EVOLUTION OF CRYSTALLINE THIN FILMS BY EVAPORATION AND CONDENSATION IN THREE DIMENSIONS

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Abstract. The morphology of crystalline thin films evolving on flat rigid substrates by condensation of extra film atoms or by evaporation of their own atoms in the surrounding vapor is studied in the framework of the theory of Stress Driven Rearrangement Instabilities (SDRI). By following the SDRI literature both the elastic contributions due to the mismatch between the film and the substrate lattices at their theoretical (free-standing) elastic equilibrium, and a curvature perturbative regularization preventing the problem to be ill-posed due to the otherwise exhibited backward parabolicity, are added in the evolution equation. The resulting Cauchy problem under investigation consists in an anisotropic mean-curvature type flow of the fourth order on the film profiles, which are assumed to be parametrizable as graphs of functions measuring the film thicknesses, coupled with a quasistatic elastic problem in the film bulks. Periodic boundary conditions are considered. The results are twofold: the existence of a regular solution for a finite period of time and the stability for all times, of both Lyapunov and asymptotic type, of any configuration given by a flat film profile and the related elastic equilibrium. Such achievements represent both the generalization to three dimensions of a previous result in two dimensions for a similar Cauchy problem, and the complement of the analysis previously carried out in the literature for the symmetric situation in which the film evolution is not influenced by the evaporation-condensation process here considered, but it is entirely due to the volume preserving surface-diffusion process, which is instead here neglected. The method is based on minimizing movements, which allow to exploit the the gradient-flow structure of the evolution equation.

1. Introduction

In this paper we study the morphological evolution of crystalline thin films deposited on flat rigid substrates by *condensation* of extra film atoms from a surrounding vapor, which results in a film growth, or by *evaporation* in such vapor of their own film atoms, which triggers instead a film dissolution. Besides on the phenomena at the film surface, the focus is on the elastic properties of the film bulk material, which is subject to the so-called *epitaxial strain* imparted by the underlying substrate. This is due to the fact that, on the one hand, the film and the substrate material are assumed to adhere at their contact interface without delamination or debonding, but on the other hand, on such interface atoms are affected by the mismatch that can occur between the atomic lattice characterizing the film free-standing elastic equilibrium and the atomic lattice characterizing the substrate elastic equilibrium.

The directions of investigation are twofold: the existence of a regular evolution for a finite period of time and the stability for all times, of both *Lyapunov* and *asymptotic type*, of any film configuration consisting of a flat profile and the related elastic equilibrium, which are established in Theorems 3.8 and 3.9, respectively. Such results can be seen both as a generalization to the three dimensional setting, which is the physically relevant one for the applications, of the results previously achieved in [32], and as an equivalent analysis to the one carried out in [14] (see also [13]) for the complementary setting in which the film evolution is not influenced by the evaporation-condensation process here considered, but it is entirely due to the volume preserving *surface-diffusion process*, which is instead

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here neglected. We notice that in our setting the film volume can suddenly change directly interfering with the elastic properties of the deposited film material.

As in [13, 14, 32] the reference framework for the modeling aspects is provided in the literature by the theory of *Stress Driven Rearrangement Instabilities* (SDRI) [3, 7, 15, 19, 35] of which thin films have historically been the SDRI foremost example, because of the interplay occurring between the film surface and elastic energy: Among the modes of stress-relief for the film material, the bulk deformation is energetically neutral with respect to the surface energy, but is payed in terms of the elastic energy, while the destabilization of the film free-boundary from the *Winterbottom optimal shape* [33, 34, 36] with surface roughness and interface instabilities positively contributes to the surface energy without storage of elastic energy. A delicate compromise between these competing stress-relief mechanisms must therefore be achieved, of which the film morphology is the result. A microscopic justification of the static model starting from atomic interactions has been provided in [29], while existence and regularity results for dimension $d = 2$ are available in [4, 8, 9, 11, 12, 18, 26, 27] and have been (partially) extended in higher dimensions for models related to the more general setting of SDRI in [6, 28].

In regard of the evolution theory, the reference goes back to W. W. Mullins [30, 31], who identify the equations describing the motion of a crystalline interface $\Gamma \subset \mathbb{R}^d$ by the evaporation-condensation process and by surface diffusion as the *motion by mean curvature*, i.e.,

$$V = -H \qquad \text{on } \Gamma, \tag{1.1}$$

and the *motion by the Laplacian of the mean curvature*, i.e.,

$$V = \Delta_\Gamma H \qquad \text{on } \Gamma, \tag{1.2}$$

respectively, where V denotes the normal velocity on Γ, H is the mean curvature on Γ, and Δ_Γ represents the tangential Laplacian along Γ. As described in [32] by taking into account the film-vapor interface anisotropy ψ and the elastic contribution Equations (1.1) and (1.2) become

$$V = -\mathrm{div}_\Gamma(D\psi(\nu)) - W(Eu) \qquad \text{on } \Gamma, \tag{1.3}$$

and

$$V = \Delta_\Gamma[\mathrm{div}_\Gamma(D\psi(\nu)) + W(Eu)] \qquad \text{on } \Gamma, \tag{1.4}$$

(see [2, 15, 22, 21, 23, 24] and [20, Remark 3.1, Section 8] for more details), where $u(\cdot, t)$ is the elastic equilibrium in the region $\Omega \subset \mathbb{R}^d$, Eu represents the *strain* and is the symmetric part of the gradient Du, W is the elastic energy density which is defined as

$$W(A) := \frac{1}{2}\mathbb{C}A : A$$

for every $A \in \mathbb{R}^{2\times 2}$ and for a symmetric positive fourth-order tensor $\mathbb{C}$, div_Γ is the tangential divergence along $\Gamma \subset \partial\Omega$, and ν is the outward unit normal of $\partial\Omega$.

We notice that (1.4) has been tackled in [16, 17] by means of fixed point techniques to prove existence and uniqueness results. More precisely, in [16] the authors study (1.4) for $d = 2$ and obtain short time existence and uniqueness, establishing the global-in-time existence for a specific class of initial data, while in [17] they prove short-time existence of a smooth solution and uniqueness for $d = 3$ (with the bulk contribution given by a forcing term, which also include the elastic setting). Furthermore, in the absence of elasticity the strategy based on *minimizing movements* has been extensively used previously in the literature to treat geometric flows of the type (1.1) and (1.2), also in order to establish global-in-time existence. Regarding the mean curvature flow, without aiming at a comprehensive description of the results by minimizing movements we just mention here that in [5] the authors obtain global-in-time existence and uniqueness for the anisotropic crystalline mean curvature flow in all dimensions for arbitrary (possibly unbounded) initial sets and for the natural mobility, while in [25] global-in-time existence of the flat solutions for a mean curvature flow with volume preserving is established.

Regarding the literature results for the Equations (1.3) and (1.4) carried out by minimizing movements in the presence of elasticity we refer to [13, 14, 32] where the setting of *evolving graphs* is considered under an extra perturbative *regularization term* added to the equations. More precisely, in [13, 14, 32] admissible film profiles are constrained to be parametrizable as graph Γ_h of functions $h : Q \times [0,T] \to [0,+\infty)$ measuring the thickness of the film, where $Q := (0,\ell)^{d-1}$ and $[0,T]$ relate, for given $\ell > 0$ and a time $T > 0$, to the spatial and the temporal variable, respectively. Furthermore, in [13, 14, 32] a regularization term is added on the basis of the argumentation already present in [2, 23, 24] to overcome the fact that Equations (1.3) and (1.4) are backward parabolic in the presence of highly anisotropic non-convex surface tension ψ, making the related Cauchy problem ill-posed. By adding such regularization term Equations (1.3) and (1.4) become

$$V = -\mathrm{div}_\Gamma(D\psi(\nu)) - W(Eu) + \varepsilon R(k_1,k_2) \qquad \text{on } \Gamma_h \tag{1.5}$$

and

$$V = \Delta_\Gamma[\mathrm{div}_\Gamma(D\psi(\nu)) + W(Eu) - \varepsilon R(k_1,k_2)] \qquad \text{on } \Gamma_h, \tag{1.6}$$

respectively, where $\varepsilon > 0$ and R is a function depending on the principal curvatures k_1, k_2 of Γ_h. By defining R (also on the basis of [10]) as

$$R(k_1,k_2) := \left(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\left(k_1^2 + k_2^2 - \frac{1}{p}H^2\right)\right) \tag{1.7}$$

(where we recall that $H := (k_1 + k_2)/2$) with $p = 2$ for $d = 2$ in [13, 32] and with $p > 2$ for $d = 3$ in [14], the backward parabolicity of the equation is avoided as surface patterns with large curvature get penalized. As such, minimizing movements are then used in [13, 14, 32] so that the *gradient-flow structure* exhibited by Equations (1.5) and (1.6) can be exploited: by considering the functional

$$\mathcal{F}(h,u_h) := \int_{\Omega_h} W(Eu)\mathrm{d}z + \int_{\Gamma_h}\left(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\right)\mathrm{d}\mathcal{H}^2,$$

where u_h denotes the elastic equilibrium in Ω_h (under the proper periodic and boundary conditions) and $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure, then Equations (1.5) and (1.6) formally coincide with the gradient flow of $\mathcal{F}$ with respect to an L^2- and H^{-1}-Riemannian structure. This allowed to establish short-time existence of a regular solution of (1.5) in [32] for $d = 2$, and of (1.6) in [13] and [14] for $d = 2$ and $d = 3$, respectively. It remains open the case of (1.5) for $d = 3$ that we intend here to tackle by also choosing, as in [14], $p > 2$ in (1.7).

Therefore, by recalling that the normal velocity parametrized as the graph of the thickness functions h is given by

$$V = \frac{1}{\sqrt{1+|Dh|^2}}\frac{\partial h}{\partial t} \tag{1.8}$$

on Γ_h, where Dh denotes the gradient with respect to the spatial coordinates, the *Cauchy problem* under investigation depending on the period of time $T > 0$ is the following:

$$\begin{cases} \frac{1}{\sqrt{1+|Dh|^2}}\frac{\partial h}{\partial t} = -\mathrm{div}_\Gamma(D\psi(\nu)) - W(Eu) & \\ \qquad\qquad +\varepsilon\left(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\left(|B|^2 - \frac{1}{p}H^2\right)\right) & \text{in } \mathbb{R}^2\times[0,T], \\ \mathrm{div}(\mathbb{C}Eu) = 0 & \text{in } \Omega_h, \\ \mathbb{C}Eu[\nu] = 0 & \text{on } \Gamma_h, \\ u(x,0,t) = (e_0^1 x_1, e_0^2 x_2, 0), & \\ h(\cdot,t) \text{ and } Du(\cdot,t) \text{ are } Q\text{-periodic}, & \\ h(\cdot,0) = h_0, & \end{cases} \tag{1.9}$$

where $e_0 := (e_0^1, e_0^2)$ with $e_0^i > 0$ is a vector which represents the mismatch between the crystalline lattices of the film and the substrate, h_0 is an admissible profile of the film at the initial time $t = 0$, and spatial Q-periodic conditions are considered. We notice that the Cauchy problem on the period of time $T > 0$ considered in [14] coincides with (1.9) if we replace the first equation in (1.9) with (1.6) (by taking into account (1.8)).

We can now more precisely detail the achieved results and we refer instead to Section 3.4 for their full characterization: In Theorem 3.8 we establish the existence of a time $T_0 > 0$ for which the

Cauchy problem (1.9) admits a solution in $[0, T_0]$, which is said to be *regular* in the meaning that the first equation in (1.9) is satisfied a.e. in $\mathbb{R}^2 \times [0, T_0]$ (see Definition 3.5), while in Theorem 3.9 we differentiate the setting of a nonconvex and of a convex surface tension ψ as in [14, Section 4] and we prove that any admissible flat configuration as from Definition 3.2 is Lyapunov stable both in the convex and nonconvex case, and we can establish that is asymptotic stable only in the convex case in analogy to the results of [14].

1.1. **Organization of the paper and description of the method.** In Section 2 we fix the notation adopted throughout the paper. In Section 3 we introduce the precise mathematical setting of the model, we introduce the minimizing-movement scheme obtained by discretizing the time interval and by defining the incremental family of minimum problems (3.5) at each discrete time, we define the stability properties of the solutions of (1.9), and we state the two main theorems of the paper, i.e., Theorems 3.8 and 3.9. In Section 4 we show the existence of minimizers for (3.5) in Theorem 4.1 so that a discrete-time evolution as in Definition 3.3 can be constructed, of which then we analyze the convergence properties in Theorems 4.2 and 4.3, that then are improved in Theorem 4.4 by selecting a specific time $T_0 > 0$. In Section 5 we prove in Theorem 3.8 the existence of a regular solution in the time interval $[0, T_0]$ by passing, in view of the previously proven convergence properties, to the discrete *Euler-Lagrange equation* (5.1) satisfied by the minimizers of the incremental minimum problems (3.5) (see Lemma 5.1), from which in Theorem 5.2 we are able to deduce a crucial uniform bound on the mean curvature. In Section 6 we prove Theorem 3.9 by following the program of [14, Section 4] that includes the differentiation of the settings in which ψ is a nonconvex and convex surface tension. Finally, in Section 7 we collect some auxiliary results often used in the paper for the Reader's convenience.

2. Notation

In this section we set the main notation used throughout the paper. The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$, and, in case $m = d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. Given a function $u\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by Du, whose components are $(Du)_{ij} := \partial_j u_i$ for $i = 1, \dots, m$ and $j = 1, \dots, d$, and when $u\colon \mathbb{R}^d \to \mathbb{R}^d$, we also denote by Eu the symmetric part of the gradient, i.e., $Eu := \frac{1}{2}(Du + Du^T)$. Given a tensor field $A\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, by $\operatorname{div} A$ we mean its divergence with respect to the rows, namely $(\operatorname{div} A)_i := \sum_{j=1}^d \partial_j A_{ij}$ for $i = 1, \dots, m$.

The norm of a generic Banach space X is denoted by $\|\cdot\|_X$ and in case X is a Hilbert space, we denote by $\langle\cdot,\cdot\rangle_X$ its inner product. In the case $X = \mathbb{R}^d$, we simplify the notation by using the symbol $\langle\cdot,\cdot\rangle$ to denote the euclidean scalar product, that is $\langle v, w\rangle := \sum_{i=1}^d v_i w_i$ for every $v, w \in \mathbb{R}^d$, while $A : B := \sum_{i,j=1}^d a_{ji} b_{ij}$ denotes the Hilbert-Schmidt product of two matrices $A, B \in \mathbb{R}^{d\times d}$. Given two matrices $A \in \mathbb{R}^{m\times d}$ and $B \in \mathbb{R}^{n\times q}$ we denote by $A \otimes B \in \mathbb{R}^{mn\times dq}$ the Kronecker product between the two matrices A and B, defined as the block matrix $A \otimes B := (a_{ij}B)_{i,j}$. Given two Banach spaces X_1 and X_2, the space of linear and continuous maps from X_1 to X_2 is denoted by $\mathscr{L}(X_1; X_2)$. For any $\mathbb{A} \in \mathscr{L}(X_1; X_2)$ and $u \in X_1$, we indicate the image of u under $\mathbb{A}$ with $\mathbb{A}u \in X_2$.

We denote the d-dimensional Lebesgue measure by $\mathcal{L}^d$ and the $(d-1)$-dimensional Hausdorff measure by $\mathcal{H}^{d-1}$. Given a bounded open set Ω with Lipschitz boundary, ν indicates the outer unit normal vector of $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$-a.e. and we employ the usual definition of Lebesgue and Sobolev spaces on Ω. The values of Sobolev functions on the boundary of their set of definition are always intended in the sense of traces. Given a set U, then $U^2 := U \times U$ with $\times$ denoting the cartesian product. Finally, given an open interval $(a, b) \subset \mathbb{R}$ and $p \in [1, \infty]$, we denote by $L^p(a, b; X)$ the space of L^p functions from (a, b) to X. We use $H^k(a, b; X)$ and $W^{k,p}(a, b; X)$ to denote the Sobolev space of functions from (a, b) to X with k weak derivatives in $L^2(a, b; X)$ and $L^p(a, b; X)$, respectively.

3. Mathematical setting and main results

In this section we introduce the model, the main definitions, and the statements of the main results.

3.1. **The variational model.** Let us define $Q := (0,\ell)^2$ with $\ell > 0$. For $p > 2$ we denote by $W^{2,p}_{\#}(Q)$ the space of all functions of $W^{2,p}(Q)$ whose Q-periodic extension belong to $W^{2,p}_{loc}(\mathbb{R}^2)$, and we define the class of admissible profiles as

$$AP := \{h \in W^{2,p}_{\#}(Q) : h(x) \geq 0\},$$

where we use the notation $x = (x_1, x_2) \in Q$.

Moreover, for $h \in W^{2,p}_{\#}(Q)$ we refer to the sets

$$\Gamma_h := \{z = (x, h(x)) : x \in Q\} \text{ and } \Omega_h := \{z = (x, y) \in Q \times \mathbb{R} : 0 < y < h(x)\} \tag{3.1}$$

as the *film profile* and the *region of the film with height* h, respectively, while the corresponding sets with Q replaced by $\mathbb{R}^2$ are denoted by $\Gamma^{\#}_h$ and $\Omega^{\#}_h$. We define the *family of periodic displacements* by

$$PD := \{u : \Omega^{\#}_h \to \mathbb{R}^3 : u(x,y) = u(x + \ell k, y) \text{ for every } (x,y) \in \Omega^{\#}_h \text{ and } k \in \mathbb{Z}^2\}$$

and the *the family of admissible displacements* by

$$AD_h := \{u \in L^2_{loc}(\Omega^{\#}_h, \mathbb{R}^3) \cap PD : u(x,0) = (e^1_0 x_1, e^2_0 x_2, 0),\ Eu_{|\Omega_h} \in L^2(\Omega_h, \mathbb{R}^3)\},$$

where $e_0 := (e^1_0, e^2_0)$, with $e^1_0, e^2_0 > 0$, is the vector representing the mismatch between the free-standing equilibrium crystalline lattice of the film and the substrate materials. Consequently, the *family of admissible configurations* is

$$X := \{(h,u) : h \in AP \text{ and } u \in AD_h\}.$$

The *elastic energy density* $W : \mathbb{R}^{2\times2}_{sym} \to [0,\infty)$ is defined by

$$W(M) := \frac{1}{2}\mathbb{C}M : M,$$

where $\mathbb{C}$ is a fourth-order tensor such that there exists $k > 0$ for which

$$\mathbb{C}M : M \geq 2kM : M \quad \text{for every } M \in \mathbb{R}^{2\times2}_{sym}.$$

Furthermore, let $\psi : \mathbb{R}^3 \to [0, +\infty)$ be a function of class C^2 on $\mathbb{R}^3 \setminus \{0\}$ and positively one-homogeneous, which satisfies

$$\frac{1}{c}|\xi| \leq \psi(\xi) \leq c|\xi| \tag{3.2}$$

for every $\xi \in \mathbb{R}^3$ and for some positive constant c.

The *configuration energy functional* $\mathcal{F} : X \to [0,\infty]$ is given by

$$\mathcal{F}(h,u) := \mathcal{W}(h,u) + \mathcal{S}(h)$$

for every admissible configuration $(h,u) \in X$, where $\mathcal{W} : X \to [0,\infty]$ and $\mathcal{S} : AP \to [0,\infty]$ represent the *elastic* and the *surface energy*, respectively. The elastic energy is defined by

$$\mathcal{W}(h,u) := \int_{\Omega_h} W(Eu)\,\mathrm{d}z$$

while the surface energy is defined by

$$\mathcal{S}(h) := \int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\right) \mathrm{d}\mathcal{H}^2,$$

where ν is the outer unit normal to Ω_h, ε is a positive constant, and $H = \operatorname{div}_{\Gamma_h} \nu$ denotes the sum of the principal curvatures of Γ_h, i.e.,

$$H = -\operatorname{div}\left(\frac{Dh}{\sqrt{1 + |Dh|^2}}\right) \quad \text{in } Q. \tag{3.3}$$

Notice that from (3.3) it follows that

$$\int_Q H\,\mathrm{d}x = 0.$$

Definition 3.1 (Elastic equilibrium)**.** We say that $\bar{u} \in AD_h$ is the *elastic equilibrium* of $\bar{h} \in AP$ if

$$\mathcal{F}(\bar{h}, \bar{u}) = \min\{\mathcal{F}(\bar{h}, u) : (\bar{h}, u) \in X\}.$$

Notice that the elastic equilibrium exixts for every $\bar{h} \in AP$ and it is unique in view of the Dirichlet condition.

Definition 3.2 (Flat configuration)**.** We say that an admissible configuration $(h, u) \in X$ is a flat configuration if $h \equiv d > 0$ and $u = u_d$, where u_d represents the elastic equilibrium in Ω_d.

3.2. **The incremental minimum problem.** Let $(h_0, u_0) \in X$ be such that $h_0 > 0$ and u_0 is the elastic equilibrium of h_0, and let Λ_0 be a positive constant such that

$$\|h_0\|_{C^1_\#(Q)} < \Lambda_0. \tag{3.4}$$

By considering a sequence $\tau_N \searrow 0$, for every $i \in \mathbb{N}$ we define inductively $(h_{i,N}, u_{i,N})$ as a solution to the following minimum problem:

$$\min\{G_{i,N}(h, u) : (h, u) \in X,\ \|Dh\|_{L^\infty(Q)} \leq \Lambda_0\}. \tag{3.5}$$

The functional $G_{i,N}$ is given by

$$G_{i,N}(h, u) := \mathcal{F}(h, u) + P_{i,N}(h), \tag{3.6}$$

with the *penalization term* $P_{i,N}$ defined by

$$P_{i,N}(h) := \frac{1}{2\tau_N} \int_{\Gamma_{h_{i-1,N}}} \left(\frac{h - h_{i-1,N}}{J_{i-1,N}} \right)^2 \mathrm{d}\mathcal{H}^2 = \frac{1}{2\tau_N} \int_Q \frac{(h - h_{i-1,N})^2}{J_{i-1,N}} \mathrm{d}x, \tag{3.7}$$

where $J_{i-1,N} := \sqrt{1 + |Dh_{i-1,N}|^2}$. It is proved in Theorem 4.1 that problem (3.5) admits a minimizer.

Definition 3.3 (Dicrete-time evolution)**.** Let $(h_0, u_0) \in X$ be an initial configuration and for $i, N \in \mathbb{N}$ let $(h_{i,N}, u_{i,N})$ be a solution to (3.5). We refer to the piecewise linear interpolation $h_N : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$, given by

$$h_N(x, t) := h_{i,N}(x) + \frac{1}{\tau_N}(t - (i-1)\tau_N)(h_{i,N}(x) - h_{i-1,N}(x)) \tag{3.8}$$

for every $(x, t) \in \mathbb{R}^2 \times [(i-1)\tau_N, i\tau_N]$ and $i \in \mathbb{N}$, as a *discrete-time evolution* of the incremental minimum problem (3.5). Moreover, we denote with $u_N(\cdot, t)$ the elastic equilibrium corresponding to $h_N(\cdot, t)$.

In Theorem 3.8 we make use of a different type of interpolation of solution to (3.5).

Definition 3.4. Let $(h_0, u_0) \in X$ be an initial configuration. Given a solution $(h_{i,N}, u_{i,N})$ for $i, N \in \mathbb{N}$ to (3.5), we define the piecewise constant interpolations $\tilde{h}_N : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$ and $\tilde{u}_N : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ in the following way:

$$\tilde{h}_N(x, t) := h_{i,N}(x) \qquad \text{for } (x, t) \in \mathbb{R}^2 \times [(i-1)\tau_N, i\tau_N) \text{ and } i \in \mathbb{N}, \tag{3.9}$$

$$\tilde{u}_N(x, y, t) := u_{i,N}(x, y) \qquad \text{for } (x, y, t) \in \Omega_{h_{i,N}} \times [(i-1)\tau_N, i\tau_N) \text{ and } i \in \mathbb{N} \tag{3.10}$$

3.3. **Solutions of the evolution problem and properties.** We now introduce the notion of the solutions to (1.9) and stability properties, that will be proven to apply under certain conditions to the flat configuration.

Definition 3.5 (Solutions of the evolution problem)**.** Let $(h_0, u_0) \in X$ be a configuration satisfying (3.4) and $T > 0$. We say that a function $h \in L^\infty(0, T; W^{2,p}_\#(Q)) \cap H^1(0, T; L^2_\#(Q))$ such that

(i) $h(\cdot, 0) = h_0(\cdot)$ in Q,
(ii) $-\mathrm{div}_\Gamma(D\psi(\nu)) - W(Eu) + \varepsilon \left(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H \left(|B|^2 - \frac{1}{p}H^2 \right) \right) \in L^2(0, T; L^2_\#(Q))$,

(iii) the equation

$$\frac{1}{J}\frac{\partial h}{\partial t} = -\operatorname{div}_\Gamma(D\psi(\nu)) - W(Eu) + \varepsilon\left(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\left(|B|^2 - \frac{1}{p}H^2\right)\right)$$

is satisfied for $\mathcal{L}^3$-a.e. in $Q\times(0,T)$,

where $J := \sqrt{1+|Dh|^2}$, $\Gamma := \Gamma_{h(\cdot,t)}$, $u(\cdot,t)$ is the elastic equilibrium in $\Omega_{h(\cdot,t)}$, and $W(Eu)$ denotes the trace of $W(Eu)$ on $\Gamma_{h(\cdot,t)}$, is a *regular solution* to (1.9) in $[0,T]$ with initial datum h_0.

Definition 3.6 (Variational solutions)**.** We say that a solution h to (1.9) (in the sense of Definition 3.5) is *variational* if there exist a time step $\tau_N \searrow 0$ and a subsequence $\{h_{N_k}\}$ of a discrete-time evolution $\{h_N\}$ of the minimum incremental problem (3.5) such that

$$h_{N_k} \xrightarrow[k\to\infty]{} h \text{ in } H^1(0,T,L^2(Q)), \qquad h_{N_k} \xrightarrow[k\to\infty]{} h \text{ in } C^{0,\beta}([0,T],C^{1,\alpha}_{\#}(Q)) \tag{3.11}$$

for every $\alpha\in(0,\frac{p-2}{p})$ and $\beta\in[0,\frac{(p+2)(p-2-\alpha p)}{8p^2})$.

Definition 3.7 (Stability of flat configurations)**.** We say that the flat configuration (d,u_d) is:

- *Lyapunov stable* if, for every $\sigma>0$, there exists $\delta(\sigma)>0$ such that, if $(h_0,u_0)\in X$ satisfies $\mathcal{L}^3(\Omega_{h_0}) = \mathcal{L}^3(\Omega_d)$ and $\|h_0-d\|_{W^{2,p}_{\#}(Q)} \le \delta(\sigma)$, then every variational solution h to (1.9) with initial datum h_0 (according to Definition 3.6) exists for all times and satisfies
$$\|h(\cdot,t)-d\|_{W^{2,p}_{\#}(Q)} \le \sigma$$
for every $t>0$;
- *Asymptotically stable* if there exists $\delta>0$ such that, if $(h_0,u_0)\in X$ satisfies $\mathcal{L}^3(\Omega_{h_0}) = \mathcal{L}^3(\Omega_d)$ and $\|h_0-d\|_{W^{2,p}_{\#}(Q)}\le\delta$, then every variational solution h to (1.9) with initial datum h_0 (according to Definition 3.6) exists for all times and satisfies
$$\|h(\cdot,t)-d\|_{W^{2,p}_{\#}(Q)} \to 0$$
as $t\to+\infty$.

3.4. **Main results.** The main results of the manuscript are twofolds: we prove the existence of a solution of (1.9) and stability properties of flat configurations.

Theorem 3.8 (Short time existence of a regular solution)**.** *Let $h_0\in AP$ with $h_0>0$ and such that h_0 satisfies (3.4). There exist $T_0>0$ and a regular solution h to (1.9) in $[0,T]$ with initial datum h_0 in the sense of Definition (3.5). Moreover, there exist a nonincreasing function g and a negligible set Z_0 such that*

$$\mathcal{F}(h(\cdot,t),u_h(\cdot,t)) = g(t) \quad \text{for every } t\in[0,T_0]\setminus Z_0 \tag{3.12}$$

and

$$\mathcal{F}(h(\cdot,t),u_h(\cdot,t)) \le g(t) \quad \text{for every } t\in Z_0. \tag{3.13}$$

We notice that the solution h and the time T_0 are given by Theorem 4.2 and Theorem 4.4, respectively.

In regard to stability, we prove that, under some convexity assumption on the density ψ, flat configurations are both Lyapunov and asymptotically stable. In particular, it is necessary that for every $\xi\in\mathbb{S}^2$ we have

$$\langle D^2\psi(\xi)w,w\rangle > 0 \quad \text{for every } w\perp\xi,\, w\neq 0, \tag{3.14}$$

and that the flat configuration (d,u_d) satisfies

$$\partial^2 G(d,u_d)[\varphi] > 0 \quad \text{for every } \varphi\in\tilde H^1_{\#}(Q)\setminus\{0\}, \tag{3.15}$$

where

$$G(h,u) = \int_{\Omega_h} W(Eu)\mathrm{d}z + \int_{\Gamma_h}\psi(\nu)\mathrm{d}\mathcal{H}^2 \quad\text{and}\quad \tilde H^1_{\#} := \left\{\varphi\in H^1_{\#}(Q) : \int_Q \varphi\mathrm{d}x = 0\right\}.$$

In the nonconvex case we can prove that if the boundary of the Wulff shape W_ψ, that is defined (depending on the nonconvex density ψ) by

$$W_\psi := \{z \in \mathbb{R}^3 : z \cdot \nu < \psi(\nu) \text{ for every } \nu \in S^2\},$$

contains a flat horizontal facet, then the flat configuration is always Lyapunov stable.

Theorem 3.9 (Stability of the flat configuration)**.** *Let $\psi : \mathbb{R}^3 \to [0, +\infty)$ be a positively one-homogeneous function satisfying (3.2). The following assertions hold true:*

(i) *if $\psi \in C^2(\mathbb{R}^3 \setminus \{0\})$ and there exist $\beta, \gamma > 0$ such that*

$$\{(x, y) \in \mathbb{R}^3 : |x| \leq \beta, y = \gamma\} \subset \partial W_\psi,$$

then, for every $d > 0$ the flat configuration (d, u_d) is Lyapunov stable;

(ii) *if $\psi \in C^3(\mathbb{R}^3 \setminus \{0\})$ and (3.14) and (3.15) are satisfied, then, (d, u_d) is Lyapunov and asymptotic stable.*

4. Incremental minimum problem: Existence and convergence

In this section, we construct by applying a minimizing movement scheme a candidate to be a solution to (1.9). In Theorem 4.1 we solve the incremental minimum problem. Then, in Theorem 4.2 and 4.3 we prove that, up to a subsequence, the discrete-time evolution $\{h_N\}$ converges to some function h as $N \to \infty$. Finally, in Theorem 4.4 we choose T_0 small enough to get the validity of (4.24) and that h_N is nonnegative.

The following result allows to verify that the incremental minimum problem is well defined, as, for each $i \in \mathbb{N}$, we can recursively find a solution to the minimum problem (3.5).

Theorem 4.1. *Let $(h_0, u_0) \in X$ be an initial configuration such that h_0 satisfies (3.4). Then, the minimum problem (3.5) admits a solution $(h_{i,N}, u_{i,N}) \in X$ for every $i \in \mathbb{N}$.*

Proof. We prove the theorem by induction. We fix $i \in \mathbb{N}$ such that $i > 1$ and we suppose to have a solution to (3.5) for $k = 1, ..., i-1$. We construct a solution to (3.5) for $k = i$. Firstly, we begin by noticing that, in view of the minimality of $(h_{k,N}, u_{k,N})$, by (3.6) and (3.7) we have that

$$\mathcal{F}(h_{k,N}, u_{k,N}) \leq G_{k,N}(h_{k,N}, u_{k,N}) \leq G_{k,N}(h_{k-1,N}, u_{k-1,N}) = \mathcal{F}(h_{k-1,N}, u_{k-1,N}), \tag{4.1}$$

from which we deduce that

$$0 \leq \inf\{G_{i,N}(h, u) : (h, u \in X)\} \leq G_{i,N}(h_{i-1,N}, u_{i-1,N}) = \mathcal{F}(h_{i-1,N}, u_{i-1,N}) \leq ... \leq \mathcal{F}(h_0, u_0). \tag{4.2}$$

As a consequence of (4.2), we can select a minimizing sequence $\{(h_n, u_n)\} \in X$ such that

$$\|Dh_n\|_{L^\infty(Q)} \leq \Lambda_0 \quad \text{and} \quad \sup\{G_{i,N}(h_n, u_n) : n \in \mathbb{N}\} < \infty.$$

By denoting with H_n the sum of principal curvatures of Γ_{h_n}, we obtain that

$$\sup\{\|H_n\|^p_{L^p(Q)} : n \in \mathbb{N}\} \leq \frac{p}{\varepsilon} \sup\{G_{i,N}(h_n, u_n) : n \in \mathbb{N}\} < \infty,$$

and so $\{H_n\}$ is bounded in $L^p(Q)$. Therefore, by Lemma 7.3 the sequence $\{h_n\}$ is bounded in $W^{2,p}_\#(Q)$. Then, up to a subsequence, $h_n \rightharpoonup h$ in $W^{2,p}_\#(Q)$ as $n \to \infty$, from which we deduce that $h_n \to h$ in $C^{1,\alpha}_\#(Q)$ for some $\alpha \in (0, 1)$ as $n \to \infty$. It follows that

$$H_n \xrightharpoonup[n\to\infty]{L^p(Q)} H = -\operatorname{div}\left(\frac{Dh}{\sqrt{1 + |Dh|^2}}\right).$$

Furthermore, by lower semicontinuity

$$\int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\right) \mathrm{d}\mathcal{H}^2 \leq \liminf_{n\to\infty} \int_{\Gamma_{h_n}} \left(\psi(\nu_n) + \frac{\varepsilon}{p}|H_n|^p\right) \mathrm{d}\mathcal{H}^2, \tag{4.3}$$

and by Fatou's Lemma we conclude that

$$P_{i,N}(h) \leq \liminf_{n\to\infty} P_{i,N}(h_n). \tag{4.4}$$

Now we extend the function u_n to $A := Q \times (-\infty, 0)$ in the following way

$$u_n(x,y) := (e_0^1 x_1, e_0^2 x_2, 0)\varphi(y)$$

for every $(x,y) \in A$, where φ is a given cut-off function such that $\varphi(y) := 1$ in $(-1, +\infty)$ and $\varphi(y) := 0$ in $(-\infty, -2)$. We consider also the set

$$A_h := \{(x,y) \in \mathbb{R}^3 : y < h(x)\}.$$

Since

$$\sup\left\{\int_{\Omega_{h_n}} |Eu_n|^2 \mathrm{d}z : n \in \mathbb{N}\right\} < \infty,$$

by reasoning as in [12, Proposition 2.2], we use Korn's inequality to get the existence of a subsequence of $\{u_n\}_n$, and of a function $u \in H^1_{loc}(A_h; \mathbb{R}^3)$ with $Eu \in L^2(A_h; \mathbb{R}^{3\times 3}_{sym})$, such that $u_n \rightharpoonup u$ in $H^1(D;\mathbb{R}^3)$ as $n \to \infty$, for every D compactly contained in A_h. Then, $(h,u) \in X$ and we also have

$$\int_{\Omega_h} W(Eu)\mathrm{d}z \le \liminf_{n\to\infty} \int_{\Omega_{h_n}} W(Eu_n)\mathrm{d}z. \tag{4.5}$$

Thanks to (4.3), (4.4), and (4.5) we can conclude that (h,u) is a minimizer of (3.5) for $k = i$. □

We now prove that the discrete-time evolution $\{h_N\}$ is uniformly bounded in $L^\infty(0,T; W^{2,p}_\#(Q)) \cap H^1(0,T;L^2(Q))$.

Theorem 4.2. *Let $(h_0, u_0) \in X$ be an initial configuration such that h_0 satisfies (3.4). For every $N, i \in \mathbb{N}$ we have*

$$\int_0^{+\infty}\int_Q \left|\frac{\partial h_N}{\partial t}\right|^2 \mathrm{d}x\mathrm{d}t \le CF(h_0,u_0), \tag{4.6}$$

$$\mathcal{F}(h_{i,N}, u_{i,N}) \le \mathcal{F}(h_{i-1,N}, u_{i-1,N}) \le \mathcal{F}(h_0,u_0), \tag{4.7}$$

$$\sup_{t\in[0,+\infty)} \|h_N(\cdot,t)\|_{W^{2,p}_\#(Q)} < +\infty, \tag{4.8}$$

for some constant $C = C(\Lambda_0) > 0$. Moreover, up to a subsequence, for every $T > 0$ we have the following convergences

$$h_N \xrightarrow[N\to\infty]{} h \text{ in } H^1(0,T,L^2(Q)), \tag{4.9}$$

$$h_N \xrightarrow[N\to\infty]{} h \text{ in } C^{0,\alpha}([0,T],L^2(Q)) \text{ for every } \alpha \in (0,\tfrac{1}{2}), \tag{4.10}$$

where the function h satisfies

$$h(\cdot,t) \in W^{2,p}_\#(Q) \text{ and } \mathcal{F}(h(\cdot,t), u_{h(\cdot,t)}) \le \mathcal{F}(h_0,u_0) \text{ for every } t \in [0,+\infty). \tag{4.11}$$

Proof. We begin by observing that (4.7) is a direct consequence of (4.1).

To prove (4.6) we notice that for each $i \in \mathbb{N}$, by (4.1) we have that

$$G_{i,N}(h_{i,N}, u_{i,N}) \le G_{i,N}(h_{i-1,N}, u_{i-1,N}) = \mathcal{F}(h_{i-1,N}, u_{i-1,N}),$$

which implies

$$P_{i,N}(h_{i,N}) \le \mathcal{F}(h_{i-1,N}, u_{i-1,N}) - \mathcal{F}(h_{i,N}, u_{i,N}). \tag{4.12}$$

Since $\|Dh_{i-1,N}\|_{L^\infty(Q)} < \Lambda_0$, by summing in (4.12) over i we get

$$\sum_{i=0}^{+\infty} \tau_N \int_Q \left(\frac{h_{i,N} - h_{i-1,N}}{\tau_N}\right)^2 \mathrm{d}x \le C(\Lambda_0)\sum_{i=0}^{+\infty}[\mathcal{F}(h_{i-1,N}, u_{i-1,N}) - \mathcal{F}(h_{i,N}, u_{i,N})] = C(\Lambda_0)\mathcal{F}(h_0,u_0), \tag{4.13}$$

with $C(\Lambda_0) := 2\sqrt{1+\Lambda_0^2}$. Furthermore, by (3.8) we have

$$\frac{\partial h_N}{\partial t} = \frac{h_{i,N} - h_{i-1,N}}{\tau_N},$$

and hence, by also (4.13) we obtain (4.6).

To prove (4.8), we notice that (4.7) implies

$$\sup\left\{\int_{\Gamma_{h_{i,N}}}|H_{i,N}|^p\mathrm{d}\mathcal{H}^2 : i,N\in\mathbb{N}\right\} < \frac{p}{\varepsilon}\mathcal{F}(h_0,u_0), \tag{4.14}$$

from which, by taking into account that $\|Dh_{i,N}\|_{L^\infty(Q)} < \Lambda_0$ and by Lemma 7.3, (4.8) follows.

Notice that (4.9) is a direct consequence of (4.6) and (4.8). It remains to prove (4.10) and (4.11). Since $h_N(x,\cdot)$ is absolutely continuous on $[0,T]$ for every $T>0$, then by Holder inequality, Fubini's Theorem, and (4.6), for every $t_1,t_2\in[0,T]$ we obtain

$$\begin{aligned}\|h_N(\cdot,t_1)-h_N(\cdot,t_2)\|_{L^2(Q)} &\le \left(\int_Q\left(\int_{t_1}^{t_2}\frac{\partial h_N}{\partial t}(x,t)\mathrm{d}t\right)^2\mathrm{d}x\right)^{\frac12}\\ &\le \left(\int_{t_1}^{t_2}\left\|\frac{\partial h_N}{\partial t}(\cdot,t)\right\|^2_{L^2(Q)}\mathrm{d}t\right)^{\frac12}(t_2-t_1)^{\frac12}\le\sqrt{CF(h_0,u_0)}(t_2-t_1)^{\frac12}.\end{aligned} \tag{4.15}$$

By using (4.15) and Ascoli-Arzelà Theorem (see e.g. [1, Proposition 3.3.1]) we get (4.10).

Finally, by (4.7), (4.9), (4.10), and the lower semicontinuity we obtain (4.11). □

In the following result the convergence of h_N to h is significantly improved, and we prove also that $\tilde{h}_N$ converges to h.

Theorem 4.3. *Let $(h_0,u_0)\in X$ be an initial configuration such that h_0 satisfies (3.4). For every $T>0$ we have the following convergences*

$$h_N \xrightarrow[N\to\infty]{} h \text{ in } C^{0,\beta}([0,T],C^{1,\alpha}_\#(Q)), \tag{4.16}$$

$$\tilde{h}_N \xrightarrow[N\to\infty]{} h \text{ in } L^\infty(0,T,C^{1,\alpha}_\#(Q)), \tag{4.17}$$

for every $\alpha\in(0,\frac{p-2}{p})$ and $\beta\in[0,\frac{(p-2-\alpha p)(p+2)}{8p^2})$. Moreover, $h(\cdot,t)\to h_0$ in $C^{1,\alpha}_\#(Q)$ as $t\to 0^+$.

Proof. Let $t_1,t_2\in[0,T]$ be such that $t_1<t_2$, then by (4.8), (4.15), and the interpolation inequality of Lemma 7.4 we have

$$\begin{aligned}&\|Dh_N(\cdot,t_2)-Dh_N(\cdot,t_1)\|_{L^\infty(Q)}\\ &\quad\le C\|D^2h_N(\cdot,t_2)-D^2h_N(\cdot,t_1)\|^{\frac{p+2}{2p}}_{L^p(Q)}\|h_N(\cdot,t_2)-h_N(\cdot,t_1)\|^{\frac{p-2}{2p}}_{L^p(Q)}\\ &\quad\le C\|h_N(\cdot,t_2)-h_N(\cdot,t_1)\|^{\frac{p-2}{2p}}_{L^p(Q)}\\ &\quad\le C\left(\|D^2h_N(\cdot,t_2)-D^2h_N(\cdot,t_1)\|^{\frac{p-2}{2p}}_{L^2(Q)}\|h_N(\cdot,t_2)-h_N(\cdot,t_1)\|^{\frac{p+2}{2p}}_{L^2(Q)}\right)^{\frac{p-2}{2p}}\\ &\quad\le C(t_2-t_1)^{\frac{p^2-4}{8p^2}}.\end{aligned} \tag{4.18}$$

Moreover, thanks to Mean Value Theorem there exists $x_0\in Q$ such that

$$h_N(x_0,t_2)-h_N(x_0,t_1)=\frac{1}{\ell^2}\int_Q(h_N(x,t_2)-h_N(x,t_1))\mathrm{d}x,$$

from which we deduce that for every $x\in Q$ we have

$$|h_N(x,t_2)-h_N(x,t_1)|\le\ell^2\|Dh_N(\cdot,t_2)-Dh_N(\cdot,t_1)\|_{L^\infty(Q)}+\frac{1}{\ell}\|h_N(\cdot,t_2)-h_N(\cdot,t_1)\|_{L^2(Q)}. \tag{4.19}$$

Now, by (4.15), (4.18), and (4.19) we get

$$\|h_N(\cdot,t_2)-h_N(\cdot,t_1)\|_{L^\infty(Q)}\le C\left[(t_2-t_1)^{\frac{p^2-4}{8p^2}}+(t_2-t_1)^{\frac12}\right]. \tag{4.20}$$

Furthermore, notice that for every $\alpha \in (0, \frac{p-2}{p})$ we have

$$[Dh_N(\cdot,t_2) - Dh_N(\cdot,t_1)]_\alpha \leq [Dh_N(\cdot,t_2) - Dh_N(\cdot,t_1)]_{\frac{p-2}{p}}^{\frac{\alpha p}{p-2}} \left(2\|Dh_N(\cdot,t_2) - Dh_N(\cdot,t_1)\|_{L^\infty(Q)}\right)^{\frac{p-2-\alpha p}{p-2}} \tag{4.21}$$

where $[\cdot]_\alpha$ denotes the α-Holder seminorm, and by Sobolev embedding and (4.8) we obtain

$$\sup\left\{\sup_{t\in[0,T]} \|h_N(\cdot,t)\|_{C^{1,\frac{p-2}{p}}_{\#}(Q)} \;:\; N \in \mathbb{N}\right\} < +\infty. \tag{4.22}$$

By (4.18), (4.21), and (4.22) we can write

$$[Dh_N(\cdot,t_2) - Dh_N(\cdot,t_1)]_\alpha \leq C(t_2 - t_1)^{\frac{(p+2)(p-2-\alpha p)}{8p^2}}. \tag{4.23}$$

Therefore, it follows from (4.18), (4.20), and (4.23), that for every $\alpha \in (0, \frac{p-2}{p})$, h_N is uniformly equicontinuous with respect to the $C^{1,\alpha}(Q)$-norm topology and that

$$\|h_N(\cdot,t_2) - h_N(\cdot,t_1)\|_{C^{1,\alpha}(Q)} \leq C(t_2 - t_1)^{\frac{(p+2)(p-2-\alpha p)}{8p^2}}.$$

In particular, by applying Ascoli-Arzelà Theorem we get both (4.16) and (4.17). Finally, by noticing that $\|h_N(\cdot,t) - h_N(\cdot,t_1)\|_{C^{1,\alpha}(Q)} \to 0$ as $t \to t_1$, we choose $t_1 = 0$ to conclude the proof. □

From now on, we assume that the initial profile h_0 is strictly positive. Thanks to this, we can use standard elliptic estimates to establish the convergence of u_N and $\tilde{u}_N$.

Theorem 4.4. *Let $(h_0, u_0) \in X$ be an initial configuration such that $h_0 > 0$ satisfies (3.4). Then there exist $T_0 > 0$ and $C > 0$ depending only on (h_0, u_0) such that:*

(i) *$h_N, h \geq C_0 > 0$ for some positive constant C_0, and*

$$\sup_{t\in[0,T_0]} \|Dh_N(\cdot,t)\|_{L^\infty(Q)} < \Lambda_0 \tag{4.24}$$

for every $N \in \mathbb{N}$;

(ii) *it holds*

$$\|Du_{i,N}\|_{C^{0,\frac{p-2}{p}}(\bar{\Omega}_{h_{i,N}};\mathbb{R}^{3\times3}_{sym})} \leq C; \tag{4.25}$$

(iii) *for every $\alpha \in (0, \frac{p-2}{p})$ and $\beta \in [0, \frac{(p+2)(p-2-\alpha p)}{8p^2})$ we have the following convergences:*

$$E(u_N(\cdot,h_N)) \xrightarrow[N\to\infty]{} E(u(\cdot,h)) \text{ in } C^{0,\beta}([0,T_0]; C^{1,\alpha}_{\#}(Q)), \tag{4.26}$$

$$E(\tilde{u}_N(\cdot,\tilde{h}_N)) \xrightarrow[N\to\infty]{} E(u(\cdot,h)) \text{ in } L^\infty(0,T_0; C^{1,\alpha}_{\#}(Q)), \tag{4.27}$$

where $u(\cdot,t)$ is the elastic equilibrium in $\Omega_{h(\cdot,t)}$.

Proof. We begin by observing that by inserting $t_2 = t$ and $t_1 = 0$ in (4.20) we can write

$$h_N(x,t) \geq \min_{x\in Q} h_0(x) - C\left[t^{\frac{p^2-4}{8p^2}} + t^{\frac{1}{2}}\right] \qquad \text{for every } (x,t) \in Q \times [0,T].$$

Therefore, since $t \mapsto t^{\frac{p^2-4}{8p^2}} + t^{\frac{1}{2}}$ is an increasing function, there exists $T_1 > 0$ such that $h_N(x,t) \geq C_0$ for every $(x,t) \in Q \times [0,T_1]$ and for some constant $0 < C_0 < \min\{h_0(x) : x \in Q\}$. As a consequence $h \geq C_0$. Now, we notice that in view of (4.18) with $t_2 = t$ and $t_1 = 0$ we get

$$\|Dh_N(\cdot,t)\|_{L^\infty(Q)} \leq \|Dh_N(\cdot,t) - Dh_0(\cdot)\|_{L^\infty(Q)} + \|Dh_0\|_{L^\infty(Q)} \leq Ct^{\frac{p^2-4}{8p^2}} + \|h_0\|_{C^1_{\#}(Q)},$$

and hence, by (3.4) we can choose $T_2 > 0$ such that

$$T_2 < \left(\frac{\Lambda_0 - \|h_0\|_{C^1_{\#}(Q)}}{C}\right)^{\frac{8p^2}{p^2-4}},$$

from which we deduce that

$$\sup_{t\in[0,T_2]}\|Dh_N(\cdot,t)\|_{L^\infty(Q)}<\Lambda_0.$$

By choosing $T_0:=\min\{T_1,T_2\}$ we conclude the proof of (i).

To prove (4.25), we use standard elliptic estimates to bound the norm of $Du_{i,N}$ in $C^{0,\alpha}(\bar{\Omega}_{h_{i,N}})$ with a constant depending only on the $C^{1,\alpha}$-norm of $h_{i,N}$ (see [18]). Then, (4.22) implies (4.25).

Finally, by (4.16), (4.17), and Lemma 7.1, (4.26) and (4.27) follows. □

5. Existence of evolution

In this section we prove that the candidate found in Section 4 is a solution to (1.9) in the sense of the Definition 3.5 in the case of short time intervals (see Theorem 3.8).

In the next lemma, thanks to the regularity given by Theorem 4.4, we derive the Euler-Lagrange equation satisfied by the minimizer $(h_{i,N},u_{i,N})$ to the problem (3.5).

Lemma 5.1. *Every minimizer $(h_{i,N},u_{i,N})$ of (3.5) satisfies the following Euler-Lagrange equation*

$$\begin{aligned}&\int_Q W(E(u_{i,N}(x,h_{i,N}(x))))\varphi\mathrm{d}x+\int_Q\langle D\psi(-Dh_{i,N},1),(-D\varphi,0)\rangle\mathrm{d}x+\frac{\varepsilon}{p}\int_Q|H_{i,N}|^p\frac{\langle Dh_{i,N},D\varphi\rangle}{J_{i,N}}\mathrm{d}x\\&-\varepsilon\int_Q|H_{i,N}|^{p-2}H_{i,N}\left[\Delta\varphi-\frac{\langle D^2\varphi Dh_{i,N},Dh_{i,N}\rangle}{J_{i,N}^2}-\frac{\Delta h_{i,N}\langle Dh_{i,N},D\varphi\rangle}{J_{i,N}^2}-2\frac{\langle D^2h_{i,N}Dh_{i,N},D\varphi\rangle}{J_{i,N}^2}\right]\mathrm{d}x\\&-3\varepsilon\int_Q|H_{i,N}|^{p-2}H_{i,N}\frac{\langle Dh_{i,N},D\varphi\rangle\langle D^2h_{i,N}Dh_{i,N},Dh_{i,N}\rangle}{J_{i,N}^4}\mathrm{d}x+\frac{1}{\tau_N}\int_Q\frac{h_{i,N}-h_{i-1,N}}{J_{i-1,N}}\varphi\mathrm{d}x=0,\end{aligned}\tag{5.1}$$

for every $\varphi\in C^2_\#(Q)$.

Proof. Since $(h_{i,N},u_{i,N})$ is the minimizer of (3.5), it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{F}(h_{i,N}(s),u_{i,N})+\frac{1}{2\tau_N}\int_Q\frac{(h_{i,N}(s)-h_{i-1,N})^2}{J_{i-1,N}}\mathrm{d}x\right)_{|s=0}=0\tag{5.2}$$

for every $\varphi\in C^2_\#(Q)$, where $h_{i,N}(s):=h_{i,N}+s\varphi$.

By considering the penalization, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2\tau_N}\int_Q\frac{(h_{i,N}(s)-h_{i-1,N})^2}{J_{i-1,N}}\mathrm{d}x\right)_{|s=0}=\frac{1}{\tau_N}\int_Q\frac{(h_{i,N}-h_{i-1,N})\varphi}{J_{i-1,N}}\mathrm{d}x.\tag{5.3}$$

For the elastic energy, we use the definition (3.1) and we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\Omega_{h_{i,N}+s\varphi}}W(Eu_{i,N}(x,y))\mathrm{d}x\mathrm{d}y\right)_{|s=0}&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_Q\int_0^{h_{i,N}(s)}W(Eu_{i,N}(x,y))\mathrm{d}y\mathrm{d}x\right)_{|s=0}\\&=\int_Q[W(Eu_{i,N}(x,h_{i,N}(x,s))\varphi(x)]_{|s=0}\mathrm{d}x=\int_Q W(Eu_{i,N}(x,h_{i,N}(x))\varphi(x)\mathrm{d}x.\end{aligned}\tag{5.4}$$

It remains to consider the surface energy. We begin by observing that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_Q\psi(-Dh_{i,N}(s),1)\mathrm{d}x\mathrm{d}y\right)_{|s=0}\\&=\int_Q[\langle D\psi(-Dh_{i,N}(s),1),(-D\varphi,0)\rangle]_{|s=0}\mathrm{d}x=\int_Q\langle D\psi(-Dh_{i,N},1),(-D\varphi,0)\rangle\mathrm{d}x,\end{aligned}\tag{5.5}$$

and that the regularization term can be treated in the following way

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\Gamma_{h_{i,N}(s)}}|H_{h_{i,N}(s)}|^p\mathrm{d}\mathcal{H}^2\right)_{|s=0}=\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_Q f_{i,N}(s)g_{i,N}(s)\mathrm{d}x\mathrm{d}y\right)_{|s=0}$$

$$= \int_Q \left[\frac{\mathrm{d}f_{i,N}}{\mathrm{d}s}(0)g_{i,N}(0) + f_{i,N}(0)\frac{\mathrm{d}g_{i,N}}{\mathrm{d}s}(0)\right]\mathrm{d}x\mathrm{d}y, \tag{5.6}$$

where

$$g_{i,N}(s) := \sqrt{1+|Dh_{i,N}(s)|^2} \quad \text{and} \quad f_{i,N}(s) := \left|\mathrm{div}\left(\frac{Dh_{i,N}(s)}{g_{i,N}(s)}\right)\right|^p$$

Notice that

$$\frac{\mathrm{d}g_{i,N}}{\mathrm{d}s}(s) = \frac{1}{g_{i,N}(s)}\langle Dh_{i,N}(s), D\varphi\rangle \Longrightarrow \frac{\mathrm{d}g_{i,N}}{\mathrm{d}s}(0) = \frac{\langle Dh_{i,N}, D\varphi\rangle}{J_{i,N}}, \tag{5.7}$$

$$\frac{\mathrm{d}f_{i,N}}{\mathrm{d}s}(s) = p|H_{h_{i,N}(s)}|^{p-2}H_{h_{i,N}(s)}\frac{\mathrm{d}}{\mathrm{d}s}H_{h_{i,N}(s)} \Longrightarrow \frac{\mathrm{d}f_{i,N}}{\mathrm{d}s}(0) = p|H_{i,N}|^{p-2}H_{i,N}\frac{\mathrm{d}}{\mathrm{d}s}(H_{h_{i,N}(s)})_{|s=0}. \tag{5.8}$$

By definition we have

$$\begin{aligned}
H_{h_{i,N}(s)} &= -\,\mathrm{div}\left(\frac{Dh_{i,N}(s)}{g_{i,N}(s)}\right) = -\sum_{k=1}^{2}\frac{\partial}{\partial x_k}\left(\frac{\frac{\partial h_{i,N}(s)}{\partial x_k}}{g_{i,N}(s)}\right)\\
&= -\sum_{k=1}^{2}\left[\frac{1}{g_{i,N}(s)}\frac{\partial^2 h_{i,N}(s)}{\partial x_k^2} + \frac{\partial h_{i,N}(s)}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{1}{g_{i,N}(s)}\right)\right]\\
&= -\sum_{k=1}^{2}\left[\frac{1}{g_{i,N}(s)}\frac{\partial^2 h_{i,N}(s)}{\partial x_k^2} - \frac{1}{2g_{i,N}^3(s)}\frac{\partial h_{i,N}(s)}{\partial x_k}\sum_{m=1}^{2}\frac{\partial}{\partial x_k}\left(\frac{\partial h_{i,N}(s)}{\partial x_m}\right)^2\right]\\
&= -\frac{\Delta h_{i,N}(s)}{g_{i,N}(s)} + \frac{\langle D^2h_{i,N}(s)Dh_{i,N}(s), Dh_{i,N}(s)\rangle}{g_{i,N}^3(s)} := A(s) + B(s),
\end{aligned}$$

and hence, we can write

$$\frac{\mathrm{d}}{\mathrm{d}s}(H_{h_{i,N}(s)})_{|s=0} = \frac{\mathrm{d}A}{\mathrm{d}s}(0) + \frac{\mathrm{d}B}{\mathrm{d}s}(0). \tag{5.9}$$

Now, we compute the derivative of A and B with respect to s. Since

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = \frac{-\Delta\varphi g_{i,N}(s) + \frac{\Delta h_{i,N}(s)}{g_{i,N}(s)}\langle Dh_{i,N}(s), D\varphi\rangle}{g_{i,N}^2(s)} = \frac{-\Delta\varphi g_{i,N}^2(s) + \Delta h_{i,N}(s)\langle Dh_{i,N}(s), D\varphi\rangle}{g_{i,N}^3(s)},$$

we get

$$\frac{\mathrm{d}A}{\mathrm{d}s}(0) = -\frac{\Delta\varphi}{J_{i,N}} + \frac{\Delta h_{i,N}\langle Dh_{i,N}, D\varphi\rangle}{J_{i,N}^3}. \tag{5.10}$$

Moreover, by setting

$$C(s) := \sum_{k=1}^{2}\frac{\partial h_{i,N}(s)}{\partial x_k}\sum_{m=1}^{2}\frac{\partial^2 h_{i,N}(s)}{\partial x_k\partial x_m}\frac{\partial h_{i,N}(s)}{\partial x_m},$$

in view of (5.7) we have

$$\begin{aligned}
\frac{\mathrm{d}B}{\mathrm{d}s}(0) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{C(s)}{g_{i,N}^3(s)}\right)_{|s=0} = \frac{\frac{\mathrm{d}C}{\mathrm{d}s}(0)g_{i,N}^3(0) - C(0)\frac{\mathrm{d}}{\mathrm{d}s}(g_{i,N}^3(s))_{|s=0}}{g_{i,N}^6(0)}\\
&= \frac{1}{J_{i,N}^3}\frac{\mathrm{d}C}{\mathrm{d}s}(0) - \frac{3\langle D^2h_{i,N}Dh_{i,N}, Dh_{i,N}\rangle\langle Dh_{i,N}(s), D\varphi\rangle}{J_{i,N}^5}.
\end{aligned} \tag{5.11}$$

It remains to study the last term C:

$$\frac{\mathrm{d}C(s)}{\mathrm{d}s} = \sum_{k=1}^{2}\frac{\partial\varphi}{\partial x_k}\sum_{m=1}^{2}\frac{\partial^2 h_{i,N}(s)}{\partial x_k\partial x_m}\frac{\partial h_{i,N}(s)}{\partial x_m} + \sum_{k=1}^{2}\frac{\partial h_{i,N}(s)}{\partial x_k}\sum_{m=1}^{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial^2 h_{i,N}(s)}{\partial x_k\partial x_m}\frac{\partial h_{i,N}(s)}{\partial x_m}\right)$$

$$
\begin{aligned}
&= \langle D^2 h_{i,N}(s) Dh_{i,N}(s), D\varphi\rangle + \sum_{k=1}^{2} \frac{\partial h_{i,N}(s)}{\partial x_k} \sum_{m=1}^{2} \left(\frac{\partial^2 \varphi}{\partial x_k \partial x_m} \frac{\partial h_{i,N}(s)}{\partial x_m} + \frac{\partial^2 h_{i,N}(s)}{\partial x_k \partial x_m} \frac{\partial \varphi}{\partial x_m} \right) \\
&= \langle D^2 h_{i,N}(s) Dh_{i,N}(s), D\varphi\rangle + \langle D^2\varphi Dh_{i,N}(s), Dh_{i,N}(s)\rangle + \langle D^2 h_{i,N}(s) D\varphi, Dh_{i,N}(s)\rangle \\
&= 2\langle D^2 h_{i,N}(s) Dh_{i,N}(s), D\varphi\rangle + \langle D^2\varphi Dh_{i,N}(s), Dh_{i,N}(s)\rangle,
\end{aligned}
$$

and hence

$$
\frac{\mathrm{d}C}{\mathrm{d}s}(0) = 2\langle D^2 h_{i,N} Dh_{i,N}, D\varphi\rangle + \langle D^2\varphi Dh_{i,N}, Dh_{i,N}\rangle. \tag{5.12}
$$

By (5.11) and (5.12) we obtain

$$
\frac{\mathrm{d}B}{\mathrm{d}s}(0) = \frac{2\langle D^2 h_{i,N} Dh_{i,N}, D\varphi\rangle + \langle D^2\varphi Dh_{i,N}, Dh_{i,N}\rangle}{J_{i,N}^3} - \frac{3\langle D^2 h_{i,N} Dh_{i,N}, Dh_{i,N}\rangle\langle Dh_{i,N}(s), D\varphi\rangle}{J_{i,N}^5}. \tag{5.13}
$$

Finally, from (5.2)-(5.10) and (5.13) the assertion follows. □

Now let $\widetilde{H}_N : \mathbb{R}^2 \times [0, T_0] \to \mathbb{R}$ be the function defined by

$$
\widetilde{H}_N(x,t) := H_{i,N}(x, h_{i,N}(x)) \qquad \text{for } t \in [(i-1)\tau_N, i\tau_N), \tag{5.14}
$$

where $H_{i,N}$ is the sum of the principal curvatures of $\Gamma_{i,N}$. Moreover, we will denote by $|B_{i,N}|^2$ the sum of the squares of the principal curvatures of $\Gamma_{i,N}$, and by $J_{i,N}$ the following quantity

$$
J_{i,N} := \sqrt{1 + |Dh_{i,N}|^2},
$$

Theorem 5.2. *Let T_0 be as in Theorem 4.4 and let $\widetilde{H}_N$ be given by (5.14). Then, there exists a constant $C > 0$ such that*

$$
\int_0^{T_0} \int_Q |D^2(|\widetilde{H}_N|^{p-2}\widetilde{H}_N)|^2 \mathrm{d}x\mathrm{d}t \le C \tag{5.15}
$$

for every $N \in \mathbb{N}$.

Proof. We divide the proof into two steps.

Step 1 In this step we prove that for every $k \ge 1$ and $\sigma \in (0, \frac{1}{p-1})$ we have

$$
|H_{i,N}|^{p-2} H_{i,N} \in W^{1,q}_{\#}(\Gamma_{i,N}) \qquad \text{and} \qquad h_{i,N} \in C^{2,\sigma}_{\#}(Q).
$$

In order to do this, we begin by showing that $h_{i,N} \in W^{2,q}_{\#}(Q)$ for every $q \ge 2$. Since $h_{i,N}$ is the solution to (3.5), thanks to Lemma 5.1 it satisfies (5.1). Now, by setting

$$
\begin{aligned}
& w := |H_{i,N}|^{p-2} H_{i,N}, \quad A := \varepsilon \left(I - \frac{Dh_{i,N} \otimes Dh_{i,N}}{J_{i,N}^2} \right), \qquad (5.16) \\
& b := \pi(D(\psi(-Dh_{i,N}, 1))) - \frac{\varepsilon}{p} |H_{i,N}|^p Dh_{i,N} \\
& \qquad + \varepsilon w \left[-\frac{\Delta h_{i,N} Dh_{i,N}}{J_{i,N}^2} - 2\frac{D^2 h_{i,N} Dh_{i,N}}{J_{i,N}^2} + 3\frac{\langle D^2 h_{i,N} Dh_{i,N}, Dh_{i,N}\rangle Dh_{i,N}}{J_{i,N}^4} \right], \\
& c := -W(E(u_{i,N}(x, h_{i,N}(x)))) - \frac{h_{i,N} - h_{i-1,N}}{\tau_N J_{i-1,N}},
\end{aligned}
$$

we can rephrase (5.1) as follows

$$
\int_Q wA : D^2\varphi \mathrm{d}x + \int_Q \langle b, D\varphi\rangle \mathrm{d}x + \int_Q c\varphi \mathrm{d}x = 0 \qquad \text{for every } \varphi \in C^\infty_{\#}(Q). \tag{5.17}
$$

By (4.8) and Theorem 4.4 we have that $A \in W^{1,p}_{\#}(Q; \mathbb{R}^{2\times 2}_{sym})$, $b \in L^1(Q; \mathbb{R}^2)$ and $c \in C^{0,\alpha}_{\#}(Q)$ for some α, and hence, since (5.17) is in particular satisfied for every $\varphi \in C^\infty_{\#}(Q)$ with $\int_Q \varphi \mathrm{d}x = 0$, we can apply Lemma 7.2 to get that $w \in L^q(Q)$ for $q \in (\frac{p}{p-1}, 2)$. By following the same argument in [14, Theorem 3.11] we obtain $|H_{i,N}|^{p-2} H_{i,N} \in W^{1,q}_{\#}(Q)$ for every $q \ge 1$, then $|H_{i,N}|^{p-1} \in C^{0,\alpha}_{\#}(Q)$ for every $\alpha \in (0,1)$. As a consequence, $H_{i,N} \in C^{0,\sigma}_{\#}(Q)$ for every $\sigma \in (0, \frac{1}{p-1})$, and hence, by Schauder estimates we deduce that $h_{i,N} \in C^{2,\sigma}_{\#}(Q)$ for every $\sigma \in (0, \frac{1}{p-1})$.

Step 2 In view of Lemma 7.3 and the fact that $\|Dh_{i,N}\|_{L^\infty(Q)} < \Lambda_0$, there exists a positive constant $C = C(q, \Lambda_0)$ such that

$$\|D^2 h_{i,N}\|_{L^q}(Q) \leq C \|H_{i,N}\|_{L^q(Q)}. \tag{5.18}$$

Furthermore, since $\Gamma_{i,N}$ is of class $C^{2,\sigma}$ then $|H_{i,N}|^{p-2}H_{i,N} \in H^2(\Gamma_{i,N})$, and this implies that $|H_{i,N}|^{p-2}H_{i,N} \in H^2(Q)$.

It is easy to verify (see [14, Theorem 3.11]) that $h_{i,N}$ satisfies the following Euler-Lagrange equation in intrinsic form

$$\begin{aligned} -\varepsilon \int_{\Gamma_{i,N}} D_{\Gamma_{i,N}}(|H_{i,N}|^{p-2}H_{i,N}) D_{\Gamma_{i,N}} \phi \, \mathrm{d}\mathcal{H}^2 + \varepsilon \int_{\Gamma_{i,N}} |H_{i,N}|^{p-2}H_{i,N} \left(|B_{i,N}|^2 - \frac{1}{p} H^2_{i,N}\right) \phi \, \mathrm{d}\mathcal{H}^2 \\ - \int_{\Gamma_{i,N}} \left[\mathrm{div}_{\Gamma_{i,N}}(D\psi(\nu_{i,N})) + W(Eu_{i,N})\right] \phi \, \mathrm{d}\mathcal{H}^2 = \int_{\Gamma_{i,N}} \frac{h_{i,N} - h_{i-1,N}}{J_{i-1,N}\tau_N} \phi \, \mathrm{d}\mathcal{H}^2, \end{aligned} \tag{5.19}$$

where $\phi := \frac{\varphi}{J_{i,N}} \circ \pi$ with $\varphi \in C^2_{\#}(Q)$ such that $\int_Q \varphi \mathrm{d}x = 0$. Now, by setting $w := |H_{i,N}|^{p-2}H_{i,N}$ we can rephrase (5.19) in the following way

$$\begin{aligned} -\int_Q \langle ADw, D\left(\frac{\varphi}{J_{i,N}}\right)\rangle J_{i,N} \mathrm{d}x + \varepsilon \int_Q w\varphi \left(|B_{i,N}|^2 - \frac{1}{p} H^2_{i,N}\right) \mathrm{d}x \\ - \int_Q \left[\mathrm{div}_{\Gamma_{i,N}}(D\psi(\nu_{i,N})) + W(Eu_{i,N})\right] \varphi \mathrm{d}x = \int_Q \frac{h_{i,N} - h_{i-1,N}}{J_{i-1,N}\tau_N} \varphi \mathrm{d}x, \end{aligned} \tag{5.20}$$

for every $\varphi \in H^1_{\#}(Q)$ with $\int_Q \varphi \mathrm{d}x = 0$, where A is defined in (5.16). We now use $\varphi = D_k \eta$ with $\eta \in H^2_{\#}(Q)$, and since

$$\frac{D_k \eta}{J_{i,N}} = D_k\left(\frac{\eta}{J_{i,N}}\right) + \frac{\eta D_k J_{i,N}}{J^2_{i,N}},$$

we can integrate by parts to get

$$\begin{aligned} &\int_Q \langle ADw, D\left(\frac{D_k\eta}{J_{i,N}}\right)\rangle J_{i,N} \mathrm{d}x \\ &\quad= \sum_{s,l} \int_Q a_{sl} D_l w D_s D_k \left(\frac{\eta}{J_{i,N}}\right) J_{i,N} \mathrm{d}x + \sum_{s,l} \int_Q a_{sl} D_l w D_s \left(\frac{\eta D_k J_{i,N}}{J^2_{i,N}}\right) J_{i,N} \mathrm{d}x \\ &\quad= -\sum_{s,l} \int_Q D_k(a_{sl} J_{i,N} D_l w) D_s \left(\frac{\eta}{J_{i,N}}\right) \mathrm{d}x - \sum_{s,l} \int_Q D_s(a_{sl} J_{i,N} D_l w) \frac{\eta D_k J_{i,N}}{J^2_{i,N}} \mathrm{d}x \\ &\quad= -\sum_{s,l} \int_Q D_k(a_{sl} J_{i,N}) D_l w D_s \left(\frac{\eta}{J_{i,N}}\right) \mathrm{d}x - \sum_{s,l} \int_Q a_{sl} D_k D_l w D_s \left(\frac{\eta}{J_{i,N}}\right) J_{i,N} \mathrm{d}x \\ &\qquad- \sum_{s,l} \int_Q D_s(a_{sl} J_{i,N}) D_l w \frac{\eta D_k J_{i,N}}{J^2_{i,N}} \mathrm{d}x - \sum_{s,l} \int_Q a_{sl} D_s D_l w \frac{\eta D_k J_{i,N}}{J_{i,N}} \mathrm{d}x \\ &\quad= -\int_Q \langle D_k(AJ_{i,N}) Dw, D\left(\frac{\eta}{J_{i,N}}\right)\rangle \mathrm{d}x - \int_Q \langle AD(D_k w), D\left(\frac{\eta}{J_{i,N}}\right)\rangle J_{i,N} \mathrm{d}x \\ &\qquad- \int_Q \langle \mathrm{div}(AJ_{i,N}), Dw\rangle \frac{\eta D_k J_{i,N}}{J^2_{i,N}} \mathrm{d}x - \int_Q A : D^2 w \frac{\eta D_k J_{i,N}}{J_{i,N}} \mathrm{d}x \end{aligned} \tag{5.21}$$

Hence, by (5.20), (5.21), and by a density argument, we obtain that

$$\int_Q \langle AD(D_k w), D\left(\frac{\eta}{J_{i,N}}\right)\rangle J_{i,N} \mathrm{d}x$$

$$
\begin{aligned}
&= -\int_Q \langle D_k(AJ_{i,N})Dw, D\left(\frac{\eta}{J_{i,N}}\right)\rangle \mathrm{d}x - \int_Q A : D^2 w \frac{\eta D_k J_{i,N}}{J_{i,N}} \mathrm{d}x \\
&- \int_Q \langle \operatorname{div}(AJ_{i,N}), Dw\rangle \frac{\eta D_k J_{i,N}}{J_{i,N}^2} - \varepsilon \int_Q w D_k\eta \left(|B_{i,N}|^2 - \frac{1}{p}H_{i,N}^2\right)\mathrm{d}x \\
&+ \int_Q \left[\operatorname{div}_{\Gamma_{i,N}}(D\psi(\nu_{i,N})) + W(Eu_{i,N})\right] D_k\eta \mathrm{d}x + \int_Q \frac{1}{J_{i-1,N}}\frac{\partial h_N}{\partial t} D_k\eta \mathrm{d}x,
\end{aligned}
$$

for every $\eta \in H^1_\#(Q)$. Therefore, by choosing $\eta = D_k w J_{i,N}$ we obtain

$$
\begin{aligned}
&\int_Q \langle AD(D_k w), D(D_k w)\rangle J_{i,N}\mathrm{d}x \\
&= -\int_Q \langle D_k(AJ_{i,N})Dw, D(D_k w)\rangle \mathrm{d}x - \int_Q A : D^2 w D_k w D_k J_{i,N}\mathrm{d}x \\
&- \int_Q \langle \operatorname{div}(AJ_{i,N}), Dw\rangle \frac{D_k w D_k J_{i,N}}{J_{i,N}} - \varepsilon \int_Q w D_k(D_k w J_{i,N}) \left(|B_{i,N}|^2 - \frac{1}{p}H_{i,N}^2\right)\mathrm{d}x \\
&+ \int_Q \left[\operatorname{div}_{\Gamma_{i,N}}(D\psi(\nu_{i,N})) + W(Eu_{i,N})\right] D_k(D_k w J_{i,N})\mathrm{d}x + \int_Q \frac{1}{J_{i-1,N}}\frac{\partial h_N}{\partial t} D_k(D_k w J_{i,N})\mathrm{d}x. \quad (5.22)
\end{aligned}
$$

Now, by summing (5.22) for $k = 1, 2$, by using Young's inequality, the ellipticity property of matrix A, and the estimate of $\operatorname{div}(AJ_{i,N})$ in terms of $D^2 h_{i,N}$, we conclude that

$$
\int_Q |D^2 w|^2 \mathrm{d}x \le C \int_Q \left(|Dw|^2 |D^2 h_{i,N}|^2 + |H_{i,N}|^{2(p+1)} + |H_{i,N}|^{2(p-1)}|D^2 h_{i,N}|^4 + \left|\frac{\partial h_N}{\partial t}\right|^2 + 1\right)\mathrm{d}x, \tag{5.23}
$$

where the constant C depends only on Λ_0, $D^2\psi$, and on the $C^{1,\alpha}$ bound of (4.25). By Young's inequality, together to (5.18) and Lemma 7.4, we have

$$
\begin{aligned}
\int_Q |H_{i,N}|^{2p-2}|D^2 h_{i,N}|^4 \mathrm{d}x &\le C\int_Q (|H_{i,N}|^{2p+2} + |D^2 h_{i,N}|^{2p+2})\mathrm{d}x \le C\int_Q |H_{i,N}|^{2p+2}\mathrm{d}x \\
&= C\int_Q |w|^{\frac{2(p+1)}{p-1}}\mathrm{d}x \le C\|D^2 w\|^{\frac{p+2}{p}}_{L^{\frac{p}{p-1}}(Q)} \|w\|^{\frac{p^2+p+2}{p(p-1)}}_{L^{\frac{p}{p-1}}(Q)} \\
&\le \frac{1}{4}\|D^2 w\|^2_{L^{\frac{p}{p-1}}(Q)} + C\|w\|^{\frac{p^2+p+2}{(p-1)(p-2)}}_{L^{\frac{p}{p-1}}(Q)} = \frac{1}{4}\|D^2 w\|^2_{L^{\frac{p}{p-1}}(Q)} + C\|H_{i,N}\|^{\frac{p^2+p+2}{p-2}}_{L^p(Q)} \\
&\le \frac{1}{4}\|D^2 w\|^2_{L^2(Q)} + C, \qquad (5.24)
\end{aligned}
$$

where in the last two inequalities we used (4.14) and

$$
L^2 \hookrightarrow L^{\frac{p}{p-1}} \tag{5.25}
$$

since $\frac{p}{p-1} < 2$, and

$$
\|w\|_{L^{\frac{p}{p-1}}(Q)} = \|H_{i,N}\|^{p-1}_{L^p(Q)}. \tag{5.26}
$$

Moreover, we have

$$
\begin{aligned}
\int_Q |Dw|^2|D^2 h_{i,N}|^2\mathrm{d}x &\le \|D^2 h\|^2_{L^{2(p-1)}(Q)}\|Dw\|^2_{L^{\frac{2(p-1)}{p-2}}(Q)} \le C\|w\|^2_{L^{\frac{2}{p-1}}(Q)}\left(\|D^2 w\|^{\frac{p}{2(p-1)}}_{L^2(Q)}\|w\|^{\frac{p-2}{2(p-1)}}_{L^2(Q)}\right)^2 \\
&= C\|D^2 w\|^{\frac{p}{p-1}}_{L^2(Q)}\|w\|^{\frac{p}{p-1}}_{L^2(Q)} \le C\|D^2 w\|^{\frac{p}{p-1}}_{L^2(Q)}\left(\|D^2 w\|^{\frac{p-2}{2p}}_{L^{\frac{p}{p-1}}(Q)}\|w\|^{\frac{p+2}{2p}}_{L^{\frac{p}{p-1}}(Q)}\right)^{\frac{p}{p-1}} \\
&\le C\|D^2 w\|^{\frac{3p-2}{2(p-1)}}_{L^2(Q)}\|w\|^{\frac{p+2}{2(p-1)}}_{L^{\frac{p}{p-1}}(Q)} \le \frac{1}{4}\|D^2 w\|^2_{L^2(Q)} + C\|w\|^{\frac{2(p+2)}{p-2}}_{L^{\frac{p}{p-1}}(Q)} \\
&= \frac{1}{4}\|D^2 w\|^2_{L^2(Q)} + C\|H_{i,N}\|^{\frac{2(p+2)(p-1)}{p-2}}_{L^p(Q)} \le \frac{1}{4}\|D^2 w\|^2_{L^2(Q)} + C, \qquad (5.27)
\end{aligned}
$$

where in the first inequality we used Holder's inequality, in the second and third ones we used Lemma 7.4, in the fourth and the fifth ones we used (5.25) and (5.26), and in the last one we used (4.14).

Finally, by (5.23), (5.24), and (5.27) we get

$$\int_Q |D^2 w|^2 \mathrm{d}x \leq C \int_Q \left(1 + \left|\frac{\partial h_N}{\partial t}\right|^2\right) \mathrm{d}x,$$

and by integrating with respect to time and by using (4.6) the assertion follows. □

Thanks to the bound (5.15) provided by the previous theorem, in the next lemma we obtain the convergence of powers of the sum of the squares of principal curvatures $\widetilde{H}_N$ defined in (5.14).

Lemma 5.3. *Let $\widetilde{H}_N$ be the function defined in (5.14). Then*

$$|\widetilde{H}_N|^p \xrightarrow[N\to\infty]{} |H|^p \quad \text{in } L^1(0,T_0;L^1(Q)), \tag{5.28}$$

$$|\widetilde{H}_N|^{p-2}\widetilde{H}_N \xrightarrow[N\to\infty]{} |H|^{p-2}H \quad \text{in } L^1(0,T_0;L^2(Q)), \tag{5.29}$$

where

$$H := -\operatorname{div}\left(\frac{Dh}{\sqrt{1+|Dh|^2}}\right)$$

and h is the function provided by Theorem 4.4.

Proof. The proof is analogous of the one in [14, Corollary 3.15] based on [14, Lemma 3.13]. □

Finally, we can prove the short time existence for (1.9).

Proof of Theorem 3.8. Fix $t \in (0, T_0)$ and let $\{i_k\}_k$ and $\{N_k\}_k$ be sequences such that $t_k := i_k \tau_{N_k} \to t$ for $k \to +\infty$. Now, by summing (5.1) for $i = 1, ..., i_k$ we obtain

$$\begin{aligned}
&\int_0^{t_k}\int_Q \tilde{W}_{N_k}\varphi \mathrm{d}x\mathrm{d}t + \int_0^{t_k}\int_Q \langle D\psi(-D\tilde{h}_{N_k},1),(-D\varphi,0)\rangle \mathrm{d}x\mathrm{d}t \\
&\quad - \varepsilon\int_0^{t_k}\int_Q |\widetilde{H}_{N_k}|^{p-2}\widetilde{H}_{N_k}\left[\Delta\varphi - \frac{\langle D^2\varphi D\tilde{h}_{N_k}, D\tilde{h}_{N_k}\rangle}{J_{i,N}^2} - \frac{\Delta\tilde{h}_{N_k}\langle D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2}\right]\mathrm{d}x\mathrm{d}t \\
&\quad - \varepsilon\int_0^{t_k}\int_Q |\widetilde{H}_{N_k}|^{p-2}\widetilde{H}_{N_k}\left[3\frac{\langle D\tilde{h}_{N_k}, D\varphi\rangle\langle D^2\tilde{h}_{N_k}D\tilde{h}_{N_k}, D\tilde{h}_{N_k}\rangle}{\tilde{J}_{N_k}^4} - 2\frac{\langle D^2\tilde{h}_{N_k}D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2}\right]\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\varepsilon}{p}\int_0^{t_k}\int_Q |\widetilde{H}_{N_k}|^p\frac{\langle D\tilde{h}_{N_k}, D\varphi\rangle}{J_{i,N}}\mathrm{d}x\mathrm{d}t + \int_0^{t_k}\int_Q \frac{1}{\tilde{J}_{N_k}(\cdot,\cdot-\tau_{N_k})}\frac{\partial h_{N_k}}{\partial t}\varphi\mathrm{d}x\mathrm{d}t = 0,
\end{aligned} \tag{5.30}$$

where h_{N_k}, $\tilde{h}_{N_k}$, and $\widetilde{H}_{N_k}$ are defined in (3.8), (3.9), and (5.14), and we define

$$\begin{aligned}
&\tilde{W}_{N_k}(x,t) := W(Eu_{i,N_k}(x, h_{i,N_k}(x))) && \text{for every } (x,t) \in \mathbb{R}^2 \times [(i-1)\tau_{N_k}, i\tau_{N_k}) \\
&\tilde{J}_{N_k}(x,t) := \sqrt{1+|D\tilde{h}_{N_k}|^2}, && \text{for every } (x,t) \in \mathbb{R}^2 \times [0,T_0].
\end{aligned}$$

Now, we claim that

$$\frac{1}{\tilde{J}_{N_k}(\cdot,\cdot-\tau_{N_k})}\frac{\partial h_{N_k}}{\partial t} \xrightarrow[k\to\infty]{} \frac{1}{J}\frac{\partial h}{\partial t} \quad \text{in } L^2(0,T_0;L^2(Q)). \tag{5.31}$$

To prove the claim, it is enough to notice that

$$\begin{aligned}
&\left|\int_0^{T_0}\int_Q \left(\frac{1}{\tilde{J}_{N_k}(\cdot,\cdot-\tau_{N_k})}\frac{\partial h_{N_k}}{\partial t} - \frac{1}{J}\frac{\partial h}{\partial t}\right)\eta \mathrm{d}x\mathrm{d}t\right| \\
&\leq \left|\int_0^{T_0}\int_Q \left(\frac{1}{\tilde{J}_{N_k}(\cdot,\cdot-\tau_{N_k})} - \frac{1}{J}\right)\frac{\partial h_{N_k}}{\partial t}\eta \mathrm{d}x\mathrm{d}t\right| + \left|\int_0^{T_0}\int_Q \left(\frac{\partial h_{N_k}}{\partial t} - \frac{\partial h}{\partial t}\right)\frac{\eta}{J}\mathrm{d}x\mathrm{d}t\right|
\end{aligned}$$

$$\leq \left\| \frac{\partial h_{N_k}}{\partial t} \right\|_{L^2(0,T_0,L^2(Q))} \left\| \frac{\eta}{\tilde{J}_{N_k}(\cdot, \cdot - \tau_{N_k})} - \frac{\eta}{J} \right\|_{L^2(0,T_0,L^2(Q))} + \left| \int_0^{T_0} \int_Q \left(\frac{\partial h_{N_k}}{\partial t} - \frac{\partial h}{\partial t} \right) \frac{\eta}{J} \mathrm{d}x\mathrm{d}t \right|,$$

for every $\eta \in L^2(0, T_0, L^2(Q))$, and hence, from (4.6), (4.9), and (4.16) it follows (5.31).

Moreover, in view of Theorems 4.3 and 4.4 we get

$$\int_0^{t_k} \int_Q \tilde{W}_{N_k} \varphi \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q W(E(u(x, h(x, s), s))\varphi \mathrm{d}x\mathrm{d}s, \tag{5.32}$$

$$\int_0^{t_k} \int_Q \langle D\psi(-D\tilde{h}_{N_k}, 1), (-D\varphi, 0)\rangle \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q \langle D\psi(-Dh, 1), (-D\varphi, 0)\rangle \mathrm{d}x\mathrm{d}s, \tag{5.33}$$

and by using again Theorem 4.3, (5.28), and (5.29), we obtain

$$\int_0^{t_k} \int_Q |\widetilde{H}_{N_k}|^p \frac{\langle D\tilde{h}_{N_k}, D\varphi\rangle}{J_{i,N}} \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q |H|^p \frac{\langle Dh, D\varphi\rangle}{J} \mathrm{d}x\mathrm{d}s, \tag{5.34}$$

$$\int_0^{t_k} \int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \left[\Delta\varphi - \frac{\langle D^2\varphi D\tilde{h}_{N_k}, D\tilde{h}_{N_k}\rangle}{J_{i,N}^2} \right] \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q |H|^{p-2} H \left[\Delta\varphi - \frac{\langle D^2\varphi Dh, Dh\rangle}{J^2} \right] \mathrm{d}x\mathrm{d}s. \tag{5.35}$$

In order to establish the convergence of the other terms, we firstly observe that by (4.8) and (4.16) we have $\Delta\tilde{h}_{N_k}(\cdot, t) \rightharpoonup \Delta h(\cdot, t)$ for every $t \in (0, T_0)$. Moreover, by (5.29) we have for a.e. $t \in (0, T_0)$ we have that $(|\widetilde{H}_{N_k}|^{p-2}\widetilde{H}_{N_k})(\cdot, t) \to (|H|^{p-2}H)(\cdot, t)$ in $L^2(Q)$, and hence

$$\int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \frac{\Delta\tilde{h}_{N_k} \langle D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2} \mathrm{d}x \xrightarrow[k\to\infty]{} \int_Q |H|^{p-2} H \frac{\Delta h \langle Dh, D\varphi\rangle}{J^2} \mathrm{d}x. \tag{5.36}$$

Since by (3.8), (4.8), and (5.29) we have

$$\left| \int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \frac{\Delta\tilde{h}_{N_k} \langle D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2} \mathrm{d}x \right| \leq C \|\Delta\tilde{h}_{N_k}\|_{L^2(Q)} \||\widetilde{H}_{N_k}|^{p-2}\widetilde{H}_{N_k}\|_{L^2(Q)} \leq C \||\widetilde{H}_{N_k}|^{p-2}\widetilde{H}_{N_k}\|_{L^2(Q)} \xrightarrow[k\to\infty]{L^1(0,T_0)} \||H|^{p-2}H\|_{L^2(Q)}, \tag{5.37}$$

the generalized Lebesgue dominated convergence theorem, together with (5.37) and (5.36), implies that

$$\int_0^{t_k} \int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \frac{\Delta\tilde{h}_{N_k} \langle D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2} \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q |H|^{p-2} H \frac{\Delta h \langle Dh, D\varphi\rangle}{J^2} \mathrm{d}x\mathrm{d}s. \tag{5.38}$$

For the remaining terms, by arguing in analogously we obtain

$$\int_0^{t_k} \int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \frac{\langle D^2\tilde{h}_{N_k} D\tilde{h}_{N_k}, D\varphi\rangle}{\tilde{J}_{N_k}^2} \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q |H|^{p-2} H \frac{\langle D^2 h Dh, D\varphi\rangle}{J^2} \mathrm{d}x\mathrm{d}s, \tag{5.39}$$

and

$$\int_0^{t_k} \int_Q |\widetilde{H}_{N_k}|^{p-2} \widetilde{H}_{N_k} \frac{\langle D\tilde{h}_{N_k}, D\varphi\rangle \langle D^2\tilde{h}_{N_k} D\tilde{h}_{N_k}, D\tilde{h}_{N_k}\rangle}{\tilde{J}_{N_k}^4} \mathrm{d}x\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^t \int_Q |H|^{p-2} H \frac{\langle Dh, D\varphi\rangle \langle D^2 h Dh, Dh\rangle}{J^4} \mathrm{d}x\mathrm{d}s. \tag{5.40}$$

Finally, by (5.31), (5.32), (5.33), (5.34), (5.35), (5.38), (5.39), and (5.40), we pass to the limit in (5.30), obtaining that the limiting function h satisfies

$$\int_0^t \int_Q \frac{1}{J}\frac{\partial h}{\partial t}\varphi \mathrm{d}x\mathrm{d}s = -\int_0^t \int_Q W(E(u(x,h(x,s),s))\varphi \mathrm{d}x\mathrm{d}s - \int_0^t \int_Q \langle D\psi(-Dh,1),(-D\varphi,0)\rangle \mathrm{d}x\mathrm{d}s$$
$$+\varepsilon \int_0^t \int_Q |H|^{p-2}H\left[\Delta\varphi - \frac{\langle D^2\varphi Dh, Dh\rangle}{J^2} - \frac{\Delta h\langle Dh, D\varphi\rangle}{J^2} - 2\frac{\langle D^2hDh, D\varphi\rangle}{J^2}\right]\mathrm{d}x\mathrm{d}s$$
$$+\varepsilon \int_0^t \int_Q \left[3|H|^{p-2}H\frac{\langle Dh, D\varphi\rangle\langle D^2hDh, Dh\rangle}{J^4} - \frac{1}{p}|H|^p\frac{\langle Dh, D\varphi\rangle}{J}\right]\mathrm{d}x\mathrm{d}s. \tag{5.41}$$

Now, by letting φ vary in a countable dense subset of $H^1_\#(Q)$ and by differentiating (5.41) with respect to t, we obtain

$$-\int_Q W(E(u(x,h(x,t),t))\varphi \mathrm{d}x - \int_Q \langle D\psi(-Dh,1),(-D\varphi,0)\rangle \mathrm{d}x - \frac{\varepsilon}{p}\int_Q |H|^p\frac{\langle Dh, D\varphi\rangle}{J}\mathrm{d}x$$
$$+\varepsilon \int_Q |H|^{p-2}H\left[\Delta\varphi - \frac{\langle D^2\varphi Dh, Dh\rangle}{J^2} - \frac{\Delta h\langle Dh, D\varphi\rangle}{J^2} - 2\frac{\langle D^2hDh, D\varphi\rangle}{J^2}\right]\mathrm{d}x$$
$$+3\varepsilon \int_Q |H|^{p-2}H\frac{\langle Dh, D\varphi\rangle\langle D^2hDh, Dh\rangle}{J^4}\mathrm{d}x = \int_Q \frac{1}{J}\frac{\partial h}{\partial t}\varphi \mathrm{d}x. \tag{5.42}$$

Since by (5.15) $|H|^{p-2}H \in L^2(0,T_0;H^2_\#(Q))$, by arguing as in Theorem 5.2 we have that (5.42) is equivalent to

$$-\varepsilon\int_{\Gamma_h} D_{\Gamma_h}(|H|^{p-2}H)D_{\Gamma_h}\phi \mathrm{d}\mathcal{H}^2 + \varepsilon\int_{\Gamma_h} |H|^{p-2}H\left(|B|^2 - \frac{1}{p}H^2\right)\phi \mathrm{d}\mathcal{H}^2$$
$$-\int_{\Gamma_h}\left[\mathrm{div}_{\Gamma_h}(D\psi(\nu)) + W(Eu)\right]\phi \mathrm{d}\mathcal{H}^2 = \int_{\Gamma_h} V\phi \mathrm{d}\mathcal{H}^2$$

for every $t \in (0,T_0)$, with $\phi := \frac{\varphi}{J}$.

Now, we want to show that the energy decreases during the evolution. Thanks to (4.7) we can say that the function $t \mapsto \mathcal{F}(\tilde{h}_N(\cdot,t), \tilde{u}_N(\cdot,t))$ is nonincreasing, where $\tilde{h}_N(\cdot,t)$ and $\tilde{u}_N(\cdot,t)$ are defined in (3.9) and (3.10). Moreover, by using (5.28) we have, up to a subsequence, that for a.e. $t \in (0,T_0)$ it holds $\widetilde{H}_N(\cdot,t) \to H(\cdot,t)$ in $L^p(Q)$. Thanks to this, we can use (4.17) and (4.27) to get $\mathcal{F}(\tilde{h}_N(\cdot,t), \tilde{u}_N(\cdot,t)) \to \mathcal{F}(h(\cdot,t), u(\cdot,t))$ as $N \to \infty$ for all such t, which implies (3.12). Now, to prove (3.13) we consider $t \in Z_0$ and $t_N \to t+$ with $t_N \notin Z_0$ as $N \to \infty$ for every $N \in \mathbb{N}$. Since $h(\cdot,t_N) \rightharpoonup h(\cdot,t)$ in $W^{2,p}_\#(Q)$ by (4.8), by using the lower semicontinuity of $\mathcal{F}$ we obtain

$$\mathcal{F}(h(\cdot,t), u(\cdot,t)) \leq \liminf_{N\to\infty} \mathcal{F}(h(\cdot,t_N), u(\cdot,t_N)) = \lim_{N\to\infty} g(t_N) = g(t+),$$

which concludes the proof. □

6. Stability of flat configurations

In this section, inspired by [14, Section 4], we prove the Lyapunov stability and the asymptotic stability of any admissible flat configuration. We prove Lyapunov stability both in convex and nonconvex case, and the asymptotic stability only in convex case. We begin with the following proposition:

Proposition 6.1. *Let h be a variational solution to (1.9) in the sense of Definition 3.6. Then, there exist a nonincreasing function g and a negligible set Z_0 such that*

$$\mathcal{F}(h(\cdot,t), u_h(\cdot,t)) = g(t) \quad \textit{for every } t \in [0,T_0]\setminus Z_0$$

and

$$\mathcal{F}(h(\cdot,t), u_h(\cdot,t)) \leq g(t) \quad \textit{for every } t \in Z_0.$$

Proof. Since h is a variational solution to (1.9) in the sense of Definition 3.6, there exists a dicrete-time evolution $\{h_N\}$ such that (3.11) holds. Thanks to this, we can repeat the arguments in Theorems 5.2, in Lemma 5.3, and in the final part of the proof of Theorem 3.8 to conclude the proof. □

Now, we show that, if a variational solution exists for every times, then there exists a sequence of times $\{t_n\} \subset (0,+\infty)$ such that $t_n \to +\infty$ and $h(\cdot,t_n)$ converges to a critical profile, which will be used to prove (ii) of Theorem 3.9.

Proposition 6.2. *Let $h_0 \in W^{2,p}_\#(Q)$ be an initial datum such that there exists a global-in-time variational solution h (in the sense of Definition (3.6)). Then, there exist a sequence $\{t_n\}_n \subset (0,+\infty)\setminus Z_0$ and a critical profile $\bar h$ of $\mathcal{F}$ such that $t_n \to +\infty$ and $h(\cdot,t_n) \to \bar h$ in $W^{2,p}_\#(Q)$.*

Proof. By (4.6) we have that

$$\int_0^{+\infty} \left\| \frac{\partial h}{\partial t} \right\|^2_{L^2(Q)} \mathrm{d}t \le C\mathcal{F}(h_0,u_0),$$

and hence, there exists $\{t_n\}_n \subset (0,+\infty)\setminus Z_0$ with $t_n \to +\infty$ such that

$$\left\| \frac{\partial h}{\partial t}(\cdot,t_n) \right\|_{L^2(Q)} \xrightarrow[n\to+\infty]{} 0.$$

Since $h \in L^\infty(0,+\infty;W^{2,p}_\#(Q)) \cap H^1(0,+\infty;L^2(Q))$, by setting $h_n := h(\cdot,t_n)$ there exists $\bar h \in W^{2,p}_\#(Q)$ such that, up to extracting a non-relabelled subsequence, we obtain that

$$h_n \xrightharpoonup[n\to+\infty]{} \bar h \quad \text{in } W^{2,p}_\#(Q).$$

Furthermore, from Lemma 7.1 it follows that

$$u_{h_n}(\cdot,h_n(\cdot)) \xrightarrow[n\to+\infty]{} u_{\bar h}(\cdot,\bar h(\cdot)) \quad \text{in } C^{1,\alpha}_\#(Q),$$

where u_{h_n} and $u_{\bar h}$ are the elastic equilibria with respect to h_n and $\bar h$, respectively. We now notice that the equation satisfied by h_n as in (5.1) is

$$\begin{aligned}
&\int_Q W(E(u_{h^n}(x,h^n(x))))\varphi \mathrm{d}x + \int_Q \langle D\psi(-Dh^n,1),(-D\varphi,0)\rangle \mathrm{d}x + \frac{\varepsilon}{p}\int_Q |H_n|^p \frac{\langle Dh^n, D\varphi\rangle}{J_n}\mathrm{d}x \\
&-\varepsilon \int_Q |H_n|^{p-2}H_n \left[\Delta\varphi - \frac{\langle D^2\varphi Dh^n, Dh^n\rangle}{J_n^2} - \frac{\Delta h^n \langle Dh^n, D\varphi\rangle}{J_n^2} - 2\frac{\langle D^2h^n Dh^n, D\varphi\rangle}{J_n^2}\right]\mathrm{d}x \\
&-3\varepsilon \int_Q |H_n|^{p-2}H_n \frac{\langle Dh^n, D\varphi\rangle\langle D^2h^n Dh^n, Dh^n\rangle}{J_n^4}\mathrm{d}x + \int_Q \frac{1}{J_n}\frac{\partial h}{\partial t}(\cdot,t_n)\varphi \mathrm{d}x = 0
\end{aligned} \tag{6.1}$$

for every $\varphi \in C^2_\#(Q)$, with $J_n = \sqrt{1+|Dh_n|^2}$ and $H_n = -\operatorname{div}\left(\frac{Dh_n}{J_n}\right)$.

By repeating the same arguments of Theorem 5.2 we obtain that

$$\int_Q |D^2(|H_n|^{p-2}H_n)|^2 \mathrm{d}x \le \int_Q \left(1 + \left|\frac{\partial h}{\partial t}(\cdot,t_n)\right|^2\right)\mathrm{d}x,$$

and hence, by arguing as in the proof of Lemma 5.3 we have

$$\begin{aligned}
|H_n|^{p-2}H_n &\xrightharpoonup[n\to+\infty]{} |\bar H|^{p-2}\bar H && \text{in } H^2_\#(Q),\\
|H_n|^{p-2}H_n &\xrightarrow[n\to+\infty]{} |\bar H|^{p-2}\bar H && \text{in } H^1_\#(Q),\\
|H_n|^p &\xrightarrow[n\to+\infty]{} |\bar H|^p && \text{in } L^1(Q),
\end{aligned}$$

where

$$\bar H = -\operatorname{div}\left(\frac{D\bar h}{\sqrt{1+|D\bar h|^2}}\right).$$

Finally, in view of such convergences, by passing to the limit in (6.1) in the same way as done in the proof of Theorem 3.8, we prove that $\bar h$ is a critical profile. □

The proof of Theorem 3.9 directly follows from the arguments already used in [14, Section 4], as the functional $\mathcal{F}$ is the same used in [14].

Proof of Theorem 3.9. The proof of assertion ((i)) is exactly the same as the proof of [14, Theorem 4.7]. The proof of assertion ((ii)) instead follows by replacing [14, Proposition 4.11] with Proposition 6.2 in the proof of [14, Theorem 4.14]. □

7. Appendix

For the proof of the following lemmas see [14].

Lemma 7.1. *Let $M < 0$ and $c_0 > 0$. Let $h_1, h_2 \in C^{1,\alpha}_{\#}(Q)$ for some $\alpha \in (0,1)$, with $\|h_i\|_{C^{1,\alpha}_{\#}(Q)} \leq M$ and $h_i \geq c_0$ for $i = 1, 2$, and let u_1 and u_2 be the corresponding elastic equilibria of h_1 and of h_2, respectively. Then*

$$\|E(u_1(\cdot, h_1(\cdot))) - E(u_2(\cdot, h_2(\cdot)))\|_{C^{1,\alpha}_{\#}(Q)} \leq C\|h_1 - h_2\|_{C^{1,\alpha}_{\#}(Q)},$$

for some constant $C > 0$ depending only on M, c_0, and α.

Lemma 7.2. *Let $p > 2$, $u \in L^{\frac{p}{p-1}}(Q)$ such that*

$$\int_Q uA : D^2\varphi \mathrm{d}x + \int_Q \langle b, D\varphi\rangle \mathrm{d}x + \int_Q c\varphi \mathrm{d}x = 0$$

for every $\varphi \in C^\infty_{\#}(Q)$ with $\int_Q \varphi \mathrm{d}x = 0$, where $A \in W^{1,p}_{\#}(Q; \mathbb{R}^{2\times 2}_{sym})$ satisfies standard uniform elliptic condition (7.2), $b \in L^1(Q; \mathbb{R}^2)$ and $c \in L^1(Q)$. Then $u \in L^q(Q)$ for every $q \in (1, 2)$. Moreover, if $b, u \operatorname{div} A \in L^r(Q; \mathbb{R}^2)$ and $c \in L^r(Q)$ for some $r > 1$, then $u \in W^{1,r}_{\#}(Q)$.

In the next lemma we denote by Lu an elliptic operator of the form

$$Lu := \sum_{i,j} a_{ij}(x)D_{ij}u + \sum_i b_i(x)D_iu \tag{7.1}$$

where all the coefficients are Q-periodic functions, the a_{ij} are continuous, and the b_i are bounded. Moreover, there exist $\lambda, \Lambda > 0$ such that

$$\Lambda|\xi|^2 \geq \sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{7.2}$$

for every $\xi \in \mathbb{R}^2$, $\sum_i |b_i| \leq \Lambda$.

Lemma 7.3. *Let $p \geq 2$. Then, there exists $C > 0$ such that for all $u \in W^{2,p}_{\#}(Q)$ we have*

$$\|D^2u\|_{L^p(Q)} \leq C\|Lu\|_{L^p(Q)},$$

where L is the differential operator defined in (7.1). The constant C depends only on p, λ, Λ, and the moduli of continuity of the coefficients a_{ij}.

Lemma 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let s, j , and m be integers such that $0 \leq s \leq j \leq m$. Let $1 \leq p \leq q < \infty$ if $(m-j)p \geq n$, and let $1 \leq p \leq q \leq \infty$ if $(m-j)p > n$. Then, there exists $C > 0$ such that*

$$\|D^jf\|_{L^q(Q)} \leq C\left(\|D^mf\|^\theta_{L^p(Q)}\|D^sf\|^{1-\theta}_{L^p(Q)} + \|D^sf\|_{L^p(Q)}\right) \tag{7.3}$$

for every $f \in W^{m,p}(\Omega)$, where

$$\theta := \frac{1}{m-s}\left(\frac{n}{p} - \frac{n}{q} + j - s\right).$$

Moreover, if Ω is a cube, $f \in W^{m,p}_{\#}(Q)$ and, if either f vanishes at the boundary or $\int_\Omega f\mathrm{d}x = 0$, then (7.3) holds in the stronger form

$$\|D^jf\|_{L^q(Q)} \leq C\|D^mf\|^\theta_{L^p(Q)}\|D^sf\|^{1-\theta}_{L^p(Q)}.$$

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