ell^{d-2} \cdot \ell^{2q} + \left(n \rho \ell^{d-2} \cdot \ell^4\right)^{q/2} \tag{5.36}$$
$$\lesssim (T\ell^{d-2})^{q/2} \ell^{2q},$$

having used that $T\ell^{d-2} \to \infty$. Such concentration bound yields that, for some constant $c = c(d) > 0$, the event

$$G_1 = \left\{\tilde{\lambda}_1(D_\ell) \geq cT\ell^d\right\} \tag{5.37}$$

has large probability. Precisely,

$$\mathbb{P}(G_1^c) \leq \mathbb{P}\left(\tilde{\lambda}_1(D_\ell) - \mathbb{E}\left[\tilde{\lambda}_1(D_\ell)\right] \leq c'T\ell^d\right) \lesssim_q \frac{(T\ell^{d-2})^{q/2} \ell^{2q}}{(T\ell^{d-2})^q \cdot \ell^{2q}} = (T\ell^{d-2})^{-q/2} \ll_\alpha T^{-\alpha}, \tag{5.38}$$

for every $\alpha > 0$, provided that we choose q sufficiently large.

We argue similarly that, for every $k \geq 2$ and any $\delta \in (0,1)$ the event

$$G_k = \left\{\lambda_k(D_\ell) \leq T\ell^d \delta\right\} \tag{5.39}$$

has large probability: for every $\alpha > 0$, it holds

$$\mathbb{P}(G_k^c) \lesssim_{k,\delta,\alpha} T^{-\alpha}, \tag{5.40}$$

Indeed, by Corollary 3.9, the strong Markov property and Lemma 3.2 it holds, for every $q \geq 1$,

$$\mathbb{E}\left[\left(I_{\{i \in S_k\}} \int_{\tau^i_{k,\ell}}^{\rho \wedge \tau^i_{k,L}} I_{\{B_s \in D_\ell\}} ds\right)^q\right] \lesssim_k (\rho \ell^{d-2})^k \cdot \ell^{2q}. \tag{5.41}$$

By Rosenthal inequality for positive variables,

$$\mathbb{E}[\lambda_k(D_\ell)^q] = \mathbb{E}\left[\left(\sum_{i \in S_k} \int_{\tau^i_{k,\ell}}^{\rho \wedge \tau^i_{k,L}} I_{\{B_s \in D_\ell\}} ds\right)^q\right] \tag{5.42}$$
$$\lesssim_k n(\rho \ell^{d-2})^k \cdot \ell^{2q} + \left(n(\rho \ell^{d-2})^k \cdot \ell^2\right)^q \lesssim T\ell^{d-2}(\rho \ell^{d-2})^{k-1} \ell^{2q},$$

having used that

$$n(\rho \ell^{d-2})^k \lesssim T\ell^{d-2}(\rho \ell^{d-2})^{k-1} \ll 1 \tag{5.43}$$

provided that we impose the condition

$$1 - \gamma(d-2) - [(d-2)\gamma - \gamma_\rho] < 0. \tag{5.44}$$

Hence,

$$\mathbb{P}(G_k^c) \lesssim_{k,\delta} \frac{T\ell^{d-2}(\rho\ell^{d-2})^{k-1}\ell^{2q}}{(T\ell^{d-2})^q\ell^{2q}} = (T\ell^{d-2})^{1-q}(\rho\ell^{d-2})^{k-1} \lesssim_\alpha T^{-\alpha}, \qquad (5.45)$$

provided that q chosen is large enough (here we use that $\rho \ll \ell^{2-d}$). Notice that if q is chosen if (5.53) holds for some k, then it also holds for $h \geq k$ (but the implicit constant may degenerate).

We also argue that, for any $\delta \in (0,1)$ the event

$$G_0 = \left\{ \lambda_0(D_\ell) \leq T\ell^d\delta \right\} \qquad (5.46)$$

has large probability: for every $\alpha > 0$, it holds

$$\mathbb{P}(G_0^c) \lesssim_{\delta,\alpha} T^{-\alpha}, \qquad (5.47)$$

Indeed, by stationarity and Lemma 3.2 it holds, for every $q \geq 1$,

$$\mathbb{E}\left[\left(I_{\{\tau_{1,\ell}^i=0\}} \int_0^{\rho\wedge\tau_{1,L}^i} I_{\{B_s\in D_\ell\}}ds \right)^q \right] \lesssim \ell^d \cdot \ell^{2q}. \qquad (5.48)$$

By Rosenthal inequality for positive variables,

$$\mathbb{E}\left[\lambda_0(D_\ell)^q\right] = \mathbb{E}\left[\left(\sum_{i\in S_0} \int_0^{\rho\wedge\tau_{1,L}^i} I_{\{B_s\in D_\ell\}}ds \right)^q \right] \\ \lesssim n\ell^d\cdot\ell^{2q} + \left(n\ell^d\cdot\ell^2 \right)^q \ll n\ell^d\ell^{2q}. \qquad (5.49)$$

having used

$$n\ell^d \sim \frac{T}{\rho}\ell^d = T^{1-\gamma_\rho-d\gamma} \ll 1, \qquad (5.50)$$

which holds true if we assume that

$$1 - \gamma_\rho - d\gamma < 0. \qquad (5.51)$$

By Markov inequality,

$$\mathbb{P}(G_0^c) \lesssim_\delta \frac{T\rho^{-1}\ell^d\ell^{2q}}{(T\ell^d)^q} = \frac{T\rho^{-1}\ell^d}{(T\ell^{d-2})^q} \lesssim T^{-\alpha} \qquad (5.52)$$

provided that q chosen is large enough.

The (possible) dependence upon k in the bounds (5.42) and (5.52) is taken into account as follows. Set

$$H_k = \left\{ \sum_{h\geq k} \lambda_h(D_\ell) \leq T\ell^d\delta \right\}, \qquad (5.53)$$

and apply Corollary 3.9, for every $q \geq 1$,

$$\mathbb{E}\left[\left(I_{\{i\in S_k\}} \sum_{h\geq k} \int_{\tau_{h,\ell}^i}^{\rho\wedge\tau_{h,L}^i} I_{\{B_s\in D_\ell\}}ds \right)^q \right] \lesssim \mathbb{P}(\tau_{k,\ell}^i \leq \rho)\rho^q \lesssim_k (\rho\ell^{d-2})^k \cdot \rho^q, \qquad (5.54)$$

having bounded from above the total mass by ρ. By Rosenthal inequality for positive variables,

$$\mathbb{E}\left[\left(\sum_{h\geq k} \lambda_h(D_\ell) \right)^q \right] \lesssim n\cdot(\rho\ell^{d-2})^k \cdot \rho^q + \left(n(\rho\ell^{d-2})^k\cdot\rho \right)^q \lesssim T\ell^{d-2}(\rho\ell^{d-2})^{k-1}\rho^q. \quad (5.55)$$

Hence, by Markov inequality and simply choosing $q = 1$,

$$\mathbb{P}(H_k^c) \lesssim_{k,\delta} \frac{T\ell^{d-2}(\rho\ell^{d-2})^{k-1}\rho}{T\ell^d} = (\rho\ell^{d-2})^{k-1}\rho/\ell^2 \tag{5.56}$$
$$\lesssim T^{-k[\gamma(d-2)-\gamma_\rho]+\gamma d} \lesssim T^{-\alpha}$$

for every $\alpha > 0$, provided that we now pick $k = k(\alpha)$ sufficiently large (recall that by assumption $\gamma(d-2) > \gamma_\rho$).

Step 5 (Removal of bad event). In this step we argue that it is sufficient to show (5.31) where expectations are restricted to the event

$$A = G_1 \cap G_{0,\delta} \cap \bigcap_{k=2}^{\bar{k}-1} G_{k,\delta} \cap H_{\bar{k},\delta}, \tag{5.57}$$

for a suitable choice of $\bar{k}$. Indeed, since we can always bound the transportation costs by $T\ell^p$, the expectation over the event A^c will contribute by a quantity not larger than

$$\mathbb{P}(A^c) \cdot T\ell^p \ll T^{1-p/(d-2)}\ell^d \tag{5.58}$$

provided that

$$\mathbb{P}(A^c) \ll T^{-p/(d-2)}\ell^{d-p} = T^{-p/(d-2)-\gamma(d-p)}, \tag{5.59}$$

which can be achieved via the union bound and the inequalities (5.37), (5.52) and (5.53) obtained in the previous step. Precisely, if we fix some $\alpha < -p/(d-2) - \gamma(d-p)$, then any $\bar{k}$ such that (5.53) holds will be a possible choice. Then, for each $k < \bar{k}$, we can apply (5.53).

Step 6 (Proof of (5.31) on A). If A holds, then we are in a position to apply Lemma 2.6 on $\Omega = D_\ell$, $\mu = \tilde{\lambda}_1$ and

$$\lambda = \lambda_0 + \sum_{k=2}^{\bar{k}-1} \lambda_k + \left(\sum_{h\geq\bar{k}} \lambda_h\right) \tag{5.60}$$

To keep the exposition simple, let us argue in the case $p > d/(d-1)$ (the case $p \leq d/(d-1)$ is discussed in a separate step below). Taking expectation, we see that to obtain (5.31) it sufficient to show, with $\nu = \lambda_0$, $\nu = \lambda_k$ for $k \in \{2,\dots,\bar{k}-1\}$, and $\nu = \sum_{h\geq\bar{k}} \lambda_h$,

$$\mathbb{E}\left[I_A\nu(D_\ell)\left(\frac{\nu(D_\ell)}{\delta\tilde{\lambda}_1(D_\ell)/\bar{k}}\right)^{p/d}\right] \leq \ell^p\mathbb{E}\left[I_A\nu(D_\ell)\left(\frac{\nu(D_\ell)}{\delta cT\ell^d/\bar{k}}\right)^{p/d}\right] \ll T^{1-p/(d-2)}\ell^d, \tag{5.61}$$

having used that $\mu(D_\ell) = \tilde{\lambda}_1(D_\ell) \geq cT\ell^d$ on A. If $\nu = \lambda_k$ with $k \geq 2$, then by (5.42) with $q = 1 + p/d$, we have

$$T^{-p/d}\mathbb{E}\left[\lambda_k(D_\ell)^{1+p/d}\right] \lesssim T^{1-p/d}\ell^d(\rho\ell^{d-2})^{k-1}\ell^{2p/d} \ll T^{1-p/(d-2)}\ell^d \tag{5.62}$$

provided that we impose the condition

$$(k-1)[\gamma(d-2)-\gamma_\rho] + 2\gamma p/d > 2p/(d(d-2)). \tag{5.63}$$

Notice that since the square bracket is positive, it is sufficient to ensure it holds for $k = 2$.

If instead $\nu = \lambda_0$, by (5.49) again with $q = 1 + p/d$, we have

$$T^{-p/d}\mathbb{E}\left[\lambda_0(D_\ell)^{1+p/d}\right] \lesssim T^{1-p/d}\rho^{-1}\ell^d\ell^{2+2p/d} \ll T^{1-p/(d-2)}\ell^d \tag{5.64}$$

provided that the additional inequality holds:

$$-\gamma_\rho + 2\gamma + 2\gamma p/d > 2p/(d(d-2)). \tag{5.65}$$

Finally, if $\nu = \sum_{h \geq \bar{k}} \lambda_h$, by (5.55) with $q = 1 + p/d$, we have

$$T^{1-p/d} \mathbb{E}\left[\left(\sum_{h \geq \bar{k}} \lambda_h(D_\ell)\right)^{1+p/d}\right] \lesssim T\ell^{d-2}(\rho\ell^{d-2})^{\bar{k}-1}\rho^{1+p/d} \ll T^{1-p/(d-2)}\ell^d \qquad (5.66)$$

provided that we impose

$$(\bar{k}-1)(\gamma(d-2) - \gamma_\rho) - 2\gamma - \gamma_\rho(1+p/d) > p/(d-2), \qquad (5.67)$$

which can be satisfied by choosing $\bar{k}$ sufficiently large. This settles the validity of (5.31).

Step 7 (Adjusting for longer first visits). Our next aim is to replace the measure $\tilde{\lambda}_1$, where the first visit is interrupted at time ρ, with the measure

$$\sum_{i \in S} \int_{\tau_{1,\ell}^i}^{\tau_{1,L}^i} \delta_{B_s^i} ds = \tilde{\lambda}_1 + \nu, \qquad (5.68)$$

where we let

$$\nu = \sum_{i \in S} \int_{\rho \wedge \tau_{1,L}^i}^{\tau_{1,L}^i} \delta_{B_s^i} ds. \qquad (5.69)$$

This is achieved by proving

$$\left| \mathbb{E}\left[W_{D_\ell}^p\left(\tilde{\lambda}_1\right)\right] - \mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i \in S} \int_{\tau_{1,\ell}^i}^{\tau_{1,L}^i} \delta_{B_s^i} ds\right)\right] \right| \ll T^{1-p/(d-2)}\ell^d, \qquad (5.70)$$

again as a consequence of Lemma 2.6. We first argue that, for every $q \geq 1$,

$$\mathbb{E}\left[I_{A^i}\left(\int_{\rho \wedge \tau_{1,L}^i}^{\tau_{1,L}^i} I_{\{B_s^i \in D_\ell\}} ds\right)^q\right] \lesssim L^2 |\log \ell|\ell^{d-2} \cdot \ell^{2q}. \qquad (5.71)$$

Indeed, let us introduce a parameter $0 \leq \tilde{\rho} \leq \rho$ and split the event into two alternatives:

$$A^i = \left\{0 < \tau_{1,\ell}^i \leq \rho\right\} = \left\{0 < \tau_{1,\ell}^i \leq \tilde{\rho}\right\} \cup \left\{\tilde{\rho} < \tau_{1,\ell}^i \leq \rho\right\} \qquad (5.72)$$

In the case the first alternative holds, then it must be $\tau_{1,L}^i - \tau_{1,\ell}^i > \rho - \tilde{\rho}$ in order to have a non null contribution. Hence, by using Proposition 3.7 and the strong Markov property, writing $\tilde{B}^i = \theta_{\tau_{1,\ell}^i} B^i$ we bound from above

$$\begin{aligned}
\mathbb{E}&\left[I_{\left\{0 < \tau_{1,\ell}^i \leq \tilde{\rho}\right\}}\left(\int_{\rho \wedge \tau_{1,L}^i}^{\tau_{1,L}^i} I_{\{B_s^i \in D_\ell\}} ds\right)^q\right] \\
&\leq (\rho\ell^{d-2})\mathbb{E}\left[I_{\left\{\tau_{D_L^c} > \rho - \tilde{\rho}\right\}}\left(\int_0^{\tau_{D_L^c}} I_{\{\tilde{B}_s^i \in D_\ell\}} ds\right)^q\right] \\
&\lesssim (\rho\ell^{d-2})\mathbb{P}\left(\tau_{D_L^c} > \rho - \tilde{\rho}\right)^{1/2}\mathbb{E}\left[\left(\int_0^{\tau_{D_L^c}} I_{\{\tilde{B}_s^i \in D_\ell\}} ds\right)^{2q}\right]^{1/2} \\
&\lesssim_q \rho\ell^{d-2}\exp\left(-c\frac{\rho - \tilde{\rho}}{L^2}\right)\ell^{2q},
\end{aligned} \qquad (5.73)$$

having used Lemma 3.8 and (3.11) in the last line. If instead the second alternative holds in (5.72), bound similarly but using only that its probability is small, again by Lemma 3.8:

$$\mathbb{E}\left[I_{\left\{0 < \tau_{1,\ell}^i \leq \tilde{\rho}\right\}}\left(\int_{\rho \wedge \tau_{1,L}^i}^{\tau_{1,L}^i} I_{\{B_s^i \in D_\ell\}} ds\right)^q\right] \leq (\rho - \tilde{\rho})\ell^{d-2}\mathbb{E}\left[\left(\int_0^{\tau_{D_L^c}} I_{\{\tilde{B}_s^i \in D_\ell\}} ds\right)^q\right]$$
$$\lesssim_q (\rho - \tilde{\rho})\ell^{d-2} \cdot \ell^{2q}. \qquad (5.74)$$

Recalling that $L = T^{-\gamma_L}$, letting $\tilde{\rho} = \rho - \tilde{c}L^2 \log(1/L)$ for a large enough constant $\tilde{c}$ yields (5.71). Rosenthal inequality for positive variables yields

$$\mathbb{E}\left[\nu(D_\ell)^q\right] \lesssim nL^2|\log\ell|\ell^{d-2} \cdot \ell^{2q} + \left(nL^2|\log\ell|\ell^{d-1}\right)^q \ell^{2q} \lesssim T\rho^{-1}L^2|\log\ell|\ell^{d-2} \cdot \ell^{2q}, \quad (5.75)$$

having used

$$nL^2|\log\ell|\ell^{d-2} \sim T\rho^{-1}L^2\ell^{d-2}|\log\ell| \ll 1 \quad (5.76)$$

provided that the following condition holds:

$$1 - \gamma_\rho - 2\gamma_L - (d-2)\gamma_\ell < 0. \quad (5.77)$$

Starting from (5.75), we now argue similarly as in the previous steps. Given $\delta \in (0,1)$, we introduce the event

$$\tilde{G}_1 = \tilde{G}_{1,\delta} = \left\{\nu(D_\ell) \leq \delta T\ell^d\right\}, \quad (5.78)$$

and argue by Markov inequality that

$$\mathbb{P}\left(\tilde{G}_1^c\right) \lesssim_\delta \frac{T\rho^{-1}L^2|\log\ell|\ell^{d-2} \cdot \ell^{2q}}{(T\ell^{d-2})^q \cdot \ell^{2q}} \ll T^{-\alpha}, \quad (5.79)$$

for every $\alpha > 0$, provided that q is chosen large enough. Using this fact, we may reduce the proof of (5.70) where the expectations are performed on the event

$$A = G_1 \cap \tilde{G}_1 \quad (5.80)$$

with G_1 as in (5.37). After applying Lemma 2.6, everything is once again reduced to prove that

$$T^{-p/d}\mathbb{E}\left[\nu(D_\ell)^{1+p/d}\right] \ll T^{1-p/(d-2)}\ell^d, \quad (5.81)$$

which is seen to be a consequence of (5.75) with $q = 1 + p/d$, provided that

$$-\gamma_\rho + 2\gamma_L + 2\gamma p/d > 2p/(d(d-2)). \quad (5.82)$$

Step 8 (Application of Lemma 4.5) Having settled (5.70), the thesis is reduced to prove that

$$\lim_{T\to\infty} \mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i\in S}\int_{\tau_{1,\ell}^i}^{\tau_{1,L}^i}\delta_{B_s^i}ds\right)\right] / \left(T^{1-p/(d-2)}|D_\ell|\right) = \mathsf{c}(\mathcal{I},d,p). \quad (5.83)$$

Let us define $\tilde{B}^i := \theta_{\tau_{1,\ell}^i}B^i$, so that we can rewrite it as

$$\lim_{T\to\infty} \mathbb{E}\left[W_{D_\ell}^p\left(\sum_{i\in S}\mu_{\tau_{D_L^c}}^{\tilde{B}^i}\tilde{B}^i\right)\right] / \left(T^{1-p/(d-2)}|D_\ell|\right) = \mathsf{c}(\mathcal{I},d,p). \quad (5.84)$$

We condition upon the event $\{S = s\}$ for a subset $s \subseteq \{1,\ldots,n\}$, i.e., the event A_i holds if and only if $i \in s$. By Proposition 3.7, the conditional hitting law ν_ρ of $B_{\tau_{1,\ell}^i}^i = \tilde{B}_0^i$ is quantitatively close to the uniform distribution on ∂D_ℓ. Moreover, each $\tilde{B}^i$ is still a Brownian motion on $\mathbb{T}^d$ and independence among the $\tilde{B}^i$'s is preserved. Furthermore, since we are considering $\tilde{B}^i$ up to $\tau_{D_L^c}\tilde{B}^i$, we can safely lift each of them to a Brownian motion on $\mathbb{R}^d$, without changing the optimal transport cost on D_ℓ.

Applying (3.11) with $q = 1$, we easily obtain that

$$\mathbb{E}\left[\sum_{i\in S}\mu_{\tau_{D_L^c}}^{\tilde{B}^i}B^i(D_\ell)\right] \lesssim n \cdot \rho\ell^{d-2}\ell^2 \lesssim T\ell^d \quad (5.85)$$

Thus,

$$\mathbb{E}\left[\sum_{i\in S}\mu_{\tau_{D_L^c}}^{\tilde{B}^i}B^i(D_\ell)\right]W_{D_\ell}^p\left(\nu_\rho, \tilde{e}_{D_\ell}\right) \lesssim T\ell^d \cdot \ell^p(\rho\ell^{d-2})|\log\ell| \ll T^{1-p/(d-2)}\ell^d, \quad (5.86)$$

provided that we impose

$$\gamma p + [\gamma(d-2) - \gamma_\rho] > p/(d-2). \tag{5.87}$$

By applying Lemma 4.5, it follows that we can replace each $\tilde{B}^i$ in (5.84) with a Brownian motion $\bar{B}^i$ with initial law $\tilde{e}_{D_\ell}$, i.e., uniform on ∂D_ℓ, and the summation is now performed over a random set S, independent of the processes $(\bar{B}^i)_{i=1}^n$.

Step 9 (Time enlargement up to ∞). In order to apply Theorem 1.1, we need to enlarge the time integration from $\tau_{D_L^c}^i$ to ∞. This is achieved along the same lines as before, i.e. by isolating a good event and applying Lemma 2.6 to prove

$$\left| \mathbb{E}\left[W_{D_\ell}^p \left(\sum_{i \in S} \mu_{\tau_{D_L^c}^i \bar{B}^i}^{\bar{B}^i} \right) \right] - \mathbb{E}\left[W_{D_\ell}^p \left(\sum_{i \in S} \mu_\infty^{\bar{B}^i} \right) \right] \right| \ll T^{1-p/(d-2)} \ell^d. \tag{5.88}$$

Indeed, if we let

$$\bar{G}_1 = \left\{ \sum_{i \in S} \mu_{\tau_{D_L^c}^i \bar{B}^i}^{\bar{B}^i}(D_\ell) \geq cT\ell^d \right\} \tag{5.89}$$

arguing as in (5.38) (it is actually simpler here because S is now independent of the $\bar{B}^i$'s, hence there is no need of the strong Markov property), we obtain that for a suitable $c = c(d) > 0$ it holds

$$\mathbb{P}\left(\bar{G}_1^c\right) \lesssim_\alpha T^{-\alpha} \tag{5.90}$$

for every $\alpha > 0$. Moreover, arguing as in the proof of (5.42), we obtain that, for $q \geq 1$,

$$\mathbb{E}\left[\left(\sum_{i \in S} \mu_\infty^{\theta_{\tau_{D_L^c}} \bar{B}^i}(D_\ell) \right)^q \right] \lesssim n \cdot \rho\ell^{d-2} \cdot \left(\frac{\ell}{L} \right)^{d-2} \cdot \ell^{2q} + \left(n \cdot \rho\ell^{d-2} \cdot \left(\frac{\ell}{L} \right)^{d-2} \right)^q \cdot \ell^{2q}$$

$$\lesssim T\ell^{d-2} \cdot \left(\frac{\ell}{L} \right)^{d-2} \cdot \ell^{2q} \tag{5.91}$$

provided that

$$T\ell^{d-2} \cdot \left(\frac{\ell}{L} \right)^{d-2} \ll 1, \tag{5.92}$$

which holds true if we impose the condition

$$\gamma(d-2) + (d-2)(\gamma_\ell - \gamma_L) > 1. \tag{5.93}$$

By Markov inequality, we see that letting

$$\bar{G}_2 = \left\{ \sum_{i \in S} \mu_\infty^{\theta_{\tau_{D_L^c}} \bar{B}^i}(D_\ell) \leq \delta T\ell^d \right\}, \tag{5.94}$$

it holds

$$\mathbb{P}\left(\bar{G}_2^c\right) \lesssim \frac{T\ell^{d-2} \cdot \left(\frac{\ell}{L} \right)^{d-2} \cdot \ell^{2q}}{(T\ell^{d-2})^q \ell^{2d}} \lesssim T^{-\alpha} \tag{5.95}$$

for every $\alpha > 0$, provided that q is chosen sufficiently large. Repeating the by now usual argument, we reduce first ourselves to the proof of (5.88) where expectations are performed on $A = \bar{G}_1 \cap \bar{G}_2$, and then applying Lemma 2.6 we see that, in order to conclude it is sufficient to show

$$T^{-p/d}\mathbb{E}\left[\left(\sum_{i \in S} \mu_\infty^{\theta_{\tau_{D_L^c}} \bar{B}^i}(D_\ell) \right)^{1+p/d} \right] \ll T^{1-p/(d-2)} \ell^d. \tag{5.96}$$

This easily follows from (5.91), provided that the condition

$$2\gamma p/d + (d-2)(\gamma - \gamma_L) > 2p/(d(d-2)) \tag{5.97}$$

holds true.

Step 10 (Application of Theorem 1.1). Thanks to (5.88), the thesis is reduced to prove that

$$\lim_{T \to \infty} \mathbb{E}\left[W_{D_\ell}^p \left(\sum_{i \in S} \mu_\infty^{\bar{B}^i} \right) \right] / T^{1-p/(d-2)} |D_\ell| = \mathsf{c}(\mathcal{I}, d, p), \tag{5.98}$$

where $(\bar{B}^i)_{i=1}^\infty$ are independent Brownian motions on $\mathbb{R}^d$, with common initial law $\tilde{e}_{D_\ell}$. Recalling also the definition of S (which is now independent of the Brownian motions), we can introduce the function

$$f(m) = \mathbb{E}\left[W_{D_1}^p \left(\sum_{i=1}^m \mu_\infty^{B^i} \right) \right], \tag{5.99}$$

where $(B^i)_{i=1}^\infty$ are independent Brownian motions on $\mathbb{R}^d$ with common initial law $\tilde{e}_{D_1}$, so that a straightforward rescaling gives

$$\mathbb{E}\left[W_{D_\ell}^p \left(\sum_{i \in S} \mu_{\tau_L^i}^{\tilde{B}^i} \right) \Big| \sharp S = m \right] = \ell^{p+2} f(m). \tag{5.100}$$

By Theorem 1.1, we have

$$\lim_{m \to \infty} f(m) / \left(\frac{m}{\mathrm{Cap}(D_1)} \right)^{1-p/(d-2)} = \mathsf{c}(\mathcal{I}, d, p) |D_1| \tag{5.101}$$

Moreover, $\sharp S$ has binomial law with parameters $n = n(T) \to \infty$, $p = p(T) \to 0$ satisfying

$$np = T\ell^{d-2}(\mathrm{Cap}(D_1) + o(1)). \tag{5.102}$$

By Remark B.2, it follows that

$$\lim_{T \to \infty} \mathbb{E}\left[f(\sharp S) \right] / \left(\frac{\mathbb{E}\left[\sharp S \right]}{\mathrm{Cap}(D_1)} \right)^{1-p/(d-2)} = \mathsf{c}(\mathcal{I}, d, p) |D_1|. \tag{5.103}$$

However, we also see that

$$\lim_{T \to \infty} T^{1-p/(d-2)} \ell^{d-2-p} / \left(\frac{\mathbb{E}\left[\sharp S \right]}{\mathrm{Cap}(D_1)} \right)^{1-p/(d-2)} = 1, \tag{5.104}$$

hence (5.96) holds.

Step 11 (Choice of the parameters). In order to complete the proof, we need to recall the multiple conditions imposed on γ, γ_ρ, γ_σ and γ_L and check that they can be all satisfied. Let us report them here for clarity: we have (5.2) and (5.23), which read

$$0 < \gamma_\sigma < \gamma_\rho < \gamma(d-2) \quad \text{and} \quad 0 < \gamma_L < \gamma. \tag{5.105}$$

Then, we have (5.8)

$$\gamma_\rho - \gamma_\sigma > p\left(1/(d-2) - \gamma \right), \tag{5.106}$$

and the similar conditions (5.44), (5.51), (5.77)

$$1 - \gamma(d-2) - [(d-2)\gamma - \gamma_\rho] < 0,$$
$$1 - \gamma_\rho - d\gamma < 0, \tag{5.107}$$
$$1 - \gamma_\rho - 2\gamma_L - (d-2)\gamma_\ell < 0.$$

Next, we have (5.63), which must be satisfied for $k = 2$ (for larger values of k it will automatically follow):

$$[\gamma(d-2) - \gamma_\rho] + 2\gamma p/d > 2p/(d(d-2)). \tag{5.108}$$

Similar conditions are (5.65), (5.82) and (5.97):

$$-\gamma_\rho + 2\gamma_L + 2\gamma p/d > 2p/(d(d-2))$$
$$-\gamma_\rho + 2\gamma_L + 2\gamma p/d > 2p/(d(d-2)) \tag{5.109}$$
$$2\gamma p/d + (d-2)(\gamma - \gamma_L) > 2p/(d(d-2)).$$

Finally, we have (5.87) and (5.93)

$$\gamma p + [\gamma(d-2) - \gamma_\rho] > p/(d-2), \quad \gamma(d-2) + (d-2)(\gamma_\ell - \gamma_L) > 1. \tag{5.110}$$

It is now an elementary exercise to check that all these conditions can be satisfied by choosing γ sufficiently close (but smaller than) $1/(d-2)$, $0 < \gamma_\sigma < \gamma_\rho$ sufficiently small, and γ_L sufficiently close to γ. This concludes the proof in the case $p > d/(d-1)$. In the following step we remark how to modify the argument in the case $p \leq d/(d-1)$.

Step 12 (The case $p \leq d/(d-1)$). In this case, we cannot apply directly Lemma 2.6 but following Remark 2.7 we need also to preliminarily fix an additional parameter $r = r(p, d) > 1$ such that $pr > d/(d-1)$, and take into account that all the applications of Lemma 2.6 will have the additional terms (2.47). For brevity, we do not perform explicitly all the computations but show e.g. what becomes of the first application of Lemma 2.6 in Step 6. The bound (5.61) must now be complemented with

$$\ell^p \mathbb{E}\left[I_A \tilde{\lambda}_1(D_\ell)^{1-1/r-p/d} \nu(D_\ell)^{1/r+p/d}\right] \ll T^{1-p/(d-2)}\ell^d, \tag{5.111}$$

with $\nu = \lambda_0$, $\nu = \lambda_k$ for $k = 2, \ldots, \bar{k} - 1$, and $\nu = \sum_{h=\bar{k}}^\infty \lambda_k$. For simplicity, let us focus on the case $\nu = \lambda_k$. If $1/r + p/d \leq 1$, then by Hölder inequality and (5.42) with $q = 1$ we obtain

$$\ell^p \mathbb{E}\left[I_A \tilde{\lambda}_1(D_\ell)^{1-1/r-p/d} \lambda_k(D_\ell)^{1/r+p/d}\right] \leq \ell^p \mathbb{E}\left[\tilde{\lambda}_1(D_\ell)\right]^{1-1/r-p/d} \mathbb{E}\left[\lambda_k(D_\ell)\right]^{1/r+p/d}$$

$$\lesssim \ell^p (T\ell^d)^{1-1/r-p/d}(T\ell^d(\rho\ell^{d-2})^{k-1})^{1/r+p/d}$$

$$= \ell^p T\ell^d(\rho\ell^{d-2})^{(k-1)(1/r+p/d)} \ll T^{1-p/(d-2)}\ell^d, \tag{5.112}$$

provided that we impose

$$\gamma p + [\gamma(d-2) - \gamma_\rho](k-1)(1/r + p/d) > p/(d-2), \tag{5.113}$$

which can be safely added to all the conditions found so far, again by choosing γ sufficiently close (but smaller) to $1/(d-2)$ (recall that $k \geq 2$). In the case $1/r + p/d > 1$, we use the fact that on A we have $\tilde{\lambda}_1(D_\ell) \gtrsim T\ell^d$, hence using (5.42) with $q = 1/r + p/d$,

$$\ell^p \mathbb{E}\left[I_A \tilde{\lambda}_1(D_\ell)^{1-1/r-p/d} \lambda_k(\Omega)^{1/r+p/d}\right] \lesssim \ell^p (T\ell^d)^{1-1/r-p/d} \mathbb{E}\left[\lambda_k(\Omega)^{1/r+p/d}\right]$$

$$\lesssim T^{-p/d}(T\ell^d)^{1-1/r} \cdot T\ell^{d-2}(\rho\ell^{d-2})^{k-1}\ell^{2(1/r+p/d)}$$

$$= T^{1-p/d}\ell^d(T\ell^{d-2})^{1-1/r}(\rho\ell^{d-2})^{k-1}\ell^{2p/d}$$

$$\ll T^{1-p/(d-2)}\ell^d, \tag{5.114}$$

provided that we impose the condition

$$2\gamma p/d - (1 - 1/r)(1 - \gamma(d-2)) + (k-1)[\gamma(d-2) - \gamma_\rho] > 2p/(d(d-2)). \tag{5.115}$$

Since r is fixed and $k \geq 2$ we see that this holds, again provided that we choose γ sufficiently close (but smaller) to $1/(d-2)$. $\qquad \square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Write for brevity $\mu_T = \mu_T^B$. Given $\bar{\gamma}$ as in Proposition 5.1, we choose $\gamma \in (\bar{\gamma}, 1/(d-2))$, set $\ell = T^{-\gamma}$ and consider the Borel family of sets $(D_\ell(z))_{z \in \mathbb{T}^d}$, which satisfies

$$\int_{\mathbb{T}^d} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} = 1. \tag{5.116}$$

By Lemma 2.2 with $E = \mathbb{T}^d$ and $\sigma(dz) = |D_\ell|^{-1} dz$, we obtain

$$W_{\mathbb{T}^d}^p (\mu_T) \leq (1+\varepsilon) \int_{\mathbb{T}^d} W_{D_\ell(z)}^p (\mu_T) \frac{dz}{|D_\ell|} + \frac{c}{\varepsilon^{(p-1)^+}} W_{\mathbb{T}^d}^p \left(\int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} \right). \tag{5.117}$$

If $p > 1$, we apply (2.29) to the second term, obtaining

$$W_{\mathbb{T}^d}^p \left(\int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} \right) \lesssim T^{1-p} \left\| \int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} - T \right\|_{L^p(\mathbb{T}^d)}^p. \tag{5.118}$$

By convexity, we bound from above

$$
\begin{aligned}
\left\| \int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} - T \right\|_{L^p(\mathbb{T}^d)}^p &= \left\| \int_{\mathbb{T}^d} \left(\frac{\mu_T(D_\ell(z))}{|D_\ell|} - T \right) \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} - T \right\|_{L^p(\mathbb{T}^d)}^p \\
&\leq \int_{\mathbb{T}^d} \left\| \left(\frac{\mu_T(D_\ell(z))}{|D_\ell|} - T \right) \chi_{D_\ell(z)} \right\|_{L^p(\mathbb{T}^d)}^p \frac{dz}{|D_\ell|} \\
&\leq \int_{\mathbb{T}^d} \left| \frac{\mu_T(D_\ell(z))}{|D_\ell|} - T \right|^p dz \\
&= |D_\ell|^{-p} \int_{\mathbb{T}^d} |\mu_T(D_\ell(z)) - T|D_\ell(z)||^p \, dz
\end{aligned} \tag{5.119}
$$

After taking expectation, consider first the case $p \geq 1$. By Lemma 3.5, we obtain that, for $T \geq 1$,

$$
\begin{aligned}
\mathbb{E} \left[W_{\mathbb{T}^d}^p \left(\int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} \right) \right] &\lesssim T^{1-p} |D_\ell|^{-p} |D_\ell|^{p \min\left\{ 1, \frac{1}{p} + \frac{1}{d} \right\}} T^{p/2} \\
&\lesssim T^{1-p/2} \ell^{-(p+d-pd)^+}.
\end{aligned} \tag{5.120}
$$

In order to ensure that this term gives an infinitesimal contribution in the limit, we need to further require that

$$\gamma < \frac{p}{(p+d-dp)^+} \left(\frac{1}{2} - \frac{1}{d-2} \right), \tag{5.121}$$

but it is simple to check that the right hand side is larger than $1/(d-2)$.

To cover the case $p \in (0, 1]$, and in particular that of $d \in \{3, 4\}$, and $p < (d-2)/2$, we use (2.18), which yields (5.120) with $p = 1$. Hence,

$$\mathbb{E} \left[W_{\mathbb{T}^d}^p \left(\int_{\mathbb{T}^d} \frac{\mu_T(D_\ell(z))}{|D_\ell|} \chi_{D_\ell(z)} \frac{dz}{|D_\ell|} \right) \right] \lesssim T^{1/2} \ll T^{1-p/(d-2)}. \tag{5.122}$$

Thus, we only need to focus on the first term in the right hand side of (5.117). By stationarity, the law of $\mu_T \llcorner D_\ell(z)$ does not depend on $z \in \mathbb{T}^d$, hence

$$\mathbb{E} \left[\int_{\mathbb{T}^d} W_{D_\ell(z)}^p (\mu_T) \frac{dz}{|D_\ell|} \right] = \int_{\mathbb{T}^d} \mathbb{E} \left[W_{D_\ell(z)}^p (\mu_T) \right] \frac{dz}{|D_\ell|} = |D_\ell|^{-1} \mathbb{E} \left[W_{D_\ell}^p (\mu_T) \right]. \tag{5.123}$$

Thus, the thesis follows by Proposition 5.1. $\qquad \square$

6. A concentration result

In this final section, we establish some concentration properties for the optimal transport cost associated to the occupation measure of a stationary Brownian motion on $\mathbb{T}^d$.

Proposition 6.1. *Let B be a stationary Brownian motion on $\mathbb{T}^d$, $d \geq 3$. Then, for every $p > 0$ such that $p < (d-2)/2$ if $d \in \{3, 4\}$ or*

$$p < \frac{d-2}{2} \cdot \frac{1+2p/d}{1+p/d}, \quad \text{if } d \geq 5, \tag{6.1}$$

it holds $\mathbb{P}$*-a.s.*

$$\lim_{T\to\infty} \left| W_{\mathbb{T}^d}^p(\mu_T^B) - \mathbb{E}\left[W_{\mathbb{T}^d}^p(\mu_T^B) \right] \right| / T^{1-p/(d-2)} = 0. \tag{6.2}$$

Notice that (6.1) is satisfied if $p \le (d-2)/2$.

Proof. The strategy is to argue as in Step 1 and Step 2 in the proof of Proposition 5.1, but on the whole $\mathbb{T}^d$, and then apply a standard concentration argument in the independent case.

Step 1 (Time splitting). We introduce two parameters γ_ρ, γ_σ, such that

$$0 < \gamma_\sigma < \gamma_\rho < 1 \tag{6.3}$$

to be further specified below, and set

$$n := \lfloor T^{1-\gamma_\rho} \rfloor, \quad \sigma := T^{\gamma_\sigma}, \quad \rho := \frac{T}{n} - \sigma. \tag{6.4}$$

and decompose

$$\mu_T = \sum_{i=0}^{n-1} \mu_\rho^{\theta_{i(\rho+\sigma)}B} + \sum_{i=0}^{n-1} \mu_\sigma^{\theta_{i(\rho+\sigma)+\rho}B} := \tilde{\mu}_T + \lambda_T \tag{6.5}$$

We trivially have

$$\tilde{\mu}_T(\mathbb{T}^d) = n\rho \sim T \quad \text{and} \quad \lambda_T(\mathbb{T}^d) = n\sigma \sim T^{1-\gamma_\rho+\gamma_\sigma}, \tag{6.6}$$

Thus,

$$\lambda_T(\mathbb{T}^d)/\tilde{\mu}_T(\mathbb{T}^d) = T^{\gamma_\sigma-\gamma_\rho} \ll 1 \tag{6.7}$$

and we are in a position to apply Lemma 2.6 with any given δ provided that T is sufficiently large. Assuming for simplicity that $p > d/(d-1)$ (to cover the case $p \le d/(d-1)$ one has to argue as in Step 12 of the proof of Proposition 5.1) we find, for $\varepsilon \in (0,1)$,

$$-W_{\mathbb{T}^d}^p(\tilde{\mu}_T) + (1+\varepsilon)^{-1} W_{\mathbb{T}^d}^p(\mu_T) \lesssim_\varepsilon T^{1-(\gamma_\rho-\gamma_\sigma)(1+p/d)} \ll T^{1-p/(d-2)}, \tag{6.8}$$

provided that we impose

$$\gamma_\rho - \gamma_\sigma > \frac{p}{d-2} \cdot \frac{d}{d+p}. \tag{6.9}$$

Using instead (2.36), we obtain, for every ε, δ sufficiently small, for some $c = c(\varepsilon, p, d)$,

$$W_{\mathbb{T}^d}^p(\tilde{\mu}_T) - (1+c\delta)(1+\varepsilon) W_{\mathbb{T}^d}^p(\mu_T) \lesssim_{\delta,\varepsilon} T^{1-(\gamma_\rho-\gamma_\sigma)(1+p/d)} \ll T^{1-p/(d-2)}, \tag{6.10}$$

again if we impose (6.9). Summing (6.8) with the expected value of (6.10) yields

$$\begin{aligned} &- W_{\mathbb{T}^d}^p(\tilde{\mu}_T) + \mathbb{E}\left[W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] \\ &\quad + (1+\varepsilon)^{-1} W_{\mathbb{T}^d}^p(\mu_T) - (1+c\delta)(1+\varepsilon)\mathbb{E}\left[W_{\mathbb{T}^d}(\tilde{\mu}_T) \right] \ll T^{1-p/(d-2)}, \end{aligned} \tag{6.11}$$

which can be rewritten after some manipulations as

$$\left(W_{\mathbb{T}^d}^p(\mu_T) - \mathbb{E}\left[W_{\mathbb{T}^d}(\tilde{\mu}_T) \right] \right) \lesssim_\varepsilon \left(W_{\mathbb{T}^d}^p(\tilde{\mu}_T) + c\delta\mathbb{E}\left[W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] \right)^+ + \mathbb{E}\left[W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] + R(T), \tag{6.12}$$

where $R(T) \ll T^{1-p/(d-2)}$. Using Theorem 1.2 (actually in a weaker form)

$$\mathbb{E}\left[W_{\mathbb{T}^d}^p(\mu_T) \right] \lesssim T^{1-p/(d-2)}, \tag{6.13}$$

hence, letting first $T \to \infty$ and then $\delta \to 0$ and finally $\varepsilon \to 0$, we obtain that $\mathbb{P}$-a.s. it holds,

$$\begin{aligned} &\limsup_{T\to\infty} \left(W_{\mathbb{T}^d}^p(\mu_T) - \mathbb{E}\left[W_{\mathbb{T}^d}(\mu_T) \right] \right) / T^{1-p/(d-2)} \\ &\quad \le \limsup_{T\to\infty} \left(W_{\mathbb{T}^d}^p(\tilde{\mu}_T) - \mathbb{E}\left[W_{\mathbb{T}^d}(\tilde{\mu}_T) \right] \right)^+ / T^{1-p/(d-2)}. \end{aligned} \tag{6.14}$$

Arguing similarly, but summing the expectation of (6.8) with (6.10) yields, after some manipulations and using (6.13), that

$$
\limsup_{T \to \infty} \left(\mathbb{E} \left[W_{\mathbb{T}^d}(\mu_T) \right] - W_{\mathbb{T}^d}^p(\mu_T) \right) / T^{1-p/(d-2)}
$$
$$
\leq \limsup_{T \to \infty} \left(\mathbb{E} \left[W_{\mathbb{T}^d}(\tilde{\mu}_T) - W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] \right)^+ / T^{1-p/(d-2)}, \tag{6.15}
$$

hence, the thesis will follow if we prove that

$$
\lim_{T \to \infty} \left| W_{\mathbb{T}^d}^p(\tilde{\mu}_T) - \mathbb{E} \left[W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] \right| / T^{1-p/(d-2)} = 0. \tag{6.16}
$$

Step 2 (Breaking dependence). We write

$$
\tilde{\mu}_T := \sum_{i=1}^n \mu_\rho^{\tilde{B}^i}, \tag{6.17}
$$

where $\tilde{B}^i = \theta_{(i-1)(\rho+\sigma)} B$ are stationary (but not independent) Brownian motions on $\mathbb{T}^d$. Our aim is to argue that we can replace the processes $\tilde{B}$ with independent (and stationary) Brownian motions $\tilde{B}'$. Before we do so, we argue that we can work on a stronger version of (6.16), namely that for every $q \geq 2$,

$$
\left\| W_{\mathbb{T}^d}^p(\tilde{\mu}_T) - \mathbb{E} \left[W_{\mathbb{T}^d}^p(\tilde{\mu}_T) \right] \right\|_{L^q} \lesssim \rho^{1/2} T^{[1-p/(d-2)]/2}, \tag{6.18}
$$

which easily implies (6.16) via Borel-Cantelli lemma, provided that we impose

$$
\gamma_\rho < 1 - p/(d-2). \tag{6.19}
$$

The interesting feature of (6.18) is that it can be equivalently restated in terms of an independent copy $\bar{B}$ of $\tilde{B}$:

$$
\left\| W_{\mathbb{T}^d}^p(\tilde{\mu}_T) - W_{\mathbb{T}^d}^p(\bar{\mu}_T) \right\|_{L^q} \lesssim \rho^{1/2} T^{[1-p/(d-2)]/2} \tag{6.20}
$$

with the notation

$$
\bar{\mu}_T := \sum_{i=1}^n \mu_\rho^{\bar{B}^i}, \tag{6.21}
$$

and $\bar{B}^i = \theta_{(i-1)(\rho+\sigma)} \bar{B}$.

We then consider the following non-negative function of $2n$ continuous curves $(\tilde{x}, \bar{x}) = (\tilde{x}^i, \bar{x}^i)_{i=1}^n$ defined on the interval $[0, \rho]$ taking values on $\mathbb{T}^d$:

$$
(\tilde{x}, \bar{x}) \mapsto F(\tilde{x}, \bar{x}) = \left| W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{x^i} \right) - W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{x^i} \right) \right|^q \tag{6.22}
$$

By (2.17), we have

$$
\sup_{(\tilde{x}, \bar{x})} F(\tilde{x}, \bar{x}) \leq T^q. \tag{6.23}
$$

By $2n$ iterated applications of (2.12), we obtain

$$
\mathsf{d}_{TV} \left(\mathbb{P}_{(\tilde{B}^i, \bar{B}^i)_{i=1}^n}, \left(\mathbb{P}_{B_{s \in [0,\rho]}} \otimes \mathbb{P}_{B_{s \in [0,\rho]}} \right)^{\otimes n} \right) \lesssim n e^{-c\sigma}. \tag{6.24}
$$

This yields in particular that we can replace the (dependent) Brownian motions $\tilde{B}$ and $\bar{B}$ in the expectation of F, with $2n$ stationary and independent Brownian motions, with an error term that is bounded from above by $T^q n e^{-c\sigma} \ll T^{-\alpha}$, for every $\alpha > 0$.

Step 3 (Concentration via Poincaré inequality). We now are left with the task of showing that

$$
\left\| W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right) - \mathbb{E} \left[W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{x^i} \right) \right] \right\|_{L^q} \lesssim \rho^{1/2} T^{[1-p/(d-2)]/2}, \tag{6.25}
$$

where the Brownian motions $(B^i)_{i=1}^n$ are now independent, and stationary, with values in $\mathbb{T}^d$. Representing each $B^i = Z^i + \tilde{B}^i$, with Z^i independent and uniform variables on $\mathbb{T}^d$ and $\tilde{B}^i$ the projection on $\mathbb{T}^d$ of a (independent) standard Brownian motion on $\mathbb{R}^d$, we can write

$$W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right) = G\left((Z^i)_{i=1}^n, (\tilde{B}^i)_{i=1}^n \right) \tag{6.26}$$

where

$$G((z^i)_{i=1}^n, (\tilde{x}^i)_{i=1}^n) = W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{z^i+\tilde{x}^i} \right). \tag{6.27}$$

The concentration bound (6.25) follows from the Poincaré inequality on the space $(\mathbb{T}^d)^n \times \left(C_0([0,\rho], \mathbb{R}^d) \right)^n$, endowed with a suitable product of uniform measure on $\mathbb{T}^d$ and Wiener measure on $C_0([0,\rho], \mathbb{R}^d)$. Indeed, since $W_{\mathbb{T}^d}^p(\cdot)$ is expressed as minimization problems, a simple approximation argument (e.g. by discretizing the paths x^i) yields that

$$(z^i)_{i=1}^n, (\tilde{x}^i)_{i=1}^n \mapsto G((z^i)_{i=1}^n, (\tilde{x}^i)_{i=1}^n) \tag{6.28}$$

is Sobolev (in the sense of Gaussian-Malliavin calculus with respect to the variables $\tilde{x}^i$'s), with (squared) modulus of the gradient estimated by

$$\left| \nabla G((z^i)_{i=1}^n, (\tilde{x}^i)_{i=1}^n) \right|^2 \lesssim (1+\rho) \int_{\mathbb{T}^d} \mathsf{d}(x,y)^{2(p-1)} d\pi^*(dx,dy), \tag{6.29}$$

where π^* denote an optimal transport plan. Since we are already assuming $p > d/(d-1) \geq 1$, it follows that, for every $q \geq 2$,

$$\mathbb{E}\left[\left| \nabla G((Z^i)_{i=1}^n, (\tilde{B}^i)_{i=1}^n) \right|^q \right] \lesssim (1+\rho)^{q/2} \mathbb{E}\left[\left(W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right) \right)^{q/2} \right]. \tag{6.30}$$

Using the same argument as in the previous step, we can estimate

$$\mathbb{E}\left[\left(W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right) \right)^{q/2} \right] \lesssim nT^{q/2}e^{-c\sigma} + \mathbb{E}\left[\left(W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right)^{q/2} \right) \right]. \tag{6.31}$$

Finally, by (2.16) with $r = q/2$ and (6.13), with exponent $pq/2$,

$$\mathbb{E}\left[\left(W_{\mathbb{T}^d}^p \left(\sum_{i=1}^n \mu_\rho^{B^i} \right)^{q/2} \right) \right] \lesssim T^{q/2-1}\mathbb{E}\left[W_{\mathbb{T}^d}^{qp/2} \left(\sum_{i=1}^n \mu_\rho^{B^i} \right) \right] \lesssim T^{q[1-p/(d-2)]/2}. \tag{6.32}$$

By Poincaré inequality, we conclude that (6.25) holds.

Step 4 (Conclusion). To conclude, we need to choose $0 < \gamma_\sigma < \gamma_\rho < 1$ such that both (6.9) and (6.19) holds true:

$$\gamma_\rho - \gamma_\sigma > \frac{p}{d-2} \cdot \frac{d}{p+d}, \quad \text{and} \quad \gamma_\rho < 1 - p/(d-2). \tag{6.33}$$

Since γ_σ can be arbitrarily small, these conditions can be satisfied provided that (6.1) holds. $\square$

Remark 6.2 (lower bounds). In the proof of Theorem 1.1 we did not prove directly that $\mathsf{c}(\mathcal{I}, p, d)$ is strictly positive, since this can be seen as a consequence of Theorem 1.2 and the fact that

$$\liminf_{T \to \infty} \mathbb{E}\left[W_{\mathbb{T}^d}^p \left(\mu_T^B \right) \right] / T^{1-p/(d-2)} > 0. \tag{6.34}$$

If $p \geq 1$, this is an immediate consequence of (1.8). For $0 < p < 1$, one can argue as follows. We start as in Step 1 of the above proof with parameters $\gamma_\rho, \gamma_\sigma$, such that

$$0 < \gamma_\sigma < \gamma_\rho < 1 \tag{6.35}$$

to be further specified, and set again

$$n := \lfloor T^{1-\gamma_\rho} \rfloor, \quad \sigma := T^{\gamma_\sigma}, \quad \rho := \frac{T}{n} - \sigma. \tag{6.36}$$

Decomposing

$$\mu_T = \sum_{i=0}^{n-1} \mu_\rho^{\theta_{i(\rho+\sigma)}B} + \sum_{i=0}^{n-1} \mu_\sigma^{\theta_{i(\rho+\sigma)+\rho}B} := \tilde{\mu}_T + \lambda_T \tag{6.37}$$

from (2.36) with $\varepsilon \to 0$, we have for some constant $c = c(p)$, that

$$W_{\mathbb{T}^d}^p \left(\mu_T^B \right) \geq W_{\mathbb{T}^d}^p \left(\mu_T^B \right) - cT^{1-\gamma_\rho+\gamma_\sigma} \tag{6.38}$$

Assuming that

$$\gamma_\rho - \gamma_\sigma > p/(d-2), \tag{6.39}$$

it is then sufficient to argue that

$$\liminf_{T \to \infty} \mathbb{E} \left[W_{\mathbb{T}^d}^p \left(\sum_{i=1}^{n} \mu_\rho^{\tilde{B}^i} \right) \right] / T^{1-p/(d-2)} > 0. \tag{6.40}$$

As in Step 2 of the above proof, we can use n applications of 2.12 to move to the case where the $\tilde{B}^i$'s are independent (and stationary) Brownian motions (this part uses that $\gamma_\sigma > 0$). Hence, by (2.19), we have

$$\mathbb{E} \left[W_{\mathbb{T}^d}^p \left(\sum_{i=1}^{n} \mu_\rho^{\tilde{B}^i} \right) \right] \geq n\rho \int_{\mathbb{T}^d} \mathbb{E} \left[\min_{i=1,\dots,n} \min_{t \in [0,\rho]} \mathsf{d}_{\mathbb{T}^d}(x, \tilde{B}_t^i)^p \right] dx$$
$$\gtrsim T\mathbb{E} \left[\min_{i=1,\dots,n} \min_{t \in [0,\rho]} \mathsf{d}_{\mathbb{T}^d}(0, \tilde{B}_t^i)^p \right] \tag{6.41}$$

having used stationarity in the last line. By the layer-cake formula and independence,

$$\mathbb{E} \left[\min_{i=1,\dots,n} \min_{t \in [0,\rho]} \mathsf{d}_{\mathbb{T}^d}(0, \tilde{B}_t^i)^p \right] \gtrsim \int_0^1 \mathbb{P} \left(\min_{t \in [0,\rho]} \mathsf{d}_{\mathbb{T}^d}(0, \tilde{B}_t^i)^p > s \right)^n ds$$
$$\int_0^{T^{-p/(d-2)}} \mathbb{P} \left(\tau_{D_{s^{1/p}}} B^i > \rho \right)^n ds \tag{6.42}$$

Using Proposition 3.7, we obtain that there exists a constant $c > 0$ such that, for each $i = 1, \dots, n$, $s \leq T^{-p/(d-2)}$, it holds

$$\mathbb{P} \left(\tau_{D_{s^{1/p}}B^i \geq \rho} \right) \geq 1 - c\rho s^{(d-2)/p} = 1 - \frac{cT s^{(d-2)/p}}{n}, \tag{6.43}$$

having also used that

$$\rho s^{(d-2)/p} \lesssim \rho T^{-1} = T^{\gamma_\rho - 1} \ll 1. \tag{6.44}$$

Using this bound, we find

$$T \int_0^{T^{-p/(d-2)}} \mathbb{P} \left(\tau_{D_{s^{1/p}}} B^i > \rho \right)^n ds \gtrsim T \int_0^{T^{-p/(d-2)}} \left(1 - \frac{cT s^{(d-2)/p}}{n} \right)^n ds \gtrsim T^{1-p/(d-2)}, \tag{6.45}$$

hence the thesis.

Appendix A. Wasserstein asymptotics for stationary measures

In this appendix, we consider a random Borel measure ν on $\mathbb{R}^d$ satisfying the following conditions:

i) (stationarity) For every $x \in \mathbb{R}^d$, it holds $\mathrm{trans}_x \nu = \nu$.
ii) (integrability) For every bounded Borel $A \subseteq \mathbb{R}^d$, $\nu(A)$ is integrable.
iii) (concentration) There exists $\alpha \in [0, d)$ such that, for every $q \geq 1$, there exists $C = C(\nu, q, \alpha) < \infty$ such that, for every Borel $A \subseteq \mathbb{R}^d$ with $\mathrm{diam}(A) \geq 1$,

$$\|\nu(A) - \mathbb{E}\left[\nu(A)\right]\|_q \leq C \, \mathrm{diam}(A)^{(d+\alpha)/2}. \tag{A.1}$$

Notice that, by stationarity and integrability, the function $A \mapsto \mathbb{E}\left[\nu(A)\right]$ is a translation invariant (σ-finite) measure hence for some constant $\lambda \in [0, \infty)$ it holds

$$\mathbb{E}\left[\nu(A)\right] = \lambda |A|. \tag{A.2}$$

Without loss of generality, we assume in what follows that $\lambda = 1$, so that $\mathbb{E}\left[\nu(A)\right] = |A|$ (if $\lambda = 0$ the statements become trivial). With this notation, we have the following result.

Theorem A.1. *Let ν be a random measure on $\mathbb{R}^d$ satisfying the conditions i), ii), iii) above. Let $p > 0$ be such that*

$$r := d - \alpha - 2 \min\{p, 1\} > 0. \tag{A.3}$$

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded connected domain with C^2 boundary (or $\Omega = Q$ a cube). Then, it holds

$$\limsup_{n \to \infty} \mathbb{E}\left[W_\Omega^p(\mathrm{dil}_{n^{-1/d}} \nu)\right] / n^{1-p/d} = \mathsf{c}_{\nu,p} |\Omega|, \tag{A.4}$$

where $\mathsf{c}_{\nu,p} \in [0, \infty)$ depends on ν and p only.

We split the proof in the next subsections, where we recall and slightly generalized ideas and tools from [7, 23, 6, 25].

A.1. **The case of a cube.** In this section, we focus on the following result.

Proposition A.2. *Let ν be a random measure on $\mathbb{R}^d$ satisfying the conditions i), ii), iii) above. Let $p > 0$ be such that*

$$r := d - \alpha - 2 \min\{p, 1\} > 0 \tag{A.5}$$

Then, the following limit exists:

$$\lim_{L \to \infty} \frac{1}{L^d} \mathbb{E}\left[W_{Q_L}^p(\nu)\right] = \mathsf{c}_{\nu,p} \in [0, \infty). \tag{A.6}$$

Moreover, there exists $C > 0$ (depending on ν and p only) such that for $L \geq 1$,

$$\mathsf{c}_{\nu,p} \leq \frac{1}{L^d} \mathbb{E}\left[W_{Q_L}^p(\nu)\right] + \frac{C}{L^{\frac{r}{2}}}. \tag{A.7}$$

Proof. By standard sub-additivity (Fekete-type) arguments, e.g. [25, Lemma 2.12], it is sufficient to prove that there exists a constant $C > 0$ such that for every $L \geq C$ and $m \in \mathbb{N}$,

$$\frac{1}{|Q_{mL}|} \mathbb{E}\left[W_{Q_{mL}}^p(\nu)\right] \leq \frac{1}{|Q_L|} \mathbb{E}\left[W_{Q_L}^p(\nu)\right] + \frac{C}{L^{\frac{r}{2}}}. \tag{A.8}$$

Starting from the cube Q_{mL}, we construct a sequence of finer and finer partitions of Q_{mL} by rectangles of moderate aspect ratios and side-length given by integer multiples of L. To simplify the notation, we define

$$f(R) = \mathbb{E}\left[\frac{1}{|R|} W_R^p(\nu)\right]. \tag{A.9}$$

We rely upon iterated applications of the following inequality: let R of moderate aspect ratio, let $\mathcal{R}$ be an admissible partition R into rectangles of moderate aspect ratios and side-lengths given by integer multiples of L. For every $\varepsilon \in [0, 1)$, we have

$$f(R) \leq (1 + \varepsilon) \sum_i \frac{|R_i|}{|R|} f(R_i) + \frac{C}{\varepsilon^{(p-1)^+}} \frac{1}{|R|^{q/(2d)}}, \tag{A.10}$$

with $C = C(p) \in (0, \infty)$ and

$$q = \begin{cases} d - \alpha - 2p & \text{if } 0 < p < 1, \\ p(d - \alpha - 2) & \text{if } p \geq 1. \end{cases} \tag{A.11}$$

Indeed, defining $\kappa = \frac{\nu(R)}{|R|}$, $\kappa_i = \frac{\nu(R_i)}{|R_i|}$, it holds

$$\mathbb{E}[\kappa_i] = \mathbb{E}[\kappa] = 1 \tag{A.12}$$

Using (2.20), we get

$$f(R) \leq (1 + \varepsilon) \sum_i \frac{|R_i|}{|R|} f(R_i) + \frac{C}{\varepsilon^{(p-1)^+}} \mathbb{E}\left[\frac{1}{|R|} W_R^p\left(\sum_i \kappa_i \chi_{R_i}, \kappa\right)\right]. \tag{A.13}$$

We then estimate the last term in the right hand side. In the case $0 < p < 1$, we use (2.18), obtaining

$$\frac{1}{|R|} W_R^p\left(\sum_i \kappa_i \chi_{R_i}, \kappa\right) \lesssim \frac{|R|^{p/d}}{|R|} \int_R \sum_i |\kappa_i - \kappa| \chi_{R_i}$$
$$\lesssim |R|^{p/d}\left(|\kappa - \mathbb{E}[\kappa]| + \sum_i |\kappa_i - \mathbb{E}[\kappa_i]|\right). \tag{A.14}$$

By (A.1), we have

$$\max\left\{\mathbb{E}[|\kappa - \mathbb{E}[\kappa]|], \max_i\{\mathbb{E}[|\kappa_i - \mathbb{E}[\kappa_i]|]\}\right\} \lesssim |R|^{\frac{(\alpha-d)}{2d}}, \tag{A.15}$$

which eventually yields (A.10).

If $p \geq 1$, we argue first by Markov inequalit and, (A.1) that for every $n \geq 1$,

$$\mathbb{P}\left(\kappa \leq \frac{1}{2}\right) \leq \mathbb{P}\left(|\kappa - \mathbb{E}[\kappa]| \geq \frac{1}{2}\right) \lesssim \|\nu(R) - \mathbb{E}[\nu(R)]\|_n^n |R|^{-n} = |R|^{n(\alpha-d)/(2d)}. \tag{A.16}$$

Choosing n sufficiently large and using (2.17), i.e.,

$$\frac{1}{|R|} W_R^p\left(\sum_i \kappa_i \chi_{R_i}, \kappa\right) \lesssim |R|^{p/d} \kappa, \tag{A.17}$$

we may reduce ourselves to the event $\{\kappa \geq \frac{1}{2}\}$. Under this condition, by (2.28), we have

$$\frac{1}{|R|} W_R^p\left(\sum_i \kappa_i \chi_{R_i}, \kappa\right) \lesssim \frac{|R|^{p/d}}{|R|} \int_R \sum_i |\kappa_i - \kappa|^p \chi_{R_i}$$
$$\lesssim |R|^{p/d}\left(|\kappa - \mathbb{E}[\kappa]|^p + \sum_i |\kappa_i - \mathbb{E}[\kappa_i]|^p\right). \tag{A.18}$$

By (A.1), we have

$$\max\left\{\mathbb{E}[|\kappa - \mathbb{E}[\kappa]|^p], \mathbb{E}[|\kappa_i - \mathbb{E}[\kappa_i]|^p]\right\} \lesssim |R|^{\frac{p(\alpha-d)}{2d}}, \tag{A.19}$$

which eventually yields (A.10).

Starting from the cube Q_{mL}, we next construct a sequence of finer and finer partitions of Q_{mL} inductively as follows. We let $\mathcal{R}_0 = \{Q_{mL}\}$. To define $\mathcal{R}_k$, let $R \in \mathcal{R}_k$. Up

to translation we may assume that $R = \prod_{i=1}^{d}[0, m_i L)$ for some $m_i \in \mathbb{N}$. We then split each interval $[0, m_i L)$ into $[0, \lfloor \frac{m_i}{2} \rfloor L) \cup [\lfloor \frac{m_i}{2} \rfloor L, m_i L)$. It is readily seen that this induces an admissible partition of R. Let us point out that when $m_i = 1$ for some i, the corresponding interval $[0, \lfloor \frac{m_i}{2} \rfloor L)$ is empty. This procedure stops after a finite number of steps K once $\mathcal{R}_K = \{Q_L + z_i, z_i \in [0, m-1]^d\}$. It is also readily seen that $2^{K-1} < m \leq 2^K$ and that for every $k \in [0, K]$ and every $R \in \mathcal{R}_k$ we have $|R| \sim (2^{K-k}L)^d$.

If $0 < p < 1$, we prove by a downward induction that, for every $k \in [0, K]$ and every $R \in \mathcal{R}_k$,

$$f(R) \leq f(Q_L) + CL^{-\frac{r}{2}} \sum_{j=k}^{K} 2^{-(K-j)r/2}. \tag{A.20}$$

This is clearly true for $k = K$. Assume that it holds true for $k + 1$. Let $R \in \mathcal{R}_k$. If $0 < p < 1$, we apply (A.10) with $\varepsilon = 0$, obtaining

$$
\begin{aligned}
f(R) &\leq \sum_{R_i \in \mathcal{R}_{k+1}, R_i \subset R} \frac{|R_i|}{|R|} f(R_i) + C \frac{1}{|R|^{q/(2d)}} \\
&\leq \sum_{R_i \in \mathcal{R}_{k+1}, R_i \subset R} \frac{|R_i|}{|R|} \left(f(Q_L) + CL^{-r/2} \sum_{j=k+1}^{K} 2^{-(K-j)r/2} \right) + CL^{-r/2} 2^{-(K-k)r/2} \\
&= f(Q_L) + CL^{-r/2} \sum_{j=k}^{K} 2^{-(K-j)r/2}.
\end{aligned}
\tag{A.21}
$$

Applying (A.20) with $k = 0$, hence $R = Q_{mL}$ yields (A.8).

In the case $p \geq 1$, we prove a slightly more involved inequality: there exists a constant $\Lambda < \infty$ (depending on p only) such that, for $R \in \mathcal{R}_k$,

$$f(R) \leq f(Q_L) + \Lambda(1 + f(Q_L))L^{-\frac{r}{2}} \sum_{j=k}^{K} 2^{-(K-j)r/2}. \tag{A.22}$$

if $\varepsilon = (2^{K-k}L)^{-r/2} \ll 1$, we get

$$
\begin{aligned}
f(R) &\leq (1 + \varepsilon) \sum_{R_i \in \mathcal{R}_{k+1}, R_i \subset R} \frac{|R_i|}{|R|} f(R_i) + \frac{C}{\varepsilon^{p-1}} \frac{1}{|R|^{\frac{pr}{2d}}} \\
&\overset{(A.22)}{\leq} (1 + \varepsilon) \left(f(Q_L) + \Lambda(1 + f(Q_L))L^{-\frac{r}{2}} \sum_{j=k+1}^{K} 2^{-(K-j)r/2} \right) + C(2^{K-k}L)^{-\frac{r}{2}} \\
&\leq f(Q_L) + \Lambda(1 + f(Q_L))L^{-\frac{r}{2}} \\
&\quad \times \left[\sum_{j=k+1}^{K} 2^{-(K-j)r/2} + 2^{-(K-k)r/2} \left(\frac{C}{\Lambda} + L^{-\frac{r}{2}} \sum_{j=k+1}^{K} 2^{-(K-j)r/2} \right) \right].
\end{aligned}
\tag{A.23}
$$

If L is large enough, then $(\sum_{j=k+1}^{K} 2^{-(K-j)r/2})L^{-r/2} \leq \frac{1}{2}$. Finally, choosing $\Lambda \geq 2C$ yields (A.22).

Applying (A.22) to $R = Q_{mL}$ and using that $\sum_{j \geq 0} 2^{-jr/2} < \infty$, we get

$$f(mL) \leq f(L) + C(1 + f(L)) \frac{1}{L^{\frac{r}{2}}}. \tag{A.24}$$

Since $f(L) \lesssim L^p$, writing that every $L \geq C$ may be written as $L = mL'$ for some $m \in \mathbb{N}$ and $L' \in [C, C+1]$, we conclude that $f(L)$ is bounded and thus (A.8) follows also in this case. $\qquad\square$

Remark A.3. We point out that as a consequence of the above result, we have

$$\mathbb{E}\left[W_Q^p(\nu)\right] \lesssim |Q|, \quad \text{for every cube } Q \subseteq \mathbb{R}^d. \tag{A.25}$$

A.2. **Whitney decomposition.** We recall some result on decomposition of domains. The first one is [6, Lemma 5.1].

Lemma A.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $\mathcal{Q} = \{Q_i\}_i$ be a Whitney partition of Ω. Then, for every $\delta > 0$ sufficiently small, letting $\mathcal{Q}_\delta = \{Q_i : \text{diam}(Q_i) \geq \delta\}$, there exists a finite family $\mathcal{R}_\delta = \{\Omega_j\}_j$ of disjoint open sets such that:*

(1) $(\Omega_k)_{k=1}^K = \mathcal{Q}_\delta \cup \mathcal{R}_\delta$ is a partition of Ω,
(2) $|\Omega_k| \sim \text{diam}(\Omega_k)^d$ for every $k = 1, \ldots, K$,
(3) if $\Omega_k \in \mathcal{Q}_\delta$, then $\text{diam}(\Omega_k) \sim \mathsf{d}(x, \Omega^c)$ for every $x \in \Omega_k$,
(4) if $\Omega_k \in \mathcal{R}_\delta$, then $\text{diam}(\Omega_k) \sim \delta$ and $\mathsf{d}(x, \Omega^c) \lesssim \delta$, for every $x \in \Omega_k$.

Here all the implicit constants depend only on the initial partition $\mathcal{Q}$ (and not on δ).

We next collect some useful bounds related to the above construction, generalizing [25, Lemma 2.2]. Define, for $\gamma \in \mathbb{R}$, $\delta \in (0, 1/2)$, the function

$$r_\gamma(\delta) := \begin{cases} 1 & \text{if } \gamma > 0, \\ |\log \delta| & \text{if } \gamma = 0, \\ \delta^\gamma & \text{if } \gamma < 0, \end{cases} \tag{A.26}$$

and notice that

$$r_\gamma(\delta) \sim \sum_{\ell \leq |\log_2 \delta|} 2^{-\ell\gamma}. \tag{A.27}$$

Then, we have the following result.

Lemma A.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $\mathcal{Q} = \{Q_i\}_i$ be a Whitney partition of Ω. Then, letting $(\Omega_k)_{k=1}^K = \mathcal{Q}_\delta \cup \mathcal{R}_\delta$ as in Lemma A.4, one has that $|\mathcal{R}_\delta| \lesssim \delta^{1-d}$ and,*

(1) for every $\alpha \in \mathbb{R}$, it holds

$$\sum_{k=1}^K \text{diam}(\Omega_k)^\alpha \lesssim r_{\alpha+1-d}(\delta), \tag{A.28}$$

(2) for every $\alpha < 0$, $k = 1, \ldots, K$, and $x \in \Omega_k$, it holds

$$\sum_j \text{diam}(\Omega_j)^\alpha \min\left\{1, \left(\frac{\text{diam}(\Omega_j)}{\mathsf{d}(x, \Omega_j)}\right)^{d-1}\right\} \lesssim (|\log(\delta)| + |\log(\text{diam}(\Omega_k))|) \delta^\alpha. \tag{A.29}$$

In the above inequalities the implicit constants depend upon $\mathcal{Q}$, d and α only.

Proof. Since the boundary of Ω is Lipschitz, it follows from the properties of the partition that, for every $x \in \Omega$ and $r \geq s \geq \delta$,

$$\sharp\{k : \Omega_k \subseteq B(x, r), \text{diam}(\Omega_k) \in [s, 2s)\} \lesssim (r/s)^{d-1}, \tag{A.30}$$

with the implicit constant depending on $\mathcal{Q}$ only. It follows that $|\mathcal{R}_\delta| \lesssim \delta^{1-d}$ and, for every $\ell \leq |\log_2 \delta|$, the number of cubes $\Omega_k \in \mathcal{Q}_\delta$ with $\text{diam}(\Omega_k) \in [2^{-\ell}, 2^{-\ell+1})$ is estimated by

$2^{\ell(d-1)}$. Therefore, for $\alpha \in \mathbb{R}$,

$$
\begin{aligned}
\sum_{k=1}^{K} \operatorname{diam}(\Omega_k)^\alpha &\lesssim \sum_{\Omega_k \in \mathcal{Q}_\delta} \operatorname{diam}(\Omega_k)^\alpha + \sum_{\Omega_k \in \mathcal{R}_\delta} \operatorname{diam}(\Omega_k)^\alpha \\
&\lesssim \sum_{\ell \leq |\log_2 \delta|} \sharp\Big\{\Omega_k \in \mathcal{Q}_\delta : \operatorname{diam}(Q_k) \in [2^{-\ell}, 2^{-\ell+1})\Big\} 2^{-\ell\alpha} + |\mathcal{R}_\delta| \cdot \delta^\alpha \quad \text{(A.31)} \\
&\lesssim \sum_{\ell \leq |\log_2 \delta|} 2^{-\ell(\alpha+1-d)} + \delta^{\alpha+1-d} \lesssim r_{\alpha+1-d}(\delta).
\end{aligned}
$$

having used that the in the summation ℓ is also bounded from below in the summation (e.g. by $-|\log_2 \operatorname{diam}(\Omega)|$). We thus obtain (A.28).

In order to prove (A.34) and (A.35) we further notice that, given Ω_k, Ω_j and $x \in \Omega_k$, we have that, for some constant $C = C(\mathcal{Q})$,

$$
\Omega_j \subseteq B(x, C \max\{\mathsf{d}(x, \Omega_j), \operatorname{diam}(\Omega_k)\}). \tag{A.32}
$$

Indeed, if $\Omega_j \in \mathcal{R}_\delta$, then $\operatorname{diam}(\Omega_j) \lesssim \delta \lesssim \operatorname{diam}(\Omega_k)$, hence (A.32) holds. If instead $\Omega_j \in \mathcal{Q}_\delta$, then we can find $y \in \Omega_j$ with $|x - y| \leq 2\mathsf{d}(x, \Omega_j)$, so that, by the triangle inequality,

$$
\mathsf{d}(y, \Omega^c) \leq |x - y| + \mathsf{d}(x, \Omega^c) \lesssim \max\{\mathsf{d}(x, \Omega_j), \operatorname{diam}(\Omega_k)\} \tag{A.33}
$$

and by property (3) in Lemma A.4 we obtain that $\operatorname{diam}(\Omega_j) \lesssim \max\{\mathsf{d}(x, \Omega_j), \operatorname{diam}(\Omega_k)\}$, yielding again the desired inclusion.

We now prove (A.29). We claim that it follows from the following inequalities, valid for any $\gamma \in \mathbb{N}$:

$$
\sum_{j : \mathsf{d}(x,\Omega_j) \leq 2^{-\gamma} \operatorname{diam}(\Omega_k)} \operatorname{diam}(\Omega_j)^\alpha \lesssim 2^{-\gamma(d-1)} \operatorname{diam}(\Omega_k)^{d-1} r_{\alpha+1-d}(\delta), \tag{A.34}
$$

and

$$
\sum_{j : \mathsf{d}(x,\Omega_j) > 2^{-\gamma} \operatorname{diam}(\Omega_k)} \frac{\operatorname{diam}(\Omega_j)^\beta}{\mathsf{d}(x, \Omega_j)^{d-1}} \lesssim |\gamma + \log(\operatorname{diam}(\Omega_k))| r_{\beta+1-d}(\delta). \tag{A.35}
$$

Indeed, we can split the summation and use (A.34) and (A.35) to get

$$
\begin{aligned}
\sum_j \operatorname{diam}(\Omega_j)^\alpha &\min\left\{1, \left(\frac{\operatorname{diam}(\Omega_j)}{\mathsf{d}(x, \Omega_j)}\right)^{d-1}\right\} \\
&\lesssim \sum_{j : \mathsf{d}(x,\Omega_j) \leq 2^{-\gamma} \operatorname{diam}(\Omega_k)} \operatorname{diam}(\Omega_j)^\alpha + \sum_{j : \mathsf{d}(x,\Omega_j) > 2^{-\gamma} \operatorname{diam}(\Omega_k)} \frac{\operatorname{diam}(\Omega_j)^{d-1+\alpha}}{\mathsf{d}(x, \Omega_j)^{d-1}} \quad \text{(A.36)} \\
&\lesssim 2^{-\gamma(d-1)} \operatorname{diam}(\Omega_k)^{d-1} r_{\alpha+1-d}(\delta) + |\gamma + \log(\operatorname{diam}(\Omega_k))| r_\alpha(\delta).
\end{aligned}
$$

Choosing γ so that $2^{-\gamma} \leq \delta \leq 2^{-\gamma+1}$ yields (A.29).

Hence, we prove (A.34) and (A.35). Let $\ell_k \leq |\log_2 \delta|$ be such that $\operatorname{diam}(\Omega_k) \in [2^{-\ell_k}, 2^{-\ell_k+1})$. Combining (A.32) and (A.30), we see that, for every $\ell \leq |\log_2 \delta|$, there are at most $2^{(\ell-\ell_k-\gamma)(d-1)}$ sets Ω_j such that $\mathsf{d}(x, \Omega_j) \leq 2^{-\gamma} \operatorname{diam}(\Omega_k)$ and $\operatorname{diam}(\Omega_j) \in [2^{-\ell}, 2^{-\ell+1})$. Therefore,

$$
\begin{aligned}
\sum_{j : \mathsf{d}(x,\Omega_j) \leq 2^{-\gamma} \operatorname{diam}(\Omega_k)} \operatorname{diam}(\Omega_j)^\alpha &\lesssim \sum_{\ell \leq |\log_2 \delta|} 2^{-\ell\alpha} 2^{(\ell-\ell_k)(d-1)} \\
&\lesssim 2^{-(\gamma+\ell_k)(d-1)} \sum_{\ell \leq |\log_2 \delta|} 2^{-\ell(\alpha+1-d)} \quad \text{(A.37)} \\
&\lesssim 2^{-\gamma(d-1)} \operatorname{diam}(\Omega_k)^{d-1} r_{\alpha+1-d}(\delta).
\end{aligned}
$$

This proves (A.34). To prove (A.35), we split dyadically,

$$
\sum_{j\,:\,\mathsf{d}(x,\Omega_j)>2^{-\gamma}\operatorname{diam}(\Omega_k)} \frac{\operatorname{diam}(\Omega_j)^\beta}{d(x,\Omega_j)^{d-1}} \lesssim \sum_{\ell\leq\ell_k+\gamma} \frac{1}{(2^{-\ell})^{d-1}} \sum_{j\,:\,\mathsf{d}(x,\Omega_j)\in[2^{-\ell},2^{-\ell+1})} \operatorname{diam}(\Omega_j)^\beta
$$

$$
\overset{(\mathrm{A.32})}{\lesssim} \sum_{\ell\leq\ell_k+\gamma} 2^{\ell(d-1)} \sum_{\Omega_j\subset B(x,C2^{-\ell})} \operatorname{diam}(\Omega_j)^\beta. \tag{A.38}
$$

Let us also notice that, if $\Omega_j \subseteq B(x,C2^{-\ell})$, then necessarily $\delta \leq \operatorname{diam}(\Omega_j) \lesssim 2^{-\ell}$ (since $\operatorname{diam}(\Omega_j)^d \sim |\Omega_j|$). Thus for ℓ' with $2^{-\ell'} \sim 2^{-\ell}$,

$$
\sum_{\Omega_j\subset B(x,C2^{-\ell})} \operatorname{diam}(\Omega_j)^\beta \lesssim \sum_{\ell'\leq u\leq|\log_2\delta|} 2^{-u\beta}\sharp\left\{\Omega_j \subseteq B(x,C2^{-\ell}) : \operatorname{diam}(\Omega_j)\in[2^{-u},2^{-u+1})\right\}
$$

$$
\overset{(\mathrm{A.30})}{\lesssim} \sum_{\ell'\leq u\leq|\log_2\delta|} 2^{-u\beta}\cdot 2^{(u-\ell)(d-1)} = 2^{-\ell(d-1)} \sum_{\ell'\leq u\leq|\log_2\delta|} 2^{-u(\beta+1-d)}
$$

$$
\lesssim 2^{-\ell(d-1)}r_{\beta+1-d}(\delta) \tag{A.39}
$$

using again that ℓ' is bounded from below by a constant depending on $\mathcal{Q}$ only. Plugging this bound in (A.38), we conclude that

$$
\sum_{j\,:\,\mathsf{d}(x,\Omega_j)>2^{-\gamma}\operatorname{diam}(\Omega_k)} \frac{\operatorname{diam}(\Omega_j)^\beta}{d(x,\Omega_j)^{d-1}} \leq \sum_{\ell\leq\ell_k+\gamma} 2^{\ell(d-1)}\cdot 2^{-\ell(d-1)}r_{\beta+1-d}(\delta)
$$

$$
\lesssim (\gamma+|\log(\operatorname{diam}(\Omega_k))|)\,r_{\beta+1-d}(\delta). \tag{A.40}
$$

This concludes the proof of (A.35). $\qquad\square$

As in [25] in the next result we rely on gradient bounds for the Green kernel $(G(x,y))_{x,y\in\Omega}$ of the Laplacian with Neumann boundary conditions:

$$
|\nabla_x G(x,y)| \lesssim |x-y|^{1-d}, \quad \text{for every } x,y\in\Omega, \tag{A.41}
$$

where the implicit constant depends uniquely on Ω. This condition is satisfied for instance if Ω is C^2 or convex, see e.g. [46]. Notice that since it is a local condition it also holds for $Q\backslash\Omega$ with Ω a C^2 open set with $d(\partial Q,\partial\Omega)>0$. We have the following bound (see [25, Lemma 2.6]).

Lemma A.6. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with Lipschitz boundary such that (A.41) holds and for $\delta>0$ sufficiently small, consider a partition $(\Omega_k)_{k=1}^K = \mathcal{Q}_\delta\cup\mathcal{R}_\delta$ as in Lemma A.4. For any $(b_k)_k\subseteq\mathbb{R}$, $\beta>0$ and $p\geq 1$,*

$$
\left\|\sum_k b_k\chi_{\Omega_k}-b\right\|_{W^{-1,p}(\Omega)} \lesssim |\log\delta|\delta^{1-\beta}\cdot\max_k\left\{|b_k|\operatorname{diam}(\Omega_k)^\beta\right\}, \tag{A.42}
$$

with $b=\sum_k b_k|\Omega_k|/|\Omega|$ and the implicit constant depends only on p, d and $\mathcal{Q}$ (not on δ nor $(b_k)_k$.).

Proof. We set

$$
f_k = \chi_{\Omega_k} - \frac{|\Omega_k|}{|\Omega|} \tag{A.43}
$$

and let ϕ_k denote the solution to the equation $\Delta\phi_k=f_k$, with null Neumann boundary conditions on Ω and use as competitor $\xi=\sum_{k=1}^K b_k\nabla\phi_k$ in the definition of the $W^{-1,p}$

norm. We get

$$\left\| \sum_k b_k \chi_{\Omega_k} - b \right\|_{W^{-1,p}(\Omega)}^p = \left\| \sum_k b_k f_k \right\|_{W^{-1,p}(\Omega)}^p \leq \int_\Omega \left| \sum_k b_k \nabla \phi_k \right|^p$$

$$\leq \left(\max_k \left\{ |b_k| \operatorname{diam}(\Omega_k)^\beta \right\} \right)^p \int_\Omega \left(\sum_k \operatorname{diam}(\Omega_k)^{-\beta} |\nabla \phi_k| \right)^p . \tag{A.44}$$

To bound the last term, we use the integral representation in terms of the Green's function,

$$\phi_k = \int_\Omega G(x,y) f_k(y) dy, \tag{A.45}$$

to obtain that, for every $x \in \Omega$,

$$|\nabla \phi_k(x)| \lesssim \min \left\{ \operatorname{diam}(\Omega_k), \frac{|\Omega_k|}{\mathsf{d}(x,\Omega_k)^{d-1}} \right\} \lesssim \operatorname{diam}(\Omega_k) \min \left\{ 1, \left(\frac{\operatorname{diam}(\Omega_k)}{\mathsf{d}(x,\Omega_k)} \right)^{d-1} \right\} . \tag{A.46}$$

Indeed, by (A.41),

$$|\nabla \phi_k(x)| \lesssim \int_{\Omega_k} \frac{dy}{|x-y|^{d-1}} + |\Omega_k| \int_\Omega \frac{dy}{|x-y|^{d-1}} \leq \int_{\{|y| \leq \operatorname{diam}(\Omega_k)\}} \frac{dy}{|y|^{d-1}} + |\Omega_k| \tag{A.47}$$
$$\lesssim \operatorname{diam}(\Omega_k).$$

Moreover, for $x \notin \Omega_k$, we get directly from (A.41),

$$|\nabla \phi_k(x)| \lesssim \frac{|\Omega_k|}{\mathsf{d}(x,\Omega_k)^{d-1}}. \tag{A.48}$$

For any $k = 1, \ldots, K$ and $x \in \Omega_k$, we then estimate

$$\sum_{j=1}^K \operatorname{diam}(\Omega_j)^{-\beta} |\nabla \phi_j(x)| \overset{(A.46)}{\lesssim} \sum_{j=1}^K \operatorname{diam}(\Omega_j)^{1-\beta} \min \left\{ 1, \left(\frac{\operatorname{diam}(\Omega_j)}{\mathsf{d}(x,\Omega_j)} \right)^{d-1} \right\} \tag{A.49}$$
$$\overset{(A.29)}{\lesssim} \left(|\log \delta| + |\log \operatorname{diam}(\Omega_k)| \right) \delta^{1-\beta}.$$

To conclude, we go back with the integration and bound from above:

$$\int_\Omega \left(\sum_{j=1}^K \operatorname{diam}(\Omega_j)^{-\beta} |\nabla \phi_j| \right)^p = \sum_{k=1}^K \int_{\Omega_k} \left(|\log \delta|^p + |\log \operatorname{diam}(\Omega_k)|^p \right) \delta^{(1-\beta)p}$$

$$\lesssim \left(|\log \delta|^p + \sum_{k=1}^K \operatorname{diam}(\Omega_k)^d |\log \operatorname{diam}(\Omega_k)|^p \right) \delta^{(1-\beta)p} \tag{A.50}$$

$$\lesssim |\log \delta|^p \delta^{(1-\beta)p},$$

by estimating $z^d |\log(z)|^p \lesssim z^{d-1/2}$ and using (A.28) with $\alpha = d - 1/2.$. $\qquad \square$

We have now all the preliminaries to address the proof of Theorem A.1.

A.3. **Upper bound.** In this section, under the assumptions of Theorem A.1, we establish the inequality

$$\limsup_{n \to \infty} n^{-1+p/d} \mathbb{E}\left[W_\Omega^p(\nu_n) \right] \leq \mathsf{c}_{\nu,p} |\Omega|. \tag{A.51}$$

Recall that we assume without loss of generality that $\mathbb{E}[\nu(A)] = |A|$. Write also $\nu_n = \operatorname{dil}_{n^{-1/d}} \nu$, i.e., $\nu_n(A) = \nu(n^{1/d} A)$, hence

$$\mathbb{E}[\nu_n(A)] = n|A|, \tag{A.52}$$

and, by (A.1),

$$\|\nu_n(A) - \mathbb{E}\left[\nu_n(A)\right]\|_q \lesssim_q \operatorname{diam}(n^{1/d}A)^{(d+\alpha)/2} = n^{\frac{d+\alpha}{2d}} \operatorname{diam}(A)^{(d+\alpha)/2}, \qquad (A.53)$$

provided that $\operatorname{diam}(n^{1/d}A) \geq 1$, a condition that surely holds if n is sufficiently large and

$$\gamma_\delta < \frac{1}{d}. \qquad (A.54)$$

This leads to the concentration inequality

$$\mathbb{P}\left(\left|\frac{\nu_n(A)}{n|A|} - 1\right| > \left(n^{1/d}\operatorname{diam}(A)\right)^{-\beta/2}\right) \lesssim_q \left(n^{\frac{\alpha-d+\beta}{2d}} \frac{\operatorname{diam}(A)^{(d+\alpha+\beta)/2}}{|A|}\right)^q, \qquad (A.55)$$

for every $q \geq 1$ and $\beta > 0$. Moreover, (A.25) becomes

$$\frac{1}{|Q|}\mathbb{E}\left[W_Q^p(\nu_n)\right] \lesssim n^{1-p/d}. \qquad (A.56)$$

We begin by fixing a Whitney decomposition $\mathcal{Q} = (Q_i)_i$ of Ω. and a positive parameter $\gamma_\delta > 0$, to be further specified below. By Lemma A.4 with $\delta = n^{-\gamma_\delta}$, if n is large enough, we have a finite Borel partition of $\Omega = \bigcup_i \Omega_i$, whose elements are collected into the two disjoint sets $\mathcal{Q}_\delta$, $\mathcal{R}_\delta$.

By (2.20), we write

$$\mathbb{E}\left[W_\Omega^p(\nu_n)\right] \leq (1+\varepsilon)\sum_i \mathbb{E}\left[W_{\Omega_i}^p(\nu_n)\right] + n\frac{C}{\varepsilon^{(p-1)^+}}\mathbb{E}\left[W_\Omega^p\left(\sum_i \kappa_i \chi_{\Omega_i}\right)\right], \qquad (A.57)$$

having defined $\kappa_i = \nu_n(\Omega_i)/(n|\Omega_i|)$. For each $\Omega_i \in \mathcal{R}_\delta$ we use the the trivial bound (2.17),

$$\mathbb{E}\left[W_{\Omega_i}^p(\nu_n)\right] \lesssim \operatorname{diam}(\Omega_i)^p \mathbb{E}\left[\nu_n(\Omega_i)\right] \lesssim \delta^{p+d}n, \qquad (A.58)$$

so that their contribution is

$$\sum_{\Omega_i \in \mathcal{R}_\delta} \mathbb{E}\left[W_{\Omega_i}^p(\nu_u)\right] \lesssim \delta^{1-d}\delta^{p+d}n = \delta^{1+p}n \ll n^{1-p/d}. \qquad (A.59)$$

provided that

$$\delta^{1+p} \ll n^{-p/d}, \quad \text{i.e.,} \quad \gamma_\delta > \frac{p}{(p+1)d}. \qquad (A.60)$$

If $\Omega_i \in \mathcal{Q}_\delta$, then Ω_i is a cube, hence we have by (A.56) that

$$\mathbb{E}\left[W_{\Omega_i}^p(\nu_n)\right] \lesssim |\Omega_i|n^{1-p/d}, \qquad (A.61)$$

where the constant does not depend on Ω_i. Using this domination and Proposition A.2, we deduce that

$$\limsup_{n \to \infty} n^{-1+p/d} \sum_{\Omega_i \in \mathcal{Q}_{\delta(n)}} \mathbb{E}\left[W_{\Omega_i}^p(\nu_n)\right] \leq \mathsf{c}_{\nu,p} \sum_{i \in \mathcal{Q}} |\Omega_i| = \mathsf{c}_{\nu,p}|\Omega|. \qquad (A.62)$$

Therefore, in order to conclude, it is sufficient to focus on the remaining term in right-hand side of (A.57) and argue that

$$\mathbb{E}\left[W_\Omega^p\left(\sum_i \kappa_i \chi_{\Omega_i}\right)\right] \ll n^{-p/d}. \qquad (A.63)$$

provided that γ_δ is suitably chosen. Writing also $\kappa = \nu_n(\Omega)/(n|\Omega|)$, we introduce an auxiliary parameter β satisfying

$$0 < \beta < d - \alpha \qquad (A.64)$$

and the event

$$A = \left\{|\kappa - 1| \leq (n^{1/d}\operatorname{diam}(\Omega))^{-\beta/2}\right\} \cap \bigcap_i \left\{|\kappa_i - 1| \leq (n^{1/d}\operatorname{diam}(\Omega_i))^{-\beta/2}\right\}, \qquad (A.65)$$

whose probability is bounded from above, via union bound and an application of (A.55) (recalling also that $|\Omega_i| \sim \text{diam}(\Omega_i)^d$):

$$\mathbb{P}(A^c) \lesssim_q n^{(\alpha-d+\beta)q/(2d)} \left(1 + \sum_i \text{diam}(\Omega_i)^{(\alpha-d+\beta)q/2}\right) \lesssim_q n^{(\alpha-d+\beta)q/(2d)} \qquad (A.66)$$

provided that q is chosen sufficiently large.By (2.17), Hölder inequality and (A.55),

$$\mathbb{E}\left[W_\Omega^p\left(\sum_i \kappa_i \chi_{\Omega_i}\right) I_{A^c}\right] \lesssim \mathbb{E}\left[\kappa I_{A^c}\right] \lesssim \mathbb{E}\left[\kappa^2\right]^{1/2} \mathbb{P}(A^c)^{1/2} \lesssim \mathbb{P}(A^c)^{1/2} \ll n^{-p/d} \quad (A.67)$$

provided that q is sufficiently large. Therefore, we are reduced to bound $W_\Omega^p\left(\sum_i \kappa_i \chi_{\Omega_i}\right)$ on the event A. If n is large enough, on A it holds $\kappa \geq 1/2$, hence by Lemma 2.3 we obtain

$$W_\Omega^p\left(\sum_i \kappa_i \chi_{\Omega_i}\right) \lesssim \left\|\sum_i (\kappa_i - \kappa)\chi_{\Omega_i}\right\|_{W^{-1,p}(\Omega)}^p \qquad (A.68)$$

Then, by the identity

$$\sum_i (\kappa_i - \kappa)\chi_{\Omega_i} = \sum_i (\kappa_i - 1)\left(\chi_{\Omega_i} - \frac{|\Omega_i|}{|\Omega|}\right), \qquad (A.69)$$

still on the event A we apply Lemma A.6 with $(b_k)_k = (\kappa_i - 1)_i$ and $\beta/2$ instead of β, so that

$$\sup_i |\kappa_i - 1| \text{diam}(\Omega_i)^{\beta/2} \leq n^{-\beta/(2d)}, \qquad (A.70)$$

obtaining

$$\left\|\sum_i (\kappa_i - \kappa)\chi_{\Omega_i}\right\|_{W^{-1,p}(\Omega)}^p \lesssim \left(|\log \delta|\, \delta^{1-\beta/2} n^{-\beta/(2d)}\right)^p. \qquad (A.71)$$

We choose γ_δ so that

$$|\log \delta| \delta^{1-\beta/2} n^{-\beta/(2d)} \ll n^{-1/d} \qquad (A.72)$$

which is ensured if

$$(\beta/2 - 1)\gamma_\delta < \frac{\beta/2 - 1}{d}, \quad \text{i.e.,} \quad \gamma_\delta < \frac{1}{d}, \qquad (A.73)$$

which is a condition we already found in (A.54). Recalling also that γ_δ must satisfy (A.60) we see that indeed indeed one can always choose such a γ_δ. This concludes the proof of (A.63), hence we settled (A.51).

A.4. **Lower bound.** Under the assumptions of Theorem A.1, we argue now that the corresponding lower bound holds:

$$\liminf_{n\to\infty} n^{-1+p/d}\mathbb{E}\left[W_\Omega^p(\nu_n)\right] \geq \mathsf{c}_{\nu,p}|\Omega|, \qquad (A.74)$$

thus concluding the proof of Theorem A.1. The main idea dates back to [7, Theorem 24]. We fix a cube $Q = Q_L$ with L so large that $\Omega \subseteq Q_{L-1}$. We set $\Omega_1 = \Omega$ and, for $i = 2, \ldots, K$ let Ω_i be the connected components of $Q\backslash\Omega$ so that $Q\backslash\Omega = \cup_{i=2}^K \Omega_i$. Notice that, for every i, either $\partial\Omega_i$ is C^2 or is the union of ∂Q and a C^2 surface. In particular each Ω_i satisfies (A.41). By Proposition A.2 and (2.20) we obtain that

$$\begin{aligned}
\mathsf{c}_{\nu,p}|Q| &= \lim_{n\to\infty} n^{-1+p/d}\mathbb{E}\left[W_Q^p(\nu_n)\right] \\
&\leq (1+\varepsilon)\liminf_{n\to\infty} n^{-1+p/d}\mathbb{E}\left[W_\Omega^p(\nu_n)\right] \\
&\quad + (1+\varepsilon)\sum_{i=2}^K \limsup_{n\to\infty} n^{-1+p/d}\mathbb{E}\left[W_{\Omega_i}^p(\nu_n)\right] + \frac{C}{\varepsilon^{(p-1)^+}}\limsup_{n\to\infty} n^{p/d}\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)\right],
\end{aligned}$$
$$(A.75)$$

with $\kappa_i = \nu_n(\Omega_i)/(n|\Omega_i|)$. Using (A.51) for $i = 2, \ldots K$, we obtain

$$\mathsf{c}_{\nu,p}|Q| \leq \liminf_{n\to\infty} n^{-1+p/d}\mathbb{E}\left[W_\Omega^p(\nu_n)\right] + \sum_{k=2}^K \mathsf{c}_{\nu,p}|\Omega_i| + \limsup_{n\to\infty} n^{p/d}\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)\right]. \quad (A.76)$$

Since $|\Omega| = |Q| - \sum_{i=2}^K |\Omega_i|$, inequality (A.74), hence the thesis, follows if we argue that

$$\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)\right] \ll n^{-p/d}. \quad (A.77)$$

We set $\kappa = \nu_n(Q)/(n|Q|)$. If $p < 1$, we use (2.18) to obtain

$$\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)\right] \leq \mathbb{E}\left[\operatorname{diam}(\Omega)^p \sum_{i=1}^K |\Omega_i|\,|\kappa_i - \kappa|\right]$$

$$\lesssim \mathbb{E}\left[|\kappa - 1|\right] + \sum_{i=1}^K \mathbb{E}\left[|\kappa_i - 1|\right] \lesssim n^{(\alpha-d)/(2d)} \ll n^{-p/d} \quad (A.78)$$

by the assumption (A.3). If $p \geq 1$, we proceed along the lines as the proof of (A.63), but here it is actually simpler since the partition is fixed (previously it depended upon n through δ). First, we set $\kappa = \nu_n(Q)/(n|Q|)$ and introduce the event

$$A = \left\{|\kappa - 1| \leq (n^{1/d}\operatorname{diam}(Q))^{-\beta/2}\right\} \cap \bigcap_{i=1}^K \left\{|\kappa_i - 1| \leq (n^{1/d}\operatorname{diam}(\Omega_i))^{-\beta/2}\right\}, \quad (A.79)$$

for some $0 < \beta < d - \alpha$. Arguing as in the proof of (A.67), we are easily reduced to prove

$$\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)I_A\right] \ll n^{-p/d}. \quad (A.80)$$

If n is sufficiently large, we have $\kappa \geq 1/2$ on A, hence by (2.29) we obtain

$$\mathbb{E}\left[W_{\Omega_i}^p(\kappa_i)I_A\right] \lesssim \sum_{i=1}^K \mathbb{E}\left[|\kappa_i - 1|^p\,I_A\right] \lesssim n^{-p\beta/(2d)}, \quad (A.81)$$

which is $\ll n^{-p/d}$ using (A.3) in this case.

Appendix B. De-Poissonization

In this section we discuss a De-Poissonization argument in order to transfer limit results from the case of the sum of a random number N_λ of measures, and that of a deterministic number $\mathbb{E}[N_\lambda] \approx \lambda$.

Proposition B.1. *Let $(\mu^i)_{i=1}^\infty$ be i.i.d. random Borel measures on $\Omega \subseteq \mathbb{R}^d$ such that*

$$\mathbb{E}\left[W_\Omega^p(\mu^1)\right] < \infty, \quad (B.1)$$

and set

$$f(n) := \mathbb{E}\left[W_\Omega^p\left(\sum_{i=1}^n \mu^i\right)\right] \quad (B.2)$$

Let N_λ denote a (further) independent Poisson random variable with mean λ. Then, for every $\alpha \in \mathbb{R}$,

$$\liminf_{n\to\infty} n^\alpha f(n) \geq \liminf_{\lambda\to\infty} \lambda^\alpha \mathbb{E}\left[f(N_\lambda)\right], \quad (B.3)$$

and

$$\limsup_{n\to\infty} n^\alpha f(n) \leq \limsup_{\lambda\to\infty} \lambda^\alpha \mathbb{E}\left[f(N_\lambda)\right]. \quad (B.4)$$

In particular, the two limits

$$\lim_{n\to\infty} n^\alpha f(n) \quad and \quad \limsup_{\lambda\to\infty} \lambda^\alpha \mathbb{E}\left[f(N_\lambda)\right] \quad (B.5)$$

exist and coincide, whenever one is known to exist.

Proof. Given $1 \leq m \leq n$, the identity

$$\frac{1}{n} \sum_{i=1}^{n} \mu^i = \binom{n}{m}^{-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \sharp I = m}} \frac{1}{m} \sum_{i \in I} \mu^i \tag{B.6}$$

in combination with (2.20) yields

$$\frac{1}{n} W_\Omega^p \left(\sum_{i=1}^{n} \mu^i \right) = W_\Omega^p \left(\binom{n}{m}^{-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \sharp I = m}} \frac{1}{m} \sum_{i \in I} \delta_{\mu^i} \right)$$
$$\leq \binom{n}{m}^{-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \sharp I = m}} \frac{1}{m} W_\Omega^p \left(\sum_{i \in I} \mu^i \right). \tag{B.7}$$

Taking expectation, we obtain that

$$\frac{f(n)}{n} \leq \frac{f(m)}{m}, \tag{B.8}$$

since $(\mu^i)_{i=1}^m$ have the same law of $(\mu^i)_{i \in I}$, if I contains m elements.

Given $\varepsilon > 0$ and $\lambda > 0$ introduce

$$A = A_{\varepsilon,\lambda} = \{\lambda(1-\varepsilon) \leq N_\lambda \leq \lambda(1+\varepsilon)\}, \tag{B.9}$$

which Markov inequality satisfies, for every $q \geq 1$, the inequality

$$\mathbb{E}\left[N_\lambda I_{A^c}\right] \lesssim_q \lambda^{-q}. \tag{B.10}$$

Using (2.20) and the trivial bound

$$W_\Omega^p \left(\sum_{i=1}^{n} \mu^i \right) \leq \sum_{i=1}^{n} W_\Omega^p \left(\mu^i \right), \tag{B.11}$$

we find that

$$\mathbb{E}\left[f(N_\Lambda) I_{A^c}\right] \leq \mathbb{E}\left[\sum_{i=1}^{N_\lambda} W_\Omega^p \left(\mu^i \right) I_{A^c} \right] = \mathbb{E}\left[N_\lambda I_{A^c}\right] \mathbb{E}\left[W_\Omega^p \left(\mu^1 \right)\right]$$
$$\lesssim_q \lambda^{-q}, \tag{B.12}$$

which is infinitesimal (even after multiplying by λ^α, if q is sufficiently large). Thus, we are reduced to bound $W_\Omega^p \left(\sum_{i=1}^{N_\lambda} \mu^i \right)$ on A, for which we use (B.8)

$$\frac{f(\lfloor \lambda(1+\varepsilon) \rfloor)}{\lfloor \lambda(1+\varepsilon) \rfloor} \leq \frac{f(N_\lambda)}{N_\lambda} \leq \frac{f(\lfloor \lambda(1-\varepsilon) \rfloor)}{\lfloor \lambda(1-\varepsilon) \rfloor}. \tag{B.13}$$

Multiplying both sides by N_λ we obtain (still on A) that

$$\frac{\lfloor \lambda(1-\varepsilon) \rfloor f(\lfloor \lambda(1+\varepsilon) \rfloor)}{\lfloor \lambda(1+\varepsilon) \rfloor} \leq f(N_\lambda) \leq \frac{\lfloor \lambda(1+\varepsilon) \rfloor f(\lfloor \lambda(1-\varepsilon) \rfloor)}{\lfloor \lambda(1-\varepsilon) \rfloor}. \tag{B.14}$$

Taking also expectation with respect to N_λ, we obtain

$$\frac{\lfloor \lambda(1-\varepsilon) \rfloor f(\lfloor \lambda(1+\varepsilon) \rfloor)}{\lfloor \lambda(1+\varepsilon) \rfloor} \mathbb{P}(A) \leq \mathbb{E}\left[f(N_\lambda) I_A\right] \leq \frac{\lfloor \lambda(1+\varepsilon) \rfloor f(\lfloor \lambda(1-\varepsilon) \rfloor)}{\lfloor \lambda(1-\varepsilon) \rfloor} \mathbb{P}(A). \tag{B.15}$$

Multiplying both sides by λ^α and letting $\lambda \to \infty$, we obtain that

$$\limsup_{\lambda \to \infty} \frac{\lambda^\alpha \lfloor \lambda(1-\varepsilon) \rfloor f(\lfloor \lambda(1+\varepsilon) \rfloor)}{\lfloor \lambda(1+\varepsilon) \rfloor} \mathbb{P}(A) \leq \limsup_{\lambda \to \infty} \lambda^\alpha \mathbb{E}\left[f(N_\lambda)\right] \tag{B.16}$$

and

$$\liminf_{\lambda \to \infty} \lambda^{\alpha} \mathbb{E}\left[f(N_{\lambda})\right] \le \liminf_{\lambda \to \infty} \frac{\lambda^{\alpha} \lfloor \lambda(1 - \varepsilon) \rfloor f(\lfloor \lambda(1 + \varepsilon) \rfloor)}{\lfloor \lambda(1 + \varepsilon) \rfloor}. \tag{B.17}$$

The thesis then follows by noticing that

$$\begin{aligned}
\limsup_{n \to \infty} n^{\alpha} f(n) &= \limsup_{\lambda \to \infty} \lfloor \lambda(1 + \varepsilon) \rfloor^{\alpha} f(\lfloor \lambda(1 + \varepsilon) \rfloor) \\
&= \frac{(1 + \varepsilon)^{\alpha - 1}}{1 - \varepsilon} \limsup_{\lambda \to \infty} \frac{\lambda^{\alpha} \lfloor \lambda(1 - \varepsilon) \rfloor f(\lfloor \lambda(1 + \varepsilon) \rfloor)}{\lfloor \lambda(1 + \varepsilon) \rfloor} \mathbb{P}(A)
\end{aligned} \tag{B.18}$$

and similarly

$$\liminf_{n \to \infty} n^{\alpha} f(n) = \frac{(1 - \varepsilon)^{\alpha - 1}}{1 + \varepsilon} \liminf_{\lambda \to \infty} \frac{\lambda^{\alpha} \lfloor \lambda(1 - \varepsilon) \rfloor f(\lfloor \lambda(1 + \varepsilon) \rfloor)}{\lfloor \lambda(1 + \varepsilon) \rfloor} \mathbb{P}(A), \tag{B.19}$$

and letting $\varepsilon \to 0$. $\qquad\square$

Remark B.2. We notice that the argument above does not depend very much on the fact that N_{λ} has Poisson law, but rather than (B.10) holds. In particular, if we replace it with a binomial variable N_T with parameters $m = m(T) \to \infty$ and $p = p(T) \to 0$ such that $\lambda := mp \to \infty$, and we assume that

$$\lim_{n \to \infty} n^{\alpha} f(n) \tag{B.20}$$

exists, then also the limit

$$\lim_{T \to \infty} \lambda^{\alpha} \mathbb{E}\left[f(N_T)\right] \tag{B.21}$$

exists and coincides with the first limit.

References

[1] M. Ajtai, J. Komlós, and G. Tusnády. "On optimal matchings." In: *Combinatorica* 4 (1984), pp. 259–264 (cit. on p. 2).

[2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Second. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008, pp. x+334 (cit. on pp. 1, 7).

[3] L. Ambrosio and F. Glaudo. "Finer estimates on the 2-dimensional matching problem". In: *J. Éc. polytech. Math.* 6 (2019), pp. 737–765 (cit. on p. 2).

[4] L. Ambrosio, F. Glaudo, and D. Trevisan. "On the optimal map in the 2-dimensional random matching problem". In: *Discrete Contin. Dyn. Syst.* 39.12 (2019), pp. 7291–7308 (cit. on pp. 2, 8).

[5] L. Ambrosio, F. Stra, and D. Trevisan. "A PDE approach to a 2-dimensional matching problem". In: *Probab. Theory Relat. Fields* 173.1-2 (2019), pp. 433–477 (cit. on p. 2).

[6] L. Ambrosio, M. Goldman, and D. Trevisan. "On the quadratic random matching problem in two-dimensional domains". In: *Electronic Journal of Probability* 27 (2022), pp. 1–35 (cit. on pp. 2–4, 22, 40, 43).

[7] F. Barthe and C. Bordenave. "Combinatorial optimization over two random point sets." In: *Séminaire de probabilités XLV*. Cham: Springer, 2013, pp. 483–535 (cit. on pp. 2, 4, 22, 40, 48).

[8] D. Benedetto and E. Caglioti. "Euclidean random matching in 2D for non-constant densities". In: *Journal of Statistical Physics* 181.3 (2020), pp. 854–869 (cit. on p. 2).

[9] S. G. Bobkov and M. Ledoux. "Transport inequalities on Euclidean spaces for non-Euclidean metrics". In: *Journal of Fourier Analysis and Applications* 26.4 (2020), p. 60 (cit. on pp. 2, 4).

[10] J. H. Boutet de Monvel and O. C. Martin. "Almost sure convergence of the minimum bipartite matching functional in Euclidean space". In: *Combinatorica* 22.4 (2002), pp. 523–530 (cit. on p. 2).

[11]   R. Capelli, S. Caracciolo, A. Di Gioacchino, and E. M. Malatesta. "Exact value for the average optimal cost of the bipartite traveling salesman and two-factor problems in two dimensions". In: *Physical Review E* 98.3 (2018), p. 030101 (cit. on p. 2).

[12]   S. Caracciolo, M. D'Achille, and G. Sicuro. "Anomalous scaling of the optimal cost in the one-dimensional random assignment problem". In: *Journal of Statistical Physics* 174.4 (2019), pp. 846–864 (cit. on p. 2).

[13]   S. Caracciolo, M. P. D'Achille, V. Erba, and A. Sportiello. "The Dyck bound in the concave 1-dimensional random assignment model". In: *Journal of Physics A: Mathematical and Theoretical* 53.6 (2020), p. 064001 (cit. on p. 2).

[14]   S. Caracciolo and G. Sicuro. "Scaling hypothesis for the Euclidean bipartite matching problem. II. Correlation functions". In: *Physical Review E* 91.6 (2015), p. 062125 (cit. on p. 2).

[15]   N. Clozeau and F. Mattesini. "Annealed quantitative estimates for the quadratic 2D-discrete random matching problem". In: *arXiv preprint arXiv:2303.00353* (2023) (cit. on p. 2).

[16]   V. Dobrić and J. E. Yukich. "Asymptotics for transportation cost in high dimensions". In: *Journal of Theoretical Probability* 8.1 (1995), pp. 97–118 (cit. on p. 2).

[17]   A. Drewitz, B. Ráth, and A. Sapozhnikov. *An introduction to random interlacements.* Springer, 2014 (cit. on pp. 2, 3, 14).

[18]   R. M. Dudley. "The speed of mean Glivenko-Cantelli convergence". In: *The Annals of Mathematical Statistics* 40.1 (1969), pp. 40–50 (cit. on p. 2).

[19]   W. Gangbo and R. J. McCann. "The geometry of optimal transportation". In: *Acta Math.* 177.2 (1996), pp. 113–161 (cit. on p. 1).

[20]   M. Goldman and M. Huesmann. "A fluctuation result for the displacement in the optimal matching problem". In: *The Annals of Probability* 50.4 (2022), pp. 1446–1477 (cit. on p. 2).

[21]   M. Goldman, M. Huesmann, and F. Otto. "A large-scale regularity theory for the Monge-Ampere equation with rough data and application to the optimal matching problem". In: *arXiv preprint arXiv:1808.09250* (2018) (cit. on p. 2).

[22]   M. Goldman, M. Huesmann, and F. Otto. "Quantitative Linearization Results for the Monge-Ampère Equation". In: *Communications on Pure and Applied Mathematics* 74.12 (2021), pp. 2483–2560 (cit. on p. 2).

[23]   M. Goldman and D. Trevisan. "Convergence of asymptotic costs for random Euclidean matching problems". In: *Probability and Mathematical Physics* 2.2 (2021), pp. 341–362 (cit. on pp. 2, 3, 22, 40).

[24]   M. Goldman and D. Trevisan. "On the concave one-dimensional random assignment problem and Young integration theory". In: *arXiv preprint arXiv:2305.09234* (2023) (cit. on pp. 2, 4).

[25]   M. Goldman and D. Trevisan. "Optimal transport methods for combinatorial optimization over two random point sets". In: *arXiv preprint arXiv:2209.14615* (2022) (cit. on pp. 2–4, 8, 9, 22, 40, 43, 45).

[26]   A. Grigor'yan and L. Saloff-Coste. "Hitting probabilities for Brownian motion on Riemannian manifolds". In: *Journal de Mathématiques Pures et Appliquées* 81.2 (2002), pp. 115–142 (cit. on pp. 14, 15, 17).

[27]   M. Huesmann, F. Mattesini, and F. Otto. "There is no stationary cyclically monotone Poisson matching in 2d". In: *arXiv preprint arXiv:2109.13590* (2021) (cit. on p. 2).

[28]   M. Huesmann, F. Mattesini, and D. Trevisan. "Wasserstein asymptotics for the empirical measure of fractional Brownian motion on a flat torus". In: *Stochastic Processes and their Applications* 155 (2023), pp. 1–26 (cit. on pp. 2, 4).

[29]   J. Jalowy. "The Wasserstein distance to the Circular Law". In: *arXiv preprint arXiv:2111.03595* (2021) (cit. on p. 2).

[30] M. Ledoux and J.-X. Zhu. "On optimal matching of Gaussian samples III". In: *Probability and Mathematical Statistics* 41 (2021) (cit. on p. 2).

[31] H. Li and B. Wu. "Wasserstein Convergence for Empirical Measures of Subordinated Fractional Brownian Motions on the Flat Torus". In: *arXiv preprint arXiv:2305.01228* (2023) (cit. on p. 2).

[32] R. J. McCann. "Exact solutions to the transportation problem on the line". In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 455.1984 (1999), pp. 1341–1380 (cit. on p. 1).

[33] G. Peyré, M. Cuturi, et al. "Computational Optimal Transport: With Applications to Data Science". In: *Foundations and Trends® in Machine Learning* 11.5-6 (2019), pp. 355–607 (cit. on p. 7).

[34] S. Port. *Brownian motion and classical potential theory.* Elsevier, 2012 (cit. on pp. 11, 17).

[35] G. Sicuro. *The Euclidean Matching Problem.* en. Springer Theses. Springer International Publishing, 2017 (cit. on p. 2).

[36] A.-S. Sznitman. "On scaling limits and Brownian interlacements". In: *Bulletin of the Brazilian Mathematical Society, New Series* 44 (2013), pp. 555–592 (cit. on pp. 2, 3).

[37] M. Talagrand. "Matching random samples in many dimensions". In: *The Annals of Applied Probability* (1992), pp. 846–856 (cit. on p. 2).

[38] M. Talagrand. *Upper and lower bounds for stochastic processes.* Vol. 60. Springer, 2014 (cit. on p. 2).

[39] C. Villani. *Optimal transport: old and new.* Vol. 338. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 2009, pp. xxii+973 (cit. on pp. 1, 7).

[40] F.-Y. Wang. "Convergence in Wasserstein distance for empirical measures of Dirichlet diffusion processes on manifolds". In: *Journal of the European Mathematical Society* (2022) (cit. on p. 2).

[41] F.-Y. Wang. "Convergence in Wasserstein Distance for Empirical Measures of Non-Symmetric Subordinated Diffusion Processes". In: *arXiv preprint arXiv:2301.08420* (2023) (cit. on p. 2).

[42] F.-Y. Wang. "Convergence in Wasserstein distance for empirical measures of semilinear SPDEs". In: *The Annals of Applied Probability* 33.1 (2023), pp. 70–84 (cit. on p. 2).

[43] F.-Y. Wang. "Precise limit in Wasserstein distance for conditional empirical measures of Dirichlet diffusion processes". In: *Journal of Functional Analysis* 280.11 (2021), p. 108998 (cit. on p. 2).

[44] F.-Y. Wang. "Wasserstein convergence rate for empirical measures on noncompact manifolds". In: *Stochastic Processes and their Applications* 144 (2022), pp. 271–287 (cit. on p. 2).

[45] F.-Y. Wang and B. Wu. "Wasserstein Convergence for Empirical Measures of Subordinated Diffusions on Riemannian Manifolds". In: *Potential Analysis* (2022), pp. 1–22 (cit. on p. 2).

[46] F.-Y. Wang and L. Yan. "Gradient estimate on convex domains and applications". In: *Proceedings of the American Mathematical Society* 141.3 (2013), pp. 1067–1081 (cit. on p. 45).

[47] F.-Y. Wang and J.-X. Zhu. "Limit theorems in Warsserstein distance for empirical measures of diffusion processes on Riemannian manifolds". In: *arXiv preprint arXiv:1906.03422* (2019) (cit. on p. 2).

M.M.: Faculty of Mathematics, HSE University, 101000 Moscow, Russia

D.T.: Dipartimento di Matematica, Università degli Studi di Pisa, 56125 Pisa, Italy
Email address: dario.trevisan@unipi.it